Università di Firenze, Università di Perugia, INdAM consorziate nel CIAFM

DOTTORATO DI RICERCA
IN MATEMATICA, INFORMATICA, STATISTICA

CURRICULUM IN MATEMATICA
CICLO XXXI

Sede amministrativa Università degli Studi di Firenze
Coordinatore Prof. Graziano Gentili

A dynamic approach to functions and their graphs: a study of students' discourse from a commognitive perspective

Settore Scientifico Disciplinare MAT/04

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Anni 2015/2018

Abstract

The purpose of this research is to study how a set of mathematical activities designed within a Dynamic Interactive Environment (DIE) influence a class of high school students' learning of functions. The study builds on findings of previous research on students' difficulties in working with real functions; these include difficulties in managing covariation of the two variables involved, that is the relationship between variations of the independent variable and variations of the dependent variable, as well as difficulties in dealing with the graphs of functions. The study involved designing a sequence of activities in a DIE that introduce functions through a dynamic approach, and it takes a discursive approach in analyzing students' interactions in this setting over a nine-week period. Data collected included: audio and video recordings of what happened in the classroom, video recordings of what happened on the students' computer screens, and the students' work on paper that was produced during the lessons and during a set of subsequent interviews. The analyses of these data lead to a model describing how dragging mediates students' discourse on functions; moreover, they shed light onto important features characterizing such discourse on functions and their graphs. The study has implications for the design of activities based on the use of DIEs for introducing real functions, highlighting the covariational aspect behind the functional dependency between the two variables.

The work involved in this thesis is structured as follows.

We situate the study within the literature, taking into account the educational issue of teaching and learning functions and studies pointing to students' common difficulties in this mathematical domain. We highlight previous research involving activities on functions presented within a DIE, focusing on covariation in relation to functional dependency.

In Chapter 2 we present the foundational elements of the Theory of Commognition, which is the theoretical framework within which we based the study. We focus on specific tools offered by this theory and on certain theoretical constructs om which we further elaborated. Indeed, since we use a DIE to represent functions, we needed to refine theoretical tools that allow to focus on dynamism and on the possibility of interacting with those that the theory of commognition describes as visual mediators of discourse.

In light of the theoretical framework, in Chapter 3 we formulate four research questions. In Chapter 4, we introduce the methodology used, including a presentation of all the activities designed for the classroom lessons and for the interviews, as well as the design principles behind the activities. Almost all of the activities involved the use of a DIE in which, thanks to the dragging tool, students could experience the dependence relation. In Chapter 5 our analyses focus on how dragging mediated students' discourse on functions, while in Chapter 6 we analyzed students' discourse comparing it to the hypothetical discourse of an expert in the same context.

In the concluding chapter, after answering the research questions, we re-contextualize the findings within the literature, highlighting the theoretical contributions of the study, which include a refinement of Sfard's notion of *visual mediator*. This is done by introducing the notion of DIM (Dynamic Interactive Mediator) and by elaborating on Ng's notion of

dragsturing by considering also cases in which there is no physical use of the dragging tool. Finally, we describe possible implications and directions for further research, which include the realization of a longer term teaching experiment, where the formal mathematical definitions of functions and their properties are introduced to students; and the design of activities with DIMs, such as those designed here, contextualized within the greater research problem of generating "good problems" aimed at achieving certain educational goals.

Acknowledgments

Many people have contributed in many ways to the production of this dissertation.

Firstly, I would like to express my sincere gratitude to my advisors, Prof. Samuele Antonini and Prof. Anna Baccaglini-Frank, whose excellent guidance led me in this three-year journey through my PhD studies. Especially, I thank you Anna for your support that on various occasions helped me to keep my focus, as well as the number of opportunities that you provided me to do research in the field, from the very beginning of my studies to the concluding stage of my thesis. I thank you Samuele for your tremendous mentorship on conducting research and writing, I always look up to your vast breadth of knowledge and experience in the field.

I would also extend my appreciation and gratitude to my referees for their valuable feedback. I thank you Prof. Nathalie Sinclair for your encouragement and kind words, for all the fruitful discussions you engaged in with me; but also for helping me to identify the weaknesses which allowed me to widen my research from various perspectives. I thank you Prof. Mirko Maracci, for your precious comments and suggestions on my work, and for all what you taught me through your insightful lectures that I had the pleasure to take.

I gratefully acknowledge the support of Università degli Studi di Firenze and GNSAGA of the Istituto Nazionale di Alta Matematica "F. Severi". I would like to thank all the faculty members of the Mathematics Department, in particular, Prof. Graziano Gentili, who has extraordinarily coordinated my PhD program. I also warmly thank all friends and colleagues in my institution for their daily support and friendship.

I want to thank, as well, all the Professors, Researchers and PhD students in math education who discussed with me, for enlightening me the first glance of research in math education and for their interest in my work. In particular, I thank you Elisa, for the sleepless nights we were working together before deadlines and presentations, for giving me the possibility to share fears and stupid questions, and for all the fun we have had on various occasions.

A special thanks go to "my" anonymous students who participated in this study, without their collaboration all this would have been impossible.

Finally, yet importantly, I thank my friends and family for providing their ongoing support throughout these years. I'm very grateful to my dear mamy, who was always there for me, every step of the way, patiently tolerating my stress and frustrations that came out at times. My great appreciation to my boyfriend, Tomi, for your unconditional love and care; I'm lucky to have you in my life.

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1 INTRODUCTION

In this chapter we contextualize our study within the literature, describing how it is situated within the educational issue of teaching and learning functions and looking at students' common difficulties in this field. In particular, we discuss how these difficulties can be related to the plurality of different representations and approaches traditionally used for teaching functions. At this regard, we focus on the possibility of implementing activities on functions within a Dynamic Interactive Environment (DIE) and on the contributions of this approach to the processes of teaching and learning. There are several studies describing the potentials of DIEs, especially, related to their implementation within a classroom setting and a review of them can be found in Sinclair & Robutti (2013). In this chapter we introduce some of these studies, taken from the literature on dragging, and we show the elements that are particularly significant with respect to the focuses of our research.

Then, after giving a deeper insight into the covariational aspect characterizing functions, we describe different graphical representations, implemented within both static and dynamic environment, highlighting their possible strengths and weaknesses for supporting students understanding of the main functions' properties.

Finally, we introduce a general version of the research questions we set out to investigate, and the main goals of the study.

1.1 CONTEXTUALIZATION OF THE RESEARCH PROBLEM WITHIN THE LITERATURE

We are interested in studying the learning process of students who are introduced to a new (for them) mathematical object, in the context of their mathematics classroom. Part of our research deals with the design of a sequence of activities on functions and graphs of functions aimed at supporting the learning process of 10^{th} grade students.

This topic has a central role both in secondary school and university mathematics and it has always entered different fields of mathematics. This fact leads to a great variety of definitions of functions, for example experts in geometry or algebra speak about transformations and homomorphism, while in calculus a function is immediately associated with its graph on the Cartesian plane. Formally, a function can even be defined without using almost any words (Sierpinska, 1992). According to the notion of function that can be found in the *Encyclopedia of Mathematics Education* (2014), there are three main aspects characterizing this mathematical object:

> "Firstly, a function is a purely mathematical entity in its own right. Depending on the level of abstraction, that entity can be introduced, for example, as either a correspondence that links every element in a given domain to one and only one element in another domain, called the codomain, or as a certain kind of relation. […] Secondly, functions have crucial roles as lenses through which other mathematical objects or theories can be viewed or connected, for instance, when perceiving arithmetic operations as functions of two variables. […] Thirdly, functions play crucial parts in the application of mathematics to and modelling of extra-mathematical situations and contexts." (p. 239)

Moreover, being able to interpret the graph of a function, as representing a function through its graph, are key elements in the mathematics education field but, at the same time, they are also basic requirements for most people, as indicated in the document *Matematica 2003* (UMI et al., 2003). All the competencies that should be developed by students attending Italian high schools are stated in a document that is called *Indicazioni Nazionali* (MIUR, 2010). In the text referring to a special kind of high schools, that in Italy are called "Licei", it is possible to see that functions, differential calculus and integrals are all notions that students are required to have learned by the end of their education. Moreover, great emphasis is given to the study of different representations of functions and to the ability of passing from one representation to another:

> "Lo studente sarà in grado di passare agevolmente da un registro di rappresentazione a un altro (numerico, grafico, funzionale), anche utilizzando strumenti informatici per la rappresentazione dei dati." [1] (MIUR, 2010, p. 24)

A similar view is expressed in the introduction to *Matematica 2003* (UMI et al., 2003), where it is highlighted the importance of building a connection between the graph of a function, its algebraic expression, its behavior and specific elements of it like the zeros and the sign. In particular, the following is expressed:

> "Uno dei maggiori obiettivi didattici di questo nucleo [relazioni e funzioni] è, infatti, l'acquisizione da parte degli alunni di un 'pensiero funzionale'. Come lo si può favorire? Con una forte connessione fra il grafico di una funzione, l'interpretazione dell'andamento, il collegamento di questo con l'espressione algebrica della funzione, gli aspetti numerici, e l'analisi di momenti particolari di questo andamento che corrispondono agli zeri (cioè alle equazioni), al segno (cioè alle disequazioni) […] Ciò non vuol dire che non si possa parlare di equazioni e sistemi indipendentemente dallo studio delle funzioni, ma che, laddove possibile, si cerchi di favorire l'interazione con la rappresentazione geometrica. Momenti particolari dell'andamento del grafico sono anche i massimi e i minimi, la crescenza e la decrescenza, il comportamento in prossimità di valori particolari; questa non è l'analisi matematica, perlomeno non è l'analisi matematica in senso classico. È lo studio qualitativo di un fenomeno. La considerazione dei fenomeni a livello qualitativo deve diventare un'abitudine mentale degli alunni e degli insegnanti, se si vuole fare in modo che le tecniche che l'alunno imparerà nel corso degli anni non siano mai oggetto di applicazione meccanica, ma frutto di riflessione sui significati nei diversi contesti proposti." [2] (UMI et al., p. 206)

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[1] "The student will be able to easily pass from a register of representation to another one (numerical, graphical, functional), even by using technological artifacts for representing data" (translated by the authors).

[2] "Indeed, one of the main didactical aims of this section [relations and functions] is the acquisition of 'functional thinking' by students. How can it be fostered? Through a strong connection between the graph of a function, the interpretation of its behavior, its link to the algebraic expression of the function, the numerical aspects, and the analysis of specific aspects of its behavior that may correspond to the zeros (that is, the equations), to the sign (that is, the inequalities) […] This does not mean that it is not possible to speak about equations and systems of equations independently from the study of functions, but, when possible, that we have to support the interplay with the geometrical representation. Other specific aspects of the behavior of a graph are maximum and minimum, monotonicity properties, the behavior in correspondence to special values; this is not calculus, at least, it is not calculus in the classical sense. It is the qualitative study of a phenomenon. The qualitative analysis of phenomena has to become a habit of mind for students and teachers, if the aim is that of

The existence of a wide variety of possible representations of functions and the fact that each of them allows to highlight certain aspects of them is an important issue that arises in this context and that we are going to deepen in this chapter.

We also observe that, especially from the Nineties on, the concept of function interested lots of researchers who started to investigate why many students, and also teachers, experience so many difficulties when dealing with functions.

1.1.1 Students' difficulties with the notion of function

There is a wide literature describing a variety of possible difficulties related to the learning of functions and graphs of functions. Some studies approach the problem from a cognitive point of view (Dubinsky, 1991; Tall & Vinner, 1981; Vinner & Dreyfus, 1989), others highlight the difference between the images one holds and the definition of the same mathematical object, called *concept image* and *concept definition* (Tall & Vinner, 1981; Vinner & Dreyfus, 1989); finally, some researchers focus on the process/object duality (Sfard, 1992; Dubinsky, 1991). Moreover, there are studies concerning the problems related to the visualization and, in particular, the visualization and the graph of a function (Monk & Nemirovsky, 1994; Goldenberg, 1995). Now we are going to describe the main issues pointed out by these different approaches to the problem.

Tall (1992) highlighted how the past experience of students, before they are introduced to the formal definition of a specific mathematical object, may affect the formation of their mental representations of the same object. Indeed, Tall argued that the words used in a definition are just one of the factors influencing students' way of thinking about a mathematical object. For example, if we consider the learning process of the mathematical object 'function', there are many students who succeed in giving a correct set theoretic definition but they are likely to use their intuitive images when they are asked general questions about functions (Vinner, 1983).

A description of the different roles played by the formal definition and the mental images of a student, about a specific mathematical object, has been initially provided by Vinner & Hershkowitz (1980) who introduced the terms *concept image* and *concept definition.* These terms were later explained by Tall & Vinner (1981) as follows:

> "we shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes […] the concept definition is a form of words used to specify that concept" (p. 152).

Vinner & Dreyfus (1989) examined which definitions and images of the mathematical object 'function' were evoked by college students and high school teachers. In particular, they described six categories for a complete characterization of students' answers when asked to define a function: correspondence, dependence relation, rule, operation, formula and representation. The concept image is the first thing evoked in our memory when we hear a term used to indicate the name of a concept, it is something that for some reason we associate with that concept and it is non-verbal, such that only at a second stage it can be translated into verbal forms. Moreover, it is possible that even people who know the set

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making all the techniques that the student will learn during his formation, not just as results of a mechanical repetition but as products coming from a reflection on the meanings, within the different contexts proposed" (translated by the authors).

theory definition of function do not provide it or a similar description of it when answering the question ‘what is a function?’ (Vinner, 2005). The study of the interplay between concept image and concept definition of a specific mathematical object, and especially of the concept images evoked by a student, can give information about the meaning that the student associates to that object. Concerning the example of functions, Dubinsky (1991) showed that

> “for most students, and indeed for many scientists, the idea of function is completely contained in the formula […] just as the concept of variable in which the student insists that x stands for a single number (which may not be known), the concept of function as formula has a very static flavor” (p. 116).

Moreover, Sfard (1992) described the answers given by a group of students, at the end of their math course at the university, which included set theory, algebra and calculus; when they were asked to say if each function can be expressed through a specific algebraic formula. She found that most of the students answered yes. Therefore, it seems that many students think of functions only in terms of their algebraic expression.

Now we are going to explain why a static conception of functions may be source of difficulties in students’ learning process. Dubinsky & Harel (1992) developed a framework distinguishing between different conceptions of functions, that can be used to analyze students’ learning process. In particular, according to their model students may have an *action view* or a *process view* of functions whose distinction is reported in Carlson & Oerthman (2005) as follows:

> “an action conception of function would involve the ability to plug numbers into an algebraic expression and calculate it. This is a static conception in that the subject will tend to think about it one step at a time (e.g., one evaluation of an expression). A student whose function conception is limited to actions might be able to form the composition of two functions, defined by algebraic expressions, by replacing each occurrence of the variable in one expression by the other expression and then simplifying; however, the students would probably be unable to compose two functions that are defined by tables or graphs.” (ibid, p. 7)

In contrast to this view,

> “a process conception of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done. When the subject has a process conception, he or she will be able, for example, to combine it with other processes, or even reverse it.” (ibid, p. 8)

Carlson & Oehrtman (2005) pointed out the necessity for students to move from what is called an action view to what is called a process view of functions. Indeed, the descriptions above highlight that students who think about a function in terms of symbolic manipulations and procedural techniques experience several difficulties in dealing with it as a general mapping of a set of input values to a set of output values. In contrast to this, having a process view may contribute to conceiving function relationships dynamically, that is, considering the change of the dependent variable respect to the change of the independent variable. These reflections suggest that students having an action view of functions may experience much more difficulties, while developing a process view seems to be not sufficient but at least fundamental for the understanding of many functions properties (Carlson, 1998; Thompson, 1994; Carlson et al., 2002). For example, Carlson & Oerthman (2005, p.9) argued that

> "the ability to coordinate function inputs and outputs dynamically is an essential reasoning ability for limits, derivatives, and definite integrals. In order to understand the definition of a limit, a student must coordinate an entire interval of output values, imagine reversing the function process, and determine the corresponding region of input values. The action of a function on these values must be considered simultaneously since another process (one of reducing the size of the neighborhood in the range) must be applied while coordinating the results. Unfortunately, most pre-calculus students do not develop beyond an action view, and even strong calculus students have a poorly developed process view that often leads only to computational proficiency (Carlson, 1998)."

Many researchers reported on possible problems encountered by students when dealing with the graphical representation of a function. For example, among students a common interpretation of the Cartesian graph of a function is as an object or a static picture of a physical situation, rather than as the trajectory of a point moving on the plane according to the covariation of two quantities, one dependent on the other. This is actually one of the main problems pointed out by Carlson (1998), in line with Thompson (1994), who stressed the importance of conceiving functions as asymmetric relations between two variables, one depending on the other. On the contrary, what happens is that very few students seem to relate the Cartesian graph to the underlying functional relationship, but they see it as made up of points and lines; indeed, their attention is often on specific points (Dubinsky, 1991; Tall, 1992). Other difficulties concerning the interpretation of the graph have been described by Tall (1996) and Barnes (1988), who found that according to many students the graph of a function should be regular and smooth. Moreover, they affirmed that the algebraic expression of a function should be given by a single formula and often, given the algebraic expression of a constant function, students did not consider it to represent a function because the dependency on x was not explicit.

It is important to notice that most of students' difficulties regarding functions can also be found among teachers. In particular, Even (1993) highlighted that many prospective teachers considered functions to be equations or formulas, they claimed that graphs of functions should be "nice" and that functions were all known. Her study also showed that people having a concept image of functions as equations or as "nice" graphs were likely to consider, for example, circumferences and ellipses to be functions.

Finally, there is a great number of more recent studies investigating students' learning of calculus, that show how for many students understanding functions and their properties is conceptually challenging. To address this, researchers developed a variety of new approaches to functions and graphs of functions, aimed at promoting students' engagement in the teaching and learning process (Jayakody, 2015; Ferrara & Ferrari, 2018). In the next section we are going to describe some of these approaches, especially focusing on that aimed at fostering a description of functions involving an analysis of changes, even when change in terms of average or instantaneous rate of change and derivative are not yet introduced to students (Sahin-Gur & Prediger, 2018).

1.1.2 Different didactical approaches to functions

We start this section with a very short description of the historical genesis of the concept of function, which had an interesting evolution that allows us to better frame different approaches traditionally used for teaching functions. In particular, this evolution of the concept has been in some cases described as a move from a *dynamic* to a *static* definition of functional dependency (Freudenthal, 1983), or from an *operational* view, as a process, to a *structural* view, as an object (Sfard, 1991).

In the attempt to briefly summarize the slow and intense process of development of the concept of function among mathematicians, it is possible to identify three main phases that characterized it (Boyer, 1946). Initially (17th century) the attention was mainly oriented towards motion and the relationships between quantities. However, the representation of these relations was mostly realized geometrically, which allowed people to convey relationships only statically; so movements were not explicitly addressed. Then Boyer described a spreading of equations, that thanks to the Descartes' creation of algebraic notation, were used to better represent motion. Through equations it was finally possible to express the variations of two quantities, whose changing values were represented by variables, by constraining the two variables one to the other. Finally, it became possible to explicitly represent the relationship between the variations of two quantities and, in particular, when the values of the two variables were linked by a dependence relation such that the values of one determined the values of the other, the relationship between them was represented by a formula or a graph. The term 'function' appeared for the first time in 1692, in a work by Leibniz and then, in the 18th century, it was used by Bernoulli to denote an analytic expression, made up of variables and constants, representing the relation between variables and by its graph having no "sharp corners" (Even, 1990, 1993).

A progressive attempt to eliminate time from the definition of function took place, until this problem was solved by Bourbaki's group that introduced a formal definition of function. At this regard we cite Frege:

> "In recent times the word 'variable' is predominant in the definitions [of function]. Consequently Analysis would have to deal with a process in time, since it takes variables into consideration. But in fact it has nothing to do with time; its applicability to occurrences in time is irrelevant… as soon as we try to mention a variable, we shall hit upon something that varies in time and thus does not belong to pure Analysis." (Frege, 1970, p. 107; see Sfard, 1991)

Dirichlet laid the foundations for function notation claiming that "a precise law of correspondence between *x* and *y* can be stated clearly" (Dirichlet, as quoted in Boyer, 1946). Moreover, he stressed that this law of correspondence could be arbitrary and that functions could be discontinuous, by proposing the famous "Dirichlet function" as an example. Only from this moment arbitrary functions started to be considered functions, thus the concept enlarged its meaning. Researchers highlighted that deep changes took place thanks to the evolution in the definition of functions, for example, Malik (1980) expressed that

> "a deep gap separates early notions of function based on an implicit sense of motion and the modern definition of function that is algebraic in spirit, appeals to discrete approach and lacks a feel for variable" (p. 492).

