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# Stacks of ramified Galois covers

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# 1 Introduction.

Let $G$ be a flat, finite group scheme finitely presented over a base scheme $S$. In this thesis we study $G$-Galois covers of very general schemes. A morphism $f\colon X \longrightarrow Y$ is called a *cover* if it is finite, flat and of finite presentation. We define a (ramified) $G$-*cover* as a cover $f\colon X \longrightarrow Y$ with an action of $G$ on $X$ such that $f$ is $G$-invariant and $f_*\mathcal{O}_X$ is fppf-locally isomorphic to the regular representation $\mathcal{O}_Y[G]$ as $\mathcal{O}_Y[G]$-comodule. This definition is a natural one: it generalizes the notion of $G$-torsors and, under suitable hypothesis, coincides with the usual definition of Galois cover when the group $G$ is constant (see for example [Par91, AP12, Eas11]). Moreover, as explained below, in the abelian case $G$-covers are tightly related to the theory of equivariant Hilbert schemes (see for example [Nak01, SP02, HS04, AB05]). We call $G$-Cov the stack of $G$-covers and the aim of this thesis will be to describe its structure.

We denote by $\mu_n$ the diagonalizable group over $\mathbb{Z}$ with character group $\mathbb{Z}/n\mathbb{Z}$. In many concrete problems, one is interested in a more direct and concrete description of a $G$-cover $f\colon X \longrightarrow Y$. This is very simple and well known when $G = \mu_2$: such a cover $f$ is given by an invertible sheaf $\mathcal{L}$ on $Y$ with a section of $\mathcal{L}^{\otimes 2}$. Similarly, when $G = \mu_3$, a $\mu_3$-cover $f$ is given by a pair $(\mathcal{L}_1, \mathcal{L}_2)$ of invertible sheaves on $Y$ with maps $\mathcal{L}_1^{\otimes 2} \longrightarrow \mathcal{L}_2$ and $\mathcal{L}_2^{\otimes 2} \longrightarrow \mathcal{L}_1$ (see [AV04, § 6]).

In general, however, there is no comparable description of $G$-covers. Very little is known when $G$ is not abelian, except for $G = S_3$ (see [Eas11]) and the case of Galois covers with groups $G = D_n, A_4, S_4$ having regular total space (see [Tok94, Tok02, Rei99]). In the non-Galois case, there exist a general description of covers of degree 3 and 4 (see [Mir85, Par89, HM99]) and of Gorenstein covers of degree $d \geq 3$ (see [CE96, Cas96]).

Even in the abelian case, the situation becomes complicated very quickly when the order of $G$ grows. The paper that inspires our work is [Par91]; here the author describes $G$-covers $X \longrightarrow Y$ when $G$ is an abelian group, $Y$ is a smooth variety over an algebraically closed field of characteristic prime to $|G|$ and $X$ is normal, in terms of certain invertible sheaves on $Y$, generalizing the description given above for $G = \mu_2$ and $G = \mu_3$.

We now outline the content of this thesis remarking the results obtained and we will follow the division in chapters.

## 1.1 Preliminaries on Galois covers.

Chapter 2 is dedicated to explaining the basic properties of $G$-covers. Our first result is:

**Theorem.** *[2.1.3, 2.1.8] The stack $G$-Cov is algebraic and finitely presented over $S$. Moreover $\mathrm{B}\,G$, the stack of $G$-torsors, is an open substack of $G$-Cov.*

As particular cases, we will study $G$-covers for the groups $G = \mu_2, \mu_3, \alpha_p$, where $p$ is a prime and $\alpha_p$ is the kernel of the Frobenius map $\mathbb{G}_a \longrightarrow \mathbb{G}_a$ over $\mathbb{F}_p$. Example 2.1.12, suggested by Prof. Romagny, shows that covers that are generically $\alpha_p$-torsors are not $\alpha_p$-cover in general. Moreover, we will prove an unexpected result, that is that $\mathrm{B}\,\alpha_p = \alpha_p$-Cov or, in other words, that every $\alpha_p$-cover is an $\alpha_p$-torsor (see 2.1.11). As we will see, the situation is completely different for Galois covers of linearly reductive groups (introduced below), even in the diagonalizable case. For instance we will show that $G$-Cov is almost never irreducible. This motivates the following definition. The *main irreducible component of* $G$-Cov*,* denoted by $\mathcal{Z}_G$, is the schematic closure of $\mathrm{B}\,G$ in $G$-Cov. Notice that, if $S$ is irreducible, then $\mathrm{B}\,G$ is irreducible as well and therefore $\mathcal{Z}_G$ is an irreducible component of $G$-Cov.

In the last part of the chapter we study examples of isomorphisms $G$-Cov $\simeq H$-Cov using the notion of bitorsor: given finite, flat and finitely presented group schemes $G$ and $H$ over a scheme $S$ a $(G,H)$-bitorsor is an $S$-scheme which is simultaneously a left $G$-torsor and a right $H$-torsor and such that the actions are compatible. Notice that the existence of a $(G,H)$-bitorsor implies that $G$ and $H$ are locally isomorphic, but the converse is false. The $(G,H)$-bitorsors correspond to isomorphisms $\mathrm{B}\,G \simeq \mathrm{B}\,H$ (see [Gir71, Chapter III, Remarque 1.6.7]). We will give a proof of this fact and we will also show that they induce isomorphisms $G$-Cov $\simeq H$-Cov (see 2.3.7 and 2.3.12). Moreover if the $G$-cover $X \longrightarrow Y$ is sent to the $H$-cover $X' \longrightarrow Y$ through one of these isomorphisms, then $X$ and $X'$ are fppf locally isomorphic over $S$, étale locally if $G$ or $H$ is étale and therefore they share many geometric properties, like reduceness, smoothness, geometrical connectedness and irreducibility and, in the étale case, regularity and regularity in codimension 1. We will use this construction when we will study $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$-covers and $S_3$-covers in the last chapter.

## 1.2 Galois covers under diagonalizable group schemes.

Chapter 3 of this thesis essentially coincides with the article [Ton13]. In it, we concentrate on the case when $G$ is a finite diagonalizable group scheme over $\mathbb{Z}$; thus, $G$ is isomorphic to a finite direct product of group schemes of the form $\mu_d$ for $d \geq 1$. We consider the dual finite abelian group $M = \mathrm{Hom}(G, \mathbb{G}_m)$ so that, by standard duality results (see [GD70]), $G$ is the fppf sheaf of homomorphisms $M \longrightarrow \mathbb{G}_m$ and a decomposition of $M$ into a product of cyclic groups yields the decomposition of $G$ into a product of $\mu_d$'s. Although Chapter 4 study Galois covers for general groups, we have decided to consider the diagonalizable case first, because in this case $G$-covers have a very explicit description in terms of sequences of invertible sheaves. Indeed a $G$-cover over $Y$ is of the form $X = \mathrm{Spec}\,\mathscr{A}$ where $\mathscr{A}$ is a quasi-coherent sheaf of algebras over $Y$ with a decomposition

$$\mathscr{A} = \bigoplus_{m \in M} \mathcal{L}_m \text{ s.t. } \mathcal{L}_0 = \mathcal{O}_Y,\ \mathcal{L}_m \text{ invertible and } \mathcal{L}_m\mathcal{L}_n \subseteq \mathcal{L}_{m+n} \text{ for all } m,n \in M \tag{1.2.1}$$

So a $G$-cover corresponds to a sequence of invertible sheaves $(\mathcal{L}_m)_{m\in M}$ with maps $\psi_{m,n}\colon \mathcal{L}_m\otimes\mathcal{L}_n \longrightarrow \mathcal{L}_{m+n}$ satisfying certain rules and our principal aim will be to simplify the data necessary to describe such covers. For instance $G$-torsors correspond to sequences where all the maps $\psi_{m,n}$ are isomorphisms. Therefore, if $G=\mu_l$, a $G$-torsor is simply given by an invertible sheaf $\mathcal{L}=\mathcal{L}_1$ and an isomorphism $\mathcal{L}^{\otimes l}\simeq\mathcal{O}$.

When $G=\mu_2$ or $G=\mu_3$ the description given above shows that the stack $G$-Cov is smooth, irreducible and very easy to describe. In the general case its structure turns out to be extremely intricate. For instance, as we will see, $G$-Cov is almost never irreducible. The existence of the 'special' irreducible component $\mathcal{Z}_G$ parallels what happens in the theory of $M$-equivariant Hilbert schemes (see [HS04, Remark 5.1]). It turns out that this theory and the theory of $G$-covers are deeply connected: given an action of $G$ on $\mathbb{A}^r$, induced by elements $\underline{m}=m_1,\dots,m_r\in M$, the equivariant Hilbert scheme $M$-Hilb $\mathbb{A}^r$, that we will denote by $M\text{-Hilb}^{\underline{m}}$ to underline the dependence on the sequence $\underline{m}$, can be viewed as the functor whose objects are $G$-covers with an equivariant closed immersion in $\mathbb{A}^r$. The forgetful map $\vartheta\colon M\text{-Hilb}^{\underline{m}}\longrightarrow G\text{-Cov}$ is smooth with geometrically irreducible fibers onto an open substack $U_{\underline{m}}$ of $G$-Cov. Moreover it is surjective, that is an atlas, provided that $\underline{m}$ contains all the elements in $M-\{0\}$ (3.2.8). This means that $U_{\underline{m}}$ and $M\text{-Hilb}^{\underline{m}}$ share several geometric properties, like connectedness, irreducibility, smoothness or reduceness. Moreover $\vartheta^{-1}(\mathcal{Z}_G)$ coincides with the main irreducible component of $M\text{-Hilb}^{\underline{m}}$, first studied by Nakamura in [Nak01].

We will prove the following results on the structure of $G$-Cov.

**Theorem.** *[3.2.14,3.2.18,3.2.19,3.2.21] When $G$ is a finite diagonalizable group scheme over $\mathbb{Z}$, the stack $G$-Cov is*

- *flat and of finite type with geometrically connected fibers,*
- *smooth if and only if $G\simeq 0,\mu_2,\mu_3,\mu_2\times\mu_2$,*
- *normal if $G\simeq\mu_4$,*
- *reducible if $|G|\geq 8$ and $G\not\simeq(\mu_2)^3$.*

*The above properties continue to hold if we replace $G$-Cov by $M\text{-Hilb}^{\underline{m}}$, provided that each nonzero element of $M$ belongs to the sequence $\underline{m}$.*

We do not know whether $G$-Cov is integral for $G\simeq\mu_5,\mu_6,\mu_7,(\mu_2)^3$. So $G$-Cov is usually reducible, its structure is extremely complicated and we have little hope of getting to a real understanding of the components not containing $\mathrm{B}\,G$. Therefore we will focus on the main irreducible component $\mathcal{Z}_G$ of $G$-Cov. The main idea behind the study of $G$-covers when $G$ is diagonalizable, inspired by the results in [Par91], is to try to decompose the multiplications $\psi_{m,n}\in\mathcal{L}_{m+n}\otimes\mathcal{L}_m^{-1}\otimes\mathcal{L}_n^{-1}$ as a tensor product of sections of other invertible sheaves. Following this idea we will construct parametrization maps $\pi_{\underline{\mathcal{E}}}\colon\mathcal{F}_{\underline{\mathcal{E}}}\longrightarrow\mathcal{Z}_G\subseteq G\text{-Cov}$, where $\mathcal{F}_{\underline{\mathcal{E}}}$ are 'nice' stacks, for example smooth and irreducible, whose objects are those decompositions.

This construction can be better understood locally, where a $G$-cover over $Y = \operatorname{Spec} R$ is just $X = \operatorname{Spec} A$, where $A$ is an $R$-algebra with an $R$-basis $\{v_m\}_{m\in M}$, $v_0 = 1$ ($\mathcal{L}_m = \mathcal{O}_Y v_m$), so that the multiplications are elements $\psi_{m,n} \in R$ such that $v_m v_n = \psi_{m,n} v_{m+n}$.

Consider $a \in R$, a collection of natural numbers $\mathcal{E} = (\mathcal{E}_{m,n})_{m,n\in N}$ and set $\psi_{m,n} = a^{\mathcal{E}_{m,n}}$. The condition that the product structure on $A = \oplus_m R v_m$ defined by the $\psi_{m,n}$ yields an associative, commutative $R$-algebra, i.e. makes $\operatorname{Spec} A$ into a $G$-cover over $\operatorname{Spec} R$, translates into some additive relations on the numbers $\mathcal{E}_{m,n}$. Call $K_+^\vee$ the set of collections $\mathcal{E}$ satisfying those relations. More generally given $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in K_+^\vee$ we can define a parametrization

$$R^r \ni (a_1, \dots, a_r) \longrightarrow \psi_{m,n} = a_1^{\mathcal{E}^1_{m,n}} \cdots a_r^{\mathcal{E}^r_{m,n}}$$

This is essentially the local behavior of the map $\pi_{\underline{\mathcal{E}}} \colon \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow G\text{-Cov}$. In the global case the elements $a_i$ will be sections of invertible sheaves.

From this point of view the natural questions are: given a $G$-cover over a scheme $Y$ when does there exist a lift to an object of $\mathcal{F}_{\underline{\mathcal{E}}}(Y)$? Is this lift unique? How can we choose the sequence $\underline{\mathcal{E}}$?

The key point is to give an interpretation to $K_+^\vee$ (that also explains this notation). Consider $\mathbb{Z}^M$ with canonical basis $(e_m)_{m\in M}$ and define $v_{m,n} = e_m + e_n - e_{m+n} \in \mathbb{Z}^M/\langle e_0\rangle$. If $p\colon \mathbb{Z}^M/\langle e_0\rangle \longrightarrow M$ is the map $p(e_m) = m$, the $v_{m,n}$ generate $\operatorname{Ker} p$. Now call $K_+$ the submonoid of $\mathbb{Z}^M/\langle e_0\rangle$ generated by the $v_{m,n}$, $K = \operatorname{Ker} p$ its associated group and also consider the torus $\mathcal{T} = \underline{\operatorname{Hom}}(\mathbb{Z}^M/\langle e_0\rangle, \mathbb{G}_m)$, which acts on $\operatorname{Spec} \mathbb{Z}[K_+]$. By construction we have that a collection of natural numbers $(\mathcal{E}_{m,n})_{m,n\in M}$ belongs to $K_+^\vee$ if and only if the association $v_{m,n} \longrightarrow \mathcal{E}_{m,n}$ defines an additive map $K_+ \longrightarrow \mathbb{N}$. Therefore, as the symbol suggests, we can identify $K_+^\vee$ with $\operatorname{Hom}(K_+, \mathbb{N})$, the dual monoid of $K_+$. Its elements will be called *rays*. More generally a monoid map $\psi\colon K_+ \longrightarrow (R, \cdot)$, where $R$ is a ring, yields a multiplication $\psi_{m,n} = \psi(v_{m,n})$ on $\bigoplus_{m\in M} R v_m$ and therefore we obtain a map $\operatorname{Spec} \mathbb{Z}[K_+] \longrightarrow \mathcal{Z}_G$. We will prove that (see 3.2.6):

**Theorem.** *We have $\mathcal{Z}_G \simeq [\operatorname{Spec} \mathbb{Z}[K_+]/\mathcal{T}]$ and $\mathrm{B}\, G \simeq [\operatorname{Spec} \mathbb{Z}[K]/\mathcal{T}]$.*

Notice that the whole $G$-Cov has a similar description as quotient, but we have to consider non cancellative monoids. We introduce the following notation: given $\alpha \in \mathbb{N}$, we set $0^\alpha = 1$ if $\alpha = 0$ and $0^\alpha = 0$ otherwise. Given $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in K_+^\vee$ we have defined a map $\pi_{\underline{\mathcal{E}}} \colon \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{Z}_G$. Notice that if $\underline{\gamma}$ is a subsequence of $\underline{\mathcal{E}}$ then $\mathcal{F}_{\underline{\gamma}}$ is an open substack of $\mathcal{F}_{\underline{\mathcal{E}}}$ and $(\pi_{\underline{\mathcal{E}}})_{|\mathcal{F}_{\underline{\gamma}}} = \pi_{\underline{\gamma}}$. The lifting problem for the maps $\pi_{\underline{\mathcal{E}}}$ clearly depends on the choice of the sequence $\underline{\mathcal{E}}$. Considering larger $\underline{\mathcal{E}}$ allows us to parametrize more covers, but also makes uniqueness of the lifting unlikely. In this direction we have proved that:

**Theorem.** *[3.1.22] Let $k$ be an algebraically closed field and suppose we have a collection $\underline{\mathcal{E}}$ whose rays generate the rational cone $K_+^\vee\mathbb{Q}$. Then the map of groupoids $\mathcal{F}_{\underline{\mathcal{E}}}(k) \longrightarrow \mathcal{Z}_G(k)$ is essentially surjective. In other words a $G$-cover of* $\operatorname{Spec} k$ *in the main component $\mathcal{Z}_G$ has a multiplication of the form $\psi_{m,n} = 0^{\mathcal{E}_{m,n}}$ for some $\mathcal{E} \in K_+^\vee$.*

On the other hand small sequences $\underline{\mathcal{E}}$ can guarantee uniqueness but not existence. The solution we have found is to consider a particular class of rays, called extremal, that have minimal non empty support. Set $\underline{\eta}$ for the sequence of all extremal rays (that is finite). Notice that extremal rays generate $K_+^\vee\mathbb{Q}$. We prove that:

**Theorem.** *[3.1.47, 3.1.48] The smooth locus $\mathcal{Z}_G^{\mathrm{sm}}$ of $\mathcal{Z}_G$ is of the form $[X_G/\mathcal{T}]$ where $X_G$ is a smooth toric variety over $\mathbb{Z}$ (whose maximal torus is $\operatorname{Spec}\mathbb{Z}[K]$). Moreover $\pi_{\underline{\eta}}\colon \mathcal{F}_{\underline{\eta}} \longrightarrow \mathcal{Z}_G$ induces an isomorphism of stacks*

$$\pi_{\underline{\eta}}^{-1}(\mathcal{Z}_G^{\mathrm{sm}}) \xrightarrow{\simeq} \mathcal{Z}_G^{\mathrm{sm}}$$

Among the extremal rays there are special rays, called smooth, that can be defined as extremal rays $\mathcal{E}$ whose associated multiplication $\psi_{m,n} = 0^{\mathcal{E}_{m,n}}$ yields a cover in $\mathcal{Z}_G^{\mathrm{sm}}$. Set $\underline{\xi}$ for the sequence of smooth extremal rays. It turns out that the theorem above holds if we replace $\underline{\eta}$ with $\underline{\xi}$.

If, given a scheme $X$, we denote by $\underline{\operatorname{Pic}}\, X$ the category whose objects are invertible sheaves on $X$ and whose arrows are arbitrary maps of sheaves, we also have:

**Theorem.** *[3.1.52] Consider a 2-commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathcal{F}_{\underline{\mathcal{E}}} \arrow[d, "\pi_{\underline{\mathcal{E}}}"] \\ Y \arrow[r] \arrow[ur, dashed] & G\text{-Cov} \end{tikzcd}$$

*where $X, Y$ are schemes and $\underline{\mathcal{E}}$ is a sequence of elements of $K_+^\vee$. If $\underline{\operatorname{Pic}}\, Y \xrightarrow{f^*} \underline{\operatorname{Pic}}\, X$ is fully faithful (resp. an equivalence) the dashed lifting is unique (resp. exists and is unique).*

In particular the theorems above allow us to conclude that:

**Theorem.** *[3.1.48, 3.1.53] Let $Y$ be a locally noetherian and locally factorial scheme. A cover $\chi \in G\text{-Cov}(Y)$ such that $\chi_{|k(p)} \in \mathcal{Z}_G^{\mathrm{sm}}(k(p))$ for any $p \in Y$ with $\operatorname{codim}_p Y \leq 1$ lifts uniquely to $\mathcal{F}_{\underline{\xi}}(Y)$.*

An interesting problem is to describe all (smooth) extremal rays. This seems very difficult and it is related to the problem of finding $\mathbb{Q}$-linearly independent sequences among the $v_{m,n} \in K_+$. A natural way of obtaining extremal rays is trying to describe $G$-covers with special properties. The first examples of them arise looking at covers with normal total space. Indeed in [Par91] the author is able to describe the multiplications yielding regular $G$-covers of a discrete valuation ring. This description, using the language introduced above, yields a sequence $\underline{\delta} = (\mathcal{E}^\phi)_{\phi\in\Phi_M}$ of smooth extremal rays, where $\Phi_M$ is the set of surjective maps $M \longrightarrow \mathbb{Z}/d\mathbb{Z}$ with $d > 1$. We will define a stratification of $G$-Cov by open substacks $\mathrm{B}\,G = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{|G|-1} = G\text{-Cov}$ and we will prove that there exists an explicitly given sequence $\underline{\mathcal{E}}$ of smooth extremal rays (defined in 3.3.40) containing $\underline{\delta}$ such that:

**Theorem.** *[3.2.41, 3.3.42] We have $U_2 \subseteq \mathcal{Z}_G^{\mathrm{sm}}$ and $\pi_{\underline{\mathcal{E}}}\colon \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{Z}_G$ induces isomorphisms of stacks*

$$\pi_{\underline{\mathcal{E}}}^{-1}(U_2) \xrightarrow{\simeq} U_2,\ \pi_{\underline{\delta}}^{-1}(U_1) = \pi_{\underline{\mathcal{E}}}^{-1}(U_1) \xrightarrow{\simeq} U_1$$

Theorem above implies that $M$-Hilb$\mathbb{A}^2$ is smooth and irreducible (3.3.43). In this way we get an alternative proof of the result in [Mac03] (later generalized in [MS10]) in the particular case of equivariant Hilbert schemes.

**Theorem.** *[3.2.42, 3.3.45] Let $Y$ be a locally noetherian and locally factorial scheme and $\chi \in G\text{-Cov}(Y)$. If $\chi_{|k(p)} \in U_1$ (resp. $\chi_{|k(p)} \in U_2$) for all $p \in Y$ with $\operatorname{codim}_p Y \leq 1$, then $\chi$ lifts uniquely to $\mathcal{F}_{\underline{\delta}}(Y)$ (resp. $\mathcal{F}_{\underline{\mathcal{E}}}(Y)$).*

Notice that $\underline{\mathcal{E}} = \underline{\delta}$ if and only if $G \simeq (\mu_2)^l$ or $G \simeq (\mu_3)^l$ (3.3.44). Finally we prove:

**Theorem.** *[3.2.43, 3.3.55] Let $Y$ be a locally noetherian and locally factorial integral scheme with $\dim Y \geq 1$ and such that $|M| \in \mathcal{O}_Y^*$. Let also $f \colon X \longrightarrow Y$ be a $G$-cover. If $X$ is regular in codimension $1$ (resp. normal crossing in codimension $1$ (see 3.3.47)) then $f$ comes from a unique object of $\mathcal{F}_{\underline{\delta}}(Y)$ (resp. $\mathcal{F}_{\underline{\gamma}}(Y)$, where $\underline{\delta} \subseteq \underline{\gamma} \subseteq \underline{\mathcal{E}}$ is an explicitly given sequence).*

Note that one can replace "regular in codimension 1" with "normal" in the above theorem because $G$-covers have Cohen-Macaulay fibers. The part concerning covers that are regular in codimension 1 is essentially a rewriting of Theorem 2.1 and Corollary 3.1 of [Par91] extended to locally noetherian and locally factorial schemes, while the last part generalizes Theorem 1.9 of [AP12].

## 1.3 Equivariant affine maps and monoidality.

In Chapter 4 we focus on the problem of describing Galois covers for a general finite, flat and of finite presentation group scheme $G$ over a given base ring $R$. This problem can be stated as follows.

**Problem 1.3.1.** Given an $R$-scheme $T$, describe $G$-covers over $T$ in terms of locally free sheaves over $T$ (without an action of $G$), maps among them and the representation theory of $G$ over the base ring $R$.

Denote by $\operatorname{Loc} T$ ($\operatorname{Loc}^G T$) the category of locally free sheaves of finite rank over $T$ (with an action of $G$). Similarly define $\operatorname{QCoh} T$ and $\operatorname{QCoh}^G T$ replacing locally free sheaves with arbitrary quasi-coherent sheaves. Given an $R$-linear functor $\Omega \colon \operatorname{Loc}^G R \longrightarrow \operatorname{QCoh} T$ a *monoidal* structure on it is given by a natural transformation $\iota_{V,W} \colon \Omega_V \otimes \Omega_W \longrightarrow \Omega_{V \otimes W}$ and an identity $1 \in \Omega_R$ satisfying certain natural conditions. This structure is called *strong* if $\iota$ is an isomorphism and $\Omega_R = \mathcal{O}_T 1$.

A common result of Tannaka's theory is that the category of $G$-torsors over $T$ is equivalent to the category of strong monoidal, symmetric, $R$-linear and exact functors $\operatorname{Loc}^G R \longrightarrow \operatorname{Loc} T$. This is part of the so called "reconstruction problem": reconstruct a group $G$ from its category of representations. See [DM82, Riv72] for the classical case when $R$ is a field and [Lur04] for the general one. This gives an answer to problem 1.3.1 in the case of $G$-torsors. In this chapter we provide a similar answer for $G$-covers and, more generally, for equivariant affine maps. The idea is to extend the above correspondence to the case of (non exact) non strong monoidal functors. In order to consider general equivariant affine maps and not only $G$-covers, it is also necessary to consider monoidal functors with values in the whole category of quasi-coherent sheaves.

This functorial point of view for $G$-covers arises naturally when trying to answer problem 1.3.1. As seen in the case of diagonalizable groups, the problem of describing $G$-covers is equivalent to the problem of understanding the possible algebra structures on the regular representation $R[G]$. This task should be easier when $R[G]$ is a sum of smaller parts, as it happens when $G$ is diagonalizable or a constant group over the complex numbers. Therefore the first problem to solve is to determine a class of group schemes for which this simplification is possible. Over a field $k$, such question has already an answer: the regular representation $k[G]$ decomposes into a product of irreducible representations if and only if the group $G$ is linearly reductive. This property is usually taken as definition of a linearly reductive group over a field. There is an alternative definition, which has the advantage of working over any base scheme: a group scheme $G$ over $R$ is called *linearly reductive* if the functor of invariants $\mathrm{QCoh}^G R \longrightarrow \mathrm{QCoh}\, R$, $\mathcal{F} \longmapsto \mathcal{F}^G$ is exact. For an introduction to this subject see [AOV08]. What we are looking for is a collection $I$ of objects in $\mathrm{Loc}^G R$ for which the $G$-equivariant maps

$$\eta_{I,\mathcal{F}} \colon \bigoplus_{V \in I} \underline{\mathrm{Hom}}^G(V, \mathcal{F}) \otimes V \longrightarrow \mathcal{F}$$

are isomorphisms for all $\mathcal{F} \in \mathrm{QCoh}\, T$ and all $R$-schemes $T$. Clearly this implies that $G$ is linearly reductive and, over an algebraically closed field $k$, that $I$ is the set of the irreducible representations. Assume that $G$ is a linearly reductive group. The result is:

**Proposition.** *[4.1.10] The maps $\eta_{I,\mathcal{F}}$ are isomorphisms for all $\mathcal{F} \in \mathrm{QCoh}\, T$ and all $R$-schemes $T$ if and only if, for all algebraically closed fields $k$ and geometric points $\mathrm{Spec}\, k \longrightarrow \mathrm{Spec}\, R$, the representations $V \otimes k$ are irreducible for all $V \in I$ and the restriction $- \otimes k$ yields a one to one correspondence between $I$ and the set of irreducible representations of $G \times k$ up to isomorphisms. When $\mathrm{Spec}\, R$ is connected, the previous condition can be checked at a fixed geometric point.*

We will say that a group $G$ admitting a set $I$ as above has a *good* representation theory and that the pair $(G, I)$ is a *good linearly reductive group* (abbreviated with *glrg*). For such a pair we will also write $I = I_G$. Notice that any $V \in I_G$ is not only irreducible, but also geometrically irreducible, that is it is irreducible after base changing to all the geometric points, and that $\underline{\mathrm{End}}^G(V) = \mathcal{O}_S$, while if $W \in I_G$ and $W \neq V$, then $\underline{\mathrm{Hom}}^G(V, W) = 0$. In particular over a field $k$, $G$ has a good representation theory if and only if all irreducible representations $V$ have trivial endomorphism rings, that is $\mathrm{End}^G(V) \simeq k$, and in this case $I_G$ is the set of irreducible representations. When the base is connected, $I_G$ is uniquely determined up to tensorization by invertible sheaves (with trivial action). Other examples of good linearly reductive groups are the diagonalizable groups over $R = \mathbb{Z}$: if $M = \mathrm{Hom}(G, \mathbb{G}_m)$ and we denote by $\mathbb{Z}_m$ the representation induced by $m \in M$, it is enough to consider the sequence $I_G = (\mathbb{Z}_m)_{m \in M}$. Notice that there exist linearly reductive that are not good, for instance $\mathbb{Z}/p\mathbb{Z}$ over a field not containing a primitive $p$-root of unity. On the other hand we prove that (étale) linearly reductive groups have a good representation theory locally in the (étale) fppf topology (see 4.1.21). Moreover, any constant group $G$ has a good representation theory over a strictly Henselian ring $R$, provided that the characteristic of the residue field of $R$ does not divide the order of $G$ (see 4.1.22).

When $G$ is a good linearly reductive group and the base scheme $\operatorname{Spec} R$ is connected, a $G$-comodule $\mathcal{F}$ over an $R$-scheme $T$ which is fppf locally the regular representation is of the form

$$\mathcal{F} \simeq \bigoplus_{V \in I_G} \mathcal{F}_V \otimes V^\vee \text{ where } \mathcal{F}_V \text{ is locally free of rank } \operatorname{rk} V \tag{1.3.1}$$

Thus $\mathcal{F}$ is determined by a sequence of locally free sheaves with prescribed ranks, namely $(\mathcal{F}_V)_{V \in I_G}$. Now the problem is to understand what additional data are needed and what conditions such data have to satisfy in oder to induce a structure of algebra over $\mathcal{F}$. A non associative ring structure on $\mathcal{F}$ is given by a collection of maps between sheaves obtained starting from the $\mathcal{F}_V, V$ for $V \in I_G$, whose form depends on how the tensor products $V \otimes W$ for $V, W \in I_G$ decompose into representations in $I_G$. Moreover it is not difficult to convince oneself that the conditions those maps have to satisfy in order to have a commutative and associative algebra strongly depend on the two ways one can decompose $(V \otimes W) \otimes Z \simeq V \otimes (W \otimes Z)$ for $V, W, Z \in I_G$ into representations in $I_G$. I have to admit that I have never been brave enough to write down those last conditions, although this should be an elementary task: it seems pretty clear that there is no hope to simplify those conditions for a general group, obtaining a really meaningful set of data. The diagonalizable case is much more simple than the general one because tensor products of representations are very easy.

The approach I propose to work around this situation is to associate with a sheaf $\mathcal{F}$ not only the sequence $(\mathcal{F}_V)_{V \in I_G}$, but a whole functor $\operatorname{Loc}^G R \longrightarrow \operatorname{Loc} T$. With a $G$-comodule $\mathcal{F}$ we associate the functor $\Omega^{\mathcal{F}} \colon \operatorname{Loc}^G R \longrightarrow \operatorname{Loc} T$ given by

$$\Omega^{\mathcal{F}}_V = (\mathcal{F} \otimes V)^G$$

Notice that, with notation from (1.3.1), $\mathcal{F}_V = \Omega^{\mathcal{F}}_V$ for all $V \in I_G$. Although we do not have a finite set of data, this approach has, at least, two advantages. The first is that, as we will see, a structure of algebra on $\mathcal{F}$ translates into natural properties on $\Omega^{\mathcal{F}}$. The second is that this point of view, without additional technicalities, allows us to consider and describe any $G$-equivariant affine map, that is any affine map $f \colon X \longrightarrow T$ with an action of $G$ on $X$ for which $f$ is invariant, and that the theory can be developed for any finite, flat and finitely presented group scheme.

So assume that $G$ is a finite, flat and finitely presented group scheme over a ring $R$. Given an $R$-scheme $T$ define $\operatorname{QAdd}^G T$ as the category of $R$-linear functors $\operatorname{Loc}^G R \longrightarrow \operatorname{QCoh} T$. Denote also by $\operatorname{QAdd}^G_R$ (resp. $\operatorname{QCoh}^G_R$) the stack (not in groupoids) whose fiber over an $R$-scheme $T$ is $\operatorname{QAdd}^G T$ (resp. $\operatorname{QCoh}^G T$). Given $\Omega \in \operatorname{QAdd}^G T$, we will show that $\Omega_{R[G]} \in \operatorname{QCoh} T$ has a natural structure of $G$-comodule and the first result we will prove is:

**Theorem.** *[4.2.4] Given an $R$-scheme $T$, the functors*

$$\begin{array}{ccc} \mathcal{F}_\Omega = \Omega_{R[G]} & \longleftarrow\!\!\shortmid & \Omega \\ \mathrm{QCoh}^G T & \longrightarrow & \mathrm{QAdd}^G T \\ \mathcal{F} & \longmapsto & \Omega^{\mathcal{F}} = (- \otimes \mathcal{F})^G \end{array}$$

*yield an equivalence between $\mathrm{QCoh}^G T$ and the full subcategory of $\mathrm{QAdd}^G T$ of left exact functors.*

The group $G$ is linearly reductive over $R$ if and only if the functors in $\mathrm{QAdd}^G T$ are left exact for all $R$-schemes $T$ (see 4.2.6). In this case we get an equivalence of stacks $\mathrm{QCoh}^G_R \simeq \mathrm{QAdd}^G_R$ and similar equivalences exist when we consider the category of finitely presented quasi-coherent sheaves or locally free sheaves of finite rank instead of the whole category of quasi-coherent sheaves. Note that the functor associated with the regular representation $\mathcal{O}_T[G]$ is the forgetful functor $V \longmapsto V \otimes \mathcal{O}_T$. In the particular case where $G$ is a good linearly reductive group, the functor $\Omega \longmapsto \Omega_{R[G]}$ has a more explicit description: given $\Omega \in \mathrm{QAdd}^G T$ there exists a natural, $G$-equivariant isomorphism

$$\Omega_{R[G]} \simeq \bigoplus_{V \in I_G} V^\vee \otimes \Omega_V$$

This shows how the above construction generalizes the isomorphism (1.3.1).

Now that we have a way to associate with a $G$-equivariant quasi-coherent sheaf $\mathcal{F}$ a functor $\Omega^{\mathcal{F}}$, the next question is what additional data $\Omega^{\mathcal{F}}$ must have to induce a structure of equivariant sheaf of algebras on $\mathcal{F}$. The answer is a symmetric, monoidal structure. Given an $R$-scheme $T$, denote by $\mathrm{QMon}^G T$ the category of functors $\Omega \in \mathrm{QAdd}^G T$ with a symmetric monoidal structure and by $\mathrm{QAlg}^G T$ the category of quasi-coherent sheaves of algebras with an action of $G$. Denote also by $\mathrm{QMon}^G_R$ (resp. $\mathrm{QAlg}^G_R$) the stack (not in groupoids) whose fiber over an $R$-scheme $T$ is $\mathrm{QMon}^G T$ (resp. $\mathrm{QAlg}^G T$).

**Theorem.** *[4.2.21] Given an $R$-scheme $T$, the functors*

$$\mathrm{QAlg}^G T \underset{*_{R[G]}}{\overset{\Omega^*}{\rightleftarrows}} \mathrm{QMon}^G T$$

*yield an equivalence between $\mathrm{QAlg}^G T$ and the full subcategory of $\mathrm{QMon}^G T$ of left exact functors.*

When $G$ is a linearly reductive group we obtain an equivalence of stacks $\mathrm{QAlg}^G_R \simeq \mathrm{QMon}^G_R$ and similar equivalences are defined if we consider finitely presented quasi-coherent sheaves or locally free sheaves of finite rank instead of all the quasi-coherent sheaves.

Note that the regular representation $\mathcal{O}_T[G]$ corresponds to the forgetful functor $V \longrightarrow V \otimes \mathcal{O}_T$ with the obvious monoidal structure. Denote by $\mathrm{LMon}^G_R$ the substack of $\mathrm{QMon}^G_R$ composed of functors with values in the category of locally free sheaves of finite rank. The answer to the initial problem 1.3.1 is the following.

**Theorem.** *[4.2.29] The association*

$$G\text{-Cov} \longrightarrow \mathrm{LMon}_R^G,\ (X \xrightarrow{f} T) \longmapsto \Omega^{f_*\mathcal{O}_X} = (f_*\mathcal{O}_X \otimes -)^G$$

*induces an equivalence onto the substack in groupoids of* $\mathrm{LMon}_R^G$ *of functors that, as $R$-linear functors, are fppf locally isomorphic to the forgetful functor. If $G$ is a good linearly reductive group and* $\operatorname{Spec} R$ *is connected, this is the substack of functors $\Omega$ such that* $\operatorname{rk}\Omega_V = \operatorname{rk} V$ *for all* $V \in \mathrm{Loc}^G R$ *(or all $V \in I_G$).*

Notice that the last part of the above Theorem is no longer true even for linearly reductive groups without a good representation theory. It fails in the simplest possible case: $G = \mathbb{Z}/3\mathbb{Z}$, $R = \mathbb{Q}$ and $T = \operatorname{Spec}\overline{\mathbb{Q}}$. In the above correspondence the stack $\mathrm{B}\,G$ of $G$-torsors is sent to the stack of symmetric, strong monoidal, $R$-linear and exact functors (see 4.2.34). We retrieve in this way the classical Tannaka's correspondence, which was also the starting point of the discussion about $G$-covers for general groups $G$ at the beginning of this section.

The above Theorem shows clearly how the constructions we have made are just a generalization of the description of $G$-covers when $G$ is a diagonalizable group (see 1.2.1). If $M = \operatorname{Hom}(G, \mathbb{G}_m)$, $G$ is a good linearly reductive group with $I_G = (\mathbb{Z}_m)_{m\in M}$ and a functor $\Omega \in \mathrm{QAdd}^G T$ for which $\Omega_V$ is locally free of rank $\operatorname{rk} V$ for all $V \in I_G$ is just given by a collection of invertible sheaves $\mathcal{L}_m = \Omega_{\mathbb{Z}_m}$, while a monoidal structure on $\Omega$, that is a ring structure over $\Omega_{\mathbb{Z}[G]} \simeq \oplus_{m\in M}{\mathbb{Z}_m}^\vee \otimes \mathcal{L}_m$, is just given by maps

$$\Omega_{\mathbb{Z}_m} \otimes \Omega_{\mathbb{Z}_n} = \mathcal{L}_m \otimes \mathcal{L}_n \longrightarrow \mathcal{L}_{m+n} = \Omega_{\mathbb{Z}_{m+n}} \simeq \Omega_{\mathbb{Z}_m \otimes \mathbb{Z}_n}$$

satisfying certain conditions.

From now on $G$ will be a linearly reductive group scheme over a base ring $R$ and, as always, we will assume that it is flat, finite and of finite presentation. We want to discuss some applications of the functorial point of view introduced above.

When $G$ is a diagonalizable group, $M = \operatorname{Hom}(G, \mathbb{G}_m)$ and the sequence $(\mathcal{L}_m, \psi_{m,n})_{m,n\in M}$ defines a $G$-cover (see 1.2.1), a classical result is that this cover is a $G$-torsor if and only the maps $\psi_{m,-m} \colon \mathcal{L}_m \otimes \mathcal{L}_{-m} \longrightarrow \mathcal{L}_0 = \mathcal{O}$ are isomorphisms for all $m \in M$ (see [GD70, Exposé VIII, Proposition 4.1 and 4.6]). In this thesis we generalize this property for more general groups. A linearly reductive group $G$ over an algebraically closed field is *solvable* (*super solvable*) if it admits a filtration by closed subgroups $0 = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$ such that, for all $i$, $H_{i+1}/H_i \simeq \mu_p$ for some prime $p$ (and $H_i \lhd G$). A linearly reductive group over a ring $R$ is solvable (super solvable) if it is so over any geometric point of $\operatorname{Spec} R$. Denote also by $\mathrm{LAlg}_R^G$ the full substack of $\mathrm{QAlg}_R^G$ of algebras that are locally free of finite rank, which is isomorphic to $\mathrm{LMon}_R^G$ via the functor $\Omega^*$. The result we prove is the following:

**Theorem.** *[4.2.42] Let $G$ be a super solvable good linearly reductive group over a ring $R$ and let $\mathscr{A} \in \mathrm{LAlg}_R^G T$, for an $R$-scheme $T$. Then $\mathscr{A} \in \mathrm{B}\,G$ if and only if $\Omega_R^{\mathscr{A}} (= \mathscr{A}^G) \simeq \mathcal{O}_T$ and for all $V \in I_G$ the maps*

$$\Omega_V^{\mathscr{A}} \otimes \Omega_{V^\vee}^{\mathscr{A}} \longrightarrow \Omega_{V\otimes V^\vee}^{\mathscr{A}} \longrightarrow \Omega_R^{\mathscr{A}} \simeq \mathcal{O}_T \tag{1.3.2}$$

*are surjective.*

Notice that the above Theorem is no longer true if we consider solvable groups, even in the constant case (see 4.2.53). The above criterion will be applied in the study of $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$-covers.

When $G$ is diagonalizable, we have seen that $G$-Cov is almost never irreducible. This bad behaviour continues in the non abelian case:

**Theorem.** *[4.3.1] If $G$ is a finite, non abelian and linearly reductive group then $G$-Cov is reducible.*

The methods used in the proof of the above Theorem neither reduce, nor are applicable to the diagonalizable case and they involve the study of more general $G$-equivariant algebras than the ones inducing $G$-covers. Moreover they allow us to construct $G$-covers outside the main irreducible component $\mathcal{Z}_G$, while the same problem is more difficult in the diagonalizable case.

Another interesting question about the theory of $G$-covers, and one of the most important from the point of view of classical algebraic geometry, is the question about regularity of $G$-covers and, more generally, about the preservation of geometrical properties: given a regular (resp. normal, regular in codimension 1) integral scheme $Y$ and a cover $f\colon X \longrightarrow Y$, understand when $X$ is regular (resp. normal, regular in codimension 1). When this happens we will call this cover regular (resp. normal, regular in codimension 1). Notice that a cover of a normal scheme that is regular in codimension 1 is normal, because a cover has Cohen-Macaulay fibers. The problem of detecting the regularity of a cover arises together with the problem of constructing such a cover. We have to admit that this last problem seems very difficult to handle in this generality, but one can hope to be able to find at least families of regular covers. Anyway in this thesis we will concentrate only on the first problem and only on the case of regularity in codimension 1. We generalize what happens in the diagonalizable case and the leading idea is the following. If $\mathscr{A}$ is a locally free algebra of finite rank over a scheme $Y$, denote by $\hat{\mathrm{tr}}_{\mathscr{A}}\colon \mathscr{A} \longrightarrow \mathscr{A}^\vee$ the map $x \longmapsto \mathrm{tr}_{\mathscr{A}}(x\cdot -)$, where $\mathrm{tr}_{\mathscr{A}}$ is the trace map $\mathscr{A} \longrightarrow \mathcal{O}_Y$. A classical result is that the algebra $\mathscr{A}$ is étale over $Y$ if and only if $\hat{\mathrm{tr}}_{\mathscr{A}}$ is an isomorphism (see [GR71, Proposition 4.10]). The idea is that, the less degenerate $\hat{\mathrm{tr}}_{\mathscr{A}}$ is, the more regular the algebra $\mathscr{A}$ should be. If $f\colon \operatorname{Spec}\mathscr{A} \longrightarrow Y$ is the associated cover, denote by $s_f \in (\det \mathscr{A})^{-2}$ the determinant of $\hat{\mathrm{tr}}_{\mathscr{A}}$, also called the discriminant section. This section is important because its zero locus is the complement of the locus where $f$ is étale. If $\mathscr{A} \in G\text{-Cov}Y$ and $G$ has a good representation theory, given $V \in I_G$ the map (1.3.2) induces a map $\Omega^{\mathscr{A}}_V \longrightarrow (\Omega^{\mathscr{A}}_{V^\vee})^\vee$ and we will denote by $s_{f,V} \in \det(\Omega^{\mathscr{A}}_V)^{-1} \otimes \det(\Omega^{\mathscr{A}}_{V^\vee})^{-1}$ the section associated with its determinant. When $G$ is an étale, good linearly reductive group the relation between the sections just introduced is given by the following isomorphism (see 4.4.6)

$$(\det \mathscr{A})^{-2} \simeq \bigotimes_{V\in I_G} (\det(\Omega^f_V)^{-1} \otimes \det(\Omega^f_{V^\vee})^{-1})^{\mathrm{rk}\, V} \text{ such that } s_f \longmapsto \bigotimes_{V\in I_G} s_{f,V}^{\otimes \mathrm{rk}\, V}$$

If we denote by $Y^{(1)}$ the set of codimension 1 points of $Y$ and by $v_q$ the valuation for $q \in Y^{(1)}$, the result we will prove is the following:

**Theorem.** *[4.4.7] Let $G$ be a finite and étale linearly reductive group over a ring $R$. Let also $Y$ be an integral, noetherian and regular in codimension 1 (resp. normal) $R$-scheme and $f\colon X \longrightarrow Y$ be a cover with a generically faithful action (see 4.4.8) of $G$ on $X$ such that $f$ is $G$-invariant and $X/G = Y$. Then the following are equivalent:*

*1) $X$ is regular in codimension 1 (resp. normal);*

*2) the geometric stabilizers of the codimension 1 points of $X$ are solvable and for all $q \in Y^{(1)}$ we have $v_q(s_f) < \operatorname{rk} G$ $(= \operatorname{rk} f)$.*

*In this case $f$ is generically a $G$-torsor, $f \in \mathcal{Z}_G(Y)$ and the stabilizers of the codimension 1 points of $X$ are cyclic. Moreover, if $G$ has a good representation theory, the above conditions are also equivalent to*

*3) the geometric stabilizers of the codimension 1 points of $X$ are solvable, $f \in G\text{-Cov}$ and for all $q \in Y^{(1)}$ and $V \in I_G$ we have $v_q(s_{f,V}) \leq \operatorname{rk} V$.*

I am strongly convinced that the above statement is still true without the hypothesis of solvability on the geometric stabilizers. Actually I am also convinced that, with some minor modifications, the first part of the statement continues to be true without the existence of a generically faithful action of a group. I think that the statement which should be true is:

**Conjecture.** *Let $R$ be a discrete valuation ring with residue field $k$ and $A$ be a finite and flat $R$-algebra. Then*

$$v_R(\det \hat{\operatorname{tr}}_A) \geq \operatorname{rk} A - |\operatorname{Spec} A \otimes_R \overline{k}|$$

*and equality holds if and only if $A$ is regular, generically étale with separable residue fields and the localizations of $A \otimes_R \overline{k}$ have ranks prime to the characteristic of $k$.*

Except for the implication "equality $\Longrightarrow$ regularity", I am able to prove the rest of the statement. When we have a generically faithful action of, say, a solvable group $G$ on $A$ one can argue by induction on $\operatorname{rk} A = \operatorname{rk} G$ considering the invariant algebra for a normal subgroup of $G$. The base case in this induction is $G = \mu_p$, for some prime $p$, where the result can be easily deduced from the theory developed for diagonalizable groups.

## 1.4 $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$-covers and $S_3$-covers.

In the last Chapter of this thesis we will study $G$-covers for the non abelian group scheme $G = \mu_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ and $S_3$-covers. In oder to simplify the exposition we work over the ring $\mathcal{R} = \mathbb{Z}[1/6]$. Denote by $\sigma \in \mathbb{Z}/2\mathbb{Z}(\mathcal{R})$ the generator and consider it also as a section of $G$ and, after choosing a transposition, as a section of $S_3$. The groups $G$ and $S_3$ are linearly reductive over $\mathcal{R}$ and they also have a good representation theory. We can choose $I_G = \{\mathcal{R}, A, V\}$, where $A = \mathcal{R}$ with the action induced by the non trivial character of $\mathbb{Z}/2\mathbb{Z}$ and $V = \operatorname{ind}_{\mu_3}^G V_1$, where $V_1$ is the $\mu_3$ representation associated with the character

$1 \in \mathbb{Z}/3\mathbb{Z}$. In the second chapter we prove that, if $H$ and $H'$ are étale locally isomorphic group schemes, then we have an isomorphism $\mathrm{B}(H \rtimes \underline{\mathrm{Aut}}\, H) \simeq \mathrm{B}(H' \rtimes \underline{\mathrm{Aut}}\, H')$ (of stacks classifying fppf torsors) (see 2.3.10). In particular, considering $H = \mu_3$ and $H' = \mathbb{Z}/3\mathbb{Z}$ over $\mathcal{R}$, we obtain an isomorphism $\mathrm{B}\, G \simeq \mathrm{B}\, S_3$. By the general theory of bitorsors described in the second chapter, we also obtain an isomorphism $G$-Cov $\simeq S_3$-Cov over $\mathcal{R}$. Thus the study of $G$-covers coincides with the study of $S_3$-covers, and, due to the nature of the isomorphism $G$-Cov $\simeq S_3$-Cov, the problems of regularity of covers also coincide. Anyway we will describe the structure of $G$-equivariant algebras only, because the representation theory of $G$ has a simpler explicit description and all the theory works over $\mathbb{Z}[1/2]$, instead of $\mathbb{Z}[1/6]$. The groups $G$ and $S_3$ can be considered the simplest non abelian linearly reductive groups. This is essentially the motivation for a detailed study of $G$-covers and $S_3$-covers.

A similar analysis of $S_3$-covers is conducted in [Eas11], where the author describes the data needed to build them in terms of linear algebra. Here, using a different approach, we recover this result and we expand it, describing particular families of $S_3$-covers, characterizing the regular ones and computing the invariants of the total space of a regular $S_3$-cover of a surface.

Using the theory developed above, a $G$-cover over an $\mathcal{R}$-scheme $T$ corresponds to an $\mathcal{R}$-linear, symmetric and monoidal functor $\Omega\colon \mathrm{Loc}^G \mathcal{R} \longrightarrow \mathrm{Loc}\, T$ such that $\mathrm{rk}\, \Omega_W = \mathrm{rk}\, W$ for all $W \in I_G$. It is easy to deduce the data needed to build a $G$-cover. Since $\Omega_{\mathcal{R}} = \mathcal{O}_T$ for general reasons, we need an invertible sheaf $\mathcal{L} = \Omega_A$ and a locally free sheaf $\mathcal{F} = \Omega_V$ of rank 2 in order to have a functor $\Omega \in \mathrm{QAdd}^G_{\mathcal{R}}$. For the monoidal structure, for all $W_1, W_2 \in I_G$ we need maps $\Omega_{W_1} \otimes \Omega_{W_2} \longrightarrow \Omega_{W_1 \otimes W_2}$. Since we are interested in commutative algebras with unity and we have relations $A \otimes A \simeq \mathcal{R}$, $A \otimes V \simeq V$ and $V \otimes V \simeq \mathcal{R} \oplus A \oplus V$, a monoidal structure on $\Omega$ is given by maps

$$\mathcal{L} \otimes \mathcal{L} \xrightarrow{m} \mathcal{O}_T, \quad \mathcal{L} \otimes \mathcal{F} \xrightarrow{\alpha} \mathcal{F}, \quad \mathcal{F} \otimes \mathcal{F} \xrightarrow{(-,-) \oplus \langle -,- \rangle \oplus \beta} \mathcal{O}_T \oplus \mathcal{L} \oplus \mathcal{F}$$

satisfying certain conditions, required for the associativity of $\Omega$. As it happens in the diagonalizable case, this is the hard part. Such conditions imply that $(-,-), \beta$ are symmetric, $\langle -,- \rangle$ is antisymmetric and that $(-,-)$ is uniquely determined by the other maps. In conclusion it turns out that a $G$-cover over $T$ is associated with a sequence $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle)$ where $\mathcal{L}$ is an invertible sheaf, $\mathcal{F}$ is a locally free sheaf of rank 2 and $m, \alpha, \beta, \langle -,- \rangle$ are maps

$$\mathcal{L}^2 \xrightarrow{m} \mathcal{O}_T, \quad \mathcal{L} \otimes \mathcal{F} \xrightarrow{\alpha} \mathcal{F}, \quad \mathrm{Sym}^2 \mathcal{F} \xrightarrow{\beta} \mathcal{F}, \quad \det \mathcal{F} \xrightarrow{\langle -,- \rangle} \mathcal{L}$$

that satisfy certain conditions. The above association will be formulated in terms of isomorphism of stacks (see 5.2.4). I do not think that further simplifications are possible in this generality. Although the data above are directly associated to a $G$-cover, they also correspond to an $S_3$-cover, as remarked above. We will identify $G$-Cov and $S_3$-Cov with the stack of data defined as above and all the results cited below, including the ones regarding the geometry of covers, continue to be true if we replace $G$-Cov by $S_3$-Cov and $G$-covers by $S_3$-covers. Anyway some general results will be stated for both $G$ and

$S_3$. The main idea followed in order to get to a better understanding of $G$-covers and $S_3$-covers is to look at particular loci of $G$-Cov, that is to look at data as above satisfying additional conditions. All those loci are interesting because they will allow to understand the geometry of $G$-Cov and $S_3$-Cov and also to describe regular $G$-covers and $S_3$-covers.

It is convenient at this point to introduce more notation. Denote by $\mathcal{C}_3$ the stack of pairs $(\mathcal{F}, \delta)$ where $\mathcal{F}$ is a locally free sheaf of rank 2 and $\delta$ is a map $\operatorname{Sym}^3 \mathcal{F} \longrightarrow \det \mathcal{F}$ and by $\mathrm{Cov}_3$ the stack of degree 3 covers, also called triple covers. It is a well known result of the theory of triple covers (see [Mir85, Par89, BV12]) that there exists an isomorphism of stacks $\mathcal{C}_3 \longrightarrow \mathrm{Cov}_3$ so defined: an object $\Phi = (\mathcal{F}, \delta) \in \mathcal{C}_3(T)$, where $T$ is an $\mathcal{R}$-scheme, induces maps $\eta_\delta \colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{O}_T$ and $\beta_\delta \colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{F}$ which define an algebra structure on the sheaf $\mathscr{A}_\Phi = \mathcal{O}_T \oplus \mathcal{F}$. Taking invariants by $\sigma \in \mathbb{Z}/2\mathbb{Z}$ we obtain a map $\pi \colon G\text{-Cov} \longrightarrow \mathrm{Cov}_3 \simeq \mathcal{C}_3$. Notice that the same procedure yields a map $S_3\text{-Cov} \longrightarrow \mathrm{Cov}_3$ and it is possible to prove that $G$-Cov and $S_3$-Cov are isomorphic over $\mathrm{Cov}_3$ (see 5.3.8). The first result on the geometry of $G$-Cov we prove is the following:

**Theorem.** *[5.3.5] The map $\pi \colon G\text{-Cov} \longrightarrow \mathrm{Cov}_3$ restricts to an isomorphism of stacks $\mathcal{U}_\omega \longrightarrow \mathrm{Cov}_3$, where $\mathcal{U}_\omega$ is the open substack of $G$-Cov where $\langle -,- \rangle \colon \det \mathcal{F} \longrightarrow \mathcal{L}$ is an isomorphism.*

In particular this gives a functorial way of extending triple covers to $G$-covers or $S_3$-covers. Looking at the global geometry we prove that:

**Theorem.** *[5.3.17, 5.3.22, 5.3.28] The stacks $G$-Cov and $S_3$-Cov are connected, non-reduced and have two irreducible components, the main one $\mathcal{Z}_G$, which coincides with the zero locus of the maps $\mathcal{L} \longrightarrow \mathcal{F}^\vee \otimes \mathcal{F} \longrightarrow \mathcal{O}_T$ and $\mathcal{F} \longrightarrow \mathcal{F}^\vee \otimes \mathcal{F} \longrightarrow \mathcal{O}_T$ induced by $\alpha$ and $\beta$ respectively, and the closed locus of $G$-Cov where $\beta = \langle -,- \rangle = 0$ and $\alpha$ is fppf locally a multiple of the identity. Moreover $\mathrm{B}\, G \subseteq G$-Cov is the open substack where $\langle -,- \rangle$ and $m$ are isomorphisms.*

For covers in $\mathcal{Z}_G$ are possible two further simplifications of the data associated with them, one for the whole $\mathcal{Z}_G$ and one that regards particular objects of $\mathcal{Z}_G$. We want to describe only the second simplification, because it will be the one used in the description of regular $G$-covers. Given $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in G$-Cov set $\mathcal{M} = \mathcal{L} \otimes \det \mathcal{F}^{-1}$ and $\omega \in \mathcal{M}$ the section corresponding to $\langle -,- \rangle$. Moreover, given an $\mathcal{R}$-scheme $T$ denote by $\mathcal{Z}_\omega(T)$ the full subcategory of $\mathcal{Z}_G(T)$ where $\mathcal{O}_T \xrightarrow{\omega} \mathcal{M}$ is injective, which means that $\omega$ yields a Cartier divisor over $T$. We will prove that $\mathcal{Z}_\omega(T)$ is isomorphic to the category whose objects are sequences $(\mathcal{M}, \mathcal{F}, \delta, \omega)$ where $(\mathcal{F}, \delta) \in \mathcal{C}_3(T)$, $\mathcal{M}$ is an invertible sheaf and $\omega \in \mathcal{M}$ is a section such that $\mathcal{O}_T \xrightarrow{\omega} \mathcal{M}$ is injective and its image contains the image of $\eta_\delta \colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{O}_T$ (see 5.3.29). In particular we see that the extensions of a triple cover $(\mathcal{F}, \delta)$ to a $G$-cover in $\mathcal{Z}_\omega(T)$ correspond bijectively to the effective Cartier divisors contained in the locus where $\eta_\delta$ is zero.

The last part of this thesis is dedicated to the study of regular $G$-covers and $S_3$-covers. Notice that it is possible to apply directly the result previously obtained on covers that are regular in codimension 1 for general groups (see 5.4.5), but what we get is a particular case of the description of regular $G$-covers we want to explain. Let $Y$ be an integral, noetherian and regular scheme. Given $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in G\text{-Cov}(Y)$ we define: $D_m$ and $D_\omega$ as the closed subschemes of $Y$ where $m\colon \mathcal{L}^2 \longrightarrow \mathcal{O}_Y$ and $\langle -,- \rangle\colon \det \mathcal{F} \longrightarrow \mathcal{L}$ are zero respectively; $Y_\alpha$ as the vanishing locus of the map $\alpha\colon \mathcal{L} \otimes \mathcal{F} \longrightarrow \mathcal{F}$. Notice that we have an inclusion $Y_\alpha \subseteq D_m$. Given $\Phi = (\mathcal{F}, \delta) \in \mathcal{C}_3(Y)$ we define: $Y_\delta$ and $D_\delta$ as the closed subschemes of $Y$ defined by $\eta_\delta\colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{O}_Y$ and the discriminant $\Delta_\Phi\colon (\det \mathcal{F})^2 \longrightarrow \mathcal{O}_Y$ respectively, where the last map is induced by the determinant of $\hat{\operatorname{tr}}_{\mathscr{A}_\Phi}\colon \mathscr{A}_\Phi \longrightarrow \mathscr{A}_\Phi{}^\vee$. Finally, given a proper, closed subscheme $Z$ of $Y$, denote by $D(Z)$ the divisorial component of $Z$ in $Y$, that is the maximum among the effective Cartier divisors contained in $Z$. The Theorem we will prove is the following.

**Theorem.** *[5.4.6, 5.4.13, 5.4.14] Let $Y$ be a regular, noetherian and integral scheme such that $\dim Y \geq 1$ and $6 \in \mathcal{O}_Y^*$. If $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in G\text{-Cov}(Y)$ then the associated $G$-cover ($S_3$-cover) $X_\chi \longrightarrow Y$ is regular if and only if the following conditions hold:*

*1) $D_m, D_\omega$ are Cartier divisors and $D_m \cap D_\omega = \emptyset$;*

*2) $Y_\alpha = \emptyset$ or $Y_\alpha$ is regular of pure codimension $2$ in $Y$;*

*3) $D_\omega$ is regular and $D_m$ is regular outside $Y_\alpha$.*

*In this case the triple cover $X_\chi/\sigma \longrightarrow Y$ (which does not depend on whether we see $X$ as a $G$-cover or $S_3$-cover) is regular and, if $(\mathcal{F}, \delta) \in \mathcal{C}_3$ is its associated object, we have: $D_\omega = D(Y_\delta)$, $D_\delta = 2D_\omega + D_m$ and $Y_\delta = D_\omega \sqcup Y_\alpha$. If $f\colon X \longrightarrow Y$ is a regular triple cover associated with $(\mathcal{F}, \delta) \in \mathcal{C}_3(Y)$, then $Y_\delta = D(Y_\delta) \sqcup Y'_\delta$, where $Y'_\delta$ is a closed subscheme of pure codimension $2$ if not empty and $D(Y_\delta)$ is regular. Finally the maps*

$$\begin{array}{ccc}
X & \longmapsto & X/\sigma \\
\{\textit{regular } G\textit{-covers over } Y\} & \longrightarrow & \\
\downarrow\wr & & \left\{ \begin{array}{c} \textit{regular triple covers } (\mathcal{F},\delta) \textit{ over } Y \\ \textit{such that } Y_\delta \textit{ is regular} \end{array} \right\} \\
\{\textit{regular } S_3\textit{-covers over } Y\} & \longrightarrow & \\
(\mathcal{O}_Y(D(Y_\delta)), \mathcal{F}, \delta, 1) & \longleftarrow\!\shortmid & (\mathcal{F}, \delta)
\end{array}$$

*are inverses of each other.*

After this classification of regular $G$-covers and $S_3$-covers two questions naturally arise: how to construct them and how the cohomological invariants of the total space related to those of the base. If $Z$ is a scheme and $\mathcal{E}$ is a coherent sheaf over it, we will say that $\mathcal{E}$ is strongly generated if, for any closed point $q \in Z$, the map $\mathrm{H}^0(Z, \mathcal{E}) \longrightarrow \mathcal{E} \otimes (\mathcal{O}_{Z,p}/m_p^2)$ is surjective. For the first question, we will prove the following result.

**Theorem.** *[5.4.16] Let $k$ be an infinite field with $\operatorname{char} k \neq 2, 3$, $Y$ be a smooth, irreducible and proper $k$-scheme with $\dim Y \geq 1$, $\mathcal{F}$ be a locally free sheaf of rank $2$ over $Y$ and set $\mathcal{E} = \underline{\operatorname{Hom}}(\operatorname{Sym}^3 \mathcal{F}, \det \mathcal{F})$. If $\mathcal{E} \otimes \overline{k}$ is strongly generated (over $Y \times \overline{k}$) then there exists $\delta \in \mathcal{E}$ such that the triple cover associated with $(\mathcal{F}, \delta) \in \mathcal{C}_3(Y)$ extends to a $G$-cover*

*($S_3$-cover) $X_\delta \longrightarrow Y$ with $X_\delta$ smooth and $Y_\delta = \emptyset$ or $\operatorname{codim}_Y Y_\delta = 2$. Moreover, if $Y$ is geometrically connected, then $X_\delta$ is geometrically connected if and only if $\det \mathcal{F} \not\simeq \mathcal{O}_Y$ and $\mathrm{H}^0(Y, \mathcal{F}) = 0$.*

When $Y$ is projective, it is possible to prove that, if $\mathcal{E}(-1)$ is globally generated, then $\mathcal{F}$ satisfies the hypothesis of strong generation in the above theorem (see 5.4.17). For instance $\mathcal{F} = \mathcal{O}_Y(-1)^2$ satisfies such hypothesis and $\det \mathcal{F} \not\simeq \mathcal{O}_Y$ and $\mathrm{H}^0(Y, \mathcal{F}) = 0$. Therefore

**Corollary.** *[5.4.18] Let $k$ be an infinite field with $\operatorname{char} k \neq 2, 3$. Then any smooth, projective and irreducible (resp. geometrically connected) $k$-scheme $Y$ with $\dim Y \geq 1$ has a $G$-cover ($S_3$-cover) $X \longrightarrow Y$ with $X$ smooth (resp. smooth and geometrically connected).*

Finally, when $Y$ is a surface over an algebraically closed field, we will compute the invariants of the total space of a regular $S_3$-cover of $Y$. The result is:

**Theorem.** *[5.4.24] Let $Y$ be a smooth, projective, integral surface over an algebraically closed field $k$ such that $\operatorname{char} k \neq 2, 3$ and $f \colon X \longrightarrow Y$ be a regular $S_3$-cover associated with $(\mathcal{F}, \delta) \in \mathcal{C}_3(Y)$. The closed subscheme $Y_\delta$ of $Y$ is the disjoint union of a smooth divisor $D$ and a finite set $Y_0$ of rational points and the surface $X$ is connected if and only if $\mathrm{H}^0(\mathcal{F}) = 0$ and $\mathcal{O}_Y(-D) \not\simeq \det \mathcal{F}$. In this case the invariants of $X$ are given by*

$$\begin{aligned}
K_X^2 &= 6K_Y^2 + 6c_1(\mathcal{F})^2 - 12c_1(\mathcal{F})K_Y - \frac{10}{3}D^2 - 4DK_Y \\
p_g(X) &= p_g(Y) + 2h^2(\mathcal{F}) + h^2(\mathcal{O}_Y(D) \otimes \det \mathcal{F}) \\
\chi(\mathcal{O}_X) &= 6\chi(\mathcal{O}_Y) - 2c_2(\mathcal{F}) + \frac{1}{2}(3c_1(\mathcal{F})^2 - 3c_1(\mathcal{F})K_Y - DK_Y - D^2) \\
|Y_0| &= 3c_2(\mathcal{F}) - \frac{2}{3}D^2
\end{aligned}$$

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## 1.6 Notation.

### General.

A *cover* is a map of schemes $f\colon X \longrightarrow Y$ which is finite, flat and of finite presentation or, equivalently, which is affine with $f_*\mathcal{O}_X$ locally free of finite rank. We will say that the cover $f$ is regular (resp. regular in codimension 1, normal, normal crossing in codimension 1) if the total space $X$ has the same property. (The definition of normal crossing in codimension 1 will be introduced later.)

If $X$ is a scheme and $p \in X$ we set $\operatorname{codim}_p X = \dim \mathcal{O}_{X,p}$ and we will denote by $X^{(1)} = \{p \in X \mid \operatorname{codim}_p X = 1\}$ the set of codimension 1 points of $X$.

Given $\alpha \in \mathbb{N}$, we will use the following convention

$$0^\alpha = \begin{cases} 1 & \alpha = 0 \\ 0 & \alpha > 0 \end{cases}$$

We denote by (Sets) the category of sets, by $\mathrm{Sch}/S$ the category of schemes over a base scheme $S$ and by (Grps) the category of groups. Given a (fppf) stack $\mathcal{X}$ over a scheme $S$ we will denote by $\mathcal{X}^{\mathrm{gr}}$ the associated stack of groupoids and, if $\mathcal{X}$ is an algebraic stack, we denote by $|\mathcal{X}|$ its associated topological space.

By a Henselian ring we always mean a noetherian local ring which is Henselian. If $A$ is a local ring we will often denote by $m_A$ is maximal ideal. A DVR will be a local discrete valuation ring.

### Sheaf Theory.

Let $S$ be a scheme. We will denote by $\mathrm{QCoh}_S$, $\mathrm{FCoh}_S$, $\mathrm{Loc}_S$ the stacks of quasi-coherent sheaves, finitely presented quasi-coherent sheaves, locally free sheaves of finite rank over $S$ respectively. Let $\mathcal{F} \in \mathrm{QCoh}\, S$. We define the functor $\mathrm{W}(\mathcal{F})\colon (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow (\mathrm{Sets})$ as

$$\mathrm{W}(\mathcal{F})(U \xrightarrow{f} S) = \mathrm{H}^0(U, f^*\mathcal{F})$$

Notice that if $\mathcal{F}$ is a locally free sheaf of finite rank, then $\mathrm{W}(\mathcal{F})$ is smooth and affine over $S$. The expression $s \in \mathcal{F}$ will always mean $s \in \mathcal{F}(S) = \mathrm{H}^0(S, \mathcal{F})$. Moreover we will denote by $V(s)$ the zero locus of $s$ in $S$, i.e. the closed subscheme associated with the sheaf of ideals $\mathrm{Ker}(\mathcal{O}_S \xrightarrow{s} \mathcal{F})$. Given an element $f = (a_1, \dots, a_r) \in \mathbb{Z}^r$ and invertible sheaves $\mathcal{L}_1, \dots, \mathcal{L}_r$ on a scheme we will use the notation

$$\underline{\mathcal{L}}^f = \bigotimes_i \mathcal{L}_i^{\otimes a_i}, \ \mathrm{Sym}^* \underline{\mathcal{L}} = \mathrm{Sym}^*(\mathcal{L}_1, \dots, \mathcal{L}_r) = \bigoplus_{g \in \mathbb{Z}^r} \underline{\mathcal{L}}^g$$

Notice also that, if $\mathcal{L}_i = \mathcal{O}_S$ for all $i$, then there is a canonical isomorphism $\underline{\mathcal{L}}^f \simeq \mathcal{O}$.

## Representation theory.

Let $S$ be a scheme. Given an affine group scheme $f\colon G \longrightarrow S$, we will denote by $\mathcal{O}_S[G] = f_*\mathcal{O}_G$ its associated Hopf algebra and by

$$\Delta_G\colon \mathcal{O}_S[G] \longrightarrow \mathcal{O}_S[G] \otimes \mathcal{O}_S[G],\ \varepsilon_G\colon \mathcal{O}_S[G] \longrightarrow \mathcal{O}_S,\ \sigma_G\colon \mathcal{O}_S[G] \longrightarrow \mathcal{O}_S[G]$$

the co-multiplication, the co-unity and the co-inverse of $G$ respectively. By an action of $G$ on a quasi-coherent sheaf $\mathcal{F}$ over $S$ we mean a left action of $G$ on $\mathrm{W}(\mathcal{F})$, which corresponds to a structure of right $\mathcal{O}_S[G]$-comodule $\mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_S[G]$. We will often also call it a $G$-comodule structure or call $\mathcal{F}$ a $G$-equivariant sheaf. By an action of $G$ on a $S$-scheme $X$ we mean a right action $X \times G \longrightarrow X$. If $X = \operatorname{Spec} \mathscr{A}$, for some $\mathcal{O}_S$-algebra $\mathscr{A}$, this means that we have a $G$-comodule structure $\mathscr{A} \longrightarrow \mathscr{A} \otimes \mathcal{O}_S[G]$ which is an algebra homomorphism, or, equivalently, such that the multiplication $\mathscr{A} \otimes \mathscr{A} \longrightarrow \mathscr{A}$ is $G$-equivariant and $1 \in \mathscr{A}^G$.

Given functors $F,\ H\colon (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow (\mathrm{Sets})$, left actions of $G$ on $F$ and $H$ induce a left action on $\underline{\mathrm{Hom}}(F, H)$ given by

$$\begin{array}{ccc} G \times \underline{\mathrm{Hom}}(F,H) & \longrightarrow & \underline{\mathrm{Hom}}(F,H) \\ (g,\varphi) & \longmapsto & g\varphi g^{-1} \end{array}$$

Let $\mathcal{F}$ be a locally free sheaf of finite rank over $S$ and $\mathcal{H} \in \operatorname{QCoh} S$ ($\operatorname{FCoh} S$). Then $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{H}) \in \operatorname{QCoh} S$ ($\operatorname{FCoh} S$) and we have a natural isomorphism

$$\mathrm{W}(\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{H})) \longrightarrow \underline{\mathrm{Hom}}(\mathrm{W}(\mathcal{F}), \mathrm{W}(\mathcal{H}))$$

In particular, actions of $G$ on $\mathcal{F}$ and $\mathcal{H}$ yield an action of $G$ on $\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{H})$. We denote by $\underline{\mathrm{Hom}}^G(\mathrm{W}(\mathcal{F}), \mathrm{W}(\mathcal{H}))$ (resp. $\underline{\mathrm{End}}^G \mathrm{W}(\mathcal{F})$, $\underline{\mathrm{Aut}}^G \mathrm{W}(\mathcal{F})$) the subfunctor of $\underline{\mathrm{Hom}}(\mathrm{W}(\mathcal{F}), \mathrm{W}(\mathcal{H}))$ (resp. $\underline{\mathrm{End}}\, \mathrm{W}(\mathcal{F})$, $\underline{\mathrm{Aut}}\, \mathrm{W}(\mathcal{F})$) given by the $G$-invariant elements, that are exactly the $G$-equivariant morphisms. In particular we have

$$\mathrm{W}(\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{H}))^G \simeq \underline{\mathrm{Hom}}(\mathrm{W}(\mathcal{F}), \mathrm{W}(\mathcal{H}))^G \simeq \underline{\mathrm{Hom}}^G(\mathrm{W}(\mathcal{F}), \mathrm{W}(\mathcal{H}))$$

The subsheaf of $G$-invariants of $\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{H})$, denoted by $\underline{\mathrm{Hom}}^G(\mathcal{F},\mathcal{H})$, coincides with the subsheaf of morphisms preserving the $G$-comodule structures. Finally set $\underline{\mathrm{End}}^G(\mathcal{F}) = \underline{\mathrm{Hom}}^G(\mathcal{F},\mathcal{F})$.

We will denote by $\mathrm{QCoh}_S^G$, $\mathrm{FCoh}_S^G$, $\mathrm{Loc}_S^G$ the stacks over $S$ of $G$-equivariant quasi-coherent sheaves, finitely presented quasi-coherent sheaves, locally free sheaves of finite rank respectively. If $\mathcal{F} \in \operatorname{QCoh} S$ we will denote by $\underline{\mathcal{F}} \in \operatorname{QCoh}^G S$ the quasi-coherent sheaf $\mathcal{F}$ with the trivial action of $G$. Moreover if $\mathcal{F} \in \operatorname{QCoh}^G S$ and $\delta\colon H \longrightarrow G$ is a morphism from a group scheme $H$ over $S$ we will denote by $\mathrm{R}_H \mathcal{F} \in \operatorname{QCoh}^H S$ the sheaf $\mathcal{F}$ with the $H$-action induced by $\delta$.

By a subgroup scheme $H$ of a flat and finitely presented group scheme $G$ we will always mean a subgroup which is a closed subscheme of $G$ and it is flat and finitely presented over the base. If $N$ is an abelian group we set $\mathrm{D}(N) = \underline{\mathrm{Hom}}_{\mathrm{groups}}(N, \mathbb{G}_m)$ for the diagonalizable group associated with it.

# 2 Preliminaries on Galois covers.

We fix a base scheme $S$ and a flat and finite group scheme $G$ finitely presented over $S$. In this chapter we want to introduce some basic definitions about $G$-covers and prove some general results. This is how the chapter is divided.

*Section 1.* We define the notion of $G$-covers and we introduce the stack $G$-Cov of $G$-covers. We will then prove that $G$-Cov is an algebraic stack containing $\mathrm{B}\,G$ as open substack and, as examples, we will describe $G$-Cov for the groups $G = \mu_2, \mu_3, \alpha_p$.

*Section 2.* We define the main irreducible component $\mathcal{Z}_G$ of $G$-Cov as the schematic closure of $\mathrm{B}\,G$ in $G$-Cov.

*Section 3.* We show that the isomorphisms $\mathrm{B}\,G \simeq \mathrm{B}\,H$ correspond to $(G,H)$-bitorsors and we will explain how they induce isomorphisms $G$-Cov $\simeq H$-Cov.

## 2.1 The stack $G$-Cov.

We start defining the regular representation of a group on itself.

**Definition 2.1.1.** The (right) *regular* action of $G$ on itself is the action given by

$$G \times G \longrightarrow G, \ (x,g) \longrightarrow x \star g = g^{-1}x$$

The *regular representation of* $G$ over $S$ is the sheaf $\mathcal{O}_S[G]$ endowed with the co-module structure $\mathcal{O}_S[G] \xrightarrow{\mu_G} \mathcal{O}_S[G] \otimes \mathcal{O}_S[G]$ induced by the right regular action of $G$ on itself. By definition $\mu_G$ is the composition

$$\mathcal{O}_S[G] \xrightarrow{\Delta} \mathcal{O}_S[G] \otimes \mathcal{O}_S[G] \xrightarrow{\text{swap}} \mathcal{O}_S[G] \otimes \mathcal{O}_S[G] \xrightarrow{\mathrm{id}\otimes\sigma} \mathcal{O}_S[G] \otimes \mathcal{O}_S[G]$$

*Remark.* We have chosen to define the regular action of $G$ on itself by $x \star g = g^{-1}x$ instead of the more usual $x \star g = xg$ because this makes computations natural in other situations. Note that however these two actions are isomorphic.

In what follows, we will denote by $\mathscr{A}$ the regular representation.

**Definition 2.1.2.** Given a scheme $T$ over $S$, a *ramified Galois cover of group* $G$, or simply a *$G$-cover,* over it is a cover $X \xrightarrow{f} T$ together with an action of $G_T$ on it such that there exists an fppf covering $\{U_i \longrightarrow T\}$ and isomorphisms of $G$-comodules

$$(f_*\mathcal{O}_X)_{|U_i} \simeq \mathscr{A}_{|U_i}$$

We will call $G$-Cov$(T)$ the groupoid of $G$-covers over $T$, where the arrows are the $G$-equivariant isomorphisms of schemes over $T$.

The $G$-covers form a stack $G$-Cov over $S$. Moreover any $G$-torsor is a $G$-cover and more precisely we have:

**Proposition 2.1.3.** *$\mathrm{B}\,G$ is an open substack of $G$-*Cov*.*

*Proof.* Given a scheme $U$ over S and a $G$-cover $X = \operatorname{Spec}\mathscr{B}$ over $U$, $X$ is a $G$-torsor if and only if the map $G \times X \longrightarrow X \times X$ is an isomorphism. This map is induced by a map $\mathscr{B}\otimes\mathscr{B} \xrightarrow{h} \mathscr{B}\otimes\mathcal{O}[G_U]$ and so the locus over which $X$ is a $G$-torsor is given by the vanishing of $\operatorname{Coker} h$, which is an open subset. $\square$

In order to prove that $G$-Cov is an algebraic stack we will present it as a quotient stack by a smooth group scheme.

**Proposition 2.1.4.** *The functor*

$$\begin{array}{rcl} (\mathrm{Sch}/S)^{\mathrm{op}} & \xrightarrow{\quad X_G \quad} & (\mathrm{Sets}) \\ T & \longmapsto & \left\{ \begin{array}{c} \textit{algebra structures on } \mathscr{A}_T \\ \textit{in the category of } G\textit{-comodules} \end{array} \right\} \end{array}$$

*is an affine scheme finitely presented over $S$.*

*Proof.* Let $T$ be a scheme over $S$. An element of $X_G(T)$ is given by maps

$$\mathscr{A}_T\otimes\mathscr{A}_T \xrightarrow{m} \mathscr{A}_T,\ \mathcal{O}_T \xrightarrow{e} \mathscr{A}_T$$

for which $\mathscr{A}$ becomes a sheaf of algebras with multiplication $m$ and identity $e(1)$ and such that $\mu$ is a homomorphism of algebras over $\mathcal{O}_T$. In particular $e$ has to be an isomorphism onto $\mathscr{A}^G = \mathcal{O}_T$. Therefore we have an inclusion $X_G \subseteq \underline{\mathrm{Hom}}(\mathrm{W}(\mathscr{A}\otimes\mathscr{A}), \mathrm{W}(\mathscr{A}))\times\mathbb{G}_m$, which turns out to be a closed immersion, since locally, after we choose a basis of $\mathscr{A}$, the above conditions translate into the vanishing of certain polynomials. $\square$

**Proposition 2.1.5.** *$\underline{\mathrm{Aut}}^G\,\mathrm{W}(\mathscr{A})$ is a smooth group scheme finitely presented over $S$.*

*Proof.* If $T$ is an $S$-scheme, the morphisms

$$\begin{array}{rcl} \varepsilon\circ\phi & \longleftarrow\!\!\!\shortmid & \phi \\ \mathcal{O}_T[G]^\vee & \longrightarrow & \underline{\mathrm{End}}^G(\mathscr{A}\otimes\mathcal{O}_T) \\ f & \longmapsto & (f\otimes\mathrm{id})\circ\Delta \end{array}$$

where $\Delta$ and $\varepsilon$ are respectively the co-multiplication and the co-unit of $\mathcal{O}_T[G]$, are inverses of each other. Since

$$\mathrm{W}(\mathcal{O}_S[G]^\vee) \simeq \underline{\mathrm{Hom}}(\mathrm{W}(\mathcal{O}_S[G]), \mathrm{W}(\mathcal{O}_S))$$

we obtain an isomorphism $\underline{\mathrm{End}}^G\,\mathrm{W}(\mathscr{A}) \simeq \mathrm{W}(\mathcal{O}_S[G]^\vee)$, so that $\underline{\mathrm{End}}^G\,\mathrm{W}(\mathscr{A})$ and its open subscheme $\underline{\mathrm{Aut}}^G\,\mathrm{W}(\mathscr{A})$ are smooth and finitely presented over $S$. $\square$

*Remark* 2.1.6. $\underline{\mathrm{Aut}}^G \mathrm{W}(\mathscr{A})$ acts on $X_G$ in the following way. Given a scheme $T$ over $S$, a $G$-equivariant automorphism $f\colon \mathscr{A}_T \longrightarrow \mathscr{A}_T$ and $(m,e) \in X_G(T)$ we can set $f(m,e)$ for the unique structure of sheaf of algebras on $\mathscr{A}_T$ such that $f\colon (\mathscr{A}_T, m, e) \longrightarrow (\mathscr{A}_T, f(m,e))$ is an isomorphism of $\mathcal{O}_T$-algebras.

**Proposition 2.1.7.** *The map* $X_G \xrightarrow{\pi} G\text{-Cov}$*, which sends a structure of algebra* $\chi \in X_G(T)$ *on* $\mathscr{A}_T$ *to the cover* $\mathrm{Spec}(\mathscr{A}_T, \chi) \longrightarrow T$ *is an* $\underline{\mathrm{Aut}}^G \mathrm{W}(\mathscr{A})$*-torsor. In particular*

$$G\text{-Cov} \simeq [X_G/\underline{\mathrm{Aut}}^G \mathrm{W}(\mathscr{A})]$$

*Proof.* Consider a cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X_G \arrow[d, "\pi"] \\ U \arrow[r, "f"'] & G\text{-Cov} \end{tikzcd}$$

where $U$ is a scheme and $f\colon Y \longrightarrow U$ is a $G$-cover. We want to prove that $P$ is an $\underline{\mathrm{Aut}}^G \mathrm{W}(\mathscr{A})$ torsor over $U$ and that the map $P \longrightarrow X_G$ is equivariant. Since $\pi$ is an fppf epimorphism, we can assume that $f$ comes from $X_G$, i.e. $f_*\mathcal{O}_Y = \mathscr{A}_U$ with multiplication $m$ and neutral element $e$. It is now easy to prove that

$$\begin{array}{rcl} \underline{\mathrm{Aut}}^G \mathrm{W}(\mathscr{A}_U) & \xrightarrow{\simeq} & P \\ h & \longmapsto & h(m,e) \end{array}$$

is a bijection and that all the other claims hold. □

Using above propositions we can conclude that:

**Theorem 2.1.8.** *The stack* $G$-Cov *is algebraic and finitely presented over* $S$.

We want now to discuss some examples of $G$-covers. The simplest possible case, is the trivial group $G$ over $\mathbb{Z}$: clearly, in this case, the $G$-covers are the isomorphisms and $G\text{-Cov} \simeq \mathrm{Spec}\,\mathbb{Z}$. Probably the following examples are more interesting.

**Example 2.1.9.** $G = \mu_2 = \mathrm{D}(\mathbb{Z}/2\mathbb{Z})$ over $\mathbb{Z}$. This is very classical. A $\mu_2$-cover over a scheme $T$ is given by an invertible sheaf $\mathcal{L}$ over $T$ with a morphism $\mathcal{L}^2 \longrightarrow \mathcal{O}_T$, where the induced $\mu_2$-cover is $\mathrm{Spec}\,\mathscr{A}$, $\mathscr{A} = \mathcal{O}_T \oplus \mathcal{L}$. In particular

$$\mu_2\text{-Cov} \simeq [\mathbb{A}^1/\mathbb{G}_m]$$

is smooth and irreducible.

**Example 2.1.10.** $G = \mu_3 = \mathrm{D}(\mathbb{Z}/3\mathbb{Z})$ over $\mathbb{Z}$. In [AV04, Lemma 6.2], the authors prove that the data consisting of invertible sheaves $\mathcal{L}_1, \mathcal{L}_2$ over a scheme $T$ and maps $\mathcal{L}_1^2 \longrightarrow \mathcal{L}_2, \mathcal{L}_2^2 \longrightarrow \mathcal{L}_1$ yields a unique algebra structure on $\mathcal{O}_T \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$. It is not difficult to see (and we will prove in the next chapter)that all $\mu_3$-covers can be built in this way. In particular

$$\mu_3\text{-Cov} \simeq [\mathbb{A}^2/\mathbb{G}_m^2]$$

is smooth and irreducible.

In the next chapter, we will see that those cases and the case $\mu_2 \times \mu_2$ are the unique ones for which $G$-Cov has a similar description if $G$ is diagonalizable (see 3.2.19 and 3.2.20). As a last example, we want to describe $\alpha_p$-covers. Remember that $\alpha_p$ is the group scheme over $\mathbb{F}_p$ representing the functor

$$\alpha_p \colon (\mathrm{Sch}/\mathbb{F}_p)^{op} \longrightarrow (\mathrm{Sets}),\ \ \alpha_p(X) = \{x \in \mathcal{O}_X \mid x^p = 0\} < \mathbb{G}_a(X)$$

or, equivalently, the kernel of the Frobenius map $\mathbb{G}_a \longrightarrow \mathbb{G}_a$. The result in this case is quite unexpected from the definition.

**Proposition 2.1.11.** *Let $p$ be a prime. We have an isomorphism of $\mathbb{F}_p$-stacks*

$$\mathrm{B}\,\alpha_p = \alpha_p\text{-Cov} \simeq [\mathbb{A}^1/\mathbb{G}_a]$$

*where the action of $\mathbb{G}_a$ on $\mathbb{A}^1$ is given by $\mathbb{A}^1 \times \mathbb{G}_a \longrightarrow \mathbb{A}^1$, $(x,y) \longmapsto x+y^p$. In particular every $\alpha_p$-cover is an $\alpha_p$-torsor and $\alpha_p$-Cov is smooth and irreducible.*

*Proof.* If $S$ is an $\mathbb{F}_p$-scheme an $\alpha_p$-action on a quasi-coherent sheaf $\mathcal{F}$ is given by a morphism $\gamma \colon \mathcal{F} \longrightarrow \mathcal{F}$ such that $\gamma^p = 0$ (see [DG70, II, § 2, 2.7]). If $(\mathcal{F},\gamma)$ and $(\mathcal{F}',\gamma')$ are $\alpha_p$-equivariant quasi-coherent sheaves, the representation $\mathcal{F} \otimes \mathcal{F}'$ is given by $\gamma \otimes \mathrm{id} + \mathrm{id} \otimes \gamma'$. In particular if $\mathcal{F}$ has an algebra structure, the multiplication is $\alpha_p$-equivariant if and only if $\gamma$ is an $\mathcal{O}_S$-derivation. In conclusion an $\alpha_p$-action on an affine scheme $X = \operatorname{Spec} \mathscr{A}$ over $S$ is given by an $\mathcal{O}_S$-derivation $\partial \colon \mathscr{A} \longrightarrow \mathscr{A}$ such that $\partial^p = 0$. Moreover it is easy to check that the regular representation is $\mathcal{O}_S[\alpha_p] = \mathcal{O}_S[x]/(x^p)$ with the usual derivation of polynomials.

We define the map $\phi \colon \mathbb{A}^1 \longrightarrow \alpha_p$-Cov induced by the $\alpha_p$-cover over $\mathbb{A}^1 = \operatorname{Spec} \mathbb{F}_p[z]$ given by $A = \mathbb{F}_p[z,y]/(y^p - z)$ with the derivation $\partial/\partial y$. This is an $\alpha_p$-torsor because the ring homomorphism

$$A[\alpha_p] = A[x]/(x^p) \longrightarrow A \otimes_{\mathbb{F}_p[z]} A \simeq A[x]/(x^p - z) \text{ given by } x \longmapsto x - y$$

is an $\alpha_p$-equivariant isomorphism. Moreover note that, if $R$ is a ring and $a, b \in R$, then the $\alpha_p$-equivariant isomorphisms

$$R[x]/(x^p - a) \xrightarrow{\psi} R[x]/(x^p - b)$$

are all of the form $\psi(x) = x + u$, where $u \in R$ is such that $a = b + u^p$. Therefore it remains to prove that $\phi$ is an epimorphism. The question is local. So let $R$ be a ring and $A \in \alpha_p$-Cov$(R)$ with an $R$-derivation $\partial$ such that $A$, as $\alpha_p$-module, is the regular representation. In particular there exists a basis $y_0, y_1, \dots, y_{p-1}$ such that $y_0 = 1$ and $\partial y_i = i y_{i-1}$, where we have set $y_{-1} = 0$. It is easy to prove by induction that

$$y_1^k - y_k \in \langle 1, y_1, \dots, y_1^{k-1} \rangle_R \text{ for all } k = 0, \dots, p-1$$

In particular we can write $A \simeq R[y]/(y^p - f(y))$ with $\partial y = 1$ and $\deg f < p$. Moreover the relation $\partial y^p = 0 = \partial f(y)$ tells us that $f \equiv b \in R$, as required. $\square$

**Example 2.1.12.** Let $k$ be a field of characteristic $p > 0$. We construct a cover $f\colon \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$ and actions of $\mu_p$ and $\alpha_p$ on $\mathbb{A}^1_k$ such that $f$ is invariant, it is a torsor for both groups over $\mathbb{G}_{m,k}$, but $f$ is not an $\alpha_p$-cover. This shows that for Galois covers the acting group is not uniquely determined by the cover, as it happens in the étale case. Moreover, the property of being a $G$-cover is not closed in general, while this is true, as we will see, for linearly reductive groups (see 4.3.6). This example has been suggested by Prof. Romagny.

As map $f$ consider the inclusion $k[x^p] \subseteq k[x] = A$. In particular $A \simeq k[x^p][y]/(y^p - x^p)$ and the action of $\mu_p$ is given by setting $\deg y = 1 \in \mathbb{Z}/p\mathbb{Z}$. It is easy to check by a direct computation that $f$ is a $\mu_p$-torsor over $\mathbb{G}_m$. The right action of $\alpha_p$ on $\mathbb{A}^1_k$ is functorially given by the expression

$$z \star s = \frac{z}{1 - sz} \text{ for } z \in \mathbb{A}^1_k(T),\ s \in \alpha_p(T),\ T \in \mathrm{Sch}/k$$

Note that the expression $z \star s = z'$ for $z, z' \in \mathbb{G}_m(T)$ is equivalent to $s = (z' - z)/zz'$ and such $s$ belongs to $\alpha_p(T)$ if and only if $z^p = z'^p$, that is $f(z) = f(z')$. In particular $f$ is an $\alpha_p$-torsor over $\mathbb{G}_m$. The map $f$ is not an $\alpha_p$-cover, or, equivalently, not an $\alpha_p$-torsor, because $0 \star s = 0$ for $0 \in \mathbb{A}^1_k(T), s \in \alpha_p(T), T \in \mathrm{Sch}/k$.

## 2.2 The main irreducible component $\mathcal{Z}_G$.

In this subsection we want to introduce what we will call the main irreducible component of $G$-Cov. In order to do that we recall what is a schematic closure and some of its properties.

**Definition 2.2.1.** Let $\mathcal{X}$ be an algebraic stack and $\mathcal{U} \subseteq \mathcal{X}$ be an open substack. We will say that $\mathcal{U}$ is schematically dense in $\mathcal{X}$ if for any factorization

$$\mathcal{U} \longrightarrow \mathcal{Z} \xrightarrow{j} \mathcal{X}$$

where $j$ is a closed immersion, $j$ is an isomorphism.

Taking into account [Gro66, Theorem 11.10.5] and extending this result to algebraic stacks by taking an atlas, we get

**Proposition 2.2.2.** *Let $\mathcal{X}$ be an algebraic stack and $\mathcal{U}$ be a quasi-compact open substack. Then there exists a minimum closed substack $\mathcal{Z}$ of $\mathcal{X}$ containing $\mathcal{U}$ and $\mathcal{U}$ is schematically dense in $\mathcal{Z}$. Moreover $\mathcal{Z}$ is the closed substack defined by the ideal* $\mathrm{Ker}(\mathcal{O}_\mathcal{X} \longrightarrow \mathcal{O}_\mathcal{U})$ *and $\mathcal{U}$ is topologically dense in $\mathcal{Z}$. Finally, if $f\colon \mathcal{X}' \longrightarrow \mathcal{X}$ is flat, then $\mathcal{X}' \times_\mathcal{X} \mathcal{U}$ is schematically dense in $\mathcal{X}' \times_\mathcal{X} \mathcal{Z}$.*

**Definition 2.2.3.** If $\mathcal{X}, \mathcal{U}$ and $\mathcal{Z}$ are as in 2.2.2, we will call $\mathcal{Z}$ the *schematic closure* of $\mathcal{U}$ in $\mathcal{X}$.

**Definition 2.2.4.** We define the stack $\mathcal{Z}_G$ as the schematic closure of $\mathrm{B}\,G$ in $G$-Cov and we will call it the *main irreducible component of* $G$-Cov.

Notice that, when the base scheme is irreducible, then $\mathcal{Z}_G$ is an irreducible component because $\mathrm{B}\,G \subseteq G\text{-Cov}$ is an irreducible open substack. The formation of $\mathcal{Z}_G$ commutes with flat base changes of the base. Thus, if $G$ is defined over a field, $\mathcal{Z}_G$ commutes with arbitrary base changes and it is therefore geometrically integral.

## 2.3 Bitorsors and Galois covers.

It is a well known result that $G$-equivariant quasi-coherent sheaves can also be thought of as quasi-coherent sheaves on the stack $\mathrm{B}\,G$. In this section we want to use this point of view in order to show examples of groups $G$ and $H$ for which $G\text{-Cov} \simeq H\text{-Cov}$. The idea is that such an isomorphism can be defined as soon as we have an isomorphism $\phi\colon \mathrm{B}\,G \longrightarrow \mathrm{B}\,H$, using the push-forward $\phi_*$ of quasi-coherent sheaves. We will meet this situation when we will study $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$-covers and $S_3$-covers: in this section we will prove that, over the ring $\mathbb{Z}[1/6]$, we have isomorphisms

$$\mathrm{B}(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{B}\,S_3 \text{ and } (\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})\text{-Cov} \simeq S_3\text{-Cov}$$

The content of this section can also be found in [Gir71, Chapter III, Remarque 1.6.7]. We start showing that isomorphisms $\mathrm{B}\,G \simeq \mathrm{B}\,H$ correspond to what we will call $(G,H)$-bitorsors. This correspondence works in great generality, that is as soon as we can talk about torsors and we will present it from a categorical point of view. We refer to [FGI+05, Part 1] for definitions and properties used in this section. In what follows we consider given a site $\mathscr{C}$, that is a category endowed with a Grothendieck topology. We assume that the site $\mathscr{C}$ satisfies the following condition: given an object $S \in \mathscr{C}$ there exists a set of coverings $\mathcal{U}$ of $S$ such that any covering of $S$ is refined by some covering in $\mathcal{U}$. This condition insures that any functor $\mathscr{C}^{op} \longrightarrow (\mathrm{Sets})$ can be sheafified and it is satisfied by the site $\mathrm{Sch}/S$, where $S$ is a scheme, with the fppf topology or the étale topology, which is the case in which we will apply the theory explained below.

We introduce now the concept of $(G,H)$-biactions and $(G,H)$-bitorsors.

**Definition 2.3.1.** Let $G\colon \mathscr{C}^{op} \longrightarrow (\mathrm{Grps})$ be a sheaf of groups. We will denote by $\mathrm{Sh}^G/\mathscr{C}$ the fibered category of sheaves over $\mathscr{C}$ with a right $G$-action. The left (resp. right) regular action of $G$ on itself is the left (resp. right) action given by

$$G \times G \longrightarrow G,\ (g,h) \longrightarrow gh$$

A left (resp. right) $G$-torsor is a sheaf $P\colon \mathscr{C}^{op} \longrightarrow (\mathrm{Sets})$ with a left action $G \times P \longrightarrow P$ (resp. right action $P \times G \longrightarrow P$) for which $P$ is locally isomorphic to $G$ endowed with the left (resp. right) regular action.

If $H$ is another sheaf of groups $\mathscr{C}^{op} \longrightarrow (\mathrm{Grps})$, a $(G,H)$-biaction on a sheaf $P\colon \mathscr{C}^{op} \longrightarrow (\mathrm{Sets})$ is a pair $(G \times P \xrightarrow{u} P, P \times H \xrightarrow{v} P)$ where $u$ and $v$ are, respectively, a left $G$-action and a right $H$-action on $P$, such that the following diagram is commutative.

$$\begin{tikzcd} G \times P \times H \arrow[r, "u\times \mathrm{id}_H"] \arrow[d, "\mathrm{id}_G \times v"] & P \times H \arrow[d, "v"] \\ G \times P \arrow[r, "u"] & P \end{tikzcd}$$

A $(G,H)$-bitorsor is a sheaf $P\colon \mathscr{C}^{op} \longrightarrow$ (Sets) with a $(G,H)$-biaction for which $P$ is both a left $G$-torsor and a right $H$-torsor. Denote by $\mathrm{B}(G,H)$ the fibered category over $\mathscr{C}$ of $(G,H)$-bitorsors.

*Remark* 2.3.2. Note that the right regular representation introduced above differs from the one introduced in 2.1.1. The above definition will be used only in this section because it will simplify the exposition. On the other hand, since the two actions are isomorphic, it is clear that the results obtained below are independent of the choice of the regular representation to use.

*Remark* 2.3.3. The fibered category $\mathrm{B}(G,H)$ is a stack over $\mathscr{C}$. This is easy to prove directly, using the fact that $(\mathrm{Sh}/\mathscr{C})$, the fibered category of sheaves of sets over $\mathscr{C}$, is a stack (see [FGI+05, Part 1, Example 4.11]). Otherwise this can be seen as consequence of the isomorphism proved in 2.3.7.

*Remark* 2.3.4. Given a left $G$-action $u\colon P\times G \longrightarrow P$ and a right $H$-action $v\colon P\times H \longrightarrow P$ , the pair $(u,v)$ is a $(G,H)$-biaction if and only if the homomorphism $G \longrightarrow \underline{\mathrm{Aut}}\, P$ induced by $u$ factors through $\underline{\mathrm{Aut}}^H P$, that is if $G$ acts through $H$-equivariant isomorphisms.

*Remark* 2.3.5. Let $G$ be a sheaf of groups over $\mathscr{C}$. If $X \in (\mathrm{Sh}^G/\mathscr{C})$ there always exists a categorical quotient $X \longrightarrow X/G$, that is a map through which all the $G$-invariant maps $X \longrightarrow Y$, where $Y$ is a sheaf, uniquely factor. Indeed the quotient $X/G$ is the sheafification of the functor $\mathscr{C}^{op} \longrightarrow$ (Sets) that associates with an object $S$ the set $X(S)/G(S)$. In particular, if $H$ is another sheaf of groups, $T \in \mathscr{C}$ and $P \in \mathrm{B}(G,H)(T)$, $X \times P$ has a right action of $G$ given by $(x,p)g = (xg, g^{-1}p)$ and we can consider the quotient $(X\times P)/G$, which has a right $H$-action induced by the one of $P$.

**Proposition 2.3.6.** *Let $G$ and $H$ be sheaves of groups over $\mathscr{C}$. Then the association*

$$\begin{array}{rcl} \mathrm{B}(G,H) & \xrightarrow{\quad\Lambda\quad} & \underline{\mathrm{Iso}}_{\mathscr{C}}((\mathrm{Sh}^G/\mathscr{C}),(\mathrm{Sh}^H/\mathscr{C})) \\ P & \longmapsto & (X \longmapsto (X\times P)/G) \end{array}$$

*is a functor of fibered categories. If $T \in \mathscr{C}$, $X \in (\mathrm{Sh}^G/\mathscr{C})(T)$, $G\times T \simeq H \times T$ and $P$ is a trivial $(G,H)$-bitorsor over $T$, then there exists a natural isomorphism $\Lambda_P(X) \simeq X$ as objects of $\mathscr{C}$.*

The proof of the above statement is not difficult and left to the reader.

**Proposition 2.3.7.** *The functor $\Lambda$ of 2.3.6 induces an isomorphism $\mathrm{B}(G,H) \longrightarrow \underline{\mathrm{Iso}}(\mathrm{B}\,G,\mathrm{B}\,H)$ whose inverse is given by $\phi \longmapsto \phi(G)$, where the left $G$-action on $\phi(G)$ is given by $G \simeq \mathrm{Aut}^G G \simeq \mathrm{Aut}^H \phi(G)$. In particular $(G,H)$-bitorsors are $H$-torsors $P$ with an isomorphism $G \simeq \underline{\mathrm{Aut}}^H P$.*

*Proof.* The only non trivial point is showing the existence of an isomorphism $\Lambda_{\phi(G)} \simeq \phi$. This is induced by the map

$$Q \times \phi(G) \simeq \underline{\mathrm{Hom}}^G(G,Q) \times \phi(G) \longrightarrow \phi(Q)$$

for $Q \in (\mathrm{Sh}^G/\mathscr{C})$, functorially in $Q$. $\square$

**Corollary 2.3.8.** *Let $G$ be a sheaf of groups over $\mathscr{C}$. Then the sheaves of groups $H$ for which there exists an isomorphism $\mathrm{B}\,H \simeq \mathrm{B}\,G$ are the sheaves $\underline{\mathrm{Aut}}^G P$ for $P \in \mathrm{B}\,G$.*

We now want to describe two examples of non trivial bitorsors.

