To logconcavity and beyond

Kazuhiro Ishige, Paolo Salani and Asuka Takatsu

Abstract

In 1976 Brascamp and Lieb proved that the heat flow preserves logconcavity and that the first positive Dirichlet eigenfunction for the Laplace operator in a bounded convex domain is logconcave. In this paper, introducing a variation of concavity, we show that the heat flow preserves stronger concavity than logconcavity and we identify the strongest concavity preserved by the heat flow.

Addresses:

K. I.: Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan.
E-mail: ishige@ms.u-tokyo.ac.jp

P. S.: Dipartimento di Matematica “U. Dini”, Università di Firenze, viale Morgagni 67/A, 50134 Firenze
E-mail: paolo.salani@unifi.it

A. T.: Department of Mathematical Sciences, Tokyo Metropolitan University, Minami-osawa, Hachioji-shi, Tokyo 192–0397, Japan
E-mail: asuka@tmu.ac.jp

1 Introduction

A nonnegative function u in $\mathbf{R}^N$ is said *logconcave* in $\mathbf{R}^N$ if

$$u((1-\mu)x+\mu x) \ge u(x)^{1-\mu}u(y)^{\mu}$$

for $\mu \in [0,1]$ and x, $y \in \mathbf{R}^N$ such that $u(x)u(y) > 0$. This is equivalent to that the set $S_u := \{x \in \mathbf{R}^N : u(x) > 0\}$ is convex and $\log u$ is concave in S_u. Logconcavity is a very useful variation of concavity and plays an important role in various fields such as PDEs, geometry, probability, statics, optimization theory and so on (see e.g. [17]). Most of its relevance, especially for elliptic and parabolic equations, is due to the fact that the Gauss kernel

$$G(x,t) := (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right) \tag{1.1}$$

is logconcave in $\mathbf{R}^N$ for any fixed $t > 0$. Indeed,

$$\log G(x,t) = -\frac{|x|^2}{4t} + \log(4\pi t)^{-\frac{N}{2}} \tag{1.2}$$

is concave in $\mathbf{R}^N$ for any fixed $t > 0$. Exploiting the logconcavity of the Gauss kernel, Brascamp and Lieb [4] proved that logconcavity is preserved by the heat flow and they also obtained the logconcavity of the first positive Dirichlet eigenfunction for the Laplace operator $-\Delta$ in a bounded convex domain. (See also [7, 13].) For later convenience, we state explicitly these two classical results below.

(a) Let u be a bounded nonnegative solution of

$$\begin{cases} \partial_t u = \Delta u & \text{in} \quad \Omega \times (0,\infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \quad \text{if} \quad \partial\Omega \neq \emptyset, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.3}$$

where Ω is a convex domain in $\mathbf{R}^N$ and u_0 is a bounded nonnegative function in Ω. Then $u(\cdot,t)$ is logconcave in Ω for any $t > 0$ if u_0 is logconcave in Ω.

(b) Let Ω be a bounded convex domain in $\mathbf{R}^N$ and λ_1 the first Dirichlet eigenvalue for the Laplace operator in Ω. If ϕ solves

$$\begin{cases} -\Delta\phi = \lambda_1 \phi & \text{in} \quad \Omega, \\ \phi = 0 & \text{on} \quad \partial\Omega, \\ \phi > 0 & \text{in} \quad \Omega, \end{cases} \tag{1.4}$$

then ϕ is logconcave in Ω.

We denote by $e^{t\Delta_\Omega}u_0$ the (unique) solution to problem (1.3). In particular, in the case of $\Omega = \mathbf{R}^N$, we write $e^{t\Delta}u_0 = e^{t\Delta_{\mathbf{R}^N}}u_0$, that is,

$$[e^{t\Delta}u_0](x) = \int_{\mathbf{R}^N} G(x-y,t)u_0(y)\,dy, \quad x \in \mathbf{R}^N,\ t > 0. \tag{1.5}$$

Logconcavity is so naturally and deeply linked to heat transfer that $e^{t\Delta}u_0$ spontaneously becomes logconcave in $\mathbf{R}^N$ even without the logconcavity of initial function u_0. Indeed, Lee and Vázquez [14] proved the following:

(c) Let u_0 be a bounded nonnegative function in $\mathbf{R}^N$ with compact support. Then there exists $T > 0$ such that $e^{t\Delta}u_0$ is logconcave in $\mathbf{R}^N$ for $t \geq T$. (See [14, Theorem 5.1].)

Due to the above reasons, logconcavity is commonly regarded as the optimal concavity for the heat flow and for the first positive Dirichlet eigenfunction for $-\Delta$.

In this paper we dare to ask the following question:

(**Q1**) Is logconcavity the strongest concavity preserved by the heat flow in a convex domain? If not, what is the strongest concavity preserved by the heat flow?

We introduce a new variation of concavity and give answers to (**Q1**). More precisely, we introduce α-logconcavity as a refinement of p-concavity at $p = 0$ (see Section 2) and show that the heat flow preserves 2-logconcavity (see Theorem 3.1). Here 2-logconcavity is stronger than usual logconcavity and we prove that 2-logconcavity is exactly the strongest concavity property preserved by the heat flow (see Theorem 3.2).

Another natural question which spontaneously arises after (**Q1**) is the following: *is logconcavity the strongest concavity shared by the solution ϕ of* (1.4) *for any bounded convex domain Ω?* We are not able to give here an exhaustive answer to this question, but we conjecture that it is negative and that also the first positive Dirichlet eigenfunction for $-\Delta$ in every convex domain is 2-logconcave. See Remark 4.2 about this.

The rest of this paper is organized as follows. In Section 2 we introduce a new variation of concavity and prove some lemmas. In particular, we show that 2-logconcavity is the strongest concavity for the Gauss kernel $G(\cdot, t)$ to satisfy. In Section 3 we state the main results of this paper. The proofs of the main results are given in Sections 4 and 5.

2 Logarithmic power concavity

For $x \in \mathbf{R}^N$ and $R > 0$, set $B(x, R) := \{y \in \mathbf{R}^N : |x - y| < R\}$. For any measurable set E, we denote by χ_E the characteristic function of E. Furthermore, for any function u in a set Ω in $\mathbf{R}^N$, we say that U is the *zero extension* of u if $U(x) = u(x)$ for $x \in \Omega$ and $U(x) = 0$ for $x \notin \Omega$. We often identify u with its zero extension U. A function $f : \mathbf{R}^N \to \mathbf{R} \cup \{-\infty\}$ is a (proper) *concave* function if $f((1-\mu)x + \mu y) \geq (1-\mu)f(x) + \mu f(y)$ for $x, y \in \mathbf{R}^N$, $\mu \in [0, 1]$ (and $f(x) > -\infty$ for at least one $x \in \mathbf{R}^N$). Here we deal with $-\infty$ in the obvious way, that is: $-\infty + a = -\infty$ for $a \in \mathbf{R}$ and $-\infty \geq -\infty$.