The mathematical definition of function that nowadays is presented in schools and textbooks follows the definition given by Bourbaki in 1939:

> "Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a functional relation in y if, for all x in E, there exists a unique y in F which is in the given relation with x." (Kleiner 1989, p. 299)

But it is usually stated in terms of Cartesian products and ordered pairs (Vinner, 2005; Thompson & Carlson, 2017). For example, we looked at the high school textbooks by Bergamini, Trifone and Barozzi (2005) and by Sasso (2015), that are among the most adopted textbooks in Italian high schools, and we noticed that the concept of function is presented through its formal definition. This definition is minimal and elegant, following the Dirichlet-Bourbaki approach. According to this approach, the definition of function includes also many correspondences that were previously not considered as functions; for example, discontinuous functions or that defined on split domains. However, most of the examples that are reported in textbooks are functions given by a formula or a Cartesian graph.

A possible consequence of considering the representation through the Cartesian plane or the formal definition is that students are brought to associate the function to a static image, which is often hard to interpret and figure out for them. Sierpinska (1988) argued that introducing functions to students starting from the formal definition may be a didactical error, because:

> "the most fundamental conception of a function is that of a relationship between variable magnitudes. If this is not developed, representations such as equations and graphs lose their meaning and become isolated from one another" (p. 572).

Moreover, according to Falcade (2003), often the crucial problem is that, even if students grip the idea of correspondence, they are not able to perceive the covariation of the two variables. To foster this understanding through a dynamic interpretation of the notion of function, Laborde & Mariotti (2001) and Falcade (2001) came up with a key idea: to consider the curve representing the function on a Cartesian plane as the trajectory of a point. They developed this idea within the study of geometrical functions in a dynamic environment, but they suggested to also implement it out of that environment in order to support an interpretation of the Cartesian graph of functions as an object incorporating the asymmetric relation of covariation among two variables.

Confrey & Smith (1995) presented a general approach for teaching exponential functions, based on their previous research where they investigated possible implications of the use of contextual problems (Confrey, 1991; Confrey & Smith, 1994), transformations and multiple representations. In particular, they described a model, which they called "splitting", that can be efficiently used to define multiplication and division, instead of recurring to repeated additions. Then they argued that a combination of "splitting" and "a covariational approach to functions" seemed to be effective for students' understanding of exponential functions. Where a *covariational approach* is intended as a way of describing situations in terms of rate of change and they presented it in contrast to a correspondence approach to functions. Actually, it happened that the distinction between 'action view' and 'process view' of function has been gradually considered as a dynamic interplay, which led the growth of several approaches emphasizing the covariational aspect of functions (Carlson et al., 2002; Thompson, 2011). As we will better explain later in this chapter, the 'covariational view' is based on the understanding of the manner in which dependent and independent variables change as well as of the coordination of their variations. In a very recent study, drawing on Thompson's theory of quantitative reasoning (Thompson, 1994), Johnson & McClintock (2018) identified students' quantitative variational reasoning as a possible factor affording their discernment of variation in unidirectional change. Where "quantitative variational reasoning" is called students' conception of attributes as something that varies and that can be measured, while "variation in unidirectional change" means that the direction of change is invariant. The studies concerning possible covariational approaches to functions are foundational for this study and we will examine in depth this part of the literature in Section 1.2.

However, as we discussed above, a very common approach for introducing functions is that of starting from the formal definition as a correspondence between two sets, but there are different uses of functions. There are several possible notations, and also different labels commonly used in mathematics addressing functions: mapping, transformation, permutation, operation, functional, operator, relation, morphism, etc.. From a didactical point of view, all these aspects together contribute to making it difficult to teach and learn 'function' as a mathematical concept. In particular, as a direct consequence of the different uses of functions, a variety of possible approaches to functions came up. For example, pointwise approaches allow to plot, read or deal with single points of the function; other approaches focus on specific intervals, such as a neighborhood of a local extremum; global approaches show the global behavior of the function within its domain (Even, 1990). Moreover, there is a variety of possible representations: functions can be introduced through diagrams with arrows, input-output tables, algebraic expressions, sets of ordered pairs. For example, working with tables of data may be an approach to functions, that involves the process of entering the data and then the coordination of the columns. From the literature we observe that since the computer has been started to be used for introducing the function concept, researchers have mainly focused on the graphical representation (Yerushalmy, 1991).

Bell & Janvier (1981) were among the first to claim that the construction of the graph of a function starting from an input-output table was not a fruitful approach for students' understanding of the underlying relations. Alternatively, they suggested to introduce graphs to students through a qualitative approach. More generally, many studies have supported the importance of going back to the first interpretation of functions, which involved variables and time, instead of stressing the use of the formal and static definition. Ayalon, Watson & Lerman (2016) highlighted some of the possible positive contributions of dealing with variables and the concept of variation for the understanding of functions. The teaching approach that they proposed provides some graphing experiences from everyday life, in an attempt to foster students' reasoning about variables and related concepts as rate of change, but also in an attempt to link these everyday situations to a more formal algebraic language.

Regardless of the approach chosen to introduce functions, we noticed that the first examples given to students are usually linear functions. Discussing this, Markovits et al. (1986, 1988) highlighted how giving emphasis to straight-line graphs, especially when introducing functions, may then bring students to draw linear graphs each time they are asked to trace a possible graph passing through certain given points. Hitt & Gonzalez-Martin (2016) reported the main contributions to research on the topic of functions and, among the others, they cited a study conducted by Aspinwall, Haciomeroglu & Presmeg (2008) where it is showed that students who were successful in calculus used a combination of visualization and analytic thinking. This was interpreted as their becoming able to dynamically transform their visual mental images, and so to construct and interpret graphs of functions.

After this overview of the main approaches that have been developed to teach functions, in the next section we are going to describe some studies existing in literature about the use of technological artifacts for representing and teaching functions. In particular, we will focus on that implementing DIEs.

1.1.3 Using technological artifacts to introduce functions

The widespread availability of computers gave rise to the design of new approaches to functions, based on representations realized by implementing technological artifacts. Initially, the use of computers in calculus lessons involved programming and implementing numerical algorithms, then software for graphical tools appeared.

Al Cuoco (1995) described innovative approaches to functions, involving computer environments, that have been designed to support students' development of computational models for representing functions. In particular, *Function Machines* was a programming language whose underlying idea was that of building input-output machines. *Logo* was another programming language that presented as a sequence of instructions being performed on a specific input to produce an output. Finally, ISETL was a language more similar to the language of mathematics, since it used notation and constructions that appeared to be more formal. He found that an approach to functions through programming in Logo gave significantly different insights from a traditional approach. For example, he observed that "students who are able to model a situation with a Logo procedure are already viewing the function at hand as a process" (ibid, p. 12). Indeed, he described Logo as providing "an extensible and playful environment in which students can build and experiment with processes, compare them, and begin to manipulate them as data" (ibid, p. 17).

Schwarz & Dreyfus (1995) created an entire curriculum on functions, using a computer microworld called *Triple Representation Model*, with the aim of supporting problem solving processes. Moreover, this model was used to foster students' use of different representations of functions, taken from several settings (tables, graphs and algebra), and also to ask them to generate other representations. The students involved in their study seemed to be able to cope with partial data about functions, to coordinate different representations, recognizing invariants in different representations belonging to the same setting.

Tall (1996) identified the main potentiality of an approach using computer graphics with the possibility to magnify the graph of a function. Indeed, by magnifying the graph it will look always less curved; going on with this process, it is possible to see it become almost straight. When it looks visibly straight, the slope of the curve representing the function is the same as the slope of the line on the screen. This type of approach involves the concept of limit, implicitly expressed during the magnification process, and at the same time it shows students a way to find an approximation of the derivative.

However, among the various technological artifacts that can be used in the teaching and learning of functions, for this study we are particularly interested in DIEs. They have been developed especially for the dynamic geometry, but in some cases they have been implemented also for representing functions. For example, Hazzan and Goldenberg (1997) proposed an approach to functions built on dynamic geometry, in contrast with the numerical approach: it consisted in analyzing geometrical constructions by looking at the functional dependency that linked the geometric objects involved, as points and lines. One of the possible consequences related to this approach, from a didactical point of view, was pointed out by Grugnetti, Marchini & Maffini (1999). They observed that, if the geometrical relations proposed can be all described through continuous functions, a risk for students could be that of considering all functions as being continuous. Another approach based on the use of dynamic geometry was developed by Falcade, Laborde and Mariotti (2007). They introduced the functional dependency by first having students experience variables' movements in an attempt to support a dynamic interpretation of the dependence relation. In particular, they implemented the activities in the software Cabri where it is possible to construct geometrical objects; with the aim of "providing a qualitative experience of covariation, and in particular an experience of functional dependency not primarily based on a numerical setting" (p. 318). Finally, Ng (2014, 2016) drew on Sfard's communicational approach to investigate changes in bilingual learners' communication about derivative, during their interactions with touchscreen-based DIEs. In particular, she highlighted the role of the DIE and students' use of gestures for conveying dynamic and temporal relationships. Exploring a construction realized within a DIE involves searching for possible relationships existing between the movements, that can be obtained through dragging and that can be perceived as variations or invariants. A high number of research studies highlights how the identification of such invariants lies at the heart of a dynamic exploration (Holzl, 1996; Arzarello, Olivero, Paola & Robutti, 2002; Olivero 2002; Healy & Hoyles 2001; Laborde, 2005; Baccaglini-Frank, Mariotti & Antonini, 2009; Leung, Baccaglini-Frank & Mariotti, 2013).

There is also a wide literature about the design of the tasks and about the role of the teacher in organizing and orchestrating the discussions, during classroom activities implemented within a DIE (Laborde, 2001; Bartolini Bussi & Mariotti, 2008; Mariotti, 2002). Moreover, some researchers investigated how a description in terms of logical dependency can be supported by explorations involving the use of the dragging tool, thanks to the difference between two possible types of motion: *direct* and *indirect* (Mariotti, 2006, 2010; Laborde, 2003). This distinction will be particularly inspiring for our study because, in a similar way, we want to investigate how students can be introduced to the functional dependency through experiences of direct and indirect dragging in DIEs.

Now we present some relevant studies concerning the characterization of different dragging modalities spontaneously used by experts during the exploration and solution of geometric open problems implemented in a DIE. Even if this literature is not specifically related to the teaching and learning of functions, it will be useful and we will refer to it in this study in order to investigate possible uses of dragging, that is a characterizing feature of DIEs, which enables to move objects on the computer screen according to different types of motion, depending on the construction.

Arzarello et al. (1998, 2002) and Olivero (2002) operated a classification of different functions of dragging in Cabri environments for describing some of their cognitive features in learning processes. Indeed, their analysis focused on the use of the dragging tool from a cognitive point of view and it highlighted how different uses of dragging depended on different aims during the solution process of a given geometric problem.

The following table shows their classification of the dragging modalities summarized.

Tag	Description
Wandering dragging	Moving the basic points on the screen randomly, without a plan, in order to discover interesting configurations or regularities in the figures
Bound dragging	Moving a semi-draggable point, which is already linked to an object
Guided dragging	Moving the basic points of a figure in order to give it a particular Shape
Dummy locus dragging	Moving a basic point so that the figure keeps a discovered property; that means you are following a hidden path even without being aware of this
Line dragging	Drawing new points on the ones that keep the regularity of the figure
Linked dragging	Linking a point to an object and moving it onto that object
Dragging test	Moving draggable or semi-draggable points in order to see whether the figure keeps the initial properties. If so, then the figure passes the test; if not, then the figure was not constructed according to the geometric properties you wanted it to have

Table 1.1. Dragging modalities (Arzarello et al., 2002)

Antonini & Martignone (2009) proposed a similar classification in the case of physical artifacts. They introduced a classification of students' utilization schemes of pantographs, that are particular mathematical machines designed for geometrical transformations. Although the differences due to the different nature of the instruments these two studies concern, there are certain similarities. Especially the common purpose is to identify students' utilization schemes in order to analyze the cognitive processes involved in the investigation of geometric problems.

A further contribution to this classification has been proposed by Baccaglini-Frank & Mariotti (2010) who gave a description of *maintaining dragging*. In particular, they defined it as follows:

> “maintaining dragging (MD) involves the recognition of a particular configuration as interesting, and the user’s attempt to induce the particular property to become an invariant during dragging” (p. 230).

The same study reveals how the combination of maintaining dragging and the activation of the trace tool on the selected base point may be particularly useful for the process of generation of conjectures during the solution of certain geometric problems.

1.2 THE COVARIATIONAL ASPECT OF FUNCTIONS

The first studies on covariational reasoning were conducted by Confrey and Thompson, who developed this theoretical construct characterizing covariation in two slightly different ways. On one hand, in terms of coordinating two variables’ values as they change (Confrey, 1994), on the other hand in terms of conceptualizing individual quantities’ values as varying and then conceptualizing two or more quantities as varying simultaneously (Thompson, 1994).

Thompson supported the idea that the concept of rate was fundamental for developing a dynamic interpretation of functions; while Confrey & Smith (1994) described a covariation approach to functions as follows:

> "it entails being able to move operationally from y_m to y_{m+1} coordinating with movement from x_m to x_{m+1}. For tables, it involves the coordination of the variation in two or more columns as one moves down (or up) the table" (p. 137).

Carlson also contributed to the characterization of functions as covariation. By drawing from Confrey and Thompson's earlier work Carlson, Jacobs, Coe, Larsen & Hsu (2002) created a framework for studying covariational reasoning and they illustrated how their framework could be used to describe students' cognitive processes when they are asked to deal with dynamic situations that involve two simultaneously changing quantities. In the covariation framework five developmental levels of mental actions were specified, such that they become more sophisticated depending on the nature of students' coordination of the values of the quantities involved. For example, students may look at the amount of change or at the direction of change, or at both of them. Moreover, they concluded that a student reached a given level of development if he showed covariational reasoning associated with that level and all lower levels.

Thompson & Carlson (2017) revised prior covariational frameworks in two ways: by attending to students' variational reasoning separately from their covariational reasoning; and by attending to how students coordinated their images of quantities' values varying, taking into account their way of reasoning variationally.

From this detailed description of the covariational aspects characterizing functions, it emerges that there are two possible ways of interpreting functional dependency: as a correspondence, which means functions as entities that accept an input and produce an output; or covariationally as a process involving two quantities varying together. The first aspect answers questions like "which $f(x)$ belongs to a specific x?" or "which x belongs to a specific $f(x)$?", while the second aspect is related to questions like "how does $f(x)$ change when x increases?" or "how do we have to change x in order to decrease $f(x)$?". Some representations of functions highlight one of these two aspects and the identification of the other one is not always immediate, but it involves a deeper interpretation of the representation.

The covariational view of functions has been found to be essential for understanding other concepts of calculus that are related to functions, such as limits and derivatives (Kaput, 1992; Cottrill et al., 1996; Saldanha & Thompson, 1998). In line with these findings, many recent studies about the teaching and learning of functions focused on the covariational aspects, and most of them also investigated possible ways of employing digital environments. Indeed, in order to provide students with opportunities to use a covariation perspective on functions, researchers designed activities involving dynamic and static Cartesian graphs (Ellis et al., 2015; Hitt & González-Martín, 2015; Johnson & McClintock, 2018). For example, Kafetzopoulos & Psycharis (2016) implemented the software Casyopée (Lagrange, 2010) to design modelling tasks involving geometrical dependencies, aimed at studying students' conceptualization of function. Nagle, Tracy, Adams & Scutella (2017) highlighted the importance of fostering students' dynamic imagery to include simultaneous movement in both the x- and y-coordinates to avoid the common conception of the limit of a function as the value reached near a certain x value. Moreover, they argued that students' attention should be lead towards the changing in both the x and y values, whatever approach is chosen, for example when looking at a table of values or a graphical representation of the function.

Falcade, Laborde & Mariotti (2007) developed a covariational approach to functions by using a DIE; their study was framed within the framework of the semiotic mediation. It was particularly interesting for us to see that a fundamental assumption of the teaching sequence that they designed was that motion constituted a basic metaphor for covariation, indeed they argued that covariation can be experienced through change and the first change that we can think of is that of space in function of time, that is *motion* (Lakoff & Nunez, 2000). Based on this assumption, they introduced the Cartesian graph of functions starting from a dynamic representation realized within the software Cabri. In particular, they supported the interpretation of the graph of a function as the trajectory of a certain point P in the plane, representing the dependent variable, in function of the variation of another point M such that M belongs to the x-axis and it represents the independent variable (Laborde, 1999). In order to develop this reading, it became necessary for them to introduce the temporal dimension and to show the covariation of P and M through the simultaneous variation of the two points. They called this interpretation as "dynamic interpretation" of a graph.

An important assumption at the core of this study is that covariation is an essential feature of the concept of function. In particular, we observe that here the term 'covariation' will always be used referring to quantities whose simultaneous variations are also related by a dependence relation. More specifically, we consider covariation to be a dynamic relation between two variables that is an asymmetric relation, because the variations of one of the variables depends on the variations of the other. This interpretation gives a bit of a different meaning to the term 'covariation' with respect to some of the studies that we described above, especially, with respect to the original description given by Confrey (1994). She referred to the rate of change of variables within the numerical context by looking at tables of values and focusing on the variations of the two variables in quantitative terms. Differently, our description of covariation is qualitative, more in line with the dynamic interpretation of the graph suggested by Falcade, Laborde & Mariotti (2007).

1.3 GRAPHICAL REPRESENTATIONS OF FUNCTIONS

The overview on students' difficulties relative to the notion of variable and to the graphical representation of functions, that we made above, brought us to conclude that a purely quantitative approach to functions, mainly involving numerical and algebraic calculations, may be a source of several problems from a didactical point of view.

In this section we are going to advance some considerations on the Cartesian graph to represent a real function of a real variable and then present studies that propose other graphical representations, designed in order to bring out the covariational aspect of the functional relation, in dynamic terms.

1.3.1 Cartesian graphs

The Cartesian graph, as mathematical object, is the set of points $(x, f(x))$, where the independent variable x belongs to the domain of the function and the dependent variable $f(x)$ is its image. Traditionally, 'Cartesian graph' is used to indicate a drawing of this set of points in the Cartesian plane and, from now on, we will always use 'Cartesian graph' referring to this representation of the set. This description suggests that a curve on the Cartesian plane representing the graph of a function is made up of points that incorporate the functional relation between the two variables in their coordinates. In particular, these coordinates are two numbers belonging to the same set (the set of real numbers) but they correspond to two points respectively belonging to two different axes: one horizontally oriented and the other vertically oriented. This splitting of the set makes the construction of the curve possible but it can also be confusing for a student who is approaching functions for the first time. A possible obstacle consists in recognizing that each point on the graph is a coordinated presentation of two pieces of information, a domain point and its image. Indeed, very often students consider the curve to be the image of the function and they identify a point on the curve as "the $f(x)$ value"; this is incorrect, since a point is defined by two values and not just one (Colacicco, Lisarelli & Antonini, 2017). In particular, this type of difficulties then influences the understanding of more advanced mathematical objects, for example limits. In the same paper, it is shown the case of a person having a degree in a scientific subject, that, by looking at the Cartesian graph of the function $f(x) = \frac{1}{x}$, affirms that the limit of the function for x tending to positive infinity is positive infinity and, while speaking, he moves one hand showing that the function 'keeps going to the right'.

Similar observations have also been pointed out by Thompson & Carlson (2017) who suggested that many of students' difficulties about the interpretation of the Cartesian graph in terms of covariation of two quantities "are grounded in their not having conceived points on a graph as multiplicative objects that represent two measurements simultaneously". Moreover, what happens is that, often, students do not associate the graphical representation in the Cartesian plane directly to functions. Indeed, this type of representation is used in math classes long time before the teaching of functions, for example, for working with circumferences and ellipses. In these cases, the roles of the two variables, x and y, are symmetrical. Therefore, a possible source of difficulty for students consists in considering functions as relations between two variables such that the order is not important in the pair (x, y).

Therefore, the Cartesian graph is an extremely rich in meaning and useful representation of real functions but, at the same time, the interpretation and manipulation of a graph requires a deep understanding of the relations existing between its elements. As we just discussed, the reconstruction of these relations is not an easy task from a cognitive point of view. For this reason, some researchers began to consider alternative graphical representations of functions and their properties.

1.3.2 Dynamic graphs

Goldenberg, Lewis & O'Keefe (1992) called DynaGraph an artifact that they created to visually realize functions, in which the domain variable is dynamically variable and it is separately presented by its image. It consists in representing two horizontal lines with one point on each line, so it develops in one dimension and originally they called these two axes 'x Line' and 'f(x) Line'.

This particular representation of a function cannot be obtained without using a DIE, where objects can be moved on the screen thanks to the dragging tool. These types of software can help students focus their mathematical thinking on bigger and often more abstract mathematical ideas than it usually happens in paper-and-pencil environments. Indeed, a dynamic algebra and geometry software allows students to manipulate and investigate constructions without being mired in the technical aspects of drawing them, as can sometimes happen if they must sketch on paper. Furthermore, constructions on paper-and-pencil context virtually force students' attention on the actual positions of objects. DIEs designed for experimentation make these features changeable and they bring students' attention to the invariant properties of the construction, that are not easy to observe in a static environment (Hazzan & Goldenberg, 1997).