**Example 2.3.9.** If $G$ and $H$ are sheaves of groups of $\mathscr{C}$ set $P = \underline{\mathrm{Iso}}(G,H)$. The maps

$$\underline{\mathrm{Aut}}\,G \times P \longrightarrow P,\ P \times \underline{\mathrm{Aut}}(H) \longrightarrow P, \text{ both given by } (\phi,\psi) \longmapsto \phi \circ \psi$$

induce a $(\underline{\mathrm{Aut}}\,G, \underline{\mathrm{Aut}}\,H)$-action on $\underline{\mathrm{Iso}}(G,H)$ and, if $G$ and $H$ are locally isomorphic, then $\underline{\mathrm{Iso}}(G,H)$ is a $(\underline{\mathrm{Aut}}\,G, \underline{\mathrm{Aut}}\,H)$-bitorsor. In particular, in this case, we obtain an isomorphism

$$\mathrm{B}\,\underline{\mathrm{Aut}}(G) \simeq \mathrm{B}\,\underline{\mathrm{Aut}}(H)$$

The second bitorsor we want to describe is a refinement of the previous one.

**Proposition 2.3.10.** *Let $G$ and $H$ be sheaves of groups $\mathscr{C}^{op} \longrightarrow (\mathrm{Grps})$ and set $P = G \times \underline{\mathrm{Iso}}(H,G)$. The maps*

$$P \times (H \rtimes \underline{\mathrm{Aut}}\,H) \longrightarrow P,\ (G \rtimes \underline{\mathrm{Aut}}\,G) \times P \longrightarrow P, \text{ both given by } (x,\phi)\cdot(y,\psi) = (x\phi(y), \phi\psi)$$

*define a $((G \rtimes \underline{\mathrm{Aut}}\,G),(H \rtimes \underline{\mathrm{Aut}}\,H))$-action on $P$ and, if $G$ and $H$ are locally isomorphic, then $P$ is a $((G \rtimes \underline{\mathrm{Aut}}\,G),(H \rtimes \underline{\mathrm{Aut}}\,H))$-bitorsor. In particular, in this case, we have an isomorphism*

$$\mathrm{B}(G \rtimes \underline{\mathrm{Aut}}\,G) \simeq \mathrm{B}(H \rtimes \underline{\mathrm{Aut}}\,H)$$

*and, if $\Lambda_P \colon (\mathrm{Sh}^{G \rtimes \underline{\mathrm{Aut}}\,G}/\mathscr{C}) \longrightarrow (\mathrm{Sh}^{H \rtimes \underline{\mathrm{Aut}}\,H}/\mathscr{C})$ is the functor defined in 2.3.6, we have a canonical isomorphism*

$$(X/\,\underline{\mathrm{Aut}}\,G) \simeq (\Lambda_P(X)/\,\underline{\mathrm{Aut}}\,H) \text{ for all } X \in (\mathrm{Sh}^{G \rtimes \underline{\mathrm{Aut}}\,G}/\mathscr{C})$$

*Proof.* A direct computation shows that the maps in the statement yield compatible actions. Moreover, if $\gamma \colon G \longrightarrow H$ is an isomorphism, it is also straightforward to check that the maps $g \longmapsto g \cdot \gamma$ and $h \longmapsto \gamma \cdot h$ are equivariant isomorphisms $G \rtimes \underline{\mathrm{Aut}}\,G \longrightarrow P$ and $H \rtimes \underline{\mathrm{Aut}}\,H \longrightarrow P$ respectively. Finally consider the map

$$\pi \colon X \times P = X \times G \times \underline{\mathrm{Iso}}(H,G) \longrightarrow X \text{ given by } \pi(x,g,\phi) = x(g,\mathrm{id}_G)$$

It is easy to check that $\pi(z(u,\psi)) = \pi(z)(1_G,\psi)$ and $\pi(z(1_H,\delta)) = \pi(z)$ for all $(u,\psi) \in G \rtimes \underline{\mathrm{Aut}}\, G$ and $(1_H,\delta) \in H \rtimes \underline{\mathrm{Aut}}\, H$. In particular $\pi$ yields a map $(\Lambda_P(X)/\underline{\mathrm{Aut}}\, H) \longrightarrow (X/\underline{\mathrm{Aut}}\, G)$. This is an isomorphism since it is so locally, i.e. when we have an isomorphism $H \xrightarrow{\phi} G$: in this case the inverse is given by $x \longrightarrow (x, 1_G, \phi)$. $\square$

**Example 2.3.11.** Consider the group $G = \mu_n$ and $H = \mathbb{Z}/n\mathbb{Z}$ over $\mathbb{Z}[1/n]$: they are étale locally isomorphic and therefore

$$\mathrm{B}(\mu_n \rtimes (\mathbb{Z}/n\mathbb{Z})^*) \simeq \mathrm{B}(\mathbb{Z}/n\mathbb{Z} \rtimes (\mathbb{Z}/n\mathbb{Z})^*)$$

In particular for $n = 3$ we get $\mathrm{B}(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{B}\, S_3$.

This is the connection with the theory of Galois covers.

**Theorem 2.3.12.** *Let $G$ and $H$ be flat, finite and finitely presented group schemes over a base scheme $S$. If $P$ is a fppf $(G,H)$-bitorsor over $S$, the functor $\Lambda_P$ of 2.3.6 induces an isomorphism*

$$\begin{array}{ccc} G\text{-Cov} & \xrightarrow{\Lambda_P} & H\text{-Cov} \\ X & \longmapsto & (X \times P)/G \end{array}$$

*In particular, given an $S$-scheme $Y$ and $X \in G\text{-Cov}(Y)$, the cover $X \longrightarrow Y$ (resp. the $S$-scheme $X$) is fppf locally isomorphic to the cover $\Lambda_P(X) \longrightarrow Y$ (resp. the $S$-scheme $\Lambda_P(X)$) and therefore they share all the properties that are local and satisfy descent in the fppf topology. Moreover if $G$ or $H$ is étale, the same conclusion follows for the étale topology.*

*Proof.* It is enough to note that, if $X \in G\text{-Cov}(Y)$ and $P$ is trivial, a section of $P$ induces an isomorphism $\mu\colon H \longrightarrow G$ and $\Lambda_P(X)$ is isomorphic to $X$ with the $H$-action obtained from its $G$-action through $\mu$. $\square$

*Remark* 2.3.13. Let $G$ and $H$ be affine group schemes with an isomorphism $\phi\colon \mathrm{B}\, G \longrightarrow \mathrm{B}\, H$. Since $\mathrm{QCoh}_S^G \simeq \mathrm{QCoh}_{\mathrm{B}\, G}$ and similarly for $H$, the pushforward $\phi_*$ induces an isomorphism $\mathrm{QCoh}_S^G \simeq \mathrm{QCoh}_S^H$. If $\phi$ corresponds to the $(G,H)$-bitorsor $P = \mathrm{Spec}\, \mathscr{A}_P$ over $S$ it is possible to check that we have an isomorphism

$$\phi_* \mathcal{F} \simeq (\mathcal{F} \otimes \mathscr{A}_P)^G$$

where the $H$-action is induced by the one over $\mathscr{A}_P$.

# 3 Galois Covers under diagonalizable group schemes.

The aim of this chapter is to study the theory of $G$-covers in the diagonalizable case. Let $G$ be a finite diagonalizable group scheme over $\mathbb{Z}$. We now briefly summarize how this chapter is divided.

*Section 1.* The stack $G$-Cov and some of its substacks, like $\mathcal{Z}_G$ and $\mathrm{B}\,G$, share a common structure, i.e. they are all of the form $\mathcal{X}_\phi = [\operatorname{Spec}\mathbb{Z}[T_+]/\mathcal{T}]$, where $T_+$ is a finitely generated commutative monoid whose associated group is free of finite rank, $\mathcal{T}$ is a torus over $\mathbb{Z}$ and $\phi\colon T_+ \longrightarrow \mathbb{Z}^r$ is an additive map, that induces the action of $\mathcal{T}$ on $\operatorname{Spec}\mathbb{Z}[T_+]$. The first section will be dedicated to the study of such stacks. As we will see many facts about $G$-Cov are just applications of general results about such stacks. For instance the existence of a special irreducible component $\mathcal{Z}_\phi$ of $\mathcal{X}_\phi$ as well as the use of $T_+^\vee = \operatorname{Hom}(T_+, \mathbb{N})$ for the study of the smooth locus of $\mathcal{Z}_\phi$ are properties that can be stated in this setting.

*Section 2.* We will explain how $G$-Cov can be viewed as a stack of the form $\mathcal{X}_\phi$ and how it is related to the equivariant Hilbert schemes. Then we will study the properties of connectedness, irreducibility and smoothness for $G$-Cov. Finally we will introduce the stratification $\mathrm{B}\,G = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{|G|-1} = G\text{-Cov}$ and we will characterize the locus $U_1$.

*Section 3.* We will study the locus $U_2$ and $G$-covers whose total space is normal crossing in codimension 1.

## 3.1 The stack $\mathcal{X}_\phi$.

In the following sections we will study the stack $G$-Cov when $G = \mathrm{D}(M)$, the diagonalizable group of a finite abelian group $M$. The structure of this stack and of some of its substacks is somehow special and in this section we will provide general constructions and properties that will be used later. To a monoid map $T_+ \xrightarrow{\phi} \mathbb{Z}^r$, we will associate a stack $\mathcal{X}_\phi$ whose objects are sequences of invertible sheaves with additional data and we will study particular 'parametrization' of these objects, defined by a map of stacks $\mathcal{F}_{\underline{\mathcal{E}}} \xrightarrow{\pi_{\underline{\mathcal{E}}}} \mathcal{X}_\phi$, where $\mathcal{F}_{\underline{\mathcal{E}}}$ will be a 'nice' stack, for instance smooth.

In this section we will consider given a commutative monoid $T_+$ together to a monoid map $\phi\colon T_+ \longrightarrow \mathbb{Z}^r$.

**Definition 3.1.1.** We define the stack $\mathcal{X}_\phi$ over $\mathbb{Z}$ as follows.

- *Objects.* An object over a scheme $S$ is a pair $(\underline{\mathcal{L}}, a)$ where:

- $\underline{\mathcal{L}} = \mathcal{L}_1, \dots, \mathcal{L}_r$ are invertible sheaves on $S$;
- $T_+ \xrightarrow{a} \operatorname{Sym}^* \underline{\mathcal{L}}$ is an additive map such that $a(t) \in \underline{\mathcal{L}}^{\phi(t)}$ for any $t \in T_+$.

- *Arrows.* An isomorphism $(\underline{\mathcal{L}}, a) \xrightarrow{\underline{\sigma}} (\underline{\mathcal{L}}', a')$ of objects over $S$ is given by a sequence $\underline{\sigma} = \sigma_1, \dots, \sigma_r$ of isomorphisms $\sigma_i \colon \mathcal{L}_i \xrightarrow{\simeq} \mathcal{L}'_i$ such that

$$\underline{\sigma}^{\phi(t)}(a(t)) = a'(t) \text{ for any } t \in T_+$$

**Example 3.1.2.** Let $f_1, \dots, f_s, g_1, \dots, g_t \in \mathbb{Z}^r$ and consider the stack $\mathcal{X}_{\underline{f},\underline{g}}$ of invertible sheaves $\mathcal{L}_1, \dots, \mathcal{L}_r$ with maps $\mathcal{O} \longrightarrow \underline{\mathcal{L}}^{f_i}$ and $\mathcal{O} \xrightarrow{\simeq} \underline{\mathcal{L}}^{g_j}$. If $T_+ = \mathbb{N}^s \times \mathbb{Z}^t$ and $\phi \colon T_+ \longrightarrow \mathbb{Z}^r$ is the map given by the matrix $(f_1|\cdots|f_s|g_1|\cdots|g_t)$ then $\mathcal{X}_{\underline{f},\underline{g}} = \mathcal{X}_\phi$.

*Notation* 3.1.3. We set

$$\mathbb{Z}[T_+] = \mathbb{Z}[x_t]_{t \in T_+}/(x_t x_{t'} - x_{t+t'}, x_0 - 1)$$

and $\mathcal{O}_S[T_+] = \mathbb{Z}[T_+] \otimes_{\mathbb{Z}} \mathcal{O}_S$. The scheme $\operatorname{Spec} \mathcal{O}_S[T_+]$ over $S$ represents the functor that associates to any scheme $U/S$ the set of additive maps $T_+ \longrightarrow (\mathcal{O}_U, \cdot)$, where $\cdot$ denotes the multiplication on $\mathcal{O}_U$. The group $\mathrm{D}(\mathbb{Z}^r)$ acts on $\operatorname{Spec} \mathbb{Z}[T_+]$ by the graduation $\deg x_t = \phi(t)$.

**Proposition 3.1.4.** *Set $X = \operatorname{Spec} \mathbb{Z}[T_+]$. The choice $\mathcal{L}_i = \mathcal{O}_X$ and*

$$\begin{array}{ccc} \underline{\mathcal{L}}^{\phi(t)} & \xrightarrow{\simeq} & \mathcal{O}_X \\ a(t) & \longleftrightarrow & x_t \end{array}$$

*induces a smooth epimorphism $X \longrightarrow \mathcal{X}_\phi$ such that $\mathcal{X}_\phi \simeq [X/\mathrm{D}(\mathbb{Z}^r)]$. In particular $\mathcal{X}_\phi$ is an algebraic stack.*

*Proof.* It is enough to note that an object of $[X/\mathrm{D}(\mathbb{Z}^r)](U)$ is given by invertible sheaves $\mathcal{L}_1, \dots, \mathcal{L}_r$ with a $\mathrm{D}(\mathbb{Z}^r)$-equivariant map $\operatorname{Spec} \operatorname{Sym}^* \underline{\mathcal{L}} \longrightarrow \operatorname{Spec} \mathbb{Z}[T_+]$ which exactly corresponds to an additive map $T_+ \longrightarrow \operatorname{Sym}^* \underline{\mathcal{L}}$ as in the definition of $\mathcal{X}_\phi$. It is easy to check that the map $X \longrightarrow [X/\mathrm{D}(\mathbb{Z}^r)] \longrightarrow \mathcal{X}_\phi$ is the one defined in the statement. $\square$

*Remark* 3.1.5. Given a map $U \xrightarrow{a} X = \operatorname{Spec} \mathbb{Z}[T_+]$, i.e. a monoid map $T_+ \xrightarrow{a} \mathcal{O}_U$, the induced object $U \xrightarrow{a} X \longrightarrow \mathcal{X}_\phi$ is the pair $(\underline{\mathcal{L}}, \tilde{a})$ where $\mathcal{L}_i = \mathcal{O}_U$ and for any $t \in T_+$

$$\begin{array}{ccc} \mathcal{O}_U & \xrightarrow{\simeq} & \underline{\mathcal{L}}^{\phi(t)} \\ a(t) & \longmapsto & \tilde{a}(t) \end{array}$$

We will denote by $a$ also the object $(\underline{\mathcal{L}}, \tilde{a}) \in \mathcal{X}_\phi(U)$.

Given two elements $a, b \colon T_+ \longrightarrow \mathcal{O}_U \in \mathcal{X}_\phi(U)$ we have

$$\operatorname{Iso}_{\mathcal{X}_\phi(U)}(a, b) = \{\sigma_1, \dots, \sigma_r \in \mathcal{O}_U^* \mid \underline{\sigma}^{\phi(t)} a(t) = b(t) \ \forall t \in T_+\}$$

**Lemma 3.1.6.** *Consider a commutative diagram*

$$\begin{tikzcd} T_+ \arrow[r, "h"] \arrow[d, "\phi"'] & T'_+ \arrow[d, "\psi"] \\ \mathbb{Z}^r \arrow[r, "g"] & \mathbb{Z}^s \end{tikzcd}$$

*where $T_+, T'_+$ are commutative monoids and $\phi$, $\psi$, $h$, $g$ are additive maps. Then we have a 2-commutative diagram*

$$\begin{tikzcd} \operatorname{Spec}\mathbb{Z}[T'_+] \arrow[r, "h^*"] \arrow[d] & \operatorname{Spec}\mathbb{Z}[T_+] \arrow[d] \\ \mathcal{X}_\psi \arrow[r, "\Lambda"] & \mathcal{X}_\phi \end{tikzcd}$$

$$(\underline{\mathcal{L}}, T'_+ \xrightarrow{a} \operatorname{Sym}^* \underline{\mathcal{L}}) \mapsto (\underline{\mathcal{M}}, T_+ \xrightarrow{b} \operatorname{Sym}^* \underline{\mathcal{M}}) \tag{3.1.1}$$

*where, for $i = 1, \dots, r$, $\mathcal{M}_i = \underline{\mathcal{L}}^{g(e_i)}$ and $b$ is the unique map such that*

$$\begin{tikzcd} T_+ \arrow[r, "b"] \arrow[d, "h"'] & \operatorname{Sym}^* \underline{\mathcal{M}} \arrow[d] & \underline{\mathcal{M}}^v \simeq \underline{\mathcal{L}}^{g(v)} \arrow[d] \arrow[dl, phantom, ""] \\ T'_+ \arrow[r, "a"'] & \operatorname{Sym}^* \underline{\mathcal{L}} & \underline{\mathcal{L}}^{g(v)} \end{tikzcd}$$

*Proof.* An easy computation shows that there is a canonical isomorphism $\underline{\mathcal{M}}^v \simeq \underline{\mathcal{L}}^{g(v)}$ for all $v \in \mathbb{Z}^r$ and so $b(t)$ corresponds under this isomorphism to $a(h(t)) \in \underline{\mathcal{L}}^{\psi(h(t))} = \underline{\mathcal{L}}^{g(\phi(t))} \simeq \underline{\mathcal{M}}^{\phi(t)}$. So the functor $\Lambda$ is well defined and we have only to check the commutativity of the second diagram in the statement. The map $\operatorname{Spec}\mathbb{Z}[T'_+] \longrightarrow \operatorname{Spec}\mathbb{Z}[T_+] \longrightarrow \mathcal{X}_\phi$ is given by trivial invertible sheaves and the additive map

$$\begin{array}{ccccc} T_+ & \to & \mathbb{Z}[T_+][x_1, \dots, x_r]_{\prod_i x_i} & \to & \mathbb{Z}[T'_+][x_1, \dots, x_r]_{\prod_i x_i} \\ t & \longmapsto & x_t x^{\phi(t)} & \longmapsto & x_{h(t)} x^{\phi(t)} \end{array}$$

Instead the map $\operatorname{Spec}\mathbb{Z}[T'_+] \longrightarrow \mathcal{X}_\psi \longrightarrow \mathcal{X}_\phi$ is given by trivial invertible sheaves and the map $b$ that makes the following diagram commutative

$$\begin{tikzcd} T_+ \arrow[r, "b"] \arrow[d, "h"'] & \mathbb{Z}[T'_+][x_1, \dots, x_r]_{\prod_i x_i} \arrow[d] & x^v \arrow[d, mapsto] \\ T'_+ \arrow[r, "a"] & \mathbb{Z}[T'_+][y_1, \dots, y_s]_{\prod_i y_i} & y^{g(v)} \\ t \arrow[r, mapsto] & x_t y^{\psi(t)} & \end{tikzcd}$$

Since $x_{h(t)}x^{\phi(t)}$ is sent to $x_{h(t)}y^{g(\phi(t))} = x_{h(t)}y^{\psi(h(t))} = a(h(t))$ we find again $b(t) = x_{h(t)}x^{\phi(t)}$. $\square$

*Remark* 3.1.7. The functor $\mathcal{X}_\psi \longrightarrow \mathcal{X}_\phi$ sends an element $a\colon T'_+ \longrightarrow \mathcal{O}_U \in \mathcal{X}_\psi(U)$ to the element $a \circ h \in \mathcal{X}_\phi(U)$. Moreover, taking into account the description given in 3.1.5, if $a, b\colon T'_+ \longrightarrow \mathcal{O}_U \in \mathcal{X}_\psi(U)$ we have

$$\begin{aligned} \mathrm{Iso}_U(a,b) &\longrightarrow \mathrm{Iso}_U(a\circ h, b\circ h) \\ \underline{\sigma} &\longmapsto \underline{\sigma}^{g(e_1)}, \dots, \underline{\sigma}^{g(e_r)} \end{aligned}$$

### 3.1.1 The main irreducible component $\mathcal{Z}_\phi$ of $\mathcal{X}_\phi$.

*Notation* 3.1.8. A monoid will be called *integral* if it satisfies the cancellation law, i.e.

$$\forall a,b,c, \qquad a+b = a+c \implies b = c$$

Let $T_+$ be a monoid. There exists, up to a unique isomorphism, a group $T$ (resp. integral monoid $T_+^{int}$) such that any monoid map $T_+ \longrightarrow S_+$, where $S_+$ is a group (resp. integral monoid), factors uniquely through $T$ (resp. $T_+^{int}$). We call it the *associated group* (resp. *associated integral monoid*) of $T_+$. Notice that if $T$ is the associated group of $T_+$, then $\mathrm{Im}(T_+ \longrightarrow T)$ can be chosen as the associated integral monoid of $T_+$. We will continue to denote by $T$ the associated group of $T_+$ and we set $T_+^{int} = \mathrm{Im}(T_+ \longrightarrow T) \subseteq T$. In particular $\langle T_+^{int}\rangle_\mathbb{Z} = T$.

From now on $T_+$ will be a finitely generated monoid whose associated group is a free $\mathbb{Z}$-module of finite rank. In order to simplify notation, we will often write $\phi\colon T \longrightarrow \mathbb{Z}^r$, meaning the extension of $\phi\colon T_+ \longrightarrow \mathbb{Z}^r$ to $T$. Anyway, the stack $\mathcal{X}_\phi$ will always be the stack $\mathcal{X}_{T_+\longrightarrow \mathbb{Z}^r}$ and when we will have to consider the stack $\mathcal{X}_{T\longrightarrow\mathbb{Z}^r}$, we will always specify a different symbol for the induced map $T \longrightarrow \mathbb{Z}$.

*Remark* 3.1.9. If $D$ is a domain, then $\mathrm{Spec}\, D[T]$ is an open subscheme of $\mathrm{Spec}\, D[T_+]$, while $\mathrm{Spec}\, D[T_+^{int}]$ is one of its irreducible components. In particular we have

**Proposition 3.1.10.** *Let $\hat{\phi}\colon T \longrightarrow \mathbb{Z}^r$ be the extension of $\phi$ and set $\phi^{int} = \hat{\phi}_{|T_+^{int}}$. Then $\mathcal{B}_\phi = \mathcal{X}_{\hat{\phi}} \longrightarrow \mathcal{X}_\phi$ is an open immersion, while $\mathcal{Z}_\phi = \mathcal{X}_{\phi^{int}} \longrightarrow \mathcal{X}_\phi$ is a closed one. Moreover $\mathcal{Z}_\phi$ is the reduced closed stack associated to the closure of $\mathcal{B}_\phi$, it is an irreducible component of $\mathcal{X}_\phi$ and*

$$\mathcal{B}_\phi \simeq [\mathrm{Spec}\,\mathbb{Z}[T]/\mathrm{D}(\mathbb{Z}^r)] \text{ and } \mathcal{Z}_\phi \simeq [\mathrm{Spec}\,\mathbb{Z}[T_+^{int}]/\mathrm{D}(\mathbb{Z}^r)]$$

**Definition 3.1.11.** With notation above we will call respectively $\mathcal{B}_\phi$ and $\mathcal{Z}_\phi$ the *principal open substack* and the *main irreducible component* of $\mathcal{X}_\phi$.

*Notation* 3.1.12. We set

$$T_+^\vee = \mathrm{Hom}(T_+, \mathbb{N}) = \{\mathcal{E} \in \mathrm{Hom}_{\mathrm{groups}}(T,\mathbb{Z}) \mid \mathcal{E}(T_+) \subseteq \mathbb{N}\}$$

We will call it the *dual monoid* of $T_+$ and we will call its elements the *rays* for $T_+$. Note that $T_+^\vee = T_+^{int\vee}$. Given $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s \in T_+^\vee$ we will denote by $\underline{\mathcal{E}}$ also the induced map $T \longrightarrow \mathbb{Z}^s$. Moreover we set

$$\operatorname{Supp} \underline{\mathcal{E}} = \{v \in T_+ \mid \exists i\ \mathcal{E}^i(v) > 0\}$$

Finally notice that the dual monoid of a group is always 0. Therefore, when $H$ is an abelian group, the dual $H^\vee$ of $H$ will always be the dual as $\mathbb{Z}$-module.

**Definition 3.1.13.** Given a sequence $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s \in T_+^\vee$ set

$$\begin{array}{ccc} \mathbb{N}^s \oplus T & \xrightarrow{\sigma_{\underline{\mathcal{E}}}} & \mathbb{Z}^s \oplus \mathbb{Z}^r \\ (e_i, 0) & \longmapsto & (e_i, 0) \\ (0, t) & \longmapsto & (\underline{\mathcal{E}}(t), -\phi(t)) \end{array}$$

where $e_1, \dots, e_s$ is the canonical basis of $\mathbb{Z}^s$. We will call $\mathcal{F}_{\underline{\mathcal{E}}} = \mathcal{X}_{\sigma_{\underline{\mathcal{E}}}}$.

*Remark* 3.1.14. An object of $\mathcal{F}_{\underline{\mathcal{E}}}$ over a scheme $U$ is given by a sequence $(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda)$ where:

- $\underline{\mathcal{L}} = \mathcal{L}_1, \dots, \mathcal{L}_r$ and $\underline{\mathcal{M}} = (\mathcal{M}_\mathcal{E})_{\mathcal{E} \in \underline{\mathcal{E}}} = \mathcal{M}_1, \dots, \mathcal{M}_s$ are invertible sheaves on $U$;
- $\underline{z} = (z_\mathcal{E})_{\mathcal{E} \in \underline{\mathcal{E}}} = z_1, \dots, z_s$ are sections $z_i \in \mathcal{M}_i$;
- for any $t \in T$, $\lambda(t) = \lambda_t$ is an isomorphism $\underline{\mathcal{L}}^{\phi(t)} \xrightarrow{\simeq} \underline{\mathcal{M}}^{\underline{\mathcal{E}}(t)}$ additive in $t$.

An isomorphism $(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \longrightarrow (\underline{\mathcal{L}}', \underline{\mathcal{M}}', \underline{z}', \lambda')$ is a pair $(\underline{\omega}, \underline{\tau})$ where $\underline{\omega} = \omega_1, \dots, \omega_r$, $\underline{\tau} = \tau_1, \dots, \tau_s$ are sequences of isomorphisms $\mathcal{L}_i \xrightarrow{\omega_i} \mathcal{L}_i'$, $\mathcal{M}_j \xrightarrow{\tau_j} \mathcal{M}_j'$ such that $\tau_j(z_j) = z_j'$ and for any $t \in T$ we have a commutative diagram

$$\begin{array}{ccc} \underline{\mathcal{L}}^{\phi(t)} & \xrightarrow{\lambda_t} & \underline{\mathcal{M}}^{\underline{\mathcal{E}}(t)} \\ {\scriptstyle \underline{\omega}^{\phi(t)}} \downarrow & & \downarrow {\scriptstyle \underline{\tau}^{\phi(t)}} \\ \underline{\mathcal{L}}'^{\phi(t)} & \xrightarrow{\lambda_t'} & \underline{\mathcal{M}}'^{\underline{\mathcal{E}}(t)} \end{array}$$

An object over $U$ coming from the atlas $\operatorname{Spec} \mathbb{Z}[\mathbb{N}^s \oplus T]$ is a pair $(\underline{z}, \lambda)$ where $\underline{z} = z_1, \dots, z_s \in \mathcal{O}_U$ and $\lambda \colon T \longrightarrow \mathcal{O}_U^*$ is a group homomorphism. Given $(\underline{z}, \lambda), (\underline{z}', \lambda') \in \mathcal{F}_{\underline{\mathcal{E}}}(U)$ we have

$$\operatorname{Iso}_U((\underline{z}, \lambda), (\underline{z}', \lambda')) = \{(\underline{\omega}, \underline{\tau}) \in (\mathcal{O}_U^*)^r \times (\mathcal{O}_U^*)^s \mid \tau_i z_i = z_i',\ \underline{\tau}^{\underline{\mathcal{E}}(t)} \lambda(t) = \underline{\omega}^{\phi(t)} \lambda'(t)\}$$

**Definition 3.1.15.** Given a sequence $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s$ of elements of $T_+^\vee$ we define the map

$$\pi_{\underline{\mathcal{E}}} \colon \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{X}_\phi$$

induced by the commutative diagram

$$\begin{tikzcd} t \arrow[d, mapsto] & T_+ \arrow[r, "\phi"] \arrow[d] & \mathbb{Z}^r \arrow[d, hook] \\ (\underline{\mathcal{E}}(t), -t) & \mathbb{N}^s \oplus T \arrow[r, "\sigma_{\underline{\mathcal{E}}}"'] & \mathbb{Z}^s \oplus \mathbb{Z}^r \end{tikzcd}$$

*Remark* 3.1.16. We can describe the functor $\pi_{\underline{\mathcal{E}}}$ explicitly. So suppose we have an object $\chi = (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}(U)$. We have $\pi_{\underline{\mathcal{E}}}(\chi) = (\underline{\mathcal{L}}, a) \in \mathcal{X}_\phi(U)$ where $a$ is given, for any $t \in T_+$, by

$$\begin{array}{ccc} \underline{\mathcal{L}}^{\phi(t)} & \xrightarrow{\lambda_t} & \underline{\mathcal{M}}^{\mathcal{E}(t)} \\ a(t) & \longmapsto & \underline{z}^{\mathcal{E}(t)} \end{array}$$

Moreover, if $(\underline{\omega}, \underline{\tau})$ is an isomorphism in $\mathcal{F}_{\underline{\mathcal{E}}}$, then $\pi_{\underline{\mathcal{E}}}(\underline{\omega}, \underline{\tau}) = \underline{\omega}$.

If $(\underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}(U)$ then $a = \pi_{\underline{\mathcal{E}}}(\underline{z}, \lambda) \in \mathcal{X}_\phi(U)$ is given by

$$\begin{array}{rcl} T_+ & \longrightarrow & \mathcal{O}_U \\ t & \longmapsto & \underline{z}^{\mathcal{E}(t)}/\lambda_t = z_1^{\mathcal{E}^1(t)} \cdots z_s^{\mathcal{E}^s(t)}/\lambda_t \end{array}$$

*Remark* 3.1.17. If $\underline{\mathcal{E}} = (\mathcal{E}^i)_{i\in I}$ is a sequence of elements of $T_+^\vee$, $J \subseteq I$ and we set $\underline{\delta} = (\mathcal{E}^j)_{j \in J}$ we can define a map over $\mathcal{X}_\phi$ as

$$\begin{array}{ccc} \mathcal{F}_{\underline{\delta}} & \xrightarrow{\rho} & \mathcal{F}_{\underline{\mathcal{E}}} \\ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) & \longmapsto & (\underline{\mathcal{L}}, \underline{\mathcal{M}}', \underline{z}', \lambda) \end{array} \quad \mathcal{M}'_i = \left\{ \begin{array}{ll} \mathcal{M}_i & i \in J \\ \mathcal{O} & i \notin J \end{array} \right. \quad z'_i = \left\{ \begin{array}{ll} z_i & i \in J \\ 1 & i \notin J \end{array} \right.$$

In fact $\rho$ comes from the monoid map $T \oplus \mathbb{N}^I \longrightarrow T \oplus \mathbb{N}^J$ induced by the projection. Moreover $\rho$ is an open immersion, whose image is the open substack of $\mathcal{F}_{\underline{\mathcal{E}}}$ of objects $(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda)$ such that $z_i$ generates $\mathcal{M}_i$ for all $i \notin J$. We will often consider $\mathcal{F}_{\underline{\delta}}$ as an open substack of $\mathcal{F}_{\underline{\mathcal{E}}}$.

**Definition 3.1.18.** Given a sequence $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s$ of elements of $T_+^\vee$ we define

$$T_+^{\underline{\mathcal{E}}} = T_+^{\mathcal{E}^1,\dots,\mathcal{E}^s} = \{v \in T \mid \forall i\, \mathcal{E}^i(v) \geq 0\}$$

We also consider the case $s = 0$, so that $T_+^{\underline{\mathcal{E}}} = T$. If we denote by $\hat{\phi}\colon T_+^{\underline{\mathcal{E}}} \longrightarrow \mathbb{Z}^r$ the extension of $\phi$, we also define $\mathcal{X}_\phi^{\underline{\mathcal{E}}} = \mathcal{Z}_\phi^{\underline{\mathcal{E}}} = \mathcal{X}_{\hat{\phi}}$.

*Remark* 3.1.19. Assume we have a monoid map $T_+ \longrightarrow T'_+$ (compatible with $\phi$ and $\phi'$) inducing an isomorphism on the associated groups. If $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s \in T'^\vee_+ \subseteq T_+^\vee$, then we have a 2-commutative diagram

$$\begin{tikzcd} \mathcal{F}'_{\underline{\mathcal{E}}} \arrow[r, "\simeq"] \arrow[d, "\pi'_{\underline{\mathcal{E}}}"'] & \mathcal{F}_{\underline{\mathcal{E}}} \arrow[d, "\pi_{\underline{\mathcal{E}}}"] \\ \mathcal{X}_{\phi'} \arrow[r] & \mathcal{X}_\phi \end{tikzcd}$$

where $\mathcal{F}'_{\underline{\mathcal{E}}}$ is the stack obtained from $T'_+$ with respect to $\underline{\mathcal{E}}$.

**Proposition 3.1.20.** *The map $\pi_{\underline{\mathcal{E}}} \colon \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{X}_\phi$ has a natural factorization*

$$\mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{X}_\phi^{\underline{\mathcal{E}}} \longrightarrow \mathcal{Z}_\phi \longrightarrow \mathcal{X}_\phi$$

*Proof.* The factorization follows from 3.1.19 taking monoid maps $T_+ \longrightarrow T_+^{int} \longrightarrow T_+^{\underline{\mathcal{E}}}$. ◻

*Remark* 3.1.21. This shows that $\pi_{\underline{\mathcal{E}}}$ has image in $\mathcal{Z}_\phi$.We will call with the same symbol $\pi_{\underline{\mathcal{E}}}$ the factorization $\mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{Z}_\phi$.

We want now to show how the rays of $T_+$ can be used to describe the objects of $\mathcal{Z}_\phi$ over a field. Using notation from 3.1.5, the result is:

**Theorem 3.1.22.** *Let $k$ be a field and $T_+ \xrightarrow{a} k \in \mathcal{X}_\phi(k)$. Then $a \in \mathcal{Z}_\phi(k)$ if and only if there exists a group homomorphism $\lambda : T \longrightarrow \overline{k}^*$ and $\mathcal{E} \in T_+^\vee$ such that*

$$a(t) = \lambda_t 0^{\mathcal{E}(t)}$$

*In particular if $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r$ generate $T_+^\vee \otimes \mathbb{Q}$ then $\pi_{\underline{\mathcal{E}}} \colon \mathcal{F}_{\underline{\mathcal{E}}}(\overline{k}) \longrightarrow \mathcal{Z}_\phi(\overline{k})$ is essentially surjective and so $\pi_{\underline{\mathcal{E}}} \colon |\mathcal{F}_{\underline{\mathcal{E}}}| \longrightarrow |\mathcal{Z}_\phi|$ is surjective. Finally, if the map $\phi \colon T \longrightarrow \mathbb{Z}^r$ is injective, we have a one to one correspondence*

$$\begin{array}{ccc} \mathcal{Z}_\phi(\overline{k})/\simeq & \xrightarrow{\gamma} & \{X \subseteq T_+ \mid X = \operatorname{Supp} \mathcal{E} \text{ for } \mathcal{E} \in T_+^\vee\} \\ a & \longmapsto & \{a = 0\} \end{array}$$

*In particular $|\mathcal{Z}_\phi| = (\mathcal{Z}_\phi(\overline{\mathbb{Q}})/\simeq) \bigsqcup [\bigsqcup_{\text{primes } p} (\mathcal{Z}_\phi(\overline{\mathbb{F}_p})/\simeq)]$.*

Before proving this Theorem we need some preliminary results, that will be useful also later.

**Definition 3.1.23.** If $T_+$ is integral, $\mathcal{E} \in T_+^\vee$ and $k$ is a field we define

$$p_\mathcal{E} = \bigoplus_{v \in T_+, \mathcal{E}(v) > 0} k x_v \subseteq k[T_+]$$

If $p \in \operatorname{Spec} k[T_+]$ we set $p^{om} = \bigoplus_{x_v \in p} k x_v$.

The suffix $(-)^{om}$ here stays for 'homogeneous', since, when $T_+ = \mathbb{N}^r$ and $k[T_+] = k[x_1, \dots, x_r]$, $p^{om}$ is a homogeneous ideal, actually a monomial ideal.

**Lemma 3.1.24.** *Let $k$ be a field and assume that $T_+$ is integral. Then:*

1) *if $\mathcal{E} \in T_+^\vee$, $p_\mathcal{E}$ is prime and $k[\{v \in T_+ \mid \mathcal{E}(v) = 0\}] \longrightarrow k[T_+] \longrightarrow k[T_+]/p_\mathcal{E}$ is an isomorphism.*

2) *If $p \in \operatorname{Spec} k[T_+]$ then $p^{om} = p_\mathcal{E}$ for some $\mathcal{E} \in T_+^\vee$.*

*Proof.* (1) It is obvious.

(2) $p^{om}$ is a prime thanks to [KR05, Proposition 1.7.12] and therefore $p^{om} = p_{\mathcal{E}}$ for some $\mathcal{E} \in T_+^\vee$ thanks to [Ogu12, Chapter I, Corollary 2.2.4]. $\square$

*Remark* 3.1.25. If $k$ is an algebraically closed field, $\phi\colon T \longrightarrow \mathbb{Z}^r$ is injective and $a, b \in \mathcal{X}_\phi(k)$ differ by a torsor, i.e. there exists $\lambda\colon T_+ \longrightarrow k^*$ such that $a = \lambda b$, then $a \simeq b$ in $\mathcal{Z}_\phi(k)$. Indeed $\lambda$ extends to a map $T \longrightarrow k^*$ and, since $k$ is algebraically closed, it extends again to a map $\lambda\colon \mathbb{Z}^r \longrightarrow k^*$.

*Proof.* (of Theorem 3.1.22) We can assume that $k$ is algebraically closed and that $T_+$ is integral, since if $a$ has an expression as in the statement then clearly $a \in \mathcal{Z}_\phi(k)$. Consider $p = \operatorname{Ker}(k[T_+] \xrightarrow{a} k)$. Thanks to 3.1.24, we can write $p^{om} = p_{\mathcal{E}}$ for some $\mathcal{E} \in T_+^\vee$. Set $T'_+ = \{v \in T_+ \mid \mathcal{E}(v) = 0\}$ and $T' = \langle T'_+ \rangle_{\mathbb{Z}}$. Since $a$ maps $T'_+$ to $k^*$, there exists an extension $\lambda\colon T' \longrightarrow k^*$. On the other hand, since $k$ is algebraically closed, the inclusion $T' \longrightarrow T$ yields a surjection

$$\operatorname{Hom}(T, k^*) \longrightarrow \operatorname{Hom}(T', k^*)$$

and so we can extend again to an element $\lambda\colon T \longrightarrow k^*$. Since one has $\operatorname{Supp}\mathcal{E} = \{a = 0\}$ by construction, it is easy to check that $a(t) = \lambda_t 0^{\mathcal{E}(t)}$ for all $t \in T_+$.

Now consider the last part of the statement and so assume $\phi\colon T \longrightarrow \mathbb{Z}^r$ injective. The map $\gamma$ is well defined thanks to above and surjective since, given $\mathcal{E} \in T_+^\vee$, one can always define $a(t) = 0^{\mathcal{E}(t)}$. For the injectivity, let $a, b \in \mathcal{Z}_\phi(k)$ be such that $\{a = 0\} = \{b = 0\}$. We can write $a(t) = \lambda_t 0^{\mathcal{E}(t)}$, $b(t) = \mu_t 0^{\mathcal{E}(t)}$, where $\lambda, \mu\colon T \longrightarrow k^*$, so that $a, b$ differ by a torsor and are therefore isomorphic thanks to 3.1.25. Finally, since any point of $|\mathcal{Z}_\phi|$ comes from an object of $\mathcal{Z}_\phi(\mathbb{Z})$, we also have the last equality. $\square$

In some cases the description of the objects of $\mathcal{F}_{\underline{\mathcal{E}}}$ can be simplified, regardless of $\underline{\mathcal{E}}$, in the sense that there exist a stack of reduced data $\mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red}}$, whose objects can be described by less data, and an isomorphism $\mathcal{F}_{\underline{\mathcal{E}}} \simeq \mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red}}$. This kind of simplification could be very useful when we have to deal with an explicit map of monoids $\phi\colon T_+ \longrightarrow \mathbb{Z}^r$, as we will see in 3.2.7. The idea is that in order to define an object $(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}$, we do not really need all the invertible sheaves $\mathcal{L}_1, \dots, \mathcal{L}_r$, because they are uniquely determined by a subset of them and the other data.

**Definition 3.1.26.** Assume $T \xrightarrow{\phi} \mathbb{Z}^r$ injective. Let $V \subseteq \mathbb{Z}^r$ be a submodule with a given basis $v_1, \dots, v_q$ and $\sigma\colon \mathbb{Z}^r \longrightarrow V$ be a map such that $(\mathrm{id} - \sigma)\mathbb{Z}^r \subseteq T$ (or equivalently $\pi = \pi \circ \sigma$ where $\pi$ is the projection $\mathbb{Z}^r \longrightarrow \operatorname{Coker}\phi$). Define $W = \langle (\mathrm{id} - \sigma)V, \sigma T \rangle \subseteq V$. Given $\underline{\mathcal{E}} = \mathcal{E}^1, \dots \mathcal{E}^l \in T_+^\vee$ consider the map

$$\begin{array}{ccc} W \oplus \mathbb{N}^l & \xrightarrow{\psi_{\underline{\mathcal{E}},\sigma}} & \mathbb{Z}^q \oplus \mathbb{Z}^l \\ (w, z) & \longmapsto & (-w, \underline{\mathcal{E}}(w) + z) \end{array}$$

We define $\mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red},\sigma} = \mathcal{X}_{\psi_{\underline{\mathcal{E}},\sigma}}$ and we call it the stack of reduced data of $\underline{\mathcal{E}}$.

**Lemma 3.1.27.** *Consider a submodule $U \subseteq \mathbb{Z}^p$, a map $\underline{\mathcal{E}}\colon U \longrightarrow \mathbb{Z}^l$ and $\tau\colon \mathbb{Z}^p \longrightarrow \mathbb{Z}^p$ such that $(\mathrm{id}-\tau)\mathbb{Z}^p \subseteq U$. Consider the commutative diagram*

$$\begin{array}{ccc}
(u,z) & U\oplus\mathbb{N}^l \xrightarrow{\tau\oplus\mathrm{id}} U\oplus\mathbb{N}^l \\
\downarrow & \psi\downarrow \qquad\qquad \downarrow\psi \\
(-u,\underline{\mathcal{E}}(u)+z) & \mathbb{Z}^p\oplus\mathbb{Z}^l \longrightarrow \mathbb{Z}^p\oplus\mathbb{Z}^l \\
& (u,z)\longmapsto(\tau u,\underline{\mathcal{E}}(u-\tau u)+z)
\end{array}$$

*Then the induced map $\varphi\colon \mathcal{X}_\psi \longrightarrow \mathcal{X}_\psi$ is isomorphic to $\mathrm{id}_{\mathcal{X}_\psi}$.*

*Proof.* Let $x_1,\dots,x_p$ be a $\mathbb{Z}$-basis of $\mathbb{Z}^p$ with $a_1,\dots,a_k \in \mathbb{N}$ such that $a_1x_1,\dots,a_kx_k$ is a $\mathbb{Z}$-basis of $U$. We want to define a natural isomorphism $\mathrm{id}_{\mathcal{X}_\psi} \xrightarrow{\omega} \varphi$. First note that it is enough to define it on the objects of $\mathcal{X}_\psi$ coming from the atlas $\operatorname{Spec}\mathbb{Z}[U\oplus\mathbb{N}^l]$, prove the naturality between such objects on a fixed scheme $T$ and for the restrictions. An object coming from the atlas is of the form $(\lambda,\underline{z})$ where $\lambda\colon U\longrightarrow\mathcal{O}_T^*$ is an additive map and $\underline{z}=z_1,\dots,z_l\in\mathcal{O}_T$. Moreover $\varphi(\lambda,\underline{z})=(\tilde{\lambda},\underline{z})$ where $\tilde{\lambda}=\lambda\circ\tau$. Let $\underline{\eta}\in \mathrm{D}(\mathbb{Z}^p)(T)$ the only elements such that $\underline{\eta}^{x_i}=\lambda(x_i-\tau x_i)$ for $i=1,\dots,p$. These objects are well defined since $(\mathrm{id}-\tau)\mathbb{Z}^p\subseteq U$. We claim that $\omega_{T,(\lambda,\underline{z})}=(\underline{\eta},\underline{1})$ is an isomorphism $(\lambda,\underline{z})\longrightarrow\varphi(\lambda,\underline{z})$ and define a natural transformation. It is an isomorphism since $1z_j=z_j$ and the condition

$$\underline{\eta}^{-u}\underline{1}^{\underline{\mathcal{E}}(u)}\lambda(u)=\lambda(\tau u)\ \forall u\in U$$

holds by construction checking it on the basis $a_1x_1,\dots,a_kx_k$ of $U$ (see 3.1.5). It is also easy to check that this isomorphisms commute with the change of basis. So it remains to prove that, if $(\underline{\sigma},\underline{\mu})$ is an isomorphism $(\lambda,\underline{z})\longrightarrow(\lambda',\underline{z}')$ then we have a commutative diagram

$$\begin{array}{ccc}
(\lambda,\underline{z}) & \xrightarrow{(\underline{\sigma},\underline{\mu})} & (\lambda',\underline{z}') \\
{\scriptstyle\omega_{T,(\lambda,\underline{z})}}\downarrow & & \downarrow{\scriptstyle\omega_{T,(\lambda',\underline{z}')}} \\
\varphi(\lambda,\underline{z}) & \xrightarrow{\varphi(\underline{\sigma},\underline{\mu})} & \varphi(\lambda',\underline{z}')
\end{array}$$

We have $\varphi(\underline{\sigma},\underline{\mu})=(\underline{\tilde{\sigma}},\underline{\tilde{\mu}})$ with $\underline{\tilde{\mu}}=\underline{\mu}$ and $\underline{\tilde{\sigma}}^{x_i}=\underline{\sigma}^{\tau x_i}\underline{\mu}^{\underline{\mathcal{E}}(x_i-\tau x_i)}$ (see 3.1.7). So it is easy to check that the commutativity in the second member holds. For the first, the condition is $\underline{\tilde{\sigma}}\underline{\eta}=\underline{\eta}'\underline{\sigma}$, which is equivalent to

$$(\underline{\tilde{\sigma}}\underline{\eta})^{x_i}=\underline{\sigma}^{\tau x_i}\underline{\mu}^{\underline{\mathcal{E}}(x_i-\tau x_i)}\lambda(x_i-\tau x_i)=(\underline{\eta}'\underline{\sigma})^{x_i}=\lambda'(x_i-\tau x_i)\underline{\sigma}^{x_i}$$

and to $\underline{\sigma}^{-(x_i-\tau x_i)}\underline{\mu}^{\underline{\mathcal{E}}(x_i-\tau x_i)}\lambda(x_i-\tau x_i)=\lambda'(x_i-\tau x_i)$ for any $i$. But, since $(\underline{\sigma},\underline{\mu})$ is an isomorphism $(\lambda,\underline{z})\longrightarrow(\lambda',\underline{z}')$, the condition

$$\underline{\sigma}^{-u}\underline{\mu}^{\underline{\mathcal{E}}(u)}\lambda(u)=\lambda'(u)\ \forall u\in U$$

has to be satisfied. ∎

**Proposition 3.1.28.** *Assume $T \xrightarrow{\phi} \mathbb{Z}^r$ injective and let $\underline{\mathcal{E}} = \mathcal{E}^1, \dots \mathcal{E}^r \in T_+^\vee$ and $\sigma$, $V$, $v_1, \dots, v_q$ be as in 3.1.26. For appropriate choices of isomorphisms $\tilde{\lambda}$ given by 3.1.6, the functors*

$$\begin{array}{ccc} ((\underline{\mathcal{N}}^{\sigma e_i} \otimes \underline{\mathcal{M}}^{\underline{\mathcal{E}}(e_i - \sigma e_i)})_{i=1,\dots,r}, \underline{\mathcal{M}}, \underline{z}, \tilde{\lambda}) & \longleftarrow\!\shortmid & (\underline{\mathcal{N}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \\ \mathcal{F}_{\underline{\mathcal{E}}} & \longleftrightarrow & \mathcal{F}_{\underline{\mathcal{E}}}^{red,\sigma} \\ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) & \longmapsto & ((\underline{\mathcal{L}}^{v_i})_{i=1,\dots,q}, \underline{\mathcal{M}}, \underline{z}, \lambda_{|W}) \end{array}$$

*are inverses of each other.*

*Proof.* Consider the commutative diagrams

$$\begin{tikzcd} W \oplus \mathbb{N}^s \arrow[r, hook] \arrow[d, "\psi"'] & T \oplus \mathbb{N}^s \arrow[d, "\phi_{\underline{\mathcal{E}}}"] \\ \mathbb{Z}^q \oplus \mathbb{Z}^s \arrow[r, hook] & \mathbb{Z}^r \oplus \mathbb{Z}^s \end{tikzcd} \qquad \begin{tikzcd} T \oplus \mathbb{N}^s \arrow[r, "\sigma \oplus \mathrm{id}"] \arrow[d, "\phi_{\underline{\mathcal{E}}}"'] & W \oplus \mathbb{N}^s \arrow[d, "\psi"] \\ \mathbb{Z}^r \oplus \mathbb{Z}^s \arrow[r] & \mathbb{Z}^q \oplus \mathbb{Z}^s \\ (x,y) \arrow[r, mapsto] & (\sigma x, \underline{\mathcal{E}}(x - \sigma x) + y) \end{tikzcd}$$

They induce functors $\Lambda \colon \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red},\sigma}$ and $\Delta \colon \mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red},\sigma} \longrightarrow \mathcal{F}_{\underline{\mathcal{E}}}$ respectively, that behave as the functors of the statement thanks to the description given in 3.1.6. Finally, applying 3.1.27, we obtain that $\Lambda \circ \Delta \simeq \mathrm{id}$ and $\Delta \circ \Lambda \simeq \mathrm{id}$. □

### 3.1.2 Extremal rays and smooth sequences.

We continue to use notation from 3.1.8. We have seen that given a collection $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in T_+^\vee$ we can associate to it a stack $\mathcal{F}_{\underline{\mathcal{E}}}$ and a 'parametrization' map $\mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{X}_\phi$. The stack $\mathcal{F}_{\underline{\mathcal{E}}}$ could be 'too big' if we do not make an appropriate choice of the collection $\underline{\mathcal{E}}$. This happens for example if the rays in $\underline{\mathcal{E}}$ are not distinct or, more generally, if a ray in $\underline{\mathcal{E}}$ belongs to the submonoid generated by the other rays in $\underline{\mathcal{E}}$. Thus we want to restrict our attention to a special class of rays, called extremal and to special sequences of them.

**Definition 3.1.29.** An *extremal* ray for $T_+$ is an element $\mathcal{E} \in T_+^\vee$ such that

- $\mathcal{E}$ has minimal non empty support, i.e. the set $\operatorname{Supp} \mathcal{E} \subseteq T_+$ is minimal in

$$(\{X \subseteq T_+ \mid X \neq \emptyset \text{ and } X = \operatorname{Supp} \delta \text{ for some } \delta \in T_+{}^\vee\}, \subseteq)$$

- $\mathcal{E}$ is normalized, i.e. $\mathcal{E} \colon T \longrightarrow \mathbb{Z}$ is surjective.

**Lemma 3.1.30.** *Assume that $T_+$ is an integral monoid and let $v_1, \dots, v_l$ be a system of generators of $T_+$. Then the extremal rays are the normalized $\mathcal{E} \in T_+^\vee - \{0\}$ such that* $\operatorname{Ker} \mathcal{E}$ *contains* $\operatorname{rk} T - 1$ $\mathbb{Q}$*-independent vectors among the $v_1, \dots, v_l$. In particular they are finitely many and they generate $\mathbb{Q}_+ T_+^\vee$.*

*Proof.* Denote by $\Omega \subseteq T_+^\vee$ the set of elements defined in the statement. From [Ful93, Section 1.2, (9)] it follows that $\mathbb{Q}_+\Omega = \mathbb{Q}_+T_+^\vee$. If $\mathcal{E} \in \Omega$ then it is an extremal ray. Indeed

$$\emptyset \neq \operatorname{Supp} \mathcal{E}' \subseteq \operatorname{Supp} \mathcal{E} \implies \exists \lambda \in \mathbb{Q}_+ \text{ s.t. } \mathcal{E}' = \lambda \mathcal{E} \implies \operatorname{Supp} \mathcal{E}' = \operatorname{Supp} \mathcal{E}$$

Conversely let $\mathcal{E}$ be an extremal ray and consider an expression

$$\mathcal{E} = \sum_{\delta \in \Omega} \lambda_\delta \delta \qquad \text{with } \lambda_\delta \in \mathbb{Q}_{\geq 0}$$

There must exists $\delta$ such that $\lambda_\delta \neq 0$. So

$$\operatorname{Supp} \delta \subseteq \operatorname{Supp} \mathcal{E} \implies \operatorname{Supp} \delta = \operatorname{Supp} \mathcal{E} \implies \exists \mu \in \mathbb{Q}_+ \text{ s.t. } \mathcal{E} = \mu \delta \implies \mathcal{E} = \delta$$

□

**Corollary 3.1.31.** *For an extremal ray $\mathcal{E}$ and $\mathcal{E}' \in T_+^\vee$ we have*

$$\operatorname{Supp} \mathcal{E}' = \operatorname{Supp} \mathcal{E} \iff \exists \lambda \in \mathbb{Q}_+ \textit{ s.t. } \mathcal{E}' = \lambda \mathcal{E} \iff \exists \lambda \in \mathbb{N}_+ \textit{ s.t. } \mathcal{E}' = \lambda \mathcal{E}$$

**Definition 3.1.32.** An element $v \in T_+$ is called *indecomposable* if whenever $v = v' + v''$ with $v', v'' \in T_+$ it follows that $v' = 0$ or $v'' = 0$.

**Proposition 3.1.33.** *$T_+^\vee$ has a unique minimal system of generators composed by the indecomposable elements. Moreover any extremal ray is indecomposable.*

*Proof.* The first claim of the statement follows from [Ogu12, Chapter I, Proposition 2.1.2] since $T_+^\vee$ is sharp, i.e. it does not contain invertible elements. For the second consider an extremal ray $\mathcal{E}$ and assume $\mathcal{E} = \mathcal{E}' + \mathcal{E}''$. We have

$$\operatorname{Supp} \mathcal{E}', \operatorname{Supp} \mathcal{E}'' \subseteq \operatorname{Supp} \mathcal{E} \implies \mathcal{E}' = \lambda \mathcal{E}, \mathcal{E}'' = \mu \mathcal{E} \text{ with } \lambda, \mu \in \mathbb{N}$$

and so $\mathcal{E} = (\lambda + \mu)\mathcal{E} \implies \lambda + \mu = 1 \implies \lambda = 0$ or $\mu = 0 \implies \mathcal{E}' = 0$ or $\mathcal{E}'' = 0$. □

**Definition 3.1.34.** A *smooth sequence* for $T_+$ is a sequence $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s \in T_+^\vee$ for which there exist elements $v_1, \dots, v_s$ in the associated integral monoid $T_+^{int}$ of $T_+$ such that

$$T_+^{int} \cap \operatorname{Ker} \underline{\mathcal{E}} \text{ generates } \operatorname{Ker} \underline{\mathcal{E}} \qquad \text{and} \qquad \mathcal{E}^i(v_j) = \delta_{i,j} \text{ for all } i, j$$

We will also say that a ray $\mathcal{E} \in T_+^\vee - \{0\}$ is *smooth* if there exists a smooth sequence as above such that $\mathcal{E} \in \langle \mathcal{E}^1, \dots, \mathcal{E}^s \rangle_\mathbb{N}$ or, equivalently, such that $\operatorname{Supp} \mathcal{E} \subseteq \operatorname{Supp} \underline{\mathcal{E}}$.

*Remark* 3.1.35. If $T_+$ is integral and $\Omega$ is a system of generators, one can always assume that $v_i \in \Omega$. Moreover we also have that $\Omega \cap \operatorname{Ker} \underline{\mathcal{E}}$ generates $\operatorname{Ker} \underline{\mathcal{E}}$.

Finally the equivalence in the last sentence of Definition 3.1.34 follows from the fact that, since $\operatorname{Ker} \underline{\mathcal{E}}$ is generated by elements in $T_+^{int}$, then the inclusion of the supports implies that $\mathcal{E}_{|\operatorname{Ker} \underline{\mathcal{E}}} = 0$ and therefore $\mathcal{E} = \sum_i \mathcal{E}(v_i)\mathcal{E}^i$.

**Lemma 3.1.36.** *Let $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r$ be a smooth sequence. Then*

$$T_+^{\underline{\mathcal{E}}} = \operatorname{Ker} \underline{\mathcal{E}} \oplus \langle v_1, \dots, v_r \rangle_{\mathbb{N}} \subseteq T \text{ where } v_1 \dots, v_r \in T_+^{int},\ \mathcal{E}^i(v_j) = \delta_{i,j}$$

*Moreover, if $z_1, \dots, z_s \in T_+^{int}$ generate $T_+^{int}$, then $\mathbb{Z}[T_+^{\underline{\mathcal{E}}}] = \mathbb{Z}[T_+^{int}]_{\prod_{\underline{\mathcal{E}}(z_i)=0} x_{z_i}}$ so that* $\operatorname{Spec} \mathbb{Z}[T_+^{\underline{\mathcal{E}}}]$ *($\mathcal{X}_\phi^{\underline{\mathcal{E}}}$) is a smooth open subscheme (substack) of* $\operatorname{Spec} \mathbb{Z}[T_+^{int}]$ *($\mathcal{Z}_\phi$).*

*Proof.* We have $T = \operatorname{Ker} \underline{\mathcal{E}} \oplus \langle v_1, \dots, v_r \rangle_{\mathbb{Z}}$ and clearly $\operatorname{Ker} \underline{\mathcal{E}} \oplus \langle v_1, \dots, v_q \rangle_{\mathbb{N}} \subseteq T_+^{\underline{\mathcal{E}}}$. Conversely if $v \in T_+^{\underline{\mathcal{E}}}$ we can write

$$v = z + \sum_i \mathcal{E}^i(v) v_i \text{ with } z \in \operatorname{Ker} \underline{\mathcal{E}} \implies v \in \operatorname{Ker} \underline{\mathcal{E}} \oplus \langle v_1, \dots, v_q \rangle_{\mathbb{N}}$$

In particular $\operatorname{Spec} \mathbb{Z}[T_+^{\underline{\mathcal{E}}}] \simeq \mathbb{A}_{\mathbb{Z}}^r \times D_{\mathbb{Z}}(\operatorname{Ker} \underline{\mathcal{E}})$ and so both $\operatorname{Spec} \mathbb{Z}[T_+^{\underline{\mathcal{E}}}]$ and $\mathcal{X}_\phi^{\underline{\mathcal{E}}}$ are smooth. Now let

$$I = \{ i \mid \underline{\mathcal{E}}(z_i) = 0 \} \text{ and } S_+ = \langle T_+^{int}, -z_i \text{ for } i \in I \rangle \subseteq T$$

We need to prove that $S_+ = T_+^{\underline{\mathcal{E}}}$. Clearly we have the inclusion $\subseteq$. For the reverse inclusion, it is enough to prove that $- \operatorname{Ker} \underline{\mathcal{E}} \cap T_+^{int} \subseteq S_+$. But if $v \in \operatorname{Ker} \underline{\mathcal{E}} \cap T_+^{int}$ then

$$v = \sum_{j=1}^s a_j z_j = \sum_{j \in I} a_j z_j \implies -v \in S_+$$

$\square$

*Remark* 3.1.37. Any subsequence of a smooth sequence is smooth too. Indeed let $\underline{\delta} = \mathcal{E}^1, \dots, \mathcal{E}^s$ a subsequence of a smooth sequence $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r$, with $r > s$. We have to prove that $\langle \operatorname{Ker} \underline{\delta} \cap T_+^{int} \rangle_{\mathbb{Z}} = \operatorname{Ker} \underline{\delta}$. Take $v \in \operatorname{Ker} \underline{\delta}$. So

$$v - \sum_{j=s+1}^r \mathcal{E}^j(v) v_j \in \operatorname{Ker} \underline{\mathcal{E}} = \langle \operatorname{Ker} \underline{\mathcal{E}} \cap T_+^{int} \rangle_{\mathbb{Z}} \subseteq \langle \operatorname{Ker} \underline{\delta} \cap T_+^{int} \rangle_{\mathbb{Z}} \implies v \in \langle \operatorname{Ker} \underline{\delta} \cap T_+^{int} \rangle_{\mathbb{Z}}$$

**Proposition 3.1.38.** *Let $\mathcal{E} \in T_+^\vee$. Then $\mathcal{E}$ is a smooth extremal ray if and only if $\mathcal{E}$ is a smooth sequence composed of one element, i.e.* $\operatorname{Ker} \mathcal{E} \cap T_+^{int}$ *generates* $\operatorname{Ker} \mathcal{E}$ *and there exists $v \in T_+$ such that $\mathcal{E}(v) = 1$.*

*In particular any element of a smooth sequence is a smooth extremal ray.*

*Proof.* We can assume $T_+$ integral. If $\mathcal{E}$ is smooth and extremal, then there exists a smooth sequence $\mathcal{E}^1, \dots, \mathcal{E}^q$ such that $\mathcal{E} \in \langle \mathcal{E}^1, \dots, \mathcal{E}^q \rangle_{\mathbb{N}}$. Since $\mathcal{E}$ is indecomposable, it follows that $\mathcal{E} = \mathcal{E}^i$ for some $i$. Conversely assume that $\mathcal{E}$ is a smooth sequence. So it is smooth by definition and it is normalized since $\mathcal{E}(v) = 1$ for some $v$. Finally an inclusion $\operatorname{Supp} \delta \subseteq \operatorname{Supp} \mathcal{E}$ for $\delta \in T_+^\vee$ means that $\delta \in \langle \mathcal{E} \rangle_{\mathbb{N}}$, as remarked in 3.1.35, and so $\operatorname{Supp} \delta = \emptyset$ or $\operatorname{Supp} \delta = \operatorname{Supp} \mathcal{E}$. $\square$

We conclude with a lemma that will be useful later.

**Lemma 3.1.39.** *Let $T_+$, $T'_+$ be integral monoids and $h\colon T \longrightarrow T'$ be a homomorphism such that $h(T_+) = T'_+$ and $\operatorname{Ker} h = \langle \operatorname{Ker} h \cap T_+ \rangle$. If $\underline{\mathcal{E}} = \mathcal{E}^1, \dots \mathcal{E}^r \in {T'_+}^{\vee}$ then*

$$\underline{\mathcal{E}} \text{ smooth sequence for } T'_+ \iff \underline{\mathcal{E}} \circ h \text{ smooth sequence for } T_+$$

*Proof.* Clearly there exist $v_i \in T'_+$ such that $\mathcal{E}^i(v_j) = \delta_{i,j}$ if and only if there exist $w_i \in T_+$ such that $\mathcal{E}^i \circ h(w_j) = \delta_{i,j}$. On the other hand we have a surjective morphism

$$\operatorname{Ker} \underline{\mathcal{E}} \circ h / \langle \operatorname{Ker} \underline{\mathcal{E}} \circ h \cap T_+ \rangle_{\mathbb{Z}} \longrightarrow \operatorname{Ker} \underline{\mathcal{E}} / \langle \operatorname{Ker} \underline{\mathcal{E}} \cap T'_+ \rangle_{\mathbb{Z}}$$

In order to conclude it is enough to prove that this map is injective. So let $v \in T$ such that

$$h(v) = \sum_j a_j z_j \text{ with } a_j \in \mathbb{Z},\ z_j \in T'_+,\ \underline{\mathcal{E}}(z_j) = 0$$

Since $h(T_+) = T'_+$, there exist $y_j \in T_+$ such that $h(y_j) = z_j$. In particular $y = \sum_j a_j y_j \in \langle \operatorname{Ker} \underline{\mathcal{E}} \circ h \cap T_+ \rangle_{\mathbb{Z}}$ and

$$v - y \in \operatorname{Ker} h = \langle \operatorname{Ker} h \cap T_+ \rangle \subseteq \langle \operatorname{Ker} \underline{\mathcal{E}} \circ h \cap T_+ \rangle$$

□

### 3.1.3 The smooth locus $\mathcal{Z}_\phi^{\mathsf{sm}}$ of the main component $\mathcal{Z}_\phi$.

**Lemma 3.1.40.** *Let $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^q$ be a smooth sequence and $\chi$ be a finite sequence of elements of $T_+^\vee$. Assume that all the elements of $\chi$ are distinct, each $\mathcal{E}^i$ is an element of $\chi$ and that for any $\delta$ in $\chi$ we have*

$$\delta \in \langle \mathcal{E}^1, \dots, \mathcal{E}^q \rangle_{\mathbb{N}} \implies \exists i\ \delta = \mathcal{E}^i$$

*As usual denote by $\pi_\chi$ the map $\mathcal{F}_\chi \longrightarrow \mathcal{X}_\phi$. Then we have an equivalence*

$$\mathcal{F}_{\underline{\mathcal{E}}} = \pi_\chi^{-1}(\mathcal{X}_\phi^{\underline{\mathcal{E}}}) \xrightarrow{\simeq} \mathcal{X}_\phi^{\underline{\mathcal{E}}}$$

*Proof.* Set $\chi = \mathcal{E}^1, \dots, \mathcal{E}^q, \eta^1, \dots, \eta^l = \underline{\mathcal{E}}, \underline{\eta}$. We first prove that $\pi_\chi^{-1}(\mathcal{X}_\phi^{\underline{\mathcal{E}}}) \subseteq \mathcal{F}_{\underline{\mathcal{E}}}$. Since they are open substacks, we can check this over an algebraically closed field $k$. Let $(\underline{z}, \lambda) \in \pi_\chi^{-1}(\mathcal{X}_\phi^{\underline{\mathcal{E}}})$ so that $a = \pi_\chi(\underline{z}, \lambda) = \underline{z}^{\underline{\mathcal{E}}}/\lambda\colon T_+ \longrightarrow k$ by 3.1.16. We have to prove that $z_{\eta_j} \neq 0$. Assume by contradiction that $z_{\eta_j} = 0$. Since we can write $a = b0^{\eta_j}$ and since $a$ extends to $T_+^{\underline{\mathcal{E}}}$ so that $a(t) \neq 0$ if $t \in T_+ \cap \operatorname{Ker} \underline{\mathcal{E}}$, we have that $\eta_j$ is 0 on $T_+ \cap \operatorname{Ker} \underline{\mathcal{E}}$. In particular

$$\operatorname{Supp} \eta^j \subseteq \operatorname{Supp} \underline{\mathcal{E}} \implies \eta^j \in \langle \mathcal{E}^1, \dots, \mathcal{E}^q \rangle_{\mathbb{N}} \implies \exists i\ \eta^j = \mathcal{E}^i$$

Thanks to 3.1.17, it is enough to prove that if $\underline{\mathcal{E}}$ is a smooth sequence such that $T_+ = T_+^{\underline{\mathcal{E}}}$ then $\pi_{\underline{\mathcal{E}}}$ is an isomorphism. By 3.1.36 we can write $T_+ = W \oplus \mathbb{N}^q$, where $W$ is a free $\mathbb{Z}$-module such that $\underline{\mathcal{E}}_{|W} = 0$ and, if we denote by $v_1, \dots, v_q$ the canonical base of $\mathbb{N}^q$, $\mathcal{E}^j(v_i) = \delta_{i,j}$. Consider the diagram

$$\begin{tikzcd} \mathbb{N}^q \oplus T \arrow[d, equal] & T_+ \arrow[d, equal] \\ \mathbb{N}^q \oplus W \oplus \mathbb{Z}^q \arrow[r, "\gamma"] \arrow[d, "\sigma_{\underline{\mathcal{E}}}"'] & W \oplus \mathbb{N}^q \arrow[d, "\phi"] \\ \mathbb{Z}^q \oplus \mathbb{Z}^r \arrow[r, "\delta"] & \mathbb{Z}^r \end{tikzcd} \qquad \begin{array}{c} \gamma(e_i) = v_i,\ \gamma_{|W} = -\mathrm{id}_W,\ \gamma(v_i) = 0 \\ \delta(e_i) = \phi(v_i),\ \delta_{|\mathbb{Z}^r} = \mathrm{id}_{\mathbb{Z}^r} \end{array}$$

One can check directly its commutativity. In this way we get a map $s\colon \mathcal{X}_\phi \longrightarrow \mathcal{F}_{\underline{\mathcal{E}}}$. Again a direct computation on the diagrams defining $s$ and $\pi_{\underline{\mathcal{E}}}$ shows that $\pi_{\underline{\mathcal{E}}} \circ s \simeq \mathrm{id}_{\mathcal{X}_\phi}$ and that the diagram inducing $G = s \circ \pi_{\underline{\mathcal{E}}}$ is

$$\begin{tikzcd} \mathbb{N}^q \oplus W \oplus \mathbb{Z}^q \arrow[r, "\alpha"] \arrow[d, "\sigma_{\underline{\mathcal{E}}}"'] & \mathbb{N}^q \oplus W \oplus \mathbb{Z}^q \arrow[d, "\sigma_{\underline{\mathcal{E}}}"] \\ \mathbb{Z}^q \oplus \mathbb{Z}^r \arrow[r, "\beta"] & \mathbb{Z}^q \oplus \mathbb{Z}^r \end{tikzcd} \qquad \begin{array}{c} \alpha(e_i) = e_i - v_i, \alpha_{|W} = \mathrm{id}_W,\ \alpha_{|\mathbb{Z}^q} = 0 \\ \beta(e_i) = \phi(v_i),\ \beta_{|\mathbb{Z}^r} = \mathrm{id}_{\mathbb{Z}^r} \end{array}$$

We will prove that $G \simeq \mathrm{id}_{\mathcal{F}_{\underline{\mathcal{E}}}}$. An object of $\mathcal{F}_{\underline{\mathcal{E}}}(A)$, where $A$ is a ring, coming from the atlas is given by $a = (\underline{z}, \lambda, \underline{\mu})\colon \mathbb{N}^q \oplus W \oplus \mathbb{Z}^q \longrightarrow A$ where $\underline{z} = (a(e_i))_i = z_1, \dots, z_q \in A$, $\lambda = a_{|W}\colon W \longrightarrow A^*$ is a homomorphism and $\underline{\mu} = (\mu(v_i))_i = \mu_1 \dots, \mu_q \in A^*$. Moreover $Ga = a \circ \alpha$ is $((z_i/\mu_i)_i, \lambda, \underline{1})$. It is now easy to check that $(\underline{\mu}, 1)\colon Ga \longrightarrow a$ is an isomorphism and that this map defines an isomorphism $G \longrightarrow \mathrm{id}_{\mathcal{F}_{\underline{\mathcal{E}}}}$. $\square$

**Corollary 3.1.41.** *If $\underline{\mathcal{E}}$ is a smooth sequence then $\pi_{\underline{\mathcal{E}}}\colon \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{Z}_\phi$ is an open immersion with image $\mathcal{X}_\phi^{\underline{\mathcal{E}}}$.*

It turns out that if $\underline{\mathcal{E}}$ is a smooth sequence, then $\mathcal{X}_\phi^{\underline{\mathcal{E}}}$ has a more explicit description:

**Proposition 3.1.42.** *Let $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r$ be a smooth sequence, $k$ be a field and $a \in \mathcal{X}_\phi(k)$. Then*

$$a \in \mathcal{X}_\phi^{\underline{\mathcal{E}}}(k) \iff \exists \mathcal{E} \in \langle \mathcal{E}^1, \dots, \mathcal{E}^r \rangle_{\mathbb{N}},\ \lambda\colon T \longrightarrow \overline{k}^* \ s.t.\ a = \lambda 0^{\mathcal{E}}$$

*Moreover if $\lambda 0^{\mathcal{E}} \in \mathcal{X}_\phi^{\underline{\mathcal{E}}}(k)$, for some $\mathcal{E} \in T_+^\vee$, $\lambda\colon T \longrightarrow \overline{k}^*$, then $\mathcal{E} \in \langle \mathcal{E}^1, \dots, \mathcal{E}^r \rangle_{\mathbb{N}}$.*

*Proof.* We can assume $k$ algebraically closed and $T_+$ integral. In this case $a \in \mathcal{X}_\phi^{\underline{\mathcal{E}}}(k)$ if and only if $a\colon T_+ \longrightarrow k$ extends to a map $\mathrm{Ker}\,\underline{\mathcal{E}} \oplus \mathbb{N}^r = T_+^{\underline{\mathcal{E}}} \longrightarrow k$. So $\Leftarrow$ holds. Conversely, from 3.1.22, we can write $a = \lambda 0^{\mathcal{E}}$ where $\lambda\colon T \longrightarrow k^*$ and $\mathcal{E} \in (T_+^{\underline{\mathcal{E}}})^\vee$. From 3.1.36 we see that $T_+^{\underline{\mathcal{E}}\vee} = \langle \mathcal{E}^1, \dots, \mathcal{E}^r \rangle_{\mathbb{N}}$. Finally, if $\lambda 0^{\mathcal{E}} \in \mathcal{X}_\phi^{\underline{\mathcal{E}}}$ for some $\mathcal{E}$, then $\mathrm{Supp}\,\mathcal{E} \subseteq \mathrm{Supp}\,\underline{\mathcal{E}}$ and we are done. $\square$

**Lemma 3.1.43.** *Let $\underline{\mathcal{E}} = (\mathcal{E}^i)_{i\in I}$ be a sequence of distinct smooth extremal rays and $\Theta$ be a collection of smooth sequences with rays in $\underline{\mathcal{E}}$. Set*

$$\mathcal{F}^{\Theta}_{\underline{\mathcal{E}}} = \left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \delta) \in \mathcal{F}_{\underline{\mathcal{E}}} \;\middle|\; \begin{array}{c} V(z_{i_1}) \cap \cdots \cap V(z_{i_s}) \neq \emptyset \\ \textit{iff } \exists \underline{\delta} \in \Theta \textit{ s.t. } \mathcal{E}^{i_1}, \dots, \mathcal{E}^{i_s} \subseteq \underline{\delta} \end{array} \right\}$$

*Then, taking into account the identification made in 3.1.17, we have*

$$\mathcal{F}^{\Theta}_{\underline{\mathcal{E}}} = \bigcup_{\underline{\delta}\in\Theta} \mathcal{F}_{\underline{\delta}}$$

*Proof.* Let $\chi = (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \bigcup_{\underline{\delta}\in\Theta} \mathcal{F}_{\underline{\delta}}(T)$, for some scheme $T$ and let $p \in V(z_{i_1}) \cap \cdots \cap V(z_{i_s})$. This means that the pullback of $\pi_{\underline{\mathcal{E}}}(\chi)$ to $\overline{k(p)}$ is given by $a = b0^{\mathcal{E}^{i_1}+\cdots+\mathcal{E}^{i_r}}$ for some $b\colon T_+ \longrightarrow \overline{k(p)}$. By definition there exists $\underline{\delta} \in \Theta$ such that $a \in \mathcal{F}_{\underline{\delta}}(\overline{k(p)})$, i.e. $a = \mu 0^{\delta}$ for some $\delta \in \langle \underline{\delta} \rangle_{\mathbb{N}}$, $\mu\colon T \longrightarrow \overline{k(p)}^*$. So

$$\operatorname{Supp} \mathcal{E}^{i_j} \subseteq \{a = 0\} = \operatorname{Supp} \delta \subseteq \operatorname{Supp} \underline{\delta} \implies \mathcal{E}^{i_j} \in \langle \underline{\delta} \rangle_{\mathbb{N}}$$

For the other inclusion, since all the $\mathcal{F}_{\underline{\delta}}$ are open substacks of $\mathcal{F}_{\underline{\mathcal{E}}}$, we can reduce the problem to the case of an algebraically closed field $k$. So let $(\underline{z}, \lambda) \in \mathcal{F}^{\Theta}_{\underline{\mathcal{E}}}(k)$ and set $J = \{i \in I \mid z_i = 0\}$. By definition of $\mathcal{F}^{\Theta}_{\underline{\mathcal{E}}}$ there exists $\underline{\delta} \in \Theta$ such that $\underline{\eta} = (\mathcal{E}^j)_{j\in J} \subseteq \underline{\delta}$ and, taking into account 3.1.17, this means that $a \in \mathcal{F}_{\underline{\eta}}(k) \subseteq \mathcal{F}_{\underline{\delta}}(k)$. □

**Definition 3.1.44.** Let $\Theta$ be a collection of smooth sequences. We define

$$X^{\Theta}_{\phi} = \bigcup_{\underline{\delta}\in\Theta} \operatorname{Spec} \mathbb{Z}[T^{\underline{\delta}}_{+}] \subseteq \operatorname{Spec} \mathbb{Z}[T_+] \text{ and } \mathcal{X}^{\Theta}_{\phi} = \bigcup_{\underline{\delta}\in\Theta} \mathcal{X}^{\underline{\delta}}_{\phi} \subseteq \mathcal{Z}_{\phi}$$

**Theorem 3.1.45.** *Let $\underline{\mathcal{E}} = (\mathcal{E}^i)_{i\in I}$ be a sequence of distinct smooth extremal rays and $\Theta$ be a collection of smooth sequences with rays in $\underline{\mathcal{E}}$. Then we have an isomorphism*

$$\mathcal{F}^{\Theta}_{\underline{\mathcal{E}}} = \pi^{-1}_{\underline{\mathcal{E}}}(\mathcal{X}^{\Theta}_{\phi}) \xrightarrow{\simeq} \mathcal{X}^{\Theta}_{\phi}$$

*Proof.* Taking into account 3.1.43, it is enough to note that

$$\pi^{-1}_{\underline{\mathcal{E}}}(\mathcal{X}^{\Theta}_{\phi}) = \pi^{-1}_{\underline{\mathcal{E}}}(\bigcup_{\underline{\delta}\in\Theta} \mathcal{X}^{\underline{\delta}}_{\phi}) = \bigcup_{\underline{\delta}\in\Theta} \mathcal{F}_{\underline{\mathcal{E}}\cap\underline{\delta}} = \bigcup_{\underline{\delta}\in\Theta} \mathcal{F}_{\underline{\delta}} \xrightarrow{\simeq} \mathcal{X}^{\Theta}_{\phi}$$

□

**Proposition 3.1.46.** *Let $\underline{\mathcal{E}} = (\mathcal{E}^i)_{i\in I}$ be a sequence of distinct smooth extremal rays and $\Theta$ be a collection of smooth sequences with rays in $\underline{\mathcal{E}}$. Then the set*

$$\Delta^{\Theta} = \{\langle \eta_1, \dots, \eta_r \rangle_{\mathbb{Q}_+} \mid \exists \underline{\delta} \in \Theta \textit{ s.t. } \eta_1, \dots, \eta_r \subseteq \underline{\delta}\}$$

*is a toric fan in $T^{\vee} \otimes \mathbb{Q}$ whose associated toric variety over $\mathbb{Z}$ is $X^{\Theta}_{\phi}$. Moreover*

$$\mathcal{X}^{\Theta}_{\phi} \simeq [X^{\Theta}_{\phi}/\mathrm{D}(\mathbb{Z}^r)]$$

*Proof.* We know that if $\underline{\eta}$ is a smooth sequence then $\operatorname{Spec}\mathbb{Z}[T_+^{\underline{\eta}}]$ is a smooth open subset of $\operatorname{Spec}\mathbb{Z}[T_+^{int}]$ and it is the affine toric variety associated to the cone $\langle\underline{\eta}\rangle_{\mathbb{Q}_+}$. It is then easy to check that $\Delta^\Theta$ is a fan whose associated toric variety is $X_\phi^\Theta$. Since $\operatorname{Spec}\mathbb{Z}[T_+^{\underline{\eta}}]$ is the equivariant open subset of $\operatorname{Spec}\mathbb{Z}[T_+^{int}]$ inducing $\mathcal{X}_\phi^{\underline{\eta}}$ in $\mathcal{Z}_\phi$, then $X^\Theta$ is the equivariant open subset of $\operatorname{Spec}\mathbb{Z}[T_+^{int}]$ inducing $\mathcal{X}_\phi^\Theta$. In particular we obtain the last isomorphism. □

**Lemma 3.1.47.** *Assume $T_+$ integral and set $\Theta$ for the set of all smooth sequences. Then $X_\phi^\Theta$ is the smooth locus of $\operatorname{Spec}\mathbb{Z}[T_+]$. In particular $\mathcal{Z}_\phi^{\mathrm{sm}} = \mathcal{X}_\phi^\Theta \simeq [X_\phi^\Theta/\mathrm{D}(\mathbb{Z}^r)]$.*

*Proof.* From 3.1.36 we know that $\operatorname{Spec}\mathbb{Z}[T_+^{\underline{\mathcal{E}}}]$ is smooth over $\mathbb{Z}$ and it is an open subset of $\operatorname{Spec}\mathbb{Z}[T_+]$. So we focus on the converse. Since $\operatorname{Spec}\mathbb{Z}[T_+]$ is flat over $\mathbb{Z}$, we can replace $\mathbb{Z}$ by an algebraically closed field $k$. Let $p \in \operatorname{Spec} k[T_+]$ be a smooth point. In particular $p^{om}$ is smooth too. If $p^{om} = 0$ then $p \in \operatorname{Spec} k[T]$ and we have done. So we can assume $p^{om} = p_\mathcal{E}$ for some $0 \neq \mathcal{E} \in T_+^\vee$ thanks to 3.1.24. We claim that there exist a smooth sequence $\mathcal{E}^1,\dots,\mathcal{E}^q$ such that $\mathcal{E} \in \langle \mathcal{E}^1,\dots,\mathcal{E}^q\rangle_\mathbb{N}$. This is enough to conclude that $p \in \operatorname{Spec} k[T_+^{\underline{\mathcal{E}}}]$ . Indeed if $x_w \in p$ for some $w \in \operatorname{Ker}\underline{\mathcal{E}} \cap T_+$ then it belongs to $p^{om} = p_\mathcal{E}$ and so $\mathcal{E}(w) > 0$, which is not our case.

So assume we have $\mathcal{E} \in T_+^\vee$ such that $p_\mathcal{E}$ is a regular point. Set $W = \langle \operatorname{Ker}\mathcal{E} \cap T_+\rangle_\mathbb{Z}$ and $T'_+ = T_+ + W$. Note that $\operatorname{Spec} k[T'_+]$ is an open subset of $\operatorname{Spec} k[T_+]$ that contains $p_\mathcal{E}$. Moreover $k[T'_+]/p_\mathcal{E} = k[W]$. Let $v_1,\dots,v_q \in T_+$ be elements such that

$$T'_+ = \langle v_1,\dots,v_q\rangle_\mathbb{N} + W \qquad \text{and} \qquad \mathcal{E}(v_i) > 0$$

with $q$ minimal. We claim that $M = p_\mathcal{E}/p_\mathcal{E}^2 \simeq k[W]^q$, where $p_\mathcal{E}$ is thought in $k[T'_+]$. Indeed $M$ is a $k$-vector space over the $x_v$, $v \in T'_+$ that satisfies: $\mathcal{E}(v) > 0$ and whenever we have $v = v' + v''$ with $v', v'' \in T'_+$ it follows that $\mathcal{E}(v') = 0$ or $\mathcal{E}(v'') = 0$. A simple computation shows that such a $v$ must be of the form $v_i + W$ for some $i$. But since we have chosen $q$ minimal we have $(v_i + W) \cap (v_j + W) = \emptyset$ if $i \neq j$. This implies that $M$ is a free $k[W]$-module with basis $x_{v_1},\dots,x_{v_q}$. This shows that $q = \operatorname{ht} p_\mathcal{E}$.

Now set $V = \langle v_1,\dots,v_q\rangle_\mathbb{Z}$. Since $V + W = T$, $\operatorname{rk} V \leq q$ and

$$k[W] \simeq k[T'_+]/p_\mathcal{E} \implies \operatorname{rk} T = \dim k[T'_+] = \operatorname{ht} p_\mathcal{E} + \dim k[W] = q + \operatorname{rk} W$$

we obtain that $v_1,\dots,v_q$ are independent. Let $\mathcal{E}^1,\dots,\mathcal{E}^q$ given by $\mathcal{E}^i(v_j) = \delta_{i,j}$ and $\mathcal{E}^i_{|W} = 0$. In particular $W = \operatorname{Ker}\underline{\mathcal{E}}$ and it is generated by elements in $T_+$. Since $\mathcal{E}_{|W} = 0$ we have

$$\mathcal{E} = \sum_{i=1}^q \mathcal{E}(v_i)\mathcal{E}^i \qquad \mathcal{E}(v_i) > 0$$

Moreover since $T_+ \subseteq T'_+$ and $\mathcal{E}^i \in T'^\vee_+$ we get that $\mathcal{E}^i \in T_+^\vee$, as required. □

**Theorem 3.1.48.** *If $\underline{\mathcal{E}}$ is a sequence of distinct indecomposable rays containing the smooth extremal rays then $\pi_{\underline{\mathcal{E}}}$ induces an equivalence*

$$\left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \delta) \in \mathcal{F}_{\underline{\mathcal{E}}} \;\middle|\; \begin{array}{c} V(z_{i_1}) \cap \cdots \cap V(z_{i_s}) = \emptyset \\ \text{if } \mathcal{E}^{i_1}, \dots \mathcal{E}^{i_s} \text{ is not a} \\ \text{smooth sequence} \end{array} \right\} = \pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{Z}_\phi^{\mathrm{sm}}) \xrightarrow{\simeq} \mathcal{Z}_\phi^{\mathrm{sm}}$$

*Proof.* Lemma 3.1.47 tells us that $\mathcal{Z}_\phi^{\mathrm{sm}} = \mathcal{X}_\phi^\Theta$, where $\Theta$ is the collection of all smooth sequences, while 3.1.40 allows us to replace $\underline{\mathcal{E}}$ with the sequence of all smooth extremal rays. Therefore it is enough to apply 3.1.45 and 3.1.46. □

**Proposition 3.1.49.** *Let $a\colon T_+ \longrightarrow k \in \mathcal{X}_\phi(k)$, where $k$ is a field. Then $a$ lies in $\mathcal{Z}_\phi^{\mathrm{sm}}$ if and only if there exists a smooth ray $\mathcal{E} \in T_+^\vee$ and $\lambda\colon T \longrightarrow \overline{k}^*$ such that $a = \lambda 0^{\mathcal{E}}$.*

*Proof.* Apply 3.1.48 and 3.1.42. □

### 3.1.4 Extension of objects from codimension 1.

In this subsection we want to explain how it is possible, in certain cases, to check that an object of $\mathcal{X}_\phi$ over a sufficiently regular scheme $X$ comes (uniquely) from $\mathcal{F}_{\underline{\mathcal{E}}}$ only checking what happens in codimension 1.

*Notation* 3.1.50. Given a scheme $X$ we will denote by $\underline{\mathrm{Pic}}\, X$ the category whose objects are invertible sheaves and whose arrows are maps between them.

**Proposition 3.1.51.** *Let $X \xrightarrow{f} Y$ be a map of schemes. If $\underline{\mathrm{Pic}}\, Y \xrightarrow{f^*} \underline{\mathrm{Pic}}\, X$ is fully faithful (resp. an equivalence) then $\mathcal{X}_\phi(Y) \xrightarrow{f^*} \mathcal{X}_\phi(X)$ has the same property.*

*Proof.* Let $(\underline{\mathcal{L}}, a), (\underline{\mathcal{L}}', a') \in \mathcal{X}_\phi(Y)$ and $\underline{\sigma}\colon f^*(\underline{\mathcal{L}}, a) \longrightarrow f^*(\underline{\mathcal{L}}', a')$ be a map in $\mathcal{X}_\phi(X)$. Any map $\sigma_i\colon f^*\mathcal{L}_i \longrightarrow f^*\mathcal{L}_i$ comes from a unique map $\tau_i\colon \mathcal{L}_i \longrightarrow \mathcal{L}_i$, i.e. $\sigma_i = f^*\tau_i$. Since

$$f^*(\underline{\tau}^{\phi(t)}(a(t))) = \underline{\sigma}^{\phi(t)}(f^*a(t)) = f^*(a'(t)) \implies \underline{\tau}^{\phi(t)}(a(t)) = a'(t)$$

$\underline{\tau}$ is a map $(\underline{\mathcal{L}}, a) \longrightarrow (\underline{\mathcal{L}}', a')$ such that $f^*\underline{\tau} = \underline{\sigma}$. We can conclude that $f^*\colon \mathcal{X}_\phi(Y) \longrightarrow \mathcal{X}_\phi(X)$ is fully faithful.

Now assume that $\underline{\mathrm{Pic}}\, Y \xrightarrow{f^*} \underline{\mathrm{Pic}}\, X$ is an equivalence. We have to prove that $\mathcal{X}_\phi(Y) \xrightarrow{f^*} \mathcal{X}_\phi(X)$ is essentially surjective. So let $(\underline{\mathcal{M}}, b) \in \mathcal{X}_\phi(X)$. Since $f^*$ is an equivalence we can assume $\mathcal{M}_i = f^*\mathcal{L}_i$ for some invertible sheaf $\mathcal{L}_i$ on $Y$. Since for any invertible sheaf $\mathcal{L}$ on Y one has that $\mathcal{L}(Y) \simeq (f^*\mathcal{L})(X)$, any section $b(t) \in \underline{\mathcal{M}}^{\phi(t)}$ extends to a unique section $a(t) \in \underline{\mathcal{L}}^{\phi(t)}$. Since

$$f^*(a(t) \otimes a(s)) = b(t) \otimes b(s) = b(t+s) = f^*(a(t+s)) \implies a(t) \otimes a(s) = a(t+s)$$

for any $t, s \in T_+$ and $a(0) = 1$, it follows that $(\underline{\mathcal{L}}, a) \in \mathcal{X}_\phi(Y)$ and $f^*(\underline{\mathcal{L}}, a) = (\underline{\mathcal{M}}, b)$. □

**Corollary 3.1.52.** *Let $X \xrightarrow{f} Y$ be a map of schemes and consider a commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathcal{F}_{\underline{\mathcal{E}}} \arrow[d, "\pi_{\underline{\mathcal{E}}}"] \\ Y \arrow[r] \arrow[ur, dashed] & \mathcal{X}_\phi \end{tikzcd}$$

*where $\underline{\mathcal{E}}$ is a sequence of elements of $T_+^\vee$. Then if $\underline{\mathrm{Pic}}\, X \xrightarrow{f^*} \underline{\mathrm{Pic}}\, Y$ is fully faithful (resp. an equivalence) the dashed lifting is unique (resp. exists).*

*Proof.* It is enough to consider the 2-commutative diagram

$$\begin{array}{ccc} \mathcal{F}_{\underline{\mathcal{E}}}(Y) & \xrightarrow{f^*} & \mathcal{F}_{\underline{\mathcal{E}}}(X) \\ {\scriptstyle \pi_{\underline{\mathcal{E}}}}\downarrow & & \downarrow{\scriptstyle \pi_{\underline{\mathcal{E}}}} \\ \mathcal{X}_\phi(Y) & \xrightarrow{f^*} & \mathcal{X}_\phi(X) \end{array}$$

and note that $f^*$ is fully faithful (resp. an equivalence) in both cases. $\square$

**Theorem 3.1.53.** *Let $X$ be a locally noetherian and locally factorial scheme, $\underline{\mathcal{E}} = (\mathcal{E}^i)_{i \in I}$ be a sequence of distinct smooth extremal rays and $\Theta$ be a collection of smooth sequences with rays in $\underline{\mathcal{E}}$. Consider the full subcategories*

$$\mathscr{C}_X^\Theta = \left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \delta) \in \mathcal{F}_{\underline{\mathcal{E}}}(X) \;\middle|\; \begin{array}{c} \mathrm{codim}_X V(z_{i_1}) \cap \cdots \cap V(z_{i_s}) \geq 2 \\ \text{if } \nexists \underline{\delta} \in \Theta \text{ s.t. } \mathcal{E}^{i_1}, \dots \mathcal{E}^{i_s} \subseteq \underline{\delta} \end{array} \right\} \subseteq \mathcal{F}_{\underline{\mathcal{E}}}(X)$$

*and*

$$\mathscr{D}_X^\Theta = \left\{ \chi \in \mathcal{X}_\phi(X) \;\middle|\; \begin{array}{c} \forall p \in X \text{ with } \mathrm{codim}_p X \leq 1 \\ \chi_{|\overline{k(p)}} \in \mathcal{X}_\phi^\Theta \end{array} \right\} \subseteq \mathcal{X}_\phi(X)$$

*Then $\pi_{\underline{\mathcal{E}}}$ induces an equivalence of categories*

$$\mathscr{C}_X^\Theta = \pi_{\underline{\mathcal{E}}}^{-1}(\mathscr{D}_X^\Theta) \xrightarrow{\simeq} \mathscr{D}_X^\Theta$$

*Proof.* We claim that

$$\mathscr{C}_X^\Theta = \{\chi \in \mathcal{F}_{\underline{\mathcal{E}}}(X) \mid \exists U \subseteq X \text{ open subset s.t. } \mathrm{codim}_X X - U \geq 2,\ \chi_{|U} \in \mathcal{F}_{\underline{\mathcal{E}}}^\Theta(U)\}$$

$\subseteq$ Taking into account the definition of $\mathcal{F}_{\underline{\mathcal{E}}}^\Theta$ in 3.1.43, it is enough to consider

$$U = X - \bigcup_{\nexists \underline{\delta} \in \Theta \text{ s.t. } \mathcal{E}^{i_1}, \dots \mathcal{E}^{i_s} \subseteq \underline{\delta}} V(z_{i_1}) \cap \cdots \cap V(z_{i_s})$$

$\supseteq$ If $p \in V(z_{i_1}) \cap \cdots \cap V(z_{i_s})$ and $\mathrm{codim}_p X \leq 1$ then $p \in U$ and again by definition of $\mathcal{F}_{\underline{\mathcal{E}}}^\Theta$ there exists $\underline{\delta} \in \Theta$ such that $\mathcal{E}^{i_1}, \dots, \mathcal{E}^{i_s} \subseteq \underline{\delta}$.

We also claim that

$$\mathscr{D}_X^\Theta = \{\chi \in \mathcal{X}_\phi(X) \mid \exists U \subseteq X \text{ open subset s.t. } \mathrm{codim}_X X - U \geq 2,\ \chi_{|U} \in \mathcal{X}_\phi^\Theta(U)\}$$

$\supseteq$ Such a $U$ contains all the codimension 1 or 0 points of $X$.

$\subseteq$ Let $\chi \in \mathscr{D}_X^\Theta$ and $X \xrightarrow{g} \mathcal{X}_\phi$ be the induced map. If $\xi$ is a generic point of $X$, we know that $f(\xi) \in |\mathcal{X}_\phi^\Theta| \subseteq |\mathcal{Z}_\phi|$. In particular $f(|X|) \subseteq |\mathcal{Z}_\phi|$. Since both $X$ and $\mathcal{Z}_\phi$ are reduced $g$ factors through a map $X \xrightarrow{g} \mathcal{Z}_\phi$. Since $\mathcal{X}_\phi^\Theta$ is an open substack of $\mathcal{Z}_\phi$, it follows that $U = g^{-1}(\mathcal{X}_\phi^\Theta)$ is an open subscheme of $X$, $\chi_{|U} \in \mathcal{X}_\phi^\Theta(U)$ and, by definition of $\mathscr{D}_X^\Theta$, $\operatorname{codim}_X X - U \geq 2$.

Taking into account 3.1.45 it is clear that $\mathscr{C}_X^\Theta = \pi_{\underline{\mathcal{E}}}^{-1}(\mathscr{D}_X^\Theta)$. We will make use of the fact that if $U \subseteq X$ is an open subscheme such that $\operatorname{codim}_X X - U \geq 2$ then the restriction yields an equivalence $\underline{\operatorname{Pic}}\, X \simeq \underline{\operatorname{Pic}}\, U$. The map $\mathscr{C}_X^\Theta \longrightarrow \mathscr{D}_X^\Theta$ is essentially surjective since, given an object of $\mathscr{D}_X^\Theta$, the associated map $X \xrightarrow{g} \mathcal{X}_\phi$ fits in a 2-commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \mathcal{F}_{\underline{\mathcal{E}}}^\Theta \subseteq \mathcal{F}_{\underline{\mathcal{E}}} \arrow[d, "\pi_{\underline{\mathcal{E}}}"] \\ X \arrow[r, "g"] & \mathcal{X}_\phi \end{tikzcd}$$

and so lifts to a map $X \longrightarrow \mathcal{F}_{\underline{\mathcal{E}}}$ thanks to 3.1.52.

It remains to show that $\mathscr{C}_X^\Theta \longrightarrow \mathscr{D}_X^\Theta$ is fully faithful. Let $\chi, \chi' \in \mathscr{C}_X^\Theta$ and $U, U'$ be the open subscheme given in the definition of $\mathscr{C}_X^\Theta$. Set $V = U \cap U'$. Taking into account 3.1.51 and 3.1.45 we have

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{F}_{\underline{\mathcal{E}}}(X)}(\chi,\chi') \arrow[r] \arrow[d, phantom, "\wr|"] & \operatorname{Hom}_{\mathcal{X}_\phi(X)}(\chi,\chi') \arrow[d, phantom, "\wr|"] \\ \operatorname{Hom}_{\mathcal{F}_{\underline{\mathcal{E}}}(V)}(\chi_{|V},\chi'_{|V}) \arrow[r] \arrow[d, phantom, "\wr|"] & \operatorname{Hom}_{\mathcal{X}_\phi(V)}(\chi_{|V},\chi'_{|V}) \arrow[d, phantom, "\wr|"] \\ \operatorname{Hom}_{\mathcal{F}^\Theta_{\underline{\mathcal{E}}}(V)}(\chi_{|V},\chi'_{|V}) \arrow[r, "\simeq"'] & \operatorname{Hom}_{\mathcal{X}^\Theta_\phi(V)}(\chi_{|V},\chi'_{|V}) \end{tikzcd}$$

$\square$

## 3.2 Galois covers for a diagonalizable group.

In this section we will fix a finite diagonalizable group scheme $G$ over $\mathbb{Z}$ and we will call $M = \operatorname{Hom}(G, \mathbb{G}_m)$ its character group. So $M$ is a finite abelian group and $G = \mathrm{D}(M)$. With abuse of notation we will write $\mathcal{O}_U[M] = \mathcal{O}_U[G_U]$ and $\mathcal{Z}_M = \mathcal{Z}_{\mathrm{D}(M)}$, the main component of $\mathrm{D}(M)$-Cov. It turns out that in this case $\mathrm{D}(M)$-covers have a nice and more explicit description.

In the first subsection we will show that $\mathrm{D}(M)\text{-Cov} \simeq \mathcal{X}_\phi$ for an explicit map $T_+ \xrightarrow{\phi} \mathbb{Z}^M/\langle e_0 \rangle$ and that this isomorphism preserves the main irreducible components of both stacks. Moreover we will study the connection between $\mathrm{D}(M)$-Cov and the equivariant Hilbert schemes $M\text{-Hilb}^{\underline{m}}$ and prove some results about their geometry.

Then we will introduce an upper semicontinuous map $|\mathrm{D}(M)\text{-Cov}| \xrightarrow{h} \mathbb{N}$ that yields a stratification by open substacks of $\mathrm{D}(M)$-Cov. We will also see that $\{h = 0\}$ coincides with the open substack of $\mathrm{D}(M)$-torsors, while $\{h \leq 1\}$ lies in the smooth locus of $\mathcal{Z}_M$ and can be described by a particular set of smooth extremal rays. This will allow us to describe normal $\mathrm{D}(M)$-covers over a locally noetherian and locally factorial scheme $X$ with $(\operatorname{char} X, |M|) = 1$.

### 3.2.1 The stack $\mathrm{D}(M)$-Cov and its main irreducible component $\mathcal{Z}_M$.

Consider a scheme $U$ and a cover $X = \operatorname{Spec} \mathscr{A}$ on it. An action of $\mathrm{D}(M)$ on it consists of a decomposition

$$\mathscr{A} = \bigoplus_{m \in M} \mathscr{A}_m$$

such that $\mathcal{O}_U \subseteq \mathscr{A}_0$ and the multiplication maps $\mathscr{A}_m \otimes \mathscr{A}_n$ into $\mathscr{A}_{m+n}$. If $X/U$ is a $\mathrm{D}(M)$-cover there exists an fppf covering $\{U_i \longrightarrow U\}$ such that $\mathscr{A}_{|U_i} \simeq \mathcal{O}_{U_i}[M]$ as $\mathrm{D}(M)$-comodules. This means that for any $m \in M$ we have

$$\forall i\ (\mathscr{A}_m)_{|U_i} \simeq \mathcal{O}_{U_i} \implies \mathscr{A}_m \text{ invertible}$$

Conversely any $M$-graded quasi-coherent algebra $\mathscr{A} = \bigoplus_{m\in M} \mathscr{A}_m$ with $\mathscr{A}_0 = \mathcal{O}_U$ and $\mathscr{A}_m$ invertible for any $m$ yields a $\mathrm{D}(M)$-cover $\operatorname{Spec} \mathscr{A}$.