Let $G = G(x, t)$ be the Gauss kernel (see (1.1)). Similarly to (1.2), for any fixed $t > 0$, it follows that

$$-\left(-\log\left[\kappa G(x,t)\right]\right)^{\frac{1}{2}} = -\left[\frac{|x|^2}{4t} - \log\left((4\pi t)^{-\frac{N}{2}}\kappa\right)\right]^{\frac{1}{2}}$$

is still concave in $\mathbf{R}^N$ for any sufficiently small $\kappa > 0$. Motivated by this, we formulate a definition of α-logconcavity ($\alpha > 0$). Let $L_\alpha = L_\alpha(s)$ be a strictly increasing function on $[0,1]$ defined by

$$L_\alpha(s) := -(-\log s)^{\frac{1}{\alpha}} \quad \text{for} \quad s \in (0,1], \qquad L_\alpha(s) := -\infty \quad \text{for} \quad s = 0.$$

Definition 2.1 *Let $\alpha > 0$.*
(i) *Let u be a bounded nonnegative function in $\mathbf{R}^N$. We say that u is α-logconcave in $\mathbf{R}^N$ if*

$$L_\alpha(\kappa u((1-\mu)x + \mu y)) \ge (1-\mu)L_\alpha(\kappa u(x)) + \mu L_\alpha(\kappa u(y)), \quad x, y \in \mathbf{R}^N, \ \mu \in [0,1],$$

for all sufficiently small $\kappa > 0$.
(ii) *Let u be a bounded nonnegative function in a convex set Ω in $\mathbf{R}^N$. We say that u is α-logconcave in Ω if the zero extension of u is α-logconcave in $\mathbf{R}^N$.*

Definition 2.1 means that, for any bounded nonnegative function u in a convex set Ω, u is α-logconcave in Ω if and only if the function $L_\alpha(\kappa U)$ is concave in $\mathbf{R}^N$ for all sufficiently small $\kappa > 0$. Due to Definition 2.1, we easily see the following properties:

- Logconcavity corresponds to 1-logconcavity;
- If u is α-logconcave in Ω for some $\alpha > 0$, then κu is also α-logconcave in Ω for any $\kappa > 0$;
- If $0 < \alpha \le \beta$ and u is β-logconcave in Ω, then u is α-logconcave in Ω;
- The Gauss kernel $G(\cdot, t)$ is 2-logconcave in $\mathbf{R}^N$ for any $t > 0$.

Furthermore, we have:

Lemma 2.1 *Let $\alpha \ge 1$. Let u be a function in a convex set Ω such that $0 \le u \le 1$ in Ω. If $L_\alpha(u)$ is concave in Ω, then $L_\alpha(\kappa u)$ is also concave in Ω for any $0 < \kappa \le 1$.*

Proof. Let x, $y \in \Omega$ and $\mu \in [0,1]$. Assume that $u(x)u(y) > 0$. Since $L_\alpha^{-1}(s) = \exp(-(-s)^\alpha)$ for $s \in (-\infty, 0]$, we find

$$\begin{aligned}\psi(\kappa) :=& \kappa^{-1} L_\alpha^{-1} \left[(1-\mu)L_\alpha(\kappa u(x)) + \mu L_\alpha(\kappa u(y))\right] \\ =& \kappa^{-1} \exp\left\{-\left[(1-\mu)(-\log \kappa u(x))^{\frac{1}{\alpha}} + \mu(-\log \kappa u(y))^{\frac{1}{\alpha}}\right]^\alpha\right\}\end{aligned}$$

for $0 < \kappa \le 1$. Since $L_\alpha(u)$ is concave in Ω, it follows that

$$(1-\mu)u(x) + \mu u(y) \ge \psi(1). \tag{2.1}$$

For a, $b > 0$ and $\gamma \in (-\infty, \infty)$, set

$$M_\gamma(a, b; \mu) := \left[(1-\mu)a^\gamma + \mu b^\gamma\right]^{\frac{1}{\gamma}}.$$

Then

$$
\begin{aligned}
\psi'(\kappa) = & -\kappa^{-2}\exp\left\{-M_{\frac{1}{\alpha}}(-\log\kappa u(x), -\log\kappa u(y);\mu)\right\} \\
& + \kappa^{-1}\exp\left\{-M_{\frac{1}{\alpha}}(-\log\kappa u(x), -\log\kappa u(y);\mu)\right\} \\
& \times\left[(1-\mu)(-\log\kappa u(x))^{\frac{1}{\alpha}} + \mu(-\log\kappa u(y))^{\frac{1}{\alpha}}\right]^{\alpha-1} \\
& \times\kappa^{-1}\left[(1-\mu)(-\log\kappa u(x))^{\frac{1-\alpha}{\alpha}} + \mu(-\log\kappa u(y))^{\frac{1-\alpha}{\alpha}}\right] \\
= & \ \kappa^{-2}\exp\left\{-M_{\frac{1}{\alpha}}(-\log\kappa u(x), -\log\kappa u(y);\mu)\right\} \\
& \times\left[-1 + M_{\frac{1}{\alpha}}(-\log\kappa u(x), -\log\kappa u(y);\mu)^{\frac{\alpha-1}{\alpha}} M_{\frac{1-\alpha}{\alpha}}(-\log\kappa u(x), -\log\kappa u(y);\mu)^{\frac{1-\alpha}{\alpha}}\right]
\end{aligned}
\tag{2.2}
$$

for $0 < \kappa \le 1$. On the other hand, since $\alpha \ge 1$, it follows that $1/\alpha \ge (1-\alpha)/\alpha$. Then the Jensen inequality yields

$$
M_{\frac{1}{\alpha}}(a, b;\mu) \ge M_{\frac{1-\alpha}{\alpha}}(a, b;\mu) \quad \text{for} \quad a, b > 0.
$$

This together with (2.2) implies that $\psi'(\kappa) \ge 0$ for $0 < \kappa \le 1$. Therefore, by (2.1) we obtain

$$
(1-\mu)u(x) + \mu u(y) \ge \psi(\kappa) = \kappa^{-1}L_\alpha^{-1}\left[(1-\mu)L_\alpha(\kappa u(x)) + \mu L_\alpha(\kappa u(y))\right] \tag{2.3}
$$

for $0 < \kappa \le 1$ in the case of $u(x)u(y) > 0$. In the case of $u(x)u(y) = 0$, by the definition of L_α we easily obtain (2.3) for $0 < \kappa \le 1$. These mean that $L_\alpha(\kappa u)$ is concave in Ω for $0 < \kappa \le 1$. Thus Lemma 2.1 follows. $\square$

Remark 2.1 (i) *Let u be a bounded nonnegative function in a convex set Ω and $\alpha > 0$. We say that u is weakly α-logconcave in Ω if*

$$
L_\alpha(\kappa U((1-\mu)x + \mu y)) \ge (1-\mu)L_\alpha(\kappa U(x)) + \mu L_\alpha(\kappa U(y)), \quad x, y \in \mathbf{R}^N,\ \mu \in [0,1],
$$

for some $\kappa > 0$. Here U is the zero extension of u. Lemma 2.1 *implies that α-logconcavity is equivalent to weak α-logconcavity in the case of $\alpha \ge 1$.*