In particular, thanks to the dragging tool it is possible to obtain two different types of movement: indirect and direct. The direct motion occurs when a basic element, for example a point generated by the point tool, is dragged by acting directly on it; while the indirect motion occurs when a construction procedure is accomplished and the motion of the elements obtained through it can be realized only by dragging the basic points from which the construction originates (Baccaglini-Frank & Mariotti, 2010).

In the case of DynaGraphs the independence of the x variable is realized by the possibility of freely dragging a point, bounded to a line (the x Line) and the resulting movement visually mediates the variation of the point within a specific domain. Whereas the dependence of the $f(x)$ variable is realized by an indirect motion: the dragging of the independent variable along its axis causes the motion of a point, bounded to another line (the f(x) Line), that could not be directly dragged. Indeed, the indirect motion preserves the properties defined by the construction, that in a DynaGraph consist in keeping the functional relation that links the dependent variable to the independent one invariant. In other words, the use of dragging tool visually mediates the experience of functional dependency which is realized by the dependence relation between two different types of motion: a direct and an indirect motion.

Moreover, the movement of points experienced through the use of the dragging tool can be visually materialized through the trace tool that, when activated on a point, allows the user to display its trajectory of movement. Although the final product of the trace tool is a static image, its use involves time and so it is possible to simultaneously grasp the pointwise and the global aspect of the product of trace tool: at the same time a sequence of positions of a moving point and the image consisting in the set of all such positions.

Looking at the literature in math education, we observed that the representation of function with parallel axes was not largely used in the teaching of functions. We found that the original idea of DynaGraphs has also been developed by Healy & Sinclair (2007) and by Sinclair, Healy & Reis Sales (2009) who used the Geometer's Sketchpad to dynamically realize the graphs of functions. In their representation the asymmetric relation between the two variables was visually realized as suggested by Goldenberg and his colleagues, by distinguishing between possible and impossible movements; there were just a few differences in the layout of the interactive files. They named the variables by A and f(A), but did not name the lines; and they added a segment linking A and f(A). Moreover, we found that Arcavi and Nachmias (1993) introduced a parallel axes representation of functions which they used for families of linear functions. However, their representation was very different from that designed by Goldenberg et al. (1992): it consisted of two parallel vertical axes, the one on the left is used for the domain, while the one on the right for the codomain and there were several "mapping segments" joining a number on the left axis with its image on the other axis. Since they proposed this representation for linear functions, it is possible to observe that the mapping segments were always intersecting at one point, except for $y = x$ where they were parallel. Therefore, the most significant difference with respect to DynaGraphs is the fact that this representation could be realized through computerized tools but also within a paper-and-pencil context.

The idea of considering the graph of functions as a representation of the dynamic relation between the variables and the attempt to foster the emergence of this dynamism is completely in line with the description of the graph given by Euler (1743). The original text has been translated by Mariotti, Laborde & Falcade (2003), who studied a method to represent geometrically the graph of numeric functions; as follows:

> “Because, then, a unlimited straight line represents a variable quantity x, let’s look for a method equally comfortable (useful) to represent any function of x geometrically. […] Thus any function of x, geometrically interpreted in this manner, will correspond to a well defined line, straight or curve, the nature of which will depend on the nature of the function.” (Euler, 1743, p.4-5, translated by Mariotti, Laborde & Falcade, 2003)

The representation suggested by Euler and developed by Mariotti and colleagues consists in building the graph of a function as the trajectory of a point M which is the extremity of a segment PM, whose other extremity P is a variable point on the x-axis that has a distance x from the origin. Then, M is on the perpendicular line to the x-axis passing through P and such that the length of the segment PM is determined by $f(x)$. A possible difficulty for students, that can originate from this approach, is about the interpretation of the trajectory that has a twofold meaning. Indeed, it is a succession of positions of a certain point moving on the plane and, at the same time, it is a static object visible on the screen.

In designing and carrying out our study we expected that the one dimensional representation of functions could foster a description of relative movements of the variables and comparisons between possible movements of the ticks on the lines. For example, when exploring one of these dynamic graphs, it is quite an easy task to recognize whether the two variables’ movements follow the same direction or opposite directions, that can be identified by an expert as an information about the monotonicity properties of the function: a function is increasing if both variables have the same direction of movement while it is decreasing if the variables have opposite directions. In a similar way, a change in direction of the dependent variable, with the independent one always following the same direction, reveals the presence of an extremum point. In addition to changes in direction, the dynamic graphs also provide information about the rate of change. For example, moving the independent variable at constant speed along the x-axis can result in constant growth, which the user feels actually observing the dependent variable having always the same increment. Similarly, moving x at a constant speed can result in accelerated growth and the user sees $f(x)$ whisking off the screen. Speaking about more advanced mathematical objects, descriptions of change in speed could be read mathematically as observations on the slope of the function, that is its derivative.

The being undefined of a function at one point is realized in the dynamic graphs by the dependent variable disappearing from the screen, as soon as the independent variable is dragged on a value for which the function is undefined and then it quickly comes back. For example, if the function has a vertical asymptote at x_0, with different limits at x_0+ and x_0-, then the dependent variable comes back from the other side of the screen. This sudden disappearing of the tick can be surprising for someone who doesn’t know which function has been defined and very often students produce original narratives to describe this behavior (Sinclair et al., 2009). A study of narratives (Bruner, 1996) emerging during students’ discussion in mathematics classroom has been conducted by Healy & Sinclair (2007), who investigated a possible relation between narrative modes of thinking and the learning of mathematics. The mathematical activities that they designed involved a computer interaction and, for example, the use of dynamic graphs.

1.4 RESEARCH QUESTIONS: A FIRST FORMULATION

We analyzed the most common student and teacher difficulties and different approaches that have been used in the teaching of functions and graphs of functions. From the literature review it emerges also that functions have been studied through different theoretical perspectives.

Our objective behind this research study is to introduce high school students to the notion of function by highlighting its covariational aspect. We want to study an approach to functions that allows to stress the dynamic aspects of the concepts of variable and dependency among variables, that so many studies have shown are hard for students to understand (Arcavi & Schoenfeld, 1988). In particular, we refer to Tall's (2009) description of calculus as the mathematical field that begins with the desire to quantify how things change, the function, the rate at which they change, the derivative, and the way in which they accumulate, the integral. According to this view, this field is fundamentally dynamic: even the calculation of static quantities, such as areas or volumes, involves dynamic processes of adding up a large number of very tiny elements. Then, at a certain point in history a transformation happened and calculus turned into rigorous definitions, developing the formal theory of mathematical analysis. For example, Solomon & O'Neill (1998) gave the following description of mathematics:

> "[mathematics] is structured around logical and not temporal relations" (p. 217).

However, we observe that expert mathematicians' reasoning is still characterized by an interplay between temporal and a-temporal attributes. In particular, Menz (2015, p. 31) highlighted that

> "There is also a huge discrepancy between the informal discourse between mathematicians and the presentation of work at a talk or in paper form."

For example, when exploring a new mathematical problem or trying some conjectures experts usually refer to time, because they speak about relations and processes. They draw sketches, use gestures and, at the same time, they use the mathematical formal vocabulary which allows them to be more concise and to directly refer to known mathematical objects. So, "doing mathematics" implies a continuous dialectic between the temporal dimension and the products of objectification. Indeed, even if formal mathematical discourse aims at eliminating time and dynamism, this does not imply that mathematicians engage in purely a-temporal modes of thinking. In particular, they seem to frequently communicate in ways that suggest they think of mathematical objects in motion (Sinclair & Gol Tabaghi, 2010). Many examples of this aspect are reported in a research study conducted by Menz (2015) who showed this dialectic between formal and less formal communication among experts:

"The speech of the expert mathematicians is tightly linked with the bodily enactments of mathematical objects, and that there is a flow of communication among the expert mathematicians. Furthermore, the discourse of the expert mathematicians often centres on diagrams and is heavily saturated with gestures of pointing, hand-pointing, touchpointing, holding, tracing, sweeping and covering up" (p. 153).On the basis of these reflections, we bring back temporality and movement in the teaching and learning of functions, by introducing students to functions and graphs through a covariational approach, according to the meaning of 'covariation' expressed in section 1.2. We hoped to have students experience the functional dependency in qualitative terms. Based on the studies conducted by Falcade et al. (2007) and Ng (2016), the idea is that of implementing a DIE to represent functions, and to investigate students' learning process. In particular, they found that introducing students to functions and graphs through the explorations in a DIE and then providing situations for them to communicate in both static and dynamic contexts was beneficial for their learning. Moreover, at the beginning of this chapter we have presented several studies highlighting difficulties that students experience regarding function relationships, if they do not have the ability to think dynamically. Indeed, as Thompson & Carlson (2017) argued, continuous variation and covariation seem to be epistemologically necessary for students and teachers to develop robust conceptions of functions. This is because the main purpose of functions is to represent how things change and in DIEs it is possible to experience variation and functional dependency in the form of motion.

Now we are going to clearly state the main focuses of our research, by giving a first formulation of the research questions that are at the core of this research study:

i. Does the dynamic representation of functions that we propose support students' experience of the dependence relation in terms of covariation? If so, what instances of covariation is it possible to identify?
ii. What is the role of dragging, if it has one, in students' learning of functions? Is it used to express covariation? If so, how?
iii. Do students attempt to relate together the dynamic and static representations of functions that we propose? If so, what recurrent features is possible to identify? And what difficulties do they eventually encounter?
iv. In students' protocols are there possible connections with the mathematical notions considered in the design of the activities? If so, how are these notions represented by students?

These questions will be recalled in the third chapter and, in light of the theoretical framework that we are going to make explicit in the next chapter, they will be reformulated.

2 THEORETICAL FRAMEWORK

In this chapter we present the foundational elements of the theory of commognition (Sfard, 2008), which is the theoretical framework upon which we based this study. We describe in particular the tools offered by this theory that are most significant for our research and also some specific theoretical constructs that we further elaborated and use in this study.

In particular, we explain how the theory of commognition describes mathematics, showing the interpretation given to mathematical objects and learning. This part is particularly relevant for us, because we are interested in studying students' learning process of functions, by conducting analyses as much objective as possible. Then, we discuss about different uses of the term 'mediation' that can be found in literature, which highly depend on the theoretical perspective adopted. This discussion allows us to clarify in which view this work can be placed, especially with respect to the use of artifacts in the teaching and learning process. Indeed, we use a Dynamic Interactive Environment (DIE) to represent functions and, at this regard, we propose a possible refinement of the theory of commognition, in light of some other studies that investigated about the use of DIEs in the teaching and learning of mathematics. Finally, we read through a commognitive lens some studies about dragging practices in DIEs, which we already presented in the previous chapter and they have been inspiring for our research. In particular, based on these studies we develop helpful tools of analysis of students' use of dragging in their discourse on functions.

2.1 THE THEORY OF COMMOGNITION

In our study we chose to use a commognitive perspective because it highlights the communicational nature of learning and it provides us with analytical tools for examining students' learning of functions. Moreover, we adopted this theory for its philosophical assumption about the relation of thinking and communicating, because it allowed us to investigate students' thinking processes by focusing just on observable forms of communication that they used.

In particular, we think that an important feature of the commognitive framework for this study is that it allows capturing fine-grained details, giving an operational description of what mathematical objects are and how they become objects of communication for students. Indeed, we also share this dialogical view according to which words take on meanings from the discourse in which they are used. In particular, Sfard's work for operationalizing concepts like 'thinking' or 'communicating' is rooted in Wittgenstein's definition of the meaning of words. He claimed that

> "for a *large* class of cases – though not for all – in which we employ the word 'meaning' it can be defined thus: the meaning of a word is its use in the language" (Wittgenstein, 1953/2003, p.18).

Moreover, the commognitive framework is based upon the social dimensions of the learning process which is identified in the interaction of a person with other individuals. This view is grounded upon the works of Wittgenstein and Vygotsky. In particular, Vygotsky (1987) highlighted the social nature of learning, giving rise to several theories of learning that give importance to the social interactions and to the sociocultural context of learning. Indeed, according to these theories individuals are considered as initially participating in activities of a group before becoming fully integrated into the group activities. Taking on this participationist view, that considers learning as inherently social and highly situated, we look at mathematical learning occurring in social contexts and situated in specific activities carefully designed to elicit mathematical discourse. Therefore, according to this perspective, students may change their way of acting and talking about mathematics through participation in mathematical activities, because there exists a strong link between mathematics learning and communication.

In the light of these preliminary observations, the main reason that leads us to ground this study on the theory of commognition is that it provides us with the lenses to objectively observe and describe the main features of students' communication (in a broad sense, as we are going to explain) about functions, eventually involving also a possible development of this communication. This type of observation gives us an insight into students' learning process of functions, which can be actually described as the process of developing a discourse on functions; and we do not have to make hypotheses and interpretations about students' thinking or try to read their minds. This is an important aspect to us since in educational research the subjective dimension is, in general, hard to be completely removed because researchers have to make a number of subjective choices in order to develop a research study. Therefore, to succeed in avoiding at least a part of this subjective dimension is positive for us as reserachers but, above all, it is crucial for the relevance of the study.

After this introduction aimed at contextualizing our choice of the theoretical background, we now delve deeper into the most defining features of Sfard's theory that highlights the communicative aspects of thinking and learning by defining thinking as an "individualized version of interpersonal communication" (Sfard, 2008, p. 81). The term *commognition*, obtained by blending the word 'communication' with the word 'cognition', stresses that interpersonal communication and cognition are considered as two manifestations of the same phenomenon. Here we have to specify that the word 'communication' is made to include all forms of communication, not just the verbal one, indeed it is defined as:

> "a collectively performed patterned activity in which one action A of an individual is followed by action B of another individual so that: 1. A belongs to a certain well-defined repertoire of actions known as communicational; 2. action B belongs to a repertoire of re-actions that fit A, that is, actions recurrently observed in conjunction with A. This latter repertoire is not exclusively a function of A, and it depends, among others, on factors such as the history of A (what happened prior to A), the situation in which A and B are performed, and the identities of the actor and re-actor" (p. 86).

Moreover, Sfard defines *discourse* a "special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions" (p. 297). It is important to emphasize that a discourse encompasses all forms of communication "whether verbal or not, whether with others or with oneself, whether synchronic like in a face-to-face conversation, or asynchronous like in exchange of letters or in reading a book" (Sfard & Lavie, 2005, p. 245). Therefore, in the definition of communication reported above, the re-action B can be also played by the same person of the action A.

Through the participation in communicational activities of a collective that practices a specific discourse, people start belonging to a *community of discourse*. More generally, human society may be divided into partially overlapping communities of discourse, whose boundaries are not clear-cut, because discourses are not stable entities that remain the same over time. In the next section we are going to focus on mathematical discourse, reporting its most defining features as special type of discourse, as laid down in the commognitive framework. Indeed, discourses are dynamic, time-dependent entities but it is possible to describe, for example, what mathematical discourse is since they preserve their identity through continuous change.

2.1.1 Mathematics as a discourse

In this study, we refer to mathematics as described by Sfard as a special type of discourse and this makes mathematics learning a process which involves individualizing and developing a mathematical communication. What makes mathematics consistently different from other discourses is that mathematical objects are discursive objects, instead of concrete objects, which means that they do not pre-exist the talk. For this reason, it becomes necessary to also examine the possible relationships between talking, drawing, gesturing and mathematical thinking. Indeed, in respect to this type of analysis, Sfard (2009) explains that language and gestures should not be counterpoised one to another, since language is any symbolic system used in communication and gestures are "the actual communication" (p. 194).

We characterized mathematics as a special discourse that creates its own objects, it is, however, possible to further discuss its main features, indeed, we can observe that people usually agree in deciding if a given discourse is mathematical or not. This is explained by Sfard through the description of four features characterizing a mathematical discourse. They are: *words use, visual mediators, narratives and routines* and we use these elements to analyze students' mathematical thinking about functions.

For a detailed characterization of a mathematical discourse, we report here how the four features are described by Sfard (2008):

1. *Words use* is a main feature of mathematical discourse, it is "an-all important matter because, being tantamount to what others call 'word meaning', it is responsible for what the user is able to say about (and thus to see in) the world" (p. 133). Moreover, more than the words themselves, that mainly signify quantities and shapes, what is important is the way the words are used. However, when students are engaged in a mathematical problem, their mathematical discourse is not limited to the use of words.
2. *Visual mediators* are "visible objects that are operated upon as a part of the process of communication" (p. 133). For instance, they can be concrete objects, images of concrete objects, symbolic artifacts or even imagined pictures. In fact, visual mediators include objects that pre-exist the discourse but also artifacts created especially for the sake of communication.
3. *Narratives* are sequences of utterances, spoken or written, that are framed as a description of objects, of relations between objects, of processes with or by objects, and they can be endorsed or rejected. Examples of endorsed narratives are definitions, theorems and proofs.
4. *Routines* are discursive patterns that repeat themselves in certain situations. "Such repetitive patterns can be seen in almost any aspect of mathematical discourse: in mathematical forms of categorizing, in mathematical modes of attending to the environment, in ways of viewing situations as 'the same' or different, which is crucial for the interlocutors' ability to apply mathematical discourse whenever appropriate" (p. 134).

In particular, we observe that this characterization can be used in order to state if a given discourse may be considered a mathematical discourse, because people can investigate if it features words use, visual mediators, endorsed narratives and routines that are characteristic of the mathematical discourse.

Firstly, we took into consideration a wide literature of studies adopting a commognitive perspective and, especially, that fousing on a teaching and learning process involving the use of a DIE (Sinclair & Yurita, 2008; Sinclair & Moss, 2012; Ng, 2016a). From these studies it emerges that an analysis of words use, visual mediators and routines used by students may allow to gain insight into their mathematical discourse, and so into their learning, when interacting within a DIE. Therefore, in this study we mainly focus on visual mediators and routines performed by students in order to gain information about their interaction with the graphs of functions in both dynamic and static environments. Moreover, we look at students' use of words, because an in-depth description of students' mathematical discourse requires also to look at this aspect. However, we do not look at just the formal words that characterize and are typical of mathematical discourses (as described in the first point above) but, in a more general sense, we are interested in the words used by students to communicate about functions. We are going to better explain this concern for words later in this chapter.

2.1.1.1 Signifiers and realization trees

As we previously observed, mathematical objects are discursive objects and this leads to mathematical communication involving continuous transitions from *signifiers* to other entities that Sfard calls *realizations* of the signifiers. In particular, "*signifiers* are words or symbols that function as nouns in utterances of discourse participants whereas the term *realization of a signifier S* refers to a perceptually accessible object that may be operated upon in the attempt to produce or substantiate narratives about S" (Sfard, 2008, p. 154).

A peculiarity of mathematical signifiers is that each of them has several possible realizations such that every endorsed narrative about a signifier can be translated into an endorsed narrative about one of its realizations. Realizations can primarily be of two forms, *visual* or *vocal*: visual realizations are those realized through written words, algebraic symbols, icons (they can be drawings or just imagined icons), concrete objects or gestures; while vocal realizations are those realized though spoken words. Moreover, the relation between a signifier and one of its realizations is symmetrical and each realization can play the role of signifier, thus, being realized. Given a signifier with all its realizations it is possible to build a *realization tree* that is:

> "a hierarchically organized set of all the realizations of a given signifier, together with the realizations of these realizations, as well as the realizations of these latter realizations, and so forth" (p. 301).

In particular, having deep and rich branches in the realization trees is important for mathematics. Indeed, for an expert mathematician it is fundamental to be able to pass from a realization of a signifier to another one of the same signifier. For this reason, it could be interesting for a teacher or a task designer to construct students' realization tree (Weingarden & Heyd-Metzuyanim, 2018; Caponi & Lisarelli, 2018), where the inclusion of a certain realization in the tree means that "in certain situations the person has been observed implementing this realization" (p. 166). However, during this process it is worth considering that realization trees have the following key features:

- they are personal constructs: different students may realize the same signifier in different ways. Even though they originate in public discourse, students construct realizations themselves and the resulting realization trees may differ in the amount of realizations but also in the nature of the realizations;
- they are a source of valuable information about students' discourse on the particular mathematical object. Indeed, studying the number of ramifications and the depth of each branch can be insightful as feedback and it may give a possible assessment about students' discourse on that mathematical object;
- they are highly situated, and this means that although a student may correctly use a certain realization in a certain situation, he may not evoke it in another situation.

At this point, a mathematical object can be described as the whole realization tree of a signifier within its discourse. In other words, it is the signifier itself and all the objects signified by its realizations. Therefore, in these terms the learning process may be expressed as the creation of new signifiers by collapsing different realizations into one, that may give rise to the creation of mathematical objects. We suggest visualizing this process as a path going through a realization tree such that it starts from the branches and goes towards the root. For example, Figure 1 shows a possible realization tree of the signifier 'function', which plays the role of root, where we have put some of its realizations. In particular, this complex process involves some consecutive steps and one of these is called the act of *saming,* that we are going to use in our analyses. This process is described as the act of calling a number of things that were not considered to be the same before, with the same name.