So the stack $\mathrm{D}(M)$-Cov can be described as follows. An object of $\mathrm{D}(M)$-Cov$(U)$ is given by a collection of invertible sheaves $\mathcal{L}_m$ for $m \in M$ with maps

$$\psi_{m,n} \colon \mathcal{L}_m \otimes \mathcal{L}_n \longrightarrow \mathcal{L}_{m+n}$$

and an isomorphism $\mathcal{O}_U \simeq \mathcal{L}_0$ satisfying the following relations:

Commutativity

$$\begin{tikzcd} \mathcal{L}_m \otimes \mathcal{L}_n \arrow[rr, "\simeq"] \arrow[dr, "\psi_{m,n}"'] & & \mathcal{L}_n \otimes \mathcal{L}_m \arrow[dl, "\psi_{n,m}"] \\ & \mathcal{L}_{m+n} & \end{tikzcd}$$

Associativity

$$\begin{tikzcd} \mathcal{L}_m \otimes \mathcal{L}_n \otimes \mathcal{L}_t \arrow[r, "\mathrm{id}\otimes\psi_{n,t}"] \arrow[d, "\psi_{m,n}\otimes\mathrm{id}"'] & \mathcal{L}_m \otimes \mathcal{L}_{n+t} \arrow[d, "\psi_{m,n+t}"] \\ \mathcal{L}_{m+n} \otimes \mathcal{L}_t \arrow[r, "\psi_{m+n,t}"] & \mathcal{L}_{m+n+t} \end{tikzcd}$$

Neutral Element

$$\begin{tikzcd} \mathcal{L}_m \arrow[r, "\simeq"] \arrow[rrr, bend right, "\mathrm{id}"'] & \mathcal{L}_m \otimes \mathcal{O}_U \arrow[r, "\simeq"] & \mathcal{L}_m \otimes \mathcal{L}_0 \arrow[r, "\psi_{m,0}"] & \mathcal{L}_m \end{tikzcd}$$

If we assume that $\mathcal{L}_m = \mathcal{O}_U v_m$, i.e. that we have sections $v_m$ generating $\mathcal{L}_m$, the maps $\psi_{m,n}$ can be thought of as elements of $\mathcal{O}_U$ and the algebra structure is given by $v_m v_n = \psi_{m,n} v_{m+n}$. In this case we can rewrite the above conditions obtaining

$$\psi_{m,n} = \psi_{n,m}, \quad \psi_{m,0} = 1, \quad \psi_{m,n}\psi_{m+n,t} = \psi_{n,t}\psi_{n+t,m} \tag{3.2.1}$$

The functor that associates to a scheme $U$ the functions $\psi \colon M \times M \longrightarrow \mathcal{O}_U$ satisfying the above conditions is clearly representable by the spectrum of the ring

$$R_M = \mathbb{Z}[x_{m,n}]/(x_{m,n} - x_{n,m}, x_{m,0} - 1, x_{m,n}x_{m+n,t} - x_{n,t}x_{n+t,m}) \tag{3.2.2}$$

In this way we obtain a Zariski epimorphism $\operatorname{Spec} R_M \longrightarrow \mathrm{D}(M)\text{-Cov}$, that we will prove to be smooth. We now want to prove that the stack $\mathrm{D}(M)$-Cov is isomorphic to a stack of the form $\mathcal{X}_\phi$.

**Definition 3.2.1.** Define $\tilde{K}_+$ as the quotient monoid of $\mathbb{N}^{M\times M}$ by the equivalence relation generated by

$$e_{m,n} \sim e_{n,m}, \quad e_{m,0} \sim 0, \quad e_{m,n} + e_{m+n,t} \sim e_{n,t} + e_{n+t,m}$$

Also define $\phi_M \colon \tilde{K}_+ \longrightarrow \mathbb{Z}^M/\langle e_0 \rangle$ by $\phi_M(e_{m,n}) = e_m + e_n - e_{m+n}$.

**Proposition 3.2.2.** *$R_M \simeq \mathbb{Z}[\tilde{K}_+]$ and there exists an isomorphism*

$$\mathcal{X}_{\phi_M} \simeq \mathrm{D}(M)\text{-Cov} \tag{3.2.3}$$

*such that* $\operatorname{Spec} \mathbb{Z}[\tilde{K}_+] \simeq \operatorname{Spec} R_M \longrightarrow \mathrm{D}(M)\text{-Cov} \simeq \mathcal{X}_{\phi_M}$ *is the map defined in 3.1.4. In particular*

$$\mathrm{D}(M)\text{-Cov} \simeq [\operatorname{Spec} R_M/\mathrm{D}(\mathbb{Z}^M/\langle e_0 \rangle)]$$

*Proof.* The required isomorphism sends $(\underline{\mathcal{L}}, \tilde{K}_+ \xrightarrow{\psi} \operatorname{Sym}^* \underline{\mathcal{L}}) \in \mathcal{X}_{\phi_M}$ to the object of $\mathrm{D}(M)$-Cov given by invertible sheaves $(\mathcal{L}'_m = \mathcal{L}_m^{-1})$ and $\psi_{m,n} = \psi(e_{m,n})$. $\square$

We want to prove that the isomorphism 3.2.3 sends $\mathcal{Z}_{\phi_M}$ to $\mathcal{Z}_M$ (see def. 2.2.4) and $\mathcal{B}_{\phi_M}$ to $\mathrm{B}\,\mathrm{D}(M)$. We need the following classical result on the structure of a $\mathrm{D}(M)$-torsor (see [GD70, Exposé VIII, Proposition 4.1 and 4.6]):

**Proposition 3.2.3.** *Let $M$ be a finite abelian group and $P \longrightarrow U$ a $\mathrm{D}(M)$-equivariant map. Then $P$ is an fppf $\mathrm{D}(M)$-torsor if and only if $P \in \mathrm{D}(M)\text{-Cov}(U)$ and all the multiplication maps $\psi_{m,n}$ are isomorphisms.*

Now consider the exact sequence

$$\begin{array}{c} 0 \longrightarrow K \rightarrow \mathbb{Z}^M/\langle e_0 \rangle \rightarrow M \longrightarrow 0 \\ e_m \longmapsto m \end{array}$$

**Definition 3.2.4.** For $m, n \in M$ we define

$$v_{m,n} = \phi_M(e_{m,n}) = e_m + e_n - e_{m+n} \in K$$

and $K_+$ as the submonoid of $K$ generated by the $v_{m,n}$. We will set $x_{m,n} = x^{v_{m,n}} \in \mathbb{Z}[K_+]$ and, for $\mathcal{E} \in K_+^\vee$, $\mathcal{E}_{m,n} = \mathcal{E}(v_{m,n})$.

**Lemma 3.2.5.** *The map*

$$\begin{array}{ccc} \tilde{K}_+ & \longrightarrow & K \\ e_{m,n} & \longmapsto & v_{m,n} \end{array}$$

*is the associated group of $\tilde{K}_+$ and $K_+$ is its associated integral monoid. In particular we have a 2-cartesian diagram*

$$\begin{tikzcd} \operatorname{Spec}\mathbb{Z}[K] \arrow[r] \arrow[d] & \operatorname{Spec}\mathbb{Z}[K_+] \arrow[r] \arrow[d] & \operatorname{Spec} R_M \arrow[d] \\ \mathrm{B}\,\mathrm{D}(M) \arrow[r] & \mathcal{Z}_M \arrow[r] & \mathrm{D}(M)\text{-Cov} \end{tikzcd}$$

*Proof.* Set $x = \prod_{m,n} x_{m,n}$. Since an object $\psi \in \operatorname{Spec} R_M(U)$ is a torsor if and only if $\psi_{m,n} \in \mathcal{O}_U^*$ for all $m, n$, it follows that $(\operatorname{Spec} R_M)_x = \mathrm{B}\,\mathrm{D}(M) \times_{\mathrm{D}(M)\text{-Cov}} \operatorname{Spec} R_M$. We want to define an inverse to $(R_M)_x \longrightarrow \mathbb{Z}[K]$. Consider the algebra $S_M$ over $R_M$ induced by the atlas map $\operatorname{Spec} R_M \longrightarrow \mathrm{D}(M)$-Cov, i.e.

$$S_M = \bigoplus_{m\in M} R_M w_m \text{ with } w_0 = 1,\ w_m w_n = x_{m,n} w_{m+n}$$

The algebra $(S_M)_x$ is a $\mathrm{D}(M)$-torsor over $(R_M)_x$ and so $w_m \in (S_M)_x^*$ for all $m$. In particular we can define a group homomorphism

$$\begin{aligned} \mathbb{Z}^M/\langle e_0\rangle &\longrightarrow (S_M)_x^* \\ e_m &\longmapsto w_m \end{aligned}$$

which restricts to a map $K \longrightarrow (R_M)_x$ that sends $v_{m,n}$ to $x_{m,n}$. In particular the map $\tilde{K}_+ \longrightarrow K$ defined in the statement gives the associated group of $\tilde{K}_+$ and has as image exactly $K_+$, which means that $K_+$ is the integral monoid associated to $\tilde{K}_+$.

In order to conclude the proof it is enough to apply 3.1.9 and 3.1.10. □

**Corollary 3.2.6.** *The isomorphism $\mathcal{X}_{\phi_M} \simeq \mathrm{D}(M)$-Cov (3.2.3) induces isomorphisms $\mathcal{B}_{\phi_M} \simeq \mathrm{B}\,\mathrm{D}(M)$ and $\mathcal{Z}_{\phi_M} \simeq \mathcal{Z}_M$. In particular $\mathcal{Z}_M$ is an irreducible component of $\mathrm{D}(M)$-Cov and*

$$\mathrm{B}\,\mathrm{D}(M) \simeq [\operatorname{Spec}\mathbb{Z}[K]/\mathrm{D}(\mathbb{Z}^M/\langle e_0\rangle)] \text{ and } \mathcal{Z}_M \simeq [\operatorname{Spec}\mathbb{Z}[K_+]/\mathrm{D}(\mathbb{Z}^M/\langle e_0\rangle)]$$

Note that the induced map $\phi_M\colon K \longrightarrow \mathbb{Z}^M/\langle e_0\rangle$ is just the inclusion and so it is injective. This means that any result obtained in section 3.1 applies naturally in the context of $\mathrm{D}(M)$-covers. In particular now we show how we can describe the objects of $\mathcal{F}_{\underline{\mathcal{E}}}$, for a sequence of rays in $\tilde{K}_+^\vee$, in a simpler way.

**Proposition 3.2.7.** *Let $M \simeq \prod_{i=1}^n \mathbb{Z}/l_i\mathbb{Z}$ be a decomposition and let $m_1, \dots, m_n$ be the associated generators. Given $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in K_+^\vee$ define $\mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red}}$ as the stack whose objects over a scheme $X$ are sequences $\underline{\mathcal{L}} = \mathcal{L}_1, \dots, \mathcal{L}_n, \underline{\mathcal{M}} = \mathcal{M}_1, \dots, \mathcal{M}_r, \underline{z} = z_1, \dots, z_r, \underline{\mu} = \mu_1, \dots, \mu_n$ where $\underline{\mathcal{L}}$, $\underline{\mathcal{M}}$ are invertible sheaves over $X$, $z_i \in \mathcal{M}_i$ and $\underline{\mu}$ are isomorphisms*

$$\mu_i\colon \mathcal{L}_i^{-l_i} \xrightarrow{\simeq} \underline{\mathcal{M}}^{\underline{\mathcal{E}}(l_i e_{m_i})} = \mathcal{M}_1^{\mathcal{E}^1(l_i e_{m_i})} \otimes \cdots \otimes \mathcal{M}_r^{\mathcal{E}^r(l_i e_{m_i})}$$

*Then we have an isomorphism of stacks*

$$\begin{array}{ccc} \mathcal{F}_{\underline{\mathcal{E}}} & \longrightarrow & \mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red}} \\ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) & \mapsto & ((\mathcal{L}_{m_i})_{i=1,\dots,n}, \underline{\mathcal{M}}, \underline{z}, (\lambda(l_i e_{m_i}))_{i=1,\dots,n}) \end{array}$$

*Proof.* We want to find $\sigma$, $V$, $v_1, \dots, v_q$ as in 3.1.26 such that $\mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red},\sigma} = \mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red}}$ and that the map in the statement coincides with the one defined in 3.1.28. Set $\delta^i \colon M \longrightarrow \{0, \dots, l_i - 1\}$ as the map such that $\pi_i(m) = \pi_i(\delta^i_m m_i)$, where $\pi_i \colon M \longrightarrow \mathbb{Z}/l_i\mathbb{Z}$ is the projection, and think of it also as a map $\delta^i \colon \mathbb{Z}^M/\langle e_0 \rangle \longrightarrow \mathbb{Z}$. Set $V = \bigoplus_{i=1}^n \mathbb{Z} e_{m_i}$, $v_i = e_{m_i}$ and $\sigma \colon \mathbb{Z}^M/\langle e_0 \rangle \longrightarrow V$ as $\sigma(e_m) = \sum_{i=1}^n \delta^i_m v_i$. Clearly $(\mathrm{id} - \sigma)\mathbb{Z}^M/\langle e_0 \rangle \subseteq K$ and $(\mathrm{id} - \sigma)V = 0$. So $W = \sigma K$. We have

$$\sigma(v_{m,n}) = \sum_{i=1}^n \delta^i_{m,n} v_i \in \bigoplus_{i=1}^n l_i \mathbb{Z} v_i$$

since $\delta^i_{m,n} \in \{0, l_i\}$ for all $i$. On the other hand $\sigma(v_{(l_i-1)m_i,m_i}) = l_i v_i$. Therefore we have $W = \bigoplus_{i=1}^n l_i \mathbb{Z} v_i$. It is now easy to check that all the definitions agree. $\square$

We now want to express the relation between $\mathrm{D}(M)$-Cov and the equivariant Hilbert scheme, that can be defined as follows. Given $\underline{m} = m_1, \dots, m_r \in M$, so that $\mathrm{D}(M)$ acts on $\mathbb{A}^r_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_r]$ with graduation $\deg x_i = m_i$, we define $M\text{-}\mathrm{Hilb}^{\underline{m}} \colon \mathrm{Sch}^{\mathrm{op}} \longrightarrow (\mathrm{Sets})$ as the functor that associates to a scheme $Y$ the set of pairs $(X \xrightarrow{f} Y, j)$ where $X \in \mathrm{D}(M)\text{-}\mathrm{Cov}(Y)$ and $j \colon X \longrightarrow \mathbb{A}^r_Y$ is an equivariant closed immersion over $Y$. Such a pair can be also thought of as a coherent sheaf of algebras $\mathscr{A} \in \mathrm{D}(M)\text{-}\mathrm{Cov}(Y)$ together with a graded surjective map $\mathcal{O}_Y[x_1, \dots, x_r] \longrightarrow \mathscr{A}$. This functor is proved to be a scheme of finite type in [HS04].

**Proposition 3.2.8.** *Let $\underline{m} = m_1, \dots, m_r \in M$. The forgetful map $\vartheta_{\underline{m}} \colon M\text{-}\mathrm{Hilb}^{\underline{m}} \longrightarrow \mathrm{D}(M)\text{-}\mathrm{Cov}$ is a smooth Zariski epimorphism onto the open substack $\mathrm{D}(M)\text{-}\mathrm{Cov}^{\underline{m}}$ of $\mathrm{D}(M)\text{-}\mathrm{Cov}$ of sheaves of algebras $\mathscr{A}$ such that, for all $y \in Y$, $\mathscr{A} \otimes k(y)$ is generated in the degrees $m_1, \dots, m_r$ as a $k(y)$-algebra. Moreover $M\text{-}\mathrm{Hilb}^{\underline{m}}$ is an open subscheme of a vector bundle over $\mathrm{D}(M)\text{-}\mathrm{Cov}^{\underline{m}}$.*

*Proof.* Let $\mathscr{A} = \oplus_{m \in M} \mathscr{A}_m \in \mathrm{D}(M)\text{-}\mathrm{Cov}$ and consider the map

$$\eta_{\mathscr{A}} \colon \operatorname{Sym}(\mathscr{A}_{m_1} \oplus \cdots \oplus \mathscr{A}_{m_r}) \longrightarrow \mathscr{A}$$

induced by the direct sum of the inclusions $\mathscr{A}_{m_i} \longrightarrow \mathscr{A}$. It is easy to check that $\eta_{\mathscr{A}}$ is surjective if and only if $\mathscr{A} \in \mathrm{D}(M)\text{-}\mathrm{Cov}^{\underline{m}}$. Therefore $\mathrm{D}(M)\text{-}\mathrm{Cov}^{\underline{m}}$ is an open substack of $\mathrm{D}(M)$-Cov and clearly contains the image of $\vartheta_{\underline{m}}$. Consider now the cartesian diagram

$$\begin{array}{ccc} F & \longrightarrow & M\text{-}\mathrm{Hilb}^{\underline{m}} \\ \downarrow & & \downarrow \vartheta_{\underline{m}} \\ T & \xrightarrow{\mathscr{A}} & \mathrm{D}(M)\text{-}\mathrm{Cov}^{\underline{m}} \end{array}$$

and let $U \xrightarrow{\phi} T$ be a map. The objects of $F(U)$ are pairs composed by a graded surjection $\mathcal{O}_U[x_1,\dots,x_r] \longrightarrow \mathscr{B}$ and an isomorphism $\mathscr{B} \simeq \phi^*\mathscr{A}$. This is equivalent to giving a graded surjection $\mathcal{O}_U[x_1,\dots,x_r] \longrightarrow \phi^*\mathscr{A}$. In this way we obtain a map

$$F \xrightarrow{g_T} \prod_i \underline{\mathrm{Hom}}_T(\mathcal{O}_T, \mathscr{A}_{m_i}) \simeq \operatorname{Spec}\operatorname{Sym}(\bigoplus_i \mathscr{A}_{m_i}^{-1})$$

We claim that this is an open immersion. Indeed given $(a_i)_i \colon U \longrightarrow \prod_i \underline{\mathrm{Hom}}_T(\mathcal{O}_T, \mathscr{A}_{m_i})$, the fiber product with $F$ is the locus where the induced graded map $\mathcal{O}_U[x_1,\dots,x_r] \longrightarrow \mathscr{A}\otimes\mathcal{O}_U$ is surjective, that is an open subscheme of $U$. In particular $F$ is smooth over $T$ and so $\vartheta_{\underline{m}}$ is smooth too. It is easy to check that it is also a Zariski epimorphism. Finally the vector bundle $\mathcal{N}$ of the statement is defined over any $U \longrightarrow \mathrm{D}(M)\text{-}\mathrm{Cov}^{\underline{m}}$ given by $\mathscr{A} = \bigoplus_m \mathscr{A}_m$ by $\mathcal{N}_{|U} = \oplus_i \mathscr{A}_{m_i}^{-1}$. □

*Remark* 3.2.9. If the sequence $\underline{m}$ contains all elements of $M-\{0\}$, then $\mathrm{D}(M)\text{-}\mathrm{Cov}^{\underline{m}} = \mathrm{D}(M)\text{-}\mathrm{Cov}$. Therefore in this case $M\text{-}\mathrm{Hilb}^{\underline{m}}$ is an atlas for $\mathrm{D}(M)\text{-}\mathrm{Cov}$.

*Remark* 3.2.10. Let $X$ be a scheme, $\mathcal{X}$ be an irreducible (resp. connected) algebraic stack and $X \xrightarrow{\pi} \mathcal{X}$ be a fppf epimorphism such that the fiber over the generic point of $\mathcal{X}$ is irreducible (resp. such that $\pi$ is geometrically connected). Then $X$ is irreducible (resp. connected). For the connectedness, if $X = U \cup V$, since $\pi$ is open and $|\mathcal{X}| = |\pi(U)| \cup |\pi(V)|$, we have $\pi(U)\cap\pi(V) \neq \emptyset$. In particular $U$ and $V$ meet a common fiber $Z$ of $\pi$. Since $Z$ is connected we can conclude that $Z\cap U\cap V \neq \emptyset$. For the irreducibility, consider a generic point $\xi \colon \operatorname{Spec} k \longrightarrow \mathcal{X}$, with $k$ algebraically closed, and denote by $Z$ the (topological) image of $\operatorname{Spec} k \times_{\mathcal{X}} X \longrightarrow X$. Note that $Z$ does not depend on the choice of the generic point and it is irreducible by hypothesis. If $V \subseteq X$ is a non-empty open subset of $X$, since $\pi$ is an open map, we can conclude that $V \cap Z \neq \emptyset$. Therefore $Z$ is dense in $X$ and $X$ is irreducible.

*Remark* 3.2.11. The map $\vartheta_{\underline{m}} \colon M\text{-}\mathrm{Hilb}^{\underline{m}} \longrightarrow \mathrm{D}(M)\text{-}\mathrm{Cov}^{\underline{m}}$ of 3.2.8 is a smooth epimorphism with geometrically connected and irreducible fibers. In particular, taking into account 3.2.10, if $\mathcal{X}$ is an algebraic stack, $\mathcal{X} \longrightarrow \mathrm{D}(M)\text{-}\mathrm{Cov}^{\underline{m}}$ is a map and we denote by $\vartheta_{\underline{m}}^{-1}(\mathcal{X})$ the base change of $\vartheta_{\underline{m}}$ we have that: $\mathcal{X}$ is connected (resp. geometrically connected, irreducible, geometrically irreducible, smooth, reduced) if and only if $\vartheta_{\underline{m}}^{-1}(\mathcal{X})$ has the same property. The same conclusions hold if we consider the atlas $\operatorname{Spec} R_M \longrightarrow \mathrm{D}(M)\text{-}\mathrm{Cov}$ instead of $\vartheta_{\underline{m}}$.

In particular, since $\mathrm{B}\,\mathrm{D}(M) \subseteq \mathrm{D}(M)\text{-}\mathrm{Cov}^{\underline{m}}$, we can conclude that $\vartheta_{\underline{m}}^{-1}(\mathcal{Z}_M)$ is the main irreducible component of $M\text{-}\mathrm{Hilb}^{\underline{m}}$.

We want now study some geometrical properties of the stack $\mathrm{D}(M)\text{-}\mathrm{Cov}$ and, therefore, of the equivariant Hilbert schemes.

*Remark* 3.2.12. The ring $R_M$ can be written as quotient of the ring $\mathbb{Z}[x_{m,n}]_{(m,n)\in J}$, where $J$ is $\{(m,n)\in M^2 \mid m,n,m+n \neq 0\}$ divided by the equivalence relation $(m,n)\sim(n,m)$, by the ideal

$$I = \begin{pmatrix} x_{m,n}x_{m+n,t} - x_{n,t}x_{n+t,m} \text{ with } m,n,t,m+n,n+t,m+n+t \neq 0 \text{ and } m \neq t, \\ x_{-m,t}x_{-m+t,m} - x_{-m,s}x_{-m+s,m} \text{ with } m,s,t \neq 0 \text{ and distinct} \end{pmatrix}$$

Indeed the first relations are trivial when one of $m, n, t$ is zero or $m = t$, while if $m+n = 0$ yield relations $x_{m,-m} = x_{-m,t}x_{-m+t,m}$. Using these last relations we can remove all the variables $x_{m,n}$ with $0 \in \{m, n, m+n\}$.

*Remark* 3.2.13. There exists a map $f\colon \tilde{K}_+ \longrightarrow \mathbb{N}$ such that for any $m, n \neq 0$ we have $f(e_{m,n}) = 1$ if $m+n \neq 0$, $f(e_{m,-m}) = 2$ otherwise. In particular $f(v) = 0$ only if $v = 0$. Moreover $f$ induces an $\mathbb{N}$-graduation on both $(R_M \otimes A)$ and $\mathbb{Z}[K_+] \otimes A$, where $A$ is a ring, such that the degree zero part is $A$ and that the elements $x_{m,n}$ with $m+n \neq 0$ are homogeneous of degree 1. The map $f$ is obtained as the composition $\tilde{K}_+ \longrightarrow K \subseteq \mathbb{Z}^M/\langle e_0 \rangle \xrightarrow{h} \mathbb{Z}$, where $h(e_m) = 1$ if $m \neq 0$.

One of the open problems in the theory of equivariant Hilbert schemes is whether those schemes are connected. As said above $M$-$\mathrm{Hilb}^{\underline{m}}$ is connected if and only if $\mathrm{D}(M)$-$\mathrm{Cov}^{\underline{m}}$ is so. What we can say here is:

**Theorem 3.2.14.** *The stack* $\mathrm{D}(M)$-Cov *is connected with geometrically connected fibers. If any non zero element of $M$ belongs to the sequence $\underline{m}$, then $M$-$\mathrm{Hilb}^{\underline{m}}$ has the same properties.*

*Proof.* It is enough to prove that $\operatorname{Spec} R_M \otimes k$ is connected for any field $k$. But $R_M \otimes k$ has an $\mathbb{N}$-graduation such that $(R_M \otimes k)_0 = k$ by 3.2.13 and it is a general fact that such an algebra does not contain non trivial idempotents. $\square$

We now want to discuss the problem of the reducibility of $\mathrm{D}(M)$-Cov.

**Definition 3.2.15.** Let $S$ be a scheme. An algebraic stack $\mathcal{X}$ is called *universally reducible* over $S$ if, for any base change $S' \longrightarrow S$, the stack $\mathcal{X} \times_S S'$ is reducible. An algebraic stack is universally reducible if it is so over $\mathbb{Z}$.

*Remark* 3.2.16. It is easy to check that $\mathcal{X}$ is universally reducible over $S$ if and only if all the fibers are reducible.

**Lemma 3.2.17.** *If there exist $m, n, t, a \in M$ such that*

*1) $m, n, t$ are distinct and not zero;*

*2) $a \neq 0, m, n, t, m-n, n-m, n-t, t-n, m-t, 2m-t, 2n-t, m+n-t, m+n-2t$;*

*3) $2a \neq m+n-t$;*

*then* $\operatorname{Spec} R_M$ *is universally reducible.*

*Proof.* Let $k$ be a field and $I = (\underline{x}^{\alpha_i} - \underline{x}^{\beta_i})$ be an ideal of $k[x_1, \dots, x_r] = k[\underline{x}]$. We will say that $\alpha \in \mathbb{N}^r$ is transformable (with respect to $I$) if there exists $i$ such that $\alpha_i \leq \alpha$ or $\beta_i \leq \alpha$. Here by $\alpha \leq \beta \in \mathbb{N}^r$ we mean $\alpha_j \leq \beta_j$ for all $j$. A direct computation shows that if $\underline{x}^\alpha - x^\beta \in I$ and $\alpha \neq \beta$, then both $\alpha$ and $\beta$ are transformable.

We will use the above notation for the ideal $I$ defining $R_M \otimes k$ as in 3.2.12. In particular the elements $\alpha_i, \beta_i \in \mathbb{N}^J$ associated to the ideal $I$ are of the form $e_{u,v} + e_{u+v,w}$ with $u, v, u+v, w, u+v+w \neq 0$.

Set $\mu = \prod_{m,n} x_{m,n}$. Since $R_M \otimes k \longrightarrow k[K_+] \subseteq k[K] = (R_M \otimes k)_\mu$, there exists $N > 0$ such that $P = \operatorname{Ker}(R_M \otimes k \longrightarrow k[K_+]) = \operatorname{Ann} \mu^N$. Our strategy will be to find an element of $P$ which is not nilpotent. Since $P$ is a minimal prime, being $\operatorname{Spec} k[K_+]$ an irreducible component of $\operatorname{Spec} R_M \otimes k$, it follows that $R_M \otimes k$ is reducible. Now consider $\alpha = e_{a,m-a} + e_{m+n-t-a,t+a-m} + e_{t+a-n,n-a}$, $\beta = e_{m+n-t-a,t+a-n} + e_{a,n-a} + e_{m-a,t+a-m} \in \mathbb{N}^J$ and $z = \underline{x}^\alpha - \underline{x}^\beta$. We will show that $\mu z = 0$, i.e. $z \in P$ and that $z$ is not nilpotent. First of all note that $z$ is well defined since for any $e_{u,v}$ in $\alpha$ or $\beta$ we have $u, v \neq 0$ and $0 \neq u + v \in \{m, n, t\}$ thanks to 1), 2). Let $S_M$ be the universal algebra over $R_M$, i.e. $S_M = \bigoplus_{m \in M} R_M v_m$ with $v_m v_n = x_{m,n} v_{m+n}$ and $v_0 = 1$. By construction we have

$$(v_a v_{m-a})(v_{m+n-t-a} v_{t+a-m})(v_{t+a-n} v_{n-a}) = \underline{x}^\alpha v_m v_n v_t =$$
$$(v_{m+n-t-a} v_{t+a-n})(v_a v_{n-a})(v_{m-a,t+a-m}) = \underline{x}^\beta v_m v_n v_t$$

So $\underline{x}^\alpha x_{m,n} x_{m+n,t} v_{m+n+t} = \underline{x}^\beta x_{m,n} x_{m+n,t} v_{m+n+t}$ and therefore $z\mu = 0$, i.e. $z \in P$.

Now we want to prove that any linear combination $\gamma = a\alpha + b\beta \in \mathbb{N}^J$ with $a, b \in \mathbb{N}$ is not transformable. First remember that each $e_{u,v}$ in $\gamma$ is such that $u + v \in \{m, n, t\}$. If we have $e_{u,v} + e_{u+v,w} \leq \gamma$ then there must exist $e_{i,j} \leq \gamma$ such that $i \in \{m, n, t\}$ or $j \in \{m, n, t\}$. Condition 2) is exactly what we need to avoid this situation and can be written as $\{a, m-a, m+n-t-a, t+a-m, t+a-n, n-a\} \cap \{m, n, t\} = \emptyset$.

In particular, if we think of $\tilde{K}_+$ as a quotient of $\mathbb{N}^J$, we have $a\alpha + b\beta = a'\alpha + b'\beta$ in $\tilde{K}_+$ if and only if they are equal in $\mathbb{N}^J$. Assume for a moment that $\alpha \neq \beta$ in $\mathbb{N}^J$. Clearly this means that $\alpha$ and $\beta$ are $\mathbb{Z}$-independent in $\mathbb{Z}^J$. Since any linear combination of $\alpha$ and $\beta$ is not transformable, it follows that $\underline{x}^\alpha, \underline{x}^\beta$ are algebraically independent over $k$ in $R_M \otimes k$ and, in particular, that $z = \underline{x}^\alpha - \underline{x}^\beta$ cannot be nilpotent. So it remains to prove that $\alpha \neq \beta$ in $\mathbb{N}^J$. Note that for any $i \in \{m, n, t\}$ there exists only one $e_{u,v}$ in $\alpha$ such that $u + v = i$ and the same happens for $\beta$. So, if $\alpha = \beta$ and since $m, n, t$ are distinct, those terms have to be equal, for instance $e_{a,m-a} = e_{m+n-t-a,t+a-n}$. But $a \neq m + n - t - a$ by 3), while $a \neq t + a - n$ since $t \neq n$. Therefore $\alpha \neq \beta$. $\square$

**Corollary 3.2.18.** *If $|M| > 7$ and $M \not\simeq (\mathbb{Z}/2\mathbb{Z})^3$ then* $\mathrm{D}(M)$-Cov *is universally reducible and the same holds for $M$-*$\mathrm{Hilb}^{\underline{m}}$*, provided that $\underline{m}$ contains all elements of $M - \{0\}$.*

*Proof.* We have to show that $R_M$ is universally reducible and so we will apply 3.2.17. If $M = C \times T$, where $C$ is cyclic with $|C| \geq 4$ and $T \neq 0$ we can choose: $m$ a generator of $C$, $n = 3m$, $t = 2m$ and $a \in T - \{0\}$. If $M$ cannot be written as above, there are four remaining cases. 1) $M \simeq \mathbb{Z}/8\mathbb{Z}$: choose $m = 2,\ n = 4,\ t = 6,\ a = 1$. 2) $M$ cyclic with $|M| > 8$ and $|M| \neq 10$: choose $m = 1,\ n = 2,\ t = 3,\ a = 5$. 3) $M \simeq (\mathbb{Z}/2\mathbb{Z})^l$ with $l \geq 4$: choose $m = e_1,\ n = e_2,\ t = e_3,\ a = e_4$. 4) $M \simeq (\mathbb{Z}/3\mathbb{Z})^l$ with $l \geq 2$: choose $m = e_1,\ n = 2e_1,\ t = e_2,\ a = m + t = e_1 + e_2$. $\square$

**Proposition 3.2.19.** $\mathrm{D}(M)$-Cov *is smooth if and only if $\mathcal{Z}_M$ is so. This happens if and only if $M \simeq \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and in these cases* $\mathrm{D}(M)$-Cov $= \mathcal{Z}_M$. *To be more precise $R_M = \mathbb{Z}[x_{m,n}]_{(m,n) \in J}$, where $J$ is the set defined in 3.2.12.*

*In particular $M$-Hilb$^{\underline{m}}$ is smooth and irreducible for any sequence $\underline{m}$ if $M$ is as above. Otherwise, if any non zero element of $M$ belongs to the sequence $\underline{m}$, $M$-Hilb$^{\underline{m}}$ is not smooth.*

*Proof.* Let $k$ be a field. Note that

$$\mathrm{D}(M)\text{-Cov smooth} \iff R_M \text{ smooth} \implies \mathcal{Z}_M \text{ smooth} \implies k[K_+]/k \text{ smooth}$$

We first prove that if $k[K_+]$ is smooth then $M$ has to be one of the groups of the statement. We have $K_+ \simeq \mathbb{N}^r \oplus \mathbb{Z}^s$ and therefore $k[K_+]$ is UFD. We will consider $k[K_+]$ endowed with the $\mathbb{N}$-graduation defined in 3.2.13. Since any of the $x_{m,n}$ has degree 1, it is irreducible and so prime. If we have a relation $x_{m,n}x_{m+n,t} = x_{n,t}x_{n+t,m}$ with $m, n, t, m+n, n+t, m+n+t \neq 0$ and $m \neq t$, then $x_{m,n} \mid x_{n,t}x_{n+t,m}$ implies that $x_{m,n} = x_{n,t}$ or $x_{m,n} = x_{n+t,m}$, which is impossible thanks to our assumptions. We will prove that if $M$ is not isomorphic to one of the group in the statement, then such a relation exists. Clearly it is enough to find this relation in a subgroup of $M$. So it is enough to consider the following cases. 1) $M$ cyclic with $|M| \geq 5$: choose $m = n = 1,\ t = 2$. 2) $M \simeq \mathbb{Z}/4\mathbb{Z}$: choose $m = 1,\ n = 2,\ t = 3$. 3) $M \simeq (\mathbb{Z}/2\mathbb{Z})^3$: choose $m = e_1,\ n = e_2,\ t = e_3$. 4) $M \simeq (\mathbb{Z}/3\mathbb{Z})^2$: choose $m = n = e_1,\ t = e_2$.

We now want to prove that when $M$ is as in the statement, then the ideal $I$ of 3.2.12 is zero. If we have a relation as in the first row, since $m \neq t$ we have $|M| \geq 3$. If $M \simeq \mathbb{Z}/3\mathbb{Z}$ then $t = 2m$ and $m + t = 0$. If $M \simeq (\mathbb{Z}/2\mathbb{Z})^2$, if $m, n, t$ are distinct then $m + n + t = 0$, otherwise $m = n$ and $m + n = 0$. If we have a relation as in the second row, since $m, t, s$ are distinct, we must have $M \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Therefore $m + t = s$ and the relation become trivial. □

**Corollary 3.2.20.** *The stack $\mathrm{D}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$-Cov is isomorphic to the stack of sequences $(\mathcal{L}_i, \psi_i)_{i=1,2,3}$, where $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are invertible sheaves and $\psi_1 \colon \mathcal{L}_2 \otimes \mathcal{L}_3 \longrightarrow \mathcal{L}_1$, $\psi_2 \colon \mathcal{L}_1 \otimes \mathcal{L}_3 \longrightarrow \mathcal{L}_2$, $\psi_3 \colon \mathcal{L}_1 \otimes \mathcal{L}_2 \longrightarrow \mathcal{L}_3$ are maps.*

*Proof.* Set $M = (\mathbb{Z}/2\mathbb{Z})^2$. Thanks to 3.2.19, we know that $\tilde{K}_+ = K_+ \simeq \mathbb{N}v_{e_1,e_2} \oplus \mathbb{N}v_{e_1,e_1+e_2} \oplus \mathbb{N}v_{e_2,e_1+e_2}$. So an object of $\mathrm{D}(M)$-Cov is given by invertible sheaves $\mathcal{L}_1 = \mathcal{L}_{e_1},\ \mathcal{L}_2 = \mathcal{L}_{e_2},\ \mathcal{L}_3 = \mathcal{L}_{e_1+e_2}$ and maps $\psi_1 = \psi_{e_2,e_1+e_2},\ \psi_2 = \psi_{e_1,e_1+e_2},\ \psi_3 = \psi_{e_1,e_2}$. □

*Remark* 3.2.21. $\mathrm{D}(\mathbb{Z}/4\mathbb{Z})$-Cov and $\mathbb{Z}/4\mathbb{Z}$-Hilb$^{\underline{m}}$, for any sequence $\underline{m}$, are integral and normal since one can check directly that $R_{\mathbb{Z}/4\mathbb{Z}} = \mathbb{Z}[x_{1,2}, x_{3,3}, x_{2,3}, x_{1,1}]/(x_{1,2}x_{3,3} - x_{2,3}x_{1,1})$. I am not able to prove that $\mathrm{D}(M)$-Cov is irreducible when $M$ is one of $\mathbb{Z}/5\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/7\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^3$. Anyway the first two cases seem to be integral thanks to a computer program, while for the last ones there are some techniques that can be used to study this problem but they are too complicated to be explained here.

### 3.2.2 The invariant $h \colon |\mathrm{D}(M)\text{-Cov}| \longrightarrow \mathbb{N}$.

In this subsection we investigate the local structure of a $\mathrm{D}(M)$-cover, especially over a local ring. In particular we will define an upper semicontinuous map $h \colon |\mathrm{D}(M)\text{-Cov}| \longrightarrow \mathbb{N}$ that measures how much a cover fails to be a torsor: the open locus $\mathrm{B}\,\mathrm{D}(M) \subseteq \mathrm{D}(M)$-Cov will be exactly the locus $\{h = 0\}$.

*Notation* 3.2.22. Given a ring $A$, we will write $B \in \operatorname{Spec} R_M(A)$ meaning that $B$ is an $M$-graded $A$-algebra with a given $M$-graded basis, usually denoted by $\{v_m\}_{m\in M}$ with $v_0 = 1$, and a given multiplication $\psi$ such that

$$B = \bigoplus_{m\in M} Av_m,\ v_m v_n = \psi_{m,n} v_{m+n}$$

We will also denote by $A^*$ the group of invertible elements of $A$. If $f\colon X \longrightarrow Y$ is an affine map of schemes and $q \in Y$, we will use the notation $\mathcal{O}_{X,q} = f_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y,q}$. In particular $X \times_Y \operatorname{Spec} \mathcal{O}_{Y,q} \simeq \operatorname{Spec} \mathcal{O}_{X,q}$. Notice that, although $\mathcal{O}_{X,q}$ is written as a localization in a point, this ring is not local in general.

**Lemma 3.2.23.** *Let $A$ be a ring and $B \in \operatorname{Spec} R_M(A)$, with graded basis $v_m$ and multiplication map $\psi$. Then the set*

$$H_\psi = H_{B/A} = \{m \in M \mid v_m \in B^*\} = \{m \in M \mid \psi_{m,-m} \in A^*\}$$

*is a subgroup of $M$. Moreover if $m, n \in M$ and $h \in H_\psi$ then $\psi_{m,n}$ and $\psi_{m,n+h}$ differs by an element of $A^*$. If $H$ is a subgroup of $H_\psi$ then $C = \bigoplus_{m\in H} Av_m$ is an element of* $\mathrm{B}\,\mathrm{D}(H)(A)$. *Moreover if $\sigma\colon M/H \longrightarrow M$ gives representatives of $M/H$ in $M$ and we set $w_m = v_{\sigma(m)}$ for $m \in M/H$ we have*

$$B = \bigoplus_{m\in M/H} Cw_m \in \operatorname{Spec} R_{M/H}(C)$$

*Finally if we denote by $\psi'$ the induced multiplication on $B$ over $C$ we have $H_{\psi'} = H_\psi/H$ and for any $m, n \in M$ $\psi'_{m,n}$ and $\psi_{m,n}$ differ by an element of $C^*$.*

*Proof.* From the relations $v_m v_{-m} = \psi_{m,-m}$, $v_m^{|M|-1} = \lambda v_{-m}$, $v_m^{|M|} = \lambda\psi_{m,-m}$, where $\lambda \in B$ and $v_m v_n = \psi_{m,n} v_{m+n}$ we see that $v_m \in B^* \iff \psi_{m,-m} \in A^*$ and that $H_\psi < M$. From 3.2.1 we get the relations $\psi_{-h,h} = \psi_{h,u}\psi_{h+u,-h}$ and $\psi_{m,n}\psi_{m+n,h} = \psi_{n,h}\psi_{m,n+h}$. So if $h \in H$ then $\psi_{h,u} \in A^*$ for any $u$ and $\psi_{m,n}$ and $\psi_{m,n+h}$ differ by an element of $A^*$.

Now consider the second part of the statement. From 3.2.3 we know that $C$ is a torsor over $A$. Since for any $m$ we have $v_m = (\psi_{h,m}/v_h)v_{\sigma(\overline{m})}$, where $h = \sigma(\overline{m}) - m \in H$ we obtain the expression of $B$ as $M/H$ graded $C$-algebra and that

$$\psi'_{m,n} = \psi_{\sigma(m),\sigma(n)}(\psi_{h,\sigma(m)+\sigma(n)}/v_h) \text{ where } h = \sigma(m+n) - \sigma(m) - \sigma(n)$$

From the above equation it is easy to conclude the proof. $\square$

**Definition 3.2.24.** Given a ring $A$ and $B \in \operatorname{Spec} R_M(A)$ we continue to use the notation $H_{B/A}$ introduced in 3.2.23 and we will call the algebra $C$ obtained for $H = H_{B/A}$ the *maximal torsor* of the extension $B/A$. If $k$ is a field and $\mathcal{E} \in K_+^\vee$ we will write $H_\mathcal{E} = H_{B/k}$, where $B$ is the algebra induced by the multiplication $0^\mathcal{E}$. In particular

$$H_\mathcal{E} = \{m \in M \mid \mathcal{E}_{m,-m} = 0\}$$

Finally if $f\colon X \longrightarrow Y \in \mathrm{D}(M)\text{-}\mathrm{Cov}(Y)$ and $q \in Y$ we define $\mathcal{H}_f(q) = H_{\mathcal{O}_{X,q}/\mathcal{O}_{Y,q}}$.

**Proposition 3.2.25.** *We have a map*

$$\begin{array}{ccc} |\mathrm{D}(M)\text{-Cov}| & \xrightarrow{\mathcal{H}} & \{\textit{subgroups of } M\} \\ B/k & \longmapsto & H_{B/k} \end{array}$$

*such that, if* $Y \xrightarrow{u} \mathrm{D}(M)\text{-Cov}$ *is given by* $X \xrightarrow{f} Y$*, then* $\mathcal{H}_f = \mathcal{H} \circ |u|$.

*Proof.* It is enough to note that if $A$ is a local ring, $B \in \mathrm{D}(M)\text{-Cov}(A)$ is given by multiplications $\psi$ and $\pi \colon A \longrightarrow A/m_A \longrightarrow k$ is a morphism, where $k$ is a field, then $\psi_{m,-m} \in A^* \iff \pi(\psi_{m,-m}) \neq 0$. □

*Remark* 3.2.26. Let $(A, m_A)$ be a local ring and $B \in \operatorname{Spec} R_M(A)$ with $M$-graded basis $\{v_m\}_{m\in M}$. Then $H_{B/A} = \mathcal{H}_{B/A}(m_A)$. If $H_{B/A} = 0$ then any $v_m$, with $m \neq 0$, is nilpotent in $B \otimes k$ and therefore $B$ is local with maximal ideal

$$m_B = m_A \oplus \bigoplus_{m\in M-\{0\}} Av_m$$

and residue field $B/m_B = A/m_A$. In particular $m_B/m_B^2$ is $M$-graded.

**Lemma 3.2.27.** *Let* $A$ *be a local ring and* $B = \bigoplus_{m\in M} Av_m \in \mathrm{D}(M)\text{-Cov}(A)$ *such that* $H_{B/A} = 0$. *If* $m_1, \dots, m_r \in M$ *then* $B$ *is generated in degrees* $m_1, \dots, m_r$ *as an* $A$*-algebra if and only if* $m_B = (m_A, v_{m_1}, \dots, v_{m_r})_B$.

*Proof.* We can write $m_B = m_A \oplus \bigoplus_{m\in M-\{0\}} Av_m$. Denote $\underline{v} = v_{m_1}, \dots, v_{m_r}$ and $\pi(\alpha) = \sum_i \alpha_i m_i$ for $\alpha \in \mathbb{N}^r$. The "only if" follows since given $l \in M - \{0\}$ there exists a relation of the form $v_l = \mu \underline{v}^\alpha$ with $\mu \in A^*$ and $\alpha \neq 0$ and so $v_l \in (m_A, v_{m_1}, \dots, v_{m_r})_B$. For the converse note that, given $l \in M - \{0\}$, $v_l \in m_B = (m_A, v_{m_1}, \dots, v_{m_r})$ means that we have a relation $v_l = \lambda v_{l'} v_{m_i}$ for some $i$, $\lambda \in A^*$ and $l' = l - m_i$. Moreover $v_l \notin A[\underline{v}]$ implies that $v_{l'} \notin A[\underline{v}]$ and $l' \neq 0$. If, by contradiction, we have such an element $l$ we can write $v_l = \mu v_{n_1} \cdots v_{n_s}$ with $n_i \in M - \{0\}$ and $s \geq |M|^2$. In particular there must exist $i$ such that $m = n_i$ appears at least $|M|$ times in this product. So $m_A \ni v_m^{|M|} \mid v_l$ and $v_l \in m_A B$, which is not the case. □

Assume we have a cover $X \xrightarrow{f} Y \in \mathrm{D}(M)\text{-Cov}(Y)$. We want to define, for any $m \in M$ a map $h_{f,m} = h_{X/Y,m} \colon Y \longrightarrow \{0, 1\}$. Let $q \in Y$ and denote by $C$ the 'maximal torsor' of $\mathcal{O}_{X,q}/\mathcal{O}_{Y,q}$ (see 3.2.24). Also let $p \in f^{-1}(q)$ and set $p_C = p \cap C$. Taking into account 3.2.26, we know that $B = (\mathcal{O}_{X,q})_p = (\mathcal{O}_{X,q})_{p_C}$ and that $B \in \mathrm{D}(M/\mathcal{H}_f(q))\text{-Cov}(C_{p_C})$ with $H_{B/C_{p_C}} = 0$. Moreover $B$ is local, $B/m_B = C_{p_C}/p_C$ and $m_B/m_B^2$ is $(M/\mathcal{H}_f(q))$-graded. If we denote by $\overline{m}$ the image of $m \in M$ in $M/\mathcal{H}_f(q)$ and by $(m_B/m_B^2)_t$ the graded pieces of $m_B/m_B^2$, where $t \in M/\mathcal{H}_f(q)$, we can define:

**Definition 3.2.28.** With notation above we set

$$h_{f,m}(q) = \begin{cases} 0 & \text{if } m \in \mathcal{H}_f(q) \\ \dim_{C_{p_C}/p_C} (m_B/m_B^2)_{\overline{m}} & \text{otherwise} \end{cases}$$

We also set

$$h_f(q) = \dim_{C_{p_C}/p_C}(m_B/m_B^2) - \dim_{C_{p_C}/p_C}(m_B/m_B^2)_0 = (\sum_{m\in M} h_{f,m}(q))/|\mathcal{H}_f(q)|$$

If $\mathcal{E} \in K_+^\vee$ we set $h_{\mathcal{E},m} = h_{f,m}$, $h_{\mathcal{E}} = h_f \in \mathbb{N}$ where $f$ is the cover $\operatorname{Spec} A \longrightarrow \operatorname{Spec} k$ and $A$ is the algebra given by multiplication $0^{\mathcal{E}}$ over some field $k$.

The following lemma shows that the value of $h_{f,m}(q)$ does not depend on the choice of the point $p \in X$ over $q \in Y$.

**Lemma 3.2.29.** *Let $(A, m_A)$ be a local ring, $B \in \mathrm{D}(M)\text{-Cov}(A)$ given by the multiplication $\psi$ and $t \in M$. Set also $h_{B/A,t} = h_{B/A,t}(m_A)$, for some choice of a prime of $B$ over $m_A$. Then $h_{B/A,t} = 1$ if and only if the following conditions are satisfied:*

- *$t \notin H_{B/A}$;*
- *for all $u, n \in M - H_{B/A}$ such that $u + n \equiv t \mod H_{B/A}$ we have $\psi_{u,n} \notin A^*$.*

*Proof.* Let $C$ be the maximal torsor of the extension $B/A$ and $p$ be a maximal prime of $B$. We use notation from 3.2.23. For any $l \in M - H_{B/A}$ we have a surjective map

$$k(p) = (m_{B_p}/pC_p)_{\overline{l}} \longrightarrow (m_{B_p}/m_{B_p}^2)_{\overline{l}}$$

and so $\dim_{k(p)}(m_{B_p}/m_{B_p}^2)_{\overline{l}} \in \{0, 1\}$, where $\overline{l}$ is the image of $l$ under the projection $M \longrightarrow M/H_{A/B}$. If we prove the last part of the statement clearly we will also have that $h_{B/A,t}$ is well defined. If $t \in H_{B/A}$ then $h_{B/A,t} = 0$, while if there exist $u, n$ as in the statement such that $\psi_{u,n} \in A^*$, then $w_{\overline{t}} \in C_p^* w_{\overline{u}} w_{\overline{n}} \subseteq m_{B_p}^2$ and again $h_{B/A,t} = 0$. On the other hand if $h_{B/A,t} = 0$ and $t \notin H_{B/A}$ then $w_{\overline{t}} \in m_{B_p}^2$ and therefore we have an expression

$$w_{\overline{t}} = bx + \sum_{\overline{u},\overline{n}\neq 0} b_{\overline{u},\overline{n}} w_{\overline{u}} w_{\overline{n}} \text{ with } b, b_{\overline{u},\overline{n}} \in B_p, x \in m_{C_p}$$

The second sum splits as a sum of products of the form $c_{s,\overline{u},\overline{n}} w_s w_{\overline{u}} w_{\overline{n}}$ with $s + \overline{u} + \overline{n} = \overline{t}$ and $c_{s,\overline{u},\overline{n}} \in C_p$. Since $C_p$ is local, one of these monomials generates $C_p w_{\overline{t}}$. In this case, if $s + \overline{u} = 0$ then $\overline{u} \in H_{B_p/C_p} = 0$ which is not the case. So we have an expression

$$w_{\overline{t}} = \lambda w_{\overline{u}} w_{\overline{n}} = \lambda \psi'_{\overline{u},\overline{n}} w_{\overline{t}} \implies \psi'_{\overline{u},\overline{n}} \in C_p^*$$

where $\overline{u}, \overline{n} \neq 0$ and $\overline{u} + \overline{n} = \overline{t}$. Since $\psi'_{\overline{u},\overline{n}}$ and $\psi_{u,n}$ differs by an element of $C^*$ thanks to 3.2.23, it follows that $\psi_{u,n} \in A^*$. $\square$

**Proposition 3.2.30.** *We have maps*

$$\begin{array}{ccc} |\mathrm{D}(M)\text{-Cov}| & \xrightarrow{h_m} & \{0,1\} \\ B/k & \longmapsto & h_{B/k,m} \end{array} \qquad \begin{array}{ccc} |\mathrm{D}(M)\text{-Cov}| & \xrightarrow{h} & \mathbb{N} \\ B/k & \longmapsto & h_{B/k} \end{array}$$

*such that, if $Y \xrightarrow{u} \mathrm{D}(M)\text{-Cov}$ is given by $X \xrightarrow{f} Y$, then $h_{f,m} = h_m \circ |u|$ and $h_f = h \circ |u|$.*

*Proof.* Taking into account 3.2.29 and 3.2.25, it is enough to note that if $A$ is a local ring, $B \in \mathrm{D}(M)\text{-Cov}(A)$ is given by multiplications $\psi$ and $\pi\colon A \longrightarrow A/m_A \longrightarrow k$ is a morphism, where $k$ is a field, then $\psi_{u,v} \in A^* \iff \pi(\psi_{u,v}) \neq 0$ and $H_{B/A} = H_{B\otimes_A k/k}$. □

**Corollary 3.2.31.** *Under the hypothesis of 3.2.27, $\{m \in M \mid h_{B/A,m} = 1\}$ is the minimum among the subsets $Q$ of $M$ such that $B$ is generated as an $A$-algebra in the degrees $Q$. In particular $B$ is generated in $h_{B/A}$ degrees.*

**Proposition 3.2.32.** *Let $(A, m_A)$ be a local ring, $B \in \mathrm{D}(M)\text{-Cov}(A)$ and $C$ the maximal torsor of $B/A$. Then*

$$h_{B/A}(m_A) = \dim_{k(p)} \Omega_{B/C} \otimes_B k(p)$$

*for any maximal prime $p$ of $B$. In particular if $(|H_{B/A}|, \operatorname{char} A/m_A) = 1$ we also have $h_{B/A}(m_A) = \dim_{k(p)} \Omega_{B/A} \otimes_B k(p)$ for any maximal prime $p$ of $B$.*

*Proof.* If $A$ is any ring and $B \in \mathrm{D}(M)\text{-Cov}(A)$ is given by basis $\{v_m\}_{m\in M}$ and multiplication $\psi$ one sees from the universal property that

$$\Omega_{B/A} = B^M/\langle e_0, v_n e_m + v_m e_n - \psi_{m,n} e_{m+n}\rangle$$

Now consider $B \in \mathrm{D}(M/H)\text{-Cov}(C)$, where $H = H_{B/A}$ and let $p$ be a maximal prime of $B$. Following the notation of 3.2.23, we have that $w_m \in p$ for any $m \in M/H - \{0\}$ and $\psi'_{m,n} \in p \iff \psi_{m,n} \in m_A$. So $\Omega_{B/C} \otimes_B k(p)$ is free on the $e_m$ for $m \in M/H - \{0\}$ such that for any $u, n \in M/H - \{0\}$, $u+n = m$ implies $\psi_{u,n} \notin A^*$, that are exactly $h_{B/A}(m_A)$ thanks to 3.2.29. □

**Corollary 3.2.33.** *The function $h$ is upper semicontinuous.*

*Proof.* Let $X \xrightarrow{f} Y$ be a $\mathrm{D}(M)$-cover and $q \in Y$. Set $r = h_f(q)$ and $H = \mathcal{H}_f(q)$. We can assume that $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$ with graded basis $\{v_m\}_{m\in M}$ and multiplication $\psi$ and that $\psi_{m,-m} \in A^*$ for any $m \in H$. Set $C = A[v_m]_{m\in H}$. The ring $C_q$ is the maximal torsor of $B_q/A_q$ and so, if $p \in X$ is a point over $q$, we have $r = \dim_{k(p)} \Omega_{B/C} \otimes_B k(p)$. Finally let $U \subseteq X$ be an open neighborhood of $p$ such that $\dim_{k(p')} \Omega_{B/C} \otimes_B k(p') \leq r$ for any $p' \in U$ and $V = f(U)$. We want to prove that $h \leq r$ on $V$. Indeed given $q' = f(p') \in V$, if $D$ is the maximal torsor of $B_{q'}/A_{q'}$, we have $C_{q'} \subseteq D \subseteq B_{q'}$. So

$$h_f(q') = \dim_{k(p')} \Omega_{B_{q'}/D} \otimes_{B_{q'}} k(p') \leq \dim_{k(p')} \Omega_{B_{q'}/C_{q'}} \otimes_{B_{q'}} k(p') \leq r$$

□

*Remark* 3.2.34. The 0 section $R_M \longrightarrow \mathbb{Z}$, i.e. the map that sends any $x_{m,n}$ with $m, n \neq 0$ to zero, induces a closed immersion

$$\underline{\mathrm{Pic}}^{|M|-1} \simeq \mathrm{B}\mathcal{T} = [\operatorname{Spec} \mathbb{Z}/\mathcal{T}] \subseteq [\operatorname{Spec} R_M/\mathcal{T}] \simeq \mathrm{D}(M)\text{-Cov}$$

where $\mathcal{T} = \mathrm{D}(\mathbb{Z}^M/\langle e_0\rangle)$.

**Proposition 3.2.35.** *The following results hold:*

1) $\{h=0\} = |\operatorname{B}\operatorname{D}(M)|$;

2) $\{h \geq |M|\} = \emptyset$;

3) $\{h = |M|-1\} = |\operatorname{B}\operatorname{D}(\mathbb{Z}^M/\langle e_0\rangle)|$ *(see 3.2.34)*

*Proof.* If $X \xrightarrow{f} Y$ is a $\operatorname{D}(M)$-torsor, clearly $h_f = 0$. So 1) and 2) follow from 3.2.31. Finally, if $B \in \operatorname{D}(M)\text{-}\operatorname{Cov}(k)$ with multiplication $\psi$, $h_{B/k} = |M|-1$ if and only if $H_{B/k} = 0$ and $h_{B/k,m} = 1$ for all $m \in M - \{0\}$. This means that $\psi_{m,n} = 0$ for any $m, n \neq 0$ by 3.2.29. □

In particular, setting $U_i = \{h \leq i\}$, we obtain a stratification $\operatorname{B}\operatorname{D}(M) = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{|M|-1} = \operatorname{D}(M)\text{-}\operatorname{Cov}$ of $\operatorname{D}(M)\text{-}\operatorname{Cov}$ by open substacks.

### 3.2.3 The locus $h \leq 1$.

In this subsection we want to describe $\operatorname{D}(M)$-covers with $h \leq 1$. This means that 'up to torsors' we have a graded $M$-algebra generated over the base ring in one degree. We will see that $\{h \leq 1\}$ is a smooth open substack of $\mathcal{Z}_M$ determined by a special class of explicit smooth extremal rays of $K_+$. This will allow us to give a description of normal $\operatorname{D}(M)$-covers over locally noetherian and locally factorial scheme $X$ with $(\operatorname{char} X, |M|) = 1$. Such a description, when $X$ is a smooth algebraic variety over an algebraic closed field $k$ was already given in [Par91, Theorem 2.1, Corollary 3.1].

*Notation* 3.2.36. Given $\mathcal{E} \in K_+^\vee$ we will write $\mathcal{E}_{m,n} = \mathcal{E}(v_{m,n})$. Since $K \otimes \mathbb{Q} \simeq \mathbb{Q}^M/\langle e_0\rangle$ we will also write $\mathcal{E}_m = \mathcal{E}(e_m) \in \mathbb{Q}$, so that $\mathcal{E}_{m,n} = \mathcal{E}_m + \mathcal{E}_n - \mathcal{E}_{m+n}$. When we will have to consider different abelian groups, we will write $K_{+M}$, $K_M$ instead of, respectively, $K_+$, $K$, in order to avoid confusion. Given a group homomorphism $\eta \colon M \longrightarrow N$ we will denote by $\eta_* \colon K_M \longrightarrow K_N$ the homomorphism such that $\eta_*(v_{m,n}) = v_{\eta(m),\eta(n)}$ for all $m, n \in M$, where $K_M$ is the group associated to $K_+$,

*Remark* 3.2.37. Let $A$ be a ring and consider a sequence $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in K_+^\vee$. An element of $\mathcal{F}_{\underline{\mathcal{E}}}(A)$ coming from the atlas (see 3.1.14) is given by a pair $(\underline{z}, \lambda)$ where $\underline{z} = z_1, \dots, z_r \in A$ and $\lambda \colon K \longrightarrow A^*$. The image of this object under $\pi_{\underline{\mathcal{E}}}$ is the algebra whose multiplication is given by $\psi_{m,n} = \lambda_{m,n}^{-1} z_1^{\mathcal{E}^1_{m,n}} \cdots z_r^{\mathcal{E}^r_{m,n}}$.

**Lemma 3.2.38.** *Let $\eta \colon M \longrightarrow N$ be a surjective morphism and $\underline{\mathcal{E}}$ be a sequence in $(K_{+N})^\vee$. Then $\underline{\mathcal{E}}$ is a smooth sequence for $N$ if and only if $\underline{\mathcal{E}} \circ \eta_*$ is a smooth sequence for $M$.*

*Proof.* We want to apply 3.1.39. Therefore we have to prove that $\eta_*(K_{+M}) = K_{+N}$, which is clear, and that $\operatorname{Ker} \eta_* = \langle \operatorname{Ker} \eta_* \cap K_{+N} \rangle$. Consider the map $f \colon \mathbb{Z}^M/\langle e_0\rangle \longrightarrow \mathbb{Z}^N/\langle e_0\rangle$ given by $f(e_m) = e_{\eta(m)}$ and set $H = \operatorname{Ker} \eta$. Clearly $f_{|K_M} = \eta_*$. It is easy to check that $G = \langle v_{m,n} \text{ for } m \in H\rangle_{\mathbb{Z}} \subseteq \operatorname{Ker} \eta^* \subseteq \operatorname{Ker} f$ and that $\operatorname{Ker} f/\operatorname{Ker} \eta_* \simeq H$. So in order to conclude, it is enough to note that the map $H \longrightarrow \operatorname{Ker} f/G$ sending $h$ to $e_h$

is a surjective group homomorphism since we have relations $e_h + e_{h'} - e_{h+h'} = v_{h,h'}$ and $e_{m+h} - e_m = e_h - v_{m,h}$ for $m \in M$ and $h, h' \in H$. $\square$

**Proposition 3.2.39.** *Let $\eta\colon M \longrightarrow \mathbb{Z}/l\mathbb{Z}$ be a surjective homomorphism with $l > 1$. Then*

$$\mathcal{E}^\eta(v_{m,n}) = \begin{cases} 0 & \text{if } \eta(m)+\eta(n) < l \\ 1 & \text{otherwise} \end{cases}$$

*defines a smooth extremal ray for $K_+$.*

*Proof.* $\mathcal{E}^\eta \in K_+^\vee$ because, if $\sigma\colon \mathbb{Z}/l\mathbb{Z} \longrightarrow \mathbb{N}$ is the obvious section, $\mathcal{E}^\eta$ is the restriction of the map $\mathbb{Z}^M/\langle e_0\rangle \longrightarrow \mathbb{Z}$ sending $e_m$ to $\sigma(\eta(m))$. In order to conclude the proof, we will apply 3.2.38 and 3.1.38. Set $N = \mathbb{Z}/l\mathbb{Z}$. One clearly has $\mathcal{E}^\eta = \mathcal{E}^{\mathrm{id}} \circ \eta_*$ and so we can assume $M = \mathbb{Z}/l\mathbb{Z}$ and $\eta = \mathrm{id}$. In this case one can check that $v_{1,1}, v_{1,2}, \dots, v_{1,l-1}$ is a $\mathbb{Z}$-base of $K$ such that $\mathcal{E}^\eta(v_{1,j}) = 0$ if $j < l-1$, $\mathcal{E}^\eta(v_{1,l-1}) = 1$. $\square$

Those particular rays have been already defined in [Par91, Equation 2.2].

*Notation* 3.2.40. If $\phi\colon \tilde{K}_+ \longrightarrow \mathbb{Z}^M/\langle e_0\rangle$ is the usual map we set $\mathcal{Z}_M^{\underline{\mathcal{E}}} = \mathcal{X}_\phi^{\underline{\mathcal{E}}}$ (see definition 3.1.18) for any sequence $\underline{\mathcal{E}}$ of elements of $K_+^\vee$. Remember that if $\underline{\mathcal{E}}$ is a smooth sequence then $\mathcal{Z}_M^{\underline{\mathcal{E}}}$ is a smooth open subset of $\mathcal{Z}_M$ (see 3.1.41) and its points have the description given in 3.1.42.

Set $\Phi_M$ for the union over all $d > 1$ of the sets of surjective maps $M \longrightarrow \mathbb{Z}/d\mathbb{Z}$.

**Theorem 3.2.41.** *Let $\underline{\mathcal{E}} = (\mathcal{E}^\eta)_{\eta\in\Phi_M}$. We have*

$$\{h \leq 1\} = \bigcup_{\eta\in\Phi_M} \mathcal{Z}_M^{\mathcal{E}^\eta}$$

*In particular $\{h \leq 1\} \subseteq \mathcal{Z}_M^{\mathrm{sm}}$ and $\pi_{\underline{\mathcal{E}}}$ induces an equivalence of categories*

$$\{(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}} \mid V(z_\eta) \cap V(z_\mu) = \emptyset \text{ if } \eta \neq \mu\} = \pi_{\underline{\mathcal{E}}}^{-1}(\{h \leq 1\}) \xrightarrow{\simeq} \{h \leq 1\}$$

*Proof.* The last part of the statement follows from the first one just applying 3.1.45 with $\Theta = \{(\mathcal{E}^\eta)\}_{\eta\in\Phi_M}$. Let $k$ be an algebraically closed field and $B \in \mathrm{D}(M)\text{-Cov}(k)$ with graded basis $\{v_m\}_{m\in M}$ and multiplication $\psi$.

$\supseteq$. Assume $B \in \mathcal{Z}_M^{\mathcal{E}^\eta}(k)$. If B is a torsor we will have $h_{B/k} = 0$. Otherwise we can write $\psi = \xi 0^{\mathcal{E}^\eta}$ for some $\xi\colon K \longrightarrow k^*$. Replacing $\operatorname{Spec} k$ by a geometrical point of the maximal torsor of $B/k$, we can assume that $M = \mathbb{Z}/d\mathbb{Z}$ and $\eta = \mathrm{id}$. In particular $H_{B/k} = 0$ and, from the definition of $\mathcal{E}^{\mathrm{id}}$, we get $B \simeq k[x]/(x^d)$. So $h_{B/k} = \dim_k m_B/m_B^2 = 1$.

$\subseteq$. Assume $h_{B/k} = 1$. Set $C$ for the maximal torsor of $B/k$ (see 3.2.24), $H = H_{B/k}$ and $l = |M/H|$. The equality $h_{B/k} = 1$ means that there exists a unique $\overline{r} \in M/H$ (where $r \in M$) such that $h_{B/k,r} = 1$ and so $C_q[v_r] = B_q \simeq C_q[x]/(x^l)$ for all (maximal) primes $q$ of $C$. In particular $B = C[v_r] \simeq C[x]/(x^l)$ and $\overline{r}$ generates $M/H$. Let $\eta\colon M \longrightarrow M/H \simeq \mathbb{Z}/l\mathbb{Z}$ be the projection. We want to prove that $B \in \mathcal{Z}_M^{\mathcal{E}^\eta}$. Replacing $k$ by a geometrical point of some fppf extension of $k$, we can assume $C = k[H]$, i.e. $v_h v_{h'} = v_{h+h'}$ if $h, h' \in H$. Finally the elements $v_h v_r^i$ for $h \in H$ and $0 \leq i < l$ define an $M$-graded basis of $B/k$ whose associated multiplication is $0^{\mathcal{E}^\eta}$. $\square$

**Theorem 3.2.42.** *Let $\underline{\mathcal{E}} = (\mathcal{E}^\eta)_{\eta\in\Phi_M}$ and let $X$ be a locally noetherian and locally factorial scheme. Consider the full subcategories*

$$\mathscr{C}^1_X = \{(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}(X) \mid \operatorname{codim}_X V(z_\eta)\cap V(z_\mu) \geq 2 \textit{ if } \eta \neq \mu\} \subseteq \mathcal{F}_{\underline{\mathcal{E}}}(X)$$

*and*

$$\mathscr{D}^1_X = \{Y \xrightarrow{f} X \in \mathrm{D}(M)\text{-}\mathrm{Cov}(X) \mid h_f(p) \leq 1 \ \forall p \in X \textit{ with } \operatorname{codim}_p X \leq 1\} \subseteq \mathrm{D}(M)\text{-}\mathrm{Cov}(X)$$

*Then $\pi_{\underline{\mathcal{E}}}$ induces an equivalence of categories*

$$\mathscr{D}^1_X = \pi_{\underline{\mathcal{E}}}^{-1}(\mathscr{C}^1_X) \xrightarrow{\simeq} \mathscr{C}^1_X$$

*Proof.* Apply 3.1.53 with $\Theta = \{(\mathcal{E}^\eta)\}_{\eta\in\Phi_M}$. □

**Theorem 3.2.43.** *Let $\underline{\mathcal{E}} = (\mathcal{E}^\eta)_{\eta\in\Phi_M}$ and let $X$ be a locally noetherian and locally factorial scheme without isolated points and $(\operatorname{char} X, |M|) = 1$, i.e. $1/|M| \in \mathcal{O}_X(X)$. Consider the full subcategories*

$$Reg^1_X = \{Y/X \in \mathrm{D}(M)\text{-}\mathrm{Cov}(X) \mid Y \textit{ regular in codimension } 1\} \subseteq \mathrm{D}(M)\text{-}\mathrm{Cov}(X)$$

*and*

$$\widetilde{Reg}^1_X = \left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}(X) \;\middle|\; \begin{array}{c} \forall \mathcal{E} \neq \delta \in \underline{\mathcal{E}} \ \operatorname{codim}_X V(z_{\mathcal{E}}) \cap V(z_\delta) \geq 2 \\ \forall \mathcal{E} \in \underline{\mathcal{E}} \forall p \in X^{(1)} \ v_p(z_{\mathcal{E}}) \leq 1 \end{array} \right\} \subseteq \mathcal{F}_{\underline{\mathcal{E}}}(X)$$

*Then we have an equivalence of categories*

$$\widetilde{Reg}^1_X = \pi_{\underline{\mathcal{E}}}^{-1}(Reg^1_X) \xrightarrow{\simeq} Reg^1_X$$

*Proof.* We will make use of 3.2.42. If $Y \xrightarrow{f} X \in Reg^1_X$, $p \in Y^{(1)}$ and $q = f(p)$ then $h_f(q) \leq \dim_{k(p)} m_p/m_p^2 = 1$. So $Reg^1_X \subseteq \mathscr{D}^1_X$. So we have only to check that $\widetilde{Reg}^1_X = \pi_{\underline{\mathcal{E}}}^{-1}(Reg^1_X) \subseteq \mathscr{C}^1_X$. Since $X$ is a disjoint union of positive dimensional, integral connected components, we can assume that $X = \operatorname{Spec} R$, where $R$ is a discrete valuation ring. Let $\chi \in \mathscr{C}^1_X$, $A/R \in \mathscr{D}^1_X$ the associated covers, $H = H_{A/R}$ and $C$ be the maximal torsor of $A/R$. We have to prove that $\chi \in \widetilde{Reg}^1_X$ if and only if $A$ is regular in codimension 1. Since $D_R(H)$ is etale over $R$ so is also $\operatorname{Spec} C$. It is so easy to check that, replacing $R$ by a localization of $C$ and $M$ with $M/H$, we can assume that $H = 0$. Since $\chi \in \mathscr{C}^1_X$, the multiplication of $A$ over $R$ is of the form $\psi = \mu z^{r\mathcal{E}^\phi}$, where $\mu \colon K \longrightarrow R^*$ is an $M$-torsor, $z$ is a parameter of $A$, $\phi \colon M \longrightarrow \mathbb{Z}/l\mathbb{Z}$ is an isomorphism and $r = v_R(z_{\mathcal{E}^\phi})$. Moreover $v_R(z_{\mathcal{E}^\psi}) = 0$ if $\psi \neq \phi$. Replacing $M$ by $\mathbb{Z}/l\mathbb{Z}$ through $\phi$ we can assume $\phi = \mathrm{id}$. Finally, since $\mu$ induces an (fppf) torsor which is etale over $R$, replacing $R$ by an etale neighborhood, we can assume $\mu = 1$. After these reductions we have $A = R[X]/(X^{|M|} - z^r)$ which is regular in codimension 1 if and only if $r = 1$. □

*Remark* 3.2.44. In the theorem above one can replace the condition 'regular in codimension 1' in the definition of $Reg^1_X$ with 'normal' thanks to Serre's conditions, since all the fibers involved are Gorenstein. Moreover note that a locally noetherian and locally factorial scheme $X$ is a disjoint union of integral connected components. Therefore an isolated point is just a connected component which is $\operatorname{Spec} k$, for a field $k$. We want to avoid this situation because regularity in codimension 1 for a cover over a field is an empty condition.

*Remark* 3.2.45. Theorem 3.2.43 is a rewriting of Theorem 2.1 and Corollary 3.1 of [Par91] extended to locally noetherian and locally factorial schemes without isolated points, where an object of $\mathcal{F}_{\underline{\mathcal{E}}}(X)$ is called a building data.

## 3.3 The locus $h \leq 2$.

In this section we want to give a characterization of the open substack $\{h \leq 2\} \subseteq \mathrm{D}(M)\text{-Cov}$ as done in 3.2.42 for $\{h \leq 1\}$. The general problem we want to solve can be stated as follows.

**Problem 3.3.1.** Find a sequence of smooth extremal rays $\underline{\mathcal{E}}$ for $M$ and a collection $\Theta$ of smooth sequences with rays in $\underline{\mathcal{E}}$ such that (see 3.2.40)

$$\{h \leq 2\} = \bigcup_{\underline{\delta} \in \Theta} \mathcal{Z}^{\underline{\delta}}_M$$

or, equivalently, such that, for any algebraically closed field $k$, the algebras $A \in \mathrm{D}(M)\text{-Cov}(k)$ with $h_{A/k} \leq 2$ are exactly the algebras associated to a multiplication of the form $\psi = \omega 0^{\mathcal{E}}$ where $\omega \colon K \longrightarrow k^*$ is a group homomorphism and $\mathcal{E} \in \langle \underline{\delta} \rangle_{\mathbb{N}}$ for some $\underline{\delta} \in \Theta$.

For example in the case $h \leq 1$ the analogous problem is solved taking $\underline{\mathcal{E}} = (\mathcal{E}^{\phi})_{\phi \in \Phi_M}$ and $\Theta = \{(\mathcal{E}) \text{ for } \mathcal{E} \in \underline{\mathcal{E}}\}$ (see 3.2.41). Once we have found a pair $\underline{\mathcal{E}}, \Theta$ as in 3.3.1 we can formally apply theorems 3.1.45 and 3.1.53. This is done in theorems 3.3.42 and 3.3.45.

Similarly to what happens in the case $h \leq 1$, we can restrict our attention to the case when $M$ is generated by two elements $m, n$ and the first problem to solve is to describe $M$-graded algebras $A$ over a field $k$ generated in these degrees $m, n$ (see 3.3.9). This is done associating with $A$ an invariant $\overline{q}_A \in \mathbb{N}$ (see 3.3.31) and this solution also suggests how to proceed for the next problem, i.e. find the sequence $\underline{\mathcal{E}}$ of problem 3.3.1.

When $M$ is any finite abelian group, it turns out that the extremal rays $\mathcal{E}$ for $M$ such that $h_{\mathcal{E}} = 2$ correspond to particular sequences of the form $\chi = (r, \alpha, N, \overline{q}, \phi)$, where $r, \alpha, N, \overline{q} \in \mathbb{N}$ and $\phi$ is a surjective map from $M$ to a group $M_{r,\alpha,N}$ generated by two elements (see 3.3.6). The sequence of smooth extremal rays "needed" to describe the substack $\{h \leq 2\}$ is composed by the "old" rays $(\mathcal{E}^{\eta})_{\eta \in \Phi_M}$ and by these new rays. Finally the smooth sequences in the family $\Theta$ of problem 3.3.1 will all be given by elements of the dual basis of particular $\mathbb{Z}$-basis of $K$ (see 3.3.34).

In the last subsection we will see (Theorem 3.3.55) that the normal crossing in codimension 1 $\mathrm{D}(M)$-covers of a locally noetherian and locally factorial scheme with no isolated points and with $(\operatorname{char} X, |M|) = 1$ can be described in the spirit of classification 3.2.43 and extending this result.

*Notation* 3.3.2. If $m \in M$ we will denote by $o(m)$ the order of $m$ in the group $M$.

### 3.3.1 Good sequences.

In this subsection we provide some general technical results in order to work with $M$-graded algebras over local rings. So we will consider given a local ring $D$, a sequence $\underline{m} = m_1, \dots, m_r \in M$ and $C \in \mathrm{D}(M)\text{-Cov}(D)$ generated in degrees $m_1, \dots, m_r$. Since $\mathrm{Pic}(D)$=0 for any $u \in M$ we have $C_u \simeq D$. Given $u \in M$, we will call $v_u$ a generator of $C_u$ and we will also use the abbreviation $v_i = v_{m_i}$. Moreover, if $\underline{A} = (A_1, \dots, A_r) \in \mathbb{N}^r$ we will also write

$$v^{\underline{A}} = v_1^{A_1} \cdots v_r^{A_r}$$

**Definition 3.3.3.** A sequence for $u \in M$ is a sequence $\underline{A} \in \mathbb{N}^r$ such that $A_1 m_1 + \cdots + A_r m_r = u$. Such a sequence will be called *good* if the map $C_{m_1}^{A_1} \otimes \cdots \otimes C_{m_r}^{A_r} \longrightarrow C_u$ is surjective, i.e. $v^{\underline{A}}$ generates $C_u$. If $r = 2$ we will talk about pairs instead of sequences.

*Remark* 3.3.4. Any $u \in M$ admits a good sequence since, otherwise, we will have $C_u = (D[v_1, \dots, v_r])_u \subseteq m_D C_u$. If $\underline{A}$ is a good sequence and $\underline{B} \leq \underline{A}$, then also $\underline{B}$ is a good sequence.

**Lemma 3.3.5.** *Let $\underline{A}$, $\underline{B}$ be two sequences for some element of $M$ and assume that $\underline{A}$ is good. Set $\underline{E} = \min(\underline{A}, \underline{B}) = (\min(A_1, B_1), \dots, \min(A_r, B_r))$ and take $\lambda \in D$. Then*

$$v^{\underline{B}} = \lambda v^{\underline{A}} \implies v^{\underline{B}-\underline{E}} = \lambda v^{\underline{A}-\underline{E}}$$

*Proof.* Clearly we have $v^{\underline{E}}(v^{\underline{B}-\underline{E}} - \lambda v^{\underline{A}-\underline{E}}) = 0$. On the other hand, since $\underline{A} - \underline{E}$ is a good sequence, there exists $\mu \in D$ such that $v^{\underline{B}-\underline{E}} = \mu v^{\underline{A}-\underline{E}}$. Since $\underline{A}$ is a good sequence, substituting we get $v^{\underline{A}}(\mu - \lambda) = 0 \implies \mu = \lambda$. $\square$

### 3.3.2 $M$-graded algebras generated in two degrees.

**Definition 3.3.6.** Given $0 \leq \alpha < N$ and $r > 0$ we set

$$M_{r,\alpha,N} = \mathbb{Z}^2 / \langle (r, -\alpha), (0, N) \rangle$$

**Proposition 3.3.7.** *A finite abelian group $M$ with two marked elements $m, n \in M$ generating it is canonically isomorphic to $(M_{r,\alpha,N}, e_1, e_2)$ where $r = \min\{s > 0 \mid sm \in \langle n \rangle\}$, $rm = \alpha n$ and $N = o(n)$. Moreover we have: $|M| = Nr$, $o(m) = rN/(\alpha, N)$ and*

$$m, n \neq 0 \text{ and } m \neq n \iff N > 1 \text{ and } (r > 1 \text{ or } \alpha > 1)$$

*Proof.* We have

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} r & 0 \\ -\alpha & N \end{smallmatrix}\right)} \mathbb{Z}^2 \longrightarrow M_{r,\alpha,N} \longrightarrow 0 \text{ exact} \implies |M_{r,\alpha,N}| = \left| \det \begin{pmatrix} r & 0 \\ -\alpha & N \end{pmatrix} \right| = rN$$

and clearly $e_1, e_2$ generate $M$. Moreover $M_{r,\alpha,N}/\langle e_2\rangle \simeq \mathbb{Z}/r\mathbb{Z}$ and therefore $r$ is the minimum such that $re_1 \in \langle e_2\rangle$. Finally it is easy to check that $N = o(e_2)$. If now $M, r, \alpha, N$ are as in the statement, there exists a unique map $M_{r,\alpha,N} \longrightarrow M$ sending $e_1, e_2$ to $m, n$. This map is an isomorphism since it is clearly surjective and $|M| = o(m)o(n)/|\langle m\rangle \cap \langle n\rangle| = o(n)r = |M_{r,\alpha,N}|$. The last equivalence in the statement is now easy to prove. □

*Notation* 3.3.8. In this subsection we will fix a finite abelian group $M$ generated by two elements $0 \neq m, n \in M$ such that $m \neq n$. Up to isomorphism, this means $M = M_{r,\alpha,N}$ with $m = e_1,\ n = e_2$ and with the conditions $0 \leq \alpha < N,\ r > 0,\ N > 1,\ (r > 1$ or $\alpha > 1)$.

We will write $d_q$ the only integer $0 < d_q \leq N$ such that $qrm + d_q n = 0$, for $q \in \mathbb{Z}$, or, equivalently, $d_q \equiv -q\alpha \mod (N)$.

**Problem 3.3.9.** Let $k$ be a field. We want to describe, up to isomorphism, algebras $A \in \mathrm{D}(M)\text{-Cov}(k)$ such that $A$ is generated in degrees $m, n$ and $H_{A/k} = 0$. Thanks to 3.2.31, this is equivalent to asking for an algebra $A$ such that $H_{A/k} = 0$ and

$$\{l \in M \mid h_{A/k,l} = 1\} \subseteq \{m, n\}$$

The solution of this problem is contained in 3.3.31.

In this subsection we will fix an algebra $A$ as in 3.3.9, we will consider given a graded basis $\{v_l\}_{l\in M}$ of $A$ and we will denote by $\psi$ the associated multiplication. Note that $H_{A/k} = 0$ means $v_m, v_n \notin A^*$.

**Definition 3.3.10.** Define

$$z = \min\{h > 0 \mid \exists i \in \mathbb{N},\ \lambda \in k \text{ such that } v_m^h = \lambda v_n^i \text{ and } hm = in\}$$

$$x = \min\{h > 0 \mid \exists i \in \mathbb{N},\ \mu \in k \text{ such that } v_n^h = \mu v_m^i \text{ and } hn = im\}$$

Denote by $0 \leq y < o(n)$, $0 \leq w < o(m)$ the elements such that $zm = yn$, $xn = wm$, by $\lambda, \mu \in k$ the elements such that $v_m^z = \lambda v_n^y$, $v_n^x = \mu v_m^w$, with the convention that $\lambda = 0$ if $v_n^y = 0$ and $\mu = 0$ if $v_m^w = 0$. Finally set $\overline{q} = z/r$ and define the map of sets

$$\begin{array}{ccc} \{0, 1, \dots, z-1\} & \xrightarrow{f} & \{0, 1, \dots, o(n)\} \\ c & \longmapsto & \min\{d \in \mathbb{N} \mid v_m^c v_n^d = 0\} \end{array}$$

We will also write $\overline{q}_A,\ z_A,\ x_A,\ y_A,\ w_A,\ \lambda_A,\ \mu_A,\ f_A$ if necessary.

We will see that $A$ is uniquely determined by $\overline{q}$ and $\lambda$ up to isomorphism.

**Lemma 3.3.11.** *Given $l \in M$ there exists a unique good pair $(a, b)$ for $l$ with $0 \leq a < z$. Moreover $0 \leq b < f(a)$.*

*Proof. Existence.* We know that there exists a good pair $(a, b)$ for $l$ and we can assume that $a$ is minimum. If $a \geq z$ we can write $v_m^a v_n^b = \lambda v_m^{a-z} v_n^{b+y}$. Therefore $\lambda \neq 0$ and $(a - z, b + y)$ is a good pair for l, contradicting the minimality of $a$. Finally $v_m^a v_n^b \neq 0$ means $b < f(a)$.

*Uniqueness.* Let $(a, b)$, $(a', b')$ be two good pairs for $l$ and assume $0 \leq a < a' < z$. So there exists $\omega \in k^*$ such that

$$v_m^a v_n^b = \omega v_m^{a'} v_n^{b'} \implies v_n^b = \omega v_m^{a'-a} v_n^{b'}$$

If $b \geq b'$ then $a' - a \geq z$ by definition of $z$, while if $b < b'$ then $v_n$ is invertible. □

**Definition 3.3.12.** Given $l \in M$ we will write the associated good pair as $(\mathcal{E}_l, \delta_l)$ with $\mathcal{E}_l < z$. We will consider $\mathcal{E}, \delta$ as maps $\mathbb{Z}^M/\langle e_0\rangle \longrightarrow \mathbb{Z}$ and, if necessary, we will also write $\mathcal{E}^A, \delta^A$.

*Notation* 3.3.13. Up to isomorphism, we can change the given basis to

$$v_l = v_m^{\mathcal{E}_l} v_n^{\delta_l}$$

so that the multiplication $\psi$ is given by

$$v_a v_b = v_m^{\mathcal{E}_a + \mathcal{E}_b} v_n^{\delta_a + \delta_b} = \psi_{a,b} v_m^{\mathcal{E}_{a+b}} v_n^{\delta_{a+b}} = \psi_{a,b} v_{a+b} \tag{3.3.1}$$

**Corollary 3.3.14.** *$f$ is a decreasing function and*

$$f(0) + \cdots + f(z-1) = |M| \tag{3.3.2}$$

*Proof.* If $(a, b)$ is a pair such that $0 \leq a < z$ and $0 \leq b < f(a)$ then $v_m^a v_n^b \neq 0$, i.e. $(a, b)$ is a good pair for $am + bn$. So

$$\sum_{c=0}^{z-1} f(c) = |\{(a, b) \mid 0 \leq a < z,\ 0 \leq b < f(a)\}| = |M|$$

□

*Remark* 3.3.15. The following pairs are good:

$$(z-1)m : (z-1, 0),\ (x-1)n : (0, x-1),\ zm = yn : (0, y),\ xn = wm : (w, 0)$$

i.e. $v_m^{z-1}, v_n^{x-1}, v_n^y, v_m^w \neq 0$. In particular $f(0) \geq x, y + 1$ and $f(c) > 0$ for any $c$. Indeed

$$\begin{aligned} v_m^{z-1} = \omega v_m^a v_n^b &\implies v_m^{z-1-a} = \omega v_n^b \implies a = z-1,\ b = 0 \\ v_m^z = \omega v_m^a v_n^b &\implies \quad v_m^{z-a} = \omega v_n^b \implies \quad a = 0,\ b = y \end{aligned}$$

where $(a, b)$ are good pairs for the given elements and, by symmetry, we get the result.

*Remark* 3.3.16. If $\lambda \neq 0$ or $\mu \neq 0$ then $x = y$, $z = w$ and $\lambda\mu = 1$. Assume for example $\lambda \neq 0$. If $y = 0$ then $v_m^z = \lambda \neq 0$ and so $v_m$ is invertible. So $y > 0$ and, since $v_n^y = \lambda^{-1} v_m^z$, we also have $y \geq x$. Now
$$0 \neq v_m^z = \lambda v_n^y = \lambda\mu v_n^{y-x} v_m^w$$
So $\mu \neq 0$ and $(y-x, w)$ is a good pair. As before $w \geq z$ and therefore
$$\lambda\mu v_n^{y-x} v_m^{w-z} = 1 \implies y = x,\ w = x \text{ and } \lambda\mu = 1$$

**Lemma 3.3.17.** *Let $a, b \in M$. We have:*

- *Assume $\mathcal{E}_{a,b} > 0$. If $\delta_{a,b} \leq 0$ then $\mathcal{E}_{a,b} \geq z$, $\delta_{a,b} \geq -y$. Moreover $\psi_{a,b} \neq 0 \iff \lambda \neq 0, \mathcal{E}_{a,b} = z, \delta_{a,b} = -y (= -x)$ and in this case $\psi_{a,b} = \lambda$.*
- *Assume $\mathcal{E}_{a,b} < 0$. Then $\mathcal{E}_{a,b} \geq -w, \delta_{a,b} \geq x$. Moreover $\psi_{a,b} \neq 0 \iff \mu \neq 0, \mathcal{E}_{a,b} = -w(= -z), \delta_{a,b} = x$ and in this case $\psi_{a,b} = \mu$.*
- *Assume $\mathcal{E}_{a,b} = 0$. Then we have $\delta_{a,b} = 0$ and $\psi_{a,b} = 1$ or $\delta_{a,b} \geq o(n)$ and $\psi_{a,b} = 0$.*

*Proof.* Set $\psi = \psi_{a,b}$. We start with the case $\mathcal{E}_{a,b} > 0$. From 3.3.1 we get
$$v_m^{\mathcal{E}_{a,b}} v_n^{\delta_a + \delta_b} = \psi v_n^{\delta_{a+b}}$$
If $\delta_{a,b} > 0$ then $v_m^{\mathcal{E}_{a,b}} v_n^{\delta_{a,b}} = \psi$ and so $\psi = 0$ since $v_m \notin A^*$. If $\delta_{a,b} \leq 0$ we instead have $v_m^{\mathcal{E}_{a,b}} = \psi v_n^{-\delta_{a,b}}$ and so $\mathcal{E}_{a,b} \geq z$. If $-\delta_{a,b} < y$ then $(0, -\delta_{a,b})$ is good. So we can write
$$v_m^{\mathcal{E}_{a,b} - z} \lambda v_n^{y + \delta_{a,b}} = \psi \implies \psi = 0$$
since $v_n$ is not invertible. If $\delta_{a,b} \leq -y$ we have
$$0 \leq \mathcal{E}_{a,b} - z < z,\ 0 \leq -\delta_{a,b} - y < f(0),\ (\mathcal{E}_{a,b} - z)m = (-\delta_{a,b} - y)n,\ v_m^{\mathcal{E}_{a,b} - z} \lambda = \psi v_n^{-\delta_{a,b} - y}$$
and so both $(\mathcal{E}_{a,b} - z, 0)$ and $(0, -\delta_{a,b} - y)$ are good pair for the same element of $M$. Therefore we must have $\mathcal{E}_{a,b} = z$, $\delta_{a,b} = -y$ and $\psi = \lambda$.

Now assume $\mathcal{E}_{a,b} = 0$. If $\delta_{a,b} < 0$ then $v_n^{-\delta_{a,b}} \psi = 1$ which is impossible. So $\delta_{a,b} \geq 0$. If $\delta_{a,b} = 0$ clearly $\psi = 1$. If $\delta_{a,b} > 0$ then $v_n^{\delta_{a,b}} = \psi$ and so $\psi = 0$ and $\delta_{a,b} \geq o(n)$.

Finally assume $\mathcal{E}_{a,b} < 0$. From 3.3.1 we get
$$v_n^{\delta_a + \delta_b} = \psi v_m^{-\mathcal{E}_{a,b}} v_n^{\delta_{a+b}}$$
We must have $\delta_{a,b} > 0$ since $v_m$ is not invertible. So $v_n^{\delta_{a,b}} = \psi v_m^{-\mathcal{E}_{a,b}}$ and $\delta_{a,b} \geq x$, from which
$$v_n^{\delta_{a,b} - x} \mu v_m^w = \psi v_m^{-\mathcal{E}_{a,b}}$$
Note that, since $0 \leq -\mathcal{E}_{a,b} \leq \mathcal{E}_{a+b} < z$, $(-\mathcal{E}_{a,b}, 0)$ is a good pair. If $w > -\mathcal{E}_{a,b}$ then $\psi = 0$. So assume $w \leq -\mathcal{E}_{a,b}$. Arguing as above we must have $\delta_{a,b} = x$, $\mathcal{E}_{a,b} = -w$ and $\psi = \mu$. $\square$

**Lemma 3.3.18.** *Define*

$$A' = k[s,t]/(s^z, s^c t^{f(c)} \, for \, 0 \leq c < z)$$

*Then* $A' \in \mathrm{D}(M)\text{-}\mathrm{Cov}(k)$ *with graduation* $\deg s = m$, $\deg t = n$ *and it satisfies the requests of 3.3.9, i.e.* $A'$ *is generated in degrees* $m, n$ *and* $H_{A'/k} = 0$. *Moreover we have*

$$\overline{q}_{A'} = \overline{q}_A, \ z_{A'} = z_A, \ y_{A'} = y_A, \ \mathcal{E}^{A'} = \mathcal{E}^A, \ \delta^{A'} = \delta^A, \ \lambda_{A'} = \mu_{A'} = 0, \ f_{A'} = f_A$$

*Proof.* Clearly the elements $s^c t^d$ for $0 \leq c < z$, $0 \leq d < f(c)$ generates $A'$ as a $k$-space. Since they are $\sum_{c=0}^{z-1} f(c) = |M|$ and they all have different degrees, it is enough to prove that any of them are non-zero. So let $(c', d')$ a pair as always. It is enough to show that $B = k[s,t]/(s^{c'+1}, t^{d'+1}) \longrightarrow A'/(s^{c'+1}, t^{d'+1})$ is an isomorphism. But $c' < z$ implies that $s^z = 0$ in $B$. If $c' < c$ then $s^c t^{f(c)} = 0$ in $B$ and finally if $c' \geq c$ then $d' + 1 \leq f(c') \leq f(c)$ and so $s^c t^{f(c)} = 0$ in B.

The algebra $A'$ is clearly generated in degrees $m, n$ and $H_{A'/k} = 0$ since $s^z = t^{f(0)} = 0$ and $z, f(0) > 0$. Moreover $s^z = 0 t^y$ implies that $z' = z_{A'} \leq z$. Assume by contradiction $z' < z$. From $0 \neq s^{z'} = \lambda' t^{y'}$ we know that $t^{y'} \neq 0$ so that $y' < f(0)$. Therefore $(\mathcal{E}_{z'm}, \delta_{z'm}) = (z', 0) = (0, y')$ and so $z' = 0$, which is a contradiction. Then $z' = z$, $y_{A'} = y' = y$. Also $s^z = 0 t^y$ and $t^y \neq 0$ imply $\lambda_{A'} = 0$ and, thanks to 3.3.16, $\mu_{A'} = 0$. Finally by construction we also have $\mathcal{E}^{A'} = \mathcal{E}$, $\delta^{A'} = \delta$ and $f_{A'} = f$. $\square$

**Lemma 3.3.19.** *We have*

$$d_{\overline{q}} = \max_{1 \leq q \leq \overline{q}} d_q$$

*Proof.* Thanks to 3.3.18 we can assume $\lambda = 0$ and, therefore, $\mu = 0$. So $v_n^x = 0$, $v_n^{x-1} \neq 0$ and $v_n^y \neq 0$ imply $y < x = f(0)$. Let $1 \leq q < \overline{q}$ and $l = qr$. We have $(\mathcal{E}_l, \delta_l) = (qr, 0)$. If $N - d_q < x = f(0)$ then we will also have $(\mathcal{E}_l, \delta_l) = (0, N - d_q)$ and so $q = 0$, which is not the case. So $N - d_q \geq x > y = N - d_{\overline{q}} \implies d_q < d_{\overline{q}}$. $\square$

**Lemma 3.3.20.** *Define* $\hat{q}$ *as the only integers* $0 \leq \hat{q} < \overline{q}$ *such that*

$$d_{\hat{q}} = \min_{0 \leq q < \overline{q}} d_q$$

*If* $\lambda = 0$ *we have* $d_{\hat{q}} \leq x = f(0)$ *and* $f(c) = \left\{ \begin{array}{ll} x & \textit{if } 0 \leq c < \hat{q}r \\ d_{\hat{q}} & \textit{if } \hat{q}r \leq c < z \end{array} \right.$

*Proof.* We want first prove that $f(c) = \min(x, d_q \text{ for } 0 \leq qr \leq c)$. Clearly we have the inequality $\leq$ since $v_n^x = v_m^{qr} v_n^{d_q} = 0$. Set $d = f(c)$ and let $(a, b)$ a good pair for $cm + dn$, so that $v_m^c v_n^d = 0 v_m^a v_n^b$. We cannot have $b \geq d$ since otherwise $v_m^c = 0$ implies $c \geq z$. If $a \geq c$ then $v_n^d = 0$ and so $d = f(c) \geq x$. Conversely if $a < c$ then $0 \leq c - a = qr \leq c < z$ and $0 < d - b = d_q \leq d = f(c)$.

We are now ready to prove the expression of $f$. Note that the pairs $(qr, d_q - 1)$, with $0 \leq q < \overline{q}$, are all the possible pairs for $-n$. So there exists a unique $0 \leq \tilde{q} < \overline{q}$ such that $(\tilde{q}r, d_{\tilde{q}} - 1)$ is good. In particular if $0 \leq q \neq \tilde{q} < \overline{q}$ we have an expression

$$v_m^{qr} v_n^{d_q - 1} = 0 v_m^{\tilde{q}} v_n^{d_{\tilde{q}} - 1} \implies \left\{ \begin{array}{lll} q < \tilde{q} & \implies & v_n^{d_q - 1} = 0 \quad \implies \quad d_q \geq x \\ q > \tilde{q} & \implies & d_q > d_{\tilde{q}} \end{array} \right.$$

Since $v_n^{d_{\tilde{q}}-1} \neq 0$ we must have $d_{\hat{q}} \leq x$. This shows that $\tilde{q} = \hat{q}$ and the expression of $f$. Finally If $\overline{q} > 1$ then $\hat{q} > 0$ and so $d_{\hat{q}} \leq x = f(0)$ since $f$ is a decreasing function. If $\overline{q} = 1$ then $\hat{q} = 0$ and so $N = d_{\hat{q}} = f(0) \leq x \leq N$. $\square$

**Definition 3.3.21.** We will continue to use notation from 3.3.20 for $\hat{q}$ and we will also write $\hat{q}_A$ if necessary.

### 3.3.3 The invariant $\overline{q}$.

**Lemma 3.3.22.** *Let $\beta, N \in \mathbb{N}$, with $N > 1$, and define $d_q^\beta = d_q$, for $q \in \mathbb{Z}$, the only integer $0 < d_q \leq N$ such that $d_q \equiv q\beta \bmod N$. Set*

$$\Omega_{\beta,N} = \{0 < q \leq o(\beta, \mathbb{Z}/N\mathbb{Z}) = N/(N,\beta) \mid d_{q'} < d_q \text{ for any } 0 < q' < q\},$$

*set $q_n$ for the $n$-th element of it and denote by $0 \leq \hat{q} < q_n$ the only number such that*

$$d_{\hat{q}} = \min_{0 \leq q < q_n} d_q$$

*Then we have relations $\hat{q}N + q_n d_{\hat{q}} - \hat{q} d_{q_n} = N$ and, if $n > 1$, $q_n = q_{n-1} + \hat{q}$, $d_{q_n} = d_{q_{n-1}} + d_{\hat{q}}$ and $d_{q_{n-1}} + d_q > N$ for $q < \hat{q}$.*

*Proof.* First of all note that all is defined also in the extremal case $\beta = 0$. In this case $\Omega_{\beta,N} = \{1\}$. Assume first $n > 1$. Set $\tilde{q} = q_n - q_{n-1}$ so that $d_{q_n} = d_{q_{n-1}} + d_{\tilde{q}}$ since $d_{q_n} > d_{q_{n-1}}$. Assume by contradiction that $\tilde{q} \neq \hat{q}$. Since $\tilde{q} < q_n$ we have $d_{\hat{q}} < d_{\tilde{q}}$. Let also $q' = q_n - \hat{q}$ and, as above, we can write $d_{q_n} = d_{q'} + d_{\hat{q}}$. Now

$$d_{q_n} - d_{q'} = d_{\hat{q}} < d_{\tilde{q}} = d_{q_n} - d_{q_{n-1}} \implies d_{q_{n-1}} < d_{q'}$$

Since $q_{n-1} \in \Omega_{\beta,N}$ we must have $q' > q_{n-1}$, which is a contradiction because otherwise, being $q' < q_n$, we must have $q' = q_n$. So $\tilde{q} = \hat{q}$. For the last relation note that, since $q_n$ is the first $q > q_{n-1}$ such that $d_q > d_{q_{n-1}}$, then $\hat{q}$ is the first such that $d_{q_{n-1}} + d_{\hat{q}} \leq N$.

Now consider the first relation. We need to do induction on all the $\beta$. So we will write $d_q^\beta$ and $q_n^\beta$ in order to remember that those numbers depend on to $\beta$. The induction statement on $1 \leq q < N$ is: for any $0 \leq \beta < N$ and for any $n$ such that $q_n^\beta \leq q$ the required formula holds. The base step is $q = 1$. In this case we have $n = 1$, $q_1 = 1$, $\hat{q} = 0$, $d_0 = N$ and the formula can be proven directly. For the induction step we can assume $q > 1$ and $n > 1$. We will write $\hat{q}_n^\beta$ for the $\hat{q}$ associated to $n$ and $\beta$. First of all note that, by the relations proved above, we can write

$$\hat{q}_n^\beta N + q_n^\beta d_{\hat{q}_n^\beta}^\beta - \hat{q}_n^\beta d_{q_n^\beta}^\beta = \hat{q}_n^\beta N + q_{n-1}^\beta d_{\hat{q}_n^\beta}^\beta - \hat{q}_n^\beta d_{q_{n-1}^\beta}^\beta$$

and so we have to prove that the second member equals $N$. If $\hat{q}_n^\beta \leq q_{n-1}^\beta$ then $\hat{q}_{n-1}^\beta = \hat{q}_n^\beta$ and the formula is true by induction on $q - 1 \geq q_{n-1}^\beta$. So assume $\hat{q}_n^\beta > q_{n-1}^\beta$ and set $\alpha = N - \beta$. Clearly we will have

$$o = o(\alpha, \mathbb{Z}/N\mathbb{Z}) = o(\beta, \mathbb{Z}/N\mathbb{Z}) \text{ and } d_q^\beta + d_q^\alpha = N \text{ for any } 0 < q < o$$

Moreover

$$d^\beta_{\hat{q}^\beta_n} < d^\beta_q \text{ for any } 0 < q < q^\beta_n \implies d^\alpha_{\hat{q}^\beta_n} > d^\alpha_q \text{ for any } 0 < q < \hat{q}^\beta_n \implies \exists l \text{ s.t. } q^\alpha_l = \hat{q}^\beta_n$$

and

$$d^\beta_{q^\beta_{n-1}} \geq d^\beta_q \text{ for any } 0 < q < q^\beta_n \implies d^\alpha_{q^\beta_{n-1}} \leq d^\alpha_q \text{ for any } 0 \leq q < q^\alpha_l = \hat{q}^\beta_n \implies \hat{q}^\alpha_l = q^\beta_{n-1}$$

Using induction on $q^\alpha_l = \hat{q}^\beta_n < q^\beta_n \leq q$ we can finally write

$$\begin{aligned} N =& \hat{q}^\alpha_l N + q^\alpha_l d^\alpha_{\hat{q}^\alpha_l} - \hat{q}^\alpha_l d^\alpha_{q^\alpha_l} = q^\beta_{n-1} N + \hat{q}^\beta_n d^\alpha_{q^\beta_{n-1}} - q^\beta_{n-1} d^\alpha_{\hat{q}^\beta_n} \\ =& q^\beta_{n-1} N + \hat{q}^\beta_n (N - d^\beta_{q^\beta_{n-1}}) - q^\beta_{n-1}(N - d^\beta_{\hat{q}^\beta_n}) = \hat{q}^\beta_n N + q^\beta_{n-1} d^\beta_{\hat{q}^\beta_n} - \hat{q}^\beta_n d^\beta_{q^\beta_{n-1}} \end{aligned}$$

□

We continue to keep notation from 3.3.8. With $d_q$ we will always mean $d^{N-\alpha}_q$ as in 3.3.22. Lemma 3.3.19 can be restated as:

**Proposition 3.3.23.** *Let $A$ be an algebra as in 3.3.9. Then $\overline{q}_A \in \Omega_{N-\alpha,N}$.*

So given an algebra $A$ as in 3.3.9 we can associate to it the number $\overline{q}_A \in \Omega_{N-\alpha,N}$. Conversely we will see that any $\overline{q} \in \Omega_{N-\alpha,N}$ admits an algebra $A$ as in 3.3.9 such that $\overline{q} = \overline{q}_A$. It turns out that all the objects $z_A$, $y_A$, $f_A$, $\mathcal{E}^A$, $\delta^A$, $\hat{q}_A$ and, if $\lambda_A = 0$, $x_A$, $w_A$ associated to $A$ only depend on $\overline{q}_A$. Therefore in this subsection, given $\overline{q} \in \Omega_{N-\alpha,N}$, we will see how to define such objects independently from an algebra $A$.