(ii) *For $0 < \alpha < 1$, α-logconcavity is not equivalent to weak α-logconcavity. Indeed, set $u(x) := \exp(-|x|^\alpha)\chi_{B(0,R)}$, where $0 < R \le \infty$. Then $L_\alpha(u) = -|x|$ is concave in $B(0,R)$. On the other hand, for any $0 < \kappa < 1$, we have*

$$
\begin{aligned}
\frac{\partial}{\partial r}L_\alpha(\kappa u(x)) &= -(-\log\kappa + |x|^\alpha)^{-1+\frac{1}{\alpha}}|x|^{-1+\alpha}, \\
\frac{\partial^2}{\partial r^2}L_\alpha(\kappa u(x)) &= (\alpha-1)(-\log\kappa + |x|^\alpha)^{-2+\frac{1}{\alpha}}|x|^{-2+2\alpha} + (1-\alpha)(-\log\kappa + |x|^\alpha)^{-1+\frac{1}{\alpha}}|x|^{-2+\alpha} \\
&= (1-\alpha)(-\log\kappa + |x|^\alpha)^{-2+\frac{1}{\alpha}}|x|^{-2+\alpha}(-\log\kappa) > 0,
\end{aligned}
$$

for $x \in B(0,R)\setminus\{0\}$, where $r := |x| > 0$. This means that $L_\alpha(\kappa u)$ is not concave in $B(0,R)$ for any $0 < \kappa < 1$.

Next we introduce the notion of F-concavity, which generalises and embraces all the notions of concavity we have already seen.

Definition 2.2 *Let Ω be a convex set in $\mathbf{R}^N$.*
(i) *A function $F : [0,1] \to \mathbf{R} \cup \{-\infty\}$ is said admissible if F is strictly increasing continuous in $(0,1]$, $F(0) = -\infty$ and $F(s) \neq -\infty$ for $s > 0$.*
(ii) *Let F be admissible. Let u be a bounded nonnegative function in Ω and U the zero extension of u. Then u is said F-concave in Ω if $0 \le \kappa U(x) \le 1$ in $\mathbf{R}^N$ and*

$$F(\kappa U((1-\mu)x + \mu y)) \ge (1-\mu)F(\kappa U(x)) + \mu F(\kappa U(y)), \quad x, y \in \mathbf{R}^N, \ \mu \in [0,1],$$

for all sufficiently small $\kappa > 0$. We denote by $\mathcal{C}_\Omega[F]$ the set of F-concave functions in Ω. Furthermore, in the case of $\Omega = \mathbf{R}^N$, we write $\mathcal{C}[F] = \mathcal{C}_\Omega[F]$ for simplicity.
(iii) *Let F_1 and F_2 be admissible. We say that F_1-concavity is stronger than F_2-concavity in Ω if $\mathcal{C}_\Omega[F_1] \subsetneq \mathcal{C}_\Omega[F_2]$.*

We recall that a bounded nonnegative function u in a convex set Ω is said *p-concave* in Ω, where $p \in \mathbf{R}$, if u is F-concave with $F = F_p$ in Ω, where

$$F_p(s) := \begin{cases} \dfrac{1}{p} s^p & \text{for } s > 0 \text{ if } p \neq 0, \\ \log s & \text{for } s > 0 \text{ if } p = 0, \\ -\infty & \text{for } s = 0. \end{cases}$$

Here 1-concavity corresponds to usual concavity while 0-concavity corresponds to usual log-concavity (in other words, 1-logconcavity). Furthermore, u is said *quasiconcave* or *$-\infty$-concave* in Ω if all superlevel sets of u are convex, while it is said *∞-concave* in Ω if u satisfies

$$u((1-\mu)x + \mu y) \ge \max\{u(x), u(y)\}$$

for x, $y \in \Omega$ with $u(x)u(y) > 0$ and $\mu \in [0,1]$. Then, by the Jensen inequality we have:

- Let $-\infty \le p \le q \le \infty$. If u is q-concave in a convex set Ω, then u is also p-concave in Ω.

Among concavity properties, apart from usual concavity, of course logconcavity has been the most deeply investigated, especially for its importance in probability and convex geometry (see for instance [6] for an overview and the series of papers [1, 2, 3, 15], which recently broadened and structured the theory of log-concave functions). Clearly, if a function u is F-concave in Ω for some admissible F, then it is quasiconcave in Ω; vice versa, if u is ∞-concave in Ω, then it is F-concave in Ω for any admissible F. These mean that quasiconcavity (resp. ∞-concavity) is the weakest (resp. strongest) conceivable concavity. Notice that α-logconcavity ($\alpha > 0$) corresponds to F-concavity with $F = L_\alpha$ and it is weaker (resp. stronger) than p-concavity for any $p > 0$ (resp. $p < 0$). Indeed, the following lemma holds.

Lemma 2.2 *Let Ω be a convex set in $\mathbf{R}^N$ and u a nonnegative bounded function in Ω.*
(i) *If u is p-concave in Ω for some $p > 0$, then u is α-logconcave in Ω for any $\alpha > 0$.*
(ii) *If u is α-logconcave in Ω for some $\alpha > 0$, then u is p-concave in Ω for any $p < 0$.*

Proof. We prove assertion (i). Let $p > 0$ and $\alpha > 0$. It suffices to prove that

$$[(1-\mu)a^p + \mu b^p]^{\frac{1}{p}} \geq \exp\left\{-\left[(1-\mu)(-\log a)^{\frac{1}{\alpha}} + \mu(-\log b)^{\frac{1}{\alpha}}\right]^{\alpha}\right\} \tag{2.4}$$

holds for all sufficiently small a, $b > 0$ and all $\mu \in [0,1]$. This is equivalent to that the inequality

$$\left(-\frac{1}{p}\log\left[(1-\mu)\tilde{a} + \mu\tilde{b}\right]\right)^{\frac{1}{\alpha}} \leq (1-\mu)\left(-\frac{1}{p}\log\tilde{a}\right)^{\frac{1}{\alpha}} + \mu\left(-\frac{1}{p}\log\tilde{b}\right)^{\frac{1}{\alpha}} \tag{2.5}$$

holds for all sufficiently small $\tilde{a} := a^p$, $\tilde{b} := b^p > 0$ and all $\mu \in [0,1]$. Inequality (2.5) follows from the fact that the function

$$s \mapsto \left(-\frac{1}{p}\log s\right)^{\frac{1}{\alpha}}$$

is convex for all sufficiently small $s > 0$. Thus (2.4) holds for all sufficiently small a, $b > 0$ and all $\mu \in [0,1]$ and assertion (i) follows. Similarly, we obtain assertion (ii) and the proof is complete. $\Box$

Lemma 2.2 implies that α-logconcavity is a refinement of p-concavity at $p = 0$.