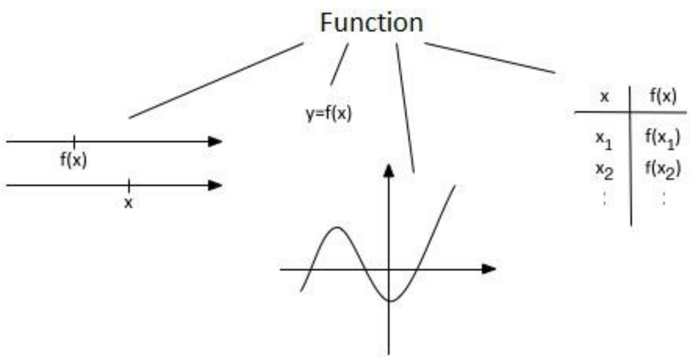


Figure 1. Example of realization tree of the signifier 'function'

As showed in Figure 1, the learning of the mathematical object 'function' can be seen as the product of a saming process involving, at least, the discursive objects: an analytic expression of two variables, a set of ordered pairs of the form $(x, f(x))$ that can be realized by a curve on the Cartesian plane passing the "vertical line test" (i.e. a vertical line cannot intersect the curve in more than one point), a dynamic graph, an input-output table. Each of these is a different realization of the signifier 'function' and can, thus, be called 'function'. Moreover, there are many other possible realizations to be considered and also the relations between them are several: an input-output table may realize the set of ordered pairs and each ordered pair is a realization of a point on the Cartesian plane, but also the opposite holds. Therefore, the act of saming, and so of *objectifying*, may be a complex process involving many discursive objects that are initially separated one from the others, then they become connected by signifier-realization relations and only at the end of this process each of them can be called by using the same term, which is the resulting mathematical object.

In this study we introduce students to functions starting from a dynamic graph with parallel axes, that is a possible realization that constitutes one of the branches in the tree in Figure 1; and we focus on their discourse about this realization, eventually investigating about the saming processes that may occur with other realizations that they may have seen in other contexts.

2.1.1.2 Learning as individualizing a discourse

In this section we give a more detailed description of how the learning process occurs, as indicated by the commognitive theory. Traditionally, during classroom activities students are introduced to new mathematical objects by teachers who already hold a rich realization tree for that signifiers and they provide students with some realizations or examples or with definitions of those signifiers. If we look at this teaching-learning process by taking a communicational approach, it can be considered as the process of changing students' discourse on specific signifiers.

So, if we are interested in investigating a learning process on functions, we need to learn how students' discourse about functions develops in relation to their interaction with the realization of function that we design. In particular, the changing in discourse develops through a process that Sfard calls *individualization,* which may be viewed as a participationist version of what Vygotsky called *internalization*. In particular, Vygotsky (1987) stated that what is being learned by an individual is something culturally and collectively produced and constantly modified. He expressed this in his famous statement about the development of an individual as that process involving higher mental functions that from the social plane move to the psychological plane.

Based on this underlying assumption, Sfard hypothesizes a four stage model of the development of students' use of words that involves the following steps.

1. *Passive use:* at this first stage the student meets the signifier for the first time and she is not able to use the word in her own speech but may utter it as a reaction to her interlocutor's utterance containing the given word.
2. *Routine-driven use*: at this stage the student is able to use the word actively in the speech, but only in a restricted number of specific routines.
3. *Phrase-driven use:* at this third stage the word becomes linked with constant phrases rather than with whole routines. The entire phrases constitute the building blocks of the student's utterances.
4. *Object-driven use:* at this stage the word is linked to a realization tree that remains relatively stable across different contexts.

By looking at this model we notice that, initially, students may start using the new (for them) signifier only in response to someone who uses it or into familiar discourses, involving some specific routines. In fact, by elaborating what Vygotsky (1987) argued, Sfard claims that the first necessary step in the process of individualizing a discourse is thoughtful imitation. Moreover, in some cases students "engage in the new type of talk while still unable to realize the new signifier in any way" (p. 178) thanks to a *mechanism of metaphor*. For example, this mechanism can take place with the individualization of a discourse on a signifier which is called with a word that is also used in colloquial discourse (e.g. 'function'). Finally, at the fourth stage of the model, students' use of the word is developed and it is in line with that of an expert. This process brings the word to be used in each discourse involving that specific mathematical object, which is denoted by the signifier and its realization tree.

For example, Nachlieli & Tabach (2012) based their study on this model to investigate students' individualizing discourse on functions and they observed students going through the first phase and entering into the second, with respect to the mathematical signifier 'function'. Moreover, they argued that words' definitions seem to play almost no role in students' decisions about the use of the words (Hershkowitz & Vinner, 1983) because the definitions provide a description of the objects, while students look for indications about how to act, that is about how to use the words in their communication. However, they did see defining as having an important role in the process of objectification. A possible conclusion is that the act of defining may occur later in the process of learning.

In this study, we focus on students' discourse on functions as it emerges during some activities that we designed in order to foster students' first discursive steps in this direction. Before and during these activities we do not give students any formal mathematical definition of 'function'. One of the consequences of this choice is, for example, that we do not discuss how students use the word 'function' during the activities, because we do not even introduce it to them. In particular, we are interested in words and verbal constructs used by students that might not be those of an expert, or that are not formal, but that their use is "close enough" to experts' mathematical discourse. Indeed, we want to investigate if students' discourse, contains possible seeds of realizations of mathematical signifiers, *mirroring a potential expert discourse* on functions and their properties. We mean by "potential expert discourse" and by "mirroring" the following.

Potential expert discourse is a discourse that we expect an expert mathematician would use for describing the same thing that students have described. This discourse is usually characterized by a formal mathematical vocabulary. On one hand we mean a discourse similar to written discourse that can be found in textbooks or journal articles, on the other hand it does not have to be necessarily completely objectified. Indeed, as discussed in section 1.4, the discourse of expert mathematicians during the explorations of problems is very often characterized by deictic words, half-finished sentences, mathematical terms that are imprecisely used and dynamism (Sinclair & Gol Tabaghi, 2010; Menz, 2015).

When an expert can recognize in a student's discourse potential expert discourse, we will also say that the student's discourse *is mirrored by* potential expert discourse. For example, we say that a description of the graph with parallel axes such as "x and f(x) move in the same direction" is mirrored by the potential expert discourse "the function is increasing" about monotonicity properties of the function. An expression like "f(x) moves on, it stays on ten and then it goes back" is mirrored by the potential expert discourse "ten is a relative maximum value". We believe that these possible connections are highly valuable in mathematical learning, because they represent some entry points into mathematical discourse for students who are not yet experts, but newcomers to the community of mathematicians.

2.1.1.3 Routines

In the section where we described mathematics as a special discourse we discussed about our focus in this study on visual mediators and routines performed by students and, in particular, we refer to a characterization of routines that has recently proposed by Lavie, Steiner and Sfard (2018). In this paper they address the relation between learning and our ability of finding and taking advantage of old memories when we have to face new situations. In particular, they explain our tendency of modeling the present actions on what we did in the past; they use the term *routine* referring to patterns of actions resulting from this process. In line with this argumentation, they describe the learning process as a "routinization of our actions". As we previously discussed in 2.1.1, their work on routines had started long before. Routines are one of the four essential features of a mathematical discourse, and Sfard & Lavie (2005) expressed the two sets of rules that allows to identify them: those telling the performer how to act and those indicating when to perform that specific routine. However, this definition revealed to not be exhaustive, because it has been showed to be difficult identifying 'the when' of some routines. Indeed, 'the how' and 'the when' of a routine are always changing and this may lead, for example, to the possibility of having different situations requiring the same routine (for which 'the when' is, then, not uniquely determined).

For this reason, Lavie, Steiner and Sfard (2018) suggest a new definition of routine that improves the previous one by making it more operational. In order to do this, they introduce the notion of *task situation*, denoting "any setting in which a person considers herself bound to act". When a person gets involved in a new task situation, she probably searches for a past situation that can be considered as sufficiently similar to the present one in order to repeat what was done then; it does not matter if it was done by herself or by another person. The researchers call *precedent* the past situation that the person identifies as acceptable and which allows her to perform in the new task situation. Moreover, they define the *task*, as understood by a person in a given task situation, as "the set of all the characteristics of the precedent events that the person considers as requiring replication". However, the search of precedents is usually restricted to a *precedent search space,* which plays the role of preselection by restricting the set of possible choices. Having identified the task, the person may wish to perform it by replicating the precedent action. The replication of a certain precedent is also necessarily selective, because there are some aspects of the past performance that are preserved and other aspects that are changed.

According to all these preliminaries, it is possible to introduce the new definition of *routine* in terms of a person's interpretation of a given task situation, so it involves both the task and the procedure being performed. We refer to this definition when we analyze students' discourse investigating which routines they perform, in order to identify a possible development of their discourse towards a discourse closer to that of an expert. Indeed, we expect that an expert mainly performs explorative routines, that now we are going to characterize. In particular, the researchers distinguish the routines by dividing them into *practical* or *discursive,* depending on the desired outcome of the performance, that is the expected change in the objects or in their relations. The distinction can be summarized as follows:

- *Practical* routine: a person interprets the task situation as requiring a change, re-organization or re-positioning of objects. For example, physical actions as biking or everyday activities as dressing
    - *ritual* if it is a process-oriented performance, highly situated, whose expected outcome consists in creating and sustaining a bond with other people;
    - *deed* if it is an outcome-oriented routine, whose expected outcome comes in the form of new, transformed or rearranged objects.

- *Discursive* routine: a person interprets the task situation as requiring a communicational action. It is a pattern that we follow while communicating with others or with ourselves
    - *ritual* if it is a process-oriented performance, highly situated, whose expected outcome consists in creating and sustaining a bond with other people;
    - *exploration* if it is an outcome-oriented routine, whose expected outcome is the production of a new endorsable statement.

It may happen that some practical routines are performed discursively rather than in the form of a physical action and also that some routines may be considered as practical in one context and as discursive in another context. It would be more proper to speak about a continuum of forms, differing one from another in the performer's ability to separate the procedure and the task: as long as a specific procedure is considered as part of the task, the routine cannot count as a full-fledged exploration; because the performer shows that a specific procedure must be used but his aim is probably not that of obtaining a particular outcome.

In the case of rituals, the performers ask themselves how they should proceed and not what they want to get, which, instead, is the question guiding the actions of the performers involved in deeds and explorations. For example, learners at an early stage, cannot yet know how a specific routine can help them in solving other problems and so they mainly worry about pleasing others, that is being like everybody else and doing whatever other people are doing. In contrast to ritual participation, Sfard and Lavie (2005) describe explorative participation for producing mathematical narratives to solve problems. Explorative participation is related more broadly to the view of mathematical learning as the process by which students gradually become able to communicate about mathematical objects.

By following these observations, in order to introduce students to functions, in this study we design activities aimed at creating opportunities for students' explorative participation during lessons. This include, for example, proposing different realizations of the signifier 'function' and encouraging students to create links between these realizations. In particular, we design different realizations of function and we try to characterize the routines performed by students when exploring them, also investigating their search for precedents, in order to identify possible changes in discourse and so in their learning. Indeed, we expect them to perform discursive routines in the form of rituals, at least during the first activities with the dynamic realization of function, and we are interested in studying a possible development of their discursive routines towards exploration. This is because we consider this development of routines as a development in the process of learning, since we think that discursive routines in the form of explorations are those mostly performed by an expert mathematician.

2.1.2 Some reflections on the word 'mediation'

As we recalled in the previous section, the two features of students' discourse on which we mainly focus in this study are routines and visual mediators. After having discussed about routines, now we analyze different uses of the term 'mediation', which is very common among the theories in educational literature, by explaining what we refer to when using this term.

Bartolini Bussi & Mariotti (2008) give a definition of 'mediation' by referring to what Hasan (2002) pointed out, that is the complex semantic structure characterizing the process of mediation involving the following participants and circumstances:

1. Someone who mediates: a mediator;
2. Something that is mediated: a content released by mediation;
3. Someone or something subjected to mediation: a mediate;
4. The circumstances for mediation: (a) the means of mediation i.e. modality; (b) the location i.e. site in which mediation might occur.

Moreover, they develop a theory that deals with mediation in relation to the use of technologies, which are widely present in the mathematical education literature. Concerning the use of technological artifacts during classroom activities, Noss and Hoyles (1996) emphasize the perspective of communication. Indeed, they identify the mediation function of the computer in the possibility of creating a channel of communication between the teacher and the students, based on a shared language.

According to the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008), an artifact may function as a *semiotic mediator*, but such a function of semiotic mediation is not automatically activated. Indeed, they describe the *semiotic potential* of an artifact as follows:

> "On the one hand, personal meanings are related to the use of the artifact, in particular in relation to the aim of accomplishing the task; on the other hand, mathematical meanings may be related to the artifact and its use. This double semiotic relationship is named the *semiotic potential of an artifact.*"
>
> (Bartolini Bussi & Mariotti, 2008, p. 754)

Then, they assume that such a semiotic mediation function of an artifact can be exploited by the expert who has the awareness of the semiotic potential of the artifact, both in terms of mathematical meanings and in terms of personal meanings. This means that the teacher acts as a mediator *using the artifact to mediate mathematical contents* to the students. Moreover, any artifact can be referred to as *tool of semiotic mediation* as long as it is intentionally used by the teacher to mediate a mathematical content through a designed didactical activity.

A different view is offered by the theory of commognition, according to which "communication mediators are often artifacts produced specially for the sake of communication" (Sfard, 2008, p. 90). Mediators are described as perceptually accessible objects with the help of which we are able to perform our actions and other individuals are prompted; so they can have auditory, visual, or even tactile effects on individuals.

Moreover, we have seen in section 2.1.1 that Sfard describes four properties which characterize mathematics as a special type of discourse and one of them is the use of *visual mediators*. In particular, she refers to them as

> "visible objects that are operated upon as a part of the process of communication. While colloquial discourses are usually mediated by images of material things existing independently of the discourse, mathematical discourses often involve symbolic artifacts, created especially for the sake of this particular form of communication" (p. 133).

This description suggests that a visual mediator provides the image with which people identify the object of their talk and it allows them to coordinate their communication. For example, gestures are considered by Sfard (2009) forms of visual mediation. They are essential for an effective mathematical communication, because “using gestures to make interlocutors’ realizing procedures in public is an effective way to help all the participants to interpret mathematical signifiers in the same way and thus to play with the same objects” (p. 198). Moreover, gestures can be realized *actually* when the signifier is present, or *virtually* when the signifier is imagined. Sfard illustrates the example of a student using different gestures to realize the signifier ‘fraction’. Since these gestures are performed in the air, they provide instances of a virtual realization.

Unlike the theory of semiotic mediation, the theory of commognition does not take into account the use of artifacts, but it provides a description of different visual mediators: concrete, iconic, symbolic and gestural; and the same mediator may be used in several ways even when the task and the result remain the same. In particular, concrete mediators may be physically manipulated; the strength of iconic and concrete mediators is that they may lead to new endorsed narratives with only a relatively small number of verbal manipulations. For example, in order to compute a certain division a person can manipulate a set of concrete objects, partitioning the whole set into the right number of equipotent subsets, or she can imagine/draw a sketch of this partition; so the computation becomes quite immediate through the use of concrete or iconic visual mediators. Otherwise she can recur to symbols and proceed by algebraic computations but, since the symbolic mediators are basically verbal, instead of visual, they require a great demand on a person’s memory. To summarize our reflections, Sfard’s approach identifies the role of mediation with something visual which is related to people communicating with other people, or with themselves, in order to better understand each other and to ensure that everyone refers to the same object. Bartolini Bussi and Mariotti mainly use the term mediation in relation to the potentiality of promoting relationships between students and mathematical knowledge, and mediation is related to the accomplishment of a task.

However, it seems that the main difference between these two theories in conceiving mediation is the role of the artifact that, according to commognition, can be used as a mediator by the teacher to better develop her discourse with a student, without necessarily referring to a mathematical object; and also by the student, in the communication with the teacher or other students. While, according to the theory of semiotic mediation, it can be a tool of semiotic mediation if it is used in a specific way by the teacher (mediator, 1 in Hasan’s list), to mediate a mathematical meaning (2 in Hasan’s list), and in this sense the artifact (part of the means of mediation, 4a in Hasan’s list) is something that links the student (mediatee, 3 in Hasan’s list) to the mathematical knowledge. This observation is strictly related to the fact that the second point of Hasan’s list seems to be considered essential for defining mediation within the theory of semiotic mediation, but not in commognition. Indeed, these theories are based on two different theoretical assumptions about the nature of mathematical objects. On the one hand, Sfard describes mathematical objects as discursive objects, so they cannot be evoked as if they existed elsewhere because they exist in the communication itself. On the other hand, in the theory of semiotic mediation, Bartolini Bussi and Mariotti speak about “mathematical meanings” that exist within a specific culture and they represent something that can be mediated.

In this study we refer to mediation as intended by Sfard, and we extend her characterization of visual mediators, in an attempt to adapt the tools offered by the theory of commognition to the cases that involve discourse in the presence of digital artifacts, which may be used as particular mediators. Specifically, we are interested in studying possible implications of the use of DIEs in classroom activities.

2.1.2.1 Static and dynamic mediation

The most defining feature of DIEs is that they allow to experience relations dynamically over time, indeed they enable us to observe and manipulate objects that move on the computer screen, changing over time. As a direct consequence of this dynamic nature, visual mediation by DIEs is significantly different from visual mediation through diagrams or figures presented in the textbooks; and we found that the theory of commognition could be refined from this point of view, because it does not specifically address the use of DIEs in the teaching and learning process.

A static visual mediator realizes a mathematical object statically, for example the drawing of a Cartesian graph on a sheet of paper realizes the mathematical signifier 'function' through a static curve; while visual mediation by a DIE may realize mathematical relationships and properties of the same mathematical object. For example, in a Cartesian graph constructed in a DIE, dragging one of the variables (the independent one) along its axis causes the movement of the other variable along the other axis and so it realizes the relationship between the variations of the two variables. Therefore, a dynamic visual mediator allows to realize the invariant properties of a mathematical signifier through motion. In particular, when interacting with DIEs, usually there is a large use of both gestures and dragging actions as visual mediators. Looking at the visual mediation of gestures, we observe that also the dynamic function of gestures has not been widely examined in literature. McNeill's (1992) categorized gestures into deictic, iconic, metaphoric and beat, by distinguishing the type of functions served by them. For example, deictic gestures serve as pointing devices, while metaphoric gestures serve to represent the mathematical objects themselves. However, in this classification it is not specified when gestures are used to convey dynamic and temporal relationships. Núnez (2003) studied how some mathematicians use hand gestures for expressing dynamic thinking of functions and continuity. His analysis shows that

> "these mathematicians are referring to fundamental dynamic aspects of the mathematical ideas they are talking about" (p. 177).

Furthermore, the words used by these mathematicians were related to motion and time; for example, they said "approaching" or "tending to" while producing metaphoric gestures as tracing the trajectory of a point. Another relevant study that showed how temporality can be evoked by the use of visual mediators like gestures was conducted by Sinclair and Gol Tabaghi (2010). They analyzed mathematicians' use of gestures to speak about the movements of some vectors, providing evidence of the time dependency of these actions. Both these studies highlight the dynamic and temporal aspects of mathematicians' thinking, focusing on the importance of gestures as visual mediators that may involve motion in the communication.

From these considerations it is clear that gestures and dragging actions can play the role of visual mediators conveying dynamic and temporal relationships, but, the commognitive theory does not yet offer a distinction between static and dynamic visual mediators. This distinction is fundamental with respect to the mathematical objects that are involved in the discourse, because different kinds of mediators allow students to communicate about different properties of the objects. Significant work in this direction has been conducted by Ng (2014, 2016) who proposed a distinction between static and dynamic visual mediation in the context of teaching derivative within both static and dynamic environment, and our purpose is to gain a deeper insight following her direction of research. Indeed, we consider the distinction between static and dynamic visual mediators to be important because the mathematical object mediated by static or dynamic visual mediators can be of completely different nature. For example, the signifier 'increasing function' can be realized by a static gesture evoking the image of a curve having positive slope or by a dynamic gesture such as the motion of a hand tracing a curve going up. For us, the main difference between the two kinds of visual mediators is that in the second case the gesture communicates temporal relationships existing behind an increasing function, as opposed to showing a possible shape of the function statically. The first aspect is fundamental in mathematical discourse, especially in the case of increasing functions, because the monotonicity properties of functions are dynamic and temporal relationships between the two variables.

In particular, in the next section we propose a characterization of a specific type of mediation, which is based on the commognitive perspective, but it takes into account the use of DIEs and their dynamic nature. The theoretical foundations of this study of the relation between visual mediators and the use of DIEs can be identified in other studies reported in the previous chapter (e.g., Sinclair & Yurita, 2008; Ng, 2016).

2.1.2.2 Definition of dynamic interactive mediators

In this section we propose a characterization of a special type of mediators that takes into account the dynamic nature of the mediation, that has been previously addressed also by Ng (2016), and the cases in which the mediator is interactive. In this way, we do not aim at replacing, but we would extend the classification operated by Sfard, by addressing the mediation that may occur when DIEs are involved in the activities. DIEs are characterized by dynamism, because there is a change over time, and interaction, because they respond to a person's manipulations.