In this subsection we will consider given an element $\overline{q} \in \Omega_{N-\alpha,N}$.

**Definition 3.3.24.** Set $\hat{q}$ for the only integer $0 \leq \hat{q} < \overline{q}$ such that $d_{\hat{q}} = \min_{0 \leq q < \overline{q}} d_q$, $q' = \overline{q} - \hat{q}$, $z = \overline{q} r$, $y = N - d_{\overline{q}}$,

$$x = \begin{cases} N - d_{q'} & \text{if } \overline{q} > 1 \\ N & \text{if } \overline{q} = 1 \end{cases}, \quad w = \begin{cases} q' r & \text{if } \overline{q} > 1 \\ 0 & \text{if } \overline{q} = 1 \end{cases}, \quad f(c) = \begin{cases} x & \text{if } 0 \leq c < \hat{q} r \\ d_{\hat{q}} & \text{if } \hat{q} r \leq c < z \end{cases}$$

We will also write $\hat{q}_{\overline{q}}$, $q'_{\overline{q}}$, $z_{\overline{q}}$, $x_{\overline{q}}$, $f_{\overline{q}}$, $y_{\overline{q}}$, $w_{\overline{q}}$ if necessary.

*Remark* 3.3.25. Using notation from 3.3.22 we have $\overline{q} = q_n$ for some $n$ and, if $n > 1$, i.e. $\overline{q} > 1$, $q_{n-1} = q'$. Note that $zm = yn$, $wm = xn$, $y < x$, $w < z$. Moreover, from 3.3.22 and from a direct computation if $\overline{q} = 1$, we obtain $zx - yw = |M|$. Finally if $\overline{q} > 1$ one has relations $\hat{q} r = z - w$ and $d_{\hat{q}} = x - y$.

**Lemma 3.3.26.** *We have that:*

1) *$f$ is a decreasing function and $\sum_{c=0}^{z-1} f(c) = |M|$;*

2) *any element $t \in M$ can be uniquely written as*

$$t = Am + Bn \text{ with } 0 \leq A < z, 0 \leq B < f(A)$$

*Proof.* 1) If $\overline{q} = 1$ it is enough to note that $\hat{q} = 0$, $d_0 = N$ and $Nr = |M|$. So assume $\overline{q} > 1$. We have $x = N - d_{q'} \geq d_{\hat{q}}$ since $d_{\overline{q}} = d_{q'} + d_{\hat{q}}$ and

$$\sum_{c=0}^{z-1} f(c) = \hat{q}rx + (\overline{q}r - \hat{q}r)d_{\hat{q}} = (z-w)x + w(x-y) = zx - wy = |M|$$

2) First of all note that the expressions of the form $Am + Bn$ with $0 \leq A < z$, $0 \leq B < f(A)$ are $\sum_{c=0}^{z-1} f(c) = |M|$. So it is enough to prove that they are all distinct. Assume we have expressions $Am + Bn = A'm + B'n$ with $0 \leq A' \leq A < z, 0 \leq B < f(A), 0 \leq B' < f(A')$.

$A' = B' = 0$, i.e. $Am + Bn = 0$. If $A = 0$ then $B = 0$ since $f(0) = x \leq N$. If $A > 0$, we can write $A = qr$ for some $0 < q < \overline{q}$. In particular $\overline{q} > 1$ and $B = d_q < f(A)$. If $q < \hat{q}$ then $f(A) = x = N - d_{q'} > d_q$ contradicting 3.3.22, while if $q \geq \hat{q}$ then $f(A) = d_{\hat{q}} \leq d_q$.

$A' = B = 0$, i.e. $Am = B'n$. If $A = 0$ then $B' = 0$ as above. If $A > 0$ we can write $A = qr$ for some $0 < q < \overline{q}$. Again $\overline{q} > 1$. In particular $B' = N - d_q < f(0) = x = N - d_{q'}$ and so $d_{q'} < d_q$, while $d_{q'} = \max_{0<q<\overline{q}} d_q$.

*General case.* We can write $(A - A')m + Bn = B'n$ and we can reduce the problem to the previous cases since if $B \geq B'$ then $B - B' \leq B < f(A) \leq f(A - A')$, while if $B < B'$ then $B' - B \leq B' < f(A') \leq f(0)$. □

**Definition 3.3.27.** Given $l \in M$ we set $(\mathcal{E}_l, \delta_l)$ the unique pair for $l$ such that $0 \leq \mathcal{E}_t < z$, $0 \leq \delta_t < f(\mathcal{E}_t)$ and we will consider $\mathcal{E}$, $\delta$ as maps $\mathbb{Z}^M/\langle e_0 \rangle \longrightarrow \mathbb{Z}$. We will also write $\mathcal{E}^{\overline{q}}$, $\delta^{\overline{q}}$ if necessary.

**Proposition 3.3.28.** *Let $A$ be an algebra as in 3.3.9. Then*

$$z_A = z_{\overline{q}_A},\ y_A = y_{\overline{q}_A},\ \hat{q}_A = \hat{q}_{\overline{q}_A},\ \mathcal{E}^A = \mathcal{E}^{\overline{q}_A},\ \delta^A = \delta^{\overline{q}_A},\ f_A = f_{\overline{q}_A}$$

*and, if $\lambda_A = 0$, then $x_A = x_{\overline{q}_A}$, $w_A = w_{\overline{q}_A}$.*

*Proof.* Set $\overline{q} = \overline{q}_A$. Then $z_A = \overline{q}r = z_{\overline{q}}$ and $z_A m = y_A n = y_{\overline{q}} n$ implies $y_A = y_{\overline{q}}$. Also $\hat{q}_A = \hat{q}_{\overline{q}}$ by definition. Taking into account 3.3.18 we can now assume $\lambda_A = 0$. We claim that all the remaining equalities follow from $x_A = x_{\overline{q}}$. Indeed clearly $w_A = w_{\overline{q}}$. Also by definition of $f_{\overline{q}}$ and thanks to 3.3.20 we will have $f_A = f_{\overline{q}}$ and therefore $\mathcal{E}^A = \mathcal{E}^{\overline{q}}$, $\delta^A = \delta^{\overline{q}}$, that conclude the proof.

We now show that $x_A = x_{\overline{q}}$. If $\overline{q} = 1$ then $\hat{q} = 0$ and so, from 3.3.20, we have $d_{\hat{q}} = N = x_A = x_1$. If $\overline{q} > 1$, by definition of $f_{\overline{q}}$ and thanks to 3.3.26 and 3.3.20, we can write

$$|M| = \sum_{c=0}^{z_{\overline{q}}-1} f_{\overline{q}}(c) = r\hat{q}_{\overline{q}}x_{\overline{q}} + (z_{\overline{q}} - \hat{q}_{\overline{q}}r)d_{\hat{q}_{\overline{q}}} = \sum_{c=0}^{z_A-1} f_A(c) = r\hat{q}_A x_A + (z_A - \hat{q}_A r)d_{\hat{q}_A}$$

and so $x_A = x_{\overline{q}}$. □

**Definition 3.3.29.** Define the $M$-graded $\mathbb{Z}[a,b]$-algebra

$$A^{\overline{q}} = \mathbb{Z}[a,b][s,t]/(s^z - at^y, t^x - bs^w, s^{\hat{q}r}t^{d_{\hat{q}}} - a^\gamma b) \text{ where } \gamma = \begin{cases} 0 & \text{if } \overline{q} = 1 \\ 1 & \text{if } \overline{q} > 1 \end{cases}$$

with $M$-graduation $\deg s = m$, $\deg t = n$. If are given elements $a_0, b_0$ of a ring $C$ we will also write $A^{\overline{q}}_{a_0,b_0} = A^{\overline{q}} \otimes_{\mathbb{Z}[a,b]} C$, where $\mathbb{Z}[a,b] \longrightarrow C$ sends $a, b$ to $a_0, b_0$.

**Proposition 3.3.30.** *$A^{\overline{q}} \in \mathrm{D}(M)\text{-Cov}(\mathbb{Z}[a,b])$, it is generated in degrees $m, n$ and $\{v_l = s^{\mathcal{E}_l}t^{\delta_l}\}_{l\in M}$ is an $M$-graded basis for it.*

*Proof.* We have to prove that, for any $l \in M$, $(A^{\overline{q}})_l = \mathbb{Z}[a,b]v_l$ and we can check this over a field $k$, i.e. considering $A = A^{\overline{q}}_{a,b}$ with $a, b \in k$. We first consider the case $a, b \in k^*$, so that $s, t \in A^*$. Let $\pi \colon \mathbb{Z}^2 \longrightarrow M$ the map such that $\pi(e_1) = m$, $\pi(e_2) = n$. The set $T = \{(a,b) \in \operatorname{Ker}\pi \mid s^a t^b \in k^*\}$ is a subgroup of $\operatorname{Ker}\pi$ such that $(z,-y), (-w,x) \in T$. Since $\det \begin{pmatrix} z & -w \\ -y & x \end{pmatrix} = zx - wy = |M|$ we can conclude that $T = \operatorname{Ker}\pi$. Therefore $v_l$ generate $(A^{\overline{q}})_l$ since for any $c, d \in \mathbb{N}$ we have $s^c t^d / v_{cm+dn} \in k^*$ and $0 \neq v_l \in A^*$.

Now assume $a = 0$. If $\overline{q} = 1$ then $\hat{q} = w = 0$, $d_{\hat{q}} = x = N$ and so $A = k[s,t]/(s^z, t^N - b)$ satisfies the requests. If $\overline{q} > 1$ it is easy to see that $v_l$ generates $A_l$. On the other hand $\dim_k A = |\{(A,B) \mid 0 \leq A < z, 0 \leq B < x, A \leq \hat{q}r \text{ or } B \leq d_{\hat{q}}\}| = zx - (z - \hat{q}r)(x - d_{\hat{q}}) = zx - yw = |M|$. The case $b = 0$ is similar. $\square$

**Theorem 3.3.31.** *Let $k$ be a field. If $\overline{q} \in \Omega_{N-\alpha,N}$ and $\lambda \in k$, with $\lambda = 0$ if $\overline{q} = N/(\alpha, N)$, then*

$$A_{\overline{q},\lambda} = k[s,t]/(s^{z_{\overline{q}}} - \lambda t^{y_{\overline{q}}}, t^{x_{\overline{q}}}, s^{\hat{q}_{\overline{q}}r}t^{d_{\hat{q}_{\overline{q}}}})$$

*is an algebra as in 3.3.9 with $\overline{q}_{A_{\overline{q},\lambda}} = \overline{q}$ and $\lambda_{A_{\overline{q},\lambda}} = \lambda$. Conversely, if $A$ is an algebra as in 3.3.9 then $\overline{q}_A \in \Omega_{N-\alpha,N}$, $\lambda_A \in k$, $\lambda_A = 0$ if $\overline{q}_A = N/(\alpha, N)$ and $A \simeq A_{\overline{q}_A,\lambda_A}$.*

*Proof.* Consider $A = A_{\overline{q},\lambda}$, which is just $A^{\overline{q}}_{\lambda,0}$. Clearly $t \notin A^*$. On the other hand $s \notin A^*$ since $y = 0 \iff z = o(m) \iff \overline{q} = N/(\alpha, N)$. Therefore $H_{A/k} = 0$ and $A$ is an algebra as in 3.3.9. Moreover clearly $\overline{q}_A \leq \overline{q}$. If by contradiction this inequality is strict, we will have a relation $s^{qr} = \omega t^{y'}$ with $0 \leq q < \overline{q}$. Since $s^{qr} = v_{qrm} \neq 0$ we will have that $t^{y'} \neq 0$ and $y' < x$, a contradiction thanks to 3.3.26. In particular $\lambda = \lambda_A$.

Now let $A$ be as in 3.3.9 and set $\overline{q} = \overline{q}_A$, $\lambda = \lambda_A$. We already know that $\overline{q} \in \Omega_{N-\alpha,N}$ (see 3.3.23). We claim that the map $A_{\overline{q},\lambda} \longrightarrow A$ sending $s, t$ to $v_m, v_n$ is well defined and so an isomorphism. Indeed we have $v_m^z = \lambda v_n^y$ by definition and, thanks to 3.3.28, we have $v_m^{\hat{q}r}v_n^{d_{\hat{q}}} = 0$ since $d_{\hat{q}} = f_A(\hat{q}r)$ and $v_n^x = 0$ since $f_A(0) = x$. Finally if $\overline{q} = N/(\alpha, N)$ then $y = y_A = 0$ and $z = o(m)$, so that $\lambda_A = v_m^{o(m)} = 0$. $\square$

**Corollary 3.3.32.** *If $k$ is an algebraically closed field then, up to graded isomorphism, the algebras as in 3.3.9 are exactly $A_{\overline{q},1}$ if $\overline{q} \in \Omega_{N-\alpha,N} - \{N/(\alpha, N)\}$ and $A_{\overline{q},0}$ if $\overline{q} \in \Omega_{N-\alpha,N}$.*

*Proof.* Clearly the algebras above cannot be isomorphic. Conversely if $\lambda \in k^*$ (and $\overline{q} < N/(\alpha, N)$) the transformation $t \longrightarrow \sqrt[y]{\lambda}t$ with $y = y_{\overline{q}}$ yields an isomorphism $A_{\overline{q},\lambda} \simeq A_{\overline{q},1}$. $\square$

### 3.3.4 Smooth extremal rays for $h \leq 2$.

In this subsection we continue to keep notation from 3.3.8, i.e. $M = M_{r,\alpha,N}$ and we will considered given an element $\overline{q} \in \Omega_{N-\alpha,N}$.

*Remark* 3.3.33. We have $z = 1 \iff \overline{q} = r = 1$ and $x = 1 \iff \overline{q} = N$. Indeed the first relation is clear, while for the second one note that, by definition of $x$ and since $N > 1$, we have $x = 1 \iff d_{q'} = N - 1 \iff \overline{q} = N/(\alpha, N), (\alpha, N) = 1$.

**Lemma 3.3.34.** *The vectors of* $K_+$

$$\begin{array}{lc} v_{cm,dn} & 0 < c < z, 0 < d < f(c) \\ v_{m,im} & 0 < i < z-1 \\ v_{n,jn} & 0 < j < x-1 \\ v_{m,(z-1)m} & \text{if } z > 1 \\ v_{n,(x-1)n} & \text{if } x > 1 \end{array} \tag{3.3.3}$$

*form a basis of* $K$. *Assume* $\overline{q}r \neq 1$ *and* $\overline{q} \neq N$, *i.e.* $z, x > 1$, *and denote by* $\Lambda, \Delta$ *the last two terms of the dual basis of 3.3.3. Then* $\Lambda, \Delta \in K_+^\vee$ *and they form a smooth sequence. Moreover* $\Lambda = 1/|M|(x\mathcal{E} + w\delta)$, $\Delta = 1/|M|(y\mathcal{E} + z\delta)$ *and*

$$\Lambda_{m,-m} = \Delta_{n,-n} = 1, \ \Lambda_{n,-n} = \left\{ \begin{array}{ll} 0 & \text{if } \overline{q} = 1 \\ 1 & \text{otherwise} \end{array} \right., \ \Delta_{m,-m} = \left\{ \begin{array}{ll} 0 & \text{if } \overline{q} = N/(\alpha, N) \\ 1 & \text{otherwise} \end{array} \right.$$

*Proof.* Note that we cannot have $z = x = 1$ since otherwise $|M| = f(0) = x = 1$, i.e. $M = 0$. The vectors of (3.3.3) are at most $\operatorname{rk} K$ since

$$\sum_{c=1}^{z-1} (f(c)-1) + z - 2 + x - 2 + 2 = \sum_{c=0}^{z-1} (f(c)-1) + z - 1 = |M| - z + z - 1 = |M| - 1 = \operatorname{rk} K$$

If $z = 1$ then (3.3.3) is $v_{n,n}, \dots, v_{n,(x-1)n}$. So $x = |M| = N$, i.e. $n$ generates $M$, and (3.3.3) is a base of $K$. In the same way if $x = 1$, then $m$ generates $M$ and (3.3.3) is a base of $K$.

So we can assume that $z, x > 1$. The functions $\mathcal{E}$ and $\delta$ define a map $\mathbb{Z}^M/\langle e_0 \rangle \xrightarrow{(\mathcal{E},\delta)} \mathbb{Z}^2$. Denote by $K'$ the subgroup of $K$ generated by the vectors in (3.3.3), except for the last two lines. We claim that $(\mathcal{E}, \delta)_{|K'} = 0$. This follows by a direct computation just observing that if we have an expression $Am + Bn$ as in 3.3.26, 2)) then $(\mathcal{E}, \delta)(e_{Am+Bn}) = (A, B)$. Consider the diagram

$$\begin{tikzcd} \overset{\sigma(e_1)=v_{m,(z-1)m}\sigma(e_2)=v_{n,(x-1)n}}{\mathbb{Z}^2} \arrow[r, "\sigma"] \arrow[rrr, bend right, "U"'] & K/K' \arrow[r, hook] & \mathbb{Z}^M/\langle e_0, K' \rangle \arrow[r, "(\mathcal{E},\delta)"] & \underset{\tau(e_1)=e_m, \tau(e_2)=e_n}{\mathbb{Z}^2} \arrow[r, "\tau"'] \arrow[rr, bend left, "\pi"] & \mathbb{Z}^M/\langle e_0, K' \rangle \arrow[r, "p"', "p(e_l)=l"] & M \end{tikzcd}$$

We have $(\mathcal{E}, \delta)(v_{m,(z-1)m}) = (z, -y)$ since $y < x = f(0)$ and $(\mathcal{E}, \delta)(v_{n,(x-1)n}) = (-w, x)$ since $w < z$. So $|\det U| = zx - yw = |M|$ and, since $\pi \circ U = 0$, $U$ is an isomorphism onto $\operatorname{Ker}\pi$. Moreover $\tau^{-1} = (\mathcal{E}, \delta)$ since $e_l \equiv \mathcal{E}_l e_m + \delta_l e_n \mod K'$. It follows that $\sigma$ is an isomorphism and so (3.3.3) is a basis of $K$.

Consider now the second part of the statement. Clearly $\Lambda, \Delta \in \langle \mathcal{E}, \delta \rangle_{\mathbb{Q}}$. Therefore we have

$$\Lambda = a\mathcal{E} + b\delta, \begin{cases} \Lambda(v_{m,(z-1)m}) = 1 = az - yb \\ \Lambda(v_{n,(x-1)n}) = 0 = xb - aw \end{cases} \implies \begin{cases} a = x/|M| \\ b = w/|M| \end{cases}$$

and the analogous relation for $\Delta$ follows in the same way. Now note that, thanks to 3.3.31 and 3.3.28, we have that $\mathcal{E} = \mathcal{E}^A$, $\delta = \delta^A$ for an algebra $A$ as in 3.3.9 with $\overline{q}_A = \overline{q}$, $\lambda_A = 0$ and sharing the same invariants of $\overline{q}$. So we can apply 3.3.17. We want to prove that $\Lambda, \Delta \in K_+^\vee$ so that they form a smooth sequence by construction. Assume first that $\mathcal{E}_{a,b} > 0$. Clearly $\Lambda_{a,b}, \Delta_{a,b} \geq 0$ if $\delta_{a,b} \geq 0$. On the other hand if $\delta_{a,b} < 0$ we know that $\mathcal{E}_{a,b} \geq z$ and $\delta_{a,b} \geq -y$ and so

$$|M|\Lambda_{a,b} = x\mathcal{E}_{a,b} + w\delta_{a,b} \geq xz - yw = |M| \text{ and } |M|\Delta_{a,b} = y\mathcal{E}_{a,b} + z\delta_{a,b} \geq yz - zy = 0$$

The other cases follows in the same way. It remains to prove the last relations. Since $-n = \hat{q}rm + (d_{\hat{q}} - 1)n$, we have $\mathcal{E}_{n,-n} = \hat{q}r$ and $\delta_{n,-n} = d_{\hat{q}}$. Using the relation $zx - wy = |M|$ the values of $\Lambda_{n,-n}$, $\Delta_{n,-n}$ can be checked by a direct computation. Similarly, considering the relations $-m = (\hat{q}r - 1)m + d_{\hat{q}}n$ if $1 < \overline{q}$, $-m = (r-1)m + (N - \alpha)n$ if $\overline{q} = 1$ and $\alpha \neq 0$, $-m = (r-1)m$ if $\alpha = 0$, we can compute the values of $\Lambda_{m,-m}$ and $\Delta_{m,-m}$. $\square$

**Proposition 3.3.35.** *The multiplication of $A^{\overline{q}}$ (see 3.3.29) with respect to the basis $v_l = v_m^{\mathcal{E}_l} v_n^{\delta_l}$ is: $a^{\mathcal{E}^\phi}$ if $\overline{q} = N$, where $\phi\colon M \xrightarrow{\simeq} \mathbb{Z}/|M|\mathbb{Z}$, $\phi(m) = 1$; $b^{\mathcal{E}^\eta}$ if $\overline{q}r = 1$, where $\eta\colon M \xrightarrow{\simeq} \mathbb{Z}/|M|\mathbb{Z}$, $\phi(n) = 1$; $a^\Lambda b^\Delta$ if $\overline{q}r \neq 1$, $\overline{q} \neq N$, where $\Lambda, \Delta$ are the rays defined in 3.3.34.*

*Proof.* In the proof of 3.3.30 we have seen that if $x = 1$ ($\overline{q} = N$), then $M = \langle m \rangle$ and $A^{\overline{q}} = \mathbb{Z}[a,b][s]/(s^{|M|} - a)$, while if $z = 1$ ($\overline{q}r = 1$) then $M = \langle n \rangle$ and $A^{\overline{q}} = \mathbb{Z}[a,b][t]/(t^{|M|} - b)$. So we can assume $x, z > 1$. Let $B$ the $\mathrm{D}(M)$-cover over $\mathbb{Z}[a,b]$ given by multiplication $\psi = a^\Lambda b^\Delta$ and denote by $\{\omega_l\}_{l \in M}$ a graded basis (inducing $\psi$). By definition of $\Lambda, \Delta$ we have $\omega_l = \omega_m^{\mathcal{E}_l} \omega_n^{\delta_l}$ for any $l \in M$ and $\psi_{m,(z-1)m} = a$, $\psi_{n,(x-1)n} = b$. Therefore

$$\omega_m^z = \omega_m \omega_{(z-1)m} = a\omega_{zm} = a\omega_{yn} = a\omega_n^y, \ \omega_n^x = \omega_n \omega_{(x-1)n} = b\omega_{xn} = b\omega_{wm} = b\omega_m^w$$

and, checking both cases $\overline{q} = 1$ and $\overline{q} > 1$, $\omega_m^{\hat{q}r} \omega_n^{d_{\hat{q}}} = \omega_{-n}\omega_n = a^{\Lambda_{n,\cdot n}} b^{\Delta_{n,\cdot n}} = a^\gamma b$. In particular we have an isomorphism $A^{\overline{q}} \longrightarrow B$ sending $v_m, v_n$ to $\omega_m, \omega_n$. $\square$

*Notation* 3.3.36. From now on M will be any finite abelian group. If $\phi\colon M \longrightarrow M_{r,\alpha,N}$ is a surjective map, $r, \alpha, N$ satisfy the conditions of 3.3.8, $\overline{q} \in \Omega_{N-\alpha,N}$ with $\overline{q}r \neq 1$, $\overline{q} \neq N$ then we set $\Lambda^{r,\alpha,N,\overline{q},\phi} = \Lambda \circ \phi_*$, $\Delta^{r,\alpha,N,\overline{q},\phi} = \Delta \circ \phi_*$, where $\Lambda, \Delta$ are the rays defined in 3.3.34 with respect to $r, \alpha, N, \overline{q}$. If $\phi = \mathrm{id}$ we will omit it.

**Definition 3.3.37.** Set

$$\Sigma_M = \left\{ (r,\alpha,N,\overline{q},\phi) \;\middle|\; \begin{array}{c} 0 \leq \alpha < N,\ r > 0,\ N > 1,\ (r > 1 \text{ or } \alpha > 1) \\ \overline{q} \in \Omega_{N-\alpha,N},\ \overline{q}r \neq 1,\ \overline{q}\alpha \not\equiv 1 \bmod N \\ \overline{q} \neq N/(\alpha,N),\ \phi\colon M \longrightarrow M_{r,\alpha,N} \text{ surjective} \end{array} \right\}$$

and $\Delta^*\colon \Sigma_M \longrightarrow \{\text{smooth extremal rays of } M\}$.

*Remark* 3.3.38. Since $e_2, e_1$ generate $M_{r,\alpha,N}$, there exist unique $r^\vee, \alpha^\vee, N^\vee$ with an isomorphism $(-)^\vee\colon M_{r,\alpha,N} \longrightarrow M_{r^\vee,\alpha^\vee,N^\vee}$ sending $e_2, e_1$ to $e_1, e_2$. One can check that $r^\vee = (\alpha, N)$, $N^\vee = rN/(\alpha,N)$ and $\alpha^\vee = \tilde{q}r$, where $\tilde{q}$ is the only integer $0 \leq \tilde{q} < N/(\alpha,N)$ such that $\tilde{q}\alpha \equiv (\alpha,N) \bmod N$.

If $A$ is an algebra as in 3.3.9 for $M_{r,\alpha,N}$, then, through $(-)^\vee$, $A$ can be thought of as a $M_{r^\vee,\alpha^\vee,N^\vee}$-cover, that we will denote by $A^\vee$, and $A^\vee$ is an algebra as in 3.3.9 with respect to $M_{r^\vee,\alpha^\vee,N^\vee}$, with $\overline{q}_{A^\vee} = x_A/(\alpha,N)$, $\lambda_{A^\vee} = \mu_A$. We can define a bijection $(-)^\vee\colon \Omega_{N-\alpha,N} - \{N/(N,\alpha)\} \longrightarrow \Omega_{N^\vee-\alpha^\vee,N^\vee} - \{N^\vee/(\alpha^\vee,N^\vee)\}$ in the following way. Given $\overline{q}$ take an algebra $A$ as in 3.3.9 for $M_{r,\alpha,N}$ with $\overline{q}_A = \overline{q}$ and $\lambda_A \neq 0$, which exists thanks to 3.3.31, and set $\overline{q}^\vee = \overline{q}_{A^\vee}$. Taking into account 3.3.16 and 3.3.28, $\overline{q}^\vee = y_{\overline{q}}/(\alpha,N)$ since $x_A = y_A = y_{\overline{q}}$ and $(-)^\vee$ is well defined and bijective since $\lambda_{A^\vee} = \mu_A = \lambda_A^{-1}$. Note that the condition $\overline{q}\alpha \equiv 1 \bmod N$ is equivalent to $r^\vee = 1$ and $\overline{q}^\vee = 1$

Finally if $\phi\colon M \longrightarrow M_{r,\alpha,N}$ is a surjective morphism then we set $\phi^\vee = (-)^\vee \circ \phi\colon M \longrightarrow M_{r^\vee,\alpha^\vee,N^\vee}$. Note that in any case we have the relation $(-)^{\vee\vee} = \mathrm{id}$. In particular, since $1^\vee = \alpha/r^\vee$, $\overline{q} = \alpha^\vee/r$ is the dual of $1 \in \Omega_{N^\vee-\alpha^\vee,N^\vee}$.

**Proposition 3.3.39.** *Let $r, \alpha, N$ be as in 3.3.8, $\overline{q} \in \Omega_{N-\alpha,N}$ with $\overline{q}r \neq 1$, $\overline{q} \neq N$ and $\phi\colon M \longrightarrow M_{r,\alpha,N}$ be a surjective map. Set $\chi = (r,\alpha,N,\overline{q},\phi)$. Then*

1) *$\overline{q} = N/(\alpha,N)$: $\Delta^\chi = \mathcal{E}^\xi$, $\xi\colon M \xrightarrow{\phi} M_{r,\alpha,N} \longrightarrow M_{r,\alpha,N}/\langle m\rangle \simeq \langle n\rangle \simeq \mathbb{Z}/(\alpha,N)\mathbb{Z}$;*
   *$\overline{q}\alpha \equiv 1 \bmod N$: $\Delta^\chi = \mathcal{E}^\zeta$, $\zeta\colon M \xrightarrow{\phi} M_{r,\alpha,N} = \langle e_1\rangle$;*

2) *$\overline{q} = 1$: $\Lambda^\chi = \mathcal{E}^\omega$, $\omega\colon M \xrightarrow{\phi} M_{r,\alpha,N} \longrightarrow M_{r,\alpha,N}/\langle n\rangle = \langle m\rangle \simeq \mathbb{Z}/r\mathbb{Z}$;*
   *$w_{\overline{q}} = 1$: $\Lambda^\chi = \mathcal{E}^\theta, \theta\colon M \xrightarrow{\phi} M_{r,\alpha,N} = \langle e_2\rangle$;*

3) *$\overline{q} > 1$ and $w_{\overline{q}} \neq 1$: $\Lambda^\chi = \Delta^{r,\alpha,N,\overline{q}-\hat{q},\phi}$.*

*In particular in the first two cases we have $h_{\Lambda^\chi} = h_{\Delta^\chi} = 1$.*

*Proof.* We can assume $M = M_{r,\alpha,N}$ and $\phi = \mathrm{id}$. The algebra associated to $0^{\Lambda^\chi}$, $0^{\Delta^\chi}$ are respectively $C_{\overline{q}} = k[s,t]/(s^z, t^x - s^w, s^{\hat{q}r}t^{d_{\hat{q}}} - 0^\gamma)$, $B_{\overline{q}} = k[s,t]/(s^z - t^y, t^x, s^{\hat{q}r}t^{d_{\hat{q}}})$ by 3.3.35.

1) If $\overline{q} = N/(\alpha,N)$, then $z = o(m)$, $y = 0$, $d_{\hat{q}} = (\alpha,N)$ and so $B_{\overline{q}} = k[s,t]/(s^{o(m)} - 1, t^{(\alpha,N)})$, the algebra associated to $0^{\mathcal{E}^\xi}$. If $\overline{q}\alpha \equiv 1 \bmod N$ then $r^\vee = (\alpha,N) = 1$ and $\overline{q} = \alpha^\vee/r$, i.e. $\overline{q}^\vee = 1$. So $y = 1$ and $B_{\overline{q}} \simeq k[s]/(s^{|M|})$, the algebra associated to $0^{\mathcal{E}^\gamma}$.

2) If $\overline{q} = 1$, then $z = r$, $\hat{q} = w = 0$, $x = d_{\hat{q}} = N$ and so $C_1 = k[s,t](t^n - 1, s^r)$, the algebra associated to $0^{\mathcal{E}^\omega}$. If $w = 1$ then $\overline{q} > 1$ and so $C_{\overline{q}} = k[t]/(t^{|M|})$, the algebra associated to $0^{\mathcal{E}^\theta}$.

3) If $\overline{q} > 1$ then $H_{C_{\overline{q}}} = 0$ and so $C_{\overline{q}}$ is an algebra as in 3.3.9. An easy computation shows that $z_{C_{\overline{q}}} = w > 1$, so that $\overline{q}_{C_{\overline{q}}} = \overline{q} - \hat{q}$ and $\lambda_{\overline{q}} = 1$. Therefore $\Lambda^\chi = \Delta^{r,\alpha,N,\overline{q}-\hat{q}}$ by 3.3.31. $\square$

**Proposition 3.3.40.** $\Sigma_M^\vee = \Sigma_M$ *and we have a bijection*

$$\Delta^*\colon \Sigma_M/(-)^\vee \longrightarrow \{\textit{smooth extremal rays } \mathcal{E} \textit{ with } h_\mathcal{E} = 2\}$$

*Proof.* $\Sigma_M^\vee \subseteq \Sigma_M$ since $\overline{q}\alpha \not\equiv 1 \bmod N$ is equivalent to $\overline{q}^\vee r^\vee \neq 1$. Now let $\mathcal{E}$ be a smooth extremal ray such that $h_\mathcal{E} = 2$ and $A$ the associated algebra over some field $k$. We can assume $H_{A/k} = H_\mathcal{E} = 0$. The relation $h_\mathcal{E} = 2$ means that there exist $0 \neq m, n \in M$, $m \neq n$ such that $A$ is generated in degrees $m, n$. So $M = M_{r,\alpha,N}$ as in 3.3.8 and $A$ is an algebra as in 3.3.9. By 3.3.31 and 3.3.39 we can conclude that there exist $\chi \in \Sigma_M$ such that $\mathcal{E} = \Delta^\chi$.

Now let $\chi = (r, \alpha, N, \overline{q}, \phi) \in \Sigma_M$. We have to prove that $h_{\Delta^\chi} = 2$ and, since $M_{r,\alpha,N} \neq 0$, assume by contradiction that $h_{\Delta^\chi} = 1$. We can assume $M = M_{r,\alpha,N}$ and $\phi = \mathrm{id}$. Note that $h_{\Delta^\chi} = 1$ means that the associated algebra $B$ is generated in degree $m$ or $n$. If $A$ is an algebra as in 3.3.9, then $A$ is generated in degree $n$ if and only if $z = 1$, that means $\overline{q}r = 1$. So $B$ is generated in degree $m$, i.e. $B^\vee$ is generated in degree $e_2 \in M_{r^\vee,\alpha^\vee,N^\vee}$, which is equivalent to $1 = z_{B^\vee} = \overline{q}^\vee r^\vee = 1$, and, as we have seen, to $\overline{q}\alpha \equiv 1$ mod N.

Now let $\chi' = (r', \alpha', N', \overline{q}', \phi') \in \Sigma_M$ such that $\mathcal{E} = \Delta^\chi = \Delta^{\chi'}$. Again we can assume $H_\mathcal{E} = 0$ and take $B, B'$ the algebras associated respectively to $\chi, \chi'$. By definition of $\Delta_*$, $\phi, \phi'$ are isomorphisms. If $g = \phi' \circ \phi^{-1}\colon M_{r,\alpha,N} \longrightarrow M_{r',\alpha',N'}$ then we have a graded isomorphism $p\colon B \longrightarrow B'$ such that $p(B_l) = B'_{g(l)}$. Therefore $g(\{e_1, e_2\}) = \{e_1, e_2\}$, i.e. $g = \mathrm{id}$ or $g = (-)^\vee$. It is now easy to show that $\chi' = \chi$ or $\chi' = \chi^\vee$. $\square$

*Notation* 3.3.41. We set $\Phi_M = \{\phi\colon M \longrightarrow \mathbb{Z}/l\mathbb{Z} \mid l > 1,\ \phi \text{ surjective}\}$, $\Theta_M^2 = \{\mathcal{E}^\phi\}_{\phi\in\Phi_M} \cup \{(\Lambda^\chi, \Delta^\chi)\}_{\chi\in\overline{\Sigma}_M}$, where $\overline{\Sigma}_M$ is the set of sequences $(r, \alpha, N, \overline{q}, \phi)$ where $r, \alpha, N \in \mathbb{N}$ satisfy $0 \leq \alpha < N, r > 0, r > 1$ or $\alpha > 1$, $\overline{q} \in \Omega_{N-\alpha,N}$ satisfy $\overline{q}r \neq 1$, $\overline{q} \neq N$ and $\phi\colon M \longrightarrow M_{r,\alpha,N}$ is a surjective map. Finally set $\underline{\mathcal{E}} = (\mathcal{E}^\phi, \Delta^\chi)_{\phi\in\Phi_M,\chi\in\Sigma_M/(-)^\vee}$.

**Theorem 3.3.42.** *Let $M$ be a finite abelian group. Then*

$$\{h \leq 2\} = \Big(\bigcup_{\phi\in\Phi_M} \mathcal{Z}_M^{\mathcal{E}^\phi}\Big) \bigcup \Big(\bigcup_{(\Lambda,\Delta)\in\Theta_M^2} \mathcal{Z}_M^{\Lambda,\Delta}\Big)$$

*In particular $\{h \leq 2\} \subseteq \mathcal{Z}_M^{\mathrm{sm}}$. Moreover $\pi_{\underline{\mathcal{E}}}\colon \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathrm{D}(M)\text{-}\mathrm{Cov}$ induces an equivalence of categories*

$$\left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}} \;\middle|\; \begin{array}{c} V(z_{\mathcal{E}^1}) \cap \cdots \cap V(z_{\mathcal{E}^r}) \neq \emptyset \textit{ iff} \\ r = 1 \textit{ or } (r = 2 \textit{ and } (\mathcal{E}^1, \mathcal{E}^2) \in \Theta_M^2) \end{array} \right\} = \pi_{\underline{\mathcal{E}}}^{-1}(h \leq 2) \xrightarrow{\simeq} \{h \leq 2\}$$

*Proof.* The expression of $\{h \leq 2\}$ follows from 3.3.31 and 3.3.35. Taking into account 3.3.40, the last part instead follows from 3.1.45 taking $\Theta = \Theta_M^2$. $\square$

In [Mac03] the authors prove that the toric Hilbert schemes associated to a polynomial algebra in two variables are smooth and irreducible. The same result is true more generally for multigraded Hilbert schemes, as proved later in [MS10]. Here we obtain an alternative proof in the particular case of equivariant Hilbert schemes:

**Corollary 3.3.43.** *If $M$ is a finite abelian group and $m,n \in M$ then $M$-$\mathrm{Hilb}^{m,n}$ is smooth and irreducible.*

*Proof.* Taking into account the diagram in 3.2.11 it is enough to note that $\mathrm{D}(M)\text{-}\mathrm{Cov}^{m,n} \subseteq \{h \leq 2\} \subseteq \mathcal{Z}_M^{\mathrm{sm}}$. $\square$

**Proposition 3.3.44.** *$\Sigma_M = \emptyset$ if and only if $M \simeq (\mathbb{Z}/2\mathbb{Z})^l$ or $M \simeq (\mathbb{Z}/3\mathbb{Z})^l$.*

*Proof.* For the only if, note that if $\phi\colon M \longrightarrow \mathbb{Z}/l\mathbb{Z}$ with $l > 3$ is surjective, then, taking $m = l-1,\ n = 1 \in \mathbb{Z}/l\mathbb{Z}$, we have $\mathbb{Z}/l\mathbb{Z} \simeq M_{1,l-1,l}$ and $(1, l-1, l, 2, \phi) \in \Sigma_M$.

For the converse set $M = (\mathbb{Z}/p\mathbb{Z})^l$, where $p = 2,3$ and, by contradiction, assume we have $(r, \alpha, N, \overline{q}, \phi) \in \Sigma_M$. In particular $\phi$ is a surjective map $M \longrightarrow M_{r,\alpha,N}$. If $e_1, e_2 \in M_{r,\alpha,N}$ are $\mathbb{F}_p$-independent then $M_{r,\alpha,N} = \langle e_1 \rangle \times \langle e_2 \rangle$, $\alpha = 0$, $\Omega_{N-\alpha,N} = \{1\}$ and therefore $\overline{q} = 1 = N/(\alpha, N)$, which implies that $\chi \notin \Sigma_M$. On the other hand, if $M_{1,\alpha,p} \simeq \mathbb{Z}/p\mathbb{Z}$, the only extremal rays for $\mathbb{Z}/p\mathbb{Z}$ are $\mathcal{E}^{\mathrm{id}}$ and, if $p = 3$, $\mathcal{E}^{-\mathrm{id}}$ since $K_{+\mathbb{Z}/p\mathbb{Z}} \simeq \mathbb{N}^{p-1}$ by 3.2.19. $\square$

**Theorem 3.3.45.** *Let $M$ be a finite abelian group and $X$ be a locally noetherian and locally factorial scheme. Consider the full subcategories*

$$\mathscr{C}_X^2 = \left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \delta) \in \mathcal{F}_{\underline{\mathcal{E}}}(X) \;\middle|\; \begin{array}{c} \operatorname{codim}_X V(z_{i_1}) \cap \cdots \cap V(z_{i_s}) \geq 2 \\ \textit{if } \nexists \underline{\delta} \in \Theta_M^2 \textit{ s.t. } \mathcal{E}^{i_1}, \dots \mathcal{E}^{i_s} \subseteq \underline{\delta} \end{array} \right\} \subseteq \mathcal{F}_{\underline{\mathcal{E}}}(X)$$

*and*

$$\mathscr{D}_X^2 = \{ Y \xrightarrow{f} X \in \mathrm{D}(M)\text{-}\mathrm{Cov}(X) \mid h_f(p) \leq 2\ \forall p \in X \textit{ with } \operatorname{codim}_p X \leq 1 \} \subseteq \mathrm{D}(M)\text{-}\mathrm{Cov}(X)$$

*Then $\pi_{\underline{\mathcal{E}}}$ induces an equivalence of categories*

$$\mathscr{C}_X^2 = \pi_{\underline{\mathcal{E}}}^{-1}(\mathscr{D}_X^2) \xrightarrow{\simeq} \mathscr{D}_X^2$$

*Proof.* Apply 3.1.53 with $\Theta = \Theta_M^2$. $\square$

*Remark* 3.3.46. In general $\{h \leq 3\}$ does not belong to the smooth locus on $\mathcal{Z}_M$. For example, if $M = \mathbb{Z}/4\mathbb{Z}$, $\mathrm{D}(M)\text{-}\mathrm{Cov} = \{h \leq 3\}$ is integral but not smooth by 3.2.19 and 3.2.21.

### 3.3.5 Normal crossing in codimension 1.

In this subsection we want describe, in the spirit of classification 3.2.43, normal crossing in codimension 1 covers of a locally noetherian and locally factorial scheme with no isolated points and with $(\operatorname{char} X, |M|) = 1$.

**Definition 3.3.47.** A scheme $X$ is normal crossing in codimension 1 if for any codimension 1 point $p \in X$ there exists a local and etale map $\widehat{\mathcal{O}}_{X,p} \longrightarrow R$, where $R$ is $k[[x]]$ or $k[[s,t]]/(st)$ for some field $k$ and $\widehat{\mathcal{O}}_{X,p}$ denote the completion of $\mathcal{O}_{X,p}$.

*Remark* 3.3.48. If $X$ is locally of finite type over a perfect field $k$, one can show that the above condition is equivalent to having an open subset $U \subseteq X$ such that $\operatorname{codim}_X X - U \geq 2$ and there exists an etale coverings $\{U_i \longrightarrow U\}$ with etale maps $U_i \longrightarrow \operatorname{Spec} k[x_1, \dots, x_{n_i}]/(x_1 \cdots x_{r_i})$ for any $i$. Anyway we will not use this property.

*Notation* 3.3.49. In this subsection we will consider a field $k$ and we will set $A = k[[s,t]]/(st)$. Given an element $\xi \in \operatorname{Aut}_k k[[x]]$ we will write $\xi_x = \xi(x)$ so that, if $p \in k[[x]]$ then $\xi(p)(x) = p(\xi_x)$. We will call $I \in \operatorname{Aut}_k k[[s,t]]$ the unique map such that $I(s) = t$, $I(t) = s$. Given $B \in k^*$ we will denote by $\underline{B}$ the automorphism of $k[[x]]$ such that $\underline{B}_x = Bx$.

Finally, given $f \in k[[x_1, \dots, x_n]]$ and $g \in k[x_1, \dots, x_n]$ the notation $f = g + \cdots$ will mean $f \equiv g \mod (x_1, \dots, x_r)^{\deg g+1}$.

The first problem to deal with is to describe the action on $A$ of a finite group $M$ and check when $A$ is a $\mathrm{D}(M)$-cover over $A^M$, assuming to have the $|M|$-roots of unity in $k$. We start collecting some general facts about $A$.

**Proposition 3.3.50.** *We have:*

1) $A = k \oplus sk[[s]] \oplus tk[[t]]$

2) *Given* $f, g \in A - \{0\}$ *then* $fg = 0$ *if and only if* $f \in sk[[s]], g \in tk[[t]]$ *or vice versa.*

3) *Any automorphism in* $\operatorname{Aut}_k A$ *is of the form* $(\xi, \eta)$ *or* $I(\xi, \eta)$ *where* $\xi, \eta \in \operatorname{Aut}_k k[[x]]$ *and* $(\xi, \eta)(f(s,t)) = f(\xi_s, \eta_t)$.

4) *If* $\xi \in \operatorname{Aut}_k k[[x]]$ *has finite order then* $\xi = \underline{B}$ *where* $B$ *is a root of unity in* $k$. *In particular if* $(\xi, \eta) \in \operatorname{Aut}_k A$ *has finite order then* $\xi = \underline{B}$, $\eta = \underline{C}$ *where* $B$, $C$ *are roots of unity in* $k$.

5) *Let* $f \in k[[x]] - \{0\}$, $B, C$ *roots of unity in* $k$. *Then* $f(Bx) = Cf(x)$ *if and only if* $C = B^r$ *for some* $r > 0$ *and, if we choose the minimum* $r$, $f \in x^r k[[x^{o(B)}]]$.

*Proof.* 1) is straightforward and 2) follows easily expressing $f$ and $g$ as in 1). For 3) note that if $\theta \in \operatorname{Aut}_k A$ then $\theta(s)\theta(t) = 0$ and apply 2). Finally 4) and 5) can be shown looking at the coefficients of $\xi_x$ and of $f$. $\square$

**Lemma 3.3.51.** *If* $M < \operatorname{Aut}_k A$ *is a finite subgroup containing only automorphisms of the form* $(\xi, \eta)$ *then* $A^M \simeq A$.

*Proof.* It is easy to show that $A^M \simeq k[[s^a, t^b]]/(s^a t^b) \simeq A$ where $a = \operatorname{lcm}\{i \mid \exists (\underline{A}, \underline{B}) \in M \text{ s.t. } \operatorname{ord} A = i\}$ and $b = \operatorname{lcm}\{i \mid \exists (\underline{A}, \underline{B}) \in M \text{ s.t. } \operatorname{ord} B = i\}$. $\square$

Since we are interested in covers of regular in codimension 1 schemes (and $A$ is clearly not regular) we can focus on subgroups $M < \operatorname{Aut}_k A$ containing some $I(\xi, \eta)$.

**Lemma 3.3.52.** *Let $M < \operatorname{Aut}_k A$ be a finite abelian group and assume that* $(\operatorname{char} k, |M|) = 1$ *and that there exists* $I(\xi,\eta) \in M$. *Then, up to equivariant automorphisms, we have* $M = \langle I(\mathrm{id}, \underline{B})\rangle$ *or, if $M$ is not cyclic,* $M = \langle(\underline{C},\underline{C})\rangle \times \langle I\rangle$ *where $\underline{B}$, $\underline{C}$ are roots of unity and $o(C)$ is even.*

*Proof.* The existence of an element of the form $I(\xi,\eta)$ in $M$ implies that $s$ and $t$ cannot be homogeneous in $m_A/m_A^2$, that $2 \mid |M|$ and therefore that $\operatorname{char} k \neq 2$.

Applying the exact functor $\operatorname{Hom}_k^M(m_A/m_A^2,-)$, we get that the surjection $m_A \longrightarrow m_A/m_A^2$ has a $k$-linear and $M$-equivariant section. This means that there exists $x,y \in m_A$ such that $m_A = (x,y)$ and $M$ acts on $x$, $y$ with characters $\chi,\zeta$. In this way we get an action of $M$ on $k[[X,Y]]$ and an equivariant surjective map $\phi\colon k[[X,Y]] \longrightarrow A$. Moreover $\operatorname{Ker}\phi = (h)$, where $h = fg$ and $f$, $g \in k[[X,Y]]$ are such that $\phi(f) = s$, $\phi(g) = t$. We can write $f = aX + bY + \cdots$, $g = cX + dY + \cdots$ with $ad - bc \neq 0$. Since $ax + by = s$ in $m_A/m_A^2$ and $s$ is not homogeneous there, we have $a,b \neq 0$. Similarly we get $c,d \neq 0$. In particular, up to normalize $f$, $g$, $x$ we can assume $b = c = d = 1$. Now $h = aX^2 + (a+1)XY + Y^2 + \cdots$ and applying Weierstrass preparation theorem [Lan02, Theorem 9.2], there exists a unique $\tilde{h} \in (h)$ such that $(\tilde{h}) = (h)$ and $\tilde{h} = \psi_0(X) + \psi_1(X)Y + Y^2$. The uniqueness of $\tilde{h}$ and the $M$-invariance of $(h)$ yield the relations $m(\tilde{h}) = \eta(m)^2\tilde{h}$,

$$m(\psi_0) = \psi_0(\chi(m)X) = \eta(m)^2\psi_0,\ m(\psi_1) = \psi_1(\chi(m)X) = \eta(m)\psi_1 \tag{3.3.4}$$

for any $m \in M$. Moreover $\tilde{h} = \mu h$ where $\mu \in k[[X,Y]]^*$ and, since the coefficient of $Y^2$ in both $h$ and $\tilde{h}$ is 1, we also have $\mu(0) = 1$. In particular $\psi_0 = aX^2 + \cdots$ and $\psi_1 = (a+1)X + \cdots$ and so $(a+1)(\chi - \zeta) = 0$ by 3.3.4. Since $s$ is not homogeneous in $m_A/m_A^2$, $\chi \neq \eta$ and $a = -1$. Since $\operatorname{char} k \neq 2$ we can write $\tilde{h} = (Y+\psi_1/2)^2 - (\psi_1^2/4 - \psi_0) = y'^2 - z'$. Note that $y', z'$ are homogeneous thanks to 3.3.4. Moreover, by Hensel's lemma, we can write $z' = X^2 + \cdots = X^2q^2$ for an homogeneous $q \in k[[x]]$ with $q(0) = 1$. So $x' = xq$ is homogeneous and $\tilde{h} = y'^2 - x'^2$. This means that we can assume $s = x - y$, $t = x + y$. In particular $\chi^2 = \eta^2$ and $M$ acts on $s,t$ as

$$m(s) = \frac{\chi+\zeta}{2}(m)s + \frac{\chi-\zeta}{2}(m)t \qquad m(t) = \frac{\chi-\zeta}{2}(m)s + \frac{\chi+\zeta}{2}(m)t$$

Consider the exact sequence

$$0 \longrightarrow H \longrightarrow M \xrightarrow{\chi/\eta} \{-1,1\} \longrightarrow 0 \tag{3.3.5}$$

If $M$ is cyclic, say $M = \langle m\rangle$, we have $\chi(m) = -\eta(m)$ and so $m = I(\underline{B},\underline{B})$, where $B = (\chi(m) - \eta(m))/2$ is a root of unity. Up to normalize $s$ we can write $m = I(\mathrm{id},\underline{B})$.

Now assume that $M$ is not cyclic. The group $H$ acts on $s$ and $t$ with the character $\chi_{|H} = \zeta_{|H}$ and this yields an injective homomorphism $\chi_{|H}\colon H \longrightarrow \{\text{roots of unity of } k\}$. So $H = \langle(\underline{C},\underline{C})\rangle$ for some root of unity $C$. The extension 3.3.5 corresponds to an element of $\operatorname{Ext}^1(\mathbb{Z}/2\mathbb{Z},H) \simeq H/2H$ that differs to the sequence $0 \longrightarrow H \longrightarrow \mathbb{Z}/2o(C)\mathbb{Z} \longrightarrow \{-1,1\} \longrightarrow 0$. So $H/2H \simeq \mathbb{Z}/2\mathbb{Z}$, $o(C)$ is even and the sequence 3.3.5 splits. We can conclude that $M = \langle(\underline{C},\underline{C})\rangle \times \langle m\rangle$, where $m = I(\underline{D},\underline{D})$ for some root of unity $D$ and $o(m) = 2$. Normalizing $s$ we can write $m = I(\mathrm{id},\underline{D}) = I$. ∎

Table 3.3.1:

| $H$ | $m,n,r,\alpha,N,\overline{q}$ | $B$ | $\mathcal{E}$ |
|---|---|---|---|
| $\mathbb{Z}/2\mathbb{Z}$ | $1,1,1,1,2,1$ | $\frac{k[[z]][U]}{(U^2-z^2)}$ | $2\mathcal{E}^{id}$ |
| $(\mathbb{Z}/2\mathbb{Z})^2$ | $(1,0),(0,1),2,0,2,1$ | $\frac{k[[z]][U,V]}{(U^2-z,V^2-z)}$ | $\mathcal{E}^{\mathrm{pr}_1}+\mathcal{E}^{\mathrm{pr}_2}$ |
| $\mathbb{Z}/2l\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ $l>1$ | $(1,0),(1,1),2,2,2l,1$ | $\frac{k[[z]][U,V]}{(U^2-V^2,V^{2l}-z)}$ | $\Delta^{2,2,2l,1}$ |
| $\mathbb{Z}/4l\mathbb{Z}$ | $1,2l+1,1,2l+1,4l,2$ | $\frac{k[[z]][U,V]}{(U^2-V^2,V^{2l+1}-zU,UV^{2l-1}-z)}$ | $\Delta^{1,2l+1,4l,2}$ |
| $\mathbb{Z}/2l\mathbb{Z}$ $l>1$ odd | $1,l+1,2,2,l,1$ | $\frac{k[[z]][U,V]}{(U^2-V^2,V^l-z)}$ | $\Delta^{2,2,l,1}$ |

**Proposition 3.3.53.** *Let $M<\operatorname{Aut}_k A$ be a finite abelian group such that $(\operatorname{char} k,|M|)=1$ and that there exists $I(\xi,\eta)\in M$. Also assume that $k$ contains the $|M|$-roots of unity. Then $A^M\simeq k[[z]]$, $A\in \mathrm{D}(M)\text{-Cov}(A^M)$ and only the following possibilities happen: there exists a row of table 3.3.1 such that $M\simeq H$ is generated by $m,n$, $H\simeq M_{r,\alpha,N}$, $A\simeq B$ as $M$-covers, where $\deg U=m$, $\deg V=n$ and $A$ over $A^M$ is given by multiplication $z^{\mathcal{E}}$. Moreover all the rays of the form $\Delta^*$ in the table satisfy $h_{\Delta^*}=2$.*

*Proof.* We can reduce the problem to the actions obtained in 3.3.52. We first consider the cyclic case, i.e. $M=\langle I(\mathrm{id},\underline{B})\rangle\simeq\mathbb{Z}/2l\mathbb{Z}$ where $l=o(B)$. There exists $E$ such that $E^2=B$. Given $0\leq r<|M|=2l$, we want to compute $A_r=\{a\in A\mid I(\mathrm{id},\underline{B})a=E^r a\}$. The condition $a=c+f(s)+g(t)\in A_r$ holds if and only if $a=0$ when $r>0$, $f(t)=E^r g(t)$ and $g(Bs)=E^r f(s)$. Moreover $f(t)=E^{-r}g(Bt)=E^{-2r}f(Bt)\implies f(Bt)=B^r f(t)$. If we denote by $\delta_r$ the only integer such that $0\leq\delta_r<l$ and $\delta_r\equiv r \bmod l$, we have that, up to constants, $A^r$ is given by elements of the form $E^r f(s)+f(t)$ for $f\in X^{\delta_r}k[[X^l]]$. Call $\beta=s^l+t^l\in A_0=A^M$ and $v_r=E^r s^{\delta_r}+t^{\delta_r}$, $v_0=1$. We claim that $A^M=A_0=k[[\beta]]$ and $v_r$ freely generates $A_r$ as an $A_0$ module. The first equality holds since $A_0$ is a domain and we have relations

$$\sum_{n\geq1}a_n s^{nl}+\sum_{n\geq1}a_n t^{nl}=\sum_{n\geq1}a_n(s^l+t^l)^n=\sum_{n\geq1}a_n\beta^n$$

while the second claim come from the relation

$$E^r s^{\delta_r}(c+h(s))+t^{\delta_r}(c+h(t))=(E^r s^{\delta_r}+t^{\delta_r})(c+h(s)+h(t)) \text{ for } h\in X^l k[[X^l]]$$

and the fact that $v_r$ is not a zero divisor in $A$.

So $A\in\mathrm{D}(M)\text{-Cov}(k[[\beta]])$ and it is generated by $v_1=Es+t$ and $v_{l+1}=-Es+t$ and so in degrees 1 and $l+1$. If $l=1$, so that $M\simeq\mathbb{Z}/2\mathbb{Z}$, $B=1$, $E=-1$ and $v_1^2=\beta^2$. This means that $A\simeq k[[\beta]][U]/(U^2-\beta^2)$ and its multiplication over $k[[\beta]]$ is given by $\beta^{2\mathcal{E}^{\mathrm{id}}}$. This is the first row. Assume $l>1$ and set $m=1$, $n=l+1$. Note that $0\neq m\neq n$ and that $M\simeq M_{r,\alpha,N}$ for some $r$, $\alpha$, $N$ that we are going to compute.

$l$ odd. We have $r=\alpha=2$ and $N=l$ since $\langle l+1\rangle=\langle 2\rangle\subseteq\mathbb{Z}/2l\mathbb{Z}$. Consider $\overline{q}=1\in\Omega_{N,N-\alpha}$ and the associated numbers are $z=r=2$, $y=\alpha=2$, $\hat{q}=0$, $d_{\hat{q}}=$

$x = N = l$, $w = 0$. Since $v_1^z = v_{l+1}^y$ and $v_{l+1}^l = \beta$, we will have $A \simeq_{k[[\beta]]} A^1_{\lambda,\mu}$ where $\lambda, \mu = 1, \beta \in k[[\beta]]$ (see 3.3.29) and therefore the multiplication is $\beta^{\Delta^{2,2,l,1}}$ by 3.3.35. This is the fifth row.

$l$ even. We have $r = 1$, $\alpha = l+1$, $N = 2l$ since $\langle l+1\rangle = \mathbb{Z}/2l\mathbb{Z}$. Since $d_1 = l-1 \equiv -\alpha$ and $d_2 = 2l-2 \equiv 2(-\alpha)$ modulo $2l$ we can consider $\overline{q} = 2 \in \Omega_{N-\alpha,N}$. The associated numbers are $z = y = 2$, $\hat{q} = 1$, $d_{\hat{q}} = l-1$, $x = N - (d_{\overline{q}} - d_{\hat{q}}) = l+1$, $w = 1 \equiv xn = (l+1)^2 \bmod 2l$. Since $v_1^z = v_{l+1}^y$, $v_{l+1}^x = \beta v_1$ and $v_1^{\hat{q}r} v_{l+1}^{d_{\hat{q}}} = \beta$, we will have $A \simeq_{k[[\beta]]} A^2_{\lambda,\mu}$ where $\lambda, \mu = 1, \beta \in k[[\beta]]$ whose multiplication is $\beta^{\Delta_{1,l+1,2l,2}}$. This is the fourth row.

Now consider the case $M = \langle(\underline{C}, \underline{C})\rangle \times \langle I\rangle$ with $o(C) = l$ even. Set $\beta = s^l + t^l$, $v_{1,0} = s + t$ and $v_{1,1} = -s + t$. Note that $v_{r,i}$ is homogeneous of degree $(r, i)$. Set $m = (1, 0)$, $n = (1, 1)$. They are generators of $M$ and so $M \simeq M_{r,\alpha,N}$ for some $r, \alpha, N$. We have $N = o(n) = l$, $r > 1$ since $\langle n\rangle \neq M$ and so $r = 2$ since $2m = 2n$. If $l = 2$ we get $\alpha = 0$ and if $l > 2$ we get $\alpha = 2$. Choose $\overline{q} = 1$ so that the associated numbers are $z = 2$, $y = \alpha$, $\hat{q} = 0$, $d_{\hat{q}} = x = N = l$, $w = 0$. As done above, it is easy to see that $A^M = k[[\beta]]$. We first consider the case $l = 2$. Since $v_{1,0}^2 = \beta$, $v_{1,1}^2 = \beta$ we get a surjection $A^1_{\beta,\beta} \longrightarrow A$ which is an isomorphism by dimesion. From the expression of $A^1_{\beta,\beta}$ we can deduce directly that the multiplication is $\beta^{\mathcal{E}^{\mathrm{pr}_1} + \mathcal{E}^{\mathrm{pr}_2}}$, where $\mathrm{pr}_i \colon (\mathbb{Z}/2\mathbb{Z})^2 \longrightarrow \mathbb{Z}/2\mathbb{Z}$ are the two projections. This is the second row.

Now assume $l > 2$. Since $v_{1,0}^2 = v_{1,1}^2$ and $v_{1,1}^l = \beta$ and arguing as above we get $A \simeq_{k[[\beta]]} A^1_{\lambda,\mu}$ where $\lambda, \mu = 1, \beta \in k[[\beta]]$ and the multiplication $\beta^{\Delta^{2,2,l,1}}$. This is the third row.

Finally the last sentence is clear by definition of $\Sigma_M$ and 3.3.40. □

*Remark* 3.3.54. If $X$ is a locally noetherian integral scheme and there exists a $\mathrm{D}(M)$-cover $Y/X$ such that $Y$ is normal crossing in codimension 1, then $X$ is defined over a field. Indeed if $\operatorname{char} \mathcal{O}_X(X) = p$ then $\mathbb{F}_p \subseteq \mathcal{O}_X(X)$. Otherwise $\mathbb{Z} \subseteq \mathcal{O}_X(X)$ and we have to prove that any prime number $q \in \mathbb{Z}$ is invertible. We can assume $X = \operatorname{Spec} R$, where $R$ is a local noetherian domain. If $\dim R = 0$ then $R$ is a field, otherwise, since $\mathrm{ht}(q) \leq 1$, we can assume $\dim R = 1$ and $R$ complete. By definition of normal crossing in codimension 1, if $Y = \operatorname{Spec} S$ and $p \in Y$ is over $m_R$ we have a flat and local map $R \longrightarrow S \longrightarrow S_p \longrightarrow B$, such that $B$ contains a field $k$. The prime $q$ is a non zero divisor in $R$ and therefore in $B$. In particular $0 \neq q \in k^* \subseteq B^*$ and $q \in R^*$.

**Theorem 3.3.55.** *Let $M$ be a finite abelian group, $X$ be a locally noetherian and locally factorial scheme with no isolated points and* $(\operatorname{char} X, |M|) = 1$. *Consider the full subcategory*

$$NC^1_X = \{Y/X \in \mathrm{D}(M)\text{-}\mathrm{Cov}(X) \mid Y \textit{ is normal crossing in codimension } 1\} \subseteq \mathrm{D}(M)\text{-}\mathrm{Cov}(X)$$

*Then $NC^1_X \neq \emptyset$ if and only if each connected component of $X$ is defined over a field. In this case define*

$$\underline{\mathcal{E}} = \begin{pmatrix} \mathcal{E}^\phi \textit{ for } \phi \colon M \longrightarrow \mathbb{Z}/l\mathbb{Z} \textit{ surjective with } l \geq 1, \\ \Delta^{2,2,l,1,\phi} \textit{ for } \phi \colon M \longrightarrow M_{2,2,l} \textit{ surjective with } l \geq 3, \\ \Delta^{1,2l+1,4l,2,\phi} \textit{ for } \phi \colon M \longrightarrow M_{1,2l+1,4l} \textit{ surjective with } l \geq 1 \end{pmatrix}$$

*and* $\mathscr{C}^1_{NC,X}$ *as the full subcategory of* $\mathcal{F}_{\underline{\mathcal{E}}}(X)$ *of objects* $(\underline{\mathcal{L}},\underline{\mathcal{M}},\underline{z},\lambda)$ *such that:*

1) *for all* $\mathcal{E}\neq\delta\in\underline{\mathcal{E}}$, $\operatorname{codim}V(z_{\mathcal{E}})\cap V(z_\delta)\geq 2$ *except for the case where* $\mathcal{E}=\mathcal{E}^\phi$, $\delta=\mathcal{E}^\psi$

$$\begin{tikzcd} & & \mathbb{Z}/2\mathbb{Z} \\ M \arrow[r, two heads] \arrow[urr, bend left, "\phi"] \arrow[drr, bend right, "\psi"'] & (\mathbb{Z}/2\mathbb{Z})^2 \arrow[ur, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \\ & & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

*in which* $v_p(z_{\mathcal{E}^\phi})=v_p(z_{\mathcal{E}^\psi})=1$ *if* $p\in Y^{(1)}\cap V(z_{\mathcal{E}^\phi})\cap V(z_{\mathcal{E}^\psi})$*;*

2) *for all* $\mathcal{E}\in\underline{\mathcal{E}}$ *and* $p\in Y^{(1)}$ $v_p(z_{\mathcal{E}})\leq 2$ *and* $v_p(z_{\mathcal{E}})=2$ *if and only if* $\mathcal{E}=\mathcal{E}^\phi$ *where* $\phi\colon M\longrightarrow\mathbb{Z}/2\mathbb{Z}$ *is surjective.*

*Then we have an equivalence of categories*

$$\mathscr{C}^1_{NC,X}=\pi^{-1}_{\underline{\mathcal{E}}}(NC^1_X)\xrightarrow{\simeq}NC^1_X$$

*Proof.* The first claim comes from 3.3.54. We will make use of 3.3.45. If $Y/X\in NC^1_Y$ and $p\in Y^{(1)}$ we have $h_{Y/X}(p)\leq\dim_{k(p)}m_p/m_p^2\leq 2$ since etale maps preserve tangent spaces and $\dim m_A/m_A^2\leq 2$. So $NC^1_X\subseteq\mathscr{D}^2_X$.

Let $\underline{\delta}$ be the sequence of smooth extremal rays used in 3.3.45. We know that $\pi^{-1}_{\underline{\delta}}(NC^1_X)\subseteq\mathscr{C}^2_X$. So we have only to prove that $\pi^{-1}_{\underline{\delta}}(NC^1_X)\subseteq\mathcal{F}_{\underline{\mathcal{E}}}(X)\subseteq\mathcal{F}_{\underline{\delta}}(X)$ and that any element $Y\in NC^1_X$ locally, in codimension 1, satisfies the requests of the theorem. Since $X$ is a disjoint union of positive dimensional, integral connected components, we can assume that $X=\operatorname{Spec}R$, where $R$ is a complete discrete valuation ring. Since $R$ contains a field, then $R\simeq k[[x]]$ . Let $\chi\in\pi^{-1}_{\underline{\mathcal{E}}}(\mathscr{D}^2_X)$ and $D$ the associated $M$-cover over $R$. Let $C$ be the maximal torsor of $D/R$ and $H=H_{D/R}$. Note that, for any maximal ideal $q$ of $C$ we have $C_q\simeq k(q)[[x]]$ since $C/R$ is etale. Moreover $\operatorname{Spec}D\in NC^1_X$ for $M$ if and only if for any maximal prime $p$ of $D$ $\operatorname{Spec}D_p\in NC^1_{\operatorname{Spec}C_q}$ for $M/H$, where $q=C\cap p$. In the same way $\chi\in\mathscr{C}^1_{NC,X}$ for $M$ if and only if, for any maximal prime $q$ of C, $\chi_{|\operatorname{Spec}C_q}\in\mathscr{C}^1_{NC,\operatorname{Spec}C_q}$ for $M/H$. We can therefore reduce the problem to the case $H_{D/R}=0$. We can also assume that $k$ contains the $|M|$-roots of unity.

First assume that $\operatorname{Spec}D\in NC^1_Y$. If $D$ is regular, the conclusion comes from 3.2.43. So assume $D$ not regular and denote by $\mu\colon R=k[[x]]\longrightarrow D$ the associated map. We know that $D/m_A=k$. By Cohen's structure theorem we can write $D=k[[y]]/I$ in such a way that $\mu_{|k}=\mathrm{id}_k$. By definition, since $D$ is local and complete, there exists an etale extension $D\longrightarrow B=L[[s,t]]/(st)$. Using the properties of complete rings, $B/D$ is finite and so $B\simeq D\otimes_k L$. Replacing the base $R$ by $R\otimes_k L$ we can assume that $D\simeq k[[s,t]]/(st)$. The function $\mu_{|k}\colon k\longrightarrow D$ extends to a map $\nu\colon D\longrightarrow D$ sending $s,t$ to itselves. This map is clearly surjective. Since $\operatorname{Spec}D$ contains 3 points, $\nu$ induces a closed immersion $\operatorname{Spec}D\longrightarrow\operatorname{Spec}D$ which is a bijection. Since $D$ is reduced $\nu$ is an isomorphism. This shows that we can write $D=A=k[[s,t]]/(st)$ in such a way that $\mu_{|k}=\mathrm{id}_k$. So $\mathrm{D}(M)\simeq\underline{M}$ acts as a subgroup of $\operatorname{Aut}_k A$ such that $A^M\simeq k[[z]]$.

In particular, by 3.3.51, there exists $I(\xi,\eta) \in M$. Up to equivariant isomorphisms the possibilities allowed are described in 3.3.53 and coincides with the ones of the statement. So $\chi \in \mathscr{C}^1_{NC,X}$.

Now assume that $\chi \in \mathscr{C}^1_{NC,X}$. By definition of $\pi_{\underline{\mathcal{E}}}$ the multiplication that defines $D$ over $R$ is something of the form $\psi = \lambda z^{\mathcal{E}}$, where $\lambda$ is an $M$-torsor and $\mathcal{E}$ is one of the ray of table 3.3.1. The case $\mathcal{E} = \mathcal{E}^\phi$ comes from 3.2.43. Since, in our hypothesis, an $M$-torsor (in the fppf meaning) is also an etale torsor, replacing the base $R$ by an etale neighborhood (that maintains the form $k[[x]]$), we can assume $\lambda = 1$. In this case, thanks to 3.3.52 and 3.3.53, we can conclude that $A \simeq k[[s,t]]/(st)$ as required. $\square$

**Corollary 3.3.56.** *Let $X$ be a locally noetherian and regular in codimension* 1 *(normal) scheme with no isolated points, $M$ be a finite abelian group with* $(\operatorname{char} X, |M|) = 1$ *and* $|M|$ *odd. If $Y/X$ is a* $\mathrm{D}(M)$*-cover and $Y$ is normal crossing in codimension* 1 *then $Y$ is regular in codimension* 1 *(normal).*

*Proof.* Since $Y/X$ has Cohen-Macaulay fibers it is enough to prove that $Y$ is regular in codimension 1 by Serre's criterion. So we can assume $X = \operatorname{Spec} R$, where $R$ is a discrete valuation ring, and apply 3.2.43 just observing that $\widetilde{Reg}^1_X = \mathscr{C}^1_{NC,X}$. $\square$

*Remark* 3.3.57. We keep notation from 3.3.55 and set $\underline{\delta} = (\mathcal{E}^\eta, \eta \colon M \longrightarrow \mathbb{Z}/d\mathbb{Z}$ surjective $, d > 1)$. We have that $\pi^{-1}_{\underline{\delta}}(NC^1_X) = \mathscr{C}^1_{NC,X} \cap \mathcal{F}_{\underline{\delta}}$, i.e. the covers $Y/X \in NC^1_X$ writable only with the rays in $\underline{\delta}$, has the same expression of $\mathscr{C}^1_{NC,X}$ but with object in $\mathcal{F}_{\underline{\delta}}$. Therefore the multiplications that yield a not smooth but with normal crossing in codimension 1 covers are only $\mathcal{E}^\phi + \mathcal{E}^\psi$, where $\phi$, $\psi$ are morphism as in 1), and $\mathcal{E}^{2\phi}$, where $\phi \colon M \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is surjective. This result can also be found in [AP12, Theorem 1.9]. In particular, if $M = (\mathbb{Z}/2\mathbb{Z})^r$, where $\underline{\delta} = \underline{\mathcal{E}}$ thanks to 3.3.44, these are the only possibilities.

# 4 Equivariant affine maps and monoidality.

The aim of this chapter is the study of $G$-covers for general groups, but with particular attention to the linearly reductive case. We now briefly summarize how this chapter is divided.

*Section 1.* We will introduce the definition of linearly reductive groups and study their representation theory. Looking for an analogous behaviour to the representation theory for groups over a field, we will introduce the notion of good linearly reductive groups (briefly glrg). We will then focus on linearly reductive groups over strictly Henselian rings and their action on finite algebras. The last part will be dedicated to the study of induction of equivariant algebras from a subgroup.

*Section 2.* We will prove the equivalence between the category of $G$-equivariant quasi-coherent sheaves of algebras over a scheme $T$ and the category of linear, left exact, symmetric monoidal functors $\operatorname{Loc}^G R \longrightarrow \operatorname{Loc} T$. The first step will be to establish a correspondence between $G$-equivariant quasi-coherent sheaves and functors as above, but without any monoidal structure and then describe how the properties of commutativity, associativity and existence of a unity translate into properties of the associated functor. We will then determine what functors correspond to $G$-covers and $G$-torsors and, when $G$ is a super solvable glrg, we will also describe a simpler criterion to distinguish $G$-torsors among $G$-equivariant algebras.

*Section 3.* In this section we will prove that $G$-Cov is reducible if $G$ is a linearly reductive and non abelian group. The proof is based on the use of what we will call rank functions, that allow us to distinguish $G$-Cov inside $\mathrm{LAlg}_R^G$ and their behaviour under induction from a subgroup.

*Section 4.* This section is dedicated to the problem of regular in codimension 1 $G$-covers. We will describe such covers using the trace map associated with an algebra and we will also discuss a possible extension of the results to the non equivariant case.

In this chapter, we will often prove statements valid over any scheme and, in order to simplify the reading, the letter $T$, if not stated otherwise, will denote a scheme over the given base.

## 4.1 Preliminaries on linearly reductive groups.

In this section we will study the representation theory of finite, linearly reductive groups. In particular we will introduce the notion of good linearly reductive groups (glrg). This class of groups has a very special representation theory, very close to the one of usual linearly reductive groups over an algebraically closed field.

We will then focus on groups over strictly Henselian rings, where their structure is simpler and finally we will discuss the properties of induction and state some useful results.

We will consider given a base scheme $S$ and a flat, finite and finitely presented group scheme $G$ over $S$.

### 4.1.1 Representation theory of linearly reductive groups.

As the section name suggests, in this section we will introduce the notion of linearly reductive groups and discuss their representation theory. In particular we will define the notion of good linearly reductive group (briefly glrg): these are the groups admitting a set of geometrically irreducible representations with which is possible to describe any equivariant quasi-coherent sheaf, analogously to what happens over an algebraically closed field. Let $G$ be a linearly reductive group. We will prove that if $G$ is defined over an algebraically closed field or if it is diagonalizable then it is a glrg. Moreover we will show that $G$ is always fppf locally a glrg and, if $G$ is étale, also étale locally. In particular any étale (and therefore constant), finite linearly reductive group defined over a strictly Henselian ring is a glrg.

In what follows $G$ will be a flat, finite and finitely presented group scheme over the given base. Before dealing with linearly reductive groups, we prove the following propositions, which will be very useful.

**Proposition 4.1.1.** *Let $X = \operatorname{Spec} \mathscr{A}$ be an affine $S$-scheme with a (right) action of $G$ and $\mathcal{F}$ be a quasi-coherent sheaf over $S$. Then we have a $G$-equivariant isomorphism*

$$\phi\colon \underline{\mathrm{Hom}}_S(X, \mathrm{W}(\mathcal{F})) \longrightarrow \mathrm{W}(\mathcal{F} \otimes \mathscr{A})$$

*If $X = G$, with the regular action on itself, we have vertical isomorphisms*

$$\begin{array}{ccc} \underline{\mathrm{Hom}}^G(G, \mathrm{W}(\mathcal{F})) & \longrightarrow & \underline{\mathrm{Hom}}(G, \mathrm{W}(\mathcal{F})) \\ \wr\| & & \wr\| \\ \mathrm{W}(\mathcal{F}) & \xrightarrow[\mathrm{W}(\nu)]{} & \mathrm{W}(\mathcal{F} \otimes \mathcal{O}_S[G]) \end{array}$$

*where $\nu\colon \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_S[G]$ is the structure map. In particular $\nu$ yields an isomorphism $\mathcal{F} \simeq (\mathcal{F} \otimes \mathcal{O}_S[G])^G$ of sheaves (without actions).*

*Proof.* Notice that we will only use that $G$ is an affine scheme. Let $\pi\colon X \longrightarrow S$ be the structure morphism. Note that if $U$ is an $S$-scheme, then $\mathrm{W}(\mathcal{F}) \times U \simeq \mathrm{W}(\mathcal{F} \otimes \mathcal{O}_U)$ as sheaves over $U$ and, since $\pi_*\pi^*\mathcal{F} \simeq \mathcal{F} \otimes \mathscr{A}$, we have that

$$(\mathrm{W}(F) \times U)(X \times U) = \mathrm{H}^0(X \times U, \pi_U^*(\mathcal{F} \otimes \mathcal{O}_U)) = \mathrm{H}^0(U, \mathcal{F} \otimes \mathscr{A} \otimes \mathcal{O}_U) = \mathrm{W}(\mathcal{F} \otimes \mathscr{A})(U)$$

where $\pi_U$ is the base change of $\pi$ to $U$. In particular, by Yoneda's lemma, the natural transformation $\phi\colon \underline{\mathrm{Hom}}_S(X, \mathrm{W}(\mathcal{F})) \longrightarrow \mathrm{W}(\mathcal{F} \otimes \mathscr{A})$ given by

$$\phi_U(X \times U \xrightarrow{\delta} \mathrm{W}(\mathcal{F}) \times U) = \delta(\mathrm{id}_{X \times U})$$

is an isomorphism. We have to show that $\phi$ is $G$-equivariant and we can assume that $S = \operatorname{Spec} R$, for some ring $R$. Denote by $\xi\colon \mathcal{F}\otimes\mathscr{A} \longrightarrow \mathcal{F}\otimes\mathscr{A}\otimes R[G]$ the action of $G$ on $\mathcal{F}\otimes\mathscr{A}$ induced by $\phi$. We want to prove that $\xi$ coincides with the structure morphism of the tensor product of representations $\mathcal{F}\otimes\mathscr{A}$. In general if $M$ is an $R$-module with an action of $G$ then the multiplication by $\mathrm{id}_G$ on $\mathrm{W}(M)(G) = M\otimes R[G]$ yields the structure map $M \longrightarrow M\otimes R[G] \xrightarrow{\mathrm{id}_G\cdot -} M\otimes R[G]$ of $M$. In particular, by definition we have

$$\xi(x) = \phi_G(\mathrm{id}_G\cdot\phi_G^{-1}(x\otimes 1)) = [\mathrm{id}_G\cdot\phi_G^{-1}(x\otimes 1)](\mathrm{id}_{X\times G}) \text{ for } x\in\mathcal{F}\otimes\mathscr{A}$$

Given $\delta\colon X\times G \longrightarrow \mathrm{W}(F)\times G$ we have

$$(\mathrm{id}_G\cdot\delta)(\mathrm{id}_{X\times G}) = \mathrm{id}_G\cdot(\delta(\mathrm{id}_{X\times G}\cdot\mathrm{id}_G))$$

Moreover $(\mathrm{id}_{X\times G}\cdot\mathrm{id}_G)\colon X\times G \longrightarrow X\times G$ is given by the (right) action of $G$ on $X$, i.e. it is the Spec of the structure map $\mu\colon \mathscr{A}\otimes\mathcal{O}_S[G] \longrightarrow \mathscr{A}\otimes\mathcal{O}_S[G]$. On the other hand, given $z\in\mathcal{F}\otimes R[G]\otimes\mathscr{A} = \operatorname{Hom}(X\times G, \mathrm{W}(\mathcal{F})\times G)$ then $\mathrm{id}_G\cdot z = (\overline{\nu}\otimes\mathrm{id}_{\mathscr{A}})(z)$, where $\overline{v}\colon \mathcal{F}\otimes R[G] \longrightarrow \mathcal{F}\otimes R[G]$ is the structure map, i.e. the $R[G]$-linear map such that $\overline{\nu}(x\otimes 1) = \nu(x)$. Finally, by definition of the Yoneda's isomorphism, we have

$$\phi_G^{-1}(x\otimes 1)(U \xrightarrow{\alpha} X\times G) = [\mathrm{W}(\mathcal{F})(\alpha)](x\otimes 1) \in (\mathrm{W}(\mathcal{F})\times G)(U) = \mathrm{W}(\mathcal{F})(U)$$

In conclusion $\phi_G^{-1}(x\otimes 1)(\operatorname{Spec}\mu) = (\mathrm{id}_{\mathcal{F}}\otimes\mu)(x\otimes 1)$. Putting everything together we get that $\xi$ is the composition

$$\mathcal{F}\otimes\mathscr{A}\otimes\mathcal{O}_S[G] \xrightarrow{\mathrm{id}_{\mathcal{F}}\otimes\mu} \mathcal{F}\otimes\mathscr{A}\otimes\mathcal{O}_S[G] \simeq \mathcal{F}\otimes\mathcal{O}_S[G]\otimes\mathscr{A} \xrightarrow{\overline{\nu}\otimes\mathrm{id}_{\mathscr{A}}} \mathcal{F}\otimes\mathcal{O}_S[G]\otimes\mathscr{A} \simeq \mathcal{F}\otimes\mathscr{A}\otimes\mathcal{O}_S[G]$$

which induces the classical co-module structure on the tensor product $\mathcal{F}\otimes\mathscr{A}$.

Now let $\mathscr{A} = \mathcal{O}_S[G]$. The map

$$\begin{aligned} \underline{\mathrm{Hom}}^G(G, \mathrm{W}(\mathcal{F})) &\xrightarrow{\eta} \mathrm{W}(\mathcal{F}) \\ \psi &\longmapsto \psi(1) \end{aligned}$$

is an isomorphism. The composition

$$\mathrm{W}(F) \xrightarrow{\eta^{-1}} \underline{\mathrm{Hom}}^G(G, \mathrm{W}(\mathcal{F})) \subseteq \underline{\mathrm{Hom}}(G, \mathrm{W}(\mathcal{F})) \xrightarrow{\phi} \mathrm{W}(\mathcal{F}\otimes\mathcal{O}_S[G])$$

yields a map $\omega\colon \mathcal{F} \longrightarrow \mathcal{F}\otimes\mathcal{O}_S[G]$ and we have to prove that $\omega = \nu$. Again we can assume that $S = \operatorname{Spec} R$, for a ring $R$. Given $x\in\mathrm{W}(\mathcal{F})(R) = \mathcal{F}$ we have

$$\phi_R(\eta_R^{-1}(x)) = [\eta_R^{-1}(x)](\mathrm{id}_G) = \mathrm{id}_G\cdot(x\otimes 1) = \overline{\nu}(x\otimes 1) = \nu(x)$$

as required. □

**Lemma 4.1.2.** *Let $R$ be a ring and $M\in\mathrm{QCoh}^G R$. Then there exists a $G$-equivariant presentation*

$$(R[G]^\vee)^{\oplus J} \longrightarrow (R[G]^\vee)^{\oplus I} \longrightarrow M \longrightarrow 0$$

*If $M$ is finitely presented, then we can choose $I$ and $J$ finite.*

*Proof.* From 4.1.1 we have an isomorphism

$$M \xrightarrow{\phi} \mathrm{Hom}^G(R[G]^\vee, M) \simeq (R[G] \otimes M)^G$$

and it is easy to check that $\phi_m(\varepsilon_G) = m$, where $m \in M$. Then

$$\bigoplus_{m \in M} \phi_m \colon (R[G]^\vee)^{\oplus M} \longrightarrow M$$

is $G$-equivariant and surjective. If $M$ is finitely presented, obviously we can assume $I$ finite. Let

$$K = \mathrm{Ker}((R[G]^\vee)^I \longrightarrow M)$$

Since $R[G]$ is locally free, $K$ is locally finitely presented and therefore finitely presented. □

**Definition 4.1.3.** The group scheme $G$ is linearly reductive over $S$ if the functor of invariants

$$(-)^G \colon \mathrm{QCoh}^G S \longrightarrow \mathrm{QCoh}\, S$$

is exact.

From now on we will assume that $G$ is linearly reductive. Remember that this condition is stable under base change and is local in the fppf topology ( see [AOV08, Proposition 2.6]).

The following lemmas are crucial in the study of the representation theory of linearly reductive groups over general schemes and they explain how invariants behave for such groups.

**Lemma 4.1.4.** *Let $\mathcal{F} \in \mathrm{QCoh}^G S$ and $\mathcal{H} \in \mathrm{QCoh}\, S$. Then the natural map*

$$\mathcal{F}^G \otimes \mathcal{H} \longrightarrow (\mathcal{F} \otimes \underline{\mathcal{H}})^G$$

*is an isomorphism.*

*Proof.* We can assume that $S = \mathrm{Spec}\, A$, for some ring $A$, and that $\mathcal{F} = \widetilde{M}$, $\mathcal{H} = \widetilde{N}$, for some $A$-modules $M$, $N$. The natural map $M^G \otimes_A N \longrightarrow (M \otimes_A N)^G$ is an isomorphism, bacause it is so when $N$ is free and in general taking a presentation of $N$, taking into account that $(-)^G$ is an exact functor. □

**Lemma 4.1.5.** *Let $\mathcal{F} \in \mathrm{FCoh}^G S$. Then the map $\mathcal{F}^G \longrightarrow \mathcal{F}$ splits locally, $\mathcal{F}^G \in \mathrm{FCoh}\, S$ and the natural map*

$$\mathrm{W}(\mathcal{F}^G) \longrightarrow \mathrm{W}(\mathcal{F})^G$$

*is an isomorphism. In particular if $\mathcal{F}$ is locally free so is $\mathcal{F}^G$.*

*Proof.* The second map in the statement is an isomorphism thanks to 4.1.4. For the local splitting and the finite presentation of $\mathcal{F}^G$, we can assume $S = \mathrm{Spec}\, A$, $\mathcal{F} = \widetilde{M}$ where $A$ is a ring and $M$ a finitely presented $A$-module. Moreover, because $(-)^G$ is invariant by any base change, we can also assume that $A$ is noetherian. In this case the splitting follows from [Mat89, Theorem 7.14]. □

We now recover the property that usually is used as definition of linearly reductive groups over a field:

**Lemma 4.1.6.** *Assume $S = \operatorname{Spec} k$, where $k$ is a field. Then any finite dimensional representation of $G$ is a direct sum of irreducible representations.*

*Proof.* Any $G$-equivariant injection $V \longrightarrow W$ of representations has a $G$-equivariant section since the maps

$$\operatorname{Hom}(W,V) \longrightarrow \operatorname{Hom}(V,V),\ \operatorname{Hom}^G(W,V) \longrightarrow \operatorname{Hom}^G(V,V)$$

are surjective. □

**Definition 4.1.7.** Given a finite, linearly reductive group $G$ over a field we denote by $I_G$ a set of representatives of the irreducible representations of $G$. We will often say that $I_G$ is the "set" of irreducible representations of $G$ or refer to such a "set", always meaning that we are choosing a set of representatives.

We want now to find sheaves over $S$ that play the role of the irreducible representations over a field.

**Definition 4.1.8.** Given $\mathcal{F} \in \operatorname{QCoh}_R^G$ and $V \in \operatorname{Loc}^G S$ we define

$$\theta_V^{\mathcal{F}} \colon V^\vee \otimes \underline{\operatorname{Hom}}^G(V^\vee, \mathcal{F}) \longrightarrow \mathcal{F},\ \theta_V^{\mathcal{F}}(x \otimes \psi) = \psi(x)$$

Note that this map is $G$-equivariant.

*Remark* 4.1.9. If $V \in \operatorname{Loc}^G S$ we have $G$-equivariant morphisms and commutative diagrams

$$\begin{tikzcd}
(u \longrightarrow \psi(u)x) \otimes v & & \phi \otimes v \arrow[r] & \phi(v) \\
& \underline{\operatorname{Hom}}^G(V,\mathcal{F}) \otimes V \arrow[r, hook] \arrow[d, phantom, "\wr"] & \underline{\operatorname{Hom}}(V,\mathcal{F}) \otimes V \arrow[r] \arrow[d, phantom, "\wr"] & \mathcal{F} \\
& (V^\vee \otimes \mathcal{F})^G \otimes V \arrow[r, hook] & V^\vee \otimes \mathcal{F} \otimes V \arrow[ur] & \psi(v)x \\
\psi \otimes x \otimes v \arrow[uuu] & & \psi \otimes x \otimes v \arrow[ur] &
\end{tikzcd}$$

Given a collection $I \subseteq \operatorname{Loc}^G S$ we have a natural, $G$-equivariant morphism

$$\eta_{I,\mathcal{F}} = \bigoplus_{V \in I} \theta_{V^\vee}^{\mathcal{F}} \colon \bigoplus_{V \in I} \underline{\operatorname{Hom}}^G(V, \mathcal{F}) \otimes V \longrightarrow \mathcal{F} \qquad \forall\, \mathcal{F} \in \operatorname{QCoh}^G S$$

**Proposition 4.1.10.** *Let $I$ be a collection of elements of $\operatorname{Loc}^G S$. The following are equivalent:*

1) *the natural maps*

$$\eta_{I,\mathcal{F}} \colon \bigoplus_{V \in I} \underline{\operatorname{Hom}}^G(V, \mathcal{F}) \otimes V \longrightarrow \mathcal{F} \qquad \forall\, \mathcal{F} \in \operatorname{FCoh}^G S$$

*are isomorphisms;*

*2) same as 1), but for any $\mathcal{F} \in \mathrm{QCoh}^G T$ and any $S$-scheme $T$.*

*3) for any algebraically closed field $k$ and geometric point $\operatorname{Spec} k \longrightarrow S$ the map*

$$I \xrightarrow{-\otimes k} I_{G_k}$$

*is well defined and bijective;*

*4) If $S$ is connected, same as 3) but for just one geometric point.*

*If $S = \operatorname{Spec} k$, then $I_G$ satisfies the above conditions if and only if $\operatorname{End}^G(V) \simeq k$ for any $V \in I_G$.*

*Proof.* Assume first that $S = \operatorname{Spec} k$, where $k$ is a field. Given $V, W \in I_G$, we have that any equivariant map $V \longrightarrow W$ is either 0 or an isomorphism. So $\eta_{I_G,V}$ is an isomorphism if and only if $\operatorname{End}^G(V) \simeq k$. Conversely, if this holds for any irreducible representation of $G$, then $\eta_{I_G,*}$ is an isomorphism for all finite representations since these are direct sums of irreducible representations.

Now consider the general case and given a geometric point $\operatorname{Spec} k \longrightarrow S$ denote $I_k = \{V \otimes k \mid V \in I\}$. Since $G$ is linearly reductive and thanks to 4.1.5, we can conclude that $\eta_{I,\mathcal{F}} \otimes k = \eta_{I_k,\mathcal{F}\otimes k}$ for any $\mathcal{F} \in \mathrm{FCoh}^G S$ and that $\underline{\mathrm{Hom}}^G(V, \mathcal{F})$ is locally free if $\mathcal{F}$ is so.

2) $\Longrightarrow$ 1) and 3) $\Longrightarrow$ 4). Obvious.

1) $\Longrightarrow$ 3). If $V \in I$, since $\eta_{I,V}$ is an isomorphism and $\underline{\mathrm{Hom}}^G(V, V) \otimes V \longrightarrow V$ is surjective, we have that $\underline{\mathrm{Hom}}^G(V, W) = 0$ if $V \neq W \in I$ and that $\underline{\mathrm{Hom}}^G(V, V)$ is an invertible sheaf. In particular $\operatorname{End}^{G_k}(V \otimes k) \simeq k$ and $V \otimes k$ is therefore irreducible for any geometric point $\operatorname{Spec} k \longrightarrow S$, so that $I_k \subseteq I_{G_k}$. For the converse let $W \in I_{G_k}$. Since $W^\vee \simeq (W^\vee \otimes k[G])^G \simeq \operatorname{Hom}^G(W, k[G])$ by 4.1.1, there exists a $G$-equivariant nonzero map $W \longrightarrow k[G] \simeq \mathcal{O}_S[G] \otimes k$, which is injective because $W$ is irreducible. On the other hand the only irreducible representations of $G_k$ appearing in $k[G]$ are the ones in $I_k$ since $\eta_{I,\mathcal{O}_S[G]} \otimes k = \eta_{I_k,k[G]}$.

3) $\Longrightarrow$ 2) We can assume that $T = \operatorname{Spec} R$, where $R$ is a ring. If $M \in \mathrm{QCoh}^G R$ is locally free of finite rank then $\eta_{I,M} \otimes k = \eta_{I_{G_k},M\otimes k}$ is an isomorphism for any geometric point $\operatorname{Spec} k \longrightarrow T$. Since both source and target of the map $\eta_{I,M}$ are locally free thanks to 4.1.5, we can conclude that it is an isomorphism. In particular $\eta_{I,R[G]^\vee}$ is an isomorphism. If $M \in \mathrm{QCoh}^G R$, thanks to 4.1.2 we have a $G$-equivariant presentation $V_1 \longrightarrow V_0 \longrightarrow M \longrightarrow 0$, where the $V_i$ are a direct sum of copies of $R[G]^\vee$. Since $\operatorname{Hom}^G(V, -) \otimes V$ is exact when $V$ is locally free, we have a commutative diagram

$$\begin{array}{ccccccc}
\bigoplus_{V\in I} \underline{\mathrm{Hom}}^G(V, V_1) \otimes V & \longrightarrow & \bigoplus_{V\in I} \underline{\mathrm{Hom}}^G(V, V_0) \otimes V & \longrightarrow & \bigoplus_{V\in I} \underline{\mathrm{Hom}}^G(V, M) \otimes V & \longrightarrow & 0 \\
\downarrow \eta_{I,V_1} & & \downarrow \eta_{I,V_0} & & \downarrow \eta_{I,M} & & \\
V_1 & \longrightarrow & V_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

The $\eta_{I,V_i}$ are isomorphisms by additivity and therefore we can conclude that $\eta_{I,M}$ is an isomorphism as well.

4) $\Longrightarrow$ 3) Let $\operatorname{Spec} k_0 \longrightarrow S$ be the given geometric point. For $V, W \in I$ we have that $\underline{\mathrm{Hom}}^G(V,W)$ are locally free and checking the rank on $k_0$, we can conclude that $\underline{\mathrm{Hom}}^G(V,W) = 0$ if $V \neq W$ and that $\underline{\mathrm{Hom}}^G(V,V)$ is invertible. In particular $I_k \subseteq I_{G_k}$ for any geometric point and therefore $\eta_{I_k,k[G]}$ is injective since $\eta_{I_{G_k},k[G]}$ is so. But

$$\begin{aligned}\dim_k \bigoplus_{V\in I} \mathrm{Hom}^G(V\otimes k, k[G]) \otimes V \otimes k &= \dim_{k_0} \bigoplus_{V\in I} \mathrm{Hom}^G(V\otimes k_0, k[G]) \otimes V \otimes k_0 \\ &= \dim_{k_0} k_0[G] = \dim_k k[G]\end{aligned}$$

so that $\eta_{I_k,k[G]}$ is an isomorphism and $I_k = I_{G_k}$. □

**Proposition 4.1.11.** *Let $I$ be a collection of elements of $\mathrm{Loc}^G S$ satisfying the conditions in 4.1.10. Then*

$$\underline{\mathrm{Hom}}^G(V,W) = 0,\ \underline{\mathrm{Hom}}^G(V,V) = \mathcal{O}_S \mathrm{id}_V \quad \textit{for all } V \neq W \in I$$

*If $S$ is connected then $I$ is uniquely determined up to tensorization by invertible sheaves (with trivial actions).*

*Proof.* Assume $S$ connected. If we tensor the sheaves in $I$ by invertible sheaves, we do not change their restrictions to the geometric points. Conversely let $I'$ be another collection satisfying the condition in 4.1.10. Given $W \in I'$, there exists $V \in I$ such that $\underline{\mathrm{Hom}}^G(V,W) \neq 0$. The sheaf $\underline{\mathrm{Hom}}^G(V,W)$ is locally free thanks to 4.1.5. Changing the base to all the geometric points of $S$, we see that the map $\underline{\mathrm{Hom}}^G(V,W) \otimes V \longrightarrow W$ has to be surjective, that $\underline{\mathrm{Hom}}^G(V,W)$ has rank 1 and that $\operatorname{rk} V = \operatorname{rk} W$. In this way we see that $V$ and $W$ differ by an invertible sheaf and that $V$ is uniquely determined by $W$.

Now consider the locally free sheaf $\underline{\mathrm{Hom}}^G(V,W)$ for $V, W \in I$. If $V \neq W$ then this sheaf is 0 because $I \longrightarrow I_{G_k}$ is bijective and therefore $V \otimes k \not\simeq W \otimes k$ for any geometric point $\operatorname{Spec} k \longrightarrow S$. Finally, if $V = W$, we see that $\mathrm{id}_V$ generates $\underline{\mathrm{Hom}}^G(V,V)$ in any geometric point. □

**Definition 4.1.12.** We will say that $G$ has a *good representation theory* over $S$ if it admits a collection $I$ as in 4.1.10. We will briefly call a *glrg* (good linearly reductive group) a pair $(G, I_G)$ where $G$ is a finite, flat, finitely presented and linearly reductive group scheme over $S$ and $I_G$ is a collection of elements as in 4.1.10. We will simply write $G$ if this will not lead to confusion. If $T \longrightarrow S$ is a map, then $G_T = G \times_S T$ with the collection of the pullbacks of the sheaves in $I_G$ is a glrg and we will always consider $G_T$ as a glrg with this particular collection.

Note that if $G$ is a glrg then any $V \in I_G$ is not only an irreducible representation, but a geometrically irreducible one. We now show two examples of glrg's.

**Example 4.1.13.** Assume $S = \operatorname{Spec} k$, where $k$ is a field. Then $G$ has a good representation theory if and only if $\mathrm{End}^G(V) \simeq k$ for all the irreducible representations of $G$ and in this case, up to isomorphism, the only collection $I$ satisfying 4.1.10 is $I_G$, the set of irreducible representations of $G$. In particular any linearly reductive group $G$ over an algebraically closed field is a glrg. Indeed if $W$ is an irreducible representation of $G$ then $\mathrm{Hom}^G(V,W) \neq 0$ for some $V \in I$, since $\eta_{I,W}$ is an isomorphism. So $W \simeq V$.

**Example 4.1.14.** If $G$ is a diagonalizable group with group of characters $M = \mathrm{Hom}_{\mathrm{grp}}(G, \mathbb{G}_m)$, then $G$ has a good representation theory over $\mathrm{Spec}\,\mathbb{Z}$ and we can choose as $I_G$ the set of representations $\mathbb{Z}_m$ given by $\mathbb{Z}_m \longrightarrow \mathbb{Z}_m \otimes \mathbb{Z}[G]$, $1 \longrightarrow 1 \otimes m$.

The definition of good linearly reductive group is just what we need in order to have a representation theory for which coherent sheaves with an action of $G$ are just, functorially, a collection of coherent sheaves. The correct statement, which easily follows from the definition of glrg and from 4.1.11, is the following:

**Proposition 4.1.15.** *If $G$ is a glrg, then the functors below are quasi-inverse equivalences of categories*

$$\begin{array}{ccc} \bigoplus_{V\in I_G} V^\vee \otimes \mathcal{F}_V & \longleftarrow\!\shortmid & (\mathcal{F}_V)_{V\in I_G} \\ \mathrm{QCoh}_S^G & \longrightarrow & \mathrm{QCoh}_S^{I_G} \\ \mathcal{F} & \longmapsto & ((V\otimes\mathcal{F})^G)_{V\in I_G} \end{array}$$

*The same statement holds if we replace* QCoh *by* FCoh *or* Loc.

**Example 4.1.16.** When $G$ is a finite diagonalizable group with group of characters $M = \mathrm{Hom}(G, \mathbb{G}_m)$ and $R = \mathbb{Z}$, since $I_G$ is in bijection with $M$, we retrieve the classical equivalence between $\mathrm{QCoh}_R^G$ and the stack of $M$-graded quasi-coherent sheaves.

The following result extends the usual result for linearly reductive groups over an algebraically closed field.

**Proposition 4.1.17.** *If $G$ is a glrg, then we have an isomorphism*

$$\mathcal{O}_S[G] \simeq \bigoplus_{V\in I_G} \underline{V}^\vee \otimes V$$

*Proof.* By 4.1.1, we have

$$\underline{\mathrm{Hom}}^G(V, \mathcal{O}_S[G]) \simeq (V^\vee \otimes \mathcal{O}_S[G])^G \simeq V^\vee$$

$\square$

We state here the subsequent lemma, although we will use it in the following sections.

**Lemma 4.1.18.** *Given $V \in \mathrm{Loc}^G R$, the composition*

$$V^\vee \otimes V \xrightarrow{\simeq} \mathrm{Hom}^G(V, R[G]) \otimes V \xrightarrow{\theta_{V^\vee}^{R[G]}} R[G] \xrightarrow{\varepsilon_G} R$$

*is the evaluation $e_V\colon V^\vee \otimes V \longrightarrow R$, $e_V(\phi\otimes v) = \phi(v)$. In particular if $G$ is a glrg we have*

$$\varepsilon_G = \bigoplus_{V\in I_G} e_V \colon \bigoplus_{V\in I_G} V^\vee \otimes V \simeq R[G] \longrightarrow R$$

*Proof.* The statement is local on $R$, so we can assume that $R[G]$ is free with basis $\{w_k\}_k$. If $\mu\colon V \longrightarrow V \otimes R[G]$ is the structure map of $V$, then the structure map $\nu\colon V^\vee \longrightarrow V^\vee \otimes R[G]$ has the expression

$$\nu(\phi) = \sum_k (\phi \otimes w_k^* \circ \mu) \otimes w_k$$

The composition in the statement can be written as

$$f_V\colon V^\vee \otimes V \xrightarrow{\nu \otimes \mathrm{id}} (V^\vee \otimes R[G]) \otimes V \simeq (V^\vee \otimes V) \otimes R[G] \xrightarrow{e_V \otimes \varepsilon_G} R$$

So

$$f_v(\phi \otimes v) = e_V \otimes \varepsilon_G(\sum_k (\phi \otimes w_k^* \circ \mu) \otimes v \otimes w_k) = \sum_k \varepsilon_G(w_k)\phi \otimes w_k^*(\mu(v)) \in R$$

Moreover we can write

$$\mu(v) = \sum_k v_k \otimes w_k \text{ and } v = \sum_k \varepsilon_G(w_k) v_k$$

and therefore $\phi \otimes w_k^*(\mu(v)) = \phi(v_k)$ and $f_v(\phi \otimes v) = \phi(v)$. □

We do not know an explicit characterization of glrg's among the linearly reductive groups. On the other hand we are going to prove that any finite, flat and finitely presented linearly reductive group $G$ is fppf locally a glrg. So, up to fppf base change, we can always assume that we have a collection $I_G$ of geometrically irreducible representations and therefore a simpler representation theory. If moreover $G$ is étale, we will show that $G$ is also étale locally a glrg. In particular we will conclude that if $G$ is étale and defined over a strictly Henselian ring then it is a glrg.