At the end of this section we show that 2-logconcavity is the strongest concavity for the Gauss kernel $G(\cdot,t)$ to satisfy. This plays a crucial role in giving an answer to the second part of (**Q1**).

Lemma 2.3 *Let F be admissible such that $G(\cdot,t)$ is F-concave in $\mathbf{R}^N$ for some $t > 0$. Then a bounded nonnegative function u in $\mathbf{R}^N$ is F-concave in $\mathbf{R}^N$ if u is 2-logconcave in $\mathbf{R}^N$ (in other words $\mathcal{C}[L_2] \subset \mathcal{C}[F]$). Furthermore,*

$$\mathcal{C}[L_2] = \bigcap_{F \in \{H\,:\,G(\cdot,t)\in\mathcal{C}[H]\}} \mathcal{C}[F] \quad \textit{for any } t > 0. \tag{2.6}$$

Proof. Assume that $G(\cdot,t)$ is F-concave in $\mathbf{R}^N$ for some $t > 0$. It follows from Definition 2.1 that the function $e^{-|x|^2}$ is F-concave in $\mathbf{R}^N$. Then we obtain the F-concavity of e^{-s^2} ($s \in \mathbf{R}$).

Let u be 2-logconcave in $\mathbf{R}^N$. By Definition 2.1 we see that $L_2(\kappa u)$ is concave in $\mathbf{R}^N$ for all sufficiently small $\kappa > 0$. Set

$$w(x) := -L_2(\kappa u(x)) = \begin{cases} \sqrt{-\log\kappa u(x)} & \text{if} \quad u(x) > 0, \\ \infty & \text{if} \quad u(x) = 0. \end{cases}$$

Then w is nonnegative and convex in $\mathbf{R}^N$, that is,

$$0 \leq w((1-\mu)x + \mu y) \leq (1-\mu)w(x) + \mu w(y) \tag{2.7}$$

for x, $y \in \mathbf{R}^N$ and $\mu \in [0,1]$. On the other hand, by F-concavity of e^{-s^2} we have

$$F\left(\kappa e^{-[(1-\mu)w(x)+\mu w(y)]^2}\right) \geq (1-\mu)F(\kappa e^{-w(x)^2}) + \mu F(\kappa e^{-w(y)^2}) \tag{2.8}$$

for all sufficiently small $\kappa > 0$. Since F is an increasing function, by (2.7) and (2.8) we obtain

$$\begin{aligned} F(\kappa^2 u((1-\mu)x + \mu y)) &= F\left(\kappa \exp(-w((1-\mu)x+\mu y)^2)\right) \\ &\ge F\left(\kappa e^{-[(1-\mu)w(x)+\mu w(y)]^2}\right) \ge (1-\mu)F(\kappa e^{-w(x)^2}) + \mu F(\kappa e^{-w(y)^2}) \\ &= (1-\mu)F(\kappa^2 u(x)) + \mu F(\kappa^2 u(y)) \end{aligned}$$

for all sufficiently small $\kappa > 0$ if $u(x)u(y) > 0$. This inequality also holds in the case of $u(x)u(y) = 0$. These imply that u is F-concave in $\mathbf{R}^N$ and

$$\mathcal{C}[L_2] \subset \bigcap_{F \in \{H \,:\, G(\cdot,t) \in \mathcal{C}[H]\}} \mathcal{C}[F]. \tag{2.9}$$

On the other hand, since $G(\cdot, t)$ is 2-logconcave, it turns out that

$$\bigcap_{F \in \{H \,:\, G(\cdot,t) \in \mathcal{C}[H]\}} \mathcal{C}[F] \subset \mathcal{C}[L_2].$$

This together with (2.9) implies (2.6). Thus Lemma 2.3 follows. $\square$

3 Main results

We are now ready to state the main results of this paper. The first one ensures that the heat flow preserves α-logconcavity with $1 \le \alpha \le 2$.

Theorem 3.1 *Let Ω be a convex domain in $\mathbf{R}^N$ and $1 \le \alpha \le 2$. Let u_0 be a bounded nonnegative function in Ω and $u := e^{t\Delta_\Omega} u_0$. Assume that $0 \le u_0 \le 1$ and $L_\alpha(u_0)$ is concave in Ω. Then $L_\alpha(u(\cdot,t))$ is concave in Ω for any $t > 0$.*

Since α-logconcavity with $\alpha > 1$ is stronger than usual logconcavity, Theorems 3.1 gives answer to the first part of (**Q1**). Furthermore, as a corollary of Theorem 3.1, we have the following.

Corollary 3.1 *Let Ω be a convex domain in $\mathbf{R}^N$. Let u_0 be a bounded nonnegative function in Ω. If $1 \le \alpha \le 2$ and u_0 is α-logconcave in Ω, then $e^{t\Delta_\Omega} u_0$ is α-logconcave in Ω for any $t > 0$.*

Next we state a result which shows that 2-logconcavity is the strongest concavity preserved by the heat flow. This addresses the second part of (**Q1**).

Theorem 3.2 *Let F be admissible and Ω a convex domain in $\mathbf{R}^N$. Assume that F-concavity is stronger than 2-logconcavity in Ω, that is, $\mathcal{C}_\Omega[F] \subsetneq \mathcal{C}_\Omega[L_2]$. Then there exists $u_0 \in \mathcal{C}_\Omega[F]$ such that*

$$e^{T\Delta_\Omega} u_0 \notin \mathcal{C}_\Omega[F] \quad \text{for some } T > 0.$$

Here the following question naturally arises:

(**Q2**) What is the weakest concavity preserved by the heat flow?

Unfortunately we have no answers to (**Q2**) and it is open. Notice that the heat flow does not necessarily preserve p-concavity for some $p < 0$. See [10, 11]. (See also [5].)

Finally we assure that $e^{t\Delta}u_0$ spontaneously becomes α-logconcave for any $\alpha \in [1, 2)$ if u_0 has compact support. This improves assertion (c).

Theorem 3.3 *Let u_0 be a bounded nonnegative function in $\mathbf{R}^N$ with compact support. Then, for any given $1 \le \alpha < 2$, there exists $T_\alpha > 0$ such that, for any $t \ge T_\alpha$, $L_\alpha(e^{t\Delta}u_0)$ is concave in $\mathbf{R}^N$, in particular, $e^{t\Delta}u_0$ is α-logconcave in $\mathbf{R}^N$.*

We conjecture that Theorem 3.3 holds true even for $\alpha = 2$, but we can not prove it here. Indeed, in our proof of Theorem 3.3, $T_\alpha \to \infty$ as $\alpha \to 2$.

In Section 4 we prove Theorems 3.1 and 3.2. Theorem 3.1 is shown as an application of [9] however the proof is somewhat tricky (see Remark 4.1). Furthermore, we prove Theorem 3.2 by the use of Lemma 2.3. In Section 5 we study the large time behavior of the second order derivatives of $e^{t\Delta}u_0$. This proves Theorem 3.3.