In particular, we define DIMs (Dynamic Interactive Mediators) digital objects constructed within a DIE that:

- can be manipulated and they give immediate feedback based on such manipulations, in the form of dynamic change;
- can be used in experts' discourse as visual mediators of mathematical objects.

We observe that a definition of DIM has been presented in Antonini, Baccaglini-Frank & Lisarelli (under review), where DIMs were constructed within touch environments.

The second point in the definition suggests that a DIM can play the role of realization of a mathematical signifier but it can also be the object of students' discourse which mirrors objectified discourse on a new (for the learner) mathematical object. In this case, students' learning can be fostered by promoting discourse that makes links between the different objects (for the learner), which are actually different realizations of a same mathematical object (for the expert).

In this sense, appropriately designed activities with DIMs may open doors to full-fledged participation in mathematical discourse, helping students construct new mathematical objects. Indeed, since we actually consider the learning process as a gradual development of the discourse, we can expect that a fundamental step in the process of learning mathematics with DIMs will be a transition from considering the DIMs objects of exploration *per se* to considering them as *realizations of mathematical objects*, as experts do.

2.2 DRAGGING

Dragging is a characterizing feature of DIEs which enables to move objects on the computer screen according to different types of motion, depending on the construction. Indeed, exploring a construction realized in a dynamic software involves searching for possible relationships existing between the movements, that can be perceived as variations or invariants. Many research studies highlight how the identification of such invariants lies at the heart of a dynamic exploration (Olivero 2002; Healy & Hoyles 2001; Laborde, 2005; Baccaglini-Frank, Mariotti & Antonini, 2009; Leung, Baccaglini-Frank & Mariotti, 2013).

There is a wide literature about the use of DIEs in the teaching and learning process, with studies grounded on different perspectives, and we have presented some of them in the previous chapter. Particularly significant with respect to our research is the study of how a description in terms of logical dependency can be supported by explorations involving the use of the dragging tool. This investigation has been conducted by some researchers that operated a distinction between two possible types of motion in DIEs: *direct* and *indirect* (Mariotti, 2006, 2010, 2015; Laborde, 2003). As we discussed in chapter 1, they highlighted that a description in terms of logical dependency is possible thanks to this difference. In a similar way, we want to analyze students' emergent discourse about functions realized in this specific dynamic context and, especially, we are interested in particular aspects of their discourse, that is, how the visual mediation of dragging is involved. However, our focus and also the theoretical framework that we use in this study are different from that of the studies that we have cited, which concern geometry and, for example, the use of dragging is analyzed under the lens of the instrumental approach, according to which dragging is an artifact supporting the task of generating a conjecture (Lopez-real & Leung, 2006; Leung, 2008). Differently, according to the theory of commognition, dragging can be used by students as visual mediator in the communication with other students or with themselves and, in particular, we expect it to dynamically mediate the communication. Because of the focus of this study on students' learning of functions, that is on their discourse about DIMs that for an expert are realizations of functions, we are interested in looking at different ways in which students use dragging as visual mediator. Indeed, this analysis can give us significant information about the main features of their emergent discourse.

In the next sections we present the tools of analysis that we developed in light of the studies on dragging practices in DIEs. In particular, we adapted to our context some tools offered by the literature (see the previous chapter) and we are going to describe them according to our theoretical lenses.

2.2.1 Different types of dragging

As we have previously discussed, a group of researchers (Arzarello et al., 1998, 2002; Olivero, 2002) described experts' development of dragging modalities while dealing with geometric open problems implemented in a DIE. In particular, they operated a classification of different functions of dragging for describing some of their cognitive features in learning processes. Indeed, their analysis focused on the use of the dragging tool from a cognitive point of view and it highlighted how students' different uses of dragging depended on their interpretation of the task situation, and they classified them depending on the students' aims.

In this study we are interested in observing different types of dragging used by students, because we design a realization of function in a DIE and our main goal is that of describing students' emerging discourse about functions in relation to their interaction with this realization. According to the theory of commognition, this study involves investigating the main features of students' discourse that are the use of words, visual mediators, routines and narratives; so, one of our purposes is looking at how different types of dragging mediate students' discourse. Since it is not possible to identify students' aim behind a dragging action, unless they do explicitly express it, in line with the theory that we adopt we now describe different types of dragging that we expect students to use for exploring the dynamic realization of functions and then (in chapter 5) we use this description as tool for analyze students' discourse.

First of all, we expect that students interacting with the dynamic realization of function designed in this study use the mediation of dragging to describe the asymmetric relation between the movements of the two variables, that move according to two different types of motion, direct and indirect. It follows that the distinction between direct and indirect dragging is a key element for us to analyze how dragging mediates students emergent discourse about functions. In particular, we call *impossible dragging* a movement of the mouse attempting to drag an object that cannot be directly moved. In fact, in DIEs the only draggable objects are the basic ones, which are the objects from which the construction originates. For example, in the case of our dynamic graphs, students' attempts to move the tick realizing the dependent variable, which cannot be directly dragged, are examples of *impossible dragging*.

Moreover, we distinguish between *continuous* and *discrete dragging.* The *continuous dragging* is the dragging of an object using a continuous movement, that can be also characterized by changes in the speed and in the direction. Indeed, it is not necessarily a movement maintaining a constant speed and always oriented in the same direction, it may be faster and then slower but there cannot be stops, except those involved in changing the direction. Likewise, but in contrast at the same time, we call *discrete dragging* the movement of an object with jumps; for example, in our dynamic realizations of function the dragging of the tick realizing the independent variable used to let it take on only whole numbers. This type of dragging is characterized by several stops and it may be associated with counting.

These three types of dragging can be objectively recognized by looking at students' dragging actions and how the mouse moves on the screen. For this study, they can be useful tool to describe the routines performed by students and to analyze students' discourse about the proposed realization of function. For example, we expect that an expert mathematician does not perform an *impossible dragging* on the dependent variable and that the mediation of *discrete dragging* is used to communicate about functions in terms of correspondence of values, while the mediation of *continuous dragging* is used to communicate about the continuous variations of the two variables simultaneously.

In addition to looking at how and on which object of the construction students physically use the dragging tool, that give information about the quality of the possible movements, we can observe the relation between a specific dragging action and the focus of students' discourse, both in terms of verbal description and gestures employed in the moment of dragging.

Indeed, it is possible to have a correspondence between the two elements, if students describe what happens to the object that they are dragging; but they can be also two different aspects, if students drag an object and their discourse is about another object of the construction that moves indirectly. By analyzing this interplay between dragging action and focus of discourse, we are able to identify other types of dragging, that we are going to use in the analysis of students' discourse. We call *wandering dragging* the explorative dragging of an object of the construction which is accompanied by a description of that action**.** For example, as soon as students open a GeoGebra file they might move the objects randomly, to explore the construction and to identify possible and impossible movements; if this dragging action is associated to description of what can or cannot be done we call it *wandering dragging*. Otherwise, if the focus of the discourse is not on the directly dragged object but on other movements happing in the construction, we call this type of dragging actions *handle dragging.* Indeed, it is the dragging of an object as if it was a handle, that is in a way that allows the student to visualize movements of other objects that are not directly draggable. We observe that this case can be characterized by a particular position of the mouse with respect to the tick: the arrow giving the mouse's position does not always overlap the dragged point, but it can drift far away from it. For example, when exploring our dynamic realization of function, it is possible that most of the dragging actions are *handle dragging* where the tick realizing the independent variable is dragged as a handle in order to make the other tick move on the screen. However, we consider the independent tick as actually being dragged as a handle only if the dragging action is combined with a verbal expression or a gesture suggesting the focus to be on an object which is not directly dragged. Moreover, if students express a conjecture about a specific movement of an object, or about a possible consequence of a dragging action, and they explicitly say that they use the dragging to test it, then we call the dragging action a *test dragging*. In this case students may express their idea about the expected movement of the objects before making the dragging action. Finally, we call *guided dragging* the moving of an object in order to obtain a particular configuration, which is expressed through verbal description or gesture by the student, before or while dragging the object. Otherwise, it is possible to recognize this type of dragging if it is used to answer to a question asking for the conditions under which a certain property holds or a certain configuration is obtained.

Differently from the other types of dragging, these four types of dragging cannot be recognized by looking only at students' dragging actions and how the mouse moves on the screen, but the relation between these elements and the object of students' discourse has to be considered.

A description of all these different types of dragging that we identified is summarized in ***Table 2.1***. We observe that even if we do not mention it for each type, in this study students' dragging actions are always *bound dragging,* that according to Arzarello et al. (2002) consists of moving a semi-draggable object (such as a point which is already linked to another object). Indeed, the only object that our students can move is the tick realizing the independent variable, which is bound to the x-axis, and in some cases they can also move the line realizing the ordinates-axis, which is bound to move up and down, maintaining the alignment of the zero with the abscissas-axis.

Tag	Description
Continuous dragging	Continuous movement
Discrete dragging	Movement with jumps, often associated with counting
Impossible dragging	Trying to move a dependent[3] object that cannot be directly dragged
Wandering dragging	Random movement, exploring the construction
Test dragging	Movement aimed at testing a possibly implicit conjecture
Handle dragging	Movement of the object as if it was a handle, in order to observe other objects' movements
Guided dragging	Movement aimed at reaching a particular configuration

Table 2.1. Different types of dragging

It is possible to notice that some of the tags that we have chosen for the types of dragging echo the classifications existing in literature. Moreover, in ***Table 2.1*** we use a double row to separate the first three types of dragging from the other four. Indeed, *continuous, discrete* and *impossible dragging* refer to the quality of the movement on the screen, as it is induced on the directly dragged object; while *wandering, test, handle* and *guided dragging* describe the use of dragging in relation to the object of students' description. The main difference between these two families is that on one hand the types of dragging belonging to the first can be also recognized by a computer that captures how the mouse moves on the screen and so objectively quantified, while on the other hand the types of dragging of the second family are strictly related to students' discourse in the very moment of dragging, and their identification involves an analysis of students' words, gestures and of their gaze. One of the potentialities of this classification involving two families of dragging modalities consists in the possibility of combining them, that allows a more complete description of students' dragging actions within a specific discourse that may involve also words and gestures. For example, when looking at students' interaction with a DIM we can distinguish a *continuous wandering dragging* from a *continuous handle dragging* and, in this way, we gain information about the routines performed and about different characteristics of students' discourse; and so, of their learning process.

[3] We use this term to identify the tick realizing the dependent variable, but we do not know if the students are aware of this dependence relation.

2.2.2 Dragsturing

A significant contribution to the literature of the dragging practices in DIEs has been developed by Ng (2014, 2016). In her work she combines the commognitive framework with a sociocultural view of learning as participation (Moskovich, 2007). In particular, she uses Sfard's definition of gestures as communicational acts and she investigates the different kinds of communicational functions of the gestures used by bilingual learners dealing with activities implemented in a DIE, about the derivative of a function.

When working within a DIE, students frequently use gestures and dragging actions and it is possible that some dragging actions are not merely dragging but also gestural communications. In particular, they can be used to communicate the dynamic features of the interactive file, as obtained by dragging. For example, if a student moves her finger along the graph of a function realized in a static environment, while speaking about the monotonicity properties of the function, he is showing a dynamic gesture. The same gesture in a dynamic environment may be represented by a dragging action of the dependent variable causing the indirect motion of the dependent variable and, eventually, of the point $(x, f(x))$ along the curve. In this case the student's action is one action subsuming both dragging and gesturing characteristics: it causes the point to be moved on the screen (dragging) and it fulfills a communicational function (gesture).

Ng (2014) refers to this type of action as *dragsturing* and she argues:

> "although I have named this action dragsturing, my purpose for naming is not solely to objectify an action into a noun, but to present the dual functions of dragging and gesturing in the dragsturing action for analyzing the students' thinking-communicating process" (p. 293).

In her analyses, Ng highlights how dragsturing emerges as a new and significant form of communication, which may give rise to new conversational patterns.

We paid particular attention to this theoretical construct because it is an effective tool for our study. Indeed, we design activities employing a DIE which allows students the dragging with one hand on the mouse, and the gesturing with the other hand. In this way, the two actions can blend together as a unique action, that is worth looking into in order to analyze the main features of students' emergent discourse. For example, during the description of the dynamic realization of function that we design, we expect that the use of dragsturing actions can help students address the movements of two variables in their communication with other students. In particular, it may happen that a student describes the movement of the independent variable as "moving fast towards here" while dragging the point realizing the independent variable along its line and stopping at a certain value. In this case the dragging action plays the role of dragsturing, because it is used also as a gesture that allows us to guess what the student means by "towards here".

3 RESEARCH QUESTIONS

In the first chapter we identified the importance of functions and graphs of functions in different fields of mathematics and highlighted the centrality of the notion of function in high schools and university mathematics. Moreover, we discussed various students' difficulties in dealing with functions, especially in interpreting graphs; the literature that we analysed allowed us to identify covariation as a fundamental aspect characterizing the notion of function. Starting from this assumption we decided to introduce students to functions by using a particular realization, implemented within a DIE, that thanks to its dynamic nature is actually a realization of covariation of the two variables, one depending on the other.

The theoretical framework elaborated and presented in the previous chapter allows us to introduce a second formulation of the research questions of this study. We want to study possible outcomes of a teaching and learning process involving this particular dynamic approach to functions and their graphs; we focus especially on how students' discourse about functions emerges through interactions with the DIMs, what characteristics it has, and how it potentially changes.

3.1 A SECOND FORMULATION OF THE RESEARCH QUESTIONS

In this section we give a specific formulation of the four research questions that guide our study, now contextualizing them through the theoretical constructs we have defined.

3.1.1 Research question 1

The first question is quite general and it develops throughout the entire study.

As we discussed, it is important to consider covariational aspects when fostering learning of functions: this was in important goal we kept in mind designing the activities introducing students to functions using a dynamic approach which involves a realization of covariation. Our dynamic approach lead to using a particular realization of functions, based on the idea of DynaGraph (Goldenberg et al., 1992), to investigate whether the activities designed using this realization actually support the emergence of students' discourse on functions in terms of covariation of two quantities, one depending on the other. Thus we ask:

Does students' discourse emerging during the proposed activity sequence involve covariation? If so, in what ways?

3.1.2 Research question 2

The possibility of dragging plays an important role in the dynamic realization of functions; indeed it is thanks to the distinction between direct and indirect dragging that it is possible to realize the dependence relation between the two variables. Moreover, in the previous chapter we proposed a characterization of visual mediation that takes into account a difference between static and dynamic visual mediators. We expect that dragging could be used by students as dynamic visual mediator in their communication about the DIM realizing function. We are interested in exploring this possibility, focussing especially on if (and if so, how) this dynamic visual mediation is used by students to communicate about covariation.

In particular, if students use the dynamic visual mediation of dragging, we want to analyze what types of dragging and possible dragsturing actions that may appear in students' discourse.

These observations lead to the second research question:

What is the role of dragging, if it has one, as a dynamic interactivemediator in students' discourse? Is it used to express covariation? If so, how?

This question is addressed in Chapter 5.

3.1.3 Research question 3

In this study, we aim to analyze students' discourse about different realizations of the mathematical signifier 'function'; we designed activities involving graphs of functions realized within both dynamic and static environments, hoping to identify any changes occurring in students' discourse when passing from one realization to another one. We are also interested in the aspects of discourse that remain invariant.

More specifically our third research question is:

What recurrent features is it possible to identify in students' discourse about the different realizations of functions that we design within dynamic and static environments? And in students' attempt to relate them?

This research question is explored in Chapter 5 and in Chapter 6.

3.1.4 Research question 4

We are interested in studying high school students' emerging discourse on the dependence relation between two variables (as a key feature of the mathematical object function); moreover, we are curious about how this discourse compares to that of an expert mathematician (that of course may also involve other mathematical signifiers related to functions). At this regard, in the previous chapter we defined *potential expert discourse* so that we can compare this to students' emerging discourse.

Therefore, in order to support the emergence and the development of students' discourse about different properties of functions, when designing the activities we choose different functions to realize with the DIMs; for example, including functions that are not everywhere defined or discontinuous.

The research question on this issue is:

How does students' discourse compare to potential expert discourse about functions and their properties? In particular, from an expert's point of view, what seeds of realizations of mathematical objects is it possible to identify in students' discourse?

This question is addressed in Chapter 6.

Finally, we observe that students' understanding of the covariational aspect of functions has been investigated (Carlson et al., 2002; Thompson & Carlson, 2017) and that specific approaches to functions, that make use of a DIE, has been developed in order to support this covariational view (Falcade et al., 2007; Ng, 2016). In the context of teaching and learning with the use of a DIE there are studies focused on students' use of dragging (Arzarello et al., 2002). However, an important difference is that in this study the theoretical perspective and the tools of analysis are different from those applied previously, which influence what we see and the results we obtain. In particular, we think that we can gain significant insights thanks to our theoretical perspective that looks at the construction of a discourse through tools that allow to deeply analyze its features. In the formulation of the research questions, the theory allows us to focus on learning in terms of ‘discourse’ and to identify possible developments in learning by comparing students’ discourse to a potential expert discourse. Moreover, the theory lets us look at students’ use of dragging as dynamic visual mediator, that give significant information about students’ discourse when interacting with the DIMs and, so, about their learning. In the concluding chapter, we will further discuss the original and innovative results that we found thanks to our theoretical lens.

4 DESIGN OF THE RESEARCH

Our study aims at investigating and describing students' learning of functions, when they are introduced to this notion through a particular dynamic approach. The main aspect that influenced our choice of what methodology to utilize is that we needed to be able to observe students' production of narratives during the solution of problems involving the realizations of functions both in dynamic and static environments. Indeed, according to our theoretical perspective, thinking and communicating are two manifestations of a same phenomenon and so, in order to investigate about cognitive processes, we have to look at students' discourse. This motivated our choice of designing open-problem activities, letting students work in pairs, conducting interviews and, in particular, the choice of using microgenetic methods (Schoenfeld, Smith & Arcavi, 1993; Chinn & Sherin, 2014; Lewis, 2017) for the analysis of the data.

4.1 MICROGENETIC METHODS

Chinn and Sherin (2014) describe microgenetic methods as analytic methods, originally coming from disciplines other than mathematics education, that can be used for studying learning processes. In particular, this methodology has been adapted to the learning sciences by Schoenfeld and diSessa at the end of the last century and it is compatible with a range of theoretical perspectives. The main underlying assumptions (Chinn & Sherin, 2014) are that

- "learning occurs continuously, and in small steps, with every moment of thought";
- "learning does not occur in a straight line, from lesser to greater understanding; it occurs parallel on multiple fronts";
- "learning events are heterogeneous, […] there are multiple kinds of learning, each requires its own study";
- "we learn from our environment, which includes, most critically, the cultural tools other individuals provide to us".

(ibid. p. 171)

The main goal at the core of a microgentic study consists in observing learning processes as they occur. Indeed, this method allows to make strong inferences on learning processes: after a long period of observation, where students conduct experiments in which they have to think aloud explaining their actions, the researchers can draw conclusions about what triggers change and about how this change occurs.

Siegler (2006, p.469) sets out the three main features of a microgenetic method:

- "observations span the period of rapidly changing competence" and so students should not have that competence yet, but they should gain proficiency in a short intensive learning session, which is designed by the researcher;
- "within this period, the density of observations is high, relative to the rate of change" and so the researcher should document all the students' trials;
- "observations are analysed intensively, with the goal of inferring the representations and processes that gave rise to them" and so the researcher should try to make inferences about the cognitive processes involved, going beyond the superficial behaviours.

Usually, a microgenetic study is not long lasting and it cannot include too many participants, especially for the density and the complexity of the analysis. For example, the focus can be individual learning within an interview setting. There are also some studies that used this methodology to observe small groups working within a classroom environment for a longer period, in these cases the risk could be that the observations miss critical learning events.

As we are going to explain in the next sections, for this study we designed a teaching experiment that took place in a classroom environment and a researcher was in the classroom setting along the whole sequence of lessons, which developed over 4 months. In order to reduce as much as possible the loss of observation of critical learning events, we used fixed and mobile cameras to record the discussions and actions of some selected pairs of students. At the end of the teaching experiment we also interviewed the pairs of students that we fellow more closely.

4.2 DATA COLLECTION

The organization of the different phases of the research was led by the research questions. A first relevant aspect that we were interested in is the birth and development of students' discourse about functions as covariation. For this reason, we wanted to work with students who had not taught about this topic yet and so we chose a 10^{th} grade class. In fact, traditionally, in Italy students are introduced to functions at grade 7^{th} or 8^{th} but they extensively work with them from grade 11^{th} on. Then, we can consider our intervention as students' first approach to functions and their properties.

The main study was conducted over 4 months (March-June, 2017) in an Italian High school for Math and Science. The participants were eighteen students, all belonging to the same class. In particular, we designed and implemented a sequence of lessons and at the end of the sequence we interviewed some students out of their mathematical schedule.