**Lemma 4.1.19.** *Let $\mathcal{X}$ be a proper and flat algebraic stack over a noetherian local ring $R$. Denote by $k$ the residue field of $R$ and consider a locally free sheaf $V_0$ of rank $n$ over $\mathcal{X} \times k$. If $\mathrm{H}^2(\mathcal{X} \times k, \underline{\mathrm{End}}(V_0)) = 0$, then there exists a locally free sheaf of rank $n$ over $\mathcal{X} \times \widehat{R}$ lifting $V_0$, where $\widehat{R}$ is the completion of $R$.*

*Proof.* Taking into account Grothendieck's existence theorem for proper stacks, we can assume that $R$ is an Artinian ring (so that $\widehat{R} \simeq R$) and that we have a lifting $\overline{V}$ of $V_0$ over $\mathcal{X} \times (R/I)$, where $I$ an ideal of $R$ such that $I^2 = 0$. Define the stack $\mathcal{Y}$ over the fppf site $\mathcal{X}_{\mathrm{fppf}}$ of $\mathcal{X}$ whose objects over $\operatorname{Spec} B \longrightarrow \mathcal{X}$ are locally free sheaves $N$ of rank $n$ over $B$ with an isomorphism $\phi\colon N \otimes (B/IB) \longrightarrow \overline{V} \otimes (B/IB)$. A section of $\mathcal{Y} \longrightarrow \mathcal{X}_{\mathrm{fppf}}$ yields a lifting of $\overline{V}$ on $\mathcal{X}$. We are going to prove that $\mathcal{Y}$ is a gerbe over $\mathcal{X}_{\mathrm{fppf}}$ banded by the sheaf of abelian groups $\pi_* \underline{\mathrm{End}}(V_0)$, where $\pi\colon \mathcal{X} \times k \longrightarrow \mathcal{X}$ is the obvious closed immersion. Since $\mathrm{H}^2(\mathcal{X}, \pi_* \underline{\mathrm{End}}(V_0)) = \mathrm{H}^2(\mathcal{X} \times k, \underline{\mathrm{End}}(V_0)) = 0$ parametrizes those gerbes (see [Gir71, Chapter IV, §3, Section 3.4]), we can then conclude that $\mathcal{Y} \longrightarrow \mathcal{X}_{\mathrm{fppf}}$ is a trivial gerbe, which means that it has a section as required.

I claim that $\overline{V}$ is trivial in the fppf topology of $\mathcal{X}$, which implies that $\mathcal{Y} \longrightarrow \mathcal{X}_{\mathrm{fppf}}$ has local sections. Indeed if $B$ is a ring and $P \longrightarrow \operatorname{Spec} B/IB$ is a $\mathrm{Gl}_n$-torsor then by standard deformation theory it extends to a smooth map $Q \longrightarrow \operatorname{Spec} B$. In particular, if we base change to $Q$, we can conclude that $P$ over $Q \times (B/IB)$ has a section, which means that it is trivial.

I also claim that two objects of $\mathcal{Y}$ over the same object of $\mathcal{X}_{\mathrm{fppf}}$ are locally isomorphic. Replacing again locally free sheaves by $\mathrm{Gl}_n$-torsors, given $\mathrm{Gl}_n$-torsors $P, Q$ over $\operatorname{Spec} B$, we have to show that an equivariant isomorphism $P \times (B/IB) \longrightarrow Q \times (B/IB)$ locally extends to an equivariant isomorphism $P \longrightarrow Q$. In particular we can assume that $P$ and $Q$ are both trivial and in this case the above property follows because $\mathrm{Gl}_n(B) \longrightarrow \mathrm{Gl}_n(B/IB)$ is surjective, being $\mathrm{Gl}_n$ smooth.

The previous two claims show that $\mathcal{Y} \longrightarrow \mathcal{X}_{\mathrm{fppf}}$ is a gerbe. We have now to check the banding and therefore to compute the automorphism group of an object $(N, \phi) \in \mathcal{Y}$ over a ring $B$. The group $\mathrm{Aut}(\chi)$ consists of the automorphism $N \xrightarrow{\lambda} N$ inducing the identity on $N/IN$. It is easy to check that the map

$$\mathrm{Hom}_B(N, IN) \longrightarrow \operatorname{Aut} \chi, \ \delta \longmapsto \mathrm{id}_N + \delta$$

is an isomorphism of groups. Since $IN = I \otimes_R N$ we have

$$\mathrm{Hom}_B(N, IN) = I \otimes \mathrm{End}_B(N) \simeq I/I^2 \otimes \mathrm{End}_B(N) \simeq \mathrm{End}_{B/m_R B}(M \otimes (B/m_R B))$$

□

**Lemma 4.1.20.** *Assume that $S = \operatorname{Spec} R$, where $R$ is a Henselian ring with residue field $k$ and let $V$ be a representation of $G$ over $\overline{k}$. If $V$ is defined over $k$ then it lifts to $R$.*

*Proof.* Since $G$ is finitely presented, we can assume that $R$ is the Henselization of a scheme of finite type over $\mathbb{Z}$. Since $G$ is linearly reductive, we have that $\mathrm{H}^2(\mathrm{B}(G \times k), -) = 0$ and, thinking $G$-representations as sheaves over $\mathrm{B}\, G$ and using 4.1.19, we obtain a lifting of $V$ to a representation over the completion $\widehat{R}$. We can then conclude using Artin approximation theorem over $R$. □

**Proposition 4.1.21.** *There exists an fppf coverings $\mathcal{U} = \{U_i \longrightarrow S\}_{i \in I}$ such that $G \times_S U_i$ is a glrg over $U_i$. If $G$ is étale there exists an étale covering with the same property.*

*Proof.* We first deal with the case $S = \operatorname{Spec} k$, where $k$ is a field. An irreducible representation $V$ of $G_{\overline{k}}$ is given by a group homomorphism $G_{\overline{k}} \longrightarrow \mathrm{Gl}(V)$. Such a morphism is defined over a finite extension $L/k$. Since $I_{G_{\overline{k}}}$ is finite we get our extension. Now assume that $G$ is étale. If $k$ is perfect we already have our result. So assume $\operatorname{char} k = p > 0$. After passing to a separable extension of $k$ we can assume $G$ constant of order prime to $p$. So $G$ is defined over $\mathbb{F}_p$, which is perfect and again we have our claim.

Now return to the general case. Since $G$ is finitely presented, we can assume $S$ to be of finite type over $\mathbb{Z}$. Let $p \in S$ and $L/k(p)$ an extension such that $G_L$ is a glrg and $L/k(p)$ is separable if $G$ is étale. There exists a flat finitely presented map $h \colon U \longrightarrow S$ such that $f^{-1}(p) \simeq \operatorname{Spec} L$. If $L/k$ is separable we can restrict $U$ and assume $h$ to be étale. This shows that we can assume that $G_{k(p)}$ is a glrg. Now let $R$ be the Henselization of $\mathcal{O}_{S,p}$. From 4.1.20 any $G_{k(p)}$ representation lifts to $R$ and, since $R$ is a direct limit of algebras whose spectrum is étale over $S$, we get the required result. □

Putting together 4.1.20 and 4.1.21 we get:

**Theorem 4.1.22.** *A constant linearly reductive group over a strict Henselian ring has a good representation theory.*

### 4.1.2 Linearly reductive groups over strictly Henselian rings.

In this section we will study the structure and the actions of a flat, finite and finitely presented linearly reductive group $G$ in the special case when the base scheme is the spectrum of a strictly Henselian ring. In particular we will describe the decomposition into unions of connected components of a finite scheme with an action of $G$ and the structure of the connected component of $G$ containing the identity.

Through this subsection we will assume $S = \operatorname{Spec} R$, where $R$ is a strictly Henselian ring. Again $G$ will be a finite, flat, linearly reductive group over $R$. We start with:

**Lemma 4.1.23.** *If $A, B$ are local $R$-algebras with $A$ finite, then $A \otimes_R B$ is local.*

*Proof.* Set $k_A$, $k_B$ for their residue fields. Since $A \otimes_R B$ is finite over $B$ it is enough to note that $k_A \otimes_{k_R} k_B$ is local since $k_A/k_R$ is purely inseparable. □

**Proposition 4.1.24.** *We have an exact sequence*

$$0 \longrightarrow G_1 \longrightarrow G \longrightarrow \underline{G} \longrightarrow 0$$

*where $G_1$ is the connected component of $G$ containing $1$ and $\underline{G}$ is a constant group. Moreover, if $p$ is the characteristic of the residue field of $R$, then $p \nmid |\underline{G}|$, $G_1$ is diagonalizable and its group of characters $\underline{\operatorname{Hom}}(G_1, \mathbb{G}_m)$ is a $p$-group. The decomposition of $G$ into connected components is of the form*

$$G = \bigsqcup_{i \in \underline{G}} G_i$$

*If $G$ acts on a finite $R$-scheme $X$, then it acts on the connected components of $X$ and this action factors through $\underline{G}$. Moreover the stabilizers of the connected components of $X$ are union of connected components of $G$.*

*Proof.* Let $G = \sqcup_{i \in I} G_i$ and $X = \sqcup_{j \in J} X_j$ be the decomposition into connected components of $G$ and $X$ respectively and let $\mu\colon X \times G \longrightarrow X$ the action of $G$ on $X$. Since $X_j \times G_i$ is connected, there exists a unique $k_{j,i} \in J$ such that $\mu(X_j \times G_i) \subseteq X_{k_{j,i}}$. Assume now that $X = G$ with the regular representation. Since $\mu$ is isomorphic to the projection $G \times G \longrightarrow G$, it is flat and finite, so $\mu(G_j \times G_i) = G_{k_{j,i}}$ is a connected component of $G$. Define a product on $I$ by $i \cdot j = k_{i,j}$. It is easy to check that $I$ is a group, whose neutral element $1 \in I$ is the index of the connected component of the identity. Set $\underline{G} = I$. The map $G \longrightarrow \underline{G}$ is surjective and the kernel is exactly $G_1$. Since both $G_1$ and $\underline{G}$ are linearly reductive, we can conclude that $G_1$ is diagonalizable and that $p \nmid |\underline{G}|$ by [AOV08, Lemma 2.20]. Set $M = \underline{\operatorname{Hom}}(G_1, \mathbb{G}_m)$ and $k$ for the residue field of $R$. If $\mathbb{Z}/q\mathbb{Z} < M$, then we have a surjective morphism $G_1 \longrightarrow \mu_{q,R}$. So $\mu_{q,R}$ has to be connected and, since it is finite and flat, $\mu_{q,k}$ is connected as well. But if $q \neq p$ then $\mu_{q,k} \simeq \mathbb{Z}/q\mathbb{Z}$. Therefore $M$ is a $p$-group.

Now return to the general case, i.e. when $X$ is a finite $R$-scheme. Since $\mu$ is an action, then $k_{*,*}$ defines an action of $\underline{G}$ on $J$. Moreover if $g \in G_i(T)$ we have that $(X_j \times T)g \subseteq X_{k_{j,i}}$ and therefore this is an equality since $J$ is finite. In particular

$$\operatorname{Stab} X_j = \bigsqcup_{i \in \underline{G} \ | \ k_{j,i}=j} G_i$$

□

*Notation* 4.1.25. We will continue to denote by $G_1$ the connected component of $G$, by $\underline{G}$ the constant group $G/G_1$ and by $M = \underline{\operatorname{Hom}}(G_1, \mathbb{G}_m)$ the group of characters of $G_1$. Given an index $i \in \underline{G}$ we will also denote by $G_i$ the connected component of $G$ corresponding to such index.

**Corollary 4.1.26.** *If $R = k$ is an algebraically closed field, then $G \longrightarrow \underline{G}$ has a unique section. In particular*

$$G \simeq G_1 \ltimes \underline{G}$$

*Proof.* Set $p = \operatorname{char} k$. If $p = 0$ then $G = \underline{G}$. So assume $p \neq 0$ and let $G_i$ be a connected component of $G$. If we prove that $|G_i(k)| = 1$ then $G(k) \longrightarrow \underline{G}$ is an isomorphism (of constant groups) and the section is unique. Since $k$ is algebraically closed, we have $G_i(k) \neq \emptyset$. In particular $G_i \simeq G_1$. But

$$G_1(k) = \operatorname{Hom}_{(\mathrm{Grps})}(M, k^*) = 0$$

since $M$ is a $p$-group. □

Now we want to study the open subgroups of $G$.

*Remark* 4.1.27. If $G$ is linearly reductive as we are assuming, then a subgroup scheme is again a finite, flat and of finite presentation linearly reductive group scheme (see [AOV08, Proposition 2.7]).

**Proposition 4.1.28.** *Let $H$ be an open and closed subgroup of $G$ and set*

$$H^i = \bigsqcup_{j \in \underline{H} i} G_j \text{ where } i \in \underline{G}$$

*The schemes $H^i$ are stable under the right action of $H$ on $G$ and they are fppf $H$-torsors. Moreover if $g \in H^i(T)$, then $H^i \times T = (H \times T)g$.*

*Proof.* If $h \in \underline{H}$ and $j \in \underline{H}i$ then $G_j \star G_h = G_{j \star h} = G_{h^{-1}j} \subseteq H^i$, where $\star$ denote the regular representation, so $H^i$ is $H$-stable. Since $G$ is flat and finite, $H^i$ has section in the fppf topology, so we have to prove only the last claim, since the multiplication by $g$ $H \times T \longrightarrow (H \times T)g$ is $H$-equivariant. Let $g \in H^i(T)$. We can assume that $g \in G_j(T)$ for $j \in \underline{H}i$. In this case it is enough to note that $(G_h \times T)g = G_{hj} \times T$. □

We state the following lemma here, although it will be used in the following sections.

**Lemma 4.1.29.** *Let $X$ be a finite $R$-scheme with an action of $G$. Then $X/G_1$ has the same connected components as $X$.*

*Proof.* We have to prove that if $(A, m_A)$ is a local and finite $R$-algebra with an action of a diagonalizable group $D(H)$, then $A_0$ is local. So we have to prove that any $x \in A_0 - m_A \cap A_0$ is invertible in $A_0$. Since $x \notin m_A$ there exists $y \in A$ such that $xy = 1$. Writing $y$ with respect to the decomposition $A = \bigoplus_{h\in H} A_h$ we get

$$y = \sum_{h\in H} y_h \implies 1 = xy = \sum_{h\in H} xy_h \implies xy_0 = 1$$

□

### 4.1.3 Induction and $G$-equivariant algebras.

One of the key points in the study of $G$-covers in the following sections is the fact that each such cover, locally (at least on a strict Henselization), can be described from an $H$-cover, where $H$ is a proper subgroup of $G$, having some extra properties. Algebraically, this procedure is obtained through an induction from $H$ to $G$. So in this section we will introduce the concept of induction from a subgroup, state some of its properties and then we will focus on induction of algebras.

Throughout this section we will assume $S = \operatorname{Spec} R$, where $R$ is a ring and $G$ will be as always a finite, flat and finitely presented group scheme over $R$.

*Remark* 4.1.30. Let $H$ be a subgroup of $G$ and $F\colon (\mathrm{Sch}/S)^{op} \longrightarrow (\mathrm{Sets})$ be a functor with a left action of $H$. Regarding $G$ as a $H$-space via the restriction of the regular representation, we define

$$\operatorname{ind}_H^G F = \underline{\operatorname{Hom}}^H(G, F)$$

We endow $\operatorname{ind}_H^G F$ with the following left action of $G$. The group $G$ acts on the right on itself through the product $G \times G \xrightarrow{m} G$ and, considering the trivial action of $G$ on $F$, we get a left action of $G$ on $\underline{\operatorname{Hom}}(G, F)$ that restricts to a left action of $G$ on $\operatorname{ind}_H^G F$.

Concretely, given $f\colon G \longrightarrow F \in \underline{\operatorname{Hom}}(G, F)$ we have that

$$f \in \operatorname{ind}_H^G F = \underline{\operatorname{Hom}}^H(G, F) \iff f(hg) = hf(g) \text{ for all } h \in H$$

and if $g \in G$ then

$$(g \star f)(t) = f(tg)$$

**Definition 4.1.31.** If $H$ is a subgroup scheme of $G$ and $\mathcal{F} \in \mathrm{FCoh}^H$ we have (see 4.1.1)

$$\mathrm{W}((\mathcal{F} \otimes \mathcal{O}[G])^H) \simeq \underline{\operatorname{Hom}}^H(G, \mathrm{W}(\mathcal{F})) = \operatorname{ind}_H^G \mathrm{W}(\mathcal{F})$$

So we can define

$$\operatorname{ind}_H^G \mathcal{F} = (\mathcal{F} \otimes \mathcal{O}[G])^H \in \mathrm{FCoh}^G$$

with the action given by the isomorphism $\mathrm{W}(\operatorname{ind}_H^G \mathcal{F}) \simeq \operatorname{ind}_H^G \mathrm{W}(\mathcal{F})$.

The following is a well known property of adjunction between induction and restriction.

**Proposition 4.1.32.** *[Jan87, section 3.3] If $H$ is a flat subgroup scheme of $G$ and $V \in \mathrm{FCoh}^G$, $W \in \mathrm{FCoh}^H$, we have an isomorphism*

$$\underline{\mathrm{Hom}}^H(\mathrm{R}_H V, W) \simeq \underline{\mathrm{Hom}}^G(V, \mathrm{ind}_H^G W)$$

We now pass to the study of induction of finite algebras with an action of $G$. From now to the end of the section $G$ will be assumed linearly reductive.

**Definition 4.1.33.** We will denote by $\mathrm{CAlg}^G R$ the category of finite $R$-algebras $A$ with a left action of $G$ on them, or, equivalently, a right action of $G$ on $\mathrm{Spec}\, A$.

**Lemma 4.1.34.** *If $R$ is strictly Henselian, $H$ is an open and closed subgroup of $G$ and $A \in \mathrm{CAlg}^H R$ then*

$$\mathrm{ind}_H^G A \simeq \prod_{i \in \underline{G/H}} B_i$$

*as rings, where the $B_i$ are fppf locally isomorphic to $A$. More precisely, if $R'$ is an $R$-algebra and $g \in G_i(R')$ then we have an induced isomorphism*

$$\begin{tikzcd} \mathrm{ind}_H^G A \otimes R' \arrow[r, "g"] \arrow[d] & \mathrm{ind}_H^G A \otimes R' \arrow[d] \\ B_i \otimes R' \arrow[r, dashed, "\simeq"] & A \otimes R' \end{tikzcd}$$

*Proof.* We will make use of 4.1.28. The inclusions $H^i \longrightarrow G$ induce an isomorphism of functors

$$\underline{\mathrm{Hom}}^H(G, \mathrm{W}(A)) \longrightarrow \prod_i \underline{\mathrm{Hom}}^H(H^i, \mathrm{W}(A))$$

So we can set $B_i$ for the coherent algebra such that $\mathrm{W}(B_i) \simeq \underline{\mathrm{Hom}}^H(H^i, \mathrm{W}(A))$. Since $H^i$ is an fppf $H$-torsor, $B_i$ is fppf locally isomorphic to $A$. For the last claim, note that $H^i \times R' = (H \times R')g$ and therefore it is enough to apply $\underline{\mathrm{Hom}}^H(-, \mathrm{W}(A))$ to the commutative diagram of $H$-spaces

$$\begin{tikzcd} H \arrow[r] \arrow[d] & G \arrow[d] & x \arrow[d, mapsto] \\ Hg \arrow[r] & G & xg \end{tikzcd}$$

$\square$

**Lemma 4.1.35.** *Let $R$ be a local ring and $A \in \mathrm{CAlg}^G R$ such that $A^G = R$. If $G$ is constant then it acts transitively on the maximal ideals of $A$.*

*Proof.* Let $p, q \in \operatorname{Spec} A$ be closed points and assume by contradiction that for any $g \in G$, $q \neq g(p)$. In particular we cannot have $q \subseteq \cup_{g\in G} g(p)$ and therefore there exists $x \in q$ such that $g(x) \notin p$ for any $g \in G$. But

$$\prod_{g\in G} g(x) \in q \cap A^G = q \cap R = m_R \subseteq p \implies \exists g \in G \mid g(x) \in p$$

$\square$

The following proposition is one of the key points in the study of the structure of covers and we will use it many times in the following sections. It roughly means that the whole algebra (over which $G$ acts) can be recovered from a local algebra (over which acts a particular subgroup of $G$) through induction. In particular it allows us to reduce problems to local algebras, when we have to deal with properties that behave well under induction.

**Proposition 4.1.36.** *Assume that $R$ is strictly Henselian and let $A \in \operatorname{CAlg}^G R$ be such that $A^G = R$ and $p \in \operatorname{Spec} A$ be a closed point. Denote by $H_p$ the stabilizer of the connected component $\operatorname{Spec} A_p$ of $\operatorname{Spec} A$. Then we have a $G$-equivariant isomorphism*

$$A \longrightarrow \operatorname{ind}_{H_p}^G A_p$$

*Proof.* Set $H = H_p$. The map $A \longrightarrow A_p$ is $H$-equivariant and therefore we get a map $A \xrightarrow{\psi} \operatorname{ind}_H^G A_p$. Write $X_q = \operatorname{Spec} A_q$ for a closed point $q$ of $\operatorname{Spec} A$. Those are the connected components of $X = \operatorname{Spec} A$. Let also $Y = \operatorname{Spec}(\operatorname{ind}_H^G A_p)$, $\operatorname{ind}_H^G A_p = \prod_{i\in \underline{G}/\underline{H}} B_i$ and $Y_i = \operatorname{Spec} B_i$. Assume $X_p G_i = X_q$, where $i \in \underline{G}$. Since $Y_1$ is mapped to $X_p$ and $Y_1 G_i = Y_i$ we have a decomposition

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d] & \operatorname{ind}_H^G A_p \arrow[d] \\ A_q \arrow[r, "\psi_q"] & B_i \end{tikzcd}$$

We have to prove that all the maps $\psi_q$ are isomorphisms and that $G$ acts transitively on the connected components of $X$.

If $X_p G_i = X_q$, $R'$ is an fppf $R$-algebra and $g \in G(R')$ we have a commutative diagram

$$\begin{tikzcd} & A \otimes R' \arrow[r, "g"] \arrow[d, "\psi\otimes R'"] \arrow[dl, bend right] & A \otimes R' \arrow[d, "\psi\otimes R'"] \\ & \operatorname{ind}_H^G A_p \otimes R' \arrow[r, "g"] \arrow[d] & \operatorname{ind}_H^G A_p \otimes R' \arrow[d] \\ A_q \otimes R' \arrow[r, "\psi_q\otimes R'"] & B_i \otimes R' \arrow[r, "u"] & A_p \otimes R' \end{tikzcd}$$

Since $G$ permutes the connected components of $X$, thanks to 4.1.34, the composition $u \circ (\psi_q \otimes R')$ is an isomorphism. Since also $u$ is an isomorphism we can conclude that $\psi_q \otimes R'$ and therefore $\psi_q$ is an isomorphism.

It remains to prove that $G$ acts transitively on the connected components of $X$. Since $Z = X/G_1$ has the same connected components as $X$ for 4.1.29, $\underline{G}$ acts on $Z$ and $Z/G = \operatorname{Spec} R$, we can assume $G = \underline{G}$. In this case the conclusion follows from 4.1.35. $\square$

## 4.2 Equivariant sheaves and functors.

Given a glrg $G$ over a ring $R$, proposition 4.1.10 tells us that a $G$-equivariant quasi-coherent sheaf $\mathcal{F}$ over an $R$-scheme $T$ is determined by a collection of quasi-coherent sheaves on $T$ indexed by $I_G$, namely $\{(V \otimes \mathcal{F})^G\}_{V \in I_G}$. Since we are mainly interested in affine maps of schemes, it is natural to ask what additional structure a collection of sheaves as above must have in order to correspond to a quasi-coherent sheaf of algebras. We will answer this question but, in order to do that, it will be convenient to associate to a sheaf $\mathcal{F}$ not only a collection, but a whole functor $\Omega^{\mathcal{F}} = (- \otimes \mathcal{F})^G$ from the category of locally free and finite $G$-representations $\operatorname{Loc}^G R$ to the category of quasi-coherent sheaves. This has the advantage of making sense for any finite, flat and finitely presented group scheme $G$. The functor $\Omega^{\mathcal{F}}$ is left exact and $R$-linear. We will show that a structure of sheaf of algebras on $\mathcal{F}$ corresponds to a structure of monoidal functor on $\Omega^{\mathcal{F}}$ and we will conclude that the category of $G$-equivariant quasi-coherent sheaves of algebras is equivalent to the category of left exact and $R$-linear monoidal functors $\operatorname{Loc}^G R \longrightarrow \operatorname{QCoh} T$. When $G$ is linearly reductive, any $R$-linear functor $\operatorname{Loc}^G R \longrightarrow \operatorname{QCoh} T$ is automatically exact, and the above correspondences hold if we consider finitely presented quasi-coherent sheaves or locally free sheaves of finite ranks instead of all the quasi-coherent sheaves.

In the last two sections we will consider the case of $G$-torsors and we will prove that, in the association above, they correspond to left exact strong monoidal functors. This result is already proved in [Lur04], and comes from a more general statement. On the other hand the proof we present here is more elementary. We will also prove a stronger result when $G$ is a super-solvable glrg (see 4.2.36), always in terms of functors.

In what follows we will consider given a flat, finite and finitely presented group scheme $G$ over the base scheme $S$. We will also assume that $S$ is affine, namely $S = \operatorname{Spec} R$, where $R$ is a ring.

### 4.2.1 Linear functors and equivariant quasi-coherent sheaves.

In this section we will show how we can pass from a $G$-equivariant quasi-coherent sheaf on an $R$-scheme $T$ to a functor $\operatorname{Loc}^G R \longrightarrow \operatorname{QCoh} T$ and conversely.

We start defining the stack of $R$-linear functors $\operatorname{Loc}^G R \longrightarrow \operatorname{QCoh}(-)$.

**Definition 4.2.1.** Given an $R$-scheme $T$ we define $\operatorname{QAdd}^G T$ as the category whose objects are $R$-linear functors

$$\Omega\colon \operatorname{Loc}^G R \longrightarrow \operatorname{QCoh} T$$

We will denote by $\operatorname{QAdd}^G_R$ the stack over $\operatorname{Sch}/R$ whose fibers are the categories

$\mathrm{QAdd}^G T$. We define the categories $\mathrm{LAdd}^G T$, $\mathrm{CAdd}^G T$ and the stacks $\mathrm{LAdd}^G_R$, $\mathrm{CAdd}^G_R$ replacing $\mathrm{QCoh}\, T$ by $\mathrm{Loc}\, T$, $\mathrm{FCoh}\, T$ respectively in the above definition.

The motivation of the notation $\mathrm{QAdd}^G$ is that Add stands for additive functors, while Q recall quasi-coherent sheaves.

Since we have to deal with additive categories that are not abelian, namely $\mathrm{Loc}^G R$, we specify here what we mean by (left) exact functors.

**Definition 4.2.2.** An additive functor $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ between additive categories is (left, right) exact if it sends short exact sequences to (left, right) short exact sequences.

*Remark* 4.2.3. Notice that, if $\mathscr{A}$ is not abelian, the definition above does not imply that an exact functor sends long exact sequences to long exact sequences.

We first state the main Theorem of this section.

**Theorem 4.2.4.** *Given an $R$-scheme $T$, we have functors*

$$\begin{array}{ccc} \mathcal{F}_\Omega = \Omega_{R[G]} & \longleftarrow\!\shortmid & \Omega \\ \mathrm{QCoh}^G T & \longrightarrow & \mathrm{QAdd}^G T \\ \mathcal{F} & \longmapsto & \Omega^{\mathcal{F}} = (- \otimes \mathcal{F})^G \end{array}$$

*Moreover $\Omega^{\mathcal{F}}$ is always left exact, there exist a natural isomorphism $\mathcal{F} \longrightarrow \Omega^{\mathcal{F}}_{R[G]}$ and a natural transformation $\Omega \longrightarrow \Omega^{\Omega_{R[G]}}$ which is an isomorphism if and only if $\Omega$ is left exact. In particular $\Omega^*$ is an equivalence onto the full subcategory of $\mathrm{QAdd}^G T$ of left exact functors.*

*Remark* 4.2.5. It is part of the statement of the Theorem that for each $\Omega \in \mathrm{QAdd}^G T$ there exists a natural action of $G$ on the quasi-coherent sheaf $\Omega_{R[G]}$. Moreover we have to warn the reader that the functor $\Omega^*$ does not extend to a map of stacks, because if $\mathcal{F} \in \mathrm{QCoh}^G T$ and $f\colon T' \longrightarrow T$ is a base change, then the natural map $f^*(\mathcal{F} \otimes V)^G \longrightarrow ((f^*\mathcal{F}) \otimes V)^G$ is not an isomorphism in general. However, assuming Theorem 4.2.4, we can prove the following.

**Proposition 4.2.6.** *The following conditions are equivalent:*

1) *$G$ is linearly reductive over $R$;*

2) *the functor of invariants $(-)^G\colon \mathrm{Loc}^G R \longrightarrow \mathrm{QCoh}\, R$ is exact;*

3) *all the $R$-linear functors $\Omega\colon \mathrm{Loc}^G R \longrightarrow \mathrm{QCoh}\, R$ are left exact*

*In this case all the $R$-linear functors $\mathrm{Loc}^G R \longrightarrow \mathrm{QCoh}\, T$ are exact and the maps defined in 4.2.4 yield isomorphisms of stacks*

$$\mathrm{QCoh}^G_R \simeq \mathrm{QAdd}^G_R \qquad \mathrm{Loc}^G_R \simeq \mathrm{LAdd}^G_R \qquad \mathrm{FCoh}^G_R \simeq \mathrm{CAdd}^G_R$$

*Proof.* We first prove that if all functors in $\mathrm{QAdd}^G T$ are left exact, then $(-)^G\colon \mathrm{QCoh}^G T \longrightarrow \mathrm{QCoh}\, T$ is exact. In particular we will have implications 3) $\implies$ 1) $\implies$ 2). Given a surjection $\phi\colon \mathcal{F} \longrightarrow \mathcal{F}'$ in $\mathrm{QCoh}^G R$ we define the functor

$$\Omega\colon \mathrm{Loc}^G R \longrightarrow \mathrm{QCoh}\, T,\ \Omega_V = \mathrm{Coker}((\mathcal{F}\otimes V)^G \xrightarrow{(\phi\otimes \mathrm{id}_V)^G} (\mathcal{F}'\otimes V)^G)$$

From 4.1.1 we see that $\Omega_{R[G]} = 0$ and from 4.2.4 we can conclude that $\Omega = 0$. In particular $\Omega_R = \mathrm{Coker}(\mathcal{F}^G \longrightarrow \mathcal{F}'^G) = 0$.

Now assume that $(-)^G\colon \mathrm{Loc}^G R \longrightarrow \mathrm{QCoh}\, R$ is exact. We want to prove that any $\Omega \in \mathrm{QAdd}^G T$ is exact, showing, in particular, implication 2) $\implies$ 3). It is enough to prove that any short exact sequence in $\mathrm{Loc}^G R$ has a $G$-equivariant splitting. Consider a short exact sequence in $\mathrm{Loc}^G R$

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

This is a split sequence in $\mathrm{Loc}\, R$. In particular $\mathrm{Hom}^G(V'',-) = (-)^G \circ \mathrm{Hom}(V'',-)$ maintains the exactness of such sequence. Therefore the map

$$\mathrm{Hom}^G(V'',V) \longrightarrow \mathrm{Hom}^G(V'',V'')$$

is surjective and a lifting of $\mathrm{id}_{V''}$ yields the required section. $\square$

*Remark* 4.2.7. Theorem 4.2.4 is no longer true if $S$ is not affine. For instance let $S$ be a proper scheme over $k$ such that $\mathrm{H}^0(\mathcal{O}_S) = k$ and consider $G = 1$ and the $\mathcal{O}_S$-linear functor

$$\Omega = \mathrm{H}^0(-)\otimes_k \mathcal{O}_S\colon \mathrm{Loc}^G S = \mathrm{Loc}\, S \longrightarrow \mathrm{QCoh}\, S$$

If $\Omega \simeq (\Omega_{\mathcal{O}_S[G]}\otimes -)^G = \mathrm{id}_{\mathrm{Loc}\, S}$ it will follow that any locally free sheaf is free. When $G$ is linearly reductive, the right class of functors to consider for a general base scheme $S$ is the one of functors $\mathrm{Loc}^G_S \longrightarrow \mathrm{QCoh}_S$. This works also in general, for non linearly reductive groups, if we restrict those stacks to the fppf site of $S$. Indeed we have to warn the reader that in general, if $\mathcal{F} \in \mathrm{QCoh}\, S$, the functor $(\mathcal{F}\otimes -)^G$ does not yield a map of stacks $\mathrm{Loc}^G_S \longrightarrow \mathrm{QCoh}_S$, even when $S$ is affine, because the invariant functor $(-)^G$ does not commute with arbitrary base changes. Anyway in this exposition we have preferred to avoid technicalities and , for instance, consider the simplest case $S$ affine.

*Remark* 4.2.8. When $G$ is a glrg, theorem 4.2.4 and 4.1.15 say that, in order to define an $R$-linear functor $\Omega\colon \mathrm{Loc}^G R \longrightarrow \mathrm{QCoh}\, T$, it is enough to give quasi-coherent sheaves $(\mathcal{F}_V)_{V\in I_G}$. We can then set

$$\Omega_W = \bigoplus_{V\in I_G} \underline{\mathrm{Hom}}^G(V,W)\otimes \mathcal{F}_V$$

Before proving theorem 4.2.4 we need some preliminary lemmas, which will be useful also in other situations.

**Lemma 4.2.9.** *Given an $R$-scheme $T$ we have equivalences of categories*

$$\begin{array}{rcl} F(R) & \longleftarrow\!\dashv & F \\ \operatorname{QCoh} T & \longrightarrow & \{R\text{-linear functors } \operatorname{Loc} R \longrightarrow \operatorname{QCoh} T\} \\ \mathcal{F} & \longmapsto & - \otimes_R \mathcal{F} \end{array}$$

*Proof.* Clearly $R \otimes_R \mathcal{F} \simeq \mathcal{F}$. On the other hand, since $F$ is $R$-linear, we can define

$$\begin{array}{rcl} V \otimes F(R) & \xrightarrow{\gamma_{F,V}} & F(V) \\ v \otimes x & \longmapsto & F_v(x) \end{array}$$

where $F_v = F(R \xrightarrow{v} V)\colon F(R) \longrightarrow F(V)$. It is straightforward to check that the maps $\gamma_{F,*} - \otimes F(R) \longrightarrow F$ are natural in $F$. So it remains to prove that it is an isomorphism. By additivity of $F$, $\gamma_{F,V}$ is an isomorphism when $V$ is free. Now let $V \in \operatorname{Loc} R$ and consider a presentation $V_1 \longrightarrow V_0 \longrightarrow V$ with $V_1, V_0$ free. We have a commutative diagram

$$\begin{array}{ccccccc} V_1 \otimes F(R) & \longrightarrow & V_0 \otimes F(R) & \longrightarrow & V \otimes F(R) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \gamma_{F,V_1}} & & \downarrow{\scriptstyle \gamma_{F,V_0}} & & \downarrow{\scriptstyle \gamma_{F,V}} & & \\ F(V_1) & \longrightarrow & F(V_0) & \longrightarrow & F(V) & \longrightarrow & 0 \end{array}$$

Since $V$ is projective, both rows are exact and since $V_1, V_0$ are free we can conclude that $\gamma_{F,V}$ is an isomorphism. $\square$

**Corollary 4.2.10.** *Let $\Omega \in \operatorname{QAdd}^G T$. Then there exists a unique natural transformation*

$$\gamma_{V,W}\colon V \otimes \Omega_W \longrightarrow \Omega_{\underline{V} \otimes W} \text{ for } V \in \operatorname{Loc} R,\ W \in \operatorname{Loc}^G R$$

*such that $\gamma_{R,W} = \mathrm{id}_{\Omega_W}$ and it is an isomorphism. Moreover $\gamma$ is natural also in $\Omega \in \operatorname{QAdd}^G T$.*

*Proof.* The functors $V \longmapsto V \otimes \Omega_W$ and $V \longmapsto \Omega_{\underline{V} \otimes W}$ from $\operatorname{Loc} R$ to $\operatorname{QCoh} T$ coincides on $V = R$. So $\mathrm{id}_{\Omega_W}$ extends to a unique natural transformation $\gamma_{-,W}\colon -\otimes \Omega_W \longrightarrow \Omega_{\underline{-} \otimes W}$, which is an isomorphism. The naturality with respect to $W \in \operatorname{Loc}^G R$ and $\Omega \in \operatorname{QAdd}^G T$ follows by a similar trick. $\square$

We are now ready to define the action of $G$ on $\Omega_{R[G]}$ for any $\Omega \in \operatorname{QAdd}^G T$.

**Lemma 4.2.11.** *The co-multiplication*

$$R[G] \xrightarrow{\Delta_G} R[G] \otimes \underline{R[G]}$$

*is $G$-equivariant and, given $\Omega \in \operatorname{QAdd}^G T$, the map*

$$\Omega_{R[G]} \xrightarrow{\Omega_{\Delta_G}} \Omega_{R[G] \otimes \underline{R[G]}} \simeq \Omega_{R[G]} \otimes R[G]$$

*defines an action of $G$ on $\Omega_{R[G]}$.*

*Proof.* The map $\Delta_G$ is $G$-equivariant since $(gh)\star t = t^{-1}gh = (g\star t)h$ for any $h, g, t \in G$, where $\star$ denotes the regular action of $G$ on itself (see 2.1.1 for the convention used). Instead the commutative diagrams that $\Omega_{\Delta_G}$ has to satisfy in order to be an action come from the following commutative diagrams of $G$-equivariant maps, after applying the functor $\Omega$.

$$\begin{tikzcd} R[G] \arrow[r, "\Delta_G"] \arrow[d, "\Delta_G"] & R[G]\otimes \underline{R[G]} \arrow[d, "\mathrm{id}\otimes\Delta_G"] \\ R[G]\otimes \underline{R[G]} \arrow[r, "\Delta_G\otimes\mathrm{id}"] & R[G]\otimes \underline{R[G]}\otimes \underline{R[G]} \end{tikzcd} \qquad \begin{tikzcd} & R[G]\otimes \underline{R[G]} \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\ R[G] \arrow[ur, "\Delta_G"] \arrow[r, "\mathrm{id}"] & R[G] \end{tikzcd}$$

□

**Lemma 4.2.12.** *Let $\Omega, \Gamma \in \mathrm{QAdd}^G T$, with $\Gamma$ left exact. Then the map*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{QAdd}^G T}(\Omega,\Gamma) &\longrightarrow \mathrm{Hom}_T(\Omega_{R[G]},\Gamma_{R[G]}) \\ \sigma &\longmapsto \sigma_{R[G]} \end{aligned}$$

*is injective and its image is composed of the morphisms $\delta\colon \Omega_{R[G]} \longrightarrow \Gamma_{R[G]}$ such that, for any $u \in \mathrm{End}^G(R[G])$, the following diagram is commutative*

$$\begin{tikzcd} \Omega_{R[G]} \arrow[r, "\delta"] \arrow[d, "\Omega_u"] & \Gamma_{R[G]} \arrow[d, "\Gamma_u"] \\ \Omega_{R[G]} \arrow[r, "\delta"] & \Gamma_{R[G]} \end{tikzcd}$$

*Moreover $\sigma\colon \Omega \longrightarrow \Gamma$ is an isomorphism if and only if $\sigma_{R[G]}$ is an isomorphism and $\Omega$ is left exact.*

*Proof.* Denote by $M$ the set of maps $\delta$ as in the statement. Clearly, if $\sigma\colon \Omega \longrightarrow \Gamma$ is a natural transformation, then $\sigma_{R[G]} \in M$. With all the $V \in \mathrm{Loc}^G R$ we associate an exact sequence

$$V_1{}^\vee \longrightarrow V_0{}^\vee \longrightarrow V^\vee \longrightarrow 0$$

as in 4.1.2. In particular the $V_i$ are direct sums of copies of the regular representation $R[G]$. Since $V$ is locally free, the dual of the above sequence is still exact and can be decomposed in two short exact sequences in $\mathrm{Loc}^G R$. In particular, since $\Gamma$ is left exact, the sequence

$$0 \longrightarrow \Gamma_V \longrightarrow \Gamma_{V_0} \longrightarrow \Gamma_{V_1}$$

is exact too.

Note also that, thanks to the additivity and $R$-linearity of $\Omega$ and $\Gamma$, a $\delta \in M$ extends uniquely to a natural transformation $\delta_*\colon \Omega \longrightarrow \Gamma$ if we restrict those functors to the full subcategory of $\mathrm{Loc}^G R$ of sheaves which are direct sums of copies of the regular representation. In particular, given $\delta \in M$ and $V \in \mathrm{Loc}^G R$ there exists a unique $\delta_V$ making the following diagram commutative

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_V & \longrightarrow & \Omega_{V_0} & \longrightarrow & \Omega_{V_1} \\ & & \downarrow \delta_V & & \downarrow \delta_{V_0} & & \downarrow \delta_{V_1} \\ 0 & \longrightarrow & \Gamma_V & \longrightarrow & \Gamma_{V_0} & \longrightarrow & \Gamma_{V_1} \end{array}$$

Here we use that the second row is exact. This shows that the map $*_{R[G]}$ in the statement is injective and tells how to extend a $\delta \in M$ to a map $\delta_* \colon \Omega \longrightarrow \Gamma$. In order to prove that such map is natural and does not depend on the choice of the exact sequence associated with $V \in \mathrm{Loc}^G R$, it is enough to note that every map $f \colon V \longrightarrow W$, where $W \in \mathrm{Loc}^G R$, extends to a diagram of $G$-equivariant maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & V & \longrightarrow & V_0 & \longrightarrow & V_1 \\ & & \downarrow f & & \downarrow f_1 & & \downarrow f_2 \\ 0 & \longrightarrow & W & \longrightarrow & W_0 & \longrightarrow & W_1 \end{array}$$

Indeed it is enough to take the dual sequences and note that $\mathrm{Hom}^G(R[G]^\vee, -) \simeq (R[G] \otimes -)^G$ is just the forgetful functor $\mathrm{QCoh}^G R \longrightarrow \mathrm{QCoh}\, R$ by 4.1.1. The last claim follows easily from the diagram above. $\square$

**Proposition 4.2.13.** *The composition*

$$\eta_V \colon V \xrightarrow{\mu_V} V \otimes R[G] \xrightarrow{\mathrm{id} \otimes \sigma_G} V \otimes R[G] \xrightarrow{swap} R[G] \otimes \underline{V} \text{ for } V \in \mathrm{Loc}^G R$$

*defines a natural transformation $\eta \colon \mathrm{id}_{\mathrm{Loc}^G R} \longrightarrow R[G] \otimes \underline{\ }$ of functors $\mathrm{Loc}^G R \longrightarrow \mathrm{Loc}^G R$ and $\eta_{R[G]} = \Delta_G$. Given an $R$-scheme $T$ this map induces a natural transformation*

$$\Omega_V \longrightarrow (\Omega_{R[G]} \otimes V)^G$$

*of functors $\mathrm{Loc}^G R \times \mathrm{QAdd}^G T \longrightarrow \mathrm{Coh}\, T$ which is an isomorphism if $V = R[G]$ or $\Omega$ is left exact. Moreover the induced map*

$$\theta_V \colon \Omega_V \otimes V^\vee \longrightarrow \Omega_{R[G]}$$

*is $G$-equivariant and it is given by*

$$\Omega_V \otimes V^\vee \simeq \Omega_V \otimes \mathrm{Hom}^G(V, R[G]) \xrightarrow{x \otimes \phi \longrightarrow \Omega_\phi(x)} \Omega_{R[G]}$$

*Proof.* The first claim is a classical result, taking into account the particular comodule structure we have put on $R[G]$. Given $\Omega \in \mathrm{QAdd}^G T$ and applying it on $\eta_V$ for any $V \in \mathrm{Loc}^G R$, we get a natural map $\delta_V \colon \Omega_V \longrightarrow \Omega_{R[G]} \otimes V$ such that $\delta_{R[G]} = \Omega_{\Delta_G}$, the comodule structure of $\Omega_{R[G]}$. If $V = R[G]$ we have a factorization

$$\Omega_{R[G]} \xrightarrow{\Omega_{\Delta_G}} (\Omega_{R[G]} \otimes R[G])^G \longrightarrow \Omega_{R[G]} \otimes R[G]$$

and, since $(\Omega_{R[G]}\otimes -)^G \longrightarrow \Omega_{R[G]}\otimes -$ is a natural transformation of left exact functors, by 4.2.12 it follows that $\delta_*$ factors through a natural transformation $\Omega \longrightarrow (\Omega_{R[G]}\otimes -)^G$.

The map $\theta_V$ in the statement can be obtained applying $\Omega$ to the map $\gamma_V\colon V\otimes \underline{V}^\vee \longrightarrow R[G]$ induced by $\eta_V$. We have to prove that the composition

$$f_V\colon V\otimes \operatorname{Hom}^G(V,R[G]) \xrightarrow{\mathrm{id}\otimes\varepsilon_G{}^\vee} V\otimes V^\vee \xrightarrow{\gamma_V} R[G]$$

is just the evaluation $x\otimes\phi \longmapsto \phi(x)$. By construction we have $\gamma_V(x\otimes\psi) = m_{R[G]}\circ(\mathrm{id}\otimes\psi)\circ\eta_V(x)$. In particular

$$f_V(x\otimes\phi) = m_{R[G]}\circ[\mathrm{id}\otimes(\varepsilon_G\circ\phi)]\circ\eta_V(x) = m_{R[G]}\circ(\mathrm{id}\otimes\varepsilon_G)\circ(\mathrm{id}\otimes\phi)\circ\eta_V(x)$$

Since $\eta_*$ is natural, we have $\mathrm{id}\otimes\phi\circ\eta_V = \eta_{R[G]}\circ\phi = \Delta_G\circ\phi$ and, since $m_{R[G]}\circ(\mathrm{id}\otimes\varepsilon_G)\circ\Delta_G = \mathrm{id}$, that

$$f_V(x\otimes\phi) = m_{R[G]}\circ(\mathrm{id}\otimes\varepsilon_G)\circ\Delta_G(\phi(x)) = \phi(x)$$

$\square$

*Proof. (of Theorem 4.2.4)* The left exactness of $\Omega^{\mathcal{F}}$ follows from the fact that any short exact sequence in $\operatorname{Loc}^G R$ is locally split in $\operatorname{Loc} R$, so that $(-\otimes\mathcal{F})$ is exact here and the fact that $(-)^G$ is left exact.

Let now $\mathcal{F}\in\operatorname{QCoh}_R^G$ with structure map $\mathcal{F}\xrightarrow{\mu}\mathcal{F}\otimes R[G]$. Thanks to 4.1.1 we have an isomorphism $\mathcal{F}\xrightarrow{\mu}(\mathcal{F}\otimes R[G])^G = \Omega^{\mathcal{F}}_{R[G]}$ and we want to prove that it is $G$-equivariant. This is equivalent to requiring that the dashed map $\alpha$ making the following diagram commutative is just $\mu$.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\mu"] \arrow[dd, dashed, "\alpha"] & (\mathcal{F}\otimes R[G])^G \arrow[r, hook] \arrow[d] & \mathcal{F}\otimes R[G] \arrow[d, "\mathrm{id}\otimes\Delta_G"] \\
 & (\mathcal{F}\otimes R[G]\otimes \underline{R[G]})^G \arrow[r, hook] \arrow[d] & \mathcal{F}\otimes R[G]\otimes\underline{R[G]} \arrow[d, "\mathrm{id}"] \\
\mathcal{F}\otimes\underline{R[G]} \arrow[r, "\mu\otimes\mathrm{id}"] & (\mathcal{F}\otimes R[G])^G\otimes\underline{R[G]} \arrow[r, hook] & \mathcal{F}\otimes R[G]\otimes\underline{R[G]}
\end{tikzcd}$$

Note that $\mu\otimes\mathrm{id}\circ\alpha = \mathrm{id}\otimes\Delta_G\circ\mu = \mu\otimes\mathrm{id}\circ\mu$ and that $\mu\otimes\mathrm{id}$ is injective. We can therefore conclude that $\alpha=\mu$.

The natural transformation $\Omega\longrightarrow(\Omega_{R[G]}\otimes -)^G$ and all the other claims are in 4.2.13. $\square$

We want now to give a different description of the functor $\Omega\longrightarrow\Omega_{R[G]} = \mathcal{F}_\Omega$ of Theorem 4.2.4 in the particular case when the group $G$ is a glrg.

**Proposition 4.2.14.** *Assume that $G$ is a glrg. Given $\Omega\in\operatorname{QAdd}_R^G$ the isomorphisms (see 4.1.17)*

$$R[G]\simeq\bigoplus_{V\in I_G}\underline{V}^\vee\otimes V \text{ and } \Omega_{R[G]}\longrightarrow\bigoplus_{V\in I_G}V^\vee\otimes\Omega_V$$

*are $G$-equivariant and the last one defines a natural isomorphism $(-)_{R[G]} \longrightarrow \bigoplus_{V\in I_G} V^\vee \otimes (-)_V$ of functors $\mathrm{QAdd}^G_R \longrightarrow \mathrm{QCoh}^G_R$.*

*Proof.* We can assume that $\Omega = \Omega^{\mathcal{F}} = (\mathcal{F}\otimes -)^G$ for some $\mathcal{F}\in \mathrm{QCoh}^G_R$. The map $\mu\colon \mathcal{F} \longrightarrow (\mathcal{F}\otimes R[G])^G$ is a $G$-equivariant isomorphism, where $\mu$ is comodule structure on $\mathcal{F}$ and its inverse is the restriction of $\mathrm{id}\otimes\varepsilon_G$, which is therefore $G$-equivariant. Thanks to 4.1.18 and using its notation, we have a commutative diagram

$$\begin{tikzcd}
(R[G]\otimes\mathcal{F})^G \arrow[r] \arrow[d] & R[G]\otimes\mathcal{F} \arrow[r, "\varepsilon_G\otimes\mathrm{id}_{\mathcal{F}}"] \arrow[d] & \mathcal{F} \arrow[d, "\mathrm{id}_{\mathcal{F}}"] \\
\bigoplus_{V\in I_G} V^\vee\otimes(V\otimes\mathcal{F})^G \arrow[r] & \bigoplus_{V\in I_G} V^\vee\otimes V\otimes\mathcal{F} \arrow[r, "\bigoplus_{V\in I_G} e_V\otimes\mathrm{id}_{\mathcal{F}}"] & \mathcal{F}
\end{tikzcd}$$

and we have to prove that the first vertical map is $G$-equivariant. But this is true because in each row the composition of the maps is a $G$-equivariant isomorphism. □

### 4.2.2 Lax monoidal functors and equivariant quasi-coherent sheaves of algebras.

In this section we want to use the association described above in order to describe the quasi-coherent sheaves of algebras that have an action of $G$ on it. We will see that a (non associative) ring structure on a sheaf $\mathcal{F}\in\mathrm{QCoh}^G$, its possible commutativity and associativity translate as natural properties of the functor $\Omega^{\mathcal{F}}$. For instance we will show that a (lax) symmetric monoidal structure over $\Omega^{\mathcal{F}}$ corresponds to a structure of associative and commutative sheaf of algebras on $\mathcal{F}$.

We start setting up some definitions:

**Definition 4.2.15.** Given an $R$-scheme $T$, a *pseudo monoidal* functor $\Omega\colon \mathrm{Loc}^G R \longrightarrow \mathrm{QCoh}\,T$ is an $R$-linear functor together with a natural transformation

$$\iota^\Omega_{V,W}\colon \Omega_V\otimes\Omega_W \longrightarrow \Omega_{V\otimes W} \text{ for any } V,W\in\mathrm{Loc}^G R$$

A pseudo monoidal functor $\Omega\colon \mathrm{Loc}^G R\longrightarrow \mathrm{QCoh}\,T$

1) is *symmetric* (commutative) if for any $V,W\in\mathrm{Loc}^G R$ the following diagram is commutative

$$\begin{tikzcd}
\Omega_V\otimes\Omega_W \arrow[r, "\iota^\Omega_{V,W}"] \arrow[d] & \Omega_{V\otimes W} \arrow[d] \\
\Omega_W\otimes\Omega_V \arrow[r, "\iota^\Omega_{W,V}"] & \Omega_{W\otimes V}
\end{tikzcd}$$

where the vertical arrows are the obvious isomorphisms;

2) is *associative* if for any $V, W, Z \in \operatorname{Loc}^G R$ the following diagram is commutative

$$\begin{tikzcd}
\Omega_V \otimes \Omega_W \otimes \Omega_Z \arrow[r, "\iota^\Omega_{V,W}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\iota^\Omega_{W,Z}"] & \Omega_{V\otimes W} \otimes \Omega_Z \arrow[d, "\iota^\Omega_{V\otimes W,Z}"] \\
\Omega_V \otimes \Omega_{W\otimes Z} \arrow[r, "\iota^\Omega_{V,W\otimes Z}"] & \Omega_{V\otimes W\otimes Z}
\end{tikzcd}$$

A unity for $\Omega$ is an element $1 \in \Omega_R$ such that, for any $V \in \operatorname{Loc}^G R$, the following diagram is commutative

$$\begin{tikzcd}
 & \Omega_R \otimes \Omega_V \arrow[r, "\iota^\Omega_{R,V}"] & \Omega_{R\otimes V} \arrow[dr] & \\
\Omega_V \arrow[ur, "1\otimes\mathrm{id}"] \arrow[rrr, "\mathrm{id}"] \arrow[dr, "\mathrm{id}\otimes 1"'] & & & \Omega_V \\
 & \Omega_V \otimes \Omega_R \arrow[r, "\iota^\Omega_{V,R}"] & \Omega_{V\otimes R} \arrow[ur] &
\end{tikzcd}$$

A *lax monoidal* functor $\Omega\colon \operatorname{Loc}^G R \longrightarrow \operatorname{QCoh} T$ is a pseudo monoidal functor that is associative and has a unity 1.

**Definition 4.2.16.** Given an $R$-scheme $T$ we define the categories

- $\operatorname{QRings} T$, whose objects are $\mathscr{A} \in \operatorname{QCoh} T$ with a map $m\colon \mathscr{A}\otimes\mathscr{A} \longrightarrow \mathscr{A}$, called the multiplication;
- $\operatorname{QRings}^G T$, whose objects are $\mathscr{A} \in \operatorname{QCoh}^G T$ with a $G$-equivariant map $m\colon \mathscr{A}\otimes\mathscr{A} \longrightarrow \mathscr{A}$;
- $\operatorname{QAlg}^G T$, whose objects are quasi-coherent sheaves $\mathscr{A} \in \operatorname{QRings}^G T$ of commutative and associative algebras with a unity $1 \in \mathscr{A}^G$;
- $\operatorname{QPMon}^G T$, whose objects are pseudo-monoidal functors $\Omega\colon \operatorname{Loc}^G R \longrightarrow \operatorname{QCoh} T$.
- $\operatorname{QMon}^G T$, whose objects are commutative lax monoidal functors $\Omega\colon \operatorname{Loc}^G R \longrightarrow \operatorname{QCoh} T$. Here we require that the morphisms maintain the unities.
- $\operatorname{Aff}^G T$, whose objects are affine schemes $X \xrightarrow{f} T$ with an action of $G$ on $X$ such that $f$ is $G$-invariant;
- $\operatorname{ffpSch}^G T$ , the full subcategory of $\operatorname{Aff}^G T$ of finite and finitely presented maps;
- $\operatorname{Cov}^G$, the full subcategory of $\operatorname{Aff}^G T$ of covers.

Replacing QCoh with Loc, FCoh we also define $\operatorname{LRings} T$, $\operatorname{LRings}^G T$, $\operatorname{LAlg}^G T$, $\operatorname{LPMon}^G T$, $\operatorname{LMon}^G T$ and $\operatorname{CRings} T$, $\operatorname{CRings}^G T$, $\operatorname{CAlg}^G T$, $\operatorname{CPMon}^G T$, $\operatorname{CMon}^G T$ respectively

We define the stacks $\operatorname{HRings}_R$, $\operatorname{HRings}^G_R$, $\operatorname{HAlg}^G_R$, $\operatorname{HPMon}^G_R$, $\operatorname{HMon}^G_R$ whose fibers over an $R$-scheme $T$ is $\operatorname{HRings} T$, $\operatorname{HRings}^G T$, $\operatorname{HAlg}^G T$, $\operatorname{HPMon}^G T$, $\operatorname{HMon}^G T$ respectively, where $H$ can be Q, C or L. We also define $\operatorname{Aff}^G_R$, $\operatorname{ffpSch}^G_R$, $\operatorname{Cov}^G_R$ as the stacks whose fibers over an $R$-scheme $T$ are respectively $\operatorname{Aff}^G T$, $\operatorname{ffpSch}^G T$, $\operatorname{Cov}^G T$.

*Remark* 4.2.17. The functors $\operatorname{Spec}\colon \mathrm{QAlg}_R^G \longrightarrow \mathrm{Aff}_R^G$ and the push forward $\mathrm{Aff}_R^G \longrightarrow \mathrm{QAlg}_R^G$ are each other's quasi-inverse and restrict to isomorphisms $\mathrm{CAlg}_R^G \simeq \mathrm{ffpSch}_R^G$ and $\mathrm{LAlg}_R^G \simeq \mathrm{Cov}^G$. Indeed, by [Gro64, Proposition 1.47], a finite quasi-coherent algebra is finitely presented as a module if and only if it is so as an algebra.

*Remark* 4.2.18. The categories $\mathrm{QAlg}^G T$, $\mathrm{QMon}^G T$ are (not full) subcategories of $\mathrm{QRings}^G T$, $\mathrm{QPMon}^G T$ respectively, because a unity for a ring or for a lax monoidal functor is unique.

The following is another application of 4.2.9.

**Lemma 4.2.19.** *Given $\Omega \in \mathrm{QPMon}_R^G$ and $V, W \in \operatorname{Loc} R$, $V', W' \in \operatorname{Loc}^G R$ we have a commutative diagram*

$$\begin{tikzcd}
\Omega_{\underline{V}\otimes V'} \otimes \Omega_{\underline{W}\otimes W'} \arrow[r] \arrow[d, "\iota_{\underline{V}\otimes V', \underline{W}\otimes W'}"] & V \otimes \Omega_{V'} \otimes W \otimes \Omega_{W'} \arrow[r] & V \otimes W \otimes \Omega_{V'} \otimes \Omega_{W'} \arrow[d, "\mathrm{id}\otimes\iota_{V',W'}"] \\
\Omega_{\underline{V}\otimes V'\otimes \underline{W}\otimes W'} \arrow[r] & \Omega_{\underline{V}\otimes\underline{W}\otimes V'\otimes W'} \arrow[r] & V \otimes W \otimes \Omega_{V'\otimes W'}
\end{tikzcd}$$

The following proposition describes how much data is needed to define a pseudo monoidal functor when the group $G$ is a glrg.

**Proposition 4.2.20.** *Assume that $G$ is glrg and define the stack $\mathcal{Y}$ whose objects are $(\mathcal{A}_V, \iota_{V,W})_{V,W\in I_G}$ where $\mathscr{A}_V \in \mathrm{QCoh}_R$ and $\iota_{V,W}$ is a map*

$$\iota_{V,W}\colon \mathscr{A}_V \otimes \mathscr{A}_W \longrightarrow \bigoplus_{\Delta\in I_G} \operatorname{Hom}^G(\Delta, V\otimes W) \otimes \mathscr{A}_\Delta$$

*Then the functor*

$$\begin{array}{ccc} \mathrm{QPMon}_R^G & \longrightarrow & \mathcal{Y} \\ (\Omega, \iota^\Omega) & \longmapsto & (\Omega_V, \Omega_V \otimes \Omega_W \xrightarrow{\iota^\Omega_{V,W}} \Omega_{V\otimes W} \simeq \bigoplus_{\Delta\in I_G} \operatorname{Hom}^G(\Delta, V\otimes W)\otimes\Omega_\Delta)_{V,W\in I_G}) \end{array}$$

*is an equivalence.*

*Proof.* By 4.2.8 and 4.2.19, we see that the map in the statement is fully faithful. We have only to show that it is essentially surjective. For simplicity, given $\Delta, W \in \operatorname{Loc}^G R$ we will write $W_\Delta = \operatorname{Hom}^G(\Delta, W)$. Let $\chi = (\mathscr{A}_V, \iota_{V,W})_{V,W\in I_G} \in \mathcal{Y}$. By 4.2.8, there exists $\Omega \in \mathrm{QAdd}_R^G$ such that $\Omega_V \simeq \mathscr{A}_V$ and it is given by

$$\Omega_W = \bigoplus_{\Delta\in I_G} W_\Delta \otimes \mathscr{A}_\Delta$$

By definition, the map $\iota_{V,W}$ yield maps $\iota_{V,W}\colon \Omega_V \otimes \Omega_W \longrightarrow \Omega_{V\otimes W}$ for any $V, W \in I_G$. Given $\Lambda, \Gamma \in \operatorname{Loc}^G R$ we define $\iota^\Omega_{\Lambda,\Delta}$ as

$$\Omega_\Lambda \otimes \Omega_\Gamma \simeq \bigoplus_{V,W\in I_G} \Lambda_V \otimes \Gamma_W \otimes \Omega_V \otimes \Omega_W \xrightarrow{\mathrm{id}\otimes\iota_{V,W}} \bigoplus_{V,W\in I_G} \Lambda_V \otimes \Gamma_W \otimes \Omega_{V\otimes W} \simeq \Omega_{\Lambda\otimes\Gamma}$$

where the last isomorphism is induced by $\bigoplus_{V,W\in I_G}\Lambda_V\otimes\Gamma_W\otimes V\otimes W\simeq\Lambda\otimes\Gamma$. It is easy to check that $\iota^\Omega$ is a natural transformation and that $(\Omega,\iota^\Omega)\in\mathrm{LPMon}_R^G$ is mapped to our starting object $\chi\in\mathcal{Y}$. $\square$

Given $\mathscr{A}\in\mathrm{QRings}^G\,T$ with multiplication $m$, we endow $\Omega^{\mathscr{A}}$ with the pseudo monoidal structure

$$(V\otimes\mathscr{A})^G\otimes(W\otimes\mathscr{A})^G\longrightarrow(V\otimes W\otimes\mathscr{A}\otimes\mathscr{A})^G\xrightarrow{(\mathrm{id}\otimes m)^G}(V\otimes W\otimes\mathscr{A})^G$$

Conversely given $\Omega\in\mathrm{QPMon}^G\,T$, we define the multiplication on $\mathcal{F}_\Omega=\Omega_{R[G]}$ by

$$\Omega_{R[G]}\otimes\Omega_{R[G]}\xrightarrow{\iota^\Omega_{R[G],R[G]}}\Omega_{R[G]\otimes R[G]}\xrightarrow{\Omega_{m_G}}\Omega_{R[G]}$$

We will denote $\mathscr{A}_\Omega$ the sheaf $\mathcal{F}_\Omega$ together with the multiplication map.

The following is the main Theorem of this section.

**Theorem 4.2.21.** *Given an $R$-scheme $T$, the functors $\Omega^*$ and $\mathcal{F}_*$ of Theorem 4.2.4 extend to functors*

$$\mathrm{QRings}^G\,T\ \underset{\mathscr{A}_*}{\overset{\Omega^*}{\rightleftarrows}}\ \mathrm{QPMon}^G\,T\mathrm{QAlg}^G\,T\ \underset{\mathscr{A}_*}{\overset{\Omega^*}{\rightleftarrows}}\ \mathrm{QMon}^G\,T$$

*Moreover there exist a natural isomorphism $\mathscr{A}\longrightarrow\Omega^{\mathscr{A}}_{R[G]}$ and a natural transformation $\Omega\longrightarrow\Omega^{\Omega_{R[G]}}$ which is an isomorphism if and only if $\Omega$ is left exact. In particular $\Omega^*$ is an equivalence onto the full subcategory of $\mathrm{QPMon}^G\,T$ ($\mathrm{QMon}^G\,T$) of left exact functors. If $G$ is linearly reductive the above functors define isomorphisms of stacks*

$$\mathrm{QRings}_R^G\simeq\mathrm{QPMon}_R^G,\ \mathrm{CRings}_R^G\simeq\mathrm{CPMon}_R^G,\ \mathrm{LRings}_R^G\simeq\mathrm{LPMon}_R^G$$

$$\mathrm{QAlg}_R^G\simeq\mathrm{QMon}_R^G,\ \mathrm{CAlg}_R^G\simeq\mathrm{CMon}_R^G,\ \mathrm{LAlg}_R^G\simeq\mathrm{LMon}_R^G$$

This theorem will be proved at the end of this section, because we need to collect several lemmas before.

*Remark* 4.2.22. Given an $R$-scheme $T$ we have a functor

$$\begin{aligned}\mathrm{QPMon}^G\,T\times\mathrm{LRings}^G\,R&\longrightarrow\mathrm{QRings}\,T\\(\Omega,(A,m))&\longmapsto(\Omega_A,\Omega_m\circ\iota^\Omega_{A,A})\end{aligned}$$

The following lemma shows that the functor $\mathscr{A}_*\colon\mathrm{QAdd}^G\,T\longrightarrow\mathrm{QRings}^G\,T$ is well defined.

**Lemma 4.2.23.** *If $\Omega\in\mathrm{QPMon}_R^G$ then $\mathscr{A}_\Omega=\Omega_{R[G]}\in\mathrm{QRings}_R^G$, i.e. the multiplication $\mathscr{A}_\Omega\otimes\mathscr{A}_\Omega\longrightarrow\mathscr{A}_\Omega$ is $G$-equivariant.*

*Proof.* Set $A = R[G]$ , $\Delta = \Delta_G \colon A \longrightarrow A \otimes R[G]$ and $m = m_G \colon A \otimes A \longrightarrow A$. We claim that the diagrams in

$$\begin{tikzcd}
\Omega_A \otimes \Omega_A \arrow[rr, "\iota_{A,A}"] \arrow[d, "\Omega_\Delta \otimes \Omega_\Delta"] & & \Omega_{A\otimes A} \arrow[r, "\Omega_m"] \arrow[d, "\Omega_{\Delta\otimes\Delta}"] & \Omega_A \arrow[ddd, "\Omega_\Delta"] \\
\Omega_{A\otimes \underline{R[G]}} \otimes \Omega_{A\otimes \underline{R[G]}} \arrow[rr, "\iota_{A\otimes R[G], A\otimes R[G]}"] \arrow[d] & & \Omega_{A\otimes \underline{R[G]} \otimes A \otimes \underline{R[G]}} \arrow[d] & \\
\Omega_A \otimes \Omega_A \otimes R[G] \otimes R[G] \arrow[dd, "\mathrm{id}\otimes m_G"'] \arrow[dr, "\iota_{A,A}\otimes \mathrm{id}"] & & \Omega_{A\otimes A \otimes \underline{R[G] \otimes R[G]}} \arrow[dl] \arrow[d, "\Omega_{\mathrm{id}\otimes m_G}"] & \\
& \Omega_{A\otimes A} \otimes R[G] \otimes R[G] \arrow[d, "\mathrm{id}\otimes m_G"'] & \Omega_{A\otimes A \otimes \underline{R[G]}} \arrow[r, "\Omega_{m\otimes \mathrm{id}}"] \arrow[dl] & \Omega_{A\otimes \underline{R[G]}} \arrow[d] \\
\Omega_A \otimes \Omega_A \otimes R[G] \arrow[r, "\iota_{A,A}\otimes \mathrm{id}"'] & \Omega_{A\otimes A} \otimes R[G] \arrow[rr, "\Omega_m \otimes \mathrm{id}"] & & \Omega_A \otimes R[G]
\end{tikzcd}$$

are commutative. Note that the outer diagram is the one required for the $G$-equivariancy of the multiplication $\Omega_A \otimes \Omega_A \longrightarrow \Omega_A$. The pentagonal diagram is commutative thanks to 4.2.19. The only non trivially commutative diagram left is the upper right rectangle. This is commutative because it is obtained applying $\Omega$ to the diagram

$$\begin{tikzcd}
A \otimes A \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "m"] & A \otimes \underline{R[G]} \otimes A \otimes \underline{R[G]} \arrow[r] & A \otimes A \otimes \underline{R[G] \otimes R[G]} \arrow[d, "m\otimes m_G"] \\
A \arrow[rr, "\Delta"] & & A \otimes \underline{R[G]}
\end{tikzcd}$$

which is commutative since $\Delta$ is a map of rings. $\square$

We have now to deal with how the properties of being commutative, associative or having a unity translate in the context of functors.

*Remark* 4.2.24. If $\Omega \in \mathrm{QPMon}^G T$ and $V, W \in \mathrm{Loc}^G R$ we have a commutative diagram

$$\begin{tikzcd}
V^\vee \otimes \Omega_V \otimes W^\vee \otimes \Omega_W \arrow[r, "\theta_V \otimes \theta_W"] \arrow[d] & \Omega_{R[G]} \otimes \Omega_{R[G]} \arrow[d, "m"] \\
V \otimes W^\vee \otimes \Omega_{V\otimes W} \arrow[r, "\theta_{V\otimes W}"] & \Omega_{R[G]}
\end{tikzcd}$$

where $\theta_*$ are the evaluation maps defined in 4.2.13 and $m$ is the multiplication.

*Remark* 4.2.25. Given an $R$-scheme $T$, the natural isomorphisms (see 4.1.1)

$$V \otimes \mathcal{O}_T \simeq (V \otimes \mathcal{O}_T[G])^G \simeq \underline{\mathrm{Hom}}^G(V^\vee, \mathcal{O}_T[G]) \text{ for } V \in \mathrm{Loc}^G R$$

are monoidal.

The following lemmas show that the functors $\mathscr{A}_*$ and $\Omega^*$ are well defined on $\mathrm{QMon}^G T$ and $\mathrm{QAlg}^G T$ respectively.

**Lemma 4.2.26.** *Let $\Omega \in \mathrm{QPMon}^G T$, set $\mathscr{A} = \Omega_{R[G]}$ with multiplication $m$ and let $V, W, Z \in \mathrm{Loc}^G R$. Using notations from 4.2.13, the commutativity of the diagram 1) (resp. 2)) in definition 4.2.15 implies the commutativity between $x, y$ (resp. associativity among $x, y, z$) for sections $x \in \operatorname{Im}\theta_V$, $y \in \operatorname{Im}\theta_W$ (resp. and $z \in \operatorname{Im}\theta_Z$). In particular if $\Omega$ is symmetric (associative) then $\mathscr{A}_\Omega$ is commutative (associative). The converses to the previous statements hold if $\Omega$ is left exact.*

*Proof.* Denote by $\iota_{V,W} \colon \Omega_V \otimes \Omega_W \longrightarrow \Omega_{V\otimes W}$ the monoidal structure on $\Omega$ and by $\mathrm{ex} \colon A \otimes B \longrightarrow B \otimes A$ the exchange map. Let $v \in \Omega_V$, $w \in \Omega_W$, $z \in \Omega_Z$ and set also

$$\xi = \iota_{V,W\otimes Z}(v \otimes \iota_{W,Z}(w \otimes z)),\ \zeta = \iota_{V\otimes W,Z}(\iota_{V,W}(v \otimes w) \otimes z)$$

$$\eta = \Omega_{\mathrm{ex}}(\iota_{V,W}(v \otimes w)),\ \mu = \iota_{W,V}(w \otimes v)$$

If $\alpha \in V^\vee, \beta \in W^\vee,\ \gamma \in Z^\vee$ set

$$x = \theta_V(v \otimes \alpha)(\theta_W(w \otimes \beta)\theta(z \otimes \gamma)),\ y = (\theta_V(v \otimes \alpha)\theta_W(w \otimes \beta))\theta(z \otimes \gamma)$$

$$a = \theta_V(v \otimes \alpha)\theta_W(w \otimes \beta),\ b = \theta_W(w \otimes \beta)\theta_V(v \otimes \alpha)$$

Thanks to 4.2.24, we see that

$$x = \theta_{V\otimes W\otimes Z}(\xi \otimes \alpha \otimes \beta \otimes \gamma),\ y = \theta_{V\otimes W\otimes Z}(\zeta \otimes \alpha \otimes \beta \otimes \gamma)$$

$$a = \theta_{W\otimes V}(\eta \otimes \beta \otimes \alpha),\ b = \theta_{W\otimes V}(\mu \otimes \beta \otimes \alpha)$$

The commutativity of the diagrams 1) and 2) coincide with the equalities $\eta = \mu$ and $\xi = \zeta$ for any $v, w, z$ respectively, which imply the equalities $a = b$ and $x = y$ for any $v, w, z, \alpha, \beta, \gamma$. So the first claim holds. For the converse, it is enough to show that if $U \in \mathrm{Loc}^G R$ and $u \in \Omega_U$ then

$$\theta_U(u \otimes \delta) = 0\ \forall \delta \in U^\vee \implies u = 0$$

But this is the injectivity of the induced map $\Omega_U \longrightarrow \Omega_{R[G]} \otimes U$, which comes from 4.2.13 since $\Omega$ if left exact.

For the last claims, it is enough to note that $\theta_{R[G]}$ is surjective. Indeed taking the element $\phi \in R[G]^\vee$ corresponding to $\mathrm{id}_{R[G]} \in \mathrm{End}^G R[G]$ we have

$$\theta_{R[G]}(x \otimes \phi) = x \text{ for } x \in \Omega_{R[G]}$$

thanks to 4.2.13. □

**Lemma 4.2.27.** *If $\Omega \in \mathrm{QPMon}^G T$, then the natural transformation $\Omega \longrightarrow \Omega^{\mathscr{A}_\Omega}$ defined in 4.2.13 is monoidal.*

*Proof.* Set $\Gamma = \Omega^{\mathscr{A}_\Omega}$. The monoidality of the map $\Omega \longrightarrow \Gamma$ is expressed by the equalities of the two maps

$$\Omega_V \otimes \Omega_W \longrightarrow \Gamma_V \otimes \Gamma_W \longrightarrow \Gamma_{V\otimes W},\ \Omega_V \otimes \Omega_W \longrightarrow \Omega_{V\otimes W} \longrightarrow \Gamma_{V\otimes W}$$

for any $V, W \in \operatorname{Loc}^G R$. Since $-\otimes U\colon \operatorname{Loc}^G R \longrightarrow \operatorname{Loc}^G R$ is an exact functor for any $U \in \operatorname{Loc}^G R$, by 4.2.12 we have only to check that the above maps coincide when $V = W = R[G]$. In this case, by applying $\Omega$ and thanks to 4.2.19, we reduce to the problem of the commutativity of the following diagram

$$\begin{tikzcd}
A\otimes A \arrow[r, "\Delta_G\otimes\Delta_G"] \arrow[d, "\omega"] & A\otimes \underline{A}\otimes A\otimes \underline{A} \simeq A\otimes A\otimes \underline{A}\otimes \underline{A} \arrow[r, "m_A\otimes \mathrm{id}"] & A\otimes \underline{A}\otimes \underline{A} \arrow[d, "\mathrm{id}\otimes\omega"] \\
A\otimes \underline{A} \arrow[rr, "\Delta_G\otimes \mathrm{id}"] & & A\otimes \underline{A}\otimes \underline{A}
\end{tikzcd}$$

where $A = R[G]$ and $\omega\colon A\otimes A \xrightarrow{\simeq} A\otimes \underline{A}$ is the tensor product of $\Delta_G$ and $\mathrm{id}\otimes 1$. But the commutativity of such diagram can be checked directly taking spectra and using the functorial point of view. $\square$

**Lemma 4.2.28.** *Let $\Omega \in \operatorname{QPMon}^G T$ and $1 \in \Omega_R$. If $1$ is a unity for $\Omega$ then it is also a unity for $\Omega_{R[G]}$. The converse holds is $\Omega$ is left exact.*

*Proof.* If 1 is a unity for $\Omega$ we will have a commutative diagram

$$\begin{tikzcd}
 & \Omega_R\otimes\Omega_{R[G]} \arrow[r] \arrow[d] & \Omega_{R\otimes R[G]} \arrow[d] \arrow[dr] & \\
\Omega_{R[G]} \arrow[ur, "1\otimes\mathrm{id}"] \arrow[dr] & \Omega_{R[G]}\otimes\Omega_{R[G]} \arrow[r] & \Omega_{R[G]\otimes R[G]} \arrow[r] & \Omega_{R[G]}
\end{tikzcd}$$

and so 1 is a left unity for $\Omega_{R[G]}$. Similarly it is also a right unity. Conversely, if $\Omega$ is left exact, the result follows easily because the isomorphism $\Omega_V \simeq \underline{\operatorname{Hom}}^G(V^\vee, \Omega_{R[G]})$ is monoidal thanks to 4.2.27. $\square$

We are finally ready to prove Theorem 4.2.21.

*Proof.* *(of Theorem 4.2.21)* The functors of 4.2.4 are well defined over $\operatorname{QRings}^G T$ and $\operatorname{QPMon}^G T$ thanks to 4.2.23. We claim that they are well defined also over $\operatorname{QAlg}^G T$ and $\operatorname{QMon}^G T$. For the unities, using their uniqueness, it is enough to apply 4.2.28 and note that the natural transformation $\Omega \longrightarrow \Omega^{\mathscr{A}_\Omega}$ over $R \in \operatorname{Loc}^G R$ is just $\Omega$ applied to the inclusion $R \longrightarrow R[G]$. Associativity and commutativity instead come from 4.2.26.

The natural transformation $\Omega \longrightarrow \Omega^{\mathscr{A}_\Omega}$ is the one defined in 4.2.13, which is monoidal thanks to 4.2.27. It remains to prove that if $\mathscr{A} \in \operatorname{QRings}^G T$ then the comodule map $\mathscr{A} \longrightarrow \Omega^{\mathscr{A}}_{R[G]} = (\mathscr{A}\otimes R[G])^G$ is a map of rings. But the commutative diagram expressing this fact is exactly the diagram expressing the $G$-equivariance of the multiplication $\mathscr{A}\otimes\mathscr{A} \longrightarrow \mathscr{A}$, since, by construction, the ring structure on $(\mathscr{A}\otimes R[G])^G$ is the one as subring of $\mathscr{A}\otimes R[G]$. $\square$

### 4.2.3 Ramified Galois covers and the forgetful functor.

We have seen that a quasi-coherent sheaf of (commutative, associative and with unity) algebras, or, equivalently, an affine map, with an action of $G$ corresponds to a monoidal functor. In this subsection we want to describe the functors associated with $G$-covers. The theorem we want to prove is the following.

**Theorem 4.2.29.** *The map of stacks*

$$\begin{array}{ccc} G\text{-Cov} & \longrightarrow & \mathrm{LMon}_R^G \\ X \xrightarrow{f} T & \longmapsto & \Omega^{f_*\mathcal{O}_X} \end{array}$$

*is well defined and yields an isomorphism between* $G$-Cov *and the substack in groupoids of* $\mathrm{LMon}_R^G$ *of functors* $\Omega$ *that, in* $\mathrm{LAdd}_R^G$*, are fppf locally isomorphic to the forgetful functor. When* $G$ *is a glrg over a connected scheme, this is also the substack in groupoids of* $\Omega \in \mathrm{LMon}_R^G$ *such that* $\operatorname{rk}\Omega_V = \operatorname{rk} V$ *for all the representations* $V \in I_G$*.*

*Remark* 4.2.30. In general it is not true that, if $\mathscr{A} \in \mathrm{LAlg}_R^G T$ is such that $\operatorname{rk}\Omega_V^{\mathscr{A}} = \operatorname{rk} V$ for all $V \in \mathrm{Loc}^G R$, then $\mathscr{A} \in G\text{-Cov}(T)$, even for linearly reductive groups. A counterexample with $G = \mathbb{Z}/3\mathbb{Z}$, $R = \mathbb{Q}$ and $T = \operatorname{Spec} k$, where $k = \overline{\mathbb{Q}}$, is $\mathscr{A} = k[x,y]/(x,y)^2$ with the action of $\mu_3 \simeq G \times k$ given by graduation $\deg x = \deg y = 1 \in \mathbb{Z}/3\mathbb{Z}$. Denote by $k_i$ the irreducible $\mu_3$-representation over $k$ induced by $i \in \mathbb{Z}/3\mathbb{Z} = \mathrm{Hom}(\mu_3, \mathbb{G}_m)$. Note that $\mathbb{Z}/3\mathbb{Z}$ has only one non trivial irreducible representation $W$ over $\mathbb{Q}$ and it satisfies $W \otimes k = k_1 \oplus k_2$. Therefore $\mathbb{Z}/3\mathbb{Z}$ is not a glrg. The functor $\delta \colon \mathrm{Loc}^{\mu_3} k \longrightarrow \mathrm{Loc}\, k$ associated with $\mathscr{A} \in \mathrm{LAlg}^{\mu_3} k$ is simply given by $\delta_{k_0} = k$, $\delta_{k_1} = k^2$, $\delta_{k_2} = 0$. In particular $\mathscr{A} \notin G\text{-Cov}(k)$ by 4.2.29. On the other hand, since $G$-representations over $\mathbb{Q}$ decompose into irreducible representations, it is easy to check that $\Omega^{\mathscr{A}} \colon \mathrm{Loc}^G \mathbb{Q} \longrightarrow \mathrm{Loc}\, k$, which is nothing else that $\Omega_V^{\mathscr{A}} = \delta_{V\otimes k}$, satisfies $\operatorname{rk}\Omega_V^{\mathscr{A}} = \operatorname{rk} V$ for all $V \in \mathrm{Loc}^G \mathbb{Q}$.

*Remark* 4.2.31. Thanks to 4.1.1, the functor $\Omega^{\mathcal{O}_T[G]} \in \mathrm{QAdd}^G T$ associated with the regular representation is just the forgetful functor

$$\mathrm{Loc}^G R \ni V \longmapsto V \otimes \mathcal{O}_T \in \mathrm{Loc}\, T$$

*Proof.* (*of Theorem* 4.2.29) We will make use of 4.2.21. Let $X \xrightarrow{f} T \in G\text{-Cov}(T)$ and set $\mathscr{A} = f_*\mathcal{O}_X \in \mathrm{LAlg}^G T$. Since $\Omega^{\mathcal{O}_T[G]}$ is the forgetful functor and taking invariants behaves well under flat base changes, we have that $\Omega^{\mathscr{A}} \colon \mathrm{Loc}^G R \longrightarrow \mathrm{QCoh}\, T$ is fppf locally the forgetful functor. In particular $\Omega^{\mathscr{A}}$ is exact and $\Omega^{\mathscr{A}} \in \mathrm{LMon}^G T$, that is $\Omega^{\mathscr{A}}$ has image in $\mathrm{Loc}\, T$. If $T' \xrightarrow{h} T$ is any base change, then $h^* \circ \Omega^{\mathscr{A}}$ is still exact because exact sequences in $\mathrm{Loc}\, T$ split locally, and therefore

$$h^* \circ \Omega^{\mathscr{A}} \simeq \Omega^{h^*\Omega^{\mathscr{A}}_{R[G]}} \simeq \Omega^{h^*\mathscr{A}}$$

So the map in the statement is well defined and the first equivalence is clear from 4.2.4. Assume now that $G$ is a glrg. We have to show that $\Omega \in \mathrm{LAdd}_R^G$ is locally the forgetful functor if and only if $\operatorname{rk}\Omega_V = \operatorname{rk} V$ for all $V \in I_G$. This is clear from 4.1.15 and 4.1.17. $\square$

### 4.2.4 Strong monoidal functors and $G$-torsors.

After the description of $G$-covers in terms of functors, it arises naturally the question of what kind of functors correspond to $G$-torsors. We will show that the answer is strong monoidal functors. Notice that, over a field, this is a classical result of the Tannakian theory (see [DM82, Riv72]). Moreover such a result has already been proved in [Lur04], as a particular case of a more general theory. In this subsection we want to give a more elementary proof, based on the results obtained in the previous sections.

Notice also that the equivalence between $G$-torsors and strong monoidal functors, in the diagonalizable case, is another well known result (see 3.2.3) and it does not require the machinery developed here or the Tannakian theory.

Thanks to 4.2.25 and 4.2.31, we have a description of the trivial $G$-torsor:

**Proposition 4.2.32.** *The functor associated with the trivial $G$-torsor $\mathcal{O}_T[G]$ is just the forgetful functor*

$$\operatorname{Loc}^G R \ni V \longmapsto V \otimes \mathcal{O}_T \in \operatorname{Loc} T$$

*with the usual monoidal structure.*

For general $G$-torsors we need the following definition.

**Definition 4.2.33.** Given an $R$-scheme define $\operatorname{LSMon}^G T$ as the full subcategory of $\operatorname{LMon}^G T$ of objects $\Omega$ that are left exact, *strong* monoidal, i.e. such that for any $V, W \in \operatorname{Loc}^G R$ the map

$$\iota^{\Omega}_{V,W} \colon \Omega_V \otimes \Omega_W \longrightarrow \Omega_{V \otimes W}$$

is an isomorphism, and such that the map $\mathcal{O}_T \longrightarrow \Omega_R$ is injective. Define also $\operatorname{LSMon}^G_R$ as the full subcategory of $\operatorname{LMon}^G_R$ whose fibers over an $R$-scheme $T$ are $\operatorname{LSMon}^G T$.

**Theorem 4.2.34.** $\operatorname{LSMon}^G_R$ *is a substack of* $\operatorname{LMon}^G_R$ *and the functors*

$$\begin{array}{ccc} \operatorname{Spec} \Omega_{R[G]} & \longleftarrow\!\!\!\shortmid & \Omega \\ \mathrm{B}_R G & \longrightarrow & \operatorname{LSMon}^G_R \\ X \xrightarrow{f} T & \longmapsto & (- \otimes f_*\mathcal{O}_X)^G \end{array}$$

*are well defined and they are each other's inverse.*

We will prove Theorem above after the following lemma.

**Lemma 4.2.35.** *Let $\Omega \in \operatorname{LMon}^G T$ be a strong monoidal functor. Then $\Omega_R = \mathcal{O}_T$ and*

$$\Omega \text{ left exact} \iff \Omega \text{ exact} \iff \operatorname{Supp} \Omega_{R[G]} = T$$

*In particular $\operatorname{LSMon}^G_R$ is a substack of $\operatorname{LMon}^G_R$.*

*Proof.* Since $\Omega_R \otimes \Omega_R \simeq \Omega_R$ and $\mathcal{O}_T \subseteq \Omega_R$, we can conclude that $\operatorname{rk} \Omega_R = 1$. But $\Omega_R$ has a structure of $\mathcal{O}_T$ algebra induced by the multiplication $R \otimes R \longrightarrow R$. So $\operatorname{Spec} \Omega_R \longrightarrow T$ is a degree one cover, which is therefore an isomorphism.

For the equivalences, consider a short exact sequence in $\operatorname{Loc}^G R$

$$\mathcal{V}_*\colon \quad 0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

Since there exists a natural isomorphism $U \otimes R[G] \simeq \underline{U} \otimes R[G]$ for $U \in \operatorname{Loc}^G R$ (see [Jan87, Part I, Example 3.7]), we see that $\mathcal{V}_* \otimes R[G]$ is a splitting sequence in $\operatorname{Loc}^G R$. In particular $\Omega_{\mathcal{V}_* \otimes R[G]}$ is exact. Moreover $\Omega_{\mathcal{V}_* \otimes R[G]} \simeq \Omega_{\mathcal{V}_*} \otimes \Omega_{R[G]}$. If $\operatorname{Supp} \Omega_{R[G]} = T$, the functor $- \otimes \Omega_{R[G]}$ is faithful exact since $\Omega_{R[G]}$ is locally free, and therefore $\Omega$ is exact. Conversely, if $\Omega$ is left exact we have $\mathcal{O}_T = \Omega_R \subseteq \Omega_{R[G]}$.

For the final statement, we have to show that the subcategory $\operatorname{LSMon}^G_R \subseteq \operatorname{LMon}^G_R$ is preserved by the pullback. This follows because $\Omega_R = \mathcal{O}_T$ and the pullback of an exact sequence of locally free of finite rank sheaves is still exact. □

*Proof.* (*of Theorem* 4.2.34) $\operatorname{LSMon}^G_R$ is a substack of $\operatorname{LMon}^G_R$ thanks to 4.2.35. Since $G$ is flat, finite and of finite presentation, the push forward functor $\mathrm{B}_R G \longrightarrow \operatorname{LAlg}^G_R$ is fully faithful with essential image the full subcategory of algebras $\mathscr{A}$ for which there exist $G$-equivariant isomorphisms of algebras $\mathscr{A} \simeq \mathcal{O}[G]$ locally in the fppf topology. In what follows we identify $\mathrm{B}_R G$ with this stack. Since taking invariants commutes with flat base change, given $\mathscr{A} \in \mathrm{B}_R G(T)$, $\Omega^{\mathscr{A}}$ is locally isomorphic to the forgetful functor $\operatorname{Loc}^G R \longrightarrow \operatorname{Loc} T$, which is strong monoidal and left exact. Thanks to 4.2.21, $\Omega_*\colon \mathrm{B}_R G \longrightarrow \operatorname{LSMon}^G_R$ is fully faithful and we have only to prove that, if $\Omega \in \operatorname{LSMon}^G T$, then $\Omega_{R[G]} \in \mathrm{B}_R G(T)$. Note that $f\colon \operatorname{Spec} \Omega_{R[G]} \longrightarrow T$ is faithfully flat and finitely presented since $\Omega_{R[G]}$ is locally free and $\mathcal{O}_T \subseteq \Omega_{R[G]}$. So it clearly has sections in the fppf topology. We therefore need to show that the map

$$\rho\colon \Omega_{R[G]} \otimes \Omega_{R[G]} \longrightarrow \Omega_{R[G]} \otimes R[G] \text{ given by } \rho(x \otimes y) = \mu(x)(y \otimes 1)$$

is an isomorphism, where $\mu$ is the comodule structure on $\Omega_{R[G]}$. Set $A = R[G]$ and consider the map

$$\omega\colon A \otimes A \xrightarrow{\Delta_G \otimes (\mathrm{id}_A \otimes 1)} (A \otimes \underline{A}) \otimes (A \otimes \underline{A}) \simeq A \otimes A \otimes \underline{A} \otimes \underline{A} \xrightarrow{m_A \otimes m_A} A \otimes \underline{A}$$

where $m_A$ is the multiplication. The map $\omega$ is a $G$-equivariant isomorphism because it corresponds to $G \times G \ni (g,h) \longrightarrow (gh, g) \in G \times G$. Moreover it is easy to check that we have a commutative diagram

$$\begin{tikzcd}
\Omega_A \otimes \Omega_A \arrow[r, "\rho"] \arrow[d] & \Omega_A \otimes R[G] \\
\Omega_{A \otimes A} \arrow[r, "\Omega_\omega"] & \Omega_{A \otimes \underline{R[G]}} \arrow[u, "\wr"']
\end{tikzcd}$$

Since $\Omega$ is strong monoidal we get the result.

□

### 4.2.5 Super solvable groups and $G$-torsors.

Where $G$ is a diagonalizable group and $\Omega \in \mathrm{LMon}^G$ we know that $\Omega$ corresponds to a $G$-torsor if and only if the maps

$$\Omega_m \otimes \Omega_n \longrightarrow \Omega_{m+n} \quad \forall m, n \in \mathrm{Hom}(G, \mathbb{G}_m)$$

are isomorphisms and $\Omega_0 = \mathcal{O}$. Here $\Omega_m = \Omega_{V_m}$, where $V_m$ is the one dimensional representation associated to $m \in \mathrm{Hom}(G, \mathbb{G}_m)$. On the other hand this condition is also equivalent to require that $\Omega_0 = \mathcal{O}$ and that the maps

$$\Omega_m \otimes \Omega_{-m} \longrightarrow \Omega_0 = \mathcal{O} \quad \forall m \in \mathrm{Hom}(G, \mathbb{G}_m)$$

are surjective (and therefore isomorphisms). We want to generalize this kind of statement for a larger class of groups, namely super solvable groups (see 4.2.36 for the definition).

In this section we will assume that $G$ is a glrg and we continue to work on a base ring $R$.

**Definition 4.2.36.** We will say that a group scheme $G$ over an algebraically closed field is *super solvable* if there exists a filtration by closed subgroups

$$1 = H_0 < H_1 < \cdots < H_r = G$$

such that $H_i \lhd G$ and $H_{i+1}/H_i \simeq \mu_p$ for some prime $p$ and for all $i$.

A finite, flat and finitely presented group scheme $G$ over a base $S$ will be called super solvable if it is so over any geometric point.

*Remark* 4.2.37. In our hypothesis, if $G$ is constant over an algebraically closed field $k$, then it is super solvable according to the above definition if and only if it is so as abstract group. Indeed, since $G$ is linearly reductive, we will have $\mathrm{char}\, k \nmid |G|$, and $\mu_q \simeq \mathbb{Z}/q\mathbb{Z}$ if $\mathrm{char}\, k \nmid q$.

*Remark* 4.2.38. Assume that $R$ is strictly Henselian. If $H$ is an open and closed normal subgroup of $G$ which is diagonalizable, then the conjugacy yields an action of $\underline{G}/\underline{H}$ on $\mathrm{Hom}(H, \mathbb{G}_m)$. In particular if $H = G_1$ we get an action of $\underline{G}$ on the group $M = \mathrm{Hom}(G_1, \mathbb{G}_m)$. Indeed $G$ acts by conjugacy on $H$ and, since $H$ is abelian, it induces an action of $\underline{G}/\underline{H} = G/H$ on $H$ and therefore on $\mathrm{Hom}(H, \mathbb{G}_m)$.

*Notation* 4.2.39. In the situation of remark 4.2.38 we will consider $\mathrm{Hom}(H, \mathbb{G}_m)$ and, in particular, $\mathrm{Hom}(G_1, \mathbb{G}_m)$, endowed by the left action of $\underline{G}$ defined above.

The following remark gives a concrete description of what a super solvable group is over an algebraically closed field.

*Remark* 4.2.40. Assume that $R = k$ is an algebraically closed field. Then $G$ is super solvable if and only if $\underline{G}$ is super solvable and there exists a filtration by subgroups

$$0 = H_0 < H_1 < \cdots < H_r = M = \mathrm{Hom}(G_1, \mathbb{G}_m)$$

such that each $H_i$ is $\underline{G}$-stable and $H_{i+1}/H_i$ is cyclic of prime order.

*Notation* 4.2.41. Given a group $G$ over a scheme $S$ and a character $\chi \in \operatorname{Hom}(G, \mathbb{G}_m)$ we will denote by $V_\chi$ the representation of $G$ on $\mathcal{O}_S$ induced by such character.

Given $\mathscr{A} \in \operatorname{CAlg}^G T$ and a representation $V \in I_G$ we set

$$\omega_V^{\mathscr{A}} : \Omega_V^{\mathscr{A}} \otimes \Omega_{V^\vee}^{\mathscr{A}} \longrightarrow \Omega_{V \otimes V^\vee}^{\mathscr{A}} \longrightarrow \Omega_R^{\mathscr{A}} = \mathscr{A}^G$$

We will also write simply $\omega_V$ instead of $\omega_V^{\mathscr{A}}$ if this will not lead to confusion.

**Theorem 4.2.42.** *Let $G$ be a super solvable glrg and let $\mathscr{A} \in \operatorname{LAlg}^G T$. Then $\mathscr{A} \in \operatorname{B} G$ if and only if $\mathscr{A}^G = \Omega_R^{\mathscr{A}} \simeq \mathcal{O}_T$ and for any representation $V \in I_G$ the map*

$$\omega_V^{\mathscr{A}} : \Omega_V^{\mathscr{A}} \otimes \Omega_{V^\vee}^{\mathscr{A}} \longrightarrow \Omega_{V \otimes V^\vee}^{\mathscr{A}} \longrightarrow \Omega_R^{\mathscr{A}} \simeq \mathcal{O}_T$$

*is surjective.*

Before proving the above Theorem, we need some preliminary results.

**Lemma 4.2.43.** *Let $G$ be a constant super solvable group, $H$ be a subgroup and $k$ be an algebraically closed field such that* $\operatorname{char} k \nmid |G|$. *If $V^H \neq 0$ for all the irreducible representations $V$ of $G$ over $k$ then $H = 0$.*

*Proof.* We will argue by induction on $|G|$. If $G = 0$ there is nothing to prove. So assume $G \neq 0$. If $K \neq 0$ is a normal subgroup of $G$ and $\phi\colon G \longrightarrow G/K$ is the projection, then $\phi(H) < G/K$ satisfies the inductive hypothesis and therefore $\phi(H) = 0$, i.e. $H \subseteq K$. In particular we can choose $K$ to be cyclic since $G$ is super solvable and we can conclude that $H$ is normal and abelian in $G$. Let $W$ be an irreducible $H$-representation. Given a system $\mathcal{R}$ of representatives of $G/H$ we can write

$$R_H \operatorname{ind}_H^G W = \bigoplus_{g \in \mathcal{R}} W_g$$

where $W_g$ is the representation of $H$ given by $W$ and the action $h \star x = ghg^{-1}x$. By hypothesis, we know that $(\operatorname{ind}_H^G W)^H \neq 0$. So there exist $g \in \mathcal{R}$, $x \in W_g$ such that $h \star x = ghg^{-1}x = x$ for any $h \in H$. Since $H$ is normal we can conclude that $W^H \neq 0$. So $H$ has only the trivial representation and therefore $H = 0$. $\square$

**Lemma 4.2.44.** *Let $M$ be an abelian $p$-group, for a prime $p$, and $G$ be a constant group acting on $M$. Assume that there exists a filtration*

$$0 = H_0 < H_1 < \cdots < H_r = M$$

*by $G$-stable subgroup such that $H_{i+1}/H_i \simeq \mathbb{Z}/p\mathbb{Z}$. Then for any proper subgroup $H$ of $M$ there exists a $G$-orbit in $M - H$.*

*Proof.* We can assume that $H$ has index $p$. In particular $pM \subseteq H$. The action of $G$ on $M$ induces an action of $G$ on $M/pM$ that has a filtration like the one of $M$. Since $H/pM$ is a proper subgroup of $M/pM$ we can assume that $pM = 0$, i.e. $M$ is a finite $\mathbb{F}_p$ vector space. Choosing a basis $e_1, \ldots, e_r$ according to the given filtration, we can assume that $G$ acts by triangular matrices. Let $H$ be a subspace of $M$ and assume that any $G$-orbit has an element in $H$. We have to prove that $H = M$. Set $e_0 = 0$ and assume by induction that $e_0, \dots, e_{j-1} \in H$ for $j \leq n$. We have that $Ge_j \cap H \neq \emptyset$. Since $G$ acts by triangular matrices we can write

$$H \ni g(e_j) = \lambda e_j + x \text{ with } \lambda \in \mathbb{F}_p^*,\ x \in < e_1, \dots, e_{j-1} >_{\mathbb{F}_p} \subseteq H$$

So $\lambda e_j \in H$ and $e_j \in H$. □

In what follows $G$ is still our glrg over the base ring $R$.

**Lemma 4.2.45.** *If $V \in I_G$ then $\operatorname{rk} V \in R^*$.*

*Proof.* Consider $W = \operatorname{Hom}(V, V)$ and note that, by 4.1.11, we have $W^G = R \cdot \mathrm{id}_V$. Since $G$ is a glrg, there exists a $G$-equivariant map $\phi \colon W \longrightarrow R$ such that $\phi(\mathrm{id}_V) = 1$. On the other hand any $G$-equivariant map $V \otimes V^\vee \longrightarrow R$ is of the form $\lambda e_V$ for $\lambda \in R$, where $e_V$ is the evaluation $e_V(v \otimes \psi) = \psi(v)$. If $r = \operatorname{rk} V$ it is easy to check that $e_V(\mathrm{id}_V) = r$. So $1 = \lambda e_V(\mathrm{id}_V) = \lambda r$. □

**Lemma 4.2.46.** *Let $\mathscr{A} \in \mathrm{LAlg}^G T$ and $V \in I_G$. Then*

$$\pi(\theta_V(\phi \otimes x)\theta_{V^\vee}(v \otimes y)) = \frac{\phi(v)}{\operatorname{rk} V}\omega_V(x \otimes y)$$

*where $\phi \in V^\vee, v \in V, x \in \Omega_V^{\mathscr{A}}, y \in \Omega_{V^\vee}^{\mathscr{A}}$, $\theta_- \colon (-)^\vee \otimes \operatorname{Hom}^G((-)^\vee, \mathscr{A}) \longrightarrow \mathscr{A}$ is the evaluation and $\pi \colon \mathscr{A} \longrightarrow \mathscr{A}^G$ is the projection according to the $G$-equivariant decomposition of $\mathscr{A}$.*

*Proof.* For any $W \in \mathrm{Loc}^G R$, the map

$$W^\vee \otimes \Omega_W = W^\vee \otimes \operatorname{Hom}^G(W^\vee, \mathscr{A}) \longrightarrow \mathscr{A} \longrightarrow \mathscr{A}^G$$

is non zero only on the factor $(W^G)^\vee \otimes \Omega_{W^G}$. Let $W = V^\vee \otimes V$ and remember that, by 4.1.11, we have $W^G = R\mathrm{id}_V$. Under the isomorphism $W \simeq W^\vee$, $\mathrm{id}_V$ is sent to the evaluation $e_V \colon V^\vee \otimes V \longrightarrow R$, while $\phi \otimes v$ to the map $\psi$ given by $\psi(\delta \otimes z) = \delta(v)\phi(z)$. The equivariant section of $R \xrightarrow{1 \longrightarrow e_V} W^\vee$ is given by $(\phi \longrightarrow \phi(\mathrm{id}_V)/\operatorname{rk} V)$ so the component of $\psi$ in $(W^\vee)^G$ is

$$e_V \psi(\mathrm{id}_V)/\operatorname{rk} V = e_V \phi(v)/\operatorname{rk} V$$

By definition $\omega_V(x \otimes y) = x \otimes y(e_V)$ and taking into account 4.2.24 we have

$$\pi(\theta_V(\phi \otimes x)\theta_{V^\vee}(v \otimes y)) = x \otimes y(e_V)\phi(v)/\operatorname{rk} V = \omega_V(x \otimes y)\phi(v)/\operatorname{rk} V$$

□

During the proof of Theorem 4.2.42, we will reduce to consider local algebras. The following lemma explains what happens in this situation.