4 Proofs of Theorems 3.1 and 3.2

Firstly we prove Theorem 3.1 and show the preservation of α-logconcavity $(1 \le \alpha \le 2)$ by the heat flow.

Proof of Theorem 3.1. Let Ω be a convex domain in $\mathbf{R}^N$. Let u_0 be a nontrivial function in Ω such that $0 \le u_0(x) \le 1$ in Ω. Then it follows from the strong maximum principle that $0 < u < 1$ in $\Omega \times (0, \infty)$. Assume that $L_\alpha(u_0)$ is concave in Ω for some $\alpha \in [1, 2]$.

<u>1st step</u>: We consider the case where Ω is a bounded smooth convex domain, $u_0 \in C(\overline{\Omega})$ and $u_0 = 0$ on $\partial\Omega$. Set

$$w(x,t) := -L_\alpha(u(x,t)) \ge 0, \qquad w_0(x) := -L_\alpha(u_0(x)) \ge 0.$$

Here w_0 is convex in Ω. Then it follows that

$$\begin{cases} w_t - \Delta w + \dfrac{1}{\gamma}\dfrac{|\nabla w|^2}{w^{\frac{\gamma-1}{\gamma}}} + \dfrac{\gamma-1}{\gamma}\dfrac{|\nabla w|^2}{w} = 0 & \text{in} \quad \Omega \times (0,\infty)\,, \\ w(x,0) = w_0(x) & \text{in} \quad \Omega\,, \\ w > 0 & \text{in} \quad \Omega \times (0,\infty)\,, \\ w(x,t) \to +\infty & \text{as} \quad \mathrm{dist}(x,\partial\Omega) \to 0 \text{ for any } t > 0\,, \end{cases} \tag{4.1}$$

where $\gamma := 1/\alpha \in [1/2, 1]$.

We prove that $w(\cdot, t)$ is convex in Ω for any $t > 0$. For this aim, we set $z := e^{-w}$ and show that $z(\cdot, t)$ is logconcave in Ω for any $t > 0$. (See Remark 4.1.) It follows from (4.1) that

$$\begin{cases} z_t - \Delta z + \dfrac{|\nabla z|^2}{z}\left[-\dfrac{1}{\gamma}(-\log z)^{-\frac{\gamma-1}{\gamma}} + \dfrac{\gamma-1}{\gamma}(\log z)^{-1} + 1\right] = 0 & \text{in} \quad \Omega \times (0,\infty)\,, \\ z(x,0) = e^{-w_0(x)} & \text{in} \quad \Omega\,, \\ z(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty)\,. \end{cases}$$

Furthermore, thanks to the convexity of w_0, we see that $z(\cdot,0) = e^{-w_0}$ is logconcave in Ω. Applying [9, Theorem 4.2, Corollary 4.2] (see also [8, Theorem 4.2]), we deduce that $z(\cdot,t)$ is logconcave in Ω for any $t > 0$ if

$$\begin{aligned} &h(s,A) := e^{-s}\left[-e^s\,\mathrm{trace}(A) + f(e^s, e^s\theta)\right] \text{ is convex} \\ &\text{for } (s,A) \in (-\infty,0) \times \mathrm{Sym}_N \text{ for any fixed } \theta \in \mathbf{R}^N. \end{aligned}$$

Here Sym_N denotes the space of real $N \times N$ symmetric matrices and

$$f(\zeta,\vartheta) := \frac{|\vartheta|^2}{\zeta}\left[-\frac{1}{\gamma}(-\log\zeta)^{-\frac{\gamma-1}{\gamma}} + \frac{\gamma-1}{\gamma}(\log\zeta)^{-1} + 1\right] \quad \text{for} \quad (\zeta,\vartheta) \in (0,1) \times \mathbf{R}^N.$$

On the other hand, for any fixed $\theta \in \mathbf{R}^N$,

$$h(s,A) = -\mathrm{trace}(A) + |\theta|^2\left[-\frac{1}{\gamma}(-s)^{-\frac{\gamma-1}{\gamma}} + \frac{\gamma-1}{\gamma}s^{-1} + 1\right]$$

is convex for $(s,A) \in (-\infty,0) \times \mathrm{Sym}_N$ if and only if $1/2 \le \gamma \le 1$. Therefore $z(\cdot,t)$ is logconcave in Ω for any $t > 0$. This implies that $u(\cdot,t)$ is α-logconcave for any $t > 0$.

<u>2nd step</u>: We consider the case where Ω is a bounded smooth convex domain. In this step we do not assume that $u_0 = 0$ on $\partial\Omega$. Since u_0 is α-logconcave in Ω, we see that $L_\alpha(u_0)$ is concave in Ω. Set

$$v_0(x) := \begin{cases} \exp(L_\alpha(u_0(x))) & \text{for} \quad x \in \Omega, \\ 0 & \text{for} \quad x \notin \Omega, \end{cases} \qquad v(x,t) := [e^{t\Delta}v_0](x),$$

for $x \in \mathbf{R}^N$ and $t > 0$. Here we let $e^{-\infty} := 0$. Then v_0 is logconcave in $\mathbf{R}^N$. We deduce from assertion (a) that $v(\cdot,t)$ is logconcave for any $t > 0$. Furthermore, we deduce from $v_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ that

$$v(\cdot,t) \text{ is a positive continuous function in } \mathbf{R}^N \text{ for any } t > 0, \tag{4.2}$$

$$\|v(t)\|_{L^\infty(\mathbf{R}^N)} < \|v_0\|_{L^\infty(\mathbf{R}^N)} \text{ for any } t > 0, \tag{4.3}$$

$$\lim_{t\to 0}\|v(t) - v_0\|_{L^1(\mathbf{R}^N)} = 0. \tag{4.4}$$

By (4.4) we can find a sequence $\{t_n\} \subset (0,\infty)$ with $\lim_{n\to\infty} t_n = 0$ such that

$$\lim_{n\to\infty} v(x,t_n) = v_0(x) \tag{4.5}$$

for almost all $x \in \mathbf{R}^N$.