A first experimentation has been carried out in 2016 (Colacicco, Lisarelli & Antonini, 2017), where we designed activities involving a one-dimensional realization of function in a DIE, similar to the DynaGraph, and we proposed them in a 10^{th} grade class. We focused on different ways, like expressions or specific terms, developed by students to express the functional dependency, in their efforts to describe the dynamic graphs of function. It was the work of thesis for a bachelor degree in which we were get involved. Even if the goals behind the research and also the theoretical framework adopted were different, for us the experimentation played the role of pilot study, that we used for setting up this study and that allowed us to better design the sequence of lessons.

4.2.1 Lessons

We designed eight lessons, lasting one hour each, and we proposed them to the whole class, one per week. Each lesson was conducted by the researcher and during these lessons the regular teacher was not present in the classroom. Students were asked to work in pairs using one computer; at the beginning of the first lesson they chose their partner and then the pairs stayed the same for all the other lessons. We also asked them to sit always in the same place in the classroom, in order to work with the same computer.

We chose to follow more closely selected pairs of students (but we did not tell them). These pairs were chosen at the end of the first lesson, due to their seeming particularly talkative and active in the discussions. Table 4.1 shows the pseudonyms of the students that we decided to follow, and the lessons in which they participated – not all students were always present, and this could not be controlled ahead of time.

Students	Lessons
Alessio and Nicco	1 – 6
Matilde and Nicco	7 – 8
Davide and Elena	1, 3 – 8
Lore and Franci	1 – 8

Table 4.1. The selected pairs of students

Each lesson was video recorded by three cameras: two were fixed in the back of the room to record students' gestures and to provide a global view of what happened in the classroom; one mobile camera was used to gain further insight at a fine-grained level into the pairs of students' discussions. The mobile camera was held by an undergraduate student helping the researcher. Moreover, we used a software that captures the actions on the computer screens and that records the audio near the computer. This software was used on the computer that Alessio and Nicco worked with for 6 lessons and then Matilde and Nicco for 2 lessons. We chose their computer because Alessio and Nicco seemed particularly talkative and not shy, but also because they were sit close to one of the fixed camera and so, we thought that in this way we could have rich and detailed data about the work of a specific pair of students along the entire sequence of lessons.

4.2.2 Interviews

At the end of the sequence of lessons, we interviewed the pairs of students who appear in Table 4.1. In particular, Alessio has been interviewed alone since he did not attend to the last two lessons. Therefore, he was given slightly different questions, which we are going to describe later.

The interviews lasted approximately 45 minutes and they were video recorded using a fixed camera, a mobile camera, held by the researcher, and the same screen-capturing software mentioned above.

4.3 THE EXPERIMENTAL SEQUENCE

The activities that we designed involve different realizations of the graph of functions, employing both the dynamic and the static context. In this section, we are going to describe the experimental sequence and analyze its characteristics in detail.

From now on, in order to slim down the narration, we are going to use the following acronyms to indicate the realizations of functions: DGp, DGpp, DGc, SGc. The first letter stands for the environment in which it has been designed (D: dynamic, S: static), while the lower case letters indicate the number and the position of the axes (p: one horizontal line, pp: two horizontal parallel lines, c: Cartesian plane). So, for example, DGpp stands for "dynamic graph with two horizontal parallel axes".

We observe that we disabled the magnetism in all the GeoGebra files, this is a property that this software allows to give to a point and makes it move on the real axis as if it has a magnet that attaches it to the whole numbers; and disabling this tool the dragging of the point is more uniform.

4.3.1 Description of the realizations of graphs used for the study

The artifact DGp is a dynamic graph created within the dynamic algebra and geometry software GeoGebra (**Figure *2***).

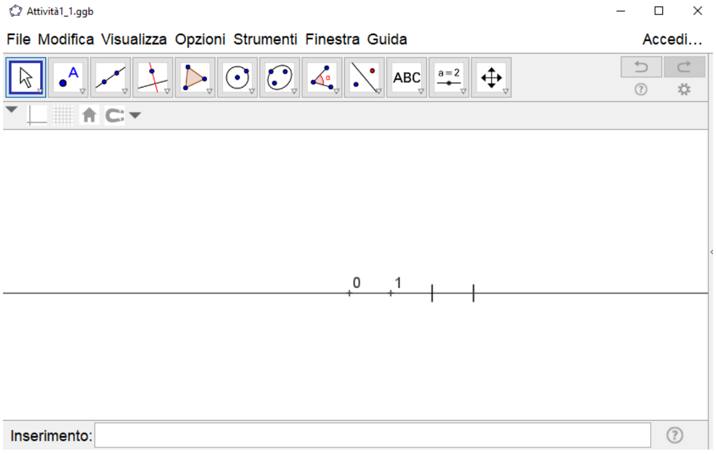


Figure 2. DGp

Unlike the DynaGraphs designed by Goldenberg et al. (1992), that we described in the first chapter, we bound the two variables on the same line in order to stress their belonging to the same set of numbers. As shown in **Figure 2**, the dynamic file contains one fixed non-numbered horizontal line, with two ticks bound to it. These ticks can be acted on, but only one of them affords direct action: one of them can always be dragged (direct motion) while the other only moves in dependence to the movements of the first one (indirect motion). The ticks have no labels and two points, 0 and 1, are marked on the line to determine the unit segment which has been placed to highlight that the line visually realizes the real numbers line.

Another realization of the function that we designed, and which is more similar to the original DynaGraph, is the dynamic graph DGpp (**Figure 3**). It can be obtained thanks to the design of the interactive files which allows to separate out the two variables, that is, it shows two copies of the real number line, each with one tick on it. In this way, the two lines containing the variables can be dragged further apart or closer together, maintaining the parallelism and the alignment of their origins. Again, the two variables move according to two different types of motion.

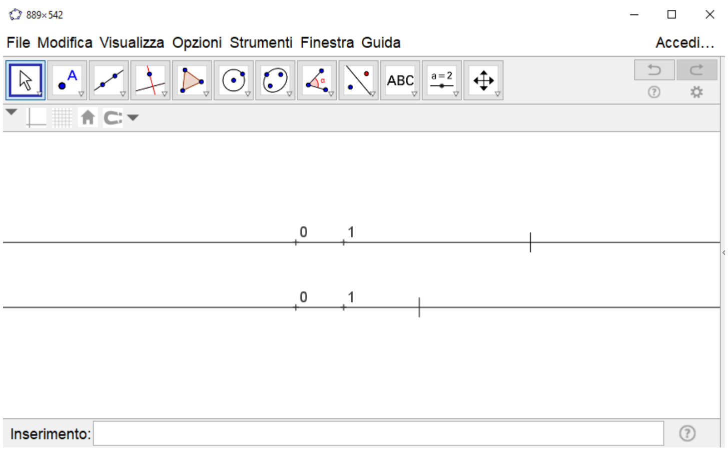


Figure 3. DGpp

The realization DGc is obtained by introducing a second dimension (**Figure 4**). In particular, by rotating the line containing the dependent variable, joining the zeros and making it orthogonal to the other line, it is possible to obtain the Cartesian numbered axes on which the two ticks are bound to move. The functional dependence between the two variables is still realized by the relation between direct and indirect motion but now they have two different directions: the tick on the abscissa axis is directly draggable, while the tick on the ordinate axis only indirectly draggable.

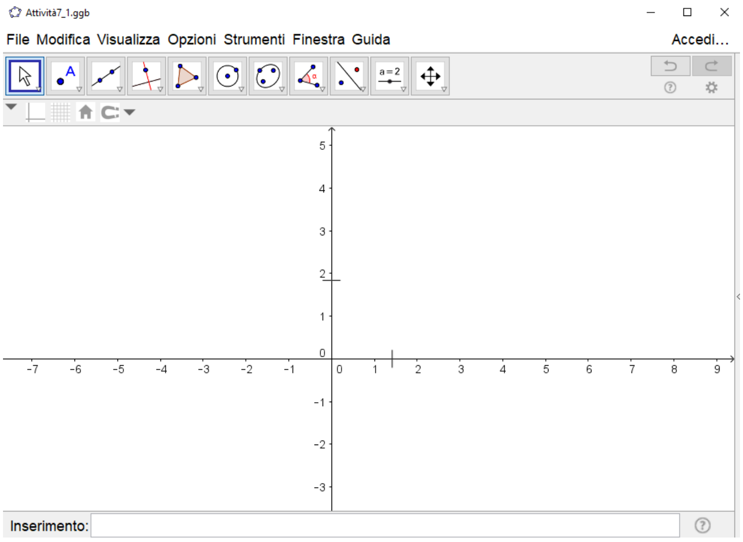


Figure 4. DGc

The tools offered by the dynamic interactive software allow to build point $(x, f(x))$ as the intersection point of the perpendicular line to the x-axis passing through x and the perpendicular line to the y-axis passing through $f(x)$; so, by dragging x, it is possible to see on the screen how $(x, f(x))$ moves in relation to the movements of x. Then, by activating the trace tool on this point and dragging the independent variable it is possible to obtain the image of the trajectory followed by $(x, f(x))$. This is the curve which realizes the graph of the function in the Cartesian plane, that we are used to seeing in textbooks and in paper-and-pencil environments in general. We call this realization of functions SGc.

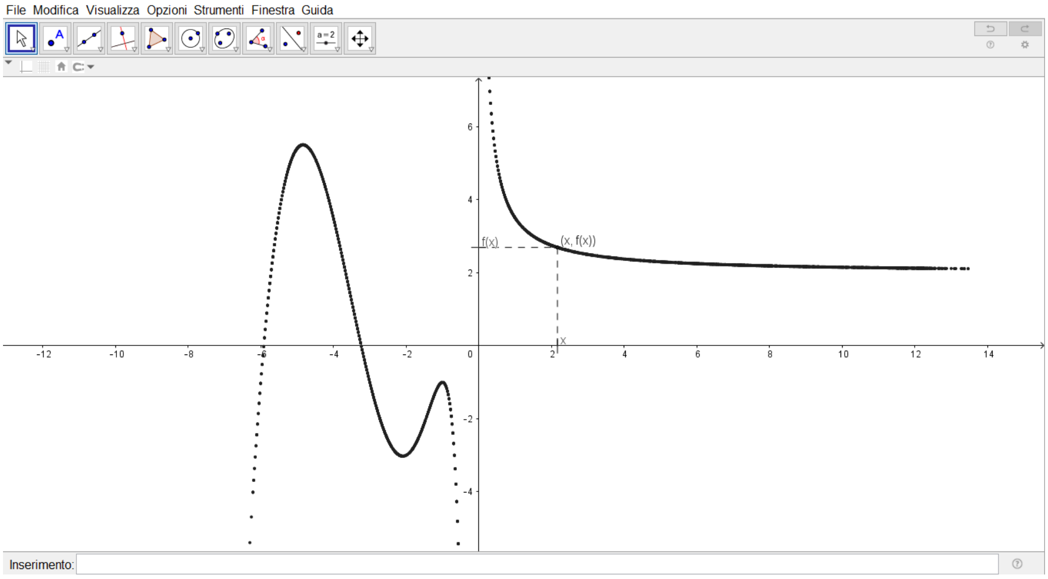


Figure 5. SGc

Traditionally, the realization SGc is the one that is used the most by the teachers to introduce students to functions and to work on functions (Section 1.1.2). For example, a common approach consists in asking students to interpret the Cartesian graph of a function or to draw it, which means to ask them to recognize the function's properties within the curve. This process involves seeing SGc as a set of points and identifying all the points belonging to the curve as pairs of coordinates, the first varying within the x-axis and the second varying within the y-axis; then these two variations have to be put in relationship with each other by imagining them happening simultaneously. In other words, the dynamism has to be collapsed into a static picture. We argue that the relation between SGc and DGc is what can make SGc actually become a realization of the signifier 'function' (**Figure 6**). Indeed, we believe that considering the static curve as the outcome of a dynamic relation between two covarying quantities, one depending on the other, is a precondition to think of SGc as a realization of the graph of a function. Otherwise a curve could simply realize a geometrical shape in the plane, as we have exhaustively discussed in Section 1.3.1.

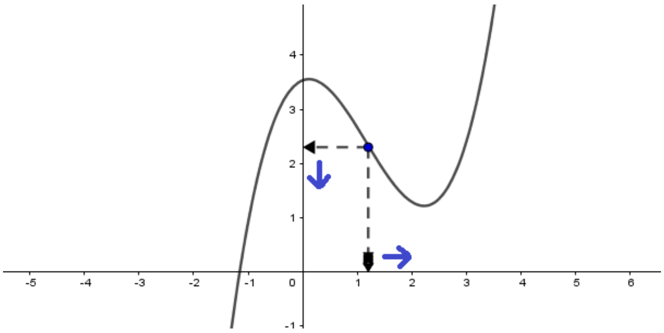


Figure 6. Relation between DGc and SGc

To sum up this section, we designed four DIMs that realize graphs of functions and all their properties and three of them are made possible by the DIE. One of their peculiarities is that each of them can also play the role of signifier and be realized by another one.

4.4 TASK DESIGN AND A PRIORI ANALYSIS OF THE ACTIVITIES

In this section we will describe the design of the research, trying to specify and explain which choices we made during the design process. Moreover, we will list all the activities that we implemented in the classroom and for each task we will provide an *a priori* analysis of the mathematical objects that we expect students will speak about. In particular, we will focus on the choice of words: if there are mathematical terms employed in non-expert ways, or if there are non-mathematical terms, in place of some formal terms, used in a coherent way for the expert. For example, the sentence "they are symmetric" used to describe two points moving in the same direction and opposite sides, but with different speeds, represents a non-expert employment of a mathematical term. Analogously, the question "How can I move $f(x)$ from 0 to 1?" can be formalized into the question "Find the pre-image of the interval [0, 1]", although it is still expressed in non-formal mathematical terms.

The task design phase of the experimental sequence took place before the beginning of the activities with the class but it was later modified after each meeting, and the sequence was re-arranged depending on the outcomes. All these modifications are explained in the following sections. For example, we named the objects in the file by using the names given by students or we decided to skip some tasks depending on the time we had left.

The activities that we designed contain all the realizations of functions described in the previous section and they were introduced following the same order that we used to list them. However, during the sequence of lessons we also supported the transitions between the different realizations, thanks to specific tasks that had the goal of building and then reinforcing the relations existing between them from a mathematical point of view. Indeed, one of our underlying goals is to lead these realizations to evolve (in the students' discourse) towards realizations of a same signifier. We think that this is an important process, because for an expert the four artifacts described above can all be considered possible realizations of the signifier 'function', but this is not necessarily the case for the students. Indeed, we worked with students who had to be introduced to functions and we decided to use these artifacts, which we expected that, at least initially, would be nothing but DIMs for the students; that is, visual mediators used in the communication with other students or with the researcher, as well as concrete mediators that they could directly manipulate. On the contrary, the ability of the expert consists in translating an approved narrative about the signifier within its different realizations, each of them having its own discourse that supports its unique set of narratives. So, the multiplicity of visual realizations broadens the communicational possibilities.

From this point of view, we can reformulate one of our goals at the core of the designed experimental sequence, that is a didactical goal, as an attempt to foster students' employment of DGp, DGpp, DGc and SGc as DIMs in their communication with themselves, with other students or with the teacher, that realize the signifier 'function' and all its properties. This has been investigated through the four research questions about students' discourse on functions as covariation of two variables, focusing on the role of dragging and on their use of the different realizations of functions proposed, in both DIE and paper-and-pencil environment. Moreover, the fourth research question addresses a possible development of students' discourse involving DGp, Dgpp, DGc and SGc along the sequence of lessons, by investigating whether a mirroring of experts' discourse occurs. Now, we are going to explain the design principles that we elaborated and applied to reach our purpose.

4.4.1 Design principles

The design principles that guided the design of this study follow two different directions: a theoretical-methodological direction; and a didactical direction. In particular, attention to the former should help us to analyze students' discourse, focusing on their learning process; while the latter supports the design of the sequence of lessons according to the didactical goal. We now introduce the design principles, underlyning for each the aspects related to the two directions.

P1) Foster students' speech aloud, supporting their discourse which includes gestures and dragging actions. In particular, foster students' discourse on specific mathematical objects related to functions.

P1a) Create conflictual situations for students who experience a mismatch between what they see and what they expect to see;

P1b) Ask for written explanations.

This principle is closely related to our view of thinking as an act of communicating in different forms – not only verbally. This principle goes in the theoretical-methodological direction, based on our theoretical assumption about learning, according to which we have to focus on students' discourse in order to gain insights about their learning.

P2) Focus on the exploration of covariation.

P2a) Do not use numbered axes, almost initially, to put the focus on variables' movements instead of their values.

This principle goes in the didactical direction. We are interested in the teaching and learning of functions as covariation because, as we explained in Section 1.2, we consider covariation to be an essential feature of the concept of function. In particular, we use this term referring to a dynamic, asymmetric relation between the variations of two variables. This qualitative description of covariation, which is in line with the dynamic interpretation of the graph suggested by Falcade, Laborde & Mariotti (2007), provides the basis for the choice of focusing on movements and looking at the numerical context at a later stage.

P3) Support continuous transitions between the different realizations in order to build and reinforce the relations existing between them, especially, the relations between dynamic and static ones.

P3a) Work on the differences and similarities between the different realizations proposed.

This principle goes both in the theoretical-methodological direction and in the didactical direction. One of our aims, that is both related to methodological and didactical choices, was that of providing students with different realizations of a same signifier and promoting their process of saming among the different realizations in order to build rich realization trees. In particular, we consider the relation between the two graphs DGc and SGc very significant for the construction of a discourse on functions, because thanks to this relation the Cartesian graph traditionally presented in schools and textbooks becomes a possible realization of functions.

P4) Give students previously constructed files where they can use basic tools like dragging and the trace mark.

This principle goes both in the theoretical-methodological direction and in the didactical direction. It goes in the former direction, because we set out to study students' discourse on dynagraphs; and constructing a dynagraph involves making explicit other realizations of the function. It goes in the didactical direction because we want to teach functions through dynagraphs and not vice versa; and the construction of a dynagraph requires previous introduction of functions.

P5) Support the development of a suitable language to communicate and to describe the realizations proposed.

> P5a) Use *ticks* instead of *points,* which is the default construction offered by the software GeoGebra, to realize the variables, to distinguish between the meanings of "one value" and "a pair of values"; because very often the misunderstanding of this fact causes difficulties;
>
> P5b) Use (or not) some mathematical formal terms in the text of the task depending on the goal of the activity and with respect to students' words choice (the teacher – here the researcher – plays a fundamental role in accomplishing this).

This is another principle closely related to our theoretical perspective, for which a student's learning process consists in individualizing the discourse of an expert and during this process a possible development occurs in the use of words that characterize each specific discourse. However, it is also connected to our didactical aim of supporting such a development of students' discourse.

P6) Minimize teacher's interventions during classroom activities.

This principle goes both in the theoretical-methodological direction and in the didactical direction. Indeed, it is related to our choice of focusing more on students rather than on the teacher, because we want to analyze their learning process as it develops. Therefore, we are interested in posing particular attention to students' interactions during the classroom activities and to the role of the teacher in the task design. For example, following principle P5b requires an important ongoing design action by the teacher, as does designing the interactive files and the activities in the first place.

P7) Set the task within a realistic context to support students' production of narratives.

This principle goes in the theoretical-methodological direction and it is in line with the first design principle that we expressed. Indeed, we are interested in students' discourse because it gives us information about their learning and so we want to foster students' production of narratives. In order to support this production, we implement some of the activities within realistic contexts involving functions, where the two variables represent specific quantities.

4.4.2 General structure of the sequence

As already discussed, we designed a series of interactive files in the dynamic algebra and geometry software GeoGebra, presenting specific activities, with the underlying goal of fostering students' emergence and development of discourse on functions and their properties. The tasks in the activities aimed at creating a challenge for the students, who then need to search for appropriate terms in order to produce a coherent discourse and communicate in the specific context of the activity. In fact, we do not expect their discourse to contain formal mathematical terms, at least initially, and this could be a source of difficulties when they are asked to describe their experience and to write down their findings. We also tried to build situations that could support the evolution of this discourse, in the direction of becoming more detached from the specific context and closer to that of an expert.

The diagram in **Figure 7** sketches out our design process for the sequence of lessons. The box contains the four DIMs, created and implemented for this study, and we separated with a vertical dotted line the realizations designed in the DIE from the realizations produced in the static environment. However, DGp, DGpp, DGc and SGc are linked together to convey that over the sequence of activities we also support continuous transitions between them. The circled part in the diagram shows the mathematical context of reference relative to which each activity has been designed, that is, the list of mathematical objects that a potential expert discourse would be about.