**Lemma 4.2.47.** *Assume that $R$ is strictly Henselian and let $A \in \mathrm{CAlg}^G R$ be a local $R$-algebra such that $A^G = R$. Then*

- *If $G = \underline{G}$ then the maximal ideal of $A$ is*

$$m_R \oplus \bigoplus_{R \neq V \in I_G} V^\vee \otimes \Omega^A_V$$

*and for any $R \neq V \in I_G$ the map $\omega^A_V$ is not surjective.*

- *If $G = G_1$ then*

$$H = \{m \in M \mid ({V_m}^\vee \otimes \Omega^A_{V_m}) \cap A^* \neq \emptyset\} = \{m \in M \mid \omega_{V_m} \text{ is surjective}\}$$

*is a subgroup of $M$, and the subalgebra $B = \oplus_{m \in H} {V_m}^\vee \otimes \Omega_{V_m} \subseteq A$ is a $D(H)$-torsor.*

*Proof.* Set $\Omega = \Omega^A$, $\omega = \omega^A$ and $k = R/m_R$. Assume $G = \underline{G}$ and let $m_A$ be the maximal ideal of $A$. Since $m_A$ is stable under the action of $\underline{G}$, it can be written as

$$m_A = \bigoplus_{V \in I_G} V^\vee \otimes \Gamma_V$$

where $\Gamma_V \subseteq \Omega_V$. In particular

$$L = A/m_A = \bigoplus_{V \in I_G} V^\vee \otimes (\Omega_V/\Gamma_V)$$

$G$ acts on $L$ and $L^G = \Omega_R/\Gamma_R = k$. Therefore $L/k$ is separable, i.e. $L = k$ and by dimension we get the first equality. Taking into account 4.2.46 we also have that if $R \neq V \in I_G$ then $\omega_V$ is not surjective.

Now assume $G = G_1 = D(M)$ and set $\Omega_m = {V_m}^\vee \otimes \Omega_{V_m}$. Note that if $\Omega_m \otimes \Omega_{-m} \longrightarrow R$ is surjective then $\Omega_m \cap A^* \neq \emptyset$ since $R$ is local. Conversely if $x \in \Omega_m \cap A^*$ let $\lambda = x^{|M|} \in R$. If $\lambda \in m_R$ then $x \in m_A$, which is not the case. Since $x^{|M|-1} \in \Omega_{-m}$ and therefore $\omega_m(x \otimes x^{|M|-1}) = x^{|M|} \in R^*$ we have that $\omega_m$ is surjective. Finally if $x \in \Omega_m \cap A^*$ and $y \in \Omega_n \cap A^*$ then $xy \in \Omega_{m+n} \cap A^*$. So $H$ is a subgroup and $B$ is a $D(H)$-torsor thanks to 3.2.3. $\square$

The following two lemmas describe how the associated functors $\Omega^*$ change when making an induction or taking invariants.

**Lemma 4.2.48.** *Let $H$ be a subgroup scheme of $G$ and assume they are both glrg. If $\mathscr{A} \in \mathrm{CAlg}^H T$, then*

$$\mathrm{ind}^G_H \mathscr{A} \simeq (\mathscr{A} \otimes R[G])^H \in \mathrm{CAlg}^G T \text{ and } \Omega^{\mathrm{ind}^G_H \mathscr{A}} = \Omega^{\mathscr{A}} \circ \mathrm{R}_H \colon \mathrm{Loc}^G R \xrightarrow{\mathrm{R}_H} \mathrm{Loc}^H R \xrightarrow{\Omega^{\mathscr{A}}} \mathrm{FCoh}\, T$$

*Proof.* We have

$$\Omega_V^{\operatorname{ind}_H^G \mathscr{A}} = \underline{\operatorname{Hom}}^G(V^\vee, \operatorname{ind}_H^G \mathscr{A}) \simeq \underline{\operatorname{Hom}}^H((\mathrm{R}_H V^\vee), \mathscr{A}) = \Omega^{\mathscr{A}}_{\mathrm{R}_H V}$$

So $\operatorname{ind}_H^G \mathscr{A} \in \operatorname{CAlg}^G T$ and it is a subring $\Omega_{R[G]}^{\operatorname{ind}_H^G \mathscr{A}} \simeq (\mathscr{A} \otimes R[G])^H \subseteq \mathscr{A} \otimes R[G]$. □

**Lemma 4.2.49.** *Let $K$ be a normal subgroup scheme of $G$ and $\mathscr{A} \in \operatorname{CAlg}^G T$. Then $\mathscr{A}^K \in \operatorname{CAlg}^{G/K} T$ and its associated functor is*

$$\operatorname{Loc}^{G/K} R \xrightarrow{\text{restriction}} \operatorname{Loc}^G R \xrightarrow{\Omega^{\mathscr{A}}} \operatorname{FCoh} T$$

*Proof.* Note that if $F\colon (\mathrm{Sch}/R)^{op} \longrightarrow (\mathrm{Sets})$ is a functor with an action of $G$, then $F^K$ is stable under the action of $G$ and therefore the map $G \longrightarrow \underline{\operatorname{Aut}} F^K$ factors through $G/K \longrightarrow \underline{\operatorname{Aut}} F^K$. So $\mathscr{A}^K \in \operatorname{CAlg}^G T$ and $\mathscr{A}^K \in \operatorname{CAlg}^{G/K} T$. Now note that if $V \in \operatorname{Loc}^{G/K} R$ then

$$\underline{\operatorname{Hom}}^G(\mathrm{R}_G V, \mathscr{A}) \simeq \underline{\operatorname{Hom}}^G(\mathrm{R}_G V, \mathscr{A}^K) = \underline{\operatorname{Hom}}^G(\mathrm{R}_G V, \mathrm{R}_G \mathscr{A}^K)$$

since $K$ acts trivially on $R_G V$. So we have to prove that the natural map

$$\underline{\operatorname{Hom}}^{G/K}(V, W) \longrightarrow \underline{\operatorname{Hom}}^G(\mathrm{R}_G\mathrm{V}, \mathrm{R}_G\mathrm{W}) \text{ for } V \in \operatorname{Loc}^{G/K} T,\ W \in \operatorname{QCoh}^{G/K} T$$

is an isomorphism. In order to do that, note that $\underline{\operatorname{Hom}}(\mathrm{R}_G V, \mathrm{R}_G W) = \mathrm{R}_G \underline{\operatorname{Hom}}(V, W)$ and that in general $(\mathrm{R}_G U)^G = U^{G/K}$ for all $G/K$-modules $U$. □

**Lemma 4.2.50.** *Let $R'$ be a local $R$-algebra, $H$ be a subgroup scheme of $G$ with a good representation theory and $\mathscr{B} \in \operatorname{CAlg}^H R'$. If we set $\mathscr{A} = \operatorname{ind}_H^G \mathscr{B}$ and we take $V \in \operatorname{Loc}^G R$ then*

$$\omega_V^{\mathscr{A}} \text{ surjective } \iff \exists \Delta \in I_H \text{ s.t. } \operatorname{Hom}^H(\Delta, V) \neq 0 \text{ and } \omega_\Delta^{\mathscr{B}} \text{ is surjective}$$

*Proof.* Let $\Omega = \Omega^{\mathscr{A}}$ and $\delta = \Omega^{\mathscr{B}}$. By 4.2.48, we know that $\Omega = \delta \circ \mathrm{R}_H$. Denote by $\iota^\Omega$, $\iota^\delta$ the natural transformations that define the monoidal structures of $\Omega$ and $\delta$ respectively. Let $V \in I_G$. If we set $V_\Delta = \operatorname{Hom}^H(\Delta, V)$ for $\Delta \in \operatorname{Loc}^H R$ we have

$$V \simeq \bigoplus_{\Delta \in I_H} V_\Delta \otimes \Delta \text{ and } V^\vee \simeq \bigoplus_{\Delta \in I_H} {V_{\Delta^\vee}}^\vee \otimes \Delta$$

The map $\iota^\Omega_{V,V^\vee}\colon \Omega_V \otimes \Omega_{V^\vee} \longrightarrow \Omega_{V \otimes V^\vee}$ factors through

$$\operatorname{id} \otimes \iota^\delta_{\Delta,\Lambda^\vee}\colon {V_\Delta}^\vee \otimes V_\Lambda \otimes \delta_\Delta \otimes \delta_{\Lambda^\vee} \longrightarrow {V_\Delta}^\vee \otimes V_\Lambda \otimes \delta_{\Delta \otimes \Lambda^\vee}$$

Now call $e_W\colon W \otimes W^\vee \longrightarrow R$ the evaluation map for any $R$-module $W$. The map $e_V$ sends any $(V_\Delta \otimes \Delta) \otimes (V_\Lambda \otimes \Lambda)^\vee$ with $\Delta \neq \Lambda$ to $0$ and restricts to

$$(V_\Delta \otimes \Delta) \otimes (V_\Delta \otimes \Delta)^\vee \simeq V_\Delta \otimes {V_\Delta}^\vee \otimes \Delta \otimes \Delta^\vee \xrightarrow{e_{V_\Delta} \otimes e_\Delta} R$$

So the composition $\omega^{\mathscr{A}}_V = \Omega_{e_V} \circ \iota^{\Omega}_{V,V^\vee} \colon \Omega_V \otimes \Omega_{V^\vee} \longrightarrow R$ is 0 on ${V_\Delta}^\vee \otimes V_\Lambda \otimes \delta_\Delta \otimes \delta_{\Lambda^\vee}$ if $\Lambda \neq \Delta$, is $e_V \otimes \omega^{\mathscr{B}}_\Delta$ if $\Lambda = \Delta$. In particular

$$\operatorname{Im}(\omega^{\mathscr{A}}_V) = \sum_{\Delta \mid V_\Delta \neq 0} \operatorname{Im}(\omega^{\mathscr{B}}_\Delta)$$

Since $R'$ is local we get the required result. $\square$

**Lemma 4.2.51.** *Let $H$ be a subgroup scheme of $G$ and $\mathscr{B} \in \operatorname{LAlg}^H T$. Then*

$$\operatorname{ind}^G_H \mathscr{B} \in \operatorname{B} G \iff \mathscr{B} \in \operatorname{B} H$$

*Proof.* If $\mathscr{B} \in \operatorname{B} H$ it is enough to note that $\operatorname{ind}^G_H \mathcal{O}[H] \simeq \mathcal{O}[G]$. So assume $\mathscr{A} = \operatorname{ind}^G_H \mathscr{B} \in \operatorname{B} G$. Since $\mathscr{B}$ is locally free, the condition of being a $H$-torsor is open, and we can assume $R = k$ and $T = \operatorname{Spec} R$, where $k$ is an algebraically closed field. If $\delta = \Omega^{\mathscr{B}}$, from 4.2.48, we know that $\Omega^{\mathscr{A}} = \delta \circ \mathrm{R}_H$. By 4.2.34, we have to prove that for any $V, W \in \operatorname{Loc}^H k$ the map $i^\delta_{V,W} \colon \delta_V \otimes \delta_W \longrightarrow \delta_{V \otimes W}$ is an isomorphism. Note that if $V = V_1 \oplus V_2$ in $\operatorname{Loc}^H k$, then $\iota^\delta_{V,W}$ is an isomorphism if and only if $\iota^\delta_{V_1,W}$ and $\iota^\delta_{V_2,W}$ are so. Therefore we must find a $H$-representation $V$, containing all the irreducible representations of $H$ and such that $\iota^\delta_{V,V}$ is an isomorphism. I claim that $V = \mathrm{R}_H k[G]$ satisfies the request. Indeed $\iota^\delta_{V,V}$ is an isomorphism since $\mathscr{A}$ is a $G$-torsor and $\Omega^{\mathscr{A}} = \delta \circ \mathrm{R}_H$. Moreover, since we have a $H$-equivariant surjective map $\mathrm{R}_H k[G] \longrightarrow k[H]$, we have that any irreducible representation of $H$ is a quotient and therefore a subrepresentation of $\mathrm{R}_H R[G]$. $\square$

**Lemma 4.2.52.** *Assume that $R$ is strictly Henselian. Let also $H$ be an open and closed normal subgroup of $G$ which is diagonalizable. Then, if $m \in \operatorname{Hom}(H, \mathbb{G}_m)$ we have a $H$-equivariant isomorphism*

$$\mathrm{R}_H \operatorname{ind}^G_H V_m \simeq \bigoplus_{g \in \underline{G}/\underline{H}} V_{g(m)}$$

*where the action of $\underline{G}/\underline{H}$ on $\operatorname{Hom}(H, \mathbb{G}_m)$ is the one given in 4.2.38. Moreover $\operatorname{ind}^G_H V_m \simeq \operatorname{ind}^G_H V_n$ if and only if there exists $g \in \underline{G}/\underline{H}$ such that $g(m) = n$.*

*Proof.* Set $\mathcal{O}_m = \mathrm{W}(V_m)$. By 4.1.28, we have a decomposition of $G$ into $H$-torsors

$$G = \bigsqcup_{i \in \underline{G}/\underline{H}} H^i$$

In particular we have

$$\operatorname{ind}^G_H \mathcal{O}_m = \underline{\operatorname{Hom}}^H(G, \mathcal{O}_m) \simeq \prod_{i \in \underline{G}/\underline{H}} \underline{\operatorname{Hom}}^H(H^i, \mathcal{O}_m)$$

Since $H^i$ is a $H$-torsor for all $i$, we see that $U_i = \underline{\operatorname{Hom}}^H(H^i, \mathcal{O}_m)$ is an invertible sheaf on $R$ and therefore $U_i \simeq \mathrm{W}(R)$. Now consider the right action of $G$ on itself given by the multiplication. Each $H^i$ is $H$ invariant since $H$ is normal, so the above decomposition is $H$-equivariant. In particular $U_i \simeq \mathcal{O}_n$ for some $n \in \mathrm{Hom}(H, \mathbb{G}_m)$. In order to compute $n$, we can assume that $R = k$ is an algebraically closed field, so that $H^i = Hi$, where we think of $i$ as an element of $H^i(k)$. Given $f\colon Hi \longrightarrow \mathcal{O}_m \in \underline{\mathrm{Hom}}^H(Hi, \mathcal{O}_m)$ and $h \in H$ we have

$$(h \star f)(ti) = m(t)f(ih) = m(t)f(ihi^{-1}i) = m(ihi^{-1})f(ti) \implies h \star f = i^{-1}(m)(h)f$$

and therefore that $n = i^{-1}(m)$.

For the last claim, a $G$-equivariant isomorphism between $\mathrm{ind}_H^G V_m$ and $\mathrm{ind}_H^G V_n$ is also $H$-equivariant and therefore $g(m) = n$ for some $g \in \underline{G}/\underline{H}$. Conversely note that two representations of $G$ are isomorphic if and only if they are so on the algebraic closure of the residue field of $R$, because the restriction $I_G \longrightarrow I_{G_k}$ is an isomorphism by hypothesis and $R$ is local. So we can again assume $R = k$ algebraically closed. Given $g \in G(k) \simeq \underline{G}$, we claim that

$$\psi\colon\ \mathrm{ind}_H^G \mathcal{O}_m = \underline{\mathrm{Hom}}^H(G, \mathcal{O}_m) \longrightarrow \underline{\mathrm{Hom}}^H(G, \mathcal{O}_{g(m)}) = \mathrm{ind}_H^G \mathcal{O}_n,\ \ \psi(f)(u) = f(g^{-1}u)$$

is $G$-equivariant isomorphism. It is well defined since

$$\psi(f)(hu) = f(g^{-1}hu) = f(g^{-1}hgg^{-1}u) = m(g^{-1}hg)f(g^{-1}u) = g(m)(h)\psi(f)(u)$$

It is $G$-equivariant since

$$(u \cdot \psi(f))(v) = \psi(f)(vu) = f(g^{-1}vu) = (u \cdot f)(g^{-1}v) = \psi(u \cdot f)(v)$$

$\square$

*Proof.* (*of theorem* 4.2.42) If $\mathscr{A} \in \mathrm{B}\,G$ then, by 4.2.34, all the maps $\omega_V$ are surjective since $\Omega_{V \otimes V^\vee} \longrightarrow \Omega_R$ is surjective. Conversely, since both conditions in the statement are open conditions, we can assume that $T = \operatorname{Spec} R$, that $R = k$ is an algebraically closed field and replace $\mathscr{A}$ by a finite $k$-algebra $A$. By 4.1.34, we can write $A \simeq \mathrm{ind}_H^G A_p$ where $p$ is a closed point of $\operatorname{Spec} A$ and $H$ is the stabilizer of $\operatorname{Spec} A_p$. In particular $A_p \in \mathrm{LAlg}^H k$ and therefore comes from a functor $\delta\colon \mathrm{Loc}^H k \longrightarrow \mathrm{Vect}_k \in \mathrm{LMon}^H k$. We want to prove first that $H = G_1$. We set $V_\Delta = \mathrm{Hom}^H(\Delta, V)$ for $V \in \mathrm{Loc}^G R$ and $\Delta \in \mathrm{Loc}^H R$. We will use 4.2.43 showing that for any $V \in I_{\underline{G}}$ we have $V^{\underline{H}} \neq 0$, so that $\underline{H} = 1$. By 4.2.50, since $\omega_V^A$ is surjective, there exists $\Delta \in I_H$ such that $V_\Delta \neq 0$ and $\omega_\Delta^{A_p}$ is surjective. We will show that $\Delta = R$. Since $R_H R_G V = R_H R_{\underline{H}} V$, we have that $V_\Lambda = 0$ if $\Lambda \in I_H - I_{\underline{H}}$. So $\Delta \in I_{\underline{H}}$. Since $A_p$ is local, we have that $A_p^{G_1} \in \mathrm{LAlg}^{\underline{H}} R$ is local thanks to 4.1.29. From 4.2.49 and 4.2.47 follows that if $\Lambda \in I_{\underline{H}}$ then $\omega_\Lambda^{A_p^{G_1}} = \omega_\Lambda^{A_p}$ and that $\omega_\Lambda^{A_p^{G_1}} = 0$ if $\Lambda \neq R$. Since $\omega_\Delta^{A_p} \neq 0$ and $\Delta \in I_{\underline{H}}$ we can conclude that $\Delta = R$ and therefore $V_\Delta = V^H = V^{\underline{H}} \neq 0$, as required.

We want now to prove that $A_p$ is a $G_1 = D(M)$-torsor. By 4.2.47, the set $Q = \{m \in M \mid \omega_{V_m}$ is surjective$\}$ is a subgroup of $M$ and, if we prove that $Q = M$, we will have that $A_p$ is a $G_1$-torsor and therefore that $A$ is a $G$-torsor. We will use 4.2.44 and 4.2.52. Given $m \in M$ we have shown that there must exist $n \in N$ such that

$$V_n \subseteq \mathrm{R}_{G_1} \operatorname{ind}_{G_1}^{G} V_m = \bigoplus_{g \in \underline{G}} V_{g(m)} \text{ and } \omega_{V_n} \text{ is surjective}$$

So given $m \in M$ there exists $g \in \underline{G}$ such that $g(m) \in Q$ and therefore $Q = M$ as required. $\square$

*Remark* 4.2.53. We want to show now an example of a constant solvable group and of a group $G$ such that $\underline{G}$ is super solvable for which 4.2.42 does not apply.

Let $k$ be an algebraically closed field and set

$$G = (\mu_2 \times \mu_2) \ltimes \mathbb{Z}/3\mathbb{Z}$$

where the action of $\mathbb{Z}/3\mathbb{Z}$ on $\operatorname{Aut}(\mu_2 \times \mu_2) \simeq \operatorname{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{GL}_2 \mathbb{F}_2$ is given by the order 3 matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

If $\operatorname{char} k \neq 2$, then $G$ is constant and solvable, if $\operatorname{char} k = 2$ then $\underline{G} = \mathbb{Z}/3\mathbb{Z}$ which is super solvable.

Set $K = (\mathbb{Z}/2\mathbb{Z})^2$ with $\mathbb{F}_2$ basis $e_1 = (1,0)$, $e_2 = (0,1)$, $H = D(K)$. Since $Ae_1 = e_2$, $A^2 e_1 = e_1 + e_2$, we see that $\mathbb{Z}/3\mathbb{Z}$ permutes the 3 subgroups of index 2 of $K$. We now describe the irreducible representations of $G$. We claim that they are

$$U = \operatorname{ind}_H^G V_{e_1},\ k,\ V_\chi,\ V_{\chi^2}$$

where $\chi \colon G \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow k^*$ is a non trivial character. We will make use of 4.2.52. If $V \in I_G$, there exists $u \in K$ such that

$$V \subseteq \operatorname{ind}_H^G V_u = V_u \oplus V_{Au} \oplus V_{A^2 u}$$

If two among $V_u, V_{Au}, V_{A^2u}$ are isomorphic then $u = Au = A^2 u$ and therefore $u = 0$. In this case

$$\operatorname{ind}_H^G V_0 = k \oplus V_\chi \oplus V_{\chi^2}$$

So assume $u \neq 0$. Since $K - \{0\}$ is a $\mathbb{Z}/3\mathbb{Z}$-orbit, $\operatorname{ind}_H^G V_{e_1} \simeq \operatorname{ind}_H^G V_{e_2} \simeq \operatorname{ind}_H^G V_{e_1+e_2}$. So we have to prove that $U = \operatorname{ind}_H^G V_{e_1}$ is irreducible. If it is not so, it will contain an irreducible representation of dimension 1 whose restriction to $H$ is not trivial. So there must exist a character $\eta \colon G \longrightarrow \mathbb{G}_m$ such that $\eta_{|H}$ is not trivial. But if $\zeta \in H = \underline{\operatorname{Hom}}(K, \mathbb{G}_m)$, we denote by $\xi \in G(k)$ the generator of $\mathbb{Z}/3\mathbb{Z}$ and $\chi = \zeta \circ A$ we have

$$A^2 = A + \mathrm{id} \implies \xi \chi \xi^{-1} \chi^{-1} = (\chi \circ A)\chi^{-1} = \zeta$$

So $\eta_{|H}(\zeta) = \eta(\xi)\eta(\chi)\eta(\xi)^{-1}\eta(\chi)^{-1} = 1$.

We are ready to explain the counterexample to 4.2.42 for the above group $G$. Consider

$$B = k[x,y]/(x^2 - 1, y^2) \in \mathrm{LAlg}^H k \text{ and } A = \operatorname{ind}_H^G B$$

where the action of $H$ on $G$ is given by $\deg x = e_1$, $\deg y = e_2$. We want to show that $\omega_V^A$ is surjective for any $V \in I_G$ but $A$ is not a $G$-torsor. Taking into account 4.2.51, $A$ is not a $G$-torsor because $B$ is not a $H$-torsor. Finally, taking into account 4.2.50, $\omega_V^A$ is surjective for any $V \in I_G$ because $\mathrm{R}_H V$ contains either the trivial representation $k$ or $V_{e_1}$ and $\omega_k^B$, $\omega_{V_{e_1}}^B$ are surjective by construction of $B$.

## 4.3 Reducibility of $G$-Cov for non abelian linearly reductive groups.

We know that, when $G$ is diagonalizable, except for some few cases of lower rank, $G$-Cov is reducible (see 3.2.18). The goal of this section is to extend this bad behavior also to all non abelian, linearly reductive groups $G$. The method we will use does not apply and does not reduce to the diagonalizable case. When $G$ is a glrg over a connected base, we will study the stacks in groupoids $(\mathrm{LAlg}_R^G)^{\mathrm{gr}}$ and $(\mathrm{LMon}_R^G)^{\mathrm{gr}}$ and we will decompose them in a disjoint union of stacks parametrized by functions $I_G \longrightarrow \mathbb{N}$, called rank functions. The stack $G$-Cov will correspond to the rank function $f_V = \mathrm{rk}\, V$. The result about reducibility of $G$-Cov is then obtained looking at the behaviour of the rank functions under induction from a subgroup.

We start stating the Theorem we will prove at the end of this section.

**Theorem 4.3.1.** *If $G$ is a finite, non abelian and linearly reductive group then $G$-*Cov *is reducible. If $G$ is defined over a connected scheme, then $G$-*Cov *is also universally reducible.*

Remember that universally reducible means reducible after any base change (see 3.2.15). Note that, if we do not assume that the base $S$ is connected, we can not conclude that $G$-Cov is universally reducible, since one can always take $G$ as disjoint union of $\mu_2$ and $S_3$ over $\mathrm{Spec}\, \mathbb{Q} \sqcup \mathrm{Spec}\, \mathbb{Q}$. On the other hand what happens when the base is not connected is clear from the following Proposition.

**Proposition 4.3.2.** *If $G$ is a linearly reductive group over a scheme $S$, then the locus of $S$ where $G$ is abelian is open and closed in $S$.*

*Proof.* Denote by $Z$ this locus. Topologically, $|Z|$ is closed in $S$, because it is the locus where the maps $G \times G \longrightarrow G$ given by $(g,h) \longmapsto gh$ and $(g,h) \longmapsto hg$ coincide and $G$ is flat and proper. We have to prove that, given an algebraically closed field $k$ and a map $\mathrm{Spec}\, k \xrightarrow{p} S$ such that $G_k = G \times k$ is abelian, there exists a fppf neighborhood of $S$ around $p$ where $G$ is abelian. By [AOV08, Theorem 2.19], we can assume that $G = \Delta \ltimes H$, where $\Delta$ is diagonalizable and $H$ is constant. If $G_k$ is abelian, then $H$ is abelian, the map $H \longrightarrow \mathrm{Aut}\, \Delta \simeq \mathrm{Aut}(\mathrm{Hom}(\Delta, \mathbb{G}_m))^{op}$ is trivial and therefore $G \simeq \Delta \times H$ is abelian. $\square$

From now on, except for the proof of Theorem 4.3.1, $G$ will be a linearly reductive group over a ring $R$ with connected spectrum. It will be clear that this is not a necessary condition, but we want to avoid technicalities like considering multivalued rank functions for a locally free sheaf. In particular any $V \in \mathrm{Loc}^G R$ has a well defined rank. As mention above, one of the main ingredient in the proof of Theorem 4.3.1 is the theory of rank functions, that we are going to introduce.

**Definition 4.3.3.** Assume that $G$ is a glrg. We will say that $\Omega \in \mathrm{LAdd}_R^G$ ($\mathscr{A} \in \mathrm{Loc}_R^G$) has *equivariant constant rank* (or is of equivariant constant rank) if for any $V \in \mathrm{Loc}^G R$ the locally free sheaf $\Omega_V$ ($\Omega_V^{\mathscr{A}} = (V \otimes \mathscr{A})^G$) has constant rank.

Given $\mathscr{A} \in \mathrm{Loc}_R^G$ or $\Omega \in \mathrm{LAdd}_R^G$ of equivariant constant rank we define the rank functions $\mathrm{rk}^{\mathscr{A}} \colon I_G \longrightarrow \mathbb{N}$, $\mathrm{rk}^{\Omega} \colon I_G \longrightarrow \mathbb{N}$ as

$$\mathrm{rk}_V^{\Omega} = \mathrm{rk}\, \Omega_V, \ \mathrm{rk}_V^{\mathscr{A}} = \mathrm{rk}_V^{\Omega^{\mathscr{A}}} = \mathrm{rk}(V \otimes \mathscr{A})^G$$

Given $f \colon I_G \longrightarrow \mathbb{N}$ we will denote by $\mathrm{Loc}_{R,f}^G$, $\mathrm{LRings}_{R,f}^G$, $\mathrm{LAlg}_{R,f}^G$, $\mathrm{LAdd}_{R,f}^G$, $\mathrm{LPMon}_{R,f}^G$, $\mathrm{LMon}_{R,f}^G$ the full substack of $\mathrm{Loc}_R^G$, $\mathrm{LRings}_R^G$, $\mathrm{LAlg}_R^G$, $\mathrm{LAdd}_R^G$, $\mathrm{LPMon}_R^G$, $\mathrm{LMon}_R^G$ of objects $\chi$ of equivariant constant rank such that $\mathrm{rk}^{\chi} = f$ respectively.

Given $f \colon I_G \longrightarrow \mathbb{N}$ we will still call $f$ the extension $f \colon \mathrm{Loc}^G R \longrightarrow \mathbb{N}$ given by

$$f_U = \sum_{V \in I_G} \mathrm{rk}(\mathrm{Hom}^G(V,U)) f_V$$

In particular if $\Omega \in \mathrm{LAdd}_{R,f}^G$ we will have $\mathrm{rk}\, \Omega_U = f_U$ for any $U \in \mathrm{Loc}^G R$.

*Remark* 4.3.4. Theorem 4.2.29 says that $\Omega^*$ induces an isomorphism $G\text{-Cov} \simeq (\mathrm{LMon}_{R,f}^G)^{\mathrm{gr}}$, where $f \colon I_G \longrightarrow \mathbb{N}$ is the rank function given by $f_V = \mathrm{rk}\, V$.

The following Theorem shows how $\mathrm{LAlg}_R^G$ can be described in terms of the rank functions.

**Theorem 4.3.5.** *Assume that $G$ is a glrg. Then*

$$(\mathrm{LAlg}_R^G)^{gr} = \bigsqcup_{f \in \mathbb{N}^{I_G}} (\mathrm{LAlg}_{R,f}^G)^{gr}$$

*Given $f \colon I_G \longrightarrow \mathbb{N}$, let $\delta \in \mathrm{LAdd}_{R,f}^G R$ be the $R$-linear functor such that $\delta_V = R^{f_V}$ for $V \in I_G$ and set*

$$X = \prod_{V,W \in I_G} \underline{\mathrm{M}}_{f_V f_W, f_{V \otimes W}}, \ \underline{\mathrm{GL}}_f = \prod_{V \in I_G} \underline{\mathrm{GL}}_{f_V}$$

*Then we have a cartesian diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Spec}\, R \arrow[d, "\delta"] \\ (\mathrm{LPMon}_{R,f}^G)^{gr} \arrow[r] & (\mathrm{LAdd}_{R,f}^G)^{gr} \end{tikzcd}$$

*where the vertical maps are $\underline{\mathrm{GL}}_f$-torsors. In particular*

$$(\mathrm{LAdd}_{R,f}^G)^{gr} \simeq \mathrm{B}(\underline{\mathrm{GL}}_f), \ (\mathrm{LPMon}_{R,f}^G)^{gr} \simeq [X/\underline{\mathrm{GL}}_f]$$

*Moreover the map* $(\mathrm{LMon}^G_{R,f})^{gr} \longrightarrow (\mathrm{LPMon}^G_{R,f})^{gr}$ *is an immersion. In particular all the stacks* $\mathrm{Loc}^G_{R,f}$, $\mathrm{LRings}^G_{R,f}$, $\mathrm{LAlg}^G_{R,f}$, $\mathrm{LAdd}^G_{R,f}$, $\mathrm{LPMon}^G_{R,f}$, $\mathrm{LMon}^G_{R,f}$ *are algebraic stacks of finite presentation over* $R$*.*

*Proof.* The first claim holds since, given $\Omega \in \mathrm{LMon}^G_R T$ for some scheme $T$ and $V \in I_G$, then $\mathrm{rk}^\Omega_V = \mathrm{rk}\,\Omega_V$ is constant on the connected components of $T$.

Now let $f \in \mathbb{N}^{I_G}$ and $\delta \in \mathrm{LAdd}^G_{R,f}$ be the $R$-linear functor given by $\delta_V = R^{f_V}$, which exists thanks to 4.2.8. By 4.2.8, we clearly have that $(\mathrm{LAdd}^G_{R,f})^{\mathrm{gr}} \simeq \mathrm{B}(\underline{\mathrm{GL}}_f)$. Now consider the forgetful map

$$(\mathrm{LPMon}^G_{R,f})^{\mathrm{gr}} \longrightarrow (\mathrm{LAdd}^G_{R,f})^{\mathrm{gr}} \text{ and } Z = (\mathrm{LPMon}^G_{R,f})^{\mathrm{gr}} \times_{(\mathrm{LAdd}^G_{R,f})^{\mathrm{gr}}} \operatorname{Spec} R$$

$Z$ is the functor that associates to an $R$-scheme $T$ all the possible pseudo monoidal structures on $\delta \otimes \mathcal{O}_T$. By 4.2.20 we have that $Z = X$. Now we have to verify that $(\mathrm{LMon}^G_{R,f})^{\mathrm{gr}} \longrightarrow (\mathrm{LPMon}^G_{R,f})^{\mathrm{gr}}$ is an immersion. First, note that this map is fully faithful, because an object in $\mathrm{LPMon}^G_R$ has at most one unity and the isomorphisms must preserve them. So $Z = (\mathrm{LMon}^G_{R,f})^{\mathrm{gr}} \times_{(\mathrm{LPMon}^G_{R,f})^{\mathrm{gr}}} X$ is a subfunctor of $X$, namely the subfunctor of the pseudo-monoidal structures $\iota_{V,W}$ that satisfy commutativity, associativity and has a unity. We have to show that $Z \longrightarrow X$ is a finitely presented immersion. We first consider the associativity. Given $V, W, Z$ and $\iota_{V,W} \in X(T)$ there are two way of forming a map

$$\delta_V \otimes \delta_W \otimes \delta_Z \otimes \mathcal{O}_T \longrightarrow \delta_{V\otimes W\otimes Z}$$

Taking the difference we get a map

$$q\colon X \longrightarrow Y = \prod_{V,W,Z} \underline{\mathrm{Hom}}(\delta_V \otimes \delta_W \otimes \delta_Z, \delta_{V\otimes W\otimes Z})$$

By functoriality, this map is a morphism of scheme, so the locus of $X$ of the $\iota_{V,W}$ that are associative is $q^{-1}(0)$, which is a closed subscheme of $X$. Moreover, it is easy to see that the map $q$ involves only a finite number of matrices defined over $R$. Therefore it is defined over some noetherian subring of $R$. In particular the locus $q^{-1}(0)$ is defined by a finite set of equations.

We can argue similarly for the commutativity. Now we have to deal with the unity. We have to describe the locus of $X$ of the $\iota_{V,W}$ such that there exists $x \in \delta_R \otimes T$ such that $\iota_{R,V}(x \otimes v) = v$ for any $V \in I_G$, $v \in V$. Consider the induced linear map

$$\zeta_\iota\colon \delta_R \otimes \mathcal{O}_T \longrightarrow \prod_{V\in I_G} \mathrm{End}(\delta_V) \otimes \mathcal{O}_T$$

The unities of $(\delta \otimes \mathcal{O}_T, \iota)$ are the elements $x \in \delta_R \otimes \mathcal{O}_T$ such that $\zeta_\iota(x) = (\mathrm{id}_{\delta_V})_{V\in I_G}$. Since we know that that the unities are unique, we can first impose the condition that $\zeta_\iota$ is injective after any base change. If we regard $\zeta_\iota$ as a matrix over $\mathcal{O}_T$, this is the locus where we have inverted the maximal minors of $\zeta_\iota$. We can now assume that there exists a maximal minor of $\zeta_\iota$ which is invertible over $T$. This means that we can write

$$\prod_{V\in I_G} \mathrm{End}(\delta_V) \otimes \mathcal{O}_T = \mathrm{Im}(\zeta_\iota) \oplus \mathcal{F}$$

If $(\mathrm{id}_V)_{V\in I_G} = z \oplus z'$, then the locus where a unity exists is exactly the zero locus of $z' \in \mathcal{F}$. $\square$

After having discussed the rank functions, we come back to our initial goal, the reducibility of $G$-Cov.

**Lemma 4.3.6.** *The stack $G$-Cov is open and closed in $(\mathrm{LAlg}_R^G)^{gr}$. In particular $\mathcal{Z}_G$ is the schematic closure of $\mathrm{B}\,G$ in $(\mathrm{LAlg}_R^G)^{gr}$.*

*Proof.* Follows from 4.1.21, 4.3.5 and 4.2.29. $\square$

The following proposition is the key in the proof of the reducibility of $G$-Cov.

**Proposition 4.3.7.** *Let $H$ be an open and closed subgroup scheme of $G$. Then if $\mathscr{B} \in \mathrm{LAlg}_R^H$, we have*

$$\mathrm{ind}_H^G \mathscr{B} \in \mathcal{Z}_G \iff \mathscr{B} \in \mathcal{Z}_H,\ \mathrm{ind}_H^G \mathscr{B} \in \mathrm{B}\,G \iff \mathscr{B} \in \mathrm{B}\,H$$

*In particular we have cartesian diagrams*

$$\begin{tikzcd}
\mathrm{B}\,H \arrow[r] \arrow[d] & \mathcal{Z}_H \arrow[r] \arrow[d] & H\text{-Cov} \arrow[d, "\mathrm{ind}_H^G"] \\
\mathrm{B}\,G \arrow[r] & \mathcal{Z}_G \arrow[r] & G\text{-Cov}
\end{tikzcd}$$

*Proof.* Consider the fiber products

$$\begin{tikzcd}
\mathcal{Y} \arrow[r] \arrow[d] & \mathcal{Z} \arrow[r] \arrow[d] & (\mathrm{LAlg}_R^H)^{\mathrm{gr}} \arrow[d, "\mathrm{ind}_H^G"] \\
\mathrm{B}\,G \arrow[r] & \mathcal{Z}_G \arrow[r] & (\mathrm{LAlg}_R^G)^{\mathrm{gr}}
\end{tikzcd}$$

It easy to show that $\mathcal{Z}\,(\mathcal{Y})$ is the substack of $(\mathrm{LAlg}_R^H)^{\mathrm{gr}}$ of algebras $\mathscr{B}$ such that $\mathrm{ind}_H^G \mathscr{B} \in \mathcal{Z}_G(\mathrm{B}\,G)$. In particular by 4.2.51, $\mathcal{Y} = \mathrm{B}\,H$ and $\mathcal{Z}_H$ is a closed subscheme of $\mathcal{Z}$. We have to prove that $\mathcal{Z}_H = \mathcal{Z}$. We claim that, if $S$ is a noetherian scheme and $\mathscr{B} \in \mathrm{LAlg}_R^H(S)$ then $\mathscr{B} \in \mathcal{Z}_H(S)$ if and only if for any strictly Henselian ring $C$ and map $\mathrm{Spec}\, C \longrightarrow S$ we have $\mathscr{B} \otimes C \in \mathcal{Z}_H(C)$. Indeed consider the base change $S' = S \times_{(\mathrm{LAlg}_R^H)^{\mathrm{gr}}} \mathcal{Z}_H \longrightarrow S$. This is a closed immersion and by hypothesis its base change to any strict Henselization of a localization of $S$ is an isomorphism. But this implies that $S' = S$ and therefore $\mathscr{B} \in \mathcal{Z}_H(S)$.

We are now going to prove that $\mathcal{Z} = \mathcal{Z}_H$. This will conclude the proof since $\mathrm{ind}_H^G$ sends $H$-Cov to $G$-Cov, because $\mathrm{ind}_H^G \mathcal{O}[H] \simeq \mathcal{O}[G]$. If $S \xrightarrow{\mathscr{B}} \mathcal{Z}$ is an fppf atlas, then this is equivalent to $\mathscr{B} \in \mathcal{Z}_H(S)$. By the remark above we have to show that if $\mathscr{B} \in \mathrm{LAlg}_R^H C$, where $C$ is a strictly Henselian ring, such that $A = \mathrm{ind}_H^G \mathscr{B} \in \mathcal{Z}_G(C)$ then $\mathscr{B} \in \mathcal{Z}_H(C)$. Note that if $X \longrightarrow \mathrm{Spec}\, C$ is an fppf map we can always assume to have a section. Indeed $\mathscr{B} \in \mathcal{Z}_H(C)$ if and only if $\mathscr{B} \otimes \mathcal{O}_X \in \mathcal{Z}_H(X)$ and again we can restrict to the strictly Henselian ring mapping to $X$. Let $Z$ be an fppf atlas of $\mathcal{Z}_G$. By remark above we can assume that $A$ comes from the atlas $Z$. Let $p \in Z$ the image of the closed point of $\operatorname{Spec} C$ under the map $\operatorname{Spec} C \xrightarrow{A} Z$ and let $D$ be the strict Henselization of $\mathcal{O}_{Z,p}$. The universal object of $Z$ induces an $A_D \in \mathcal{Z}_G(D)$ and since $\operatorname{Spec} D \longrightarrow Z$ is flat, the open subset of $\operatorname{Spec} D$ where $A_D$ is a $G$-torsor in schematically dense in $\operatorname{Spec} D$. Since $\operatorname{Spec} C \longrightarrow Z$ factors through $D$ we have a map $D \longrightarrow C$ such that $A_D \otimes C \simeq A$. By hypothesis $A = \operatorname{ind}_H^G B$ and, since $C$ is strictly Henselian, by 4.1.36 we can assume $B$ to be local. If $p \in \operatorname{Spec} A_D$ is the image of the closed point of $\operatorname{Spec} B$ under the map $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A \longrightarrow \operatorname{Spec} A_D$ by 4.1.36 we can write $A_D \simeq \operatorname{ind}_{H_p}^G (A_D)_p$ where $H_p$ is the stabilizer of the connected component $\operatorname{Spec}(A_D)_p$ in $\operatorname{Spec} A_D$. Let $V \subseteq \operatorname{Spec} D$ be the open locus where $(A_D)_p$ is a $H_p$-torsor. This is exactly the locus where $A_D$ is a $G$-torsor thanks to 4.2.51. So $V$ is schematically dense in $\operatorname{Spec} D$. Since $(A_D)_p \otimes C$ is local by 4.1.23 and it is a factor of $A \simeq A_D \otimes C$, $\operatorname{Spec}(A_D)_p \otimes C$ is the connected component of $\operatorname{Spec} A$ containing $\operatorname{Spec} B$. We are in the situation

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[rr, "\alpha"] \arrow[d, "i"] & & \operatorname{Spec}(A_D)_p \otimes C \arrow[d, "j"] \\
\operatorname{Spec} A \arrow[r] \arrow[rr, bend left, "\beta"] & \operatorname{Spec} A_D \otimes C \arrow[r] & \operatorname{Spec} \operatorname{ind}_{H_p \otimes C}^G (A_D)_p \otimes C
\end{tikzcd}$$

Since $G$ permutes the connected component of $\operatorname{Spec} A_D$, we see that $H_p \otimes C$ is the stabilizer of $\operatorname{Spec}(A_D)_p \otimes C$ and therefore $H \subseteq H_p \otimes C$. Since $H$ is open and closed in $G$, we have that $B$ is a factor of $A$ and therefore is a localization of $A$. It follows that $\alpha$ is an isomorphism and, since $i, j, \beta$ are $H$-equivariant, that is $H$-equivariant. So we have a $H$-equivariant isomorphism $B \simeq (A_D)_p \otimes C$. We are going to prove that $H = H_p \otimes C$. Since $- \otimes_D C$ maintains the connected components of $G$, there exists an open and closed subgroup $H' \subseteq H_p$ such that $H' \otimes C = H$. In particular we have

$$(A_D)_p^{H'} \otimes C \simeq ((A_D)_p \otimes C)^H \simeq B^H \simeq (\operatorname{ind}_H^G B)^G = A^G = C$$

since $A \in \mathcal{Z}_G(C)$. Since $(A_D)_p^{H'}$ is a locally free algebra over $D$, it follows that $(A_D)_p^{H'} = D$. If $q \in V$, i.e. $(A_D)_p$ is a a $H_p$-torsor over $q \in \operatorname{Spec} D$, then the base change to $\overline{k(q)}$ of $\operatorname{Spec}(A_D)_p$ is $H_p$ and $H_p/H' \simeq \operatorname{Spec} \overline{k(q)}$. So $H_p$ and $H'$ has the same connected components and therefore $H_p = H'$ and $H_p \otimes C = H$. It remains to prove that $(A_D)_p \in \mathcal{Z}_{H_p}(D)$. We have cartesian diagrams

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \operatorname{Spec} D \arrow[d, "(A_D)_p"] \\
\mathrm{B}\, H_p \arrow[r] & \mathcal{Z}_{H_p} \arrow[r] & (\mathrm{LAlg}_R^{H_p})^{\mathrm{gr}}
\end{tikzcd}$$

Since $V$ is schematically dense in $\operatorname{Spec} D$, it follows that $Z = \operatorname{Spec} D$ as required. $\square$

**Lemma 4.3.8.** *[MM03] A constant group whose proper subgroups are abelian is solvable.*

We are ready for the proof of Theorem 4.3.1. The idea is that we can reduce to a minimal non abelian subgroup of $G$ and assume that it is constant. By lemma 4.3.8 $G$ is solvable. In this case we will find a subgroup $H$ of $G$ and an algebra $\mathscr{A} \in \mathrm{LAlg}^H k$ such that $\mathscr{A} \notin H\text{-Cov}(k)$ and $\mathrm{ind}_H^G \mathscr{A} \in G\text{-Cov}(k)$. Up to some details, if $G$-Cov is irreducible, then $|G\text{-Cov}| = |\mathcal{Z}_G|$ and therefore $\mathscr{A} \in \mathcal{Z}_H(k) \subseteq H\text{-Cov}(k)$ by 4.3.7, obtaining a contradiction.

*Proof.* (*of Theorem* 4.3.1) If the base scheme is not connected, then clearly $G$-Cov is reducible. By 3.2.16 and 4.3.2, we can assume that $S = \operatorname{Spec} k$, where $k$ is a field and, since in this case $\mathcal{Z}_G$ is geometrically integral, we can also assume $k = \overline{k}$. In particular $G$ is a glrg. Let $H$ be an open and closed subgroup of $G$. We claim that if one of the following statement is fulfilled, then $G$-Cov is reducible:

1) $H$-Cov is reducible
2) there exists $f\colon I_H \longrightarrow \mathbb{N}$ whose extension $f\colon \mathrm{Loc}^H k \longrightarrow \mathbb{N}$ is such that $f_{\mathrm{R}_H V} = \operatorname{rk} V$ for any $V \in I_G$ and there exists $\Delta \in I_H$ such that $f_\Delta \neq \operatorname{rk}\Delta$

Note that $G$-Cov is irreducible if and only if $\mathcal{Z}_G(k) = G\text{-Cov}(k)$. Assume that $H$-Cov is reducible and, by contradiction, that $G$-Cov is irreducible. If $B \in H\text{-Cov}(k)$ then $\mathrm{ind}_H^G B \in G\text{-Cov}(k) = \mathcal{Z}_G(k)$ and so $B \in \mathcal{Z}_H(k)$. Therefore $H$-Cov is irreducible.

Now let $f\colon I_H \longrightarrow \mathbb{N}$ as in 2) and let $\delta \in \mathrm{LAdd}^H k$ be the unique $R$-linear functor such that $\delta_\Delta = k^{f_\Delta}$ for any $\Delta \in I_H$. Note that by hypothesis $f_R = 1$. Consider

$$F = \bigoplus_{R \neq \Delta \in I_H} \Delta^\vee \otimes \delta_\Delta,\ \ B = k \oplus F$$

If we set $F^2 = 0$ we obtain a structure of algebra on $B$ such that $B \in \mathrm{LAlg}^H_{k,f} k$. We claim that $A = \mathrm{ind}_H^G B \in G\text{-Cov}(k)$. Indeed $\Omega^B = \delta$, $\Omega^A = \Omega^B \circ \mathrm{R}_H$ and therefore

$$\operatorname{rk}\Omega^A_V = \operatorname{rk}\Omega^B_{\mathrm{R}_H V} = f_{\mathrm{R}_H V} = \operatorname{rk} V$$

We also claim that $A \notin \mathcal{Z}_G(k)$, that implies that $G$-Cov is reducible. Indeed

$$A = \mathrm{ind}_H^G B \in \mathcal{Z}_G(k) \implies B \in \mathcal{Z}_H(k) \implies B \in H\text{-Cov}(k)$$

by 4.3.7, which is not the case because there exists by hypothesis $\Delta \in I_H$ such that $\operatorname{rk}\Omega^B_\Delta = f_\Delta \neq \operatorname{rk}\Delta$.

We return now to the original statement. And we argue by induction on the rank of $G$. As base case we take the case in which there exists a normal and abelian subgroup $H$ of $G$ such that $G/H \simeq \mathbb{Z}/p\mathbb{Z}$ for some prime $p$. We first show how to reduce to this case. Since $G$ is non abelian, we have that $\underline{G}$ is a non trivial group. We start reducing to the case where $\underline{G}$ is solvable. If $\underline{G}$ is abelian we are already in this case. Otherwise take a minimal non abelian subgroup $K$ of $\underline{G}$. All the proper subgroups of $K$ are abelian and therefore $K$ is solvable thanks to 4.3.8. If we call $\phi\colon G \longrightarrow \underline{G}$, then $G' = \phi^{-1}(K)$ is a non abelian open and closed subgroup of $G$ such that $\underline{G'} \simeq K$ is solvable and we can therefore reduce to it. Now, if $\underline{G}$ is solvable, it has a surjective map $\underline{G} \longrightarrow \mathbb{Z}/p\mathbb{Z}$. So there exists an open and closed normal subgroup $H$ of $G$ such that $G/H \simeq \mathbb{Z}/p\mathbb{Z}$. If $H$ is non abelian we can lower the rank and since $\underline{H} < \underline{G}$ is solvable, we can continue the induction reaching the claimed base case.

So assume to have a surjection $G \longrightarrow \mathbb{Z}/p\mathbb{Z}$ for some prime $p$ such that the kernel $H$ is abelian. In particular $H$ is diagonalizable and set $N = \mathrm{Hom}(H, \mathbb{G}_m)$. We will construct an $f \colon I_H \longrightarrow \mathbb{N}$ as in 2). The group $G/H \simeq \mathbb{Z}/p\mathbb{Z}$ acts on $H$ and on $N = \mathrm{Hom}(H, \mathbb{G}_m)$ by conjugation as explained in 4.2.38. Let $\mathcal{R}$ be a set of representatives of $N/(\mathbb{Z}/p\mathbb{Z})$. Note that, since $p$ is prime, an element $n \in N$ is fixed or its orbit $o(n)$ has order $p$. We claim that if $V \in I_G$ there exists a unique $m \in \mathcal{R}$ such that

$$\mathrm{R}_H V = V_m^{\operatorname{rk} V} \text{ with } |o(m)| = 1 \text{ or } V = \operatorname{ind}_H^G V_m \text{ with } |o(m)| = p$$

Indeed there exists $m \in N$ such that $V \subseteq \operatorname{ind}_H^G V_m$. Remember that, by 4.2.52, given $n, n' \in N$ we have

$$\mathrm{R}_H \operatorname{ind}_H^G V_n = \bigoplus_{g \in \mathbb{Z}/p\mathbb{Z}} V_{g(n)} \text{ and } (\operatorname{ind}_H^G V_n \simeq \operatorname{ind}_H^G V_{n'} \iff n' \in o(n))$$

So we can assume $m \in \mathcal{R}$. Moreover such an $m$ is unique since if $V \subseteq \operatorname{ind}_H^G V_{m'}$, $\mathrm{R}_H V$ is the sum of some $V_n$ that are in the orbits of both $m$ and $m'$. In particular, if $|o(m)| = 1$, then $\operatorname{ind}_H^G V_m = V_m^p$ and therefore $\mathrm{R}_H V = V_m^{\operatorname{rk} V}$. So assume $|o(m)| = p$. Given $W \in \mathrm{Loc}^G k$ ($\mathrm{Loc}^H k$) and $g \in G(k)$ call $W_g$ the representation of $G$ ($H$) that has $W$ as underlying vector space, while the action of $G$ ($H$) is given by $t \star x = (g^{-1}tg)x$. Note that by definition $(V_n)_g = V_{g(n)}$. In particular the multiplication by $g^{-1}$ on $V$ yields a $G$-equivariant isomorphism $V \simeq V_g$ and therefore $V_n \subseteq \mathrm{R}_H V$ implies that $V_{g(n)} \subseteq \mathrm{R}_H V$. Since $|o(m)| = p$ we can conclude that $V = \operatorname{ind}_H^G V_m$. Define

$$f_{V_n} = \begin{cases} |o(n)| & \text{if } n \in \mathcal{R} \\ 0 & \text{otherwise} \end{cases}$$

We claim that $f$ satisfies the requests of 2). Indeed if $V \in I_G$ and there exists $m \in \mathcal{R}$ such that $V = V_m^{\operatorname{rk} V}$ with $|o(m)| = 1$ then $f_{\mathrm{R}_H V} = \operatorname{rk} V f_{V_m} = \operatorname{rk} V$. Otherwise there exists $m \in \mathcal{R}$ with $|o(m)| = p$ such that

$$V = \operatorname{ind}_H^G V_m \implies f_{\mathrm{R}_H V} = \sum_{g \in \mathbb{Z}/p\mathbb{Z}} f_{V_{g(m)}} = p = \operatorname{rk} V$$

Finally note that if $n \in \mathcal{R}$ is such that $|o(n)| = p$ then $f_{V_n} = p \neq 1 = \operatorname{rk} V_n$. So we have to show that such an $n$ exists. If by contradiction this is false, then the actions of $\mathbb{Z}/p\mathbb{Z}$ on $N$ and $H$, as well as the action of $G$ on $H$ by conjugation are trivial. So $H$ commutes with all the elements of $G$. Let $g \in G(k) \simeq \underline{G}$ be not in $H$. Any element of $G(T)$ can be written as $hg^i$ with $h \in H(T)$ and $0 \leq i < p$. It is straightforward to check that two such elements commute and that therefore $G$ is abelian, which is not the case. $\square$

## 4.4 Regularity in codimension 1.

In this section we want to address the following question: given a discrete valuation ring $R$ and $A \in \operatorname{LAlg}^G R$, what are the conditions that ensure that $A$ is regular? We will see that one of those conditions will be that $A \in G\text{-Cov}(R)$. This problem translates in the following, more geometrical problem: given a normal, noetherian scheme $S$ and a $G$-cover $X \longrightarrow S$, what are the conditions that ensure that $X$ is normal too? The idea is to look at the map $\widehat{\operatorname{tr}}\colon A \longrightarrow A^\vee$ induced by the trace map: $\widehat{\operatorname{tr}}$ is an isomorphism if and only if $A$ is étale and the less degenerate $\widehat{\operatorname{tr}}$ is, the more regular the algebra $A$ should be. We will explain what this 'less' degenerate means. At the end, we will also discuss a possible extension to covers without an action of a group.

In this section we fix a (étale) locally constant and finite group scheme $G$ over a ring $R$ such that $\operatorname{rk} G \in R^*$. This means exactly that $G$ is a finite and étale linearly reductive group over $R$. We require this last condition because we want $G$-torsors to be regular (over regular base).

*Notation* 4.4.1. If $\mathcal{P}$ is a locally free sheaf and $\eta\colon \mathcal{P}\otimes\mathcal{P} \longrightarrow \mathcal{O}_T$ is a map, we will denote by $\widehat{\eta}\colon \mathcal{P} \longrightarrow \mathcal{P}^\vee$ the associated map. If $\mathcal{P}$ is also an algebra and $\phi \in \mathcal{P}^\vee$ we will also set $\widehat{\phi} = \widehat{\eta}$ where

$$\eta\colon \mathcal{P}\otimes\mathcal{P} \xrightarrow{m} \mathcal{P} \xrightarrow{\phi} \mathcal{O}_T$$

where $m$ is the multiplication of $\mathcal{P}$. Given a basis $\beta = \{x_1, \dots, x_s\}$ of $\mathcal{P}$, the matrix associated with $\eta$ is $(\eta(x_i \otimes x_j))_{i,j}$, which is also the matrix representing $\widehat{\eta}$ with respect to the basis $\beta$ and its dual.

**Definition 4.4.2.** Let $\mathscr{A} \in \operatorname{LAlg}^G T$. We define

$$\operatorname{tr}_{\mathscr{A}/\mathcal{O}_T}\colon \mathscr{A} \longrightarrow \mathcal{O}_T \text{ and } \widehat{\operatorname{tr}}_{\mathscr{A}/\mathcal{O}_T}\colon \mathscr{A} \longrightarrow \mathscr{A}^\vee$$

the trace map and its associated map respectively. We also set

$$\mathcal{Q}^{\mathscr{A}/\mathcal{O}_T} = \operatorname{Coker} \widehat{\operatorname{tr}}_{\mathscr{A}/\mathcal{O}_T} \text{ and } e^{\mathscr{A}/\mathcal{O}_T} = \operatorname{l}(\mathcal{Q}^{\mathscr{A}/\mathcal{O}_T})$$

where l is the length, and, if $G$ is a glrg, $\mathscr{A}^G = \mathcal{O}_T$ and $V \in \operatorname{Loc}^G R$,

$$\mathcal{Q}_V^{\mathscr{A}/\mathcal{O}_T} = \operatorname{Coker}(\Omega_V^{\mathscr{A}} \xrightarrow{\xi_V} (\Omega_{V^\vee}^{\mathscr{A}})^\vee) \text{ and } e_V^{\mathscr{A}/\mathcal{O}_T} = \operatorname{l}(\mathcal{Q}_V^{\mathscr{A}/\mathcal{O}_T})$$

where $\xi_V$ is the map induced by $\omega_V\colon \Omega_V \otimes \Omega_{V^\vee} \longrightarrow \Omega_{V\otimes V^\vee} \longrightarrow \Omega_R = \mathscr{A}^G = \mathcal{O}_T$. We will also omit the superscript $*/\mathcal{O}_T$ when it will be clear what is the base scheme.

*Notation* 4.4.3. If $R$ is a local ring with residue field $k$ and $Q$ is an $R$-module, we will say that $Q$ is defined over the closed point of $R$ if $m_R Q = 0$. This condition is equivalent to the fact that the map $Q \longrightarrow Q \otimes k$ is an isomorphism or that $Q = i_* Q'$, where $i\colon \operatorname{Spec} k \longrightarrow \operatorname{Spec} R$ is the closed point, for some $k$-vector space $Q'$.

The Theorem we will prove is the following:

**Theorem 4.4.4.** *Let $R$ be a DVR and $G$ be a finite and étale linearly reductive group scheme over $R$. Let also $A \in \mathrm{LAlg}^G R$ be such that $A^G = R$ and that the action of $G$ on $A$ is generically faithful over $R$ (see 4.4.8). Then $A$ is regular if and only if the geometric stabilizers of $A$ are solvable and one of the following conditions holds:*

*1) $e^A < \mathrm{rk}\, A$;*

*2) $\mathcal{Q}^A$ is defined over the closed point of $R$.*

*In this case $A$ is generically a $G$-torsor, $A \in \mathcal{Z}_G(R)$ and, given a closed point $p$ of $A$, its geometric stabilizer $H_p$ is cyclic and we have*

$$e^A = \mathrm{rk}\, A - |\operatorname{Spec}(A \otimes_R \overline{k})| = \mathrm{rk}\, G(1 - \frac{1}{\mathrm{rk}\, H_p})$$

*If $G$ is a glrg, then conditions 1) and 2) can be replaced respectively by*

*3) $e^A_V \leq \mathrm{rk}\, V$ for all $V \in I_G$;*

*4) $\mathcal{Q}^A_V$ is defined over the closed point of $R$ for all $V \in I_G$.*

We will define what a faithful action and a geometric stabilizer means in this context. The proof of the above Theorem is at the end of the section, because we prefer to collect first the necessary lemmas and definitions. Anyway, before doing so, we want to state a global version of Theorem above.

**Definition 4.4.5.** Let $S$ be a scheme, $Y$ be an $S$-scheme and $f\colon X \longrightarrow Y$ be a cover. We define the section $s_f \in (\det f_*\mathcal{O}_X)^{-2}$ as the section induced by the determinant of the trace map

$$\widehat{\mathrm{tr}}_{f_*\mathcal{O}_X/\mathcal{O}_Y}\colon f_*\mathcal{O}_X \longrightarrow f_*\mathcal{O}_X{}^\vee$$

If $G$ acts on $X$ and $f$ is $G$-invariant set $\Omega^f_V = (f_*\mathcal{O}_X \otimes V)^G$ for $V \in \mathrm{Loc}^G S$. Moreover if $G$ is a glrg over $S$ and $f \in G$-Cov then, for any $V \in I_G$, since $\mathrm{rk}\, V = \mathrm{rk}\, \Omega^f_V$, we define $s_{f,V} \in \det(\Omega^f_V)^{-1} \otimes \det(\Omega^f_{V^\vee})^{-1}$ as the section induced by the determinant of

$$\Omega^f_V \longrightarrow \Omega^f_{V^\vee}{}^\vee \Longleftarrow \Omega^f_V \otimes \Omega^f_{V^\vee} \longrightarrow \Omega^f_{V\otimes V^\vee} \longrightarrow \Omega^f_{\mathcal{O}_S} = (f_*\mathcal{O}_X)^G = \mathcal{O}_Y$$

The following Proposition, proved in 4.4.14, explains the relations among the sections just introduced.

**Proposition 4.4.6.** *Assume that $G$ is a glrg over $S$ and let $Y$ be an $S$-scheme and $f\colon X \longrightarrow Y \in G$-Cov. Then there exists an isomorphism*

$$(\det f_*\mathcal{O}_X)^{-2} \simeq \bigotimes_{V \in I_G} (\det(\Omega^f_V)^{-1} \otimes \det(\Omega^f_{V^\vee})^{-1})^{\mathrm{rk}\, V} \text{ such that } s_f \longmapsto \bigotimes_{V\in I_G} s_{f,V}^{\otimes \mathrm{rk}\, V}$$

Given a regular in codimension 1 scheme $Y$ and a codimension 1 point $q$ of $Y$ we denote by $v_q$ the discrete valuation associated with $\mathcal{O}_{Y,q}$.

**Theorem 4.4.7.** *Let $S$ be a scheme and $G$ be a finite and étale linearly reductive group over $S$. Let also $Y$ be an integral, noetherian and regular in codimension 1 (resp. normal) $S$-scheme and $f\colon X \longrightarrow Y$ be a cover with a generically faithful action of $G$ on $X$ such that $f$ is $G$-invariant and $X/G = Y$. Then the following are equivalent:*

1) *$X$ is regular in codimension 1 (resp. normal);*
2) *the geometric stabilizers of the codimension 1 points of $X$ are solvable and for all $q \in Y^{(1)}$ we have $v_q(s_f) < \operatorname{rk} G$.*

*In this case $f$ is generically a $G$-torsor, $X \in \mathcal{Z}_G(Y)$ and the stabilizers of the codimension 1 points of $X$ are cyclic. Moreover if $G$ is a glrg over $S$ the above conditions are also equivalent to*

3) *the geometric stabilizers of the codimension 1 points of $X$ are solvable, $f \in G\text{-}\mathrm{Cov}$ and for all $q \in Y^{(1)}$ and $V \in I_G$ we have $v_q(s_{f,V}) \leq \operatorname{rk} V$.*

Note that the geometric stabilizers are automatically solvable if $G$ has this property. We will show how to obtain the above theorem as corollary of Theorem 4.4.4 as soon as we have introduced the definitions of faithful action and geometric stabilizer for an étale group scheme.

**Definition 4.4.8.** By a faithful action of a group scheme $G$ on a scheme $X$ we mean an action such that the associated morphism of functors $G \longrightarrow \underline{\mathrm{Aut}}\, X$ is injective. When both $G$ and $X$ are defined over a scheme $S$, we will say that the action of $G$ on $X$ is generically faithful over $S$ if it is faithful over a dense open subset of $S$. We will often omit to specify the base scheme $S$ when it will be clear from the context.

*Remark* 4.4.9. If $G$ and $X$ are covers of a scheme $S$, then the locus in $S$ where $G$ acts faithfully on $X$ is open. Moreover, if $G$ is constant and $S$ is integral then the action of $G$ on $X$ is generically faithful if and only if the map of sets $G \longrightarrow \mathrm{Aut}\, X$ is injective. Indeed, if $f\colon X \longrightarrow S$ is the structure morphism, $\underline{\mathrm{Aut}}\, X$ is a locally closed subscheme of the vector bundle $\underline{\mathrm{End}}_S(f_*\mathcal{O}_X)$. In particular the kernel $H$ of the map $G \longrightarrow \underline{\mathrm{Aut}}\, X$ is a closed subscheme of $G$, so that $H \longrightarrow S$ is a finite group scheme. In particular the locus where the zero section $S \longrightarrow H$ is an isomorphism, which is open, is the locus where $G$ acts faithfully on $X$. When $G$ is constant, $S$ is integral and we write $X = \operatorname{Spec} \mathscr{A}$ and $k(S)$ for the field of fractions of $S$, the action is generically faithful over $S$ if and only if $G \times k(S) \longrightarrow \underline{\mathrm{Aut}}(\mathscr{A} \otimes k(S))$ is injective, which is equivalent to the injectivity of the map of sets $G \longrightarrow \mathrm{Aut}(\mathscr{A} \otimes k(S))$, because the maps $\mathrm{Aut}(\mathscr{A} \otimes k(S)) \longrightarrow \mathrm{Aut}(\mathscr{A} \otimes \mathcal{O}_U)$ are injective for all $k(S)$-schemes $U$. Finally, since $\mathrm{Aut}\, \mathscr{A} \longrightarrow \mathrm{Aut}(\mathscr{A} \otimes k(S))$ is injective, we can also conclude that $G \longrightarrow \mathrm{Aut}\, \mathscr{A}$ is injective if and only if $G \longrightarrow \mathrm{Aut}(\mathscr{A} \otimes k(S))$ is so.

**Lemma 4.4.10.** *Assume that $R$ is reduced and let $A \in \mathrm{LAlg}^G R$. Then $A$ is generically a $G$-torsor if and only if it is generically étale, the action of $G$ is generically faithful and $A^G = R$. In this case $\operatorname{rk} A = |G|$ and the action of $G$ is faithful.*

*Proof.* Assume that $A$ is generically a $G$-torsor. Since $G$ is étale it is generically étale. Moreover, since $R$ is reduced and thanks to 4.3.6, we have $A \in \mathcal{Z}_G(R) \subseteq G\text{-Cov}(R)$ and therefore $A^G = R$. To prove the faithfulness, we can assume that $A$ is the regular representation, since the injectivity of $G \longrightarrow \underline{\mathrm{Aut}}\, X$, where $X = \operatorname{Spec} A$, is an fppf local condition. The result then follows from the fact that for the regular representation the map $G \longrightarrow \underline{\mathrm{Aut}}_R \mathrm{W}(R[G])$ is injective.

For the converse we can reduce to the case where $R$ is an algebraically closed field by looking at the generic points. In particular $G$ is constant and the claim is a classical result in the theory of étale Galois covers. $\square$

We now introduce the concept of geometric stabilizer of a point.

**Definition 4.4.11.** Let $\mathscr{A} \in \mathrm{LAlg}_R^G T$ such that $\mathscr{A}^G = \mathcal{O}_T$ and $p \in \operatorname{Spec} \mathscr{A}$. We define the geometric stabilizer $H_p$ of $p$ as

$$H_p = \{g \in G_{\overline{k(p)}} \mid g(\overline{p}) = \overline{p}\}$$

where $\overline{p} \in \operatorname{Spec} \mathscr{A} \otimes \overline{k(p)}$ is the point given by $\mathscr{A} \otimes_{\mathcal{O}_T} \overline{k(p)} \longrightarrow \overline{k(p)} \otimes_{\mathcal{O}_T} \overline{k(p)} \longrightarrow \overline{k(p)}$.

**Proposition 4.4.12.** *The geometric stabilizer is invariant by base change, i.e. if we have a cartesian diagram*

$$\begin{array}{ccc} p' & \longmapsto & p \\ \operatorname{Spec} \mathscr{A}' & \longrightarrow & \operatorname{Spec} \mathscr{A} \\ \downarrow & & \downarrow \\ T' & \longrightarrow & T \end{array}$$

*then $H_{p'} \simeq H_p$ under the map $G_{\overline{k(p')}} \longrightarrow G_{\overline{k(p)}}$. Moreover if $G$ is constant then the image of $H_p$ under the map $G_{\overline{k(p)}} \longrightarrow G$ is*

$$\{g \in G \mid g(p) = p \text{ and the induced map } k(p) \longrightarrow k(p) \text{ is the identity}\}$$

*Proof.* Assume $G$ constant and let $K_p$ be the group defined in the last part of the proposition. We need to prove that $K_p$ is invariant by base change since $K_{\overline{p}} = H_p$ where $\overline{p} \in \operatorname{Spec} \mathscr{A} \otimes \overline{k(p)}$ is as in 4.4.11 (also if $G$ is not necessarily constant). If $g \in G$, we have a commutative diagram

$$\begin{array}{ccccc} k(p) & \longrightarrow & k(p) \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} & \longrightarrow & k(p') \\ \downarrow \alpha & & \downarrow \alpha\otimes\mathrm{id} & & \downarrow \beta \\ k(g(p)) & \longrightarrow & k(g(p)) \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} & \longrightarrow & k(g(p')) \end{array}$$

where $\alpha, \beta$ are the maps induced by $g \in \operatorname{Aut} \mathscr{A}, \operatorname{Aut} \mathscr{A}'$ respectively. If $g \in K_{p'}$ then $g(p) = p$, $\beta = \mathrm{id}$, $\alpha = \beta_{|k(p)} = \mathrm{id}$ and therefore $g \in K_p$. Conversely if $g \in K_p$, so that $\alpha = \mathrm{id}$, then $\alpha \otimes \mathrm{id} = \mathrm{id}$. In particular $g(p') = p'$ and $\beta = \mathrm{id}$, i.e. $g \in H_{p'}$. $\square$

*Proof.* (*proof of Theorem 4.4.7 assuming Theorem 4.4.4*). First note that if $Y$ is normal then $X$ is regular in codimension 1 if and only if it is normal, because $f$ has Cohen-Macaulay fibers. All the statements in the Theorem are local in the codimension 1 points of $X$ and $Y$. Therefore we can assume that $Y$ is the spectrum of a DVR $R$ and that $X = \operatorname{Spec} A$, where $A \in \mathrm{LAlg}^G R$. In order to conclude it is enough to note that, in this case, $e^A = v_R(s_f)$ and, if $G$ is a glrg and $A \in G\text{-Cov}$, $e^A_V = v_R(s_{f,V})$. □

Now that we have collected all the needed definitions, we can start proving all the lemmas required for the proof of Theorem 4.4.4.

*Notation* 4.4.13. Given $\mathscr{A} \in \mathrm{LAlg}^G T$ we will denote by $\mathcal{P}^{\mathscr{A}/\mathcal{O}_T} = \operatorname{Ker} \mathrm{tr}^{\mathscr{A}/\mathcal{O}_T}$ and by $\sigma^{\mathscr{A}/\mathcal{O}_T} \colon \mathcal{P}^{\mathscr{A}/\mathcal{O}_T} \otimes \mathcal{P}^{\mathscr{A}/\mathcal{O}_T} \longrightarrow \mathcal{O}_T$ the restriction of $\mathrm{tr}^{\mathscr{A}/\mathcal{O}_T} \circ m_{\mathscr{A}}$. Again we will simply write $\mathcal{P}^{\mathscr{A}}$ and $\sigma^{\mathscr{A}}$ when $T$ will be given.

We now show how to describe the trace map in the case of an algebra with an action of a glrg.

**Lemma 4.4.14.** *Let $\mathscr{A} \in \mathrm{LAlg}^G T$. Then $\mathrm{tr}^{\mathscr{A}} \colon \mathscr{A} \longrightarrow \mathcal{O}_T$ is $G$-equivariant. If we assume that $G$ is a glrg, that $\mathscr{A}^G = \mathcal{O}_T$ and that $\operatorname{rk} \mathscr{A} = \operatorname{rk} G$, then we also have*

$$\mathcal{P}^{\mathscr{A}} = \operatorname{Ker} \mathrm{tr}^{\mathscr{A}} = \bigoplus_{R \neq V \in I_G} V^\vee \otimes \Omega_V^{\mathscr{A}} \text{ and } \mathcal{Q}^{\mathscr{A}} = \bigoplus_{V \in I_G} V^\vee \otimes \mathcal{Q}_V^{\mathscr{A}}$$

*Moreover Proposition 4.4.6 is true.*

*Proof.* In order to prove that $\mathrm{tr}^{\mathscr{A}}$ is $G$-equivariant we can work locally in the étale topology and assume $T$ affine and $G$ constant. In this case it is clear that $\mathrm{tr}^{\mathscr{A}}(g(x)) = x$ for any $g \in G$ and $x \in \mathscr{A}$. Now assume that $G$ is a glrg. We will have that

$$\operatorname{Ker} \mathrm{tr}^{\mathscr{A}} = \bigoplus_{V \in I_G} V^\vee \otimes \Gamma_V \text{ with } \Gamma_V \subseteq \Omega_V^{\mathscr{A}}$$

Note that since $G$ is linearly reductive, $\operatorname{rk} \mathscr{A} = \operatorname{rk} G$ is invertible in $\mathcal{O}_T^*$ and therefore $\mathrm{tr}^{\mathscr{A}} \colon \mathscr{A} \longrightarrow \mathcal{O}_T$ is surjective. So

$$\mathcal{O}_T = \bigoplus_{V \in I_G} V^\vee \otimes (\Omega_V^{\mathscr{A}} / \Gamma_V)$$

is a $G$-equivariant decomposition and therefore $\Gamma_V = \Omega_V^{\mathscr{A}}$. Let $V, W \in I_G$. By construction the product of elements of $V^\vee \otimes \Omega_V$ and $W^\vee \otimes \Omega_W$ lies in $\operatorname{Ker} \mathrm{tr}^{\mathscr{A}}$, i.e. has no component in $\mathscr{A}^G = \Omega_R^{\mathscr{A}}$, except for the case when $W = V^\vee$. So the trace map $\mathscr{A} \longrightarrow \mathscr{A}^\vee$ is the direct sum of the maps induced by $\delta_V \colon V^\vee \otimes \Omega_V \otimes V \otimes \Omega_{V^\vee} \longrightarrow \mathscr{A} \otimes \mathscr{A} \longrightarrow \mathscr{A} \xrightarrow{\mathrm{tr}_{\mathscr{A}}} \mathcal{O}_T$. We have seen that $\mathrm{tr}_{\mathscr{A}} = (\operatorname{rk} G)\pi$, where $\pi$ is the projection according to the $G$-equivariant decomposition of $\mathscr{A}$. By 4.2.46, the map $\delta_V$ is given by

$$V^\vee \otimes \Omega_V \otimes V \otimes \Omega_{V^\vee} \simeq V^\vee \otimes V \otimes \Omega_V \otimes \Omega_{V^\vee} \xrightarrow{u(e_{V^\vee} \otimes \omega_V)} \Omega_R$$

where $e_* \colon (*) \otimes (*)^\vee \longrightarrow R$ is the evaluation map and $u = \operatorname{rk} G / \operatorname{rk} V$, which is invertible by 4.2.45. So the map induced by the above morphism is exactly $u(\mathrm{id}_{V^\vee} \otimes \xi_V) \colon V^\vee \otimes \Omega_V^{\mathscr{A}} \longrightarrow V^\vee \otimes (\Omega_{V^\vee}^{\mathscr{A}})^\vee$, as required. For the last claim, if $\mathscr{A} \in G\text{-Cov}$, it is enough to note that $\det \xi_V$ induces the section $s_{f,V}$, where $f$ is the map $\operatorname{Spec} \mathscr{A} \longrightarrow T$. □

One of the key points in the proof of Theorem 4.4.4 is the local case. We are going now to focus on it.

**Lemma 4.4.15.** *Let $R$ be a strictly Henselian DVR, $A \in \mathrm{LAlg}^G R$ such that $A$ is local, $A^G = R$ and $\operatorname{rk} A = |G|$. Then*

$$\mathcal{Q}^A = \operatorname{Coker}(\mathcal{P}^A \xrightarrow{\widehat{\sigma^A}} (\mathcal{P}^A)^\vee), \ \operatorname{Im} \widehat{\sigma^A} \subseteq m_R (\mathcal{P}^A)^\vee, \ m_A = m_R \oplus \mathcal{P}^A, \ \operatorname{tr}_{A/R}(m_A) \subseteq m_R$$

*Moreover the following conditions are equivalent*

1) *$\widehat{\sigma^A}$ is surjective onto $m_R(\mathcal{P}^A)^\vee$;*
2) *$\mathcal{Q}^A$ is defined over the closed point of $R$;*
3) *$e^A = \operatorname{rk} A - 1$*
4) *$e^A < \operatorname{rk} A$.*

*Proof.* Since $G$ is linearly reductive, $\operatorname{rk} A = \operatorname{rk} G \in R^*$and the trace map $\operatorname{tr}^A$ is surjective and since $G$ is étale and $R$ strictly Henselian, $G$ is constant. In particular $\mathcal{P}^A$ is free and $A = R \oplus \mathcal{P}^A$. Moreover $m_A = \mathcal{P}^A \oplus m_R$ thanks to 4.4.14 and 4.2.47 and therefore $\operatorname{tr}^A(m_A) \subseteq m_R$. This shows that $\widehat{\sigma^A}$ has image in $\operatorname{Hom}(\mathcal{P}^A, m_R) = m_R(\mathcal{P}^A)^\vee$. Since $\widehat{\operatorname{tr}}^A \colon A \longrightarrow A^\vee$ is the sum of the multiplication $R \xrightarrow{\operatorname{rk} A} R^\vee$ and the map $\widehat{\sigma^A}$, we have that $\mathcal{Q}^A = \operatorname{Coker} \widehat{\sigma^A}$ and that $\mathcal{Q}^A$ is defined over the closed point of $R$ if and only if $m_R(\mathcal{P}^A)^\vee \subseteq \operatorname{Im} \sigma_A$, i.e. the equivalence between 1) and 2) holds. In this case $\mathcal{Q}^A = (\mathcal{P}^A)^\vee / m_R(\mathcal{P}^A)^\vee \simeq (R/m_R)^{|G|-1}$, which shows 2) $\implies$ 3) $\implies$ 4). So it remains to prove that 4) $\implies$ 1). Let $\pi \in R$ be a uniformizer. Since $\widehat{\sigma^A}$ has image in $\pi(\mathcal{P}^A)^\vee$ we can set $u = \widehat{\sigma^A}/\pi \colon \mathcal{P}^A \longrightarrow (\mathcal{P}^A)^\vee$ and we have to prove that $u$ is an isomorphism. Note that, by construction, $e^A = v_R(\det \widehat{\sigma^A})$ and so

$$0 \leq v_R(\det u) = e^A - \operatorname{rk} A + 1 < 1 \implies \det u \in R^* \implies u \text{ isomorphism}$$

$\square$

**Lemma 4.4.16.** *Let $R$ be a DVR, $P$ be a free $R$-module and $\eta \colon P \otimes P \longrightarrow R$ be an $R$-linear map. Then if $Q \subseteq P$ is a saturated submodule and we assume that both $\widehat{\eta}$ and $\widehat{\eta_{|Q\otimes Q}}$ are injective then $Q^\perp = \{x \in P \mid \eta(x \otimes y) = 0$ for all $y \in Q\}$ is saturated too and $Q \oplus Q^\perp = P$.*

*Proof.* It easy to check that $Q^\perp$ is a submodule of $P$ which is saturated. Moreover

$$Q \cap Q^\perp = \operatorname{Ker}(\widehat{\eta_{|Q\otimes Q}}) = 0 \implies Q \oplus Q^\perp \subseteq P$$

Since both $Q$ and $Q^\perp$ are saturated, we have only to prove that $\operatorname{rk} Q^\perp = \operatorname{rk} P - \operatorname{rk} Q$. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & Q^\perp \arrow[r] \arrow[d, dashed, "\alpha"] & P \arrow[r] \arrow[d, "\widehat{\eta}"] & P/Q^\perp \arrow[r] \arrow[d, dashed, "\beta"] & 0 \\ 0 \arrow[r] & P/Q^\vee \arrow[r] & P^\vee \arrow[r] & Q^\vee \arrow[r] & 0 \end{tikzcd}$$

The first row is exact since $Q^\perp$ is saturated, the second one because $Q$ is so. Note that $Q^\perp = \operatorname{Ker}(P \xrightarrow{\widehat{\eta}} P^\vee \longrightarrow Q^\vee)$ and therefore the maps $\beta$ and $\alpha$ are well defined and $\beta$ is injective. By the snake lemma we get that $\operatorname{rk}\operatorname{Coker}\beta = 0$ and the desired equality. $\square$

*Remark* 4.4.17. If $R$ is a DVR and $A \in \operatorname{LAlg}^G R$ then $A$ is generically étale if and only if $e^A < \infty$ or $\operatorname{rk}\mathcal{Q}^A = 0$. Indeed those conditions are all equivalent to the condition that $\widehat{\operatorname{tr}}_{A\otimes k(R)}$ is an isomorphism.

The following lemma is the hard part in the proof of Theorem 4.4.4.

**Lemma 4.4.18.** *Assume that $G$ is a solvable group and let $R$ be a strictly Henselian DVR. Let also $A \in \operatorname{LAlg}^G R$ be a local algebra such that $A^G = R$ and that the action of $G$ on $A$ is generically faithful. Then $A$ is a DVR if one of the following conditions holds:*

- *$\mathcal{Q}^A$ is defined over the closed point of $R$;*
- *$e^A < \operatorname{rk} A$.*

*Proof.* Note that if one of the above conditions is satisfied, then $A$ is generically étale and so generically a $G$-torsor thanks to 4.4.17 and 4.4.10. In particular $\operatorname{rk} A = \operatorname{rk} G$ and by 4.4.15 the two conditions in the statement are equivalent. We will make use of 4.2.47 and we will argue by induction on $\operatorname{rk} G$. Note that $G$ is constant and that by 4.1.22 $G$ is a glrg. We first consider the base case $G \simeq \mathbb{Z}/p\mathbb{Z}$, where $p$ is a prime. There exists a basis $\{v_i\}_{i\in G^*}$ of $A$ such that $v_iv_j = \psi_{i,j}v_{i+j}$, with $\psi_{i,j} \in R$. Set $e_{i,j} = v_R(\psi_{i,j})$, where $v_R$ is the valuation of $R$. The associativity conditions yield relations

$$\psi_{n,t}\psi_{n+t,s} = \psi_{t,s}\psi_{t+s,n} \implies e_{n,t} + e_{n+t,s} = e_{t,s} + e_{t+s,n} \implies e_{n,-n} = e_{-n,s} + e_{s-n,n} \tag{4.4.1}$$

From 4.4.14, we have

$$e^A = \sum_{i\neq 0} e_{i,-i}$$

Since $A$ is local we have that $e_{i,-i} > 0$ for all $i \neq 0$. In particular $e^A < p$ implies that $e_{i,-i} = 1$ for all $i \neq 0$. From (4.4.1) we see that $e_{n,t} \in \{1, 0\}$ for all $n, t$. Note that $m_A/m_A^2$ is $G$-equivariant and by contradiction assume that $\dim_k m_A/m_A^2 > 1$. Since $e_{n,-n} = 1$, $v_nv_{-n}$ generates the maximal ideal $m_R$ of $R$ and therefore $m_RA \subseteq m_A^2$ and $(m_A/m_A^2)_0 = 0$. Moreover $\dim_k(m_A/m_A^2)_n \in \{1, 0\}$ for all $n$, because, if $n \neq 0$, $v_n \in m_A$ and $m_Rv_n \subseteq m_A^2$. So there exists $n \neq t \in G^*$ such that $v_n, v_t \notin m_A^2$ and $n, t \neq 0$. Note that if $u + s = n$, with $u, s \neq 0$, then $e_{u,s} = 1$ since otherwise

$$v_uv_s = \psi_{u,s}v_n \text{ with } \psi_{u,s} \in R^* \implies v_n \in m_A^2$$

and similarly if $u + s = t$. So set $u = n - t$ and consider the relation

$$e_{-u,t+u} + e_{t,u} = e_{u,-u} = 1$$

obtained from (4.4.1). We have that $-u, t+u = n, t, u$ are all non zero and by the remark above we get

$$e_{-u,t+u} = e_{t,u} = 1 \implies 2 = 1$$

which is impossible.

We now come back to the general case. If $G$ is simple, then we are in the base case. Otherwise take a normal subgroup $0 \neq H \neq G$ of $G$ and let $B = A^H$.

We want first to prove that $B$ is a DVR using the inductive hypothesis on $B/R$. Since $H$ is normal, we have that $B \in \mathrm{LAlg}^{G/H} R$ and that it is generically a $G/H$-torsor. From 4.2.49 and 4.4.14 we have that

$$\mathcal{P}^B = \bigoplus_{R \neq W \in I_{G/H}} W^\vee \otimes \Omega^A_W \subseteq \mathcal{P}^A \cap B = \bigoplus_{R \neq V \in I_G} (V^\vee)^H \otimes \Omega^A_V$$

Since $B = R \oplus \mathcal{P}^B \subseteq R \oplus \mathcal{P}^A \cap B = B$ we can conclude that $\mathcal{P}^B = \mathcal{P}^A \cap B$. Moreover $B$ is local. Indeed if $x \in B \cap A^*$ and $y \in A$ is such that $xy = 1$ we will have

$$1 = \prod_{h \in H} h(xy) = x^{|H|}(\prod_{h \in H} h(y)) \implies x \in B^*$$

We can apply 4.4.16, since $\mathcal{P}^B$ is saturated in $\mathcal{P}^A$ and both $\widehat{\sigma^A}$ and $\widehat{\sigma^B}$ are injective since $A$ and $B$ are generically étale. So $\mathcal{P}^A = \mathcal{P}^B \oplus (\mathcal{P}^B)^\perp$. Moreover $\widehat{\sigma^B}$ and $\widehat{\sigma^A}_{|\mathcal{P}^B}$ differ only by the multiplication of $\mathrm{rk}\, A/\mathrm{rk}\, B = |G|/|H| \in R^*$ since $\mathcal{P}^B = \mathcal{P}^A \cap B$. We can conclude that $\mathcal{Q}^{B/R}$ is a submodule of $\mathcal{Q}^{A/R}$ and it is therefore defined over the closed point of $R$. We can now apply the inductive hypothesis on $B/R$ and conclude that $B$ is a DVR.

We are going now to apply inductive hypothesis on $A/B$. Note that $B$ is strictly Henselian since $B/m_B = R/m_R$ and $B/R$ is finite. We clearly have that $A$ is generically a $H$-torsor on $B$. Since $k(B) \simeq B \otimes_R k(R)$, $A \otimes_R k(R)$ is free over $k(B)$ and $A \subseteq A \otimes_R k(R)$, we see that $A$ is a $B$-module without torsion and therefore free. This means that $A \in \mathrm{LAlg}^H B$. In order to apply the inductive hypothesis on $A$ and conclude that it is a DVR, we have to show that the image of the map $\widehat{\mathrm{tr}}_{A/B} \colon A \longrightarrow \mathrm{Hom}_B(A, B)$ contains $m_B \mathrm{Hom}_B(A, B)$, thanks to 4.4.15. Since $A$ is free over $B$ and $B$ is free over $R$ we have the relations

$$\mathrm{tr}_{B/R} \circ \mathrm{tr}_{A/B} = \mathrm{tr}_{A/R} \implies \psi \circ \widehat{\mathrm{tr}}_{A/B} = \widehat{\mathrm{tr}}_{A/R}$$

where $\psi \colon \mathrm{Hom}_B(A, B) \longrightarrow \mathrm{Hom}_R(A, R)$ is the map induced by $\mathrm{tr}_{B/R} \colon B \longrightarrow R$. We start proving that $\psi$ is injective. Let $\phi \colon A \longrightarrow B$ be such that $\psi(\phi) = 0$. This means that $\mathrm{Im}\, \phi \subseteq \mathcal{P}^B = \mathrm{Ker}\, \mathrm{tr}_{B/R}$. If $\mathrm{Im}\, \phi \neq 0$, since it is an ideal of $B$, we will have $\mathrm{Im}\, \phi = m_B^t$ for some $t$. In particular $\mathrm{Im}\, \phi \cap R \neq 0$, while we know that $\mathcal{P}^B \cap R = 0$. So it remains to prove that if $y \in m_B$ and $\phi \in \mathrm{Hom}_B(A, B)$ then $\xi = \psi(y\phi) \in \mathrm{Im}\, \widehat{\mathrm{tr}}_{A/R}$. Remember that $m_R \mathrm{Hom}_R(A, R)$ is contained in $\mathrm{Im}\, \widehat{\mathrm{tr}}_{A/R}$ since $\mathcal{Q}^{\mathscr{A}}$ is defined over the closed point of $R$. Let $\pi \in R$ be an uniformizing element. We have

$$\forall x \in A\ \xi(x) = \psi(y\phi)(x) = \mathrm{tr}_{B/R}(y\phi(x)) \in m_R \implies \xi = \pi(\xi/\pi) \in m_R \mathrm{Hom}_R(A, R)$$

since $\mathrm{tr}_{B/R}(m_B) \subseteq m_R$, thanks to 4.4.15. $\square$

**Lemma 4.4.19.** *Assume that $R$ is a DVR, $A \in \mathrm{LAlg}^G R$ and call $R^{sh}$ the strict Henselization of $R$. Then $A$ is regular (generically a $G$-torsor) if and only if $A \otimes_R R^{sh}$ is so.*

*Proof.* $A \otimes_R R^{sh}$ is faithfully flat over $A$ and it is a direct limit of étale extensions of $A$. In particular $\operatorname{Spec} A \times_R R^{sh} \longrightarrow \operatorname{Spec} A$ is surjective and the dimensions of the tangent spaces remain constant. So $A$ is regular if and only if $A \otimes_R R^{sh}$ is so. Since $R^{sh}$ is a domain and the condition of being a $G$-torsor is open, we get that $A$ is generically a $G$-torsor if and only if $A \otimes_R R^{sh}$ is so. □

**Lemma 4.4.20.** *Assume that $R$ is a DVR and let $A \in \mathrm{LAlg}^G R$ such that $A^G = R$ and that the action of $G$ is generically faithful. If $A$ is regular then it is generically a $G$-torsor and the geometric stabilizer of a closed point of $A$ is cyclic.*

*Proof.* Thanks to 4.4.19 we can pass to the strict Henselization of $R$ and assume that $G$ is constant. Let $p \in \operatorname{Spec} A$ be a closed point, $k$ be the residue field of $R$ and $H$ be the stabilizer of $\operatorname{Spec} A_p$ in $\operatorname{Spec} A$. By 4.1.36, we know that $\operatorname{ind}_H^G A_p \simeq A$. In particular $A_p \in \mathrm{LAlg}^H R$, it is a DVR and $A_p^H = R$. From 4.2.47 we see that $k(p) = k$ and therefore $H$ is the geometric stabilizer of $p$ and it is given by $H = \{g \in G \mid g(p) = p\}$. Moreover $A_p \otimes k(R)$ is a field such that $(A_p \otimes k(R))^H = k(R)$ and therefore a separable extension of $k(R)$. So $A \otimes k(R)/k(R)$ is étale and therefore a $G$-torsor thanks to 4.4.10. We can conclude that $A_p$ is generically a $H$-torsor by 4.3.7 and, again by 4.4.10, that $H \subseteq \operatorname{Aut}_R A_p$. Consider the map

$$H \longrightarrow \operatorname{Aut}_k m_p/m_p^2 \simeq k^*$$

We will prove that it is injective, which implies that $H$ is cyclic. Let $h \in H \subseteq \operatorname{Aut}_R A_p$ such that $h_{|m_p/m_p^2} = \mathrm{id}$ and $\pi$ be a uniformizing of $A_p$. We first show that $h(\pi) = \pi$. If $h(\pi) \neq \pi$, we can write

$$h(\pi) = \pi + u\pi^k \mod m_p^{k+1} \text{ with } u \in R^*,\ k > 1$$

An easy induction shows that $h^n(\pi) = \pi + nu\pi^k \mod m_p^{k+1}$. If $n$ is the order of $h$, so that $h^n(\pi) = \pi$, we will have $nu = 0$ and therefore $u = 0$ since $\operatorname{char} k \nmid n$. So $h_{|m_p} = \mathrm{id}$. If $a \in A_p^*$ then there exists $r \in R$ such that $a - r \in m_p$ and we have

$$h(a) = h(a - r) + h(r) = a - r + r = a$$

□

We are now ready for the proof of 4.4.4.

*Proof.* (*of Theorem* 4.4.4) We first consider the last part of the statement, i.e. the case when $G$ is a glrg. From 4.4.14 we have that

$$\mathcal{Q}^A = \bigoplus_{V \in I_G} V^\vee \otimes \mathcal{Q}_V^A \text{ and } e^A = \sum_{V \in I_G} \operatorname{rk} V \cdot e_V^A$$

In particular 2) $\iff$ 4) and, thanks to 4.4.17 and 4.4.10, we also have that any of the conditions $1), 2), 3), 4)$ implies that $A$ is generically a $G$-torsor and $A \in G\text{-Cov}R$. In particular $\operatorname{rk} A = \operatorname{rk} G \in R^*$, $\mathcal{Q}^A_R = 0$ and $\operatorname{rk} \Omega^A_V = \operatorname{rk} V$. The existence of a surjective map ${\Omega^A_{V^\vee}}^\vee \longrightarrow \mathcal{Q}^A_V$ tells us that 4) $\implies$ 3), while $\mathcal{Q}^A_R = 0$ tells us that 3) $\implies$ 1).

We now consider the general case. Note that the length does not change when passing to the strict Henselization of $R$ and an $R$-module is schematically supported on the closed point of $R$ if and only if it is so on the strict Henselization of $R$. Thanks to 4.4.19, we can assume that $R$ is strictly Henselian and therefore that $G$ is constant. We want now to reduce to the case in which $A$ is local. Denote by $k$ the residue field of $R$ and let $p \in \operatorname{Spec} A$ be a closed point and $H$ be the stabilizer of the connected component $\operatorname{Spec} A_p$ of $\operatorname{Spec} A$. By 4.1.36, we know that $\operatorname{ind}_H^G A_p \simeq A$. In particular $A_p^H = R$, $H$ is the geometric stabilizer $H_p$ of $p$ since $k(p) = k$ by 4.2.47, $A_p \in \operatorname{LAlg}^H R$ and $A$ is generically a $G$-torsor if and only if $A_p$ is generically a $H$-torsor. Since $R$ is strictly Henselian we have that

$$A = \prod_{q \in \operatorname{Spec}_m A} A_q,\ \mathcal{Q}^A = \bigoplus_{q \in \operatorname{Spec}_m A} \mathcal{Q}^{A_q},\ e^A = \sum_{q \in \operatorname{Spec}_m A} e^{A_q} = |\operatorname{Spec}_m A| e^{A_p}$$

where the last equality holds since $A_q \simeq A_p$ for any $q \in \operatorname{Spec}_m A$, thanks to 4.1.35. Note also that

$$|G|/|H| = |\operatorname{Spec}_m A| = |\operatorname{Spec} A \otimes_R k| = |\operatorname{Spec} A \otimes_R \overline{k}|$$

since $k(q) = k$ for any $q \in \operatorname{Spec}_m A$. We can therefore assume $A$ to be local and generically a $G$-torsor and that $G$ is solvable. Lemma 4.4.18 assure us that if the conditions in the statement are fulfilled then $A$ is a DVR and $e^A = \operatorname{rk} A - 1$. So assume that $A$ is regular and therefore that $G$ is cyclic, thanks to 4.4.20. In this case

$$A = \bigoplus_{n \in G^*} R v_n,\ v_n v_t = \psi_{n,t} v_{n+t'} \text{ with } \psi_{n,t} \in R - \{0\} \text{ and } m_A = \bigoplus_{n \neq 0} R v_n$$

From 3.2.43, we can write $\psi_{n,t} = \lambda_{n.t} z^{\mathcal{E}_{n,t}}$ with $\lambda_{n,t} \in R^*$, $z \in m_R - m_R^2$ and $\mathcal{E}$ is as in 3.2.39. In particular $\mathcal{E}_{m,-m} = 1$ for all $0 \neq m \in G^*$. If $V_m$ is the one dimensional representation associated with $m \in G^*$ we see that $\mathcal{Q}^A_{V_m} = R/(z^{\mathcal{E}_{m,-m}})$. In particular condition 4), and therefore all the others thanks to the initial discussion about glrg, are satisfied. $\square$

*Remark* 4.4.21. In 4.4.4 the conditions 1) $e^A < \operatorname{rk} A$ and 2) $\mathcal{Q}^A$ defined over the closed point of $R$ do not depend on the action of $G$. This suggests the following question: given a finite and flat algebra $A$ over a DVR, is it true that $A$ is a generically étale regular algebra if and only if it satisfies 1) or 2)?

The answer to that question is negative for condition 1). Indeed take a not regular local $R$-algebra $B$, a local regular one $C$ with $e^C = \operatorname{rk} C - 1$ and define $A = B \times C^r$. The algebra $A$ is not regular but

$$e^A = e^B + r \operatorname{rk} C - r < \operatorname{rk} B + r \operatorname{rk} C = \operatorname{rk} A \iff r > e^B - \operatorname{rk} B$$

Anyway this is not a satisfying answer, since it is clear that in this general situation a right condition on $e^A$ has to take in consideration the value of $|\operatorname{Spec} A \otimes_R \overline{k}|$, where $k$ is the residue field of $R$. So one can change condition 1) with the more strong $e^A \leq \operatorname{rk} A - |\operatorname{Spec} A \otimes_R \overline{k}|$.

In general also the converse is false, if we do not assume some condition on $\operatorname{rk} A$. Consider for example a DVR $R$ with uniformizer $\pi$ such that $0 \neq 2 \in m_R$ and consider

$$f(x) = x^2 - 2x - \pi \text{ and } A = R[x]/(f)$$

I claim that $A$ is regular, generically étale over $R$ with residue field $R/m_R$ but $e^A \geq \operatorname{rk} A = 2$. It is generically étale since $A \otimes k(R)$ is either $k(R)^2$ or a field, in which case is a separable extension of $k(R)$ because $f$ is separable. It is local because $A \otimes k = k[x]/(x^2)$ and it is regular because its maximal ideal $(\pi, x)$ is clearly generated by $x$. On the other hand

$$\operatorname{tr}_A x = 2,\ \operatorname{tr}_A x^2 = 4 + 2\pi \implies \det \widehat{\operatorname{tr}}_A = 4(1+\pi) \implies e^A = v_R(\det \widehat{\operatorname{tr}}^A) = 2v_R(2) \geq 2$$

There are other possible remarks in this direction and, at the end, a very reasonable conjecture is the following

**Conjecture 4.4.22.** *Let $R$ be a DVR with residue field $k$ and $A$ be a finite and flat $R$-algebra. Then*

$$e^A \geq \operatorname{rk} A - |\operatorname{Spec} A \otimes_R \overline{k}|$$

*and the following conditions are equivalent:*

1) *$A$ is regular, generically étale with separable residue fields and the localizations of $A \otimes_R \overline{k}$ have ranks prime to the characteristic* $\operatorname{char} k$*;*

2) *the equality holds in the inequality above;*

3) *$\mathcal{Q}^A$ is defined over the closed point of $R$.*

Currently I am able to prove the inequality above, the equivalence of 2) and 3), the implication 1) $\implies$ 2) and 2) $\implies$ 1), except for the regularity of $A$.

# 5 $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$-covers and $S_3$-covers.

In this chapter we describe Galois covers for the group $G = \mu_3 \ltimes \mathbb{Z}/2\mathbb{Z}$ defined over the ring $\mathcal{R} = \mathbb{Z}[1/2]$ and for the group $S_3$ defined over $\mathcal{R}_3 = \mathbb{Z}[1/6]$. This is a summary of the division of the chapter.

*Section 1.* This section is dedicated to the description of the representation theory of the group $G$ over the ring $\mathcal{R}$. We will show that $G$ is a good linearly reductive group, that $\mathrm{B}\, G \simeq \mathrm{B}\, S_3$ over $\mathcal{R}_3$ and we will describe the geometrically irreducible representations of $G$ and their tensor products.

*Section 2.* We will describe the global data needed to define a $G$-cover. Such data will be given in terms of linear algebra, that is as a collection of locally free sheaves and maps between them satisfying the commutativity of certain diagrams. We will express those conditions in terms of local equations.

*Section 3.* In this section we describe the geometry of $G$-Cov and $S_3$-Cov and families of $G$-covers with additional properties. We will prove that $G$-Cov ($S_3$-Cov over $\mathcal{R}_3$) has exactly two irreducible components and we will describe them in terms of vanishing of maps between coherent sheaves. Such results will require the study of particular open substacks of $G$-Cov and in particular of $\mathrm{B}\, G$.

*Section 4.* We will give a characterization of regular $G$-covers and regular $S_3$-covers in terms of properties of closed subschemes of the base associated with the data defining them. In particular we will prove an equivalence between regular $G$-covers, regular $S_3$-covers and regular triple covers satisfying a codimension 2 condition. We will then show how it is possible to construct such covers and we will compute the invariants of the total space of a regular $S_3$-cover of a smooth surface over an algebraically closed field.

## 5.1 Preliminaries and notation.

In this chapter we will work over the ring $\mathcal{R}$ of integers with 2 inverted, that is $\mathcal{R} = \mathbb{Z}[1/2]$. Sometimes we will also need to have 3 invertible in the base scheme and we will denote by $\mathcal{R}_3$ the ring of integers with 6 inverted, that is $\mathcal{R}_3 = \mathbb{Z}[1/6]$. In all the chapter the symbol $G$ will denote the group scheme $G = \mu_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ defined over $\mathcal{R}$, where the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mu_3$ or, equivalently, $\mathbb{Z}/3\mathbb{Z}$ is given by the inversion. Note that, in this case, $\mu_2 \simeq \mathbb{Z}/2\mathbb{Z}$. Moreover, over $\mathcal{R}_3[x]/(x^2+x+1)$, we have $G \simeq S_3$. In particular, by 2.3.11 and 2.3.12, we have isomorphisms

$$\mathrm{B}\, G \simeq \mathrm{B}\, S_3 \text{ and } G\text{-Cov} \simeq S_3\text{-Cov}$$

over the ring $\mathcal{R}_3$. Therefore the study of $G$-covers coincides with the study of $S_3$-covers over $\mathcal{R}_3$. We have preferred to study $G$-covers directly, because the representation theory of $G$ has a simpler description and $G$, as we will see below, is a glrg over $\mathcal{R}$, while $S_3$ is not even linearly reductive over this base. On the other hand we will remark the results for $G$-covers that can be traduced in results for $S_3$-covers.