Let η solve

$$-\Delta\eta = 1 \quad \text{in} \quad \Omega, \qquad \eta > 0 \quad \text{in} \quad \Omega, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega.$$

Then η is 1/2-concave in Ω (see e.g. [12, Theorem 4.1]), which implies that $\log\eta$ is concave in Ω and $\log\eta \to -\infty$ as $\mathrm{dist}\,(x,\partial\Omega) \to 0$. By (4.2) and (4.3) we can find a sequence $\{m_n\} \subset (1,\infty)$ with $\lim_{n\to\infty} m_n = \infty$ such that

$$V_n(x) := \log v(x,t_n) + m_n^{-1}\log\eta(x)$$

is continuous and concave in Ω and

$$\sup_{x\in\Omega} V_n(x) \le \operatorname{ess\,sup}_{x\in\Omega} \log v_0.$$

Furthermore, by (4.5) we have

$$\begin{aligned}
&\lim_{n\to\infty} V_n(x) = \log v_0(x) = L_\alpha(u_0(x)) \quad \text{for almost all } x\in\Omega,\\
&V_n(x) \to -\infty \quad \text{as} \quad \operatorname{dist}(x,\partial\Omega)\to 0.
\end{aligned}$$

Then the function $u_{0,n}(x) := L_\alpha^{-1}(V_n(x))$ satisfies

$$0 \le u_{0,n} \le 1 \text{ in } \Omega,\ u_{0,n} = 0 \text{ on } \partial\Omega \text{ and } \lim_{n\to\infty} u_{0,n}(x) = u_0(x) \text{ for almost all } x\in\Omega. \tag{4.6}$$

Furthermore, $u_{0,n}$ is continuous on $\overline{\Omega}$ and $L_\alpha(u_{0,n})$ is concave in Ω. Let

$$u_n(x,t) := [e^{t\Delta_\Omega} u_{0,n}](x) = \int_\Omega G_\Omega(x,y,t) u_{0,n}(y)\, dy,$$

where $G_\Omega = G_\Omega(x,y,t)$ is the Dirichlet heat kernel in Ω. Then, by (4.6) we apply the Lebesgue dominated convergence theorem to obtain

$$\lim_{n\to\infty} u_n(x,t) = \int_\Omega G_\Omega(x,y,t) u_0(y)\, dy = u(x,t), \quad x\in\Omega,\ t>0. \tag{4.7}$$

On the other hand, by the argument in 1st step we see that $L_\alpha(u_n(\cdot,t))$ is concave in Ω for any $t>0$. Then we deduce from (4.7) that $L_\alpha(u(\cdot,t))$ is also concave in Ω for any $t>0$. Thus Theorem 3.1 follows in the case where Ω is a bounded smooth convex domain.

3rd step: We complete the proof of Theorem 3.1. There exists a sequence of bounded convex smooth domains $\{\Omega_n\}_{n=1}^\infty$ such that

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots, \qquad \bigcup_{n=1}^\infty \Omega_n = \Omega.$$

(This is for instance a trivial consequence of [16, Theorem 2.7.1]).

For any $n = 1, 2, \dots$, let $u_n := e^{t\Delta_{\Omega_n}}(u_0\chi_{\Omega_n})$. The argument in 2nd step implies that $L_\alpha(u_n(\cdot,t))$ is concave in Ω_n for any $t>0$. Furthermore, by the comparison principle we see that

$$\begin{aligned}
&u_n(x,t) \le u_{n+1}(x,t) \le u(x,t) \quad && \text{in} \quad \Omega_n\times(0,\infty),\\
&u(x,t) = \lim_{n\to\infty} u_n(x,t) && \text{in} \quad \Omega\times(0,\infty).
\end{aligned}$$

Then we observe that $L_\alpha(u(\cdot,t))$ is concave in Ω for any $t>0$. Thus Theorem 3.1 follows. □

Remark 4.1 *Sufficient conditions for the concavity of solutions to parabolic equations were discussed in* [9, *Section* 4.1]*. However we can not apply the arguments in* [9, *Section* 4.1] *to show the concavity of* $-w(\cdot,t)$*, because assumption* (F3) *with* $p=1$ *in* [9] *is not satisfied for the equation satisfied by* $-w$*.*

Remark 4.2 *Theorem* 3.1 *implies that* $e^{t\Delta_\Omega}\chi_\Omega(x)$ *is* 2*-logconcave (with respect to* x*) for every* $t>0$*. As it is well known, by the eigenfunction expansion of solutions and the regularity theorems for the heat equation, we have*

$$\lim_{t\to\infty} e^{\lambda_1 t}[e^{t\Delta_\Omega}\chi_\Omega](x) = c\phi(x)/\|\phi\|_{L^2(\Omega)}$$

uniformly on $\overline{\Omega}$*, where* λ_1 *and* ϕ *are as in assertion* (b) *of the Introduction and*

$$c = \int_\Omega \phi(x)\,dx \Big/ \|\phi\|_{L^2(\Omega)} > 0\,.$$

Then we may think to obtain the 2*-logcocanvity of* ϕ *just by letting* $t\to+\infty$ *and using the preservation of* 2*-logconcavity by pointwise convergence. Unfortunately this approach does not work, since the parameter* κ *of Definition* 2.1 *for* $e^{t\Delta_\Omega}\chi_\Omega(x)$ *may tend to* 0 *as* t *tends to* $+\infty$*, while* 2*-logconcavity is preserved only if* κ *remains strictly positive.*

Proof of Corollary 3.1. Corollary 3.1 directly follows from Theorem 3.1, Definition 2.1 and the linearity of the heat equation. □

At the end of this section we prove Theorem 3.2 with the aid of Lemma 2.3.

Proof of Theorem 3.2. Let us consider the case of $\Omega = \mathbf{R}^N$. Since F is stronger than 2-logconcavity, by Lemma 2.3 we see that $G(\cdot,t)\not\in\mathcal{C}[F]$ for any $t>0$. Then, for any $\epsilon>0$, there exist $\kappa\in(0,\epsilon)$, $\mu\in[0,1]$ and $x_1,x_2\in\mathbf{R}^N$ such that

$$F\left(\kappa e^{-|(1-\mu)x_1+\mu x_2|^2}\right) - (1-\mu)F\left(\kappa e^{-|x_1|^2}\right) - \mu F\left(\kappa e^{-|x_2|^2}\right) < 0\,. \tag{4.8}$$

Let K be a bounded convex set in $\mathbf{R}^N$ such that $|K|>0$ and set $u := e^{t\Delta}\chi_K$. Since χ_K is ∞-concave, we see that χ_K is F-concave.

On the other hand, it follows from (1.5) that

$$\lim_{t\to\infty} t^{\frac{N}{2}}\|u(t) - |K|G(t)\|_{L^\infty(\mathbf{R}^N)} = 0.$$

This implies that

$$\lim_{t\to\infty}(4\pi t)^{\frac{N}{2}}|K|^{-1}u(2\sqrt{t}\xi,t) = e^{-|\xi|^2}, \qquad \xi\in\mathbf{R}^N. \tag{4.9}$$

For any $t\geq 1$ and $i=1,2$, set $\xi_i^t := 2\sqrt{t}x_i$. Since F is continuous in $(0,1]$, by (4.8) and (4.9) we have

$$\begin{aligned}
&F\left((4\pi t)^{\frac{N}{2}}|K|^{-1}\kappa u((1-\mu)\xi_1^t+\mu\xi_2^t,t)\right)\\
&\qquad - (1-\mu)F\left((4\pi t)^{\frac{N}{2}}|K|^{-1}\kappa u(\xi_1^t,t)\right) - \mu F\left((4\pi t)^{\frac{N}{2}}|K|^{-1}\kappa u(\xi_2^t,t)\right)\\
&\to F\left(\kappa e^{-|(1-\mu)x_1+\mu x_2|^2}\right) - (1-\mu)F\left(\kappa e^{-|x_1|^2}\right) - \mu F\left(\kappa e^{-|x_2|^2}\right) < 0
\end{aligned}$$

as $t\to\infty$. Since ϵ is arbitrary, we see that

$$u(\cdot,T) = e^{T\Delta}\chi_K \text{ is not } F\text{-concave for all sufficiently large } T. \tag{4.10}$$

Thus F-concavity is not preserved by the heat flow in $\mathbf{R}^N$.