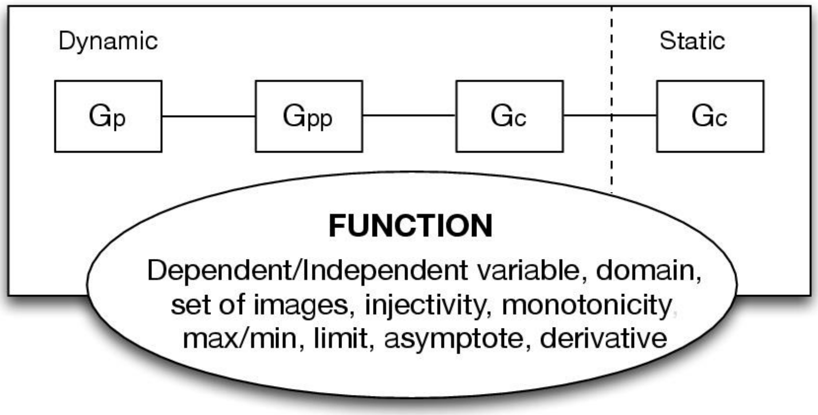


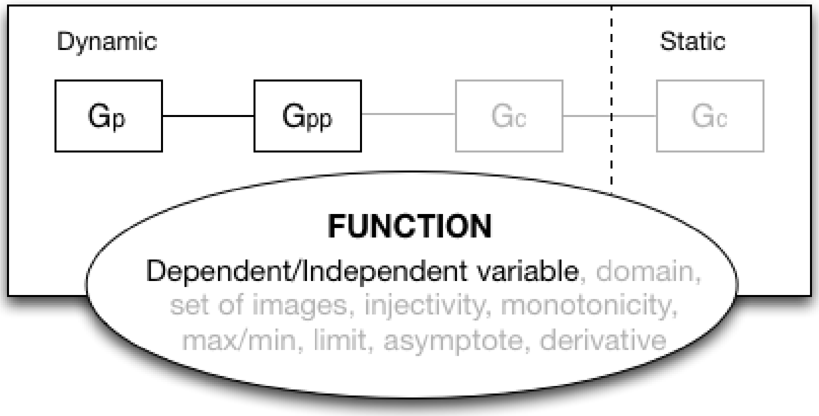
Figure 7. Diagram for the task design

An innovative and interesting aspect of this diagram is that from an expert's point of view it can be seen as a possible realization tree of the signifier 'function', where this term includes the functional dependence that links two variables and also all its properties. Indeed, each DIM can play the role of realization of a function and, at the same time, it can play the role of signifier that is realized by one of the other DIVs.

4.4.3 Task design of each lesson

In the following section we are going to describe in detail how the sequence of lessons has been designed. In particular, for each lesson we are going to illustrate: a copy of the diagram presented in **Figure 7** where we highlighted only the parts involved in the design of that lesson (the realizations used and the mathematical contents we wanted to introduce), the main goals of the lesson, the setting of the lesson, the list of the activities and, if there are any, the aspects that have been modified during the implementation of the sequence, with respect to the first design phase.

4.4.3.1 First lesson



Goal: to start to develop a language that enables to speak about the dependence relationship existing between the two variables.

Students work in pairs on pre-designed GeoGebra interactive files, using the dragging tool that is offered by the software. The task is the same for all the files and it is: *Explore the construction, identify and describe possible movements by using the dragging tool and write down your own observations on a sheet of paper.*

It is an open task that focuses on students' descriptions, it asks them to speak and discuss with their computer-mate and then elaborate and re-organize their ideas writing them down briefly. This leads students to have to choose which words to use in translating from the oral to the written form, which is an important aspect in the construction process of the discourse.

Our hypothesis was that some students could encounter difficulties in working with the new chosen DIM; but we thought that this would not be an obstacle for the implementation of the activities. In none of the files did the students have to construct anything using the tools offered by the software; they only had to move objects on the screen with the dragging tool.

Activity1_1: realization DGp of the function $f(x) = -x + 5$.

We decided to start with a linear function because it is defined everywhere and the movement of the dependent variable along its axis does not present any peculiar characteristics. Indeed, we wanted to bring the attention to the investigation of (im)possible movements and to the asymmetric relation between the two variables, because only one of them can be directly dragged. We thought that a more advanced function might lead the exploration to focus on describing the quality of movements and the speed of the two ticks. After having worked on this task, students had to share with the rest of the class their descriptions in order to compare them with the observations of other students and discuss them. The goal was to let the different descriptions of covariation emerge, highlighting possible differences in the choice of words, labels and gestures. Since the two variables are bound to the same axis, that is horizontal, our hypothesis was that students would be more likely to use "tick 1 and tick 2" or "first dash and second dash" or "A and B" to name the variables rather than x and y.

Activity1_2: realization DGp of the function $f(x) = |x|$.

We thought that this particular function could cause an inner conflict for students when they have to investigate the movement of the two variables for positive values of the independent one, since for these values the two ticks are perfectly overlapped and they move together. This aspect can even be surprising for students. After the initial exploration of the interactive file, each pair of students had to explain its observations to the whole class; we expected that someone would argue that one of the variables disappears while others would see the two variables staying close together. In this case, students would be involved in the important process of defending and supporting their own interpretation in front of the rest of the class.

In order to have feedback from the interactive file itself, some elements of the construction can be changed. For example, someone might propose to modify, hide or build some objects and so, to discover the actual behaviour of the two variables. Now is the right moment to suggest to students the introduction of a second copy of the real number line in order to bound the two ticks to two different lines. With respect to the diagram in **Figure 7**, this means passing from the realization DGp of the function to its realization DGpp and this activity plays a central role for the construction of this relation. Indeed, it shows the gains of having the two lines with the variables separated, which is relevant for laying the foundation for the construction of the Cartesian plane, where the domain and the range of the function are presented separately from one another.

Activity1_3: realization DGpp of the function $f(x) = |x|$.

As soon as the file is opened, it looks the same as Activity1_2, since the two lines are overlapped and the function is the same. But its design allows the student who is interacting with it to separate out the two lines by dragging one of them vertically, up or down, as previously described. This means that students can play with the realization of the function passing from DGp to DGpp and from DGpp to DGp. We chose not to start with them separated and have them fixed, in order to highlight that they are both copies of the real number line. We consider this fact mathematically important since these students will work, with us and during their education in general, with real functions having real values. So, we let students decide how to place the two lines, depending on the task and the exploration they have planned to do. For example, there are some cases where it is better to have the two lines separated, such as the absolute value function, but other cases where the opposite seems to occur.

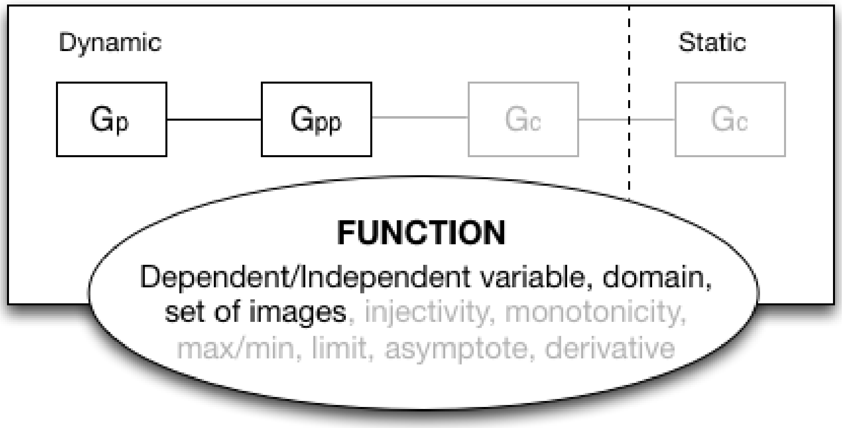
Activity1_4: realization DGp of the function $f(x) = \sqrt{x}$.

This function can generate a new conflict for students that is very similar to the previous one, but this time separating the two variables can be useful for discovering where and how the dependent variable moves when the independent one varies within the negative numbers. So our assumption is that the last two functions realized by DGp can lead students to similar reactions due to the fact that on a half-line of the real axis just one of the two ticks is visible, the one that is directly draggable. The mathematical reasons behind these two examples of "strange" behaviour of $f(x)$ are completely different because in the first case $f(x) = x$ for all $x \geq 0$, while in the second case the function is not defined for negative values of x. This difference can be appreciated by separating the two variables and seeing them on two separate axes, that is passing from DGp to DGpp, because it makes clear whether the dependent one is visible or not.

Design choices developed during the implementation of the sequence of activities:

- Activity1_4 was not given to students due to time restrictions.

4.4.3.2 Second lesson



Goal: to proceed with the construction of a language that enables to speak about the dependence relationship existing between the two variables and, in the attempt of this development, to explore some properties of functions, such as their domain and range.

As already discussed, the tasks have been designed to support the production of discourses that an expert mathematician would recognize as possible realizations of the properties considered. However, we expected the communication between students to be highly situated with respect to the particular context and that an important role for the success of this communication was played by the DGpp as a DIM.

As in the first lesson, students work in pairs on the pre-designed interactive files, using the dragging tool and, eventually, the trace tool that are offered by the software. Moreover, if someone asks for numbers on the lines s/he is allowed to activate the grid.

Activity2_1: realization DGpp of the function $f(x) = e^{x-1} + \frac{1}{25}$. The task is the same as in the previous activities: *Explore the construction, identify and describe possible movements by using the dragging tool and write down your observations on a sheet of paper.*

The function has been defined so that it is always greater than x, this means that there exists no x such that $f(x) \leq x$, but the value of $f(x)$ is very close to 1 for x tending to 1 (since the function is continuous and $f(1) = \frac{26}{25}$). So, during the solution of this problem we would bring students' attention to the movements of the variables in a neighborhood of $x = 1$. Indeed, we think that in the attempt to determine and describe the mutual position of the two variables in this neighborhood, students should need to express the dependence relation between $f(x)$ and x. This is one of the examples where it is probably more useful to keep the two lines overlapped, since this makes the comparison between the values of the variables easier. Moreover, this function has been chosen because it is appropriate for letting another mathematical object into the light, which we wanted to introduce during this lesson: the set of images. Indeed, $f(x)$ only takes on strictly positive values and we expect this to be observed by students during their explorations of the file.

Then, we designed two different activities in which the tasks are one the inverse of the other: in Activity2_2 an interval belonging to the range of a function is fixed and we ask to find out where x has to vary in order to obtain that interval as image; in Activity2_3 an interval belonging to the domain of a function is fixed and we ask to find out its image. However, it is not possible to deal with the domain and the set of images separately, especially within the one-dimensional realization of the function. In particular, in order to identify the domain of a function, given one of its dynamic realizations, the only possibility is to look for where the dependent variable exists. In general, we can already notice how in each activity several aspects characterizing the function are intertwined; so, we expected to find some of them realized in students' discourse, even if we did not consider them among the goals of that activity.

As we are going to see, these were the first two activities whose focus is not only on the behavior of the variables and their possible movements but also on their values.

Activity2_2: realization DGpp of the function $f(x) = \sqrt{x+3} - 2$, with the same task as before.

After a while, we projected the second part of the task on the screen of an interactive white board (IWB) for the class. Activity2_2bis consists of the following questions:

1) *Is it possible to have $f(x) = -3$? How?*
2) *Is it possible to have $f(x) = 3$? How?*
3) *How can you move $f(x)$ from 0 to 1?*
   Explain your answers on a sheet of paper.

We chose this particular function because we think that the presence of the square root could foster students' discourse on the domain. But we did not want to put too much emphasis on the zero; so, we did not use the function $y = \sqrt{x}$. Indeed, the zero is often considered by students as a special case where a function can have a strange behaviour. For the same reason we also decided to translate the function's output by -2.

Concerning the questions, their projection on the IWB follows a first phase of students' exploration and description of the graph, because we supposed that giving them the questions from the beginning of the activity could orient their explorations. The first and the second questions are the same, except for the numbers, which are chosen so that one is the opposite of the other and one gives an affirmative answer while the other a negative one. Indeed, our aim was to bring students' attention to critical values for the function, in order to speak about its domain. Moreover, the use of words in the construction of these two questions does not refer to the dynamism of the particular realization; this is done to investigate differences and similarities between students' discourse and a potential expert discourse mirrored. This activity can also support the production of discourse on the set of images of the function, since the dependent variable is always larger than or equal to -2; the first question helps bringing the attention to this fact. The third question is more focused on the dynamic features of the file and, in particular, it concerns the dependent variable's movement. It can be seen also from the words used to express the interval "from 0 to 1" which convey the idea of motion, differently from the realization [0, 1] of the same interval that would have statically expressed the same thing. Indeed, the underlying goal was to foster students' exploration of the covariation between the two variables and to highlight a foundational aspect characterizing the domain of a function: it expresses the possible values of the independent variable. We think that this fact has to be stressed because of the realization which allows the user to drag the independent variable along the entire real numbers line, without restrictions; so a realization of the signifier 'domain of the function' is a description of the possible values for x in order to visualize $f(x)$.

Activity2_3: realization DGpp of the function $f(x) = \sqrt{(x^2-1)(x^2-4)}$, the task is the same as in the first activity.

After a while, we projected the second part of the task to the class again using the IWB. Activity2_3bis consists of the following questions:

1) *Is it possible to have $f(x) = 4$? How?*
2) *Is it possible to have $f(x) = -4$? How?*
3) *By dragging x from -1 to 1, what are all the possible values that $f(x)$ can assume? Explain your answers on a sheet of paper.*

The function is not everywhere-defined and we consider this to be a good starting point to speak about the domain; the function is also non-injective, which could support students' observations about the set of images of the function.

The design of the first and the second questions has been already discussed for the previous activity, while the third question is about the image of an interval. A tool offered by the software GeoGebra that can be useful to answer to this question is the Trace tool. If the students are not used to working with the software, we did not expect them to use this tool spontaneously, in this case it could be suggested to them to activate it on $f(x)$. An affordance of this tool is to show all the values that $f(x)$ takes on when x varies. Interestingly, this is done by eliminating the temporal dimension: the result of the process is a static object, the set of all the positions touched by $f(x)$, which realizes the set of images of a specific interval of the domain.

Finally, the non-injectivity of the function allows to highlight the difference between the dynamic succession of values which the dependent variable takes on when the independent one moves from one value to another one, and the static set of the values which the dependent variable takes on. This is because in the first case we consider the fact that $f(x)$ takes on the same value twice, which needs to be noticed in time, while in the second case the temporal dimension is not taken into account.

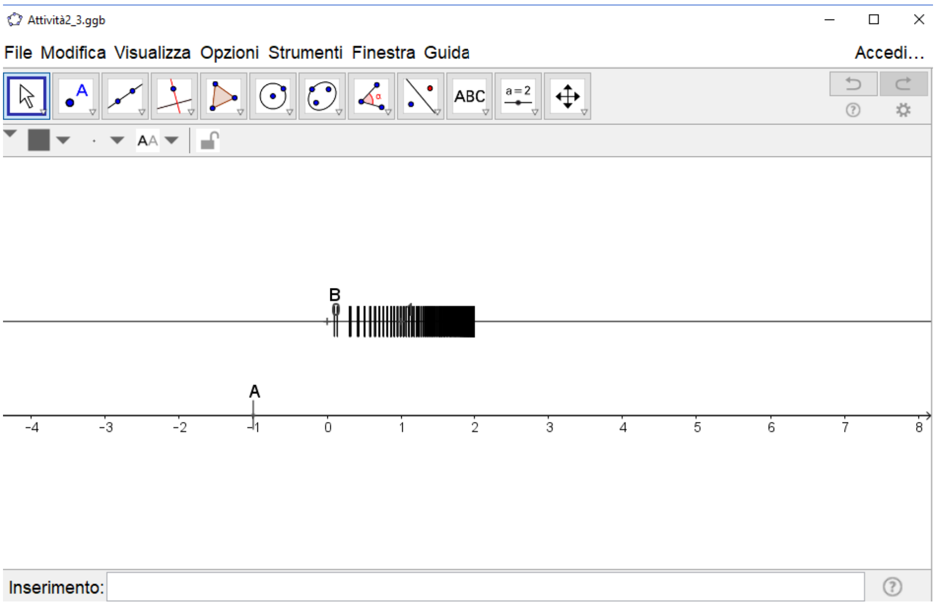
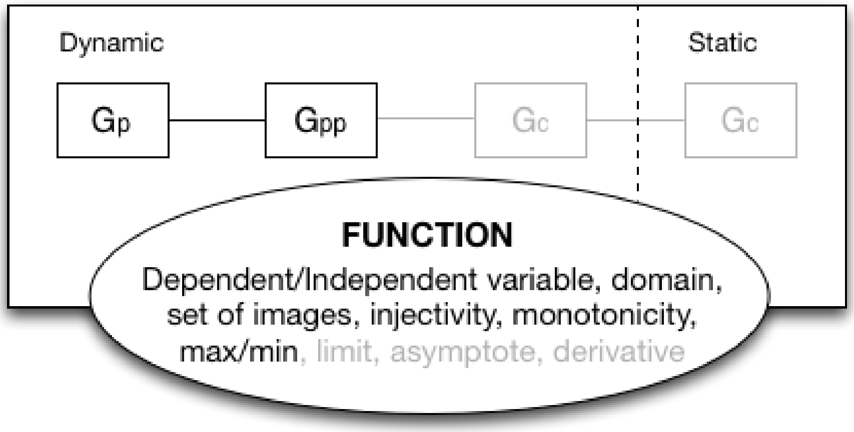


Figure 8. After having dragged A within [-1, 1], with the Trace activated on B

Design choices developed during the implementation of the sequence of activities:

- In all the activities and in the GeoGebra files the independent and dependent variables were labelled A and B respectively, because these were the names used and shared by the class during the previous lesson.

4.4.3.3 Third lesson



Goal: to reinforce the branch that links the signifier "dependent/independent variable" to its realization within the GeoGebra file "directly/indirectly draggable tick" and to go on with the focus on the domain and the set of images of a function. Moreover, while solving problems that involve these mathematical objects, to explore the injectivity that leads to observe the existence of maximum/minimum points.

Students work in pairs on pre-designed GeoGebra interactive files, using some of the tools offered by the software: Dragging and, eventually, Trace.

Activity3_1: realization DGpp of the function $f(x) = x + \frac{3}{x-3}$. The first task is: *Explore the construction, identify and describe possible movements by using the dragging tool and write down your own observations on a sheet of paper.*

The second part of the task, Activity3_1bis, is projected to the class by using the IWB and it consists of the following questions:

1) *Is it possible to have $f(x) = -1$? How?*
   Is it possible to have $f(x) = 1$? How?
2) *Which are all the values that $f(x)$ can assume?*
3) *Which of these values is it possible to obtain in 0; 1; 2; 3… different ways?*

We chose this function because it is not everywhere defined and it has a vertical asymptote. In the one-dimensional case this asymptote can be identified as follows: the dependent variable disappears from one side of the screen and it reappears from the other side. We know from the literature that students are usually surprised by this fact and they tend to produce creative narratives to interpret it (Healy & Sinclair, 2007).

Moreover, this function is non-injective and it has a relative maximum and a relative minimum. These aspects can be noticed by students if they observe that the dependent variable takes on every value twice, except at the two critical points. The third question is posed to promote this kind of exploration. The choice of the function could also support discourse on the intervals of monotonicity as a direct result of the description of maximum/minimum points, because at these points the dependent variable changes its direction of movement on the line.

The first and the second questions aim at the identification of domain and set of images. In order to identify the set of images of the function, and to answer the second question, it could be useful to activate the Trace tool on $f(x)$, as discussed for the last lesson.

Activity3_2: realization DGpp of a function defined *ad hoc*. The task is: *describe and comment all the information you can obtain about the charge level of a mobile phone battery, over time, during a 48 hours interval.*

We created a partially defined function that could ostensibly represent two life cycles of a battery. In particular, we bound x to the interval [0, 48] of the real number line and we used a quadratic function for the discharge processes and a linear function for the charge processes. Moreover, we designed the file so that during the first discharge process the battery reaches 0%, but during the second discharge process it stops at 8%. This way there are values that $f(x)$ takes on two, three or four times.

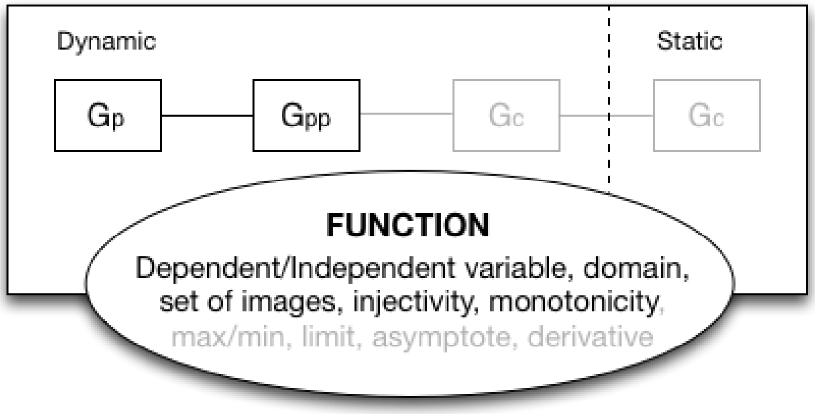
This problem is very similar to the previous one from a mathematical point of view, especially for the non-injectivity of the function and for its critical points, and also for the fact that the function is not everywhere defined. In this case the independent variable stops as soon as it reaches the extremities of the domain so it's highlighted by the construction. The main difference is that in the second case there is a realistic context. It could be interesting to investigate possible changes in students' explorations and discourse that can be related to the changing of the context.