We want to describe the representation theory of $G$. Set $\sigma \in \mathbb{Z}/2\mathbb{Z}(\mathcal{R})$ for the generator of $\mathbb{Z}/2\mathbb{Z}$. We will also think of $\sigma$ as an element of $G(\mathcal{R})$ and as a given transposition of $S_3(\mathcal{R})$. Set also $V_0 = \mathcal{R}, V_1, V_2$ for the representations of $\mu_3$ corresponding to its characters in $\mathbb{Z}/3\mathbb{Z}$. Moreover consider the set $I_G$ of $G$-representations

$$\mathcal{R},\ A = V_\chi,\ V = \operatorname{ind}_{\mu_3}^G V_1$$

where $\chi\colon G \longrightarrow \mathbb{G}_m$ is induced by the non trivial character of $\mathbb{Z}/2\mathbb{Z}$. Since 2 is invertible, it is easy to check that $(G, I_G)$ is a good linearly reductive group over $\mathcal{R}$: it is linearly reductive because extension of a diagonalizable group and an étale constant group of order invertible in $\mathcal{R}$ and it has a good representation theory because the representations in $I_G$ restrict to the irreducible representations of $S_3$ over the geometric point $\operatorname{Spec} \overline{\mathbb{Q}} \longrightarrow \operatorname{Spec} \mathcal{R}$ (see 4.1.10). We will consider the following basis $1 \in \mathcal{R}$, $1_A \in A$ and $v_1, v_2 \in V$ such that $v_i \in V_i$. Moreover since $\sigma$ exchanges $V_1$ and $V_2$, we will also assume that $\sigma(v_1) = v_2, \sigma(v_2) = v_1$. Now we describe the tensor products of the representations in $I_G$. We have

$$A \otimes A \simeq \mathcal{R},\ 1_A \otimes 1_A \longrightarrow 1 \text{ and } A \otimes V \simeq V,\ 1_A \otimes v_1 \longrightarrow -v_1, 1_A \otimes v_2 \longrightarrow v_2$$

and, if we set $v_{ij} = v_i \otimes v_j \in V \otimes V$,

$$\mathcal{R} \oplus A \oplus V \simeq V \otimes V,\ 1 \longrightarrow v_{12} + v_{21}, 1_A \longrightarrow v_{21} - v_{12}, v_1 \longrightarrow v_{22}, v_2 \longrightarrow v_{11}$$

Finally note that the $G$-equivariant projection $V \otimes V \longrightarrow \mathcal{R},\ v_{ij} \longrightarrow 1 - \delta_{ij}$, where $\delta_{ij}$ is the Kronecker symbol, yields an isomorphism

$$V \simeq V^\vee,\ v_1 \longrightarrow v_2^*, v_2 \longrightarrow v_1^*$$

Since we will have to deal with locally free sheaves of rank two, we recall here the following fact about them.

*Remark* 5.1.1. If $\mathcal{F}$ is a locally free sheaf of rank 2 over a scheme $T$, the canonical map $\mathcal{F} \otimes \mathcal{F} \longrightarrow \det \mathcal{F}$ induces an isomorphism

$$\mathcal{F} \simeq \mathcal{F}^\vee \otimes \det \mathcal{F}$$

If $y, z$ is a basis of $\mathcal{F}$, then the above map is given by $y \longrightarrow -z^* \otimes (y \wedge z)$, $z \longrightarrow y^* \otimes (y \wedge z)$.

In this chapter, we will often prove statements valid over any scheme and, in order to simplify the reading, the letter $T$, if not stated otherwise, will denote a scheme over the given base, that is $\mathcal{R}$ or $\mathcal{R}_3$.

## 5.2 Global description of $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$-covers.

In this section we want to describe the data needed to define a $G$-cover over any $\mathcal{R}$-scheme. The idea will be to use the results of the previous chapter and describe particular lax, symmetric monoidal functors $\operatorname{Loc}^G \mathcal{R} \longrightarrow \operatorname{Loc} T$. We first introduce such data and then we will show their relationship with $G$-covers. We remark here that the global description obtained here, although with a different notation, has already been introduced in [Eas11].

*Notation* 5.2.1. In this section only, by an $\mathcal{O}_T$-algebra or a sheaf of $\mathcal{O}_T$-algebras we will mean a locally free sheaf of (non associative) rings $\mathscr{A}$ over $T$ with a unity $1 \in \mathscr{A}$.

We define the stack $\mathcal{Y}$ over $\mathcal{R}$ whose objects are sequences $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle)$ where: $\mathcal{L}$ is an invertible sheaf, $\mathcal{F}$ is a rank 2 locally free sheaf and $m, \alpha, \beta, \langle -, - \rangle$ are maps

$$\mathcal{L}^2 \xrightarrow{m} \mathcal{O}_T,\ \mathcal{L} \otimes \mathcal{F} \xrightarrow{\alpha} \mathcal{F},\ \operatorname{Sym}^2 \mathcal{F} \xrightarrow{\beta} \mathcal{F},\ \det \mathcal{F} \xrightarrow{\langle -,- \rangle} \mathcal{L}$$

With an object $\chi \in \mathcal{Y}$ as above we associate the map $(-,-)_\chi \colon \mathcal{F} \otimes \mathcal{F} \longrightarrow \mathcal{O}_T$ given by

$$(-,-)_\chi \colon \mathcal{F} \otimes \mathcal{F} \simeq \mathcal{F}^\vee \otimes \det \mathcal{F} \otimes \mathcal{F} \xrightarrow{\mathrm{id} \otimes \langle -,- \rangle \otimes \mathrm{id}} \mathcal{F}^\vee \otimes \mathcal{L} \otimes \mathcal{F} \xrightarrow{\mathrm{id} \otimes \alpha} \mathcal{F}^\vee \otimes \mathcal{F} \longrightarrow \mathcal{O}_T$$

where we are using the canonical isomorphism $\mathcal{F} \simeq \mathcal{F}^\vee \otimes \det \mathcal{F}$. Notice that, although we are using the symbol $(-,-)$ of a symmetric product, $(-,-)_\chi$ is not necessarily symmetric. Moreover we also associate with $\chi$ the maps $\gamma_\chi, \gamma'_\chi \colon \mathcal{F} \otimes \mathcal{F} \longrightarrow \mathcal{O}_T \oplus \mathcal{L}$ given by

$$\gamma_\chi = (-,-)_\chi + \langle -,- \rangle,\ \gamma'_\chi = (-,-)_\chi - \langle -,- \rangle$$

When $\chi$ is given, we will simply write $(-,-), \gamma, \gamma'$ or $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, (-,-), \langle -,- \rangle) \in \mathcal{Y}$. Moreover we set

$$\mathscr{A}_\chi = \mathcal{O}_T \oplus \mathcal{L} \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \text{ with } \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$$

**Proposition 5.2.2.** *Given $\chi \in \mathcal{Y}$ as above, the sheaf $\mathscr{A}_\chi$ has a unique $G$-comodule structure such that $\mathcal{F}_0 = \mathcal{O}_T \oplus \mathcal{L}, \mathcal{F}_1, \mathcal{F}_2$ define the $\mu_3$-action and $\sigma$ acts as $-\mathrm{id}_\mathcal{L}$ on $\mathcal{L}$ and induces $\mathrm{id}_\mathcal{F} \colon \mathcal{F}_1 \longrightarrow \mathcal{F}_2$.*

This Proposition will be proved in the next section. We endow $\mathscr{A}_\chi$ with a structure of sheaf of $\mathcal{O}_T$-algebras given by the maps

$$\mathcal{L}^2 \xrightarrow{m} \mathcal{O}_T,\ \mathcal{F}_1 \otimes \mathcal{L} \simeq \mathcal{L} \otimes \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_1,\ \mathcal{F}_2 \otimes \mathcal{L} \simeq \mathcal{L} \otimes \mathcal{F}_2 \xrightarrow{-\alpha} \mathcal{F}_2$$

$$\mathcal{F}_1 \otimes \mathcal{F}_1 \xrightarrow{\beta} \mathcal{F}_2,\ \mathcal{F}_2 \otimes \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_1,\ \mathcal{F}_1 \otimes \mathcal{F}_2 \xrightarrow{\gamma_\chi} \mathcal{O}_T \oplus \mathcal{L},\ \mathcal{F}_2 \otimes \mathcal{F}_1 \xrightarrow{\gamma'_\chi} \mathcal{O}_T \oplus \mathcal{L}$$

We are implicitly assuming that the maps $\mathcal{O}_T \otimes \mathscr{A}_\chi, \mathscr{A}_\chi \otimes \mathcal{O}_T \longrightarrow \mathscr{A}_\chi$ are just the usual isomorphisms, or, in other words, that $1 \in \mathcal{O}_T$ is a unity for $\mathscr{A}_\chi$.

We want now to give a list of equations involving the maps $m, \alpha, \beta, \langle -,- \rangle$, which we will show are the relationships needed for the associativity of $\mathscr{A}_\chi$. Such equations will be 'local' relations and therefore we introduce the following notation:

*Notation* 5.2.3. When we fix a generator $t$ of $\mathcal{L}$, the maps $m, \alpha, \beta, \langle -,- \rangle$ will be thought of as: $m \in \mathcal{O}_T$, given by $m(t \otimes t)$; $\alpha \colon \mathcal{F} \longrightarrow \mathcal{F}$, given by "$\alpha(u) = \alpha(t \otimes u)$"; $\langle -,- \rangle \colon \det \mathcal{F} \longrightarrow \mathcal{O}_S$, given by "$\langle u, v \rangle = \langle u, v \rangle t$". When we will say that some particular relation among the maps $m, \alpha, \beta, \langle -,- \rangle$ locally holds, this will always mean that such relation holds as soon as basis $t$ and $y, z$ of, respectively, $\mathcal{L}$ and $\mathcal{F}$ are given.

The equations are:

$$\alpha^2 = m \mathrm{id}_{\mathcal{F}} \tag{5.2.1}$$

$$\langle \alpha(u), v \rangle = \langle \alpha(v), u \rangle \tag{5.2.2}$$

$$\alpha(\beta(u \otimes v)) = -\beta(u \otimes \alpha(v)) \tag{5.2.3}$$

$$\beta(\beta(u \otimes v) \otimes w) = \langle \alpha(w), v \rangle u - \langle w, v \rangle \alpha(u) \tag{5.2.4}$$

$$\langle u, \beta(v \otimes w) \rangle = \langle w, \beta(u \otimes v) \rangle \tag{5.2.5}$$

Moreover note that, locally, we also have the relation

$$(x, y)_\chi = \langle \alpha(y), x \rangle \tag{5.2.6}$$

Taking into account the definition of $\mathrm{LRings}_{\mathcal{R}}^G$ given in 4.2.15, we will prove the following Theorem.

**Theorem 5.2.4.** *The map of stacks*

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathrm{LRings}_{\mathcal{R}}^G \\ \chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) & \longmapsto & \mathscr{A}_\chi \end{array}$$

*is well defined and induces an isomorphism between the substack of $\mathcal{Y}$ of objects that locally satisfy the relations (5.2.1), (5.2.2), (5.2.3), (5.2.4), (5.2.5) and $G$-*Cov *(where a cover is thought of as its corresponding sheaf of algebras).*

The next section is devoted to the proof of the above Theorem.

*Notation* 5.2.5. Assuming Theorem 5.2.4, we will identify the stack $\mathcal{Y}$ with $G$-Cov and, over $\mathcal{R}_3$, with $S_3$-Cov. Therefore we will also use the expression $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in G$-Cov and similarly for $S_3$.

### 5.2.1 From functors to algebras.

The goal of this section is to deduce what data is needed to define a $G$-cover and to prove Theorem 5.2.4. Consider the map $f \colon \mathrm{Loc}^G \mathcal{R} \longrightarrow \mathbb{N}$ given by $f_W = \mathrm{rk}\, W$. By Theorem 4.2.29 and remark 4.3.4, we need to describe the symmetric lax monoidal functors $\Omega \colon \mathrm{Loc}^G \mathcal{R} \longrightarrow \mathrm{Loc}\, T$ such that $\mathrm{rk}^\Omega = f$, i.e. $(\mathrm{LMon}_{\mathcal{R},f}^G)^{\mathrm{gr}}$. We will proceed in the following way. We will identify $\mathcal{Y}$ with a closed substack of $(\mathrm{LPMon}_{\mathcal{R},f}^G)^{\mathrm{gr}}$ in such a way that, for any $\chi \in \mathcal{Y}$, the algebra $\mathscr{A}_\chi$, as defined in the previous section, is isomorphic to the algebra $\chi_{R[G]}$, where $\chi$ is thought of as a pseudo-monoidal functor. For convenience, set $\mathcal{X}$ for the full substack of $(\mathrm{LPMon}^G_{\mathcal{R},f})^{\mathrm{gr}}$ of functors $\Omega$ such that $\Omega_\mathcal{R} = \mathcal{O}_T$ and $1 \in \Omega_\mathcal{R}$ is a unity for $\Omega$.

By 4.2.6, an $\mathcal{R}$-linear functor $\Omega$ such that $\mathrm{rk}^\Omega = f$ and that $\Omega_\mathcal{R} = \mathcal{O}_T$ is just given by

$$\Omega_A = \mathcal{L},\ \Omega_V = \mathcal{F}$$

where $\mathcal{L}$ is an invertible sheaf and $\mathcal{F}$ is a rank 2 locally free sheaf, both over $T$. The corresponding $G$-equivariant sheaf is

$$\mathscr{A}_\Omega = \mathcal{R}^\vee \otimes \mathcal{O}_T \oplus A^\vee \otimes \mathcal{L} \oplus V^\vee \otimes \mathcal{F}$$

A pseudo-monoidal structure on $\Omega$ for which $1 \in \mathcal{O}_T = \Omega_\mathcal{R}$ is a unity is given by maps

$$\mathcal{L} \otimes \mathcal{L} \xrightarrow{m} \mathcal{O}_T,\ \mathcal{L} \otimes \mathcal{F} \xrightarrow{\alpha} \mathcal{F},\ \mathcal{F} \otimes \mathcal{L} \xrightarrow{\hat{\alpha}} \mathcal{F},\ \mathcal{F} \otimes \mathcal{F} \xrightarrow{\eta_1 \oplus \eta_2 \oplus \beta} \mathcal{O}_T \oplus \mathcal{L} \oplus \mathcal{F}$$

The stack $\mathcal{X}$ can be therefore thought of as the stack whose objects are sequences $(\mathcal{L}, \mathcal{F}, m, \alpha, \hat{\alpha}, \eta_*, \beta)$ as above. In particular $\mathcal{Y}$ can be embedded in $\mathcal{X}$ by sending $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in \mathcal{Y}$ to the sequence $(\mathcal{L}, \mathcal{F}, m, \alpha, \hat{\alpha}, (-,-)_\chi, \langle -,- \rangle, \beta)$, where $\hat{\alpha}$ is obtained from $\alpha$ exchanging the factors in the source.

Given $\Omega \in \mathcal{X}$, we want now to describe the algebra $\mathscr{A}_\Omega$. It will be convenient to introduce the following notation:

$$\mathcal{L}_1 = A^\vee \otimes \mathcal{L},\ \mathcal{Q}_0 = \mathcal{O}_T \oplus \mathcal{L}_1,\ \mathcal{Q}_1 = V_2{}^\vee \otimes \mathcal{F},\ \mathcal{Q}_2 = V_1{}^\vee \otimes \mathcal{F}$$

In particular

$$\mathscr{A}_\Omega = \mathcal{O}_T \oplus \mathcal{L}_1 \oplus \mathcal{Q}_1 \oplus \mathcal{Q}_2$$

where $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2$ are the sheaves induced by the $\mu_3$ action on $\mathscr{A}_\Omega$, while $\sigma$ is $-\mathrm{id}_{\mathcal{L}_1}$ on $\mathcal{L}_1$ and induces the isomorphism $\mathcal{Q}_1 \longrightarrow \mathcal{Q}_2$, $v_2^* \otimes u \longrightarrow v_1^* \otimes u$. We want to describe the multiplication on $\mathscr{A}_\Omega$ starting from the maps $m, \alpha, \hat{\alpha}, \eta_*, \beta$. The sheaf $\mathcal{Q}_0$ is a $\mathbb{Z}/2\mathbb{Z}$-cover and the associated map

$$\mathcal{L}_1 \otimes \mathcal{L}_1 \xrightarrow{\mu} \mathcal{O}_T$$

is just given by $\mu(1_A^* \otimes x \otimes 1_A^* \otimes y) = \xi(x \otimes y)$. Then we have the maps

$$\mathcal{L}_1 \otimes \mathcal{Q}_1 \xrightarrow{\zeta} \mathcal{Q}_1,\ \mathcal{L}_1 \otimes \mathcal{Q}_2 \xrightarrow{\zeta'} \mathcal{Q}_2$$

that are given by

$$\zeta(1_A^* \otimes x \otimes v_2^* \otimes y) = v_2^* \alpha(x \otimes y),\ \zeta'(1_A^* \otimes x \otimes v_1^* \otimes y) = -v_1^* \alpha(x \otimes y)$$

The multiplications of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with $\mathcal{L}$ are just obtained exchanging the factors above and replacing $\alpha$ by $\hat{\alpha}$. Finally we have maps

$$\mathcal{Q}_1 \otimes \mathcal{Q}_1 \xrightarrow{\lambda} \mathcal{Q}_2,\ \mathcal{Q}_2 \otimes \mathcal{Q}_2 \xrightarrow{\lambda'} \mathcal{Q}_1,\ \mathcal{Q}_1 \otimes \mathcal{Q}_2 \xrightarrow{\delta} \mathcal{O}_T \oplus \mathcal{L}_1,\ \mathcal{Q}_2 \otimes \mathcal{Q}_1 \xrightarrow{\delta'} \mathcal{O}_T \oplus \mathcal{L}_1$$

that are given by

$$\lambda(v_2^* \otimes x \otimes v_2^* \otimes y) = v_1^* \beta(x \otimes y),\ \lambda'(v_1^* \otimes x \otimes v_1^* \otimes y) = v_2^* \beta(x \otimes y)$$

$$\delta(v_2^* \otimes x \otimes v_1^* \otimes y) = \eta_1(x \otimes y) + 1_A^* \eta_2(x \otimes y),\ \delta'(v_1^* \otimes x \otimes v_2^* \otimes y) = \eta_1(x \otimes y) - 1_A^* \eta_2(x \otimes y)$$

Now define

$$\mathscr{B}_\Omega = \mathcal{O}_T \oplus \mathcal{L} \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \text{ with } \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F} \text{ and } \mathcal{F}_0 = \mathcal{O}_T \oplus \mathcal{L}$$

The isomorphisms $\mathcal{R} \simeq A^\vee$, $\mathcal{R} \simeq V_1{}^\vee$, $\mathcal{R} \simeq V_2{}^\vee$ induce an isomorphism $\mathcal{B}_\Omega \simeq \mathscr{A}_\Omega$ of coherent sheaves. The $G$-comodule structure inherited by $\mathscr{B}_\Omega$ is given by: $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$ yields the $\mu_3$-action, while $\sigma \in \mathbb{Z}/2\mathbb{Z}$ acts as $-\mathrm{id}_\mathcal{L}$ over $\mathcal{L}$ and induces $\mathrm{id}_\mathcal{F} \colon \mathcal{F}_1 = \mathcal{F} \longrightarrow \mathcal{F} = \mathcal{F}_2$ over $\mathcal{F}_1$. The $\mathcal{O}_T$-algebra structure inherited by $\mathscr{B}_\Omega$ is given by

$$\mathcal{L}^2 \xrightarrow{m} \mathcal{O}_T,\ \mathcal{L} \otimes \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_1,\ \mathcal{L} \otimes \mathcal{F}_2 \xrightarrow{-\alpha} \mathcal{F}_2,\ \mathcal{F}_1 \otimes \mathcal{L} \xrightarrow{\hat{\alpha}} \mathcal{F}_1,\ \mathcal{F}_2 \otimes \mathcal{L} \xrightarrow{-\hat{\alpha}} \mathcal{F}_2$$

$$\mathcal{F}_1 \otimes \mathcal{F}_1 \xrightarrow{\beta} \mathcal{F}_2,\ \mathcal{F}_2 \otimes \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_1,\ \mathcal{F}_1 \otimes \mathcal{F}_2 \xrightarrow{\gamma} \mathcal{O}_T \oplus \mathcal{L},\ \mathcal{F}_2 \otimes \mathcal{F}_1 \xrightarrow{\gamma'} \mathcal{O}_T \oplus \mathcal{L}$$

where $\gamma = \eta_1 + \eta_2$ and $\gamma' = \eta_1 - \eta_2$. This shows that given $\chi \in \mathcal{Y} \subseteq \mathcal{X}$ we have that $\mathscr{A}_\chi$, as defined in the previous section, coincides with $\mathscr{B}_\chi$ in $\mathrm{LRings}_\mathcal{R}^G$. In particular this proves 5.2.2.

**Commutativity conditions.** Given $\Omega \in \mathcal{X}$, we want to read the symmetry of $\Omega$ on the associated sequence $(\mathcal{L}, \mathcal{F}, m, \alpha, \hat{\alpha}, \eta_*, \beta)$. Since $\mathcal{L}$ is invertible there are no conditions for the commutativity of the first map $m$. The map $\hat{\alpha} \colon \mathcal{F} \otimes \mathcal{L} \longrightarrow \mathcal{F}$ is clearly obtained from $\alpha$ with an exchange. Finally note that the exchange isomorphism $V \otimes V \simeq V \otimes V$ is the identity on both $R$ and $V$, while it is minus the identity on $A$. So the symmetry for the map $\Omega_V \otimes \Omega_V \longrightarrow \Omega_{V \otimes V}$ is equivalent to the symmetry of $\eta_1$ and $\beta$ and to the antisymmetry of $\eta_2$. When $\Omega \in \mathcal{X}$ is symmetric we will use the notation

$$(-,-) = \eta_1 \colon \mathrm{Sym}^2 \mathcal{F} \longrightarrow \mathcal{O}_T,\ \langle -,- \rangle = \eta_2 \colon \det \mathcal{F} \longrightarrow \mathcal{L}$$

The associated sequence will be $(\mathcal{L}, \mathcal{F}, m, \alpha, \beta, (-,-), \langle -,- \rangle)$, where we omit $\hat{\alpha}$ because it is determined by $\alpha$.

**Associativity conditions.** Given a symmetric $\Omega \in \mathcal{X}$, we want to read the associativity conditions on the associated sequence $(\mathcal{L}, \mathcal{F}, m, \alpha, \beta, (-,-), \langle -,- \rangle)$. We can (and it is also convenient to) understand such conditions working directly on the algebra $\mathscr{B}_\Omega$. Actually, we will proceed by listing some diagrams that must commute when $\mathscr{B}_\Omega$ is associative and then we will show that their commutativity is indeed enough to imply the associativity of $\mathscr{B}_\Omega$.

We will make use of the notation introduced in 5.2.3.

- $$\begin{array}{ccc} \mathcal{L} \otimes \mathcal{L} \otimes \mathcal{F}_1 & \xrightarrow{m \otimes \mathrm{id}} & \mathcal{O}_S \otimes \mathcal{F}_1 \\ \downarrow \mathrm{id} \otimes \alpha & & \downarrow \mathrm{id} \\ \mathcal{L} \otimes \mathcal{F}_1 & \xrightarrow{\alpha} & \mathcal{F}_1 \end{array}$$

  Locally we obtain the condition (5.2.1).

- $$\begin{tikzcd}
\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{L} \arrow[r, "\gamma\otimes\mathrm{id}"] \arrow[dd, "\mathrm{id}\otimes-\alpha"] & (\mathcal{O}_S \oplus \mathcal{L}) \otimes \mathcal{L} \\
 & \mathcal{L} \otimes \mathcal{L} \oplus \mathcal{L} \arrow[u, "\wr"] \arrow[d, "m\oplus\mathrm{id}"] \\
\mathcal{F}_1 \otimes \mathcal{F}_2 \arrow[r, "\gamma"] & \mathcal{O}_S \oplus \mathcal{L}
\end{tikzcd}$$

  The commutativity of this diagram is locally equivalent to $(u, \alpha(v)) = -m\langle u, v\rangle$, $(u, v) = -\langle u, \alpha(v)\rangle$ and, assuming (5.2.1), to (5.2.2) and $(u, v) = \langle \alpha(v), u\rangle$.

- $$\begin{tikzcd}
\mathcal{F}_1 \otimes \mathcal{F}_1 \otimes \mathcal{L} \arrow[r, "\beta\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\alpha"] & \mathcal{F}_2 \otimes \mathcal{L} \arrow[d, "-\alpha"] \\
\mathcal{F}_1 \otimes \mathcal{F}_1 \arrow[r, "\beta"] & \mathcal{F}_2
\end{tikzcd}$$

  The commutativity of this diagram is locally equivalent to (5.2.3).

- $$\begin{tikzcd}
\mathcal{F}_2 \otimes \mathcal{F}_2 \otimes \mathcal{F}_1 \arrow[r, "\beta\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\gamma'"] & \mathcal{F}_1 \otimes \mathcal{F}_1 \arrow[dd, "\beta"] \\
\mathcal{F}_2 \otimes (\mathcal{O}_S \oplus \mathcal{L}) & \\
\mathcal{F}_2 \oplus \mathcal{F}_2 \otimes \mathcal{L} \arrow[u, "\wr"] \arrow[r, "\mathrm{id}\oplus(-\alpha)"] & \mathcal{F}_2
\end{tikzcd}$$

  The commutativity of this diagram, assuming that $(u, v) = \langle \alpha(v), u\rangle$, is locally equivalent to (5.2.4).

- $$\begin{tikzcd}
\mathcal{F}_1 \otimes \mathcal{F}_1 \otimes \mathcal{F}_1 \arrow[r, "\mathrm{id}\otimes\beta"] \arrow[d, "\beta\otimes\mathrm{id}"] & \mathcal{F}_1 \otimes \mathcal{F}_2 \arrow[d, "\gamma"] \\
\mathcal{F}_2 \otimes \mathcal{F}_1 \arrow[r, "\gamma'"] & \mathcal{O}_S \oplus \mathcal{L}
\end{tikzcd}$$

  Since $\gamma'(u \otimes v) = \gamma(v \otimes u)$, the commutativity of this diagram is locally equivalent to (5.2.5) and the analogous one for $(-, -)$, which however follows from (5.2.1), (5.2.2), (5.2.3) and (5.2.5), assuming $(u, v) = \langle \alpha(v), u\rangle$. Indeed

  $$\begin{aligned}
(w, \beta(u \otimes v)) &= \langle \alpha(\beta(u \otimes v)), w\rangle = -\langle \beta(\alpha(u) \otimes v), w\rangle = \langle w, \beta(\alpha(u) \otimes v)\rangle \\
&= \langle w, \beta(v \otimes \alpha(u))\rangle = \langle \alpha(u), \beta(w \otimes v)\rangle = (u, \beta(v \otimes w))
\end{aligned}$$

In order to prove that the associativity conditions we have introduced are enough to deduce the associativity of $\mathscr{B}_\Omega$, we need the following remark.

*Remark* 5.2.6. Let $A$ be a commutative (but not necessary associative) ring and $x, y, z \in A$. If
$$(xy)z = x(yz) \text{ and } (yx)z = y(xz)$$
then all the permutations of $x, y, z$ satisfy associativity. Indeed
$$y(zx) = (yx)z = x(yz) = (yz)x,\ \ z(xy) = (yx)z = y(xz) = (zx)y$$
$$(zy)x = x(yz) = (xy)z = z(yx),\ \ (xz)y = y(zx) = (yz)x = x(zy)$$

*Proof.* (*of Theorem* 5.2.4) The map in the statement is the composition $\mathcal{Y} \hookrightarrow (\mathrm{LPMon}^{G}_{\mathcal{R},f})^{\mathrm{gr}} \xrightarrow{\mathscr{A}^*} (\mathrm{LRings}^{G}_{\mathcal{R}})^{\mathrm{gr}}$ and it is fully faithful. Call $\overline{\mathcal{Y}}$ the full substack of $\mathcal{Y}$ satisfying the conditions of the statement. By Theorem 4.2.29 and remark 4.3.4, we need to prove that $(\mathrm{LMon}^{G}_{\mathcal{R},f})^{\mathrm{gr}} = \overline{\mathcal{Y}}$.

$(\mathrm{LMon}^{G}_{\mathcal{R},f})^{\mathrm{gr}} \subseteq \overline{\mathcal{Y}}$. Let $\Omega = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, (-,-), \langle -,- \rangle) \in (\mathrm{LMon}^{G}_{\mathcal{R},f})^{\mathrm{gr}}$, where we are using the description for symmetric functors, and $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in \mathcal{Y}$. One of the associativity conditions, namely the local conditions $(u, v) = \langle \alpha(v), u \rangle$, tells us that $(-,-) = (-,-)_\chi$, thanks to (5.2.6). In particular $\Omega = \chi \in \mathcal{Y}$. In order to conclude that $\Omega \in \overline{\mathcal{Y}}$, it is enough to note that the conditions in the statement are all associativity conditions for $\Omega$, which are satisfied because $\Omega$ is associative.

$\overline{\mathcal{Y}} \subseteq (\mathrm{LMon}^{G}_{\mathcal{R},f})^{\mathrm{gr}}$. Let $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in \overline{\mathcal{Y}} \subseteq \mathcal{Y}$. The relations (5.2.2) and (5.2.6) imply that $\mathscr{A}_\chi$ is a commutative $\mathcal{O}_T$-algebra. We need to show that $\mathscr{A}_\chi$ is associative and we will use 5.2.6. Given $A, B, C \in \{\mathcal{O}_S, \mathcal{L}, \mathcal{F}_1, \mathcal{F}_2\}$ we will say that $(A, B, C)$ holds if $a(bc) = (ab)c$ for all $a \in A, b \in B, c \in C$. Since $\sigma \in \mathbb{Z}/2\mathbb{Z}$ induces a ring automorphism of $\mathscr{A}_\chi$, if $(A, B, C)$ holds then $(\sigma(A), \sigma(B), \sigma(C))$ holds and, moreover, if also $(B, A, C)$ holds then all the permutations of $(A, B, C)$ and $(\sigma(A), \sigma(B), \sigma(C))$ hold.

Clearly $(\mathcal{L}, \mathcal{L}, \mathcal{L})$ holds. Condition (5.2.1) insures that $(\mathcal{L}, \mathcal{L}, \mathcal{F}_1)$, $(\mathcal{L}, \mathcal{L}, \mathcal{F}_2)$ and all their permutations hold. Condition (5.2.2) says that all the permutations of $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{L})$ hold, while condition (5.2.3) tells us that all the permutations of $(\mathcal{F}_1, \mathcal{F}_1, \mathcal{L})$ and $(\mathcal{F}_2, \mathcal{F}_2, \mathcal{L})$ hold. The relation (5.2.4) implies that $(\mathcal{F}_1, \mathcal{F}_1, \mathcal{F}_2)$, $(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_1)$ and all their permutations hold. Finally (5.2.5) says that $(\mathcal{F}_1, \mathcal{F}_1, \mathcal{F}_1)$ and $(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_2)$ hold. It is now easy to check that we have obtained all the possible triples. □

### 5.2.2 Local analysis.

Let $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in \mathcal{Y}$ and assume that $t \in \mathcal{L}$ is a generator and that $y, z$ is a basis of $\mathcal{F}$. The aim of this subsection is to translate conditions (5.2.1), (5.2.2), (5.2.3), (5.2.4) and (5.2.5), writing all the maps $\alpha, \beta, \langle -,- \rangle$ with respect to the given basis. In particular we will use notation from 5.2.3, so that $m \in \mathcal{O}_T$, $\alpha$ is a map $\mathcal{F} \longrightarrow \mathcal{F}$ and $\langle -,- \rangle \colon \det \mathcal{F} \longrightarrow \mathcal{O}_T$.

*Notation* 5.2.7. Write
$$\beta(y^2) = ay + bz,\ \ \beta(yz) = cy + dz,\ \ \beta(z^2) = ey + df,\ \ \langle y, z \rangle = \omega,\ \ \alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

We start computing useful relations in order to impose the associativity conditions

$$(y,y) = -C\omega,\ (y,z) = -D\omega,\ (z,y) = A\omega,\ (z,z) = B\omega$$

$$\begin{array}{ll} \beta(\beta(y^2)y) = (a^2+bc)y + b(a+d)z & \beta(\beta(z^2)y) = (ea+fc)y + (eb+fd)z \\ \beta(\beta(y^2)z) = (ac+be)y + (ad+bf)z & \beta(\beta(z^2)z) = e(c+f)y + (ed+f^2)z \end{array}$$

$$\beta(\beta(yz)y) = \beta(\beta(zy)y) = c(a+d)y + (cb+d^2)z$$

$$\beta(\beta(yz)z) = \beta(\beta(zy)z) = (c^2+de)y + d(c+f)z$$

$$\begin{array}{ll} \langle y, \beta(y^2)\rangle = b\omega & \langle y, \beta(z^2)\rangle = f\omega \\ \langle z, \beta(y^2)\rangle = -a\omega & \langle z, \beta(z^2)\rangle = -e\omega \\ \langle y, \beta(yz)\rangle = \langle y, \beta(zy)\rangle = d\omega & \langle z, \beta(yz)\rangle = \langle z, \beta(zy)\rangle = -c\omega \end{array}$$

If we set $\Gamma(u,v,w) = \langle \alpha(w), v\rangle u - \langle w, v\rangle \alpha(u)$ we have

$$\begin{array}{ll} \Gamma(y,y,y) = -C\omega y & \Gamma(z,y,y) = -C\omega z \\ \Gamma(y,y,z) = \omega(A-D)y + \omega C z & \Gamma(z,y,z) = B\omega y \\ \Gamma(y,z,y) = -\omega C z & \Gamma(z,z,y) = -B\omega y + \omega(A-D)z \\ \Gamma(y,z,z) = B\omega y & \Gamma(z,z,z) = B\omega z \end{array}$$

$$\begin{array}{rcl} \alpha(\beta(y^2)) + \beta(\alpha(y)y) & = & (2aA+bB+cC)y + (C(a+d)+b(A+D))z \\ \alpha(\beta(yz)) + \beta(\alpha(y)z) & = & (2cA+dB+eC)y + (C(c+f)+d(A+D))z \\ \alpha(\beta(zy)) + \beta(\alpha(z)y) & = & (B(a+d)+c(A+D))y + (2dD+bB+cC)z \\ \alpha(\beta(z^2)) + \beta(\alpha(z)z) & = & (B(c+f)+e(A+D))y + (2fD+eC+dB)z \end{array}$$

From the above relations we can conclude:

**Lemma 5.2.8.** *The object $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in \mathcal{Y}$ belongs to $G$-*Cov *if and only if the following relations hold.*

$$\begin{array}{lcc} (5.2.1) & \iff & m = A^2+BC,\ (A-D)(A+D) = B(A+D) = C(A+D) = 0 \\ (5.2.2) & \iff & \omega(A+D) = 0 \\ (5.2.3) & \iff & \left\{ \begin{array}{c} (2aA+bB+cC) = (2cA+dB+eC) = 0 \\ C(a+d)+b(A+D) = C(c+f)+d(A+D) = 0 \\ B(a+d)+c(A+D) = B(c+f)+e(A+D) = 0 \\ a(A+D)-D(a+d) = c(A+D)-D(c+f) = 0 \end{array} \right. \\ (5.2.4) & \iff & \left\{ \begin{array}{c} a^2+bc = -\omega C,\ ac+be = \omega(A-D),\ c^2+de = B\omega \\ (a-d)(a+d) = b(a+d) = c(a+d) = 0 \\ (c-f)(c+f) = d(c+f) = e(c+f) = 0 \\ a(a+d)+b(c+f) = e(a+d)+c(c+f) = 0 \end{array} \right. \\ (5.2.5) & \iff & \omega(a+d) = \omega(c+f) = 0 \end{array} \tag{5.2.7}$$

*Remark* 5.2.9. It is not true in general that some of $(a+d)$, $(c+f)$, $(A+D)$ is 0. For instance over the ring $k[x]/(x^2)$ we have a $G$-cover given by

$$a = b = c = d = e = f = \omega = A = B = C = D = x,\ m = 0$$

On the other hand if we are on a reduced ring $R$ then $(a+d) = (c+f) = 0$. Indeed over every domain we have

$$(a+d) \neq 0 \implies a = d, c = b = 0 \text{ and } a(a+d) + b(c+f) = 0 \implies a = 0$$

so $a + d = 2a = 0$. The same argument shows that $c + f = 0$.

*Notation* 5.2.10. Given $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle) \in \mathcal{Y}$, a basis $y, z$ of $\mathcal{F}$ and a generator $t \in \mathcal{L}$, we will denote by

$$a_\chi, b_\chi, c_\chi, d_\chi, e_\chi, f_\chi, \omega_\chi, A_\chi, B_\chi, C_\chi, D_\chi, m_\chi$$

the data associated with $\chi$ as above. We will always omit the $-_\chi$ if this will not lead to confusion.

## 5.3 Geometry of $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$-Cov and $S_3$-Cov.

The aim of this section is to describe the geometry of the stack $G$-Cov and, as a consequence, of $S_3$-Cov. Clearly this is related to the problem of the description of $G$-covers: we will individuate three smooth open substacks of $G$-Cov, that is families of $G$-covers with a certain global or local property. In some cases this will allow us to describe $G$-covers using less data than needed to build a general $G$-cover, that is the objects $\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle$. For instance when $\langle -, - \rangle \colon \det \mathcal{F} \longrightarrow \mathcal{L}$ is an isomorphism, we will show that the data $\mathcal{F}, \beta$ determine all the others. This is the first locus we will describe, denoted by $\mathcal{U}_\omega$. The other two will be the locus $\mathcal{U}_\alpha$ where $\alpha$ is never a multiple of the identity and the locus $\mathcal{U}_\beta$ where $\beta$ is never zero. Although in those cases we do not have a global description of $G$-covers belonging to these families, what we have is a local description that will be extremely useful also in the next sections, for instance because it will turn out that any $G$-cover between smooth varieties belongs to the locus where $\beta$ is never zero.

Unluckily, the three loci described above do not cover $G$-Cov, actually we will see that they are contained in the main irreducible component $\mathcal{Z}_G$. On the other hand they almost cover $\mathcal{Z}_G$: the complement of their union in $\mathcal{Z}_G$ is formed by the "zero covers", that is the $G$-covers where $m = \alpha = \beta = \langle -, - \rangle = 0$, and therefore, topologically, we have missed only "one point", actually one point for each characteristic. The stack $\mathcal{Z}_G$ is an irreducible component of $G$-Cov and Theorem 4.3.1 tells us that $G$-Cov is reducible. We will show more: the complementary $\mathcal{Z}_2$ of the union of $\mathcal{U}_\omega$, $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ in the whole $G$-Cov is another irreducible component. Therefore $G$-Cov and, over $\mathcal{R}_3$, $S_3$-Cov have exactly two irreducible components. Although the covers in $\mathcal{Z}_2$ are highly degenerate, the stack $\mathcal{Z}_2$ has a very simple description and a very simple geometry, for instance it is smooth, while $\mathcal{Z}_G$ is not. Over a field, $\mathcal{Z}_2$ is topologically composed of two points and $(G\text{-Cov}) - \mathcal{Z}_G \simeq \mathrm{B}\,\mathrm{Gl}_2$. The last subsection is dedicated to the study of the irreducible component $\mathcal{Z}_G$ and we will show that, in this case, the maps $m, \alpha, \beta, \langle -, - \rangle$ are uniquely determined by $\beta$ and $\langle -, - \rangle$.

### 5.3.1 Triple covers and the locus where $\langle -,- \rangle \colon \det \mathcal{F} \longrightarrow \mathcal{L}$ is an isomorphism.

In this subsection we want to describe a smooth open substack of $G$-Cov, more precisely the locus $\mathcal{U}_\omega$ of objects $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle)$ such that $\langle -,- \rangle \colon \det \mathcal{F} \longrightarrow \mathcal{L}$ is an isomorphism.

**Definition 5.3.1.** Define $\mathcal{C}_3$ as the stack whose objects are pairs $(\mathcal{F}, \delta)$ where $\mathcal{F}$ is a locally free sheaf of rank 2 and $\delta$ is a map

$$\delta \colon \operatorname{Sym}^3 \mathcal{F} \longrightarrow \det \mathcal{F}$$

Notice that $\mathcal{C}_3$ is a smooth stack over $\mathbb{Z}$ because it is a vector bundle over $\mathrm{B}\,\mathrm{Gl}_2$, which is the stack of rank 2 locally free sheaves. We will show that $\mathcal{U}_\omega$ is isomorphic to the stack $\mathcal{C}_3$. This also explains the reason of the section name: it is a classical result (see [Mir85, BV12, Par89]) that, over $\mathcal{R}_3$, the stack $\mathcal{C}_3$ is isomorphic to the stack $\mathrm{Cov}_3$ of degree 3 covers. We will show that, in this case, the map $\mathrm{Cov}_3 \simeq \mathcal{C}_3 \longrightarrow \mathcal{U}_\omega$ is a section of the map $G\text{-Cov} \longrightarrow \mathrm{Cov}_3$, obtained taking invariants by $\sigma \in \mathbb{Z}/2\mathbb{Z} \subseteq G$.

We now give an alternative description of $\mathcal{C}_3$ and we will need the following notation.

*Notation* 5.3.2. Given locally free sheaves $\mathcal{N}$ and $\mathcal{F}$ over $T$ and a map $\zeta \colon \mathcal{N} \otimes \mathcal{F} \longrightarrow \mathcal{F}$ we will call trace of $\zeta$ the composition $\operatorname{tr} \zeta \colon \mathcal{N} \longrightarrow \mathcal{F}^\vee \otimes \mathcal{F} \longrightarrow \mathcal{O}_T$. We will also denote by $\underline{\mathrm{Hom}}_{\mathrm{tr}=0}(\mathcal{N} \otimes \mathcal{F}, \mathcal{F})$ the subsheaf of $\underline{\mathrm{Hom}}(\mathcal{N} \otimes \mathcal{F}, \mathcal{F})$ of maps whose trace is 0. Notice that, when $\mathcal{N} = \mathcal{O}_T$ and $\mathcal{F}$ is free, the trace we have just defined is the usual trace of a associated matrix.

*Notation* 5.3.3. Given $\beta \colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{F}$ we will use the notation

$$\operatorname{tr} \beta = \operatorname{tr}(\mathcal{F} \otimes \mathcal{F} \longrightarrow \operatorname{Sym}^2 \mathcal{F} \xrightarrow{\beta} \mathcal{F}) \colon \mathcal{F} \longrightarrow \mathcal{O}_T$$

If $y, z$ is a basis of $\mathcal{F}$ and $\beta(y^2) = ay + bz$, $\beta(yz) = cy + dz$, $\beta(z^2) = ey + fz$, then $(\operatorname{tr} \beta)(y) = a + d$, $(\operatorname{tr} \beta)(z) = c + f$.

It is easy to check (see also [Mir85, BV12]) that if $\beta \colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{F}$ is a map such that $\operatorname{tr} \beta = 0$ there exists a dashed map $\delta$:

$$\begin{tikzcd}
\operatorname{Sym}^2 \mathcal{F} \otimes \mathcal{F} \arrow[r, "\beta\otimes\mathrm{id}"] \arrow[d] & \mathcal{F} \otimes \mathcal{F} \arrow[d] \\
\operatorname{Sym}^3 \mathcal{F} \arrow[r, dashed, "\delta"] & \det \mathcal{F}
\end{tikzcd}$$

This association yields an isomorphism

$$\begin{array}{ccc}
\underline{\mathrm{Hom}}_{\mathrm{tr}=0}(\operatorname{Sym}^2 \mathcal{F}, \mathcal{F}) & \longrightarrow & \underline{\mathrm{Hom}}(\operatorname{Sym}^3 \mathcal{F}, \det \mathcal{F}) \\
\begin{pmatrix} a & c & e \\ b & -a & -c \end{pmatrix} & \longmapsto & \begin{pmatrix} -b & a & c & e \end{pmatrix}
\end{array} \tag{5.3.1}$$

where the last row describes how this map behaves if a basis $y, z$ of $\mathcal{F}$ is chosen, where we have considered $y^3, y^2z, yz^2, z^3$ as basis of $\operatorname{Sym}^3 \mathcal{F}$. So $\mathcal{C}_3$ can also be described as the stack of pairs $(\mathcal{F}, \beta)$ where $\mathcal{F}$ is a locally free sheaf of rank 2 and $\beta\colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{F}$ is a map such that $\operatorname{tr} \beta = 0$.

*Notation* 5.3.4. We will denote the correspondence (5.3.1) by $\beta \longmapsto \delta_\beta$ and $\delta \longmapsto \beta_\delta$.

Now let $(\mathcal{F}, \delta) \in \mathcal{C}_3$. Define $\eta_\delta$ as the map

$$\operatorname{Sym}^2 \mathcal{F} \xrightarrow{u} \Lambda^2 \operatorname{Sym}^2 \mathcal{F} \otimes \Lambda^2 \mathcal{F}^\vee \xrightarrow{v} \mathcal{O}_S \tag{5.3.2}$$

where $v$ is induced by $\Lambda^2\beta_\delta\colon \Lambda^2 \operatorname{Sym}^2 \mathcal{F} \longrightarrow \Lambda^2 \mathcal{F}$ and $u$ is induced by

$$\begin{array}{ccc} \Lambda^2\mathcal{F} \otimes \operatorname{Sym}^2 \mathcal{F} & \longrightarrow & \Lambda^2 \operatorname{Sym}^2 \mathcal{F} \\ (x_1 \wedge x_2) \otimes x_3x_4 & \longmapsto & -x_1x_3 \wedge x_2x_4 - x_1x_4 \wedge x_2x_3 \end{array}$$

and $\alpha_\delta\colon \det \mathcal{F} \otimes \mathcal{F} \longrightarrow \mathcal{F}$ as the map induced by

$$\det \mathcal{F} \otimes \mathcal{F}^\vee \otimes \mathcal{F} \xrightarrow{\simeq} \mathcal{F} \otimes \mathcal{F} \xrightarrow{\eta_\delta/2} \mathcal{O}_S$$

using the canonical isomorphism $\det \mathcal{F} \otimes \mathcal{F}^\vee \simeq \mathcal{F}$. Finally define $m_\delta\colon (\det \mathcal{F})^2 \longrightarrow \mathcal{O}_T$ as minus the map induced by

$$(\det \mathcal{F})^2 \otimes \det \mathcal{F} \simeq \det(\det \mathcal{F} \otimes \mathcal{F}) \xrightarrow{\det \alpha_\delta} \det \mathcal{F}$$

Denote by $\operatorname{Cov}_3$ the stack of degree 3 covers, or, equivalently, the stack of locally free sheaves of $\mathcal{O}_T$algebras of rank 3. Denote also by $\mathcal{U}_\omega$ the open substack of $G$-Cov of objects $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle)$ such that $\langle -, - \rangle\colon \det \mathcal{F} \longrightarrow \mathcal{L}$ is an isomorphism. The theorem we want to prove is:

**Theorem 5.3.5.** *The maps of stacks*

$$\begin{array}{ccc} (\mathcal{F}, \delta) & \longmapsto & (\det \mathcal{F}, \mathcal{F}, m_\delta, \alpha_\delta, \beta_\delta, \mathrm{id}_{\det \mathcal{F}}) \\ \mathcal{C}_3 & \xrightarrow{\qquad\Lambda\qquad} & \mathcal{U}_\omega \\ (\mathcal{F}, \delta_\beta) & \longleftarrow\!\shortmid & (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle) \end{array}$$

*are well defined and they are inverses of each other. In particular $\mathcal{U}_\omega$ is a smooth open substack of $G$-Cov. Moreover, over $\mathcal{R}_3$, the composition $\operatorname{Cov}_3 \simeq \mathcal{C}_3 \xrightarrow{\Lambda} G\text{-Cov}$ is a section of the map $G\text{-Cov} \longrightarrow \operatorname{Cov}_3$ obtained by taking invariants by $\sigma \in \mathbb{Z}/2\mathbb{Z}$ and the same result hold if we replace $G$-Cov by $S_3$-Cov.*

We will prove this theorem at the end of this section, after collecting some useful remarks.

*Remark* 5.3.6. If $y, z$ is a basis of $\mathcal{F}$, we identify $\det \mathcal{F} \simeq \mathcal{O}_T$ using the generator $y \wedge z \in \det \mathcal{F}$ and we write $\delta$ as

$$\delta(y^3) = -b,\ \delta(y^2 z) = a,\ \delta(yz^2) = c,\ \delta(z^3) = e \tag{5.3.3}$$

then we have expressions

$$\eta_\delta(y^2) = 2(a^2 + bc),\ \eta_\delta(yz) = ac + be,\ \eta_\delta(z^2) = 2(c^2 - ae) \tag{5.3.4}$$

$$2\alpha_\delta(y) = \eta_\delta(yz)y - \eta_\delta(y^2)z,\ 2\alpha_\delta(z) = \eta_\delta(z^2)y - \eta_\delta(yz)z$$

In particular, if $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta_\delta, \langle -, - \rangle) \in G\text{-Cov}$, then, by (5.2.6) and (5.2.7), we have

$$\eta_\delta = 2(-,-)_\chi \tag{5.3.5}$$

We now want to show the relationship between $\mathcal{C}_3$ and $\mathrm{Cov}_3$. The reader can refer to [Mir85, BV12] for details and proofs.

*Remark* 5.3.7. If $\Phi = (\mathcal{F}, \delta) \in \mathcal{C}_3$ and we set $\mathscr{A}_\Phi = \mathcal{O}_T \oplus \mathcal{F}$, we can endow $\mathscr{A}_\Phi$ by a structure of $\mathcal{O}_T$-algebras given by

$$\mathrm{Sym}^2 \mathcal{F} \xrightarrow{\eta_\delta + \beta_\delta} \mathscr{A}_\Phi$$

This association defines a map of stacks $\mathcal{C}_3 \longrightarrow \mathrm{Cov}_3$. This map is an isomorphism if 3 is inverted in the base scheme. Indeed if $\mathscr{A} \in \mathrm{Cov}_3$, the trace map $\mathrm{tr}_{\mathscr{A}/\mathcal{O}_T} \colon \mathscr{A} \longrightarrow \mathcal{O}_T$ is surjective and we can write $\mathscr{A} = \mathcal{O}_S \oplus \mathcal{F}$, where $\mathcal{F} = \ker \mathrm{tr}_{\mathscr{A}}$. The multiplication of $\mathscr{A}$ induces a map $\beta \colon \mathrm{Sym}^2 \mathcal{F} \longrightarrow \mathcal{F}$ such that $\mathrm{tr}\,\beta = 0$ and therefore a $\delta \colon \mathrm{Sym}^3 \mathcal{F} \longrightarrow \det \mathcal{F}$ such that $\beta_\delta = \beta$.

Now let $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle) \in G\text{-Cov}$. It's easy to see that

$$\mathscr{A}_\chi^\sigma = \{a \oplus 0 \oplus x_1 \oplus x_2 \mid a \in \mathcal{O}_T,\ x_1 = x_2 \in \mathcal{F}\}$$

where $\sigma \in \mathbb{Z}/2\mathbb{Z} \subseteq G$. The map

$$\begin{array}{ccc} \mathcal{O}_T \oplus \mathcal{F} & \longrightarrow & \mathscr{A}^\sigma \\ a \oplus x & \longmapsto & a \oplus 0 \oplus x \oplus x \end{array}$$

is an isomorphism of $\mathcal{O}_S$-modules and the induced algebra structure on $\mathcal{O}_T \oplus \mathcal{F}$ is given by

$$\beta \colon \mathrm{Sym}^2 \mathcal{F} \longrightarrow \mathcal{F} \text{ and } 2(-,-) \colon \mathrm{Sym}^2 \mathcal{F} \longrightarrow \mathcal{O}_T$$

Notice that it is not true in general that $\mathcal{F} = \ker \mathrm{tr}_{\mathscr{A}^\sigma}$, also if 3 is inverted: this equality holds if and only if $\mathrm{tr}\,\beta = 0$, for instance over any reduced scheme or, as we will see in the next sections, over the principal irreducible component $\mathcal{Z}_G \subseteq G\text{-Cov}$. In this case $(\mathcal{F}, \delta_\beta) \in \mathcal{C}_3$ and $\mathscr{A}_\chi^\sigma \simeq \mathscr{A}_{(\mathcal{F}, \delta_\beta)}$. Moreover we get a well defined map of stacks $\{\mathrm{tr}\,\beta = 0\} \longrightarrow \mathcal{C}_3$, where $\{\mathrm{tr}\,\beta = 0\}$ is the closed substack of $G$-Cov where $\mathrm{tr}\,\beta = 0$. When 3 is inverted, such map, composed by $\mathcal{C}_3 \longrightarrow \mathrm{Cov}_3$, extends to a map of stacks $G\text{-Cov} \longrightarrow \mathrm{Cov}_3$, by taking invariants by $\sigma$.

*Remark* 5.3.8. Proposition 2.3.10 tells us that, over $\mathcal{R}_3$, the isomorphism $G$-Cov $\simeq$ $S_3$-Cov preserves the quotient by $\sigma \in \mathbb{Z}/2\mathbb{Z}$, that is we have a commutative diagram

$$\begin{tikzcd} G\text{-Cov} \arrow[rr, "\simeq"] \arrow[dr] & & S_3\text{-Cov} \arrow[dl] \\ & \mathrm{Cov}_3 & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, mapsto] & \mathscr{A} \arrow[d, mapsto] \\ X/\sigma & \mathscr{A}^\sigma \end{tikzcd}$$

We are ready for the proof of the main theorem of this subsection.

*Proof.* (*of Theorem* 5.3.5) We need to prove that $\Lambda$ is well defined. Let $\Phi = (\mathcal{F}, \delta) \in \mathcal{C}_3$. We have that $\chi = \Lambda(\Phi) \in \mathcal{Y}$ and we have to prove that $\chi$ satisfies the conditions of 5.2.8. We can therefore work locally and assume we have a basis $y, z$ of $\mathcal{F}$. If we write $\delta$ as in (5.3.3), the parameters associated to $\chi$ (see 5.2.10) are

$$a, b, c, d = -a, e, f = -c, \omega = 1, A = -D = (ac + be)/2,\ B = c^2 - ae,\ C = -a^2 - bc$$

It is easy to check that all the conditions in 5.2.8 are satisfied. Moreover

$$m_\delta = A^2 + BC = -(AD - BC) = -\det \alpha$$

So $\Lambda(\Phi) \in G$-Cov and, by definition, $\Lambda(\Phi) \in \mathcal{U}_\omega$.

Conversely, if $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in \mathcal{U}_\omega$, taking into account relations (5.2.8) and the fact that locally $\omega$ is invertible, we have that $\operatorname{tr} \beta = 0$. So $(\mathcal{F}, \delta_\beta) \in \mathcal{C}_3$. Denote by $\Delta$ the map $\Delta \colon \mathcal{U}_\omega \longrightarrow \mathcal{C}_3$ defined in the statement. Clearly $\Delta \circ \Lambda \simeq \mathrm{id}$. For the converse, consider

$$(\langle -,- \rangle, \mathrm{id}_\mathcal{F}) \colon \Lambda \circ \Delta(\chi) = (\det \mathcal{F}, \mathcal{F}, m_\delta, \alpha_\delta, \beta, \mathrm{id}_{\det \mathcal{F}}) \longrightarrow (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) = \chi \text{ where } \delta = \delta_\beta$$

In order to conclude that the association above defines an isomorphism $\Lambda \circ \Delta \simeq \mathrm{id}$, it is enough to prove that it is well defined. The only non trivial condition to check is the commutativity of the following diagram.

$$\begin{tikzcd} \det \mathcal{F} \otimes \mathcal{F} \arrow[r, "\alpha_\delta"] \arrow[d, "\langle -,- \rangle \otimes \mathrm{id}_\mathcal{F}"'] & \mathcal{F} \arrow[d, "\mathrm{id}_\mathcal{F}"] \\ \mathcal{L} \otimes \mathcal{F} \arrow[r, "\alpha"] & \mathcal{F} \end{tikzcd}$$

Working locally, the commutativity of the above diagram is equivalent to the condition $\alpha_\delta = \omega \alpha$, which can be easily verified.

Now assume we are over $\mathcal{R}_3$. The map $G$-Cov $\longrightarrow \mathrm{Cov}_3 \simeq \mathcal{C}_3$ extends the map $\mathcal{U}_\omega \longrightarrow \mathcal{C}_3$ defined in the statement. Therefore $\mathrm{Cov}_3 \simeq \mathcal{C}_3 \longrightarrow \mathcal{U}_\omega \subseteq G$-Cov is a section of such map. $\square$

### 5.3.2 The locus where $\alpha\colon \mathcal{L}\otimes\mathcal{F} \longrightarrow \mathcal{F}$ is never a multiple of the identity.

Define $\mathcal{U}_\alpha$ as the full substack of $G$-Cov of objects $\chi = (\mathcal{L},\mathcal{F},m,\alpha,\beta,\langle -,-\rangle)$ such that $\alpha\colon \mathcal{L}\otimes\mathcal{F} \longrightarrow \mathcal{F}$ is never a multiple of the identity, i.e. such that $\alpha$ is not a multiple of the identity after some base change. We want to prove the following:

**Theorem 5.3.9.** *Let $R = \mathcal{R}[m,a,b]$. Then*

$$(R, R^2, m, \alpha, \beta, \langle -,-\rangle)$$

*where*

$$\alpha = \begin{pmatrix} 0 & m \\ 1 & 0 \end{pmatrix}; \quad \begin{array}{l} \beta(e_1^2) = ae_1 + be_2 \\ \beta(e_1e_2) = -mbe_1 - ae_2; \\ \beta(e_2^2) = mae_1 + mbe_2 \end{array} \quad \begin{array}{l} \langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0 \\ \langle e_1, e_2\rangle = -\langle e_2, e_1\rangle = mb^2 - a^2 \end{array}$$

*is an object of $G$-Cov$(R)$. The induced map $\mathbb{A}^3 \longrightarrow G$-Cov is a smooth Zariski epimorphism onto $\mathcal{U}_\alpha$. In particular $\mathcal{U}_\alpha$ is a smooth open substack of $G$-Cov.*

Before proving this Theorem we need two lemmas.

**Lemma 5.3.10.** *Let $\mathcal{F}$ be a locally free sheaf of rank 2, $\mathcal{L}$ be an invertible sheaf, both over $T$ and $\alpha\colon \mathcal{L}\otimes\mathcal{F} \longrightarrow \mathcal{F}$ be a map. Let also $k$ be a field, $\operatorname{Spec} k \longrightarrow T$ be a map and $p \in T$ the induced point. If $\alpha \otimes k$ is not a multiple of the identity, then there exists a Zariski open neighborhood $V$ of $p$ in $T$ and $y \in \mathcal{F}_{|V}$ such that $\mathcal{L}_{|V} = \mathcal{O}_V t$ and $y, \alpha(t\otimes y)$ is a basis of $\mathcal{F}_{|V}$.*

*Proof.* If the statement is true when $T = \operatorname{Spec} k'$, for some field $k'$, then it follows in general by Nakayama's lemma. So assume that $T = \operatorname{Spec} k$ and, by contradiction, that such a basis does not exist. It is easy to deduce that any vector of $\mathcal{F}$ is an eigenvector for $\alpha$. By standard linear algebra we can conclude that $\alpha$ is a multiple of the identity. □

**Lemma 5.3.11.** *Let $\chi = (\mathcal{L},\mathcal{F},m,\alpha,\beta,\langle -,-\rangle) \in \mathcal{Y}$ and $y \in \mathcal{F}$ be such that $\mathcal{L} = \mathcal{O}_T$ and $y, z = \alpha(y)$ is a basis of $\mathcal{F}$. Then $\chi \in G$-Cov if and only if the associated parameters (see 5.2.10) of $\chi$ with respect to the basis $y, z$ are*

$$a, b, c = -mb, d = -a, e = ma, f = mb, \omega = mb^2 - a^2, A = D = 0, B = m, C = 1$$

*In this case $\chi \in \mathcal{U}_\alpha$.*

*Proof.* First of all note that, if the associated parameters of $\chi$ are as above, then they satisfy equations (5.2.7). Therefore $\chi \in G$-Cov and, by definition, $\alpha$ is nowhere a multiple of the identity, i.e. $\chi \in \mathcal{U}_\alpha$. Consider now the inverse implication and denote by $a, b, c, d, e, f, \omega, A, B, C, D$ the parameters associated to $\chi$ with respect to the basis $y, z$ of $\mathcal{F}$. By definition of $y, z$ we have $A = 0$ and $C = 1$. Moreover $m = A^2 + BC = B$ and

$$C(A + D) = 0 \implies D = 0$$

$$b(A+D)+C(a+d) = d(A+D)+C(c+f) = 0 \implies d=-a,\ f=-c$$
$$(2aA+bB+cC) = (2cA+dB+eC) = 0 \implies c=-mb, e=ma$$
$$a^2+bc = -\omega C \implies \omega = mb^2-a^2$$

□

*Proof.* (*of theorem* 5.3.9). By 5.3.11 and 5.3.10, $\mathcal{U}_\alpha$ is an open substack of $G$-Cov, $\chi \in \mathcal{U}_\alpha(R)$ and the induced map $\pi\colon \mathbb{A}^3 \longrightarrow \mathcal{U}_\alpha$ is a Zariski epimorphism. It remains to prove that $\pi$ is smooth. Let $T \xrightarrow{\chi} \mathcal{U}_\alpha$ be a map and consider the fiber product $Z = T \times_{\mathcal{U}_\alpha} \mathbb{A}^3$. We have to show that $Z$ is smooth over $T$. In order to do that, since $\pi$ is a Zariski epimorphism, we can assume to have $(m,a,b) \in \mathcal{O}_T$ such that $\pi(m,a,b) = \chi$. Let $V$ be a $T$-scheme. An element of the set $Z(V)$ is a sequence $\Phi = (m',a',b',\lambda,u,v,w,z) \in \mathcal{O}_V^8$ such that, if we set $\psi_\Phi = \begin{pmatrix} u & v \\ w & z \end{pmatrix}$, then $\psi_\Phi \in \mathrm{Gl}_{2,V}$, $\lambda \in \mathcal{O}_V^*$ and $(\lambda,\psi_\Phi)$ is an isomorphism $\pi(m',a',b') \longrightarrow \pi(m,a,b)$. We claim that the map of $T$-schemes

$$i\colon Z \longrightarrow \mathbb{A}^2_T \times \mathbb{G}_m,\ \ i(m',a',b',\lambda,u,v,w,z) = (\lambda,u,w)$$

is an open immersion. If we set

$$v(u,w) = \lambda m w,\ \ z(u,w) = \lambda w$$

the condition $\psi_\Phi^{-1} \circ \alpha \circ (\lambda \otimes \psi_\Phi)(e_1) = e_2$ is equivalent to $v = v(u,w)$, $z = z(u,w)$. Since $\lambda, \psi_\Phi$ determine $m',a',b'$ and $\lambda,u,w$ determine $\lambda,\psi_\Phi$, we can conclude that $i$ is a monomorphism. Define $U \subseteq \mathbb{A}^2_T \times \mathbb{G}_m$ as the open subscheme where $uz(u,w) - v(u,w)w$ is invertible. This is just the expression of $\det \psi_\Phi$. Therefore $i(Z) \subseteq U$. Consider now $\xi = (\lambda,u,w) \in U$ and define $\psi_\xi = \begin{pmatrix} u & v(u,w) \\ w & z(u,w) \end{pmatrix}$. Note that by construction $\psi_\xi \in \mathrm{Gl}_{2,V}$. In particular there exists $\chi' = (\mathcal{O}_V, \mathcal{O}_V^2, m', \alpha', \beta', \langle -,- \rangle') \in \mathcal{U}_\alpha(V)$ such that $(\lambda,\psi_\xi)\colon \chi' \longrightarrow \pi(m,a,b)$ is an isomorphism. Since by construction

$$\alpha'(e_1) = \psi_\Phi^{-1} \circ \alpha \circ (\lambda \otimes \psi_\Phi)(e_1) = e_2$$

from 5.3.11 we see that there exists $m',a',b' \in \mathcal{O}_V$ such that $\pi(m',a',b') = \chi'$. In particular $\Phi = (m',a',b',\lambda,u,v(u,w),w,z(u,w)) \in Z(V)$ and $i(\Phi) = \xi$. □

### 5.3.3 The locus where $\beta\colon \mathrm{Sym}^2 \mathcal{F} \longrightarrow \mathcal{F}$ is never zero.

In this subsection we work over $\mathcal{R}_3 = \mathbb{Z}[1/6]$. Define $\mathcal{U}_\beta$ as the full substack of $G$-Cov of objects $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle)$ such that $\beta\colon \mathrm{Sym}^2 \mathcal{F} \longrightarrow \mathcal{F}$ is never zero, i.e. such that $\beta$ is not zero after some base change. We want to prove the following:

**Theorem 5.3.12.** *Let $R = \mathcal{R}_3[\omega, A, C]$. Then*

$$(R, R^2, m, \alpha, \beta, \langle -, - \rangle)$$

*where*

$$\alpha = \begin{pmatrix} A & \omega C^2 \\ C & -A \end{pmatrix}; \quad \begin{array}{l} \beta(e_1^2) = e_2, \ \beta(e_1 e_2) = -\omega C e_1, \ \beta(e_2^2) = 2\omega A e_1 + \omega C e_2 \\ \langle e_1, e_2 \rangle = \omega, \ m = A^2 + \omega C^3 \end{array}$$

*is an object of $G$-*Cov$(R)$*. The associated map $\mathbb{A}^3 \longrightarrow G$-*Cov *is a smooth zariski epimorphism onto $\mathcal{U}_\beta$. In particular $\mathcal{U}_\beta$ is a smooth open substack of $G$-*Cov*.*

Before proving this theorem we need two lemmas.

**Lemma 5.3.13.** *Let $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle) \in G$-*Cov *(resp. $\beta \colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{F}$), $k$ be a field, $\operatorname{Spec} k \longrightarrow T$ be a map and $p \in T$ the induced point. If $\beta \otimes k \neq 0$ (resp. $\beta \otimes k \neq 0$, $(\operatorname{tr} \beta) \otimes k = 0$ (see 5.3.3)) then there exists a Zariski open neighborhood $V$ of $p$ in $T$ and $y \in \mathcal{F}_{|V}$ such that $y, \beta(y^2)$ is a basis of $\mathcal{F}_{|V}$.*

*Proof.* If the statement is true when $T = \operatorname{Spec} k'$, for some field $k'$, then it follows in general by Nakayama's lemma. So assume that $T = \operatorname{Spec} k$ and, by contradiction, that such a basis does not exist. Notice that if $\chi \in G$-Cov is given, then $(\operatorname{tr} \beta) \otimes k = 0$ thanks to 5.2.9. Choosing a basis of $\mathcal{F}$ we can write

$$\beta = \begin{pmatrix} a & c & e \\ b & -a & -c \end{pmatrix}$$

The condition that $y, \beta(y^2)$ are dependent for all $y \in \mathcal{F}$ is equivalent to

$$bu^3 - 3au^2 v - 3cuv^2 - ev^3 = 0 \qquad \forall u, v \in k$$

In particular, choosing $(u, v) \in \{(1,0), (0,1), (1,1), (1,-1))$, we see that $b = e = 3a = 3c = 0$ and therefore $\beta = 0$, since 3 is invertible. $\square$

**Lemma 5.3.14.** *Let $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle) \in \mathcal{Y}$ and $y \in \mathcal{F}$ be such that $\mathcal{L} = \mathcal{O}_T$ and $y, z = \beta(y^2)$ is a basis of $\mathcal{F}$. Then $\chi \in G$-*Cov *if and only if the associated parameters (see 5.2.10) of $\chi$ with respect to the basis $y, z$ are*

$$a = 0, b = 1, c = -\omega C, d = 0, e = 2\omega A, f = \omega C, \omega, A, B = \omega C^2, C, D = -A$$

*In this case $\chi \in \mathcal{U}_\beta$.*

*Proof.* First of all, it is easy to check that, if the associated parameters of $\chi$ are the ones listed in the statement, then they satisfy equations (5.2.7). Therefore $\chi \in G$-Cov and, since $\beta(y^2) \neq 0$ after all base changes, $\chi \in \mathcal{U}_\beta$.

Assume now that $\chi \in G$-Cov. By definition of the basis $y, z$, we have $a = 0$ and $b = 1$. Using relations (5.2.7), we also have

$$b(a + d) = a(a + d) + b(c + f) = 0 \implies d = -a = 0, \ f = -c$$

$$\begin{gathered} b(A+D)+C(a+d)=0 \implies D=-A \\ a^2+bc=-\omega C,\ ac+be=2\omega A \implies c=-\omega C,\ e=2\omega A \\ 2aA+bB+cC=0 \implies B=\omega C^2 \end{gathered}$$

□

*Proof.* (*of theorem* 5.3.12). From 5.3.13 and 5.3.14 we see that $\mathcal{U}_\beta$ is an open substack of $G$-Cov, that $(R, R^2, m, \alpha, \beta, \langle -,- \rangle) \in \mathcal{U}_\alpha(R)$ and that its induced map $\pi\colon \mathbb{A}^3 \longrightarrow \mathcal{U}_\beta$ is a Zariski epimorphism. It remains to prove that $\pi$ is smooth. Let $T \xrightarrow{\chi} \mathcal{U}_\beta$ be a map and consider the fiber product $Z = T \times_{\mathcal{U}_\beta} \mathbb{A}^3$. We have to show that $Z$ is smooth over $T$. In order to do that, since $\pi$ is a Zariski epimorphism, we can assume to have $(\omega, A, C) \in \mathcal{O}_T$ such that $\pi(\omega, A, C) = \chi$. Let $V$ be a $T$-scheme. An element of the set $Z(V)$ is a sequence $\Phi = (\omega', A', B', \lambda, u, v, w, z) \in \mathcal{O}_V^8$ such that, if we set $\psi_\Phi = \begin{pmatrix} u & v \\ w & z \end{pmatrix}$, then $\psi_\Phi \in \mathrm{Gl}_{2,V}$, $\lambda \in \mathcal{O}_V^*$ and $(\lambda, \psi_\Phi)$ is an isomorphism $\pi(\omega', A', C') \longrightarrow \pi(\omega, A, C)$. We claim that the map of $T$-schemes

$$i\colon Z \longrightarrow \mathbb{A}^2_T \times \mathbb{G}_m,\ i(\omega', A', C', \lambda, u, v, w, z) = (\lambda, u, w)$$

is an open immersion. If we set

$$v(u,w) = 2\omega w(Aw - Cu),\ z(u,w) = u^2 + \omega C w^2$$

the condition $\psi_\Phi^{-1} \circ \beta \circ (\mathrm{Sym}^2 \psi_\Phi)(e_1^2) = e_2$ is equivalent to $v = v(u,w)$, $z = z(u,w)$. Since $\lambda, \psi_\Phi$ determine $\omega', A', C'$ and $\lambda, u, w$ determine $\lambda, \psi_\Phi$, we can conclude that $i$ is a monomorphism. Define $U \subseteq \mathbb{A}^2_T \times \mathbb{G}_m$ as the open subscheme where $uz(u,w) - v(u,w)w$ is invertible. This is just the expression of $\det \psi_\Phi$. Therefore $i(Z) \subseteq U$. Consider now $\xi = (\lambda, u, w) \in U$ and define $\psi_\xi = \begin{pmatrix} u & v(u,w) \\ w & z(u,w) \end{pmatrix}$. Note that by construction $\psi_\xi \in \mathrm{Gl}_{2,V}$. In particular there exists $\chi' = (\mathcal{O}_V, \mathcal{O}_V^2, m', \alpha', \beta', \langle -,- \rangle') \in \mathcal{U}_\beta(V)$ such that $(\lambda, \psi_\xi)\colon \chi' \longrightarrow \pi(\omega, A, C)$ is an isomorphism. Since by construction

$$\beta'(e_1^2) = \psi_\Phi^{-1} \circ \beta \circ (\mathrm{Sym}^2 \psi_\Phi)(e_1^2) = e_2$$

from 5.3.14 we see that there exists $\omega', A', C' \in \mathcal{O}_V$ such that $\pi(\omega', A', C') = \chi'$. In particular $\Phi = (\omega', A', C', \lambda, u, v(u,w), w, z(u,w)) \in Z(V)$ and $i(\Phi) = \xi$. □

### 5.3.4 The regular representation and the stack of torsors $\mathrm{B}(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$.

We want to describe the regular representation $\mathcal{R}[G]$, as an algebra, and the stack $\mathrm{B}\,G$ of $G$-torsors. By 4.2.32, the $G$-algebra $\mathcal{R}[G]$ is associated with the forgetful functor $\Omega\colon \mathrm{Loc}^G \mathcal{R} \longrightarrow \mathrm{Loc}\,\mathcal{R}$. By the theory of representation of $G$ and taking into account how we have associated with a functor the object of $G$-Cov, it is easy to deduce that the sequence $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in G\text{-Cov}(\mathcal{R})$ associated with $\Omega$ is given by

$$\mathcal{L} = A, \mathcal{F} = V;\ \begin{array}{l} \alpha(1_A \otimes v_1) = -v_1 \\ \alpha(1_A \otimes v_2) = v_2 \end{array};\ \begin{array}{l} \beta(v_1^2) = v_2,\ \beta(v_1 v_2) = 0,\ \beta(v_2^2) = v_1 \\ \langle v_1, v_2 \rangle = (-1/2)1_A,\ m(1_A \otimes 1_A) = 1 \end{array}$$

where $I_G = \{\mathcal{R}, A, V\}$.

**Definition 5.3.15.** Given $\Phi = (\mathcal{F}, \delta) \in \mathcal{C}_3$, we define the discriminant map $\Delta_\Phi \colon (\det \mathcal{F})^2 \longrightarrow \mathcal{O}_T$ as the determinant of the map $\mathcal{F} \longrightarrow \mathcal{F}^\vee$ induced by $\eta_\delta \colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{O}_T$. Given $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in G\text{-Cov}$ we define the map $\Delta_\chi \colon (\det \mathcal{F})^2 \longrightarrow \mathcal{O}_T$ as the determinant of the map $\mathcal{F} \longrightarrow \mathcal{F}^\vee$ induced by $(-,-)_\chi \colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{O}_T$.

*Remark* 5.3.16. The map $\Delta_\chi$ coincides with minus the composition

$$(\det \mathcal{F})^2 \xrightarrow{\langle -,- \rangle^{\otimes 2}} \mathcal{L}^2 \xrightarrow{m} \mathcal{O}_T$$

Moreover, if $\operatorname{tr} \beta = 0$, then $\Delta_{(\mathcal{F}, \delta_\beta)} = 4\Delta_\chi$ thanks to 5.3.5. For the first claim, we can argue locally, i.e. choosing a basis $y, z$ of $\mathcal{F}$, setting $\mathcal{L} = \mathcal{O}_T$ and considering the parameters associated with $\chi$. In this case

$$\Delta_\chi = (y,y)(z,z) - (y,z)^2 = -BC\omega^2 - A^2\omega^2 = -\omega^2 m$$

**Theorem 5.3.17.** *An object $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in G\text{-Cov}$ (resp. $\in S_3\text{-Cov}$ over $\mathcal{R}_3$) corresponds to a $G$-torsor (resp. $S_3$-torsor) if and only if the maps*

$$m \colon \mathcal{L}^2 \longrightarrow \mathcal{O}_T, \ \langle -,- \rangle \colon \det \mathcal{F} \longrightarrow \mathcal{L}$$

*are isomorphisms, or, equivalently, $\Delta_\chi \colon (\det \mathcal{F})^2 \longrightarrow \mathcal{O}_T$ is an isomorphism. In this case: $\alpha$ is an isomorphism, $\beta, (-,-)$ are surjective and $\operatorname{tr} \beta = \operatorname{tr} \alpha = 0$. Moreover $\mathrm{B}\, G \subseteq \mathcal{U}_\omega, \mathcal{U}_\alpha$ and, over $\mathcal{R}_3$, $\mathrm{B}\, G \subseteq \mathcal{U}_\beta$. Finally the map $\Lambda$ of Theorem 5.3.5 is an isomorphism from the full substack of $\mathcal{C}_3$ of objects $\Phi$ such that $\Delta_\Phi$ is an isomorphism to $\mathrm{B}\, G$.*

*Proof.* The claims about $S_3$ follows from the same claims about $G$ because, over $\mathcal{R}_3$, the isomorphism $G\text{-Cov} \simeq S_3\text{-Cov}$ preserves the torsors. Let $\Omega$ be the functor associated to $\chi$. Since $G$ is super solvable, by Theorem 4.2.42, $\chi$ corresponds to a $G$-torsor if and only if $m \colon \mathcal{L}^2 = \Omega_A \otimes \Omega_A \longrightarrow \mathcal{O}_T$ and $(-,-) \colon \mathcal{F} \otimes \mathcal{F} = \Omega_V \otimes \Omega_{V^\vee} \longrightarrow \mathcal{O}_T$ are surjective. The first condition says that $m$ is an isomorphism and, in particular, that $\alpha$ is an isomorphism. Moreover it is easy to check, locally, that in this case $(-,-)$ is surjective if and only $\langle -,- \rangle$ is an isomorphism. By definition of $\Delta_\chi$, this map is an isomorphism if and only if both $m$ and $\langle -,- \rangle$ are isomorphisms. Except for the last sentence of the statement, all the other claimed properties follow by checking them on $\mathcal{R}[G]$.

Since $\mathrm{B}\, G \subseteq \mathcal{U}_\omega$ and $\Lambda$ is an isomorphism, we get an isomorphism $\Lambda^{-1}(\mathrm{B}\, G) \longrightarrow \mathrm{B}\, G$ and, by 5.3.16, $\Lambda^{-1}(\mathrm{B}\, G)$ is the substack of $\mathcal{C}_3$ of objects $\Phi$ such that $\Delta_\Phi$ is an isomorphism. $\square$

*Remark* 5.3.18. Let $\Phi = (\mathcal{F}, \delta) \in \mathcal{C}_3$, $\mathscr{A}_\Phi = \mathcal{O}_T \oplus \mathcal{F}$ be the algebra associated with $\Phi$ (see 5.3.7) and assume to work over $\mathcal{R}_3$. Then $\det \mathscr{A}_\Phi \simeq \det \mathcal{F}$ and the determinant of the map $\mathscr{A}_\Phi \longrightarrow {\mathscr{A}_\Phi}^\vee$ induced by the trace map $\operatorname{tr}_{\mathscr{A}_\Phi} \colon \mathscr{A}_\Phi \longrightarrow \mathcal{O}_T$ coincides with $\Delta_\Phi$. In particular $\mathscr{A}_\Phi$ is étale if and only if $\Delta_\Phi$ is an isomorphism.

**Corollary 5.3.19.** *Set $\mathcal{F} = \mathcal{R}^2$ with basis $e_1, e_2$ and consider $\delta \colon \operatorname{Sym}^3 \mathcal{F} \longrightarrow \det \mathcal{F}$ given by $\delta(e_2^3) = -\delta(e_1^3) = 1$ and $\delta(e_1 e_2^2) = \delta(e_1^2 e_2) = 0$. Then*

$$G \simeq \underline{\mathrm{Aut}}_{\mathcal{C}_3}(\mathcal{F}, \delta)$$

*Assume now that the base scheme is $\mathcal{R}_3$. Then the map $\mathrm{B}\,G \longrightarrow \mathrm{Cov}_3$, obtained by taking invariants by $\sigma \in \mathbb{Z}/2\mathbb{Z}$, is an isomorphism onto the locus $Et_3$ of étale degree 3 covers. In particular*

$$G \simeq \underline{\mathrm{Aut}}_{\mathrm{Cov}_3}(\mathcal{R}_3[t]/(t^3-1))$$

*Proof.* For the first claim, it is enough to note that $\Lambda(\mathcal{F},\delta)$ induces the $G$-cover $\mathcal{R}[G]$. For the second one, assume we work over $\mathcal{R}_3$ and let $\Phi = (\mathcal{F},\delta) \in \mathcal{C}_3$. Thanks to 5.3.18, the map $\Lambda$ of Theorem 5.3.5 yields an isomorphism $\mathrm{Et}_3 \longrightarrow \mathrm{B}\,G$, whose inverse is the map $\mathrm{B}\,G \longrightarrow \mathrm{Cov}_3$ of the statement. In particular $G \simeq \underline{\mathrm{Aut}}_{\mathrm{Cov}_3}\,\mathcal{R}_3[G]^\sigma$ and it is easy to check that $\mathcal{R}[G]^\sigma \simeq \mathcal{R}[t]/(t^3-1)$. $\square$

*Remark* 5.3.20. The above corollary gives an alternative proof of the fact that $\mathrm{B}\,G \simeq \mathrm{B}\,S_3$ over $\mathcal{R}_3$ (see 2.3.11). Indeed $\mathrm{B}\,G \simeq \mathrm{Et}_3$ and it is a classical result that $\mathrm{Et}_3 \simeq \mathrm{B}\,S_3$. Moreover we see that the $S_3$-torsor $P$ corresponding to $\mathcal{R}_3[t]/(t^3-1) \in \mathrm{Et}_3$ is a $(G,S_3)$-torsor over $\mathcal{R}_3$ (see 2.3.1).

### 5.3.5 Irreducible components of $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$-Cov and $S_3$-Cov.

In this subsection we want to prove that $G$-Cov and, over $\mathcal{R}_3$, $S_3$-Cov have exactly two irreducible components. Moreover we will show that they are universally reducible and nonreduced. We will also describe the irreducible component of $G$-Cov that is not the principal one, that is $\mathcal{Z}_G$.

**Definition 5.3.21.** Let $\mathcal{X} \longrightarrow T$ be an algebraic stack over $T$. We will say that $\mathcal{X}$ is *universally not reduced* over $T$ if for any base change $T' \longrightarrow T$ the stack $\mathcal{X} \times_T T'$ is not reduced.

The theorem we want to prove is:

**Theorem 5.3.22.** *Let $R = \mathcal{R}[a,b,c,d,e,f,\omega,A,B,C,D]/(relations)$ where the relations are the ones given in (5.2.7). Then $\chi = (R, R^2, m, \alpha, \beta, \langle -,- \rangle)$ where*

$$\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix};\ \begin{array}{l} e_1^2 = ae_1 + be_2,\ e_1e_2 = ce_1 + de_2,\ e_2^2 = ee_1 + fe_2 \\ \langle e_1, e_2 \rangle = \omega,\ m = A^2 + BC \end{array}$$

*is an object of $G$-$\mathrm{Cov}(R)$ and the associated map $\mathrm{Spec}\,R \longrightarrow G$-$\mathrm{Cov}$ is a $(\mathbb{G}_m \times \mathrm{Gl}_2)$-torsor. In particular*

$$G\text{-}\mathrm{Cov} \simeq [\mathrm{Spec}\,R/(\mathbb{G}_m \times \mathrm{Gl}_2)]$$

*The stack $G$-Cov (resp. $S_3$-Cov) is geometrically connected, universally not reduced and universally reducible over $\mathcal{R}$ (resp. $\mathcal{R}_3$) and it has two irreducible components that are geometrically irreducible.*

*The minimal primes of $R$ are*

$$P_1 = (a+d, c+f, A+D),\ P_2 = (a,b,c,d,e,f,\omega,B,C,A-D)$$

*and the irreducible components of $G$-Cov are $\mathcal{Z}_G$ and*

$$\mathcal{Z} = \{\beta = \langle -,- \rangle = 0 \text{ and } \alpha \text{ is (fppf) locally a multiple of the identity}\}$$

*Moreover we have isomorphisms*

$$[\mathbb{A}^1/\mathbb{G}_m] \times \mathrm{B}\,\mathrm{Gl}_2 \simeq \{(\mathcal{L},\mathcal{F},\mu) \mid \mu\colon \mathcal{L} \longrightarrow \mathcal{O}\} \xrightarrow{\quad\simeq\quad} \mathcal{Z}$$
$$(\mathcal{L},\mathcal{F},\mu) \longmapsto (\mathcal{L},\mathcal{F},\mu^2,\mu\otimes \mathrm{id}_{\mathcal{F}},0,0)$$

Before proving this theorem, we need some lemmas.

**Lemma 5.3.23.** *Let $\mathcal{L},\mathcal{F}$ be respectively an invertible sheaf and a rank 2 locally free sheaf. Then*

$$\underline{\mathrm{Hom}}(\mathcal{L},\mathcal{O}_S) \xrightarrow{-\otimes\mathcal{F}} \underline{\mathrm{Hom}}(\mathcal{L}\otimes\mathcal{F},\mathcal{F})$$

*is a an isomorphism onto the locus of maps $\alpha\colon \mathcal{L}\otimes\mathcal{F} \longrightarrow \mathcal{F}$ that are (fppf) locally a multiple of the identity.*

*Proof.* Clearly the map $-\otimes\mathcal{F}$ is injective and has the right image. So we have to prove that given $\alpha\colon \mathcal{L}\otimes\mathcal{F}\longrightarrow\mathcal{F}$ which is locally a multiple of the identity, there exists $\mathcal{L}\xrightarrow{\mu}\mathcal{O}_S$ such that $\alpha=\mu\otimes\mathrm{id}$. Set

$$\mu\colon \mathcal{L}\longrightarrow \mathcal{L}\otimes\mathcal{F}\otimes\mathcal{F}^\vee \xrightarrow{\alpha\otimes\mathrm{id}} \mathcal{F}\otimes\mathcal{F}^\vee \longrightarrow \mathcal{O}_S$$

We want to prove that $\alpha=(\mu/2)\otimes\mathrm{id}$. We can assume $\mathcal{L}=\mathcal{O}_S$, $\mathcal{F}=\mathcal{O}_S^2$ and $\alpha=\lambda\mathrm{id}$ whit $\lambda\in\mathcal{O}_S$. It is easy to check that $\mu$ is the multiplication for $2\lambda$, so that $\alpha=(\mu/2)\otimes\mathrm{id}$. □

**Lemma 5.3.24.** *Let $k$ be a field. Then, up to isomorphism, the only local $k$-algebras $A$ with $\dim_k A=3$ and $\dim_k m_A=2$ are*

$$A=k[x]/(x^3) \text{ and } A=k[x,y]/(x^2,xy,y^2)$$

*Proof.* Let $W=\operatorname{Ann} m_A$. We have $0\subsetneq W\subseteq m_A$. Assume first that $\dim_k W=1$ and let $x\in m_A-W$. We want to prove that $1,x,x^2$ is a basis of $A$. In this case we will have $A\simeq k[X]/(X^3)$. Consider an expression

$$a+bx+cx^2=0$$

Since $x\in m_A$, we have $a=0$. In particular $(b+cx)\in m_A$ because it is a zero divisor and again we can conclude that $b=0$. Finally $x\notin W$ implies $x^2\neq 0$ and therefore $c=0$. Assume now $\dim_k W=2$, i.e. $W=\operatorname{Ann} m_A$. If $x,y$ is a basis of $m_A$ then we have a surjective map

$$k[X,Y]/(X^2,XY,Y^2)\longrightarrow A$$

which is an isomorphism by dimension. □

**Lemma 5.3.25.** *Let $R=\mathcal{R}[b,\omega,A]/(Ab,A\omega)$. Then*

$$(R,R^2,m,\alpha,\beta,\langle -,-\rangle)$$

*where*

$$\alpha=\begin{pmatrix} A & 0\\ 0 & A\end{pmatrix};\quad \begin{array}{l}\beta(e_1^2)=be_2,\ \beta(e_1e_2)=\beta(e_2^2)=0\\ \langle e_1,e_2\rangle=\omega,\ m=A^2\end{array}$$

*is an object of $G$-$\mathrm{Cov}(R)$. The induced map $\operatorname{Spec} R\longrightarrow G$-$\mathrm{Cov}$ is topologically surjective onto the locus $|G\text{-}\mathrm{Cov}|-|\mathcal{U}_\alpha|$.*

*Proof.* Set $\chi$ for the object defined in the statement. A direct computation on the relations (5.2.7) shows that $\chi \in G\text{-Cov}(R)$. On the other hand, since $\alpha$ is globally a multiple of the identity, we also have that the image of $\operatorname{Spec} R$ in $|G\text{-Cov}|$ lies outside $|\mathcal{U}_\alpha|$.

Now we have to prove that the map in the statement is topologically surjective. Let $k$ be a field and $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in G\text{-Cov}(k)$ such that $\chi \notin \mathcal{U}_\alpha$. Consider the $k$-algebra $\mathscr{A}^\sigma \simeq k \oplus \mathcal{F} = \mathcal{B}$, where $\sigma \in \mathbb{Z}/2\mathbb{Z}$ and where the multiplication is given by $\operatorname{Sym}^2 \mathcal{F} \xrightarrow{2(-,-)_\chi + \beta} k \oplus \mathcal{F}$. The vector space $\mathcal{F} \subseteq \mathcal{B}$ is an ideal because $\alpha = \lambda \mathrm{id}$ for some $\lambda \in k$ and $(u,v)_\chi = \langle \alpha(v), u \rangle = \lambda \langle v, u \rangle = 0$, because it is both symmetric and antisymmetric. Therefore $\mathcal{F}$ is a maximal ideal of $\mathcal{B}$ and $\mathcal{B}/\mathcal{F} = k$. In particular, from 5.3.24, $\mathcal{B}$ is isomorphic to either $k[x]/(x^3)$ or $k[x,y]/(x^2, xy, y^2)$. So there exists a basis $y, z$ of $\mathcal{F}$ such that $\beta(y^2) = by$, $\beta(yz) = \beta(z^2) = 0$ and therefore the parameters associates with $\chi$ with respect to this basis satisfy

$$a = c = d = e = f = B = C = 0, D = A = \lambda$$

□

*Proof.* (*of Theorem* 5.3.22) The results about $S_3$-Cov follow from the same results about $G$-Cov. Note that $\operatorname{Spec} R$ represents the functor of $G$-equivariant structures of commutative, associative $\mathcal{R}$-algebras over $\mathcal{R}[G]$. In particular the group $\mathcal{H} = \underline{\mathrm{Aut}}_{G,1} \mathcal{R}[G]$ of the $G$-equivariant isomorphisms of $G$-comodules preserving $1 \in \mathcal{R}$ acts on $\operatorname{Spec} R$ and it is easy to verify that $G$-Cov is the quotient stack of $\operatorname{Spec} R$ by this group. Finally the representation theory of $G$ tell us that $\mathcal{H} \simeq \mathbb{G}_m \times \mathrm{Gl}_2$ and therefore that $G\text{-Cov} \simeq [\operatorname{Spec} R/\mathbb{G}_m \times \mathrm{Gl}_2]$.

If $\mathcal{R}'$ is an $\mathcal{R}$-algebra, thanks to 5.2.9, we know that $a+d, c+f$ belongs to all the prime ideals of $R \otimes_\mathcal{R} \mathcal{R}'$, but $a+d, c+f \neq 0$ in $R \otimes_\mathcal{R} \mathcal{R}'$. Therefore $R$ and $G$-Cov are universally not reduced. Since all the relations in (5.2.7) are homogeneous, $R$ is a $\mathbb{N}$-graded $\mathcal{R}$-algebra such that $R_0 = \mathcal{R}$. In particular $\operatorname{Spec} \mathcal{R}$ and therefore $G$-Cov are geometrically connected.

We now focus on the irreducible components of $G$-Cov. Let $\mathcal{R}'$ be a domain over $\mathcal{R}$, $R' = R \otimes_\mathcal{R} \mathcal{R}'$ and continue to denote by $P_1, P_2$ the ideal in the statement of $R'$. Notice that $R'/P_2 = \mathcal{R}'[A]$. In particular $P_2$ is prime, $A + D \notin P_2$ and therefore $P_1 \subsetneq P_2$. Now let $P$ be a prime ideal of $R'$. We want to show that $P_1 \subseteq P$ or $P_2 \subseteq P$. If $A + D \in P$ then $P_1 \subseteq P$. If $A + D \notin P$, then, taking into accounts (5.2.7) and the fact that $a+d, c+f \in P$, it is easy to check that $P_2 \subseteq P$. Since $R'_\omega \neq 0$, $\omega$ is not nilpotent, so there exists a minimal prime $P'$ such that $\omega \notin P'$. In particular $P_2 \subsetneq P'$ and therefore $P_1 \subseteq P'$. If by contradiction $P'$ is the only minimal prime, i.e. $\operatorname{Spec} R'$ is irreducible, then $P' \subseteq P_2$ and therefore $P_1 \subseteq P_2$, which is not the case. In particular there exists a minimal prime $P''$ such that $P'' \subseteq P_2$. Again if $P_1 \subseteq P''$ we find a contradiction, so $P_2 \subseteq P''$ and $P_2 = P''$ is a minimal prime. So $\operatorname{Spec} R$ is reducible and, by 3.2.10, the same conclusion holds for $G$-Cov. Moreover, having considered an arbitrary base change to a domain, it also follows that $\operatorname{Spec}(R/P_2)$ and the closed substack of $G$-Cov induced, which is $\mathcal{Z}$, are geometrically an irreducible component. We want now to show that

$G$-Cov has exactly 2 irreducible components, namely $\mathcal{Z}_G$ and $\mathcal{Z}$. In particular, by 3.2.10, it follows that $P'$ and $P_2$ are the only minimal primes of $R$. Moreover, since $P'$ is the only minimal prime over $P_1$, we can also conclude that $\sqrt{P_1} = P'$.

Let $\mathcal{Z}_2$ be the closed substack $\mathcal{Z}$ defined in the statement and $\mathcal{Z}_1$ the closed substack where $\alpha$ vanishes. If $R'$ is the algebra defined in 5.3.25, we see that $|\mathcal{Z}_1|$ and $|\mathcal{Z}_2|$ are the image of, respectively, $\operatorname{Spec}(R'/(A))$ and $\operatorname{Spec}(R'/(b,\omega))$ under the map $\operatorname{Spec} R' \longrightarrow G\text{-Cov}$. In particular $|\mathcal{Z}_1|$ and $|\mathcal{Z}_2|$ are irreducible and $|G\text{-Cov}| = |\mathcal{U}_\alpha| \cup |\mathcal{Z}_1| \cup |\mathcal{Z}_2|$. Denote by $\mathcal{U}_3$ the open locus where 3 is invertible, i.e. $\mathcal{U}_3 = \operatorname{Spec} \mathcal{R}_3 \times_{\mathcal{R}} G\text{-Cov}$ and by $\mathcal{U}_\beta$ the locus where $\beta$ is never 0 and 3 is invertible. Although we are working on $\mathcal{R}$, $\mathcal{U}_\beta \subseteq \mathcal{U}_3$ is exactly the stack considered in 5.3.12. Since $\mathrm{B}\, G \subseteq \mathcal{U}_\alpha$, $\emptyset \neq \mathrm{B}\, G \cap \mathcal{U}_3 \subseteq \mathcal{U}_\beta$ and both $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ are irreducible thanks to 5.3.9 and 5.3.12, we can conclude that $\overline{|\mathcal{U}_\alpha|} = \overline{|\mathcal{U}_\beta|} = |\mathcal{Z}_G|$. On the other hand $|\mathcal{U}_\beta| \cap |\mathcal{Z}_1| \neq \emptyset$, because it contains the algebra locally given by $\alpha = \langle -,- \rangle = 0$ and $a = c = d = e = f = 0$, $b = 1$, which is well defined thanks to 5.3.25. Therefore $|\mathcal{Z}_1| \subseteq \overline{|\mathcal{U}_\beta|} = |\mathcal{Z}_G|$. In particular $|G\text{-Cov}| = |\mathcal{Z}_G| \cup |\mathcal{Z}|$ and, because it is reducible, $\mathcal{Z}_G$ and $\mathcal{Z}$ are the only irreducible components of $G$-Cov.

The last isomorphisms follows from 5.3.23. In order to prove that $\mathcal{Z}_G$ is geometrically irreducible, it is enough to prove that, if $k$ is a field over $\mathcal{R}$, then $|\mathcal{Z}_G \times k| \cap |\mathcal{Z} \times k| \subseteq |\mathcal{Z}_{G \times k}|$. The stacks $\mathcal{Z}_G \times k$ and $\mathcal{Z} \times k$ are induced by $\operatorname{Spec}(R/\sqrt{P_1}) \otimes k$ and $\operatorname{Spec}(R/P_1) \otimes k$ respectively, whose intersection is the point where $m = \alpha = \beta = \langle -,- \rangle = 0$. But $\operatorname{Spec} R \otimes k$ is a cone with this point as vertex and therefore any irreducible component of $\operatorname{Spec} R \otimes k$ must contain it.

Finally the fact that $P_1$ is a prime can be checked using the software Macaulay2: if

$$I = (2aA + bB + cC, 2cA - aB + eC, a^2 + bc + \omega C, ac + be - 2\omega A, c^2 - ae - B\omega)$$

as ideal of $R' = \mathbb{Z}[a, b, c, e, \omega, A, B, C]$, we have $R/P_1 \simeq (R'/I)_2$, where $(-)_2$ denotes the localization by $2 \in \mathbb{Z}$, and Macaulay2 tells us that $I$ is a prime ideal of $R'$.

□

### 5.3.6 The main irreducible components $\mathcal{Z}_{(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})}$ and $\mathcal{Z}_{S_3}$.

In this subsection we want to give a more precise description of the irreducible component $\mathcal{Z}_G$ of $G$-Cov and, consequently, of $\mathcal{Z}_{S_3} \subseteq S_3\text{-Cov}$ over $\mathcal{R}_3$. In particular we will have to consider maps $\mathcal{L} \otimes \mathcal{F} \longrightarrow \mathcal{F}$ whose trace is zero (see 5.3.2) and we first want to describe them.

*Remark* 5.3.26. Let $\mathcal{N}$ and $\mathcal{F}$ be respectively an invertible sheaf and a locally free sheaf of rank 2. Given a map $\alpha \colon \mathcal{N} \otimes \mathcal{F} \longrightarrow \mathcal{F}$, we have a factorization

$$\begin{tikzcd}
s \otimes p \otimes q \arrow[r, mapsto] & \alpha(s \otimes q)p - \alpha(s \otimes p)q \\
\mathcal{N} \otimes \mathcal{F} \otimes \mathcal{F} \arrow[r] \arrow[dr] & \mathrm{Sym}^2 \mathcal{F} \\
& \mathcal{N} \otimes \det \mathcal{F} \arrow[u, dashed]
\end{tikzcd}$$

that yields an isomorphism

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathrm{tr}=0}(\mathcal{N} \otimes \mathcal{F}, \mathcal{F}) &\longrightarrow \underline{\mathrm{Hom}}(\mathcal{N} \otimes \det \mathcal{F}, \mathrm{Sym}^2 \mathcal{F}) \\ \begin{pmatrix} u & v \\ w & -u \end{pmatrix} &\longmapsto (v \quad -2u \quad -w) \end{aligned} \tag{5.3.6}$$

where the last row describes the behaviour of the map if $\mathcal{N} = \mathcal{O}_T$ and a basis $y, z$ of $\mathcal{F}$ is given.

We want to introduce also a different description. In order to do that we note that the pairing $\mathrm{Sym}^2 \mathcal{F} \otimes \mathrm{Sym}^2(\mathcal{F}^\vee) \longrightarrow \mathcal{O}_T$ defined by $uv \otimes \xi\eta \longmapsto \xi(u)\eta(v) + \eta(u)\xi(v)$ induces an isomorphism

$$\begin{aligned} \mathrm{Sym}^2 \mathcal{F}^\vee &\longrightarrow (\mathrm{Sym}^2 \mathcal{F})^\vee \\ (y^*)^2, y^* z^*, (z^*)^2 &\longmapsto 2(y^2)^*, (yz)^*, 2(z^2)^* \end{aligned}$$

where, again, the last row expresses the behaviour of the map if a basis $y, z$ of $\mathcal{F}$ is given. The composition

$$\mathrm{Sym}^2 \mathcal{F} \simeq \mathrm{Sym}^2(\mathcal{F}^\vee \otimes \det \mathcal{F}) \simeq \mathrm{Sym}^2 \mathcal{F}^\vee \otimes (\det \mathcal{F})^2 \simeq (\mathrm{Sym}^2 \mathcal{F})^\vee \otimes (\det \mathcal{F})^2$$

yields an isomorphism

$$\begin{aligned} \underline{\mathrm{Hom}}(\mathcal{N} \otimes (\det \mathcal{F})^2, \mathrm{Sym}^2 \mathcal{F}) &\xrightarrow{\check{-}} \underline{\mathrm{Hom}}(\mathrm{Sym}^2 \mathcal{F}, \mathcal{N}^{-1}) \\ (u \quad v \quad w) &\longmapsto (2w \quad -v \quad 2u) \end{aligned} \tag{5.3.7}$$

where the last row describes the behaviour of the map if $\mathcal{N} = \mathcal{O}_T$ and a basis $y, z$ of $\mathcal{F}$ is given.

*Notation* 5.3.27. We continue to keep notation introduced above: given $\zeta \colon \mathcal{N} \otimes \det \mathcal{F}^2 \longrightarrow \mathrm{Sym}^2 \mathcal{F}$ we will denote by $\check{\zeta} \colon \mathrm{Sym}^2 \mathcal{F} \longrightarrow \mathcal{N}^{-1}$ the associated map. Moreover we will also denote by $\hat{-}$ the inverse of $\check{-}$: given $\eta \colon \mathrm{Sym}^2 \mathcal{F} \longrightarrow \mathcal{N}^{-1}$ the associated map will be $\hat{\eta} \colon \mathcal{N} \otimes (\det \mathcal{F})^2 \longrightarrow \mathrm{Sym}^2 \mathcal{F}$.

Remarks above motivate the introduction of the following stack. Define the stack $\mathcal{X}$ whose objects are sequences $(\mathcal{M}, \mathcal{F}, \delta, \zeta, \omega)$ where $\mathcal{M}$ is an invertible sheaf, $\mathcal{F}$ is a locally free sheaf of rank 2, $\omega$ is a section of $\mathcal{M}$ and $\delta, \zeta$ are maps

$$\delta \colon \mathrm{Sym}^3 \mathcal{F} \longrightarrow \det \mathcal{F}, \ \zeta \colon (\det \mathcal{F})^2 \otimes \mathcal{M} \longrightarrow \mathrm{Sym}^2 \mathcal{F}$$

satisfying the following conditions:

1) the composition

$$(\det \mathcal{F})^2 \otimes \mathcal{M} \otimes \mathcal{F} \xrightarrow{\zeta \otimes \mathrm{id}} \mathrm{Sym}^2 \mathcal{F} \otimes \mathcal{F} \longrightarrow \mathrm{Sym}^3 \mathcal{F} \xrightarrow{\delta} \det \mathcal{F} \tag{5.3.8}$$

is zero;

2) the composition

$$\mathrm{Sym}^2 \mathcal{F} \xrightarrow{\check{\zeta}} \mathcal{M}^{-1} \xrightarrow{\omega^\vee} \mathcal{O}_S \tag{5.3.9}$$

coincides with $\eta_\delta$ (see 5.3.27 and (5.3.2)).