Next we consider the case of $\Omega \neq \mathbf{R}^N$. We can assume, without loss of generality, that $0 \in \Omega$ and $K \subset \Omega$. For $n = 1, 2, \dots$, set $\Omega_n := n\Omega$. Then

$$\lim_{n\to\infty} [e^{t\Delta_{\Omega_n}} \chi_K](x) = [e^{t\Delta} \chi_K](x) \tag{4.11}$$

for any $x \in \mathbf{R}^N$ and $t > 0$. Let T' be a sufficiently large constant. By (4.10) and (4.11) we observe that $e^{T'\Delta_{\Omega_n}} \chi_K$ is not F-concave for all sufficiently large n. Since

$$[e^{n^2 t\Delta_{\Omega_n}} \chi_K](nx) = [e^{t\Delta_\Omega} \chi_{n^{-1}K}](x), \qquad x \in \Omega,\ t > 0,$$

we see that $e^{n^{-2}T'\Delta_\Omega} \chi_{n^{-1}K}$ is not F-concave for all sufficiently large n. Combining the fact that $\chi_{n^{-1}K}$ is F-concave, we see that F-concavity is not preserved by the heat flow in Ω. Thus Theorem 3.2 follows. $\Box$

5 Proof of Theorem 3.3

We modify the arguments in the proof of [14, Theorem 5.1] and prove Theorem 3.3.

Proof of Theorem 3.3. Let u_0 be a nontrivial bounded nonnegative function in $\mathbf{R}^N$ such that $\operatorname{supp} u_0 \subset B(0, R)$ for some $R > 0$. Without loss of generality, we can assume that

$$\int_{\mathbf{R}^N} y_i u_0(y)\, dy = 0, \qquad i = 1, \dots, N. \tag{5.1}$$

Let $u := e^{t\Delta} u_0$. It follows from (1.1) and (1.5) that

$$0 < u \le \min \left\{ \|u_0\|_{L^\infty(\mathbf{R}^N)}, (4\pi t)^{-\frac{N}{2}} \|u_0\|_{L^1(\mathbf{R}^N)} \right\} \tag{5.2}$$

for $(x, t) \in \mathbf{R}^N \times (0, \infty)$. In particular, $0 < u(x, t) < 1$ in $\mathbf{R}^N \times (T, \infty)$ for some $T > 0$.

Let $\gamma := 1/\alpha \in (1/2, 1]$. For the proof of Theorem 3.3, it suffices to prove that

$$v(x, t) := (-\log u(x, t))^\gamma$$

is convex in $\mathbf{R}^N$ for all sufficiently large t. By (1.1), (1.5) and (5.1), for $i = 1, \dots, N$, we have

$$\begin{aligned} \frac{u_{x_i}(x, t)^2}{u(x, t)^2} &= \left[-\frac{x_i}{2t} + \frac{1}{2t} X_i \right]^2 = \frac{x_i^2}{4t^2} - \frac{x_i}{2t^2} X_i + \frac{1}{4t^2} X_i^2, \\ \frac{u_{x_i x_i}(x, t)}{u(x, t)} &= -\frac{1}{2t} + \frac{x_i^2}{4t^2} - \frac{x_i}{2t^2} X_i + \frac{1}{4t^2} Y_i, \end{aligned} \tag{5.3}$$

for $(x, t) \in \mathbf{R}^N \times (0, \infty)$, where

$$X_i := \frac{1}{u} \int_{\mathbf{R}^N} y_i G(x - y, t) u_0(y)\, dy, \qquad Y_i := \frac{1}{u} \int_{\mathbf{R}^N} y_i^2 G(x - y, t) u_0(y)\, dy.$$

It follows from $\operatorname{supp} u_0 \subset B(0, R)$ that

$$|X_i| \le R, \qquad 0 \le Y_i \le R^2. \tag{5.4}$$

Since

$$v_{x_i} = -\gamma(-\log u)^{-(1-\gamma)}\frac{u_{x_i}}{u},$$
$$v_{x_ix_i} = -\gamma(1-\gamma)(-\log u)^{-(2-\gamma)}\frac{(u_{x_i})^2}{u^2} + \gamma(-\log u)^{-(1-\gamma)}\frac{(u_{x_i})^2}{u^2} - \gamma(-\log u)^{-(1-\gamma)}\frac{u_{x_ix_i}}{u},$$

by (5.3) we obtain

$$\begin{aligned}\frac{2t}{\gamma}(-\log u)^{1-\gamma}v_{x_ix_i} &= 2t\left[-\frac{u_{x_ix_i}}{u} + \frac{(u_{x_i})^2}{u^2}\right] - 2t(1-\gamma)(-\log u)^{-1}\frac{(u_{x_i})^2}{u^2} \\ &= 1 + \frac{1}{2t}X_i^2 - \frac{1}{2t}Y_i - (1-\gamma)(-\log u)^{-1}\left[\frac{x_i^2}{2t} - \frac{x_i}{t}X_i + \frac{1}{2t}X_i^2\right]\end{aligned}$$

for $(x,t) \in \mathbf{R}^N \times (T,\infty)$. Since $\lim_{t\to\infty}\|u(t)\|_{L^\infty(\mathbf{R}^N)} = 0$ (see (5.2)), taking a sufficiently large T if necessary, we have

$$\frac{2t}{\gamma}(-\log u)^{1-\gamma}v_{x_ix_i} \geq 1 + \frac{1}{4t}X_i^2 - \frac{1}{2t}Y_i - (1-\gamma)(-\log u)^{-1}\left[\frac{x_i^2}{2t} - \frac{x_i}{t}X_i\right] \tag{5.5}$$

for $(x,t) \in \mathbf{R}^N \times (T,\infty)$.

Let $0 < \epsilon < 1$. By (5.2) we take a sufficiently large T so that

$$(-\log u)^{-1} \leq \left(-\log\left[(4\pi t)^{-\frac{N}{2}}\|u_0\|_{L^1(\mathbf{R}^N)}\right]\right)^{-1} \leq \left(\frac{N}{4}\log t\right)^{-1} \tag{5.6}$$

for $(x,t) \in \mathbf{R}^N \times (T,\infty)$. We consider the case where $(x,t) \in \mathbf{R}^N \times (T,\infty)$ with $|x|^2 \leq \epsilon t\log t$. By (5.4) and (5.6) we have

$$\begin{aligned}(-\log u)^{-1}\left[\frac{x_i^2}{2t} - \frac{x_i}{t}X_i\right] &\leq \left(\frac{N}{4}\log t\right)^{-1}\left[\frac{|x|^2}{2t} + R\frac{|x|}{t}\right] \\ &\leq \left(\frac{N}{4}\log t\right)^{-1}\left[\frac{\epsilon\log t}{2} + R\epsilon^{\frac{1}{2}}t^{-\frac{1}{2}}(\log t)^{\frac{1}{2}}\right] \\ &= \frac{2}{N}\epsilon + \frac{4}{N}R\epsilon^{\frac{1}{2}}t^{-\frac{1}{2}}(\log t)^{-\frac{1}{2}}.\end{aligned} \tag{5.7}$$

By (5.4), (5.5) and (5.7), taking a sufficiently small $\epsilon > 0$ and a sufficiently large T if necessary, we obtain

$$\frac{2t}{\gamma}(-\log u)^{1-\gamma}v_{x_ix_i} \geq 1 - \frac{R^2}{2t} - (1-\gamma)\left[\frac{2}{N}\epsilon + \frac{4}{N}R\epsilon^{\frac{1}{2}}t^{-\frac{1}{2}}(\log t)^{-\frac{1}{2}}\right] \geq \frac{1}{2} \tag{5.8}$$

for $(x,t) \in \mathbf{R}^N \times (T,\infty)$ with $|x|^2 \leq \epsilon t\log t$.

We consider the case where $(x,t) \in \mathbf{R}^N \times (T,\infty)$ with $|x|^2 > \epsilon t\log t$. Let δ be a positive constant to be chosen later. Since $\operatorname{supp} u_0 \subset B(0,R)$, by (1.5), taking a sufficiently large T if necessary, we have

$$u(x,t) \leq (4\pi t)^{-\frac{N}{2}}\exp\left(-\frac{|x|^2}{4(1+\delta)t}\right)\|u_0\|_{L^1(\mathbf{R}^N)}.$$

This implies that

$$(-\log u)^{-1} \le \left[\frac{N}{2}\log(4\pi t) + \frac{|x|^2}{4(1+\delta)t} - \log \|u_0\|_{L^1(\mathbf{R}^N)}\right]^{-1} \le \frac{4(1+\delta)t}{|x|^2}.$$

It follows from (5.4) and (5.5) that

$$\begin{aligned}\frac{2t}{\gamma}(-\log u)^{1-\gamma}v_{x_ix_i} &\ge 1 - \frac{R^2}{2t} - (1-\gamma)\frac{4(1+\delta)t}{|x|^2}\left[\frac{|x|^2}{2t} + R\frac{|x|}{t}\right] \\ &\ge 1 - 2(1-\gamma)(1+\delta) - \frac{R^2}{2t} - (1-\gamma)R\frac{4(1+\delta)}{(\epsilon t\log t)^{\frac{1}{2}}}.\end{aligned} \tag{5.9}$$

Since $1/2 < \gamma \le 1$, taking a sufficiently small $\delta > 0$, we see that $1 - 2(1-\gamma)(1+\delta) \ge \delta$. Then, by (5.9), taking a sufficiently large T if necessary, we obtain

$$\frac{2t}{\gamma}(-\log u)^{1-\gamma}v_{x_ix_i} \ge \frac{\delta}{2} \tag{5.10}$$

for $(x,t) \in \mathbf{R}^N \times (T,\infty)$ with $|x|^2 > \epsilon t \log t$. Combining (5.8) and (5.10), we deduce that $v(\cdot,t)$ is convex in $\mathbf{R}^N$ for $t \ge T$. Therefore we see that $L_\alpha(u(\cdot,t))$ is concave in $\mathbf{R}^N$ for $t \ge T$. This together with Lemma 2.1 implies that $u(\cdot,t)$ is α-logconcave in $\mathbf{R}^N$ for $t \ge T$. Thus Theorem 3.3 follows. □

Acknowledgements. The first author was supported in part by the Grant-in-Aid for Scientific Research (A)(No. 15H02058) from Japan Society for the Promotion of Science. The second author has been partially supported by INdAM through a GNAMPA Project. The third author was supported in part by the Grant-in-Aid for Young Scientists (B)(No. 15K17536) and the Grant-in-Aid for Scientific Research (C)(No. 16KT0132).

References

[1] S. Artstein-Avidan, B. Klartag and V. Milman, The Santaló point of a function, and a functional form of the Santaló inequality, Mathematika **51** (2004), 33–48.

[2] S. Artstein-Avidan and V. Milman, A characterization of the concept of duality, Electron. Res. Announc. Math. Sci. **14** (2007), 42–59.

[3] S. Artstein-Avidan and V. Milman, The concept of duality in convex analysis, and the characterization of the Legendre transform, Ann. of Math. **169** (2009), 661–674.

[4] H. J. Brascamp and E. H. Lieb, On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log-concave functions, and with an application to the diffusion equation, J. Funct. Anal. **22** (1976), 366–389.

[5] A. Chau and B. Weinkove, Counterexamples to quasiconcavity for the heat equation, International Mathematics Research Notices, rny243, https://doi.org/10.1093/imrn/rny243

[6] A. Colesanti, Log-Concave Functions, in: Carlen E., Madiman M., Werner E. (eds) Convexity and Concentration. The IMA Volumes in Mathematics and its Applications, vol 161. Springer, New York, NY, 2017.

[7] G. Deslauriers and S. Dubuc, Logconcavity of the cooling of a convex body, Proc. Amer. Math. Soc. **74** (1979), 291–294.

[8] A. Greco and B. Kawohl, Log-concavity in some parabolic problems, Electron. J. Differential Equations **1999** (1999), 1–12.

[9] K. Ishige, K. Nakagawa and P. Salani, Spatial concavity of solutions to parabolic systems, to appear in Ann. Scuola Norm. Sup. Pisa Cl. Sci.

[10] K. Ishige and P. Salani, Is quasi-concavity preserved by heat flow?, Arch. Math. **90** (2008), 450–460.

[11] K. Ishige and P. Salani, Convexity breaking of the free boundary for porous medium equations, Interfaces Free Bound. **12** (2010), 75–84.

[12] A. U. Kennington, Power concavity and boundary value problems, Indiana Univ. Math. J. **34** (1985), 687–704.

[13] N. J. Korevaar, Convex solutions to nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. J. **32** (1983), 603–614.

[14] K.-A. Lee and J. L. Vázquez, Geometrical properties of solutions of the porous medium equation for large times, Indiana Univ. Math. J. **52** (2003), 991–1016.

[15] V. Milman, Geometrization of probability. In: Geometry and Dynamics of Groups and Spaces, Progr. Math. **265**, M. Kapranov et al. (eds.), Birkhäuser (2008).

[16] R. Schneider, Convex bodies: the Brunn-Minkowski theory. Second expanded edition. Encyclopedia of Mathematics and its Applications, 151. Cambridge University Press, Cambridge, 2014.

[17] A. Saumard and J. A. Wellner, Log-concavity and strong log-concavity: A review, Stat. Surv. **8** (2014), 45–114.