Given an object $\chi = (\mathcal{M}, \mathcal{F}, \delta, \zeta, \omega) \in \mathcal{X}$ we define $\mathcal{L}_\chi = \mathcal{M} \otimes \det \mathcal{F}$, $\alpha_\chi \colon \mathcal{L}_\chi \otimes \mathcal{F} \longrightarrow \mathcal{F}$ the map obtained from $\zeta$ using (5.3.6), $m_\chi = -\det \alpha_\chi \colon \mathcal{L}_\chi^2 \longrightarrow \mathcal{O}_T$, $\beta_\chi = \beta_\delta \colon \mathrm{Sym}^2 \mathcal{F} \longrightarrow \mathcal{F}$ (see (5.3.1)) and finally $\langle -, - \rangle_\chi = \omega \otimes \mathrm{id}_{\det \mathcal{F}} \colon \det \mathcal{F} \longrightarrow \mathcal{M} \otimes \det \mathcal{F} = \mathcal{L}_\chi$.

Remembering the notation introduced in 5.3.3, we want to prove the following Theorem.

**Theorem 5.3.28.** *The main irreducible component $\mathcal{Z}_G$ of $G$-Cov is the closed locus of $G$-Cov of objects $(\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle)$ such that*

$$\mathrm{tr}\, \alpha \colon \mathcal{L} \longrightarrow \mathcal{O}_T \text{ and } \mathrm{tr}\, \beta \colon \mathcal{F} \longrightarrow \mathcal{O}_T$$

*vanish. Moreover we have an isomorphism of stacks*

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Z}_G \\ \chi = (\mathcal{M}, \mathcal{F}, \delta, \zeta, \omega) & \longmapsto & (\mathcal{L}_\chi, \mathcal{F}, m_\chi, \alpha_\chi, \beta_\chi, \langle -, - \rangle_\chi) \end{array}$$

Notice that, again, the above result continue to hold if we replace $\mathcal{Z}_G$ by $\mathcal{Z}_{S_3}$ and we assume to work over $\mathcal{R}_3$. Before proving this Theorem we want to give an explicit description of the objects of $\mathcal{X}$ (and a posteriori of $\mathcal{Z}_G$) such that $\omega \in \mathcal{M}$ is an effective Cartier divisor, i.e. the map $\mathcal{O}_T \xrightarrow{\omega} \mathcal{M}$ is injective. This will be helpful when we will have to study $G$-covers whose total space is regular.

Given a scheme $T$, denote by $\mathcal{Z}_\omega(T)$ the category whose objects are sequences $(\mathcal{M}, \mathcal{F}, \delta, \omega)$, where $\mathcal{M}$ is an invertible sheaf, $\mathcal{F}$ is a rank 2 locally free sheaf, $\delta$ is a map $\delta \colon \mathrm{Sym}^3 \mathcal{F} \longrightarrow \det \mathcal{F}$ and $\omega$ is a section of $\mathcal{M}$ such that $\mathcal{O}_S \xrightarrow{\omega} \mathcal{M}$ is injective and such that the zero locus of $\eta_\delta \colon \mathrm{Sym}^2 \mathcal{F} \longrightarrow \mathcal{O}_S$ contains the zero locus of $\omega$, or, equivalently, such that

$$\mathrm{Im}\, \eta_\delta \subseteq \mathrm{Im}(\mathcal{M}^{-1} \xrightarrow{\omega^\vee} \mathcal{O}_T)$$

With an object $\chi = (\mathcal{M}, \mathcal{F}, \delta, \omega) \in \mathcal{Z}_\omega(T)$ we associate the unique map $\zeta_\chi \colon (\det \mathcal{F})^2 \otimes \mathcal{M} \longrightarrow \mathrm{Sym}^2 \mathcal{F}$ such that the associated map $\check{\zeta}_\chi$ (see 5.3.27) is a factorization as in

$$\begin{tikzcd} \mathrm{Sym}^2 \mathcal{F} \arrow[r, dashed, "\check{\zeta}_\chi"] \arrow[dr, "\eta_\delta"'] & \mathcal{M}^{-1} \arrow[d, "\omega^\vee"] \\ & \mathcal{O}_S \end{tikzcd}$$

**Theorem 5.3.29.** *Let $T$ be a scheme. The association*

$$\begin{array}{ccc} \mathcal{Z}_\omega(T) & \longrightarrow & \mathcal{Z}_G(T) \\ \chi = (\mathcal{M}, \mathcal{F}, \delta, \omega) & \longmapsto & (\mathcal{M}, \mathcal{F}, \delta, \zeta_\chi, \omega) \end{array}$$

*is an equivalence of categories onto the full subcategory of $\mathcal{Z}_G(S)$ where $\omega\colon \mathcal{O}_S \longrightarrow \mathcal{M}$ is injective.*

By Theorem 5.3.28, we can define a forgetful map $\Delta\colon \mathcal{Z}_G \longrightarrow \mathcal{C}_3$ (see 5.3.1), that, when 3 is invertible, is the restriction of the map $G$-Cov $\longrightarrow$ $\mathrm{Cov}_3$ obtained by taking invariants by $\sigma \in \mathbb{Z}/2\mathbb{Z}$. We have already seen (5.3.5) that $\Delta$ has a section. On the other hand, if $\chi = (\mathcal{F}, \delta) \in \mathcal{C}_3(T)$ and we denote by $\mathcal{Z}_\chi$ the fiber of $\mathcal{Z}_\omega(T) \longrightarrow \mathcal{C}_3(T)$, i.e. the subcategory of $\mathcal{Z}_\omega(T)$ of objects $\phi \in \mathcal{Z}_\omega(T)$ such that $\Delta(\phi) = \chi$ and morphisms $\Phi \xrightarrow{\psi} \Phi'$ such that $\Delta(\psi) = \mathrm{id}_\chi$ , from 5.3.29, it easy to deduce the following:

**Corollary 5.3.30.** *Let $T$ be a scheme and $\chi = (\mathcal{F}, \delta) \in \mathcal{C}_3(T)$. Then $\mathcal{Z}_\chi$ is a set, i.e. there exists at most one isomorphism between two of its objects, and the following maps*

$$\begin{array}{ccc} \mathrm{Im}(\mathcal{M}^{-1} \xrightarrow{\omega} \mathcal{O}_T) & \longleftarrow\!\!\shortmid & (\mathcal{M}, \mathcal{F}, \delta, \omega) \\ \left\{ \begin{array}{c} \textit{invertible sheaves of ideals } \mathcal{N} \subseteq \mathcal{O}_T \\ \textit{such that } \mathrm{Im}\, \eta_\delta \subseteq \mathcal{N} \end{array} \right\} & \longrightarrow & \mathcal{Z}_\chi \\ \mathcal{N} & \longmapsto & (\mathcal{N}^{-1}, \mathcal{F}, \delta, 1) \end{array}$$

*are inverses of each other.*

We will prove Theorems 5.3.28 and 5.3.29 just after the following lemma.

**Lemma 5.3.31.** *Let $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in \mathcal{Y}$ such that $\mathrm{tr}\,\alpha = \mathrm{tr}\,\beta = 0$ and set $\mathcal{M} = \mathcal{L} \otimes \det \mathcal{F}^{-1}$. Let also $\zeta\colon \mathcal{M} \otimes (\det \mathcal{F})^2 \longrightarrow \mathrm{Sym}^2 \mathcal{F}$ the map associated with $\alpha$ through the isomorphism (5.3.6) and $\delta = \delta_\beta\colon \mathrm{Sym}^3 \mathcal{F} \longrightarrow \det \mathcal{F}$ (see (5.3.1)). If $\mathcal{L} = \mathcal{O}_T$, $y, z$ is a basis of $\mathcal{F}$ and we use notation from 5.2.10, we have equivalences*

$$\textit{the map (5.3.8) is zero} \iff \beta \circ \zeta = 0 \iff \left\{ \begin{array}{l} 2aA + bB + cC = 0 \\ 2cA + eC - aB = 0 \end{array} \right.$$

$$\textit{the map (5.3.9) coincides with } \eta_\delta \iff \left\{ \begin{array}{c} a^2 + bc = -\omega C \\ ac + be = 2\omega A \\ c^2 - ae = B\omega \end{array} \right.$$

*Proof.* The conditions $\mathrm{tr}\,\alpha = \mathrm{tr}\,\beta = 0$ means that $a + d = c + f = A + D = 0$. Note that we have expressions

$$\zeta = By^2 - 2Ayz - Cz^2, \ \check{\zeta} = -2C(y^2)^* + 2A(yz)^* + 2B(z^2)^*$$

thanks to (5.3.6) and (5.3.7). In particular

$$\beta(\zeta) = (aB - 2cA - eC)y + (2aA + bB + cC)z$$

and, by definition of $\delta_\beta$, the composition 5.3.8 is given by

$$(\beta(\zeta) \wedge y)y^* + (\beta(\zeta) \wedge z)z^* = -(2aA + bB + cC)y^* + (aB - 2cA - eC)z^*$$

Therefore the first equivalence is clear. For the second one, note that the map 5.3.9 is just $\omega\check{\zeta}$. Therefore the last equivalence easily follows taking into account the expression of $\eta_\delta$ given in 5.3.4. $\square$

*Proof.* (*of theorem 5.3.28*) The result follows easily from 5.3.22, 5.3.31 and the local conditions (5.2.7), taking into account that $\operatorname{tr}\alpha = \operatorname{tr}\beta = 0$ means that, locally, the relations $a + d = c + f = A + D = 0$ hold. □

*Notation* 5.3.32. We identify the stack $\mathcal{X}$ with the stack $\mathcal{Z}_G$. Given $\chi = (\mathcal{M}, \mathcal{F}, \delta, \zeta, \omega) \in \mathcal{Z}_G$ we will continue to denote by $\mathcal{L}_\chi, m_\chi, \alpha_\chi, \beta_\chi, \langle -, - \rangle_\chi, (-, -)_\chi$ the objects associated with $\chi$. Moreover we will often omit the $(-)_\chi$ if this will not lead to confusion.

*Remark* 5.3.33. Given $\chi = (\mathcal{M}, \mathcal{F}, \delta, \zeta, \omega) \in \mathcal{Z}_G$, we want to show an alternative way of retrieving the map $m_\chi \colon \mathcal{L}_\chi^2 = \mathcal{M}^2 \otimes (\det \mathcal{F})^2 \longrightarrow \mathcal{O}_S$, which will be useful in the next chapter. Indeed it is easy to check locally that the following composition is just $-4m_\chi$.

$$\mathcal{M}^2 \otimes (\det \mathcal{F})^2 \xrightarrow{\mathrm{id}_\mathcal{M} \otimes \zeta} \mathcal{M} \otimes \operatorname{Sym}^2 \mathcal{F} \xrightarrow{\mathrm{id}_\mathcal{M} \otimes \check{\zeta}} \mathcal{M} \otimes \mathcal{M}^{-1} \simeq \mathcal{O}_T$$

*Proof.* (*of Theorem* 5.3.29) If $(\mathcal{M}, \mathcal{F}, \delta, \zeta, \omega) \in \mathcal{Z}_G(T)$ is such that $\omega$ is injective, then $\chi = (\mathcal{M}, \mathcal{F}, \delta, \omega) \in \mathcal{Z}_\omega(T)$ and $\zeta = \zeta_\chi$, because by definition $\omega^\vee \circ \check{\zeta} = \eta_\delta$. Conversely, given $\chi = (\mathcal{M}, \mathcal{F}, \delta, \omega) \in \mathcal{Z}_\omega(T)$, we need to prove that $\Phi = (\chi, \zeta_\chi) \in \mathcal{Z}_G(T)$. By construction the condition $\omega^\vee \circ \check{\zeta}_\chi = \eta_\delta$ is satisfied. Given $\beta = \beta_\delta$ and taking into account 5.3.31, it remains to prove that $\beta \circ \zeta_\chi = 0$. Note that the composition

$$(\det \mathcal{F})^2 \xrightarrow{\omega \otimes \mathrm{id}_{(\det \mathcal{F})^2}} \mathcal{M} \otimes (\det \mathcal{F})^2 \xrightarrow{\zeta_\chi} \operatorname{Sym}^2 \mathcal{F}$$

is just $\hat{\eta_\delta}$ (see 5.3.27). Since $\mathcal{O}_T \xrightarrow{\omega} \mathcal{M}$ is injective, we only need to show that $\beta \circ \hat{\eta_\delta} = 0$. Taking into account the local descriptions (5.3.7) and 5.3.4, locally we have:

$$\begin{aligned}
\beta(\hat{\eta_\delta}) &= (c^2 - ae)(ay + bz) - (ac + be)(cy - az) + (a^2 + bc)(ey - cz) \\
&= [(c^2 - ae)a - (ac + be)c + (a^2 + bc)e]y + \\
&\quad [(c^2 - ae)b + (ac + be)a - (a^2 + bc)c]z = 0
\end{aligned}$$

□

## 5.4 Normal and Regular $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$-covers and $S_3$-covers.

In this section we want to study the following problem: given a regular in codimension 1 (resp. normal, regular) integral, noetherian scheme $Y$ describe the $G$-covers $X \longrightarrow Y$ such that $X$ is regular in codimension 1 (resp. normal, regular). Since, as we will see, this problem is related to the same problem for degree 3 covers, we will simply call them triple covers. Moreover, because we want that $G$-torsors of regular schemes are regular $G$-covers, in this chapter we assume that $G$ is étale, that is we assume that the base ring is $\mathcal{R}_3$. In other words we will work with schemes $Y$ such that $6 \in \mathcal{O}_Y^*$. Notice that, under this assumption, the above problems for $G$-covers are equivalent to the same problems for $S_3$-covers.

*Remark* 5.4.1. The isomorphism $G$-Cov $\simeq S_3$-Cov preserves the regularity of covers, that is if $X \longrightarrow Y$ is a $G$-cover and $X' \longrightarrow Y$ is the associated $S_3$-cover then, if $Y$ is regular in codimension 1 (resp. normal, regular) then $X$ has the same property if and only if $X'$ has it. Moreover, if $Y$ is defined over a scheme $S$ then $X$ is smooth (geometrically connected) over $S$ if and only if $X'$ is so. Indeed $G$ and $S_3$ are étale locally isomorphic over $\mathcal{R}_3$ and the same conclusion holds for $X$ and $X'$ (see 2.3.12). Moreover the properties of being regular in codimension 1, normal, regular, smooth and geometrically connected are all local and they all satisfy descent in the étale topology.

We continue to keep notation for which $G$-Cov and $S_3$-Cov are identified with the stack of data $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle)$ describing them. In particular anything that is defined starting from $\chi$ is automatically associated with the corresponding $G$-cover and $S_3$-cover. Moreover, as in the other sections of this chapter, we continue to use $G$-Cov instead of $S_3$-Cov, but here this is really just a notation for the stack of data $\chi$. We introduce some loci associated with a $G$-cover or an $S_3$-cover.

**Definition 5.4.2.** Given $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle) \in G\text{-Cov}(Y)$ we define:

- $D_m$ as the closed subscheme of $Y$ defined by $m\colon \mathcal{L}^2 \longrightarrow \mathcal{O}_Y$;
- $D_\omega$ as the closed subscheme of $Y$ defined by $\langle -, - \rangle\colon \det \mathcal{F} \longrightarrow \mathcal{L}$;
- $Y_\alpha$ as the zero locus of the map $\alpha\colon \mathcal{L} \otimes \mathcal{F} \longrightarrow \mathcal{F}$, that coincides with the closed subscheme of $Y$ defined by $\mathcal{L} \otimes \mathcal{F} \otimes \mathcal{F}^\vee \xrightarrow{\alpha \otimes \mathrm{id}} \mathcal{F} \otimes \mathcal{F}^\vee \longrightarrow \mathcal{O}_Y$.

Notice that we have an inclusion $Y_\alpha \subseteq D_m$.

*Notation* 5.4.3. Let $Y$ be a scheme. Given $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle) \in G\text{-Cov}(Y)$ we continue to keep notation introduced in section 5.2. In particular we have the map $(-,-)_\chi\colon \mathrm{Sym}^2 \mathcal{F} \longrightarrow \mathcal{O}_Y$ and $\mathscr{A}_\chi$ will denote the associated $G$-equivariant algebra. We will also denote by $X_\chi = \mathrm{Spec}\, \mathscr{A}_\chi$ and by $f_\chi\colon X_\chi \longrightarrow Y$ the associated $G$-cover. When $\chi \in \mathcal{Z}_G(Y)$ we will also write $\chi = (\mathcal{M}, \mathcal{F}, \delta, \zeta, \omega)$ as in 5.3.28 and denote by $\mathcal{L}_\chi, m_\chi, \alpha_\chi, \beta_\chi, \langle -, - \rangle_\chi$ its associated objects. When we have also that $\mathcal{O}_Y \xrightarrow{\omega} \mathcal{M}$, or, equivalently $\langle -, - \rangle_\chi$, is injective, i.e. $\chi \in \mathcal{Z}_\omega(Y)$, we will write $\chi = (\mathcal{M}, \mathcal{F}, \delta, \omega)$ as in 5.3.29 and denote by $\zeta_\chi$ its associated object. We will often omit the $(-)_\chi$ if this will not lead to confusion. When $\mathrm{tr}\, \beta = 0$, (condition that holds as soon as $Y$ is reduced) we will denote by $\delta = \delta_\beta$ and $\eta_\delta$ the maps introduced in 5.3.1 and 5.3.2 respectively. Remember that $\eta_\delta = 2(-,-)_\chi$ (see 5.3.4).

Since we will work mostly over regular local rings, we also introduce the following notation and remarks.

*Notation* 5.4.4. Let $(R, m_R, k)$ be a local ring and $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle) \in G\text{-Cov}(R)$, so that $\mathscr{A} = \mathscr{A}_\chi = R \oplus \mathcal{L} \oplus \mathcal{F}_1 \oplus \mathcal{F}_2$ is its associated algebra. We will denote by $t$ a generator of $\mathcal{L}$ and by $y, z$ a basis of $\mathcal{F}$ and we will use the parameters associated with $\chi$ as in 5.2.10. We set $\mathscr{A}_0 = R \oplus \mathcal{L}$, which is an algebra such that $t^2 = m$, and $y_1, z_1$ and $y_2, z_2$ the basis of $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively equal to $y, z \in \mathcal{F}$. The structure of algebra of $\mathscr{A}$ is given by

$$ty_1 = Ay_1 + Cz_1,\ tz_1 = By_1 + Dz_1,\ ty_2 = -Ay_2 - Cz_2,\ tz_2 = -By_1 - Dz_1$$

$$y_1^2 = ay_2+bz_2,\ y_1z_1 = cy_2-az_2,\ z_1^2 = ey_2-cz_2,\ y_2^2 = ay_1+bz_1,\ y_2z_2 = cy_1-az_1,\ z_2^2 = ey_1-cz_1$$

$$y_1y_2 = -C\omega,\ y_1z_2 = A\omega+\omega t,\ z_1z_2 = B\omega$$

In particular the schemes $D_m$, $D_\omega$, $Y_\alpha$ are the Spec of, respectively, $R/(m)$, $R/(\omega)$, $R/(A,B,C,D)$.

### 5.4.1 Normal and regular in codimension 1 $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$-covers.

In this subsection we want to describe $G$-covers of regular in codimension 1 (resp. normal) schemes whose total space is regular in codimension 1 (resp. normal) and we will apply the general theory developed in section 4.4. In particular the following result is a direct corollary of 4.4.7. Such result can be recovered by the results proved in the following sections, where we will describe regular $G$-covers.

**Theorem 5.4.5.** *Let $Y$ be an integral, noetherian and regular in codimension 1 (resp. normal) scheme such that $\dim Y \geq 1$ and $6 \in \mathcal{O}_Y^*$, let $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in G\text{-}\mathrm{Cov}(Y)$ and denote by $f\colon X \longrightarrow Y$ the associated $G$-cover (resp. $S_3$-cover). Then $X$ is regular in codimension 1 (resp. normal) if and only if*

$$\operatorname{codim} D_m \cap D_\omega \geq 2 \text{ and } D_m, D_\omega \text{ are regular in codimension 1 Cartier divisors}$$

*In this case $f\colon X \longrightarrow Y$ is generically a $G$-torsor (resp. $S_3$-torsor).*

*Proof.* Notice that it is enough to prove the statement only for the group $G$. Remember that $G$ is a glrg over $\mathcal{R}$ and $I_G = \{\mathcal{R}, A, V\}$. We want to apply Theorem 4.4.7. Denote by $\Omega$ the functor associated with $\chi$. Note that the action of $G$ on $X_\chi$ is generically faithful because we are assuming $\chi \in G\text{-}\mathrm{Cov}$. We have to consider the maps $\mathcal{L} = \Omega_A \longrightarrow {\Omega_{A^\vee}}^\vee = \mathcal{L}^\vee$and $\mathcal{F} = \Omega_V \longrightarrow {\Omega_{V^\vee}}^\vee = \mathcal{F}^\vee$ induced by $m\colon \mathcal{L}^2 \longrightarrow \mathcal{O}_Y$ and $(-,-)_\chi\colon \mathcal{F} \otimes \mathcal{F} \longrightarrow \mathcal{O}_Y$ and their determinants $s_{f,A} = m \in \mathcal{L}^{-2}$ and $s_{f,V} \in (\det \mathcal{F})^{-2}$. Theorem 4.4.7 tells us that $X_\chi$ is regular in codimension 1 (resp. normal) if and only if $v_p(s_{f,A}) \leq 1$ and $v_p(s_{f,V}) \leq 2$ for all $p \in Y^{(1)}$. We are using the convention for which $v_p(0) = \infty$, so that the previous conditions also implies that $s_{f,A}, s_{f,V} \neq 0$. Notice that $s_{f,V} = s_\omega^2 \otimes s_{f,A} \in (\det \mathcal{F}^{-1} \otimes \mathcal{L})^2 \otimes \mathcal{L}^{-2}$, where $s_\omega = \langle -,- \rangle_\chi \in \det \mathcal{F}^{-1} \otimes \mathcal{L}$ and that the zero loci of $s_{f,A}$ and $s_\omega$ are respectively $D_m$ and $D_\omega$. Therefore $v_p(s_{f,A}) \leq 1$ and $v_p(s_{f,V}) \leq 2$ for all $p \in Y^{(1)}$ means that $D_m \cap D_\omega \cap Y^{(1)} = \emptyset$ and that $D_m$ and $D_\omega$ are regular over the codimension 1 points of $Y$. In this case $f$ is generically a $G$-torsor again thanks to 4.4.7. □

### 5.4.2 Regular $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$-covers and $S_3$-covers.

In this subsection we are mainly interested in regular $G$-covers and $S_3$-covers. The Theorem we will prove is the following.

**Theorem 5.4.6.** *Let $Y$ be a regular, noetherian and integral scheme such that $\dim Y \geq 1$ and $6 \in \mathcal{O}_Y^*$ and let $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in G\text{-}\mathrm{Cov}(Y)$. Then its associated $G$-cover ($S_3$-cover) is regular if and only if the following conditions hold:*

*1) $D_m, D_\omega$ are Cartier divisors and $D_m \cap D_\omega = \emptyset$;*

*2) $Y_\alpha = \emptyset$ or $Y_\alpha$ is regular and has pure codimension 2 in $Y$;*

*3) $D_\omega$ is regular and $D_m$ is regular outside $Y_\alpha$.*

*Moreover in this case $\chi \in \mathcal{U}_\beta$, the locus where $\beta$ is never zero.*

In the following lemmas we will work over a regular local ring $R$ and we will consider given an object $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle) \in G\text{-Cov}(R)$.

**Lemma 5.4.7.** *The map $(-,-)\colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow R$ is surjective if and only if $\omega$ is invertible and $\alpha \otimes k \neq 0$. Moreover in this case $\chi$ lies in the locus where $\alpha$ is never a multiple of the identity and $\operatorname{Spec} \mathscr{A} \longrightarrow \operatorname{Spec} \mathscr{A}_0$ is étale.*

*Proof.* We have that $(y, y) = -C\omega$, $(y, z) = (z, y) = A\omega$, $(z, z) = B\omega$. So the first claim holds. Assume now by contradiction that $\alpha \otimes k$ is a multiple of the identity. So over $k$ we have $B = C = 0$ and $A = D \neq 0$. On the other hand from (5.2.7) we see that $\omega(A + D) = 0$ and so $A = D = 0$. In particular we can consider $y, z = \alpha(y)$ as basis of $\mathcal{F}$. So

$$y_1^3 = y_1(ay_2 + bz_2) = -\omega(a - bt) \in \mathscr{A}^* \implies \mathscr{A} \in \mathrm{B}\,\mu_3(\mathscr{A}_0)$$

since $\omega = mb^2 - a^2 = -(a - bt)(a + bt) \in \mathscr{A}^*$. □

**Lemma 5.4.8.** *Assume that $m$ is invertible, that $\alpha$ is never a multiple of the identity and that a $\lambda \in R$ such that $\lambda^2 = m$ is given. Then $\mathscr{A}_\chi \simeq \mathscr{B}^2$ as $R$-algebras, where $\mathscr{B} = R \oplus Rv_1 \oplus Rv_2 \in \mu_3\text{-Cov}(R)$ satisfies: $v_1^2 = (a + \lambda b)v_2$, $v_2^2 = (a - \lambda b)v_1$. Moreover we have $v_1 v_2 = -\omega$, $v_1^3 = -(a + \lambda b)\omega$, $v_2^3 = -(a - \lambda b)\omega$.*

*Proof.* We consider a basis of $\mathcal{F}$ of the form $y, z = \alpha(y)$. Set $u = \frac{1}{2} + \frac{t}{2\lambda} \in \mathscr{A}_0$ and note that it is an idempotent. Since $u$ is $\mu_3$-invariant, $\sigma \in \mathbb{Z}/2\mathbb{Z} \subseteq G$ yields an isomorphism of $R$-algebras $\mathscr{B} = \mathscr{A}/u\mathscr{A} \simeq \mathscr{A}/(1 - u)\mathscr{A}$, since $\sigma(u) = 1 - u$. In particular $\mathscr{A} \simeq \mathscr{B}^2$. Since $\mathscr{A}_0/u\mathscr{A}_0 \simeq R$, we have that $\mathscr{B} \in \mu_3\text{-Cov}(R)$ and its decomposition is given by

$$\mathscr{B} = R \oplus \mathcal{F}_1/u\mathcal{F}_1 \oplus \mathcal{F}_2/u\mathcal{F}_2$$

An easy computation shows that $uz_1 = \lambda u y_1$ and that $2uz_1 = \lambda y_1 + z_1$. In particular $\mathcal{F}_1/u\mathcal{F}_1$ is generated by $v_1 = y_1$ and $z_1 = -\lambda v_1$. Similarly we get that $v_2 = y_2$ generates $\mathcal{F}_2/u\mathcal{F}_2$ and $z_2 = \lambda v_2$. It is now easy to deduce the desired relations. □

**Lemma 5.4.9.** *Assume that $\dim R \geq 1$ and that $m$ is invertible or the map $(-,-)\colon \mathcal{F} \otimes \mathcal{F} \longrightarrow \mathcal{O}_S$ is surjective. Then $\mathscr{A}_\chi$ is regular if and only if one of the following conditions holds: both $\omega$ and $m$ are invertible; $m$ is invertible and $\omega \in m_R - m_R^2$; $\omega$ is invertible and $m \in m_R - m_R^2$. In all of those cases $\alpha \otimes k$ is not a multiple of the identity and $\beta \otimes k \neq 0$.*

*Proof.* We first want to show that we can assume that $\alpha \otimes k$ is not a multiple of the identity. Since $\omega \operatorname{tr} \alpha = 0$ and, if $\mathscr{A}_\chi$ is regular then $\mathscr{A}_\chi \in \mathcal{Z}_G(R)$ thanks to 4.4.10, we can conclude that $\operatorname{tr} \alpha = 0$, taking into account that $R$ is a domain and 5.3.28. On the other hand, since $\operatorname{tr}\alpha = A + D$, then $\alpha \otimes k$ is a multiple of the identity if and only if $\alpha \otimes k = 0$, that cannot happen if $m$ is invertible or $(-,-)$ is surjective.

*Case*: *$m$ invertible.* Without loss of generality, we can assume $m = \lambda^2$ for some $\lambda \in R$. From 5.4.8, $\mathscr{A}_\chi \simeq \mathscr{B}^2$ and so $\mathscr{A}_\chi$ is regular if and only if $\mathscr{B}$ is so. If $\omega$ is invertible then $\mathscr{B}$ is étalè over $R$, otherwise $\mathscr{B}$ is regular if and only if

$$-(a-\lambda b)\omega = (a-\lambda b)^2(a+\lambda b) = (a-\lambda b)(a+\lambda b)^2 \in m_R - m_R^2$$

This happens if and only if only one between $(a-\lambda b), (a+\lambda b)$ is in $m_R - m_R^2$, which is equivalent to $\omega \in m_R - m_R^2$.

*Case*: *$(-,-)$ surjective.* By 5.4.7, $\operatorname{Spec}\mathscr{A}_\chi \longrightarrow \operatorname{Spec}\mathscr{A}_0$ is étalè. Therefore $\mathscr{A}_\chi$ is regular if and only if $\mathscr{A}_0$ is so, which is equivalent to the condition: $m$ invertible or $m \in m_R - m_R^2$.

Finally note that, since $\alpha \otimes k$ is not a multiple of the identity, $\omega = mb^2 - a^2$ and so

$$\beta \otimes k = 0 \iff a, b \in m_R \implies \omega \in m_R^2$$

$\square$

**Lemma 5.4.10.** *Assume that* $\dim R \geq 1$, *that $m$ is not invertible and that* $(-,-)\colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow R$ *is not surjective. If $\mathscr{A}_\chi$ is regular, then $\beta \otimes k \neq 0$, $\alpha \otimes k = 0$ and $\omega$ is invertible. If such conditions are satisfied and we choose a basis $y, z = \beta(y^2)$ of $\mathcal{F}$ as in 5.3.12, then $\mathscr{A}_\chi$ is regular if and only if $A, C$ are independent in $m_R/m_R^2$.*

*Proof.* Note that $\mathscr{A}_\chi$ is local with maximal ideal $m_{\mathscr{A}_\chi} = m_R \oplus \mathcal{L} \oplus \mathcal{F}_1 \oplus \mathcal{F}_2$. Set $\mathscr{A} = \mathscr{A}_\chi$, $\mathcal{T} = m_{\mathscr{A}}/m_{\mathscr{A}}^2$, $\mathscr{A}_k = \mathscr{A} \otimes k$, $\mathcal{T}_k = m_{\mathscr{A}_k}/m_{\mathscr{A}_k}^2$. We claim that there exists an exact sequence

$$0 \longrightarrow m_R/(m_R \cap m_{\mathscr{A}}^2) \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}_k \longrightarrow 0$$

Indeed $\operatorname{Ker}(\mathcal{T} \longrightarrow \mathcal{T}_k) = m_R\mathscr{A}/(m_R\mathscr{A} \cap m_{\mathscr{A}}^2) = Q$ and $m_R/(m_R \cap m_{\mathscr{A}}^2) \hookrightarrow Q$. On the other this map is surjective since $m_R\mathcal{L} \oplus m_R\mathcal{F}_1 \oplus m_R\mathcal{F}_2 \subseteq m_{\mathscr{A}}^2 \cap m_R\mathscr{A}$. We also have $m_R \cap m_{\mathscr{A}}^2 = (m_R^2, m, \operatorname{Im}(-,-)_\chi)$ because $m_{\mathscr{A}}^2$ is a sub $G$-comodule of $\mathscr{A}$. Since $\dim \mathscr{A} = \dim R$, if we denote by $W$ the $k$-vector subspace of $m_R/m_R^2$ generated by $m$ and $\operatorname{Im}(-,-)_\chi$, we can conclude that

$$\mathscr{A} \text{ regular} \iff \dim_k \mathcal{T}_k \leq \dim_k W$$

Notice that

$$\mathcal{T}_k = Q_0 \oplus Q_1 \oplus Q_2 \text{ with } Q_0 \simeq k/(\omega) \text{ and } Q_1 \simeq Q_2 \simeq \mathcal{F} \otimes k/(\operatorname{Im}(\alpha \otimes k), \operatorname{Im}(\beta \otimes k))$$

Indeed $\mathcal{T}_k$ is a $G$-comodule and also a $\mu_3$-comodule, thus the decomposition with the $Q_i$. Moreover $\sigma \in \mathbb{Z}/2\mathbb{Z}$ yields an isomorphism $Q_1 \simeq Q_2$ and the relation $y_1z_2 - z_1y_2 = 2\omega t$ implies that $Q_0 = \mathcal{L}/(\omega t)$. The last isomorphism follows because $m_{\mathscr{A}_k}^2 \cap (\mathcal{F}_1 \otimes k) = (\mathcal{F}_2^2, \mathcal{L}\mathcal{F}_1)$.

Assume that $\beta \otimes k = 0$. In particular $\operatorname{Im}(-,-)_\chi \subseteq m_R^2$ and therefore $W = \langle m \rangle$. On the other hand, since $m \in m_R$, $\alpha \otimes k$ is not surjective and therefore $Q_1, Q_2 \neq 0$.

In conclusion $\dim_k W \leq 1$, while $\dim_k \mathcal{T}_k \geq 2$ and we see that $\mathscr{A}$ cannot be regular if $\beta \otimes k = 0$.

We now assume that $\beta \otimes k \neq 0$ and, thanks to 5.3.13, we choose a basis $y, z = \beta(y^2)$ for $\mathcal{F}$ as in the statement. Moreover, thanks to 5.3.14, the parameters associated with $\chi$ satisfy $a = 0$, $b = 1$, $c = -\omega C$, $e = 2\omega A$, $B = \omega C^2$, $D = -A$ and $m = A^2 + \omega C^3$. Since $\beta(y^2) = z$ we see that $\mathcal{Q}_1 \simeq k/(A, \omega C)$. Moreover by definition $W = \langle A^2, \omega A, \omega C\rangle$. If $\omega \in m_R$, it follows that $A \in m_R$, since $m \in m_R$. In this case $\dim_k W \leq 1$, while $\dim_k \mathcal{T}_k \geq 2$. In particular if $\mathscr{A}$ is regular then $\omega$ is invertible. By the hypothesis on $(-,-)$ and thanks to 5.4.7, it also follows that $\alpha \otimes k = 0$. If we assume that $\omega$ is invertible and that $\alpha \otimes k = 0$, we get $\dim_k \mathcal{T}_k = 2$ and $W = \langle A, C\rangle$. In this case $\mathscr{A}$ is regular if and only if $\dim_k W \geq 2$, which exactly means that $A, C$ are independent in $m_R/m_R^2$. □

*Proof.* (*of Theorem* 5.4.6) It is enough to prove the statement for the group $G$. Denote by $f_\chi\colon X_\chi \longrightarrow Y$ the $G$-cover associated with $\chi$. Notice that $D_m, D_\omega$ are Cartier divisor if and only if $f_\chi$ generically $G$-torsor thanks to 5.3.17, condition true when $X_\chi$ is regular, by 4.4.20. In what follows we therefore assume this condition. In particular $Y_\alpha \subsetneq Y$. Since $\dim Y \geq 1$, we can reduce the problem to the case when $Y = \operatorname{Spec} R$, where $R$ is a local, regular ring with $\dim R \geq 1$. In particular we use notation from 5.4.4. We also set $I_\alpha = (A, B, C, D)$. The conditions in the statement become:

1) $m, \omega \neq 0$; $m \notin m_R$ or $\omega \notin m_R$;
2) if $I_\alpha \neq R$ then $R/I_\alpha$ is regular and $\operatorname{ht} I_\alpha = 2$;
3) $\omega \notin m_R^2$; $m \notin m_R^2$ if $I_\alpha = R$.

We split the proof in two parts, according to lemmas 5.4.9 and 5.4.10.

*$m$ invertible or $(-,-)$ surjective.* Note that both conditions imply that $I_\alpha = R$ by 5.4.7 and the result easily follows from 5.4.9.

*$m$ not invertible and $(-,-)$ not surjective.* If $\mathscr{A}_\chi$ is regular the result easily follows from 5.4.10. Taking into account the same lemma, we have only to show that conditions 1), 2), 3) imply that $\omega$ is invertible and that $\beta \otimes k \neq 0$. The first one is clear by 1), since $m \in m_R$. In particular $(-,-)$ not surjective implies that $\alpha \otimes k = 0$, by 5.4.7. Assume by contradiction that $\beta \otimes k = 0$. By the relations 5.2.7, we see that $A = -D, B, C \in m_R^2$. In particular $I_\alpha \subseteq m_R^2$ and therefore $R/I_\alpha$ is not regular, against condition 2). □

### 5.4.3 Regular $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$-covers, $S_3$-covers and triple covers.

In this subsection we want to compare regular $G$-covers and regular $S_3$-covers with regular triple covers. In particular we will show that any regular $G$-cover ($S_3$-covers) induces a regular triple cover, by taking invariants by $\sigma \in \mathbb{Z}/2\mathbb{Z}$. Conversely a regular triple cover can be extended to a $G$-cover ($S_3$-cover), provided that a certain codimension 2 condition is fulfilled. We will also show how it is possible to construct regular $G$-covers ($S_3$-cover) of a smooth variety. We start introducing some loci associated to a triple cover.

**Definition 5.4.11.** Let $Y$ be a scheme and $\Phi = (\mathcal{F}, \delta) \in \mathcal{C}_3(Y)$. We define

- $Y_\delta$ as the closed subscheme of $Y$ defined by $\eta_\delta \colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{O}_Y$;
- $D_\delta$ as the closed subscheme of $Y$ defined by the map $\Delta_\Phi \colon (\det \mathcal{F})^2 \longrightarrow \mathcal{O}_Y$ introduced in 5.3.15.

When $Y$ is integral we denote by $\operatorname{Cov}_3^{\mathrm{nd}}(Y)$ the full subcategory of $\operatorname{Cov}_3(Y)$ of objects $(\mathcal{F}, \delta)$ such that $Y_\delta$ is a proper subscheme of $Y$, or, equivalently, such that $\eta_\delta \neq 0$.

The suffix nd in the symbol $\operatorname{Cov}_3^{\mathrm{nd}}(Y)$ stands for 'not degenerate'. Indeed, for a triple cover $f \colon X \longrightarrow Y$ associated with $(\mathcal{F}, \delta) \in \operatorname{Cov}_3$, the closed subscheme of $Y$ where $\eta_\delta = 0$ coincides, topologically, with the locus where $f$ has triple points (see 5.3.24). Moreover, the complementary of the locus $D_\delta$ is the étale locus of $f$ (see 5.3.18).

We now want to state three Theorems we want to prove in this section. In oder to do so we have to introduce the divisorial component of a subscheme:

**Definition 5.4.12.** If $Y$ is a locally factorial, noetherian and integral scheme and $Z$ is a proper closed subscheme, there exists a maximum among the effective Cartier divisors contained in $Z$. Such divisor will be denoted by $D(Z)$ and called the *divisorial component* of $Z$ in $Y$.

The first Theorem we will prove shows an alternative description of regular $G$-covers, starting from the more simple description of $\mathcal{Z}_\omega$ (see 5.3.29).

**Theorem 5.4.13.** *Let $Y$ be a regular, noetherian and integral scheme such that $\dim Y \geq 1$ and $6 \in \mathcal{O}_Y^*$. Then the association*

$$\begin{array}{ccc} \operatorname{Cov}_3^{nd}(Y) & \xrightarrow{\quad\Gamma\quad} & \mathcal{Z}_\omega(Y) \\ (\mathcal{F}, \delta) & \longmapsto & (\mathcal{O}_Y(D(Y_\delta)), \mathcal{F}, \delta, 1) \end{array}$$

*is a fully faithful section of the projection $G\text{-}\operatorname{Cov}(Y) \longrightarrow \operatorname{Cov}_3(Y)$ and all the regular $G$-covers are in the essential image of $\Gamma$. Moreover, if $\chi = \Gamma(\mathcal{F}, \delta)$, then $D = D_\delta - 2D(Y_\delta)$ is a Cartier divisor if $D_\delta$ is so and the associated $G$-cover $X_\chi \longrightarrow Y$ is regular if and only if $Y_\delta$ is regular, $D_\delta$ is a Cartier divisor, $D \cap D(Y_\delta) = \emptyset$, $Y_\delta \cap D$ is empty or has pure codimension 2 and $D$ is regular outside $Y_\delta$. In this case $D_\omega = D(Y_\delta)$, $D = D_m$, $Y_\alpha = Y_\delta \cap D_m$ and $Y_\delta = D_\omega \sqcup Y_\alpha$.*

The following Theorem shows how the correspondence among regular $G$-covers, regular $S_3$-covers and regular triple covers behaves.

**Theorem 5.4.14.** *Let $Y$ be a regular, noetherian and integral scheme such that $\dim Y \geq 1$ and $6 \in \mathcal{O}_Y^*$. If $f \colon X \longrightarrow Y$ is a regular triple cover, then $Y_\delta = D(Y_\delta) \sqcup Y'_\delta$, where $Y'_\delta$ is a closed subscheme of pure codimension 2 if not empty and $D(Y_\delta)$ is regular. Moreover $f$ extends to a regular $G$-cover ($S_3$-covers) if and only if $Y'_\delta$ is regular. More precisely the maps $G\text{-}\operatorname{Cov}, S_3\text{-}\operatorname{Cov} \longrightarrow \operatorname{Cov}_3$ obtained by quotienting by $\sigma \in \mathbb{Z}/2\mathbb{Z}$ induce isomorphisms*

$$X \longmapsto X/\sigma$$

$$\begin{array}{c} \{\textit{regular } G\textit{-covers over } Y\} \\ \downarrow\wr \\ \{\textit{regular } S_3\textit{-covers over } Y\} \end{array} \longrightarrow \left\{ \begin{array}{c} \textit{regular triple covers } (\mathcal{F}, \delta) \textit{ over } Y \\ \textit{such that } Y_\delta \textit{ is regular} \end{array} \right\}$$

*Moreover the inverse of the upper morphism is the functor $\Gamma$ introduced in 5.4.13.*

We will also prove an existence theorem for regular $G$-covers and $S_3$-covers, but we need the following definition first.

**Definition 5.4.15.** Let $k$ be a field, $Y$ be a $k$-scheme and $\mathcal{E}$ be a quasi-coherent sheaf over $Y$. The sheaf $\mathcal{E}$ is called *strongly generated* if, for any closed point $q \in Y$, the map

$$\mathrm{H}^0(Y, \mathcal{E}) \longrightarrow \mathcal{E} \otimes (\mathcal{O}_{Y,p}/m_p^2)$$

is surjective. The sheaf $\mathcal{E}$ is called *geometrically strongly generated* if it is strongly generated over the geometric fiber $Y \times \overline{k}$.

The last Theorem we want to prove is the following:

**Theorem 5.4.16.** *Let $k$ be an infinite field with $\operatorname{char} k \neq 2, 3$, $Y$ be a smooth, irreducible and proper $k$-scheme with $\dim Y \geq 1$ and $\mathcal{F}$ be a locally free sheaf of rank 2 over $Y$. If $\mathcal{E} = \underline{\mathrm{Hom}}(\mathrm{Sym}^3 \mathcal{F}, \det \mathcal{F})$ is geometrically strongly generated then there exists $\delta \in \mathcal{E}$ such that the triple cover associated with $(\mathcal{F}, \delta) \in \mathcal{C}_3(Y)$ extends to a $G$-cover ($S_3$-cover) $X_\delta \longrightarrow Y$ with $X_\delta$ smooth and $Y_\delta = \emptyset$ or $\operatorname{codim}_Y Y_\delta = 2$. Moreover, if $Y$ is geometrically connected, then $X_\delta$ is geometrically connected if and only if $\det \mathcal{F} \not\simeq \mathcal{O}_Y$ and $\mathrm{H}^0(Y, \mathcal{F}) = 0$.*

The following Proposition shows that the hypothesis of the above Theorem can be easily satisfied.

**Proposition 5.4.17.** *Let $Y$ be a projective scheme over a field and $\mathcal{E}$ be a coherent sheaf over $Y$. Then*

$$\mathcal{E}(-1) \text{ globally generated } \implies \mathcal{E} \text{ geometrically strongly generated}$$

In particular if $Y$ and $\mathcal{F}$ are as in Theorem 5.4.16, with $Y$ projective, then $\mathcal{F}(-n)$ satisfies its hypothesis for $n \gg 0$. Moreover considering $\mathcal{F} = \mathcal{O}_Y(-1)^2$, so that $\mathcal{E} = (\mathrm{Sym}^3 \mathcal{F})^\vee \otimes \det \mathcal{F} \simeq \mathcal{O}_Y(1)^3$, we obtain:

**Corollary 5.4.18.** *Let $k$ be an infinite field with $\operatorname{char} k \neq 2, 3$. Then any smooth, projective and irreducible (resp. geometrically connected) $k$-scheme $Y$ with $\dim Y \geq 1$ has a $G$-cover ($S_3$-cover) $X \longrightarrow Y$ with $X$ smooth (resp. smooth and geometrically connected).*

The rest of this section is dedicated to the proofs of what claimed above. Note that all the results for $S_3$-covers are just a consequence of the same results for $G$-covers, thanks to 5.4.1 and 5.3.8. Therefore we will focus only on $G$-covers. We want now to argue why the concept of divisorial component introduced above is well defined.

*Remark* 5.4.19. Let $Y$ be a locally factorial, noetherian and integral scheme and $Z$ be a proper closed subscheme, defined by the sheaf of ideals $\mathcal{I}$. We want to show that the divisorial component of $Z$ in $Y$ exists, or, equivalently, that the set of invertible sheaves of ideals of $\mathcal{O}_Y$ containing $\mathcal{I}$ has a minimum $\mathcal{L}^{\mathcal{I}}$. Moreover we prove that, if $p \in Y$ and $\mathcal{I}_p = (f_1, \dots, f_r) \subseteq \mathcal{O}_{Y,p}$, then $\mathcal{L}^{\mathcal{I}}_p = (\gcd(f_1, \dots, f_r))$. The key point for proving this fact is that if $\mathcal{J}, \mathcal{N}$ are sheaves of ideals of $Y$, with $\mathcal{N}$ invertible, then

$$\mathcal{J} \subseteq \mathcal{N} \iff \mathcal{J}_q \subseteq \mathcal{N}_q \text{ for all } q \in Y^{(1)}$$

In particular it is easy to check that $D(Z)$ is induced by the Weil divisor

$$\sum_{q \in Y^{(1)}} \dim_{k(q)}(\mathcal{O}_{Y,q}/\mathcal{I}_q)q$$

*Remark* 5.4.20. Let $\Phi = (\mathcal{F}, \delta) \in \mathcal{C}_3(Y)$. Its associated algebra is $\mathscr{A}_\Phi = \mathcal{O}_Y \oplus \mathcal{F}$ with multiplication $\beta = \beta_\delta \colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{F}$ and $\eta_\delta \colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{O}_Y$ (see 5.3.7). The map

$$(\det \mathcal{F})^2 \xrightarrow{\hat{\eta}_\delta} \operatorname{Sym}^2 \mathcal{F} \xrightarrow{\eta_\delta} \mathcal{O}_Y$$

coincides with the map $\Delta_\Phi \colon (\det \mathcal{F})^2 \longrightarrow \mathcal{O}_Y$ defining $D_\delta$. Indeed $\operatorname{tr}_{\mathscr{A}_\Phi}(u \cdot_{\mathscr{A}_\Phi} v) = 3\eta_\delta(uv)$ for all $u, v \in \mathcal{F}$ and, if $y, z$ is a basis of $\mathcal{F}$, then

$$\eta_\delta(\hat{\eta}_\delta) = \eta_\delta(y^2)\eta_\delta(z^2) - \eta_\delta(yz)^2$$

Since $Y_\delta$ is defined by the ideal $(\eta_\delta(y^2), \eta_\delta(yz), \eta_\delta(z^2))$ we see that $Y_\delta \subseteq D_\delta$ and, if $Y$ is locally factorial, noetherian and integral and $D_\delta$ is a Cartier divisor, then $D_\delta - 2D(Y_\delta)$ is an effective Cartier divisor.

Assume now we have an extension $(\mathcal{L}, \mathcal{F}, m, \alpha, \beta_\delta, \langle -, - \rangle) \in G\text{-Cov}$ of $(\mathcal{F}, \delta)$, with its associated parameters. Since $\eta_\delta = 2(-,-)_\chi$ (see 5.3.5) we have that $\eta_\delta(y^2) = -2C\omega$, $\eta_\delta(yz) = 2A\omega$, $\eta_\delta(z^2) = 2B\omega$. Since $A\omega = D\omega$, we have that $Y_\alpha, D_\omega \subseteq Y_\delta$, that $|Y_\delta| = |Y_\alpha| \cup |D_\omega|$ and, if $Y$ is regular, noetherian and integral and $D_\omega$ is a Cartier divisor, that $D_\omega \subseteq D(Y_\delta)$. Moreover by 5.3.16, we also obtain

$$\eta_\delta(\hat{\eta}_\delta) = -4\omega^2 m$$

Therefore $D_m, D_\omega \subseteq D_\delta$, $|D_\delta| = |D_\omega| \cup |D_m|$, $D_\delta$ is a Cartier divisor if and only if $D_m$ and $D_\omega$ are so and in this case $D_\delta = D_m + 2D_\omega$.

*Proof.* (*of Theorem* 5.4.13) By definition of $\operatorname{Cov}_3^{\mathrm{nd}}(Y)$ and thanks to 5.3.30 we see that $\Gamma$ is well defined and fully faithful. By 5.4.6 we also see that all the regular $G$-covers of $Y$ belong to $\mathcal{Z}_\omega(Y)$. Now let $\chi = (\mathcal{M}, \mathcal{F}, \delta, \omega) \in \mathcal{Z}_\omega(Y)$. Notice that, if $X_\chi$ is regular, then $D_\delta$ is Cartier and $Y_\delta \subseteq D_\delta$ by 5.4.20. In particular $(\mathcal{F}, \delta) \in \operatorname{Cov}_3^{\mathrm{nd}}(Y)$. Therefore assume that this condition holds and notice that $\Gamma(\mathcal{F}, \delta) \simeq \chi$ is equivalent to $D_\omega = D(Y_\delta)$. Assume that $X_\chi$ is regular or that: $\chi = \Gamma(\mathcal{F}, \delta)$ and the conditions in the statement are satisfied. By 5.4.6 and 5.4.20 we can conclude noting that: $D_m, D_\omega, D_\delta$ are Cartier divisors and $D = D_m = D_\delta - 2D_\omega$; $D_m \cap D_\omega = \emptyset$; $Y_\delta = D_\omega \sqcup Y_\alpha$ and $Y_\delta \cap D_m = Y_\alpha$ is regular of pure codimension 2 if not empty, which also implies that $D_\omega = D(Y_\delta)$. $\square$

*Remark* 5.4.21. Let $R$ be a regular ring with $\dim R \geq 1$, $r(t) = t^3 + gt + h \in R[t]$ and set $\Delta_r = 4g^3 + 27h^2$. Adapting [Mir85, Lemma 5.1] to our situation we have that $R[t]/(r(t))$ is regular if and only if either:

$$g \in m_R, h \notin m_R^2 \text{ or } g \notin m_R, \Delta_r \notin m_R^2$$

*Proof.* (*of Theorem* 5.4.14) We can reduce the problem to the case where $Y = \operatorname{Spec} R$, where $R$ is a local, regular ring. Let $\Phi = (\mathcal{F}, \delta) \in \mathcal{C}_3(R)$ be a triple cover, $\chi = (\mathcal{M}, \mathcal{F}, \delta, \omega) \in \mathcal{Z}_\omega(R)$ an extension of $\Phi$. Notice that any object of $\mathcal{C}_3$ has an extension to $\mathcal{Z}_\omega(R)$. If $\Phi$ corresponds to a regular triple cover then it has Gorenstein fibers and, by [BV12, Theorem 3.1], $\delta$ is never zero. On the other hand, if $X_\chi$ is regular, then $\chi \in \mathcal{U}_\beta$, i.e. $\delta$ is never zero, by 5.4.6. Therefore we can assume that $\delta$ is never zero and choose a basis $y, z = \beta(y^2)$ of $\mathcal{F}$. In particular, from 5.3.14 and 5.4.20, we see that

$$\beta(yz) = -\omega C y,\ \beta(z^2) = 2\omega A y + \omega C z,\ \eta_\delta(y^2) = -2\omega C,\ \eta_\delta(yz) = 2\omega A,\ \eta_\delta(z^2) = 2\omega^2 C^2$$

and $m = A^2 + \omega C^3$. If $\mathscr{A}_\Phi = R \oplus \mathcal{F}$ is the algebra associated to $\Phi$ and we set

$$r(t) = t^3 + 3\omega C t - 2\omega A$$

a direct computation shows that $r(y) = 0$ and therefore that $\mathscr{A}_\Phi \simeq R[t]/(r(t))$. Moreover the discriminant is

$$\Delta_r = 4 \cdot 27\omega^3 C^3 + 4 \cdot 27\omega^2 A^2 = 4 \cdot 27\omega^2 m$$

and defines the locus $D_\delta$ by 5.3.14. Notice that if $\eta_\delta = 0$, then $r(t) = t^3$ and $\mathscr{A}_\phi$ is not regular, while if $X_\chi$ is regular then $\eta_\delta \neq 0$ thanks to 5.4.13. We can therefore assume that $\Phi \in \operatorname{Cov}_3^{\mathrm{nd}}(R)$. We split the proof in two cases and in both we will use 5.4.21 and 5.4.20. In particular set $g = 3\omega C$ and $h = -2\omega A$.

*$\mathscr{A}_\Phi$ regular and $\chi = \Gamma(\mathcal{F}, \delta)$.* We will use that $\Delta_r \in m_R^2$ implies that $g \in m_R$ and $h \notin m_R^2$. By definition we have that $D_\omega = D(Y_\delta)$, i.e. that $\gcd(A, C) = 1$. Notice that we cannot have $m, \omega \in m_R$ or $\omega \in m_R^2$, since otherwise $\Delta_r, h \in m_R^2$. Therefore $D_\omega$ is regular and $D_m \cap D_\omega = \emptyset$. In particular, since $|Y_\delta| = |D_\omega| \cup |Y_\alpha|$ and $Y_\alpha \subseteq D_m$, we have $Y_\delta = D_\omega \sqcup Y_\alpha$. If $m \in m_R^2$, then $\Delta_r \in m_R^2$. In particular $g \in m_R$ and therefore $h \in m_R$, which shows that $D_m$ is regular outside $Y_\delta$. We have to show that $Y_\alpha$ has pure codimension 2 if not empty. So assume that $A, C \in m_R$. In particular $g, h \in m_R$, so $\Delta_r \in m_R^2$ and therefore $h \notin m_R^2$. Moreover $m \in m_R$ and, as we have seen, $\omega \notin m_R$. We can conclude that $A \in m_R - m_R^2$. In particular $R/(A)$ is a regular domain of dimension $\dim R - 1$. If $\operatorname{ht}(A, C) \neq 2$, we must have that $C \in (A)$, that cannot happen since $\gcd(A, C) = 1$. In conclusion if $\mathscr{A}_\Phi$ is regular then all the conditions required by the regularity of $X_\chi$ (see 5.4.13) are satisfied, except for the regularity of $Y_\alpha$.

*$X_\chi$ regular.* We will make use of 5.4.13. In particular from it we see that $\Gamma(\mathcal{F}, \delta) = \chi$ and we have to prove that $\mathscr{A}_\Phi$ is regular. Assume by contradiction that this is false. Therefore we must have $g \in m_R, h \in m_R^2$ or $g \notin m_R, \Delta_r \in m_R^2$. If the last condition is satisfied, then $\omega, C \notin m_R$ and therefore $m \in m_R^2$. Since $(A, C) = R$, in this case

$D_m$ is not regular. So assume $g \in m_R, h \in m_R^2$. Note that in particular $\Delta_R \in m_R^3$. In particular, if $\omega \in m_R$, then $m \notin m_R$ and therefore $\omega \in m_R^2$, while $D_\omega$ is regular. So $\omega \notin m_R$ and therefore $C \in m_R$ and $A \in m_R^2$. But this cannot happen because $A, C$ are independent in $m_R/m_R^2$. $\square$

**Lemma 5.4.22.** *Let $k$ be an algebraically closed field, let $(R, m_R, k)$ be a regular local ring with $\dim R \geq 1$ and denote by $\overline{(-)}\colon R \longrightarrow \overline{R} = R/m_R^2$ the projection. Set $\mathcal{F} = R^2$, $\overline{\mathcal{F}} = \overline{R}^2$ and given $\delta\colon \operatorname{Sym}^3 \mathcal{F} \longrightarrow \det \mathcal{F}$ denote by $\mathscr{A}_\delta$ the $S_3$-cover over $R$ obtained as in 5.4.13. Finally set*

$$\overline{V} = \operatorname{Hom}_{\overline{R}}(\operatorname{Sym}^3 \overline{\mathcal{F}}, \det \overline{\mathcal{F}}),\ \overline{W} = \{\gamma \in \overline{V} \mid \exists \operatorname{Sym}^3 \mathcal{F} \xrightarrow{\delta} \det \mathcal{F} \text{ s.t. } \overline{\delta} = \gamma \text{ and } \mathscr{A}_\delta \text{ not regular}\}$$

*Then $\overline{W}$ is a closed subscheme of the $k$-vector space $\overline{V}$ with $\operatorname{codim}_{\overline{V}} \overline{W} \geq \dim R + 1$.*

*Proof.* Since $\delta$ varies in the arguments below, if $\delta = (-a, b, c, e)$ with respect to some basis of $\mathcal{F}$, all the parameters associated with $\delta$ (and its associated $S_3$-cover) will be thought of as (polynomial) functions in $a, b, c, e$. Moreover we fix a $k$-basis $1, \overline{x}_1, \dots, \overline{x}_s$ of $\overline{R}$, where $x_1, \dots, x_r \in m_R$ form a basis of $m_R/m_R^2$. Moreover $r \in R$ will be denoted by

$$r = r_0 \cdot 1 + r_1\overline{x}_1 + \cdots r_s\overline{x}_s \text{ where } s = \dim R,\ r_i \in k$$

Notice that, if $\delta, \delta' \in \operatorname{Hom}(\operatorname{Sym}^3 \mathcal{F}, \det \mathcal{F})$ are such that $\overline{\delta}$ and $\overline{\delta}'$ differs by an automorphism of $\overline{\mathcal{F}}$, i.e. there exists $\phi \in \operatorname{Gl}(\overline{\mathcal{F}})$ such that $(\det \phi)\delta' = \delta \circ \operatorname{Sym}^3 \phi$, then $\mathscr{A}_\delta$ is regular if and only if $\mathscr{A}_{\delta'}$ is so. Indeed the previous condition means that $\mathscr{A}_\delta \otimes \overline{R} \simeq \mathscr{A}_{\delta'} \otimes \overline{R}$. Given $y \in \overline{\mathcal{F}}$ set

$$\overline{V}_y = \{\gamma \in \overline{V} \mid y, \beta_\gamma(y^2) \text{ is a basis of } \overline{\mathcal{F}}\}$$

Notice that the $\overline{V}_y$ are open subsets of $\overline{V}$, not empty if $y_0 \neq 0$, and that, thanks to 5.3.13, their union covers $\overline{V} - \{\gamma \in \overline{V} \mid \beta_\gamma \otimes k = 0\}$. Since $\{\gamma \in \overline{V} \mid \beta_\gamma \otimes k = 0\} \subseteq \overline{W}$ has codimension 4 in $\overline{V}$, it is enough to prove that

$$\operatorname{codim}_{\overline{V}_y} W \cap \overline{V}_y \geq s + 1$$

for all $y \in \overline{\mathcal{F}}$ such that $y_0 \neq 0$. Consider now the map $p\colon \overline{V}_y \longrightarrow \overline{R}^2$ that sends $\gamma$ to the parameters $c, e$ associated with $\gamma$ with respect the basis $y, \beta_\gamma(y^2)$. Note that a $\gamma \in \overline{V}_y$ differs by an automorphism of $\overline{\mathcal{F}}$ from $(-1, 0, p(\gamma))$. In particular if we set

$$\overline{W}' = \{(\overline{c}, \overline{e}) \in \overline{R}^2 \mid (-1, 0, \overline{c}, \overline{e}) \in \overline{W}\}$$

then $p^{-1}(\overline{W}') = \overline{W} \cap \overline{V}_y$. Now denote by $U$ the open subspace of $\overline{\mathcal{F}}$ of $z$ such that $y, z$ is a basis of $\overline{\mathcal{F}}$. The map

$$V_y \longrightarrow U \times \overline{R}^2 \text{ given by } \gamma \longmapsto (\beta_\gamma(y^2), p(\gamma))$$

is an isomorphism whose inverse sends $z \in U, u, v \in \overline{R}$ to the $\gamma \in \overline{V}_y$ given by $(-1, 0, u, v)$ with respect to the basis $y, z$ of $\mathcal{F}$. In particular we can reduce again the problem and prove that $\operatorname{codim}_{\overline{R}^2} \overline{W}' \geq s + 1$. We want to show that $\overline{W}' = W_1 \cup W_2$, where

$$W_1 = \{e^2 = 4c^3 \text{ and } c_0, e_0 \neq 0\},\ W_2 = \{c_0 = e_0 = 0 \text{ and } c, e \text{ dependent in } m_{\overline{R}}/m_{\overline{R}^2}\}$$

Remember that the parameter $m$ associated with $(-1,0,c,e)$ is given by $(e^2-4c^3)/4$ and, taking into account Theorem 5.4.6, the only non trivial point to show in the equality above is the following: if $c,e \in R$ are such that $e^2 = 4c^3$ then there exists $c', e' \in R$ such that $\overline{e}' = \overline{e}$, $\overline{c}' = \overline{c}$ and $e'^2 \neq 4c'^3$. Assume by contradiction that this is not possible. Notice that $e^2 = 4c^3$ implies that $c,e$ are invertible or that $e = c = 0$. Indeed if $c,e \neq 0$ and $c \in m_R^t - m_R^{t+1}$, $e \in m_R^l - m_R^{l+1}$ then $2t = 3l$, which implies $t = l = 0$. Consider $e' = e + w$, with $w \in m_R^2 - m_R^3$, $c' = c$. If $e = 0$ we are fine. Otherwise, modulo $m_R^3$, we get the equality $2ew = 0$ and also in this case we get a contradiction.

If we write $c = c_0 + c'$, $e = e_0 + e'$ then $W_1$ is contained in the irreducible component of the locus $\{e_0^2 = 4c_0^3,\ e_0 e' = 6c_0 c'\}$ which is not $\{c_0 = e_0 = 0\}$, that has codimension $s+1$. Finally it is easy to check that also $W_2$ has codimension $\geq s+1$. $\square$

*Remark* 5.4.23. Let $Y$ be a proper, smooth and geometrically connected scheme with $\dim Y \geq 1$ over a field $k$ with $\operatorname{char} k \neq 2,3$ and $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -,- \rangle) \in G\text{-Cov}(Y)$ such that $X_\delta$ is regular. Then $X_\delta$ is geometrically connected if and only if $\mathcal{L} \not\simeq \mathcal{O}_Y$ and $\mathrm{H}^0(Y, \mathcal{F}) = 0$. Indeed $X_\delta$ is geometrically connected if and only if $\mathrm{H}^0(X_\delta \times \overline{k}, \mathcal{O}_{X_\delta \times \overline{k}}) = \overline{k}$, which means $\mathrm{H}^0(Y \times \overline{k}, \mathcal{L} \otimes \overline{k}) = \mathrm{H}^0(Y \times \overline{k}, \mathcal{F} \otimes \overline{k}) = 0$. On the other hand, this is also equivalent to $\mathrm{H}^0(Y, \mathcal{L}) = \mathrm{H}^0(Y, \mathcal{F}) = 0$ and, since $m$ induces an injective map $\mathcal{L} \longrightarrow \mathcal{L}^{-1}$, we also have that $\mathrm{H}^0(Y, \mathcal{L}) \neq 0$ is equivalent to $\mathcal{L} \simeq \mathcal{O}_Y$.

*Proof.* (of Proposition 5.4.17) We can assume that $k$ is algebraically closed. Note that the property of being strongly generated is stable for direct sums and quotients. Moreover, if $i\colon Y \longrightarrow \mathbb{P}_k^m$ is the closed immersion defined by $\mathcal{O}_Y(1)$, then $\mathcal{E}$ is strongly generated if and only if $i_*\mathcal{E}$ is so. Since we have surjective maps $\mathcal{O}_Y(1)^N \longrightarrow \mathcal{E}$ by hypothesis and $\mathcal{O}_{\mathbb{P}_k^m}(1) \longrightarrow i_*\mathcal{O}_Y(1)$, we only need to prove that $\mathcal{O}_{\mathbb{P}_k^m}(1)$ is strongly generated. This last condition means that, if $p$ is a maximal ideal of $R = k[x_1, \dots, x_m]$, then $R/p^2$ is generated as $k$-vector space by the elements $1, x_1, \dots, x_m$. This property clearly holds because $p = (x_1 - a_1, \dots, x_m - a_m)$ for some $a_i \in k$. $\square$

*Proof.* (of Theorem 5.4.16) Denote by $V$ the vector bundle over $k$ associated with $\mathrm{H}^0(Y, \mathcal{E})$ and by $g\colon Y \times V \longrightarrow Y$ and $\pi\colon Y \times V \longrightarrow V$ the projections. By definition of $V$, there exists $\chi = (g^*\mathcal{F}, \mu) \in \mathcal{C}_3(Y \times V)$ such that, for any $\operatorname{Spec} k \xrightarrow{\delta} V$, we have $(\mathrm{id}_Y \times \delta)^*(g^*\mathcal{F}, \mu) = (\mathcal{F}, \delta)$. Consider the $G$-cover $f_\chi\colon X_\chi \longrightarrow Y \times V$ associated with $\chi$ as in 5.3.5 and let $U \subseteq V$ the smooth locus of the flat map $\pi \circ f_\chi\colon X_\chi \longrightarrow V$. We claim that we have to prove that $U \neq \emptyset$, so that we will assume $k$ algebraically closed. Indeed, since $k$ is infinite, there will exists $\delta \in U(k)$. If $f_\delta\colon X_\delta \longrightarrow Y$ is the base change $f_{\chi,\delta}\colon X_{\chi,\delta} \longrightarrow Y \times \{\delta\}$ then, by construction, $X_\delta$ is smooth and, taking into account 5.4.14, $\operatorname{codim}_Y Y_\delta = 2$ if $Y_\delta \neq \emptyset$. Moreover $\det \mathcal{F} \simeq \mathcal{L}$ and the last claim about connectedness follows by 5.4.23.

Given $\delta \in V$ we will denote by $f_\delta\colon X_\delta \longrightarrow Y$ the base change of $f_\chi$ over $Y \times \{\delta\}$. Since $\pi \circ f_\chi$ is flat, given $p \in X(k)$ and $\delta = \pi(f_\chi(p))$ we have

$$X \text{ regular in } p \iff X_\delta \text{ regular in } p$$

In particular, if $Z_X \subseteq X$ is the singular locus of $X$ and $Z = f_\chi(Z_X)$, then $(q, \delta) \in Z(k)$ if and only if $X_\delta$ has a singular point over $q \in Y$. Moreover $\pi(Z)$ is the complementary of $U$ in $V$. Therefore it is enough to prove that $\dim Z \leq \dim V - 1$,

If $q \in Y(k)$, then $g^{-1}(q) \cap Z \subseteq V$ is the locus of $\delta \in V(k)$ such that $X_\delta$ is not regular over $q$. In particular, if we denote by $\phi_q$ the map

$$\phi_q \colon V \longrightarrow \mathcal{E} \otimes (\mathcal{O}_{Y,q}/m_q^2)$$

and by $\overline{W}_q$ the subspace of $\mathcal{E} \otimes (\mathcal{O}_{Y,q}/m_q^2)$ of elements $\gamma$ such that there exists $\delta \in \phi_q^{-1}(\gamma)$ for which $X_\delta$ is not regular over $q$, then $\phi_q^{-1}(\overline{W}_q) = g^{-1}(q) \cap Z$. Indeed if $\delta, \delta' \in V$ are such that $\phi_q(\delta) = \phi_q(\delta')$, then $X_\delta$ is regular over $q$ if and only if $X_{\delta'}$ is so. Moreover $\overline{W}_q$ is contained in the locus defined in 5.4.22, where $R = \mathcal{O}_{Y,q}$, and therefore $\operatorname{codim}_{\mathcal{E}\otimes(\mathcal{O}_{Y,q}/m_q^2)} \overline{W}_q \geq \dim Y + 1$. Since $\phi_q$ is, by hypothesis, a surjective linear map of vector spaces, we also have $\operatorname{codim}_V(g^{-1}(q) \cap Z) \geq \dim Y + 1$. Therefore $\dim(g^{-1}(q) \cap Z) \leq \dim V - \dim Y - 1$ and

$$\dim Z \leq \dim Y + \dim V - \dim Y - 1 = \dim V - 1$$

□

### 5.4.4 Invariants of regular $S_3$-covers of surfaces.

The aim of this subsection is to compute the invariants of regular $S_3$-covers of surfaces over an algebraically closed field. Here and in the rest of the section by a surface over a field, we mean a projective, smooth and integral scheme of dimension 2. The result is:

**Theorem 5.4.24.** *Let $Y$ be a surface over an algebraically closed field $k$ such that $\operatorname{char} k \neq 2, 3$ and $f \colon X \longrightarrow Y$ be a regular $S_3$-cover associated with $(\mathcal{F}, \delta) \in \mathcal{C}_3$ as in 5.4.13. The closed subscheme $Y_\delta$ of $Y$ defined by the map $\eta_\delta \colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{O}_Y$ is the disjoint union of a divisor $D$ and a finite set $Y_0$ of rational points and $X$ is connected, that is a surface, if and only if $\mathrm{H}^0(\mathcal{F}) = 0$ and $\mathcal{O}_Y(-D) \not\simeq \det \mathcal{F}$. In this case the invariants of $X$ are given by*

$$\begin{aligned}
K_X^2 &= 6K_Y^2 + 6c_1(\mathcal{F})^2 - 12c_1(\mathcal{F})K_Y - \frac{10}{3}D^2 - 4DK_Y \\
p_g(X) &= p_g(Y) + 2h^2(\mathcal{F}) + h^2(\mathcal{O}_Y(D) \otimes \det \mathcal{F}) \\
\chi(\mathcal{O}_X) &= 6\chi(\mathcal{O}_Y) - 2c_2(\mathcal{F}) + \frac{1}{2}(3c_1(\mathcal{F})^2 - 3c_1(\mathcal{F})K_Y - DK_Y - D^2) \\
|Y_0| &= 3c_2(\mathcal{F}) - \frac{2}{3}D^2
\end{aligned}$$

Before proving this Theorem we need several lemmas.

**Lemma 5.4.25.** *Let $S$ be a finite disjoint union of integral schemes, $(\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle) \in \mathcal{Z}_G(S)$ and assume that $\beta$ is never zero, that $\langle -, - \rangle = 0$ and that $m$ is an isomorphism. Then there exist an isomorphism $\iota \colon \mathcal{L} \longrightarrow \mathcal{O}_S$ whose square is $m$ and a decomposition*

*$\mathcal{F} = \mathcal{H}_1 \oplus \mathcal{H}_2$ into invertible sheaves such that $\beta_{|\mathcal{H}_1^2}$ is an isomorphism $\mathcal{H}_1^2 \simeq \mathcal{H}_2$ and $\beta_{|\mathcal{H}_2^2}, \beta_{|\mathcal{H}_1\otimes\mathcal{H}_2} = 0$. In particular*

$$\operatorname{Coker}\beta \simeq \mathcal{H}_1 \text{ and } \det\mathcal{F} \simeq \mathcal{H}_1^3$$

*Proof.* What we will really prove is that $\operatorname{Im}\beta$ is an invertible sheaf, that the map $\mathcal{L} \longrightarrow \underline{\operatorname{End}}(\mathcal{F})$ induced by $\alpha$ yields an isomorphism $\iota\colon \mathcal{L} \longrightarrow \underline{\operatorname{End}}(\operatorname{Im}\beta) \simeq \mathcal{O}_S$, that $\iota^2 = m$, that $\alpha' = \alpha \circ (\iota^{-1} \otimes \mathrm{id}_{\mathcal{F}})\colon \mathcal{F} \longrightarrow \mathcal{F}$ is an isomorphism such that $\alpha'^2 = \mathrm{id}$ and that the decomposition into eigenspaces of $\alpha'$ is $\mathcal{F} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_i$ are invertible sheaves and $\mathcal{H}_2 = \operatorname{Im}\beta$. Those conditions and the requests of the statement can be checked locally.

So assume that $S$ is integral, that $\mathcal{L} = \mathcal{O}_S$ and that we have a basis $y, z = \beta(y^2)$ of $\mathcal{F}$. By 5.3.14, we have $B = \omega C^2 = 0$, $D = -A$ and therefore $A^2 = m$. Set $\iota = -A$, so that $\alpha' = \alpha/\iota$. Replacing $y$ by $y + (C/2A)z$, we get new parameters, although $A$ remains the same, such that $B = C = 0$, i.e. such that $\alpha$ is diagonal. Therefore $\alpha'^2 = \mathrm{id}$ and $\mathcal{H}_1 = \langle y\rangle, \mathcal{H}_2 = \langle z\rangle$ are the eigenspaces of $\alpha'$ with respect to the eigenvalues $-1, 1$ respectively. From relations (5.2.7) we see that $a = c = be = 0$. Since $z \in \operatorname{Im}\beta$, we get $e = 0$, $b \in \mathcal{O}_S^*$. In particular $\mathcal{H}_2 = \operatorname{Im}\beta$, which is invertible, $\beta_{|\mathcal{H}_1\otimes\mathcal{H}_2\oplus\mathcal{H}_2^2} = 0$ and $\beta_{|\mathcal{H}_1^2}$ is an isomorphism $\mathcal{H}_1^2 \simeq \operatorname{Im}\beta = \mathcal{H}_2$. □

We fix an integral, regular and noetherian scheme $Y$ with $\dim Y \geq 1$ and an object $\chi \in \mathscr{Z}_\omega(Y)$, $\chi = (\mathcal{M}, \mathcal{F}, \delta, \omega)$, whose induced $G$-cover, denoted by $f_\chi\colon X_\chi \longrightarrow Y$, is regular. We also denote by $\mathscr{A} = \mathscr{A}_\chi$ the algebra associated with $\chi$, i.e. $\mathscr{A} = f_{\chi*}\mathcal{O}_{X_\chi}$, and by

$$\mathcal{L}, m, \zeta, \alpha, \beta, \langle -, -\rangle, (-, -) \qquad D_m, D_\omega, D_\delta, Y_\alpha, Y_\delta$$

the objects associated with $\chi$ according to the inclusion $\mathscr{Z}_\omega(Y) \longrightarrow G\text{-}\mathrm{Cov}(Y)$ (see 5.3.29) and the closed subschemes of $Y$ defined in 5.4.2 and 5.4.11 respectively. We will often make use of Theorems 5.4.6, 5.4.13 and 5.4.14, which yield several conditions on the closed subschemes introduced above. In particular notice that $\beta$ is never zero. Therefore we will often consider basis of $\mathcal{F}$ of the form $y, \beta(y^2)$, thanks to 5.3.13, and the correspondent parameters associated with $\chi$, given in 5.3.14.

We are going to describe two exact sequences over $Y$, as the first step in the computation of the invariants of $X_\chi$.

*Remark* 5.4.26. If $i\colon Z \longrightarrow Y$ is a closed immersion of schemes defined by the sheaf of ideals $\mathcal{I}$ and $\mathcal{Q}$ is a coherent sheaf on $Y$ such that $\mathcal{Q}_p \simeq \mathcal{O}_{Y,p}/\mathcal{I}_p$ for any $p \in Y$, then $i^*\mathcal{Q}$ is an invertible sheaf on $Z$ and $\mathcal{Q} \simeq i_*i^*\mathcal{Q}$.

**Lemma 5.4.27.** *We have an exact sequence*

$$0 \longrightarrow (\det\mathcal{F})^2 \otimes \mathcal{M} \xrightarrow{\zeta} \operatorname{Sym}^2\mathcal{F} \xrightarrow{\beta} \mathcal{F} \longrightarrow i_*\mathcal{H} \oplus j_*\mathcal{Q} \longrightarrow 0 \tag{5.4.1}$$

*where $i\colon D_\omega \longrightarrow Y$, $j\colon Y_\alpha \longrightarrow Y$ are the immersions, $\mathcal{H}, \mathcal{Q}$ are invertible sheaves on $D_\omega$ and $Y_\alpha$ respectively and $\mathcal{H}^3 \simeq i^*\det\mathcal{F}$.*

*Proof.* We will prove that $\operatorname{Coker}\beta$ is schematically supported over $Y_\delta = D_\omega \sqcup Y_\alpha$ and therefore of the form $i_*\mathcal{H} \oplus j_*\mathcal{Q}$ where $i$ and $j$ are as in the statement and $\mathcal{H}, \mathcal{Q}$ are coherent sheaves on $D_\omega$ and $Y_\alpha$ respectively. By 5.4.25 we can then conclude that $\mathcal{H}^3 \simeq i^* \det \mathcal{F}$, since $D_\omega$ is regular. We can therefore work locally, i.e. assuming that $Y = \operatorname{Spec} R$, where $R$ is a local regular ring, and that we have a basis $y, z = \beta(y^2)$ of $\mathcal{F}$. In particular

$$\mathcal{F}/\operatorname{Im}\beta \simeq (Ry \oplus Rz)/(z, cy, ey - cz) \simeq R/(c, e)$$

So $\mathcal{F}/\operatorname{Im}\beta$ is an invertible sheaf on $Y_\delta = D_\omega \sqcup Y_\alpha$ and therefore can be written as $\mathcal{F}/\operatorname{Im}\beta \simeq i_*\mathcal{H} \oplus j_*\mathcal{Q}$, where $\mathcal{H}, \mathcal{Q}$ are invertible sheaves as in the statement.

By 5.3.31, we know that $\beta(\zeta) = 0$. By 5.3.6, we can write $\zeta = By^2 - 2Ayz - Cz^2$. If $\zeta = 0$, then $m = A^2 + \omega C^3 = 0$, which is not the case. It remains to check that $\operatorname{Ker}\beta$ is generated by $\zeta$. Note that $\beta$ is generically surjective, since $\chi$ is generically a $G$-torsor and thanks to 5.3.17. Therefore we have that $\operatorname{Ker}\beta = (k(R)\zeta) \cap \operatorname{Sym}^2 \mathcal{F} \subseteq \operatorname{Sym}^2 \mathcal{F} \otimes k(R)$ and we need to prove that if $u\zeta \in R^3$, with $u \in k(R)$, then $u \in R$. If $A$ or $C$ is invertible this is clear. If $A, C \in m_R$, then they are independent in $m_R/m_R^2$ and therefore different primes. Write $u = v/w$ with $v, w$ coprimes. By hypothesis, we have relations $vA = wr, vC = wr'$, with $r, r' \in R$. If $w$ is not invertible, any prime dividing $w$ will also divide $A$ and $C$, which is a contradiction. $\square$

In order to introduce the second exact sequence, we introduce the following notation. Set $\mathscr{B} = \mathscr{A}^\sigma \simeq \mathcal{O}_Y \oplus \mathcal{F}$ and $\pi' \colon X' = \operatorname{Spec}\mathscr{B} \longrightarrow Y$. The map $X_\chi \longrightarrow X'$ is a degree 2 cover and we denote by $\Delta$ the invertible sheaf over $\mathscr{B}$ inducing it.

*Remark* 5.4.28. Notice that $\Delta \subseteq \mathscr{A}$ is the eigenspace of $\sigma \in \mathbb{Z}/2\mathbb{Z}$ relative to $-1$. In particular

$$\mathcal{L} \oplus \mathcal{F} \simeq \Delta = \{0 \oplus s \oplus x \oplus (-x) \mid s \in \mathcal{L},\ x \in \mathcal{F}\} \subseteq \mathscr{A} = \mathcal{O}_Y \oplus \mathcal{L} \oplus \mathcal{F}_1 \oplus \mathcal{F}_2$$

Similarly to how we have identified $\mathscr{B} = \mathscr{A}^\sigma$ with $\mathcal{O}_Y \oplus \mathcal{F}$ (see 5.3.7), we will always identify $\mathcal{L} \oplus \mathcal{F}$ with $\Delta$ through the map $s \oplus x \longmapsto 0 \oplus s \oplus x \oplus (-x)$. Note that $\mathscr{A}_\chi = \mathscr{B} \oplus \Delta$. In particular we get a multiplication map $\Delta \otimes_{\mathscr{B}} \Delta \longrightarrow \mathscr{B}$ and it is easy to check that

$$m \oplus \alpha \oplus (\beta - \eta_\delta) \colon \operatorname{Sym}^2_{\mathcal{O}_Y} \Delta = \mathcal{L}^2 \oplus \mathcal{L} \otimes \mathcal{F} \oplus \operatorname{Sym}^2 \mathcal{F} \longrightarrow \Delta \otimes_{\mathscr{B}} \Delta \longrightarrow \mathscr{B}$$

Note that, in the above composition, the first map is surjective, while the second is injective because $f_\chi \colon X_\chi \longrightarrow Y$ is generically a $G$-torsor. Therefore we will identify the sheaf $\Delta \otimes_{\mathscr{B}} \Delta$ with the image of $m \oplus \alpha \oplus (\beta - \eta_\delta)$ in $\mathscr{B} = \mathcal{O}_Y \oplus \mathcal{F}$.

**Lemma 5.4.29.** *We have exact sequences*

$$0 \longrightarrow \mathcal{Q} \longrightarrow \Delta \otimes_{\mathscr{B}} \Delta \longrightarrow i_*\mathcal{O}_{D_\omega} \longrightarrow 0, \ 0 \longrightarrow \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow i_*\mathcal{K} \longrightarrow 0 \quad (5.4.2)$$

*where $i \colon D_\omega \longrightarrow Y$ is the inclusion and $\mathcal{K}$ is an invertible sheaf on $D_\omega$ such that $\mathcal{K}^3 \simeq i^* \det \mathcal{F}$.*

*Proof.* As explained in 5.4.28, $\Delta \otimes_{\mathscr{B}} \Delta$ can be identified to the image $\mathcal{G}$ of $m \oplus \alpha \oplus (\beta - \eta_\delta)$ in $\mathscr{B} = \mathcal{O}_Y \oplus \mathcal{F}$. Set

$$\mathcal{Q} = \operatorname{Im}((\beta - \eta_\delta) \oplus \alpha \colon \operatorname{Sym}^2 \mathcal{F} \oplus \mathcal{L} \otimes \mathcal{F} \longrightarrow \mathscr{B}) \subseteq \mathcal{G},\ \mathcal{H} = \operatorname{Coker}((\beta - \eta_\delta) \colon \operatorname{Sym}^2 \mathcal{F} \longrightarrow \mathcal{Q})$$

and $\mathcal{T} = \operatorname{Coker}(\mathcal{Q} \longrightarrow \mathcal{G})$. We have exact sequences

$$0 \longrightarrow \mathcal{Q} \longrightarrow \Delta \otimes_{\mathscr{B}} \Delta \longrightarrow \mathcal{T} \longrightarrow 0,\ \operatorname{Sym}^2 \mathcal{F} \xrightarrow{\beta - \eta_\delta} \mathcal{Q} \longrightarrow \mathcal{H} \longrightarrow 0$$

We first prove that $\mathcal{T} \simeq i_* \mathcal{O}_{D_\omega}$. By definition we have a surjective map $\mathcal{L}^2 \xrightarrow{m} \mathcal{G} \longrightarrow \mathcal{T}$ whose kernel is $m^{-1}(\mathcal{Q})$. Since locally $\hat{\eta_\delta} = \omega\zeta$ by 5.3.9, $\beta(\zeta) = 0$ by 5.3.31 and $\eta_\delta(\hat{\eta_\delta}) = -4\omega^2 m$ by 5.4.20 we have that $(\beta - \eta_\delta)(\hat{\eta_\delta}) = 4\omega^2 m \in \mathcal{Q}$, that $\omega^2 \in m^{-1}(\mathcal{Q})$ and therefore that $|\operatorname{Supp} \mathcal{T}| \subseteq |D_\omega|$. So $\mathcal{T}$ is supported in the locus where $m$ and therefore $\alpha$ are isomorphisms, in which we have $\mathcal{G} = \mathscr{B}$, $\mathcal{Q} = \operatorname{Im} \eta_\delta \oplus \mathcal{F}$, which yields the desired result.

We now consider $\mathcal{H}$. Assume that we have already proved that $\mathcal{H} \simeq i_* \mathcal{K}$, where $\mathcal{K}$ is an invertible sheaf over $D_\omega$. We want to prove that $\mathcal{K}^3 \simeq i^* \det \mathcal{F}$. Let

$$\overline{\mathcal{Q}} = \operatorname{Im}((\operatorname{Sym}^2 \mathcal{F} \oplus \mathcal{L} \otimes \mathcal{F}) \otimes \mathcal{O}_{D_\omega} \xrightarrow{(\beta - \eta_\delta) \oplus \alpha} \mathscr{B} \otimes \mathcal{O}_{D_\omega})$$

Since, on $D_\omega$, $\alpha$ is an isomorphism and $\eta_\delta = 0$, we get $\overline{\mathcal{Q}} = \mathcal{F} \otimes \mathcal{O}_{D_\omega}$. Moreover we have a surjective map $\mathcal{Q} \otimes \mathcal{O}_{D_\omega} \longrightarrow \overline{\mathcal{Q}}$ and a commutative diagram

$$\begin{tikzcd}
\operatorname{Sym}^2 \mathcal{F} \otimes \mathcal{O}_{D_\omega} \arrow[r] \arrow[dr, "\beta \otimes \mathcal{O}_{D_\omega}"'] & \mathcal{Q} \otimes \mathcal{O}_{D_\omega} \arrow[r] \arrow[d] & \mathcal{H} \otimes \mathcal{O}_{D_\omega} \simeq \mathcal{K} \arrow[r] \arrow[d] & 0 \\
 & \mathcal{F} \otimes \mathcal{O}_{D_\omega} \arrow[r] & \widehat{\mathcal{K}} &
\end{tikzcd}$$

where $\widehat{\mathcal{K}} = \operatorname{Coker}(\beta \otimes \mathcal{O}_{D_\omega})$. Thanks to 5.4.25, $\widehat{\mathcal{K}}$ is an invertible sheaf on $D_\omega$ such that $\widehat{\mathcal{K}}^3 \simeq i^* \det \mathcal{F}$ and the surjective map $\mathcal{K} \longrightarrow \widehat{\mathcal{K}}$ is an isomorphism. In order to prove that $\mathcal{H} \simeq i_* \mathcal{K}$, we can work on a regular local ring $R$. Considering a basis $y, z = \beta(y^2)$ of $\mathcal{F}$ and basis $1, y, z$ of $\mathcal{O}_Y \oplus \mathcal{F}$ we have

$$\beta - \eta_\delta = \begin{pmatrix} -2c & -e & -2c^2 \\ 0 & c & e \\ 1 & 0 & -c \end{pmatrix},\ e = 2\omega A,\ c = -\omega C,\ B = \omega C^2,\ m = A^2 + \omega C^3$$

In particular $\det(\beta - \eta_\delta) = 4c^3 - e^2 = -4\omega^2 m$ and therefore $\beta - \eta_\delta$ is injective. In particular if both $\omega$ and $m$ are invertible, then $\beta - \eta_\delta$ is an isomorphism and therefore $\mathcal{H} = 0$. In particular we can assume that $\omega$ or $m$ is not invertible. By definition we have a surjective map $\mathcal{L} \otimes \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow \mathcal{H}$ whose kernel is $\alpha^{-1}(\operatorname{Im}(\beta - \eta_\delta))$. Given $x = uy + vz \in \mathcal{L} \otimes \mathcal{F}$ we want to check when there exists $x' = wy^2 + gyz + hz^2 \in \operatorname{Sym}^2 \mathcal{F}$ such that $(\beta - \eta_\delta)(x') = \alpha(x)$, i.e.

$$u(Ay + Cz) + v(By - Az) = (-2cw - eg - 2c^2 h) + (gc + he)y + (w - hc)z$$

which translates in the system of 3 equations

$$-2cw - eg - 2c^2h = 0,\ uA + vB = gc + he,\ uC - vA = w - hc$$

We first get $w = uC + hc - vA$ and our equations become

$$-4c^2h - 2ucC + 2vcA - eg = 0,\ uA + vB = gc + he$$

First note that if $e = 0 \in m_R$, then $A = 0$ and if $C \in m_R$ then $R/(A, C)$ cannot have codimension 2. So $C$ is invertible and, since $4\omega^2 m = -4c^3$, $\omega$, $c$ and $m$ differs by an invertible element. From $D_\omega \cap D_m = \emptyset$, we can conclude that both $\omega$ and $m$ are invertible. Therefore we have $e \neq 0$. We can write

$$h = (uA + vB - gc)/e$$

and substituting in the first equation we get

$$g(4c^3 - e^2) = u(4c^2A + 2ceC) + v(4c^2B - 2ceA)$$

Now note that $4c^3 - e^2 = -4\omega^2 m$, $4c^2A + 2ceC = 0$ and $4c^2B - 2ceA = 4\omega^2 Cm$ and so the above equation become $4\omega^2 m(g + Cv)$ whose unique solution is $g = -Cv$. In particular $vB - gc = 0$ and our last equation is $h = uA/e = u/2\omega$. So $\alpha(uy + vz)$ is in the image of $\beta - \eta_\delta$ if and only if $\omega \mid u$, which implies that $\mathcal{H} \simeq R/(\omega)$. □

From now on we assume that $Y$ is a surface over an algebraically closed field $k$. We write $\mu = c_1(\mathcal{M}) = D_\omega$, $c_1 = c_1(\mathcal{F})$, $c_2 = c_2(\mathcal{F})$ and $K_Y = K$, the canonical divisor of $Y$.

*Remark* 5.4.30. We have $\mu c_1 = -\mu^2$. Indeed from $D_m \cap D_\omega = \emptyset$, we get $\mu c_1(\mathcal{L}) = 0$. On the other hand we have $\mathcal{L} \simeq \mathcal{M} \otimes \det \mathcal{F}$.

**Lemma 5.4.31.** *Let $\mathcal{H}$ be an invertible sheaf on $D_\omega$ such that $\mathcal{H}^3 \simeq \det \mathcal{F}^l \otimes \mathcal{O}_{D_\omega}$. Then*

$$\chi(\mathcal{H}) = -\frac{2l+3}{6}\mu^2 - \frac{\mu K}{2}$$

*Proof.* Let $\mu = D_\omega = D_1 + \cdots + D_s$ be the decomposition into smooth integral components and set $\mathcal{H}_i = \mathcal{H} \otimes \mathcal{O}_{D_i}$ and $i \colon D_\omega \longrightarrow Y$ for the inclusion. Thanks to 5.4.25, $i^*\mathcal{L} \simeq \mathcal{O}_{D_\omega}$ and therefore $i^* \det \mathcal{F} \simeq i^*\mathcal{M}^{-1}$. In particular

$$\mathcal{H}_i^3 \simeq (\det \mathcal{F})^l \otimes \mathcal{O}_{D_i} \simeq \mathcal{M}^{-l} \otimes \mathcal{O}_{D_i} \simeq (\mathcal{O}_Y(-D_i) \otimes \mathcal{O}_{D_i})^l \implies \deg_{D_i} \mathcal{H}_i = -lD_i^2/3$$

By adjunction formula we also have $2(g(D_i) - 1) = D_i^2 + D_i.K$ and moreover, by Riemann-Roch,

$$\chi_{D_i}(\mathcal{H}_i) = -lD_i^2/3 - D_i^2/2 - D_i.K/2 = -\frac{2l+3}{6}D_i^2 - \frac{D_i.K}{2}$$

Summing over all $i$ and taking into account the relation $\mu^2 = D_\omega^2 = \sum_i D_i^2$ we get the result. □

*Remark* 5.4.32. Let $\mathcal{E}$ be a locally free sheaf of rank $r$ and $\mathcal{L}$ be an invertible sheaf over $Y$. We recall the following well known formulas for the Chern classes. In particular the last one is Riemann-Roch for surfaces.

$$c_1(\mathcal{E}\otimes\mathcal{L}) = rc_1(\mathcal{L}) + c_1(\mathcal{E}), c_2(\mathcal{E}\otimes\mathcal{L}) = \frac{r(r-1)}{2}c_1(\mathcal{L})^2 + (r-1)c_1(\mathcal{L})c_1(\mathcal{E}) + c_2(\mathcal{E})$$

$$c_1(\operatorname{Sym}^2\mathcal{E}) = 3c_1(\mathcal{E}), c_2(\operatorname{Sym}^2\mathcal{E}) = 2c_1(\mathcal{E})^2 + 4c_2(\mathcal{E}) \text{ if } r = 2$$

$$\chi(\mathcal{E}) = \frac{c_1(\mathcal{E})^2 - 2c_2(\mathcal{E}) - c_1(\mathcal{E})K}{2} + r\chi(\mathcal{O}_Y)$$

**Lemma 5.4.33.** *We have*

$$|Y_\alpha| = 3c_2 - \frac{2}{3}\mu^2$$

*Proof.* Consider the operator

$$\Phi = 2(\chi - \chi(\mathcal{O}_Y)\operatorname{rk})$$

on coherent sheaves, which is additive on exact sequences. We will apply it on the exact sequence (5.4.1). We have $c_1(\det\mathcal{F}^2\otimes\mathcal{M}) = 2c_1+\mu$ and

$$\Phi(\det\mathcal{F}^2\otimes\mathcal{M}) = (2c_1+\mu)^2-(2c_1+\mu)K = 4c_1^2+\mu^2+4\mu c_1-2c_1K-\mu K = 4c_1^2-2c_1K-3\mu^2-\mu K$$

where we have used that $\mu c_1 = -\mu^2$. By 5.4.31, we get $\Phi(i_*\mathcal{H}) = 2\chi(\mathcal{H}) = -5\mu^2/3-\mu K$. Moreover

$$\Phi(\operatorname{Sym}^2\mathcal{F}) = 9c_1^2 - 2(2c_1^2+4c_2) - 3c_1K = 5c_1^2 - 8c_2 - 3c_1K$$

Finally, since $Y_\alpha$ is regular of dimension 0, we obtain

$$2|Y_\alpha| = \Phi(j_*\mathcal{Q}) = \Phi(\det\mathcal{F}^2\otimes\mathcal{M}) - \Phi(\operatorname{Sym}^2\mathcal{F}) + \Phi(\mathcal{F}) - \Phi(\mathcal{H}) = 6c_2 - 4\mu^2/3$$

□

**Lemma 5.4.34.** *We have*

$$2\chi(\Delta\otimes_{\mathscr{B}}\Delta) = 5c_1^2 - 8c_2 - 3c_1K - 8\mu^2/3 - 2\mu K + 6\chi(\mathcal{O}_Y)$$

*Proof.* Consider the exact sequence 5.4.2. Taking into account 5.4.31, the result follows from the relations $2\chi(\operatorname{Sym}^2\mathcal{F}) = 5c_1^2 - 8c_2 - 3c_1K + 6\chi(\mathcal{O}_Y)$, $2\chi(\mathcal{O}_{D_\omega}) = -\mu^2 - \mu k$, $2\chi(\mathcal{K}) = -5\mu^2/3 - \mu k$. □

**Lemma 5.4.35.** *Let $f\colon X \longrightarrow X'$ be a degree 2 cover between surfaces and write $f_*\mathcal{O}_X = \mathcal{O}_{X'}\oplus\mathcal{W}$, where $\mathcal{W}$ is the invertible sheaf inducing $f$. Then*

$$K_X^2 = 2K_{X'}^2+2c_1(\mathcal{W})^2-4c_1(\mathcal{W})K_{X'},\ p_g(X) = p_g(X')+h^2(\mathcal{W}),\ \chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X'})+\chi(\mathcal{W})$$

*Proof.* The last two formulas are clear. Therefore we focus on the first. Set $\mathscr{B} = f_*\mathcal{O}_X$. The map $\mathcal{W}^{-1} \longrightarrow \mathscr{B}^\vee \simeq f_*\omega_{X/X'}$ induces a map

$$f^*\mathcal{W}^{-1} \longrightarrow f^*f_*\omega_{X/X'} \longrightarrow \omega_{X/X'}$$

We want to prove that this map is surjective and therefore an isomorphism. Locally, $\mathcal{W} = \mathcal{O}_{X'}t$, $t^2 = m \in \mathcal{O}_{X'}$, $\mathcal{W}^{-1} = \mathcal{O}_{X'}t^*$. Since $t \cdot t^* = 1^*$, where $t^* \in \mathscr{B}^\vee$, we see that $t^*$ generates $\mathscr{B}^\vee$ as a $\mathscr{B}$-module. Thus $f^*\mathcal{W}^{-1} \simeq \omega_{X/X'}$ and

$$\omega_X \simeq f^*\omega_{X'}\otimes\omega_{X/X'} \simeq f^*(\omega_{X'}\otimes\mathcal{W}^{-1}) \implies f_*\omega_X^{-1} \simeq \omega_{X'}^{-1}\otimes\mathcal{W}\otimes\mathscr{B} \simeq \omega_{X'}^{-1}\otimes\mathcal{W}\oplus\omega_{X'}^{-1}\otimes\mathcal{W}^2$$

Now note that if $Z$ is a surface and $D$ is a divisor of it, by Riemann-Roch formula we have

$$2\chi(nD - K_Z) = n^2D^2 - 3nDK_Z + 2K_Z^2 + 2\chi(\mathcal{O}_Z)$$

If we write $\mathcal{W} = \mathcal{O}_Y(C)$, the result comes from the following relations

$$\chi(-K_X) = \chi(f_*\omega_X^{-1}) = \chi(C - K_{X'}) + \chi(2C - K_{X'}) = (5C^2 - 9CK_{X'})/2 + 2K_{X'}^2 + 2\chi(\mathcal{O}_{X'})$$
$$\chi(-K_X) = K_X^2 + \chi(\mathcal{O}_X) = K_X^2 + \chi(\mathcal{W}) + \chi(\mathcal{O}_{X'}) = K_X^2 + (C^2 - CK_{X'})/2 + 2\chi(\mathcal{O}_{X'})$$

□

**Lemma 5.4.36.** *Let $f\colon X' \longrightarrow Y$ be a degree 3 cover between surfaces induced by $(\mathcal{F}, \delta) \in \mathcal{C}_3(Y)$. Then we have*

$$K_{X'}^2 = 3K_Y^2 - 4c_1(\mathcal{F})K_Y + 2c_1(\mathcal{F})^2 - 3c_2(\mathcal{F}),\ \chi(\mathcal{O}_{X'}) = \chi(\mathcal{O}_Y) + \chi(\mathcal{F}),\ p_g(X') = p_g(Y) + h^2(\mathcal{F})$$

*Proof.* The last two formulas are clear. The first one instead is proved in [Par89, Corollary 8.3] or [Mir85, Proposition 10.3]. □

*Proof.* (*of Theorem* 5.4.24) The claim about connectedness follows from 5.4.23. The formula for $|Y_0|$ is given in 5.4.33, while the formula for $p_g$ is clear since $f_*\mathcal{O}_X \simeq \mathcal{O}_Y \oplus (\mathcal{M} \otimes \det \mathcal{F}) \oplus \mathcal{F} \oplus \mathcal{F}$. This relation, Riemann-Roch formula and the computation

$$\chi(\mathcal{L}) = \chi(\mathcal{M}\otimes\det\mathcal{F}) = ((\mu+c_1)^2 - (\mu+c_1)K)/2 + \chi(\mathcal{O}_Y) = (-\mu^2 - \mu k + c_1^2 - c_1K)/2 + \chi(\mathcal{O}_Y)$$

yields the formula for $\chi(\mathcal{O}_X)$. Therefore we focus on the formula for $K_X^2$. The map $f\colon X \longrightarrow Y$ factors as $X \xrightarrow{f_2} X' \xrightarrow{f_3} Y$ where $f_2$, $f_3$ are covers of degree $2, 3$ respectively and $X'$ is a surface, thanks to 5.4.14. Moreover, by definition, $f_2$ is induced by the invertible sheaf $\Delta$ on $X'$, while $f_3$ is induced by $(\mathcal{F}, \delta) \in \mathcal{C}_3$. Notice that $X'$ is a complete smooth surface. Set $\Delta = \mathcal{O}_{X'}(-C)$, where $C$ is a divisor over $X'$ and set $\Delta^2 = \Delta\otimes_{\mathcal{O}_{X'}}\Delta$. Recall that $\Delta \simeq \mathcal{L} \oplus \mathcal{F}$. By 5.4.36 and 5.4.35 we have

$$K_X^2 = 2K_{X'}^2 + 2C^2 + 4CK_{X'} = 6K^2 - 8c_1K + 4c_1^2 - 6c_2 + 2C^2 + 4CK_{X'} \tag{5.4.3}$$

By Riemann-Roch and the definition of the intersection product we also get

$$2(\chi(\Delta) - \chi(\mathcal{O}_{X'})) = C^2 + CK_{X'},\ C^2 = \chi(\Delta^2) - 2\chi(\Delta) + \chi(\mathcal{O}_{X'})$$

In particular $CK_{X'} = 4\chi(\Delta) - \chi(\Delta^2) - 3\chi(\mathcal{O}_{X'})$. Putting everything together and using 5.4.36 and 5.4.34 we obtain

$$\begin{aligned} 2C^2 + 4CK_{X'} &= -2\chi(\Delta^2) + 12\chi(\Delta) - 10\chi(\mathcal{O}_{X'}) = -2\chi(\Delta^2) + 12\chi(\mathcal{L}) + 2\chi(\mathcal{F}) - 10\chi(\mathcal{O}_Y) \\ &= -5c_1^2 + 8c_2 + 3c_1K + 8\mu^2/3 + 2\mu K - 6\chi(\mathcal{O}_Y) - 6\mu^2 - 6\mu K + 6c_1^2 - 6c_1K + \\ & 12\chi(\mathcal{O}_Y) + c_1^2 - 2c_2 - c_1K + 4\chi(\mathcal{O}_Y) - 10\chi(\mathcal{O}_Y) \\ &= 2c_1^2 + 6c_2 - 4c_1K - 10\mu^2/3 - 4\mu K \end{aligned}$$

Substituting in 5.4.3 we get the desired result. $\square$

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