The scale used for the two lines is not the same, so we did not expect students to overlap them – which would mean switching from the realization DGpp to DGp.

Design choices developed during the implementation of the sequence of activities:

- In Activity3_1 the independent and dependent variables were labelled A and B respectively, because these were the names used and shared by the class during the previous lessons.

4.4.3.4 Fourth lesson



Goal: to speak about the mathematical object 'set of images of a function', highlighting the difference between the set (a static object) and the trajectory of the dependent variable while moving the independent one in an interval of the domain (a dynamic process that includes time). In order to reach this main goal, we designed some activities that ask students to compare two functions. Another purpose of these tasks was to promote students' discourse on domain and monotonicity properties.

Students work in pairs on predesigned GeoGebra interactive files, using some of the tools offered by the software: Dragging and Trace.

First of all, we observe that the one-dimensional realization of two functions needs the construction of four parallel lines, one for each variable. This means that by dragging one independent variable the user can only move one dependent variable, that is its image. But the tasks ask for a comparison between the two functions, so we expected students to give answers separately for the two functions and then to try to match their answers in order to obtain only one valid for both functions.

Activity4_1: realization DGpp of the two functions $f(x) = \frac{1}{2}x(x+4)(x-2)$ and $g(x) = e^{x-1} - 5$. The task is: *compare and describe which the possible movements for f(x) and g(x) are. Then, determine whether it is possible that:*

1. *f(x) is greater than g(x)*
2. *f(x) is smaller than g(x)*
3. *f(x) and g(x) have the same value*
   And if it possible, for which x values does this happen?

The first request is to compare movements, as we usually asked in these activities, whereas the three questions are standard, similar to those we can find in an Italian textbook. This allows investigation of whether students translate their descriptions of movements of variables, that we expected to appear in the first answer, into static intervals to which the independent variable belongs.

The functions are defined so that they take on the same value for one x-value and, from a certain value of x on, one is always greater than the other one. With respect to the mathematical context of reference these functions are appropriate with respect to the goals of this lesson because $f(x)$ is non-injective, so it has different intervals of monotonicity where its behavior changes, while $g(x)$ tends to -5 for x tending towards negative infinity and it is an always increasing function.

Activity4_2: realization DGpp of the two functions $f(x) = x^2$ *and* $g(x) = |x| + \frac{3}{2}$. The task is: *choose, if possible, an interval where x_1 and x_2 can vary, in order for the set of values which $f(x_1)$ takes on and the set of values which $g(x_2)$ takes on to be disjoint.*

We chose these examples because they are always positive and non-injective functions, features that could remind students of the exploration that took place during the last lesson about finding the values that a function takes on zero, one or two times. We wanted two functions taking on the same value, for at least one x-value, in order to focus on the difference between functions having disjoint sets of images and functions having non-intersecting graphs. Indeed, to solve the task it is not enough to choose an interval of the domain which does not contain $\boldsymbol{x}$ such as $f(\boldsymbol{x}) = g(\boldsymbol{x})$, but we need to have that "for every x_1, x_2 in the interval $f(x_1) \neq g(x_2)$". This means that, given an interval where the independent variable varies, it does not matter for which value in this interval (when) the dependent variable takes on a fixed value in the codomain, but whether it takes on this value at all. By looking at the DGpp without considering the temporal dimension characterizing the variation of the independent variable we could clearly visualize and verify this property, and this can be done for example by activating the Trace tool on $f(x_1)$ and $g(x_2)$ and dragging x_1, x_2 within a certain interval.

We expected students to overlap the lines where x_1, x_2 move, as shown in the figure below.

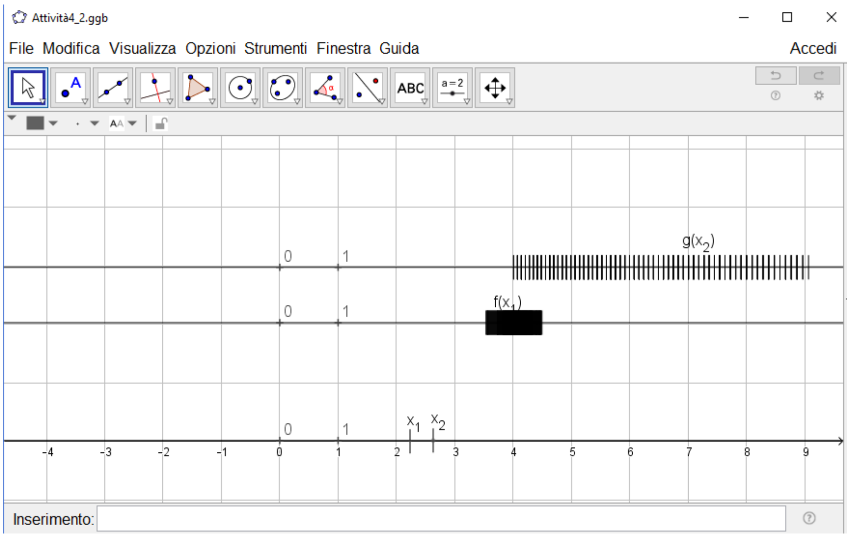


Figure 9. Example of dragging x_1, x_2 in [2, 3] with activated trace. The functions have non-disjoint sets of images but different for every x in [2, 3].

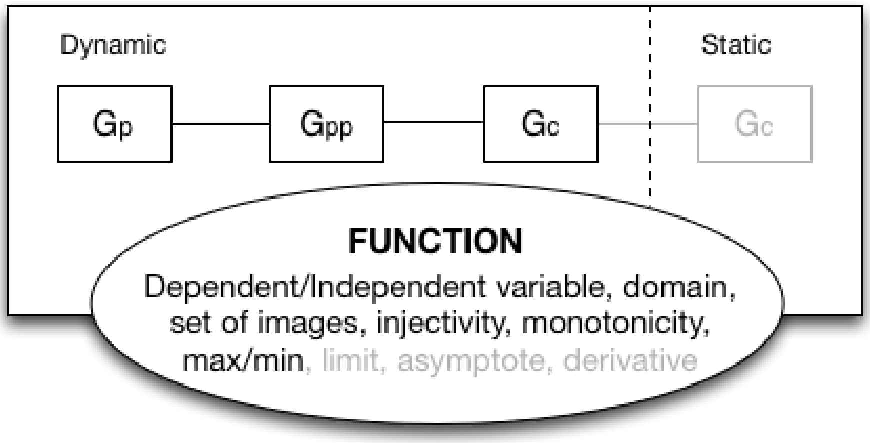
Design choices developed during the implementation of the sequence of activities:

- We replaced the first activity with another one, that is very similar to it from a mathematical point of view, but which involves a realistic context. This is because

we noticed that students developed richer and more creative narratives when they had to solve problems where they could find some relations with their experiences.
Activity4_1: realization DGpp of two functions, which contains four parallel lines and the grid is activated in order to have all the lines numbered. The task is: *describe and comment all the information you can obtain about Aldo's (T_A) and Bianca's (T_B) telephone plans, which are expressed in euros, depending on the time spent to call, which is expressed in hours. Then, compare the two plans.*
We defined two functions in this way: one linear function and a partially defined function that is initially constant and then it becomes a step function. They seem quite realistic in representing phone fees. Moreover, these functions take on the same value for two different x-values, that can support discourse on the slope, and so the derivative, of the functions. Indeed, we expected students to observe which dependent variable is the fastest or the slowest in a fixed interval before and after their meeting.
The task requires a comparison, for example students could find which function is the greatest, or the smallest for a fixed point of x_1 and x_2, which in terms of the phone fees means to find the most expensive, or cheapest, fee for a person who spends that amount of time calling.
This problem could also foster discourse on the set of images, for example by setting a particular range of time calling and asking how much Aldo and Bianca should pay.

- In Activity4_2 the variables $x_1, f(x_1), x_2, g(x_2)$ were labelled A, B, C and D respectively, because these were the names used and shared by the class during the previous lessons.

4.4.3.5 Fifth lesson



Goal: to introduce students to the dynamic realization of a function on the Cartesian plane, and to work on the construction of the link that relates it to the already known realizations of the function and its properties.

Students work in pairs on predesigned GeoGebra interactive files, using some of the tools offered by the software: Dragging and, possibly, Trace.

Activity5_1: realization DGc of the function $f(x) = -x + 5$. The task is: *Explore the construction, identify and describe possible movements by using the dragging tool and write down your own observations on a sheet of paper.*

This is the same function we used for Activity1_1, because we wanted a linear function to let students explore the realization DGc for the first time and we wanted a function that they had already met within the one dimensional graph DGp, in order to investigate changes or patterns in their discourse. The main goal of this first activity was to work on the covariation of the two variables in two dimensions. There is a great difference with the previous realizations because one variable moves horizontally while the other one now moves vertically, so we expected that dealing with both variations simultaneously could be somewhat harder.

Moreover, we asked students to compare this file with Activity5_1bis, the realization DGpp of the same function, and to find similarities and differences. This second task was designed to build the relation between DGpp and DGc.

We expected students to name the variables "x, y" (if it has not happened yet), as soon as they see the Cartesian plane, and also that someone might ask for the point $(x, f(x))$; in that case we would investigate what they suggest doing in order to visualize it or how they imagine its movement. So, for the moment we would ask them why they consider that point to be important, explaining that it is an interesting idea but that we would speak about it next time.

Activity5_2: realization DGc of the function $f(x) = \sqrt{x+3} - 2$. The task is:

1. *Is it possible to have $f(x) = 3$? If yes, how?*
2. *Is it possible to have $f(x) = -3$? If yes, how?*
3. *Which are all the possible values for $f(x)$? And for which x values are they taken on?*
   Explain your answers on a sheet of paper.

This is Activity2_2 with some modifications: the realization of the function is two- instead of one-dimensional and the third question is more general. We decided to propose some functions and some tasks again, in order to analyze possible changes in students' approach and discourse and to let students explore all the mathematical objects within the new realization of the function. In particular, this example supports the exploration of the dependence relation, the domain and the set of images, as discussed for Activity2_2. We re-formulated the third question in a more general and formal mathematical way to support the evolution of the discourse.

Finally, we designed a problem that could reproduce a realistic situation.

Activity5_3: realization DGc of the function *ad hoc* defined $f(x) = -\frac{x^2}{25} + x + 1$, where the x-axis is labelled "liter" and the y-axis is labelled "ton". The task is: *describe and comment all the information that you can obtain about the seasonal trend of the citrus production of a farm, depending on the quantity of fertilizer which is used.*

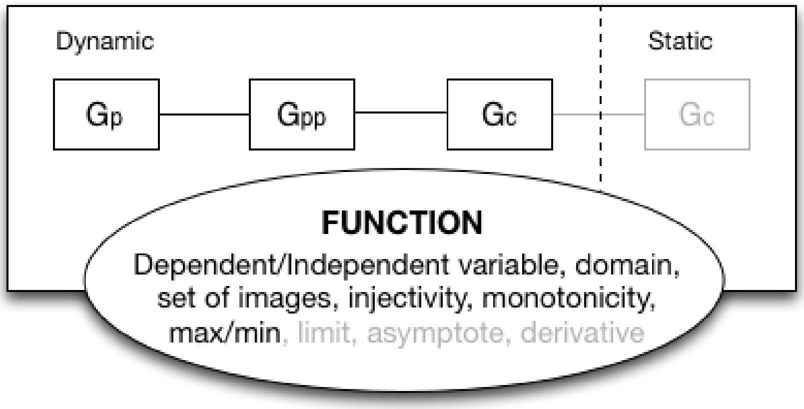
We rotated and translated a parabola in order to have a fertilizer's optimal quantity for the production of the farm, while for a higher quantity the citrus production decreases. The mathematical objects that could emerge from this activity are the following. The domain, indeed we restricted x to the interval $[0, \frac{25+5\sqrt{29}}{2}]$ and so the tick stops when it reaches one of the two endpoints that have been chosen in this way: $x = 0$ because using a negative quantity of fertilizer has no meaning in a real context; $x = \frac{25+5\sqrt{29}}{2}$ because $f\left(\frac{25+5\sqrt{29}}{2}\right) = 0$ and for bigger x-values the function is negative, that is, the farm produces a negative amount of citrus and, again, it has no meaning in a real context. The set of images, indeed $f(x)$'s movement is limited, and the intervals of monotonicity of the function that increases and then decreases; its maximum point and the fact that it is a non-injective function which takes on almost every value twice.

As we can notice, the same task could be given in a static context, because there are no references to the specific dynamic environment. This is designed to not influence students' words choice and discourse construction.

Design choices developed during the implementation of the sequence of activities:

- In all the activities the independent and dependent variables were labelled A and B, respectively, because these were the names used and shared by the class during the previous lessons.

4.4.3.6 Sixth lesson



Goal: to develop students' use of words and expressions to describe the mathematical object 'function'. In particular, to reinforce the branches between different realizations that we can find in students' discourse on the dependence relation between two variables and all the mathematical properties of functions met at this time.

Students work in pairs on predesigned GeoGebra interactive files, using some of the tools offered by the software: Dragging and, potentially, Trace.

In order to reach the main goal of this lesson students need to talk and produce narratives. We designed a problem where two functions are involved and it asks for the composition. In particular, students were engaged in working with two separated files where the dependent variable of one of them should play the role of the independent variable in the other file. In this way they could gain some information about the compound function.

Activity6_1: realization DGc of a function defined *ad hoc*, where the x-axis is labelled "month" and the y-axis is labelled "m^3". The first task is: *describe and comment on all the* *information that you can obtain about the quantity of water that has to be used every month to irrigate a garden for a year, that has 100 m^2 of grass.*

We used a partially defined function whose domain is the interval [0, 12) and for which $f(x) = 0$ for every integer x belonging to this interval. Moreover, we set it up so that during the summer, for $x \in (5, 7), f(x)$ reaches the maximum value.

Activity6_1bis: realization DGc of a function defined *ad hoc*, where the x-axis is labelled "m^3" and the y-axis is labelled "euro". The task is: *describe and comment on all the information that you can obtain about the price of water with respect to its consumption.*

The domain of this function is the set of real positive numbers and it is defined by using three linear functions, that model the water's fee (according to information we found on the web).

As for the last lesson, someone could ask for the point $(x, f(x))$ and in that case we would investigate what they suggest doing in order to visualize it or how they imagine it might move. In that case, for the time being, we would ask them why they consider that point to be important, explaining that it is an interesting idea that we will speak about next time.

Then the second part of the task is projected onto the IWB; it contains the following questions: *The owner of a garden is worried about the amounts he has to pay to irrigate his garden that has 100 m^2 of grass, and he has the following doubts:*

- *a) Will I ever have to pay less than 5 euros for the irrigation of my garden? If yes, when? If not, why not?*
- *b) Will I ever have to pay more than 50 euros for the irrigation of my garden? If yes, when? If not, why not?*

By keeping the files containing Activity6_1 and Activity6_1bis open, answer the two questions on a sheet of paper.

Then, describe how the cost of the water varies in time, for the owner of the garden that each month has to pay for the garden's irrigation (we do not consider any monthly fixed fees).

Questions a) and b) are designed to have students use both the files simultaneously, in order to gain some information about the compound function. For example, in order to answer the first question, we expected that they might open Acctivity6_1bis and drag x to obtain $f(x) \leq 5$ and discover how many m^3 there are.

Finally, we explicitly asked for a description of this function, in qualitative terms. We notice that the tasks never use the formal expression 'compound function'; indeed we do not think our students have ever seen this mathematical object in class, but by using the dynamic realization designed and the discursive approach proposed they should be able to make sense of it and deal with it.

The following problem is taken from the literature and re-arranged for the specific dynamic environment.

Activity6_2: realization DGc of two functions, where there is one independent variable bound to the x–axis and two variables depending on it bound to the y–axis. The task is: *Here are represented the speeds of two different cars with respect to time, they started the race at the same position and they move in the same direction. What can we infer about the mutual position of the two cars at different times?*

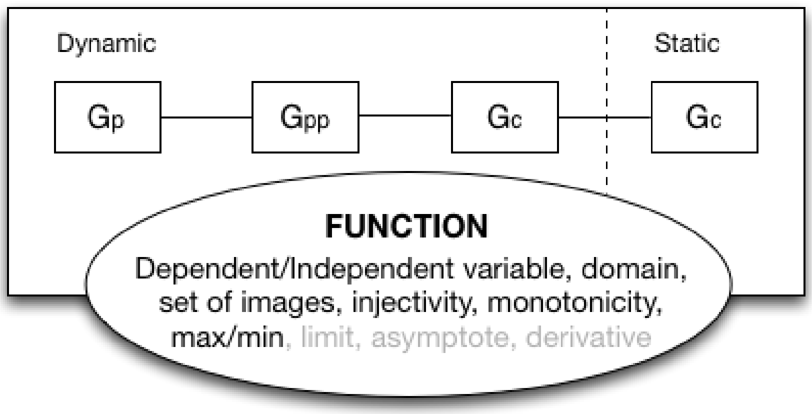
We defined two speed-time functions $v_1(t), v_2(t)$, that model a possible real situation if we consider the units $\frac{km}{h}$ on the y-axis and m on the x-axis. For small t values $v_1<v_2$, then this relation swaps and the cars have a constant speed. The most interesting point is where the two cars have the same speed, because it is usually interpreted by students as the point where the cars physically meet each other (Carlson et al., 2002).

As we can notice, the same task could be given in a static context, because there is no reference to the specific dynamic environment. This was done in order not to influence students' choice of words and discourse construction.

Design choices developed during the implementation of the sequence of activities:

- Activity6_2 was not proposed in class, to leave more time for the compound function problem that better fitted within the sequence of lessons that were actually proposed to our students. Indeed, Activity6_2 is first of all a modeling problem whose focus is the fact that a speed-time graph is represented and typically treated as a space-time graph: this aspect was not among our priorities at this point of the experimental sequence.

4.4.3.7 Seventh lesson



Goal: to introduce the realization SGc of the function by working on the branches that link it to the other realizations and, in particular, to deal with the point $(x, f(x))$, studying its trajectory in the Cartesian plane.

Students work in pairs both in the dynamic and static environments. Indeed, there are some tasks in GeoGebra interactive files, where students have to use Dragging and Trace tools and some other tools offered by the software to make small constructions in the same file (for example points and perpendicular lines), but there are other tasks that involve paper and pencil answers.

The first and second activities support the transition between SGc and DGc, so the construction of a relation between two realizations of the same signifier.

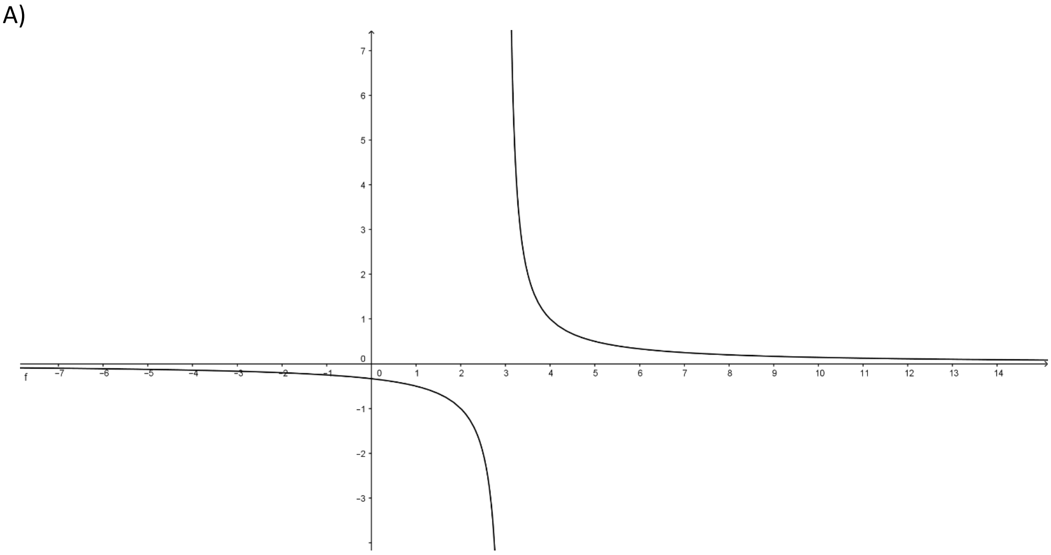
Activity7_1: realization DGc of the function $f(x) = \frac{1}{10}\left(\frac{x}{2} + 4\right)(x+1)(x-2) + \frac{5}{2}$. The task is: *draw on a sheet of paper the trajectory of the point* $(x, f(x))$.

This is a non-injective cubic function with only one real zero, because we wanted to try to avoid the strategy that consists in finding some specific coordinates and then connecting them with a curve to obtain the graph of the (usually known) function. In this case the intersection points with the x-axis, at least, are not so simple to discover. Instead, we tried to foster the description of the dynamic aspects, for example imagining the trajectory of point $(x, f(x))$. We observe that this point is not visualized on the computer screen, so the task requires paying attention to the variations of both variables at the same time and then sketching a curve on the sheet of paper that realizes this covariation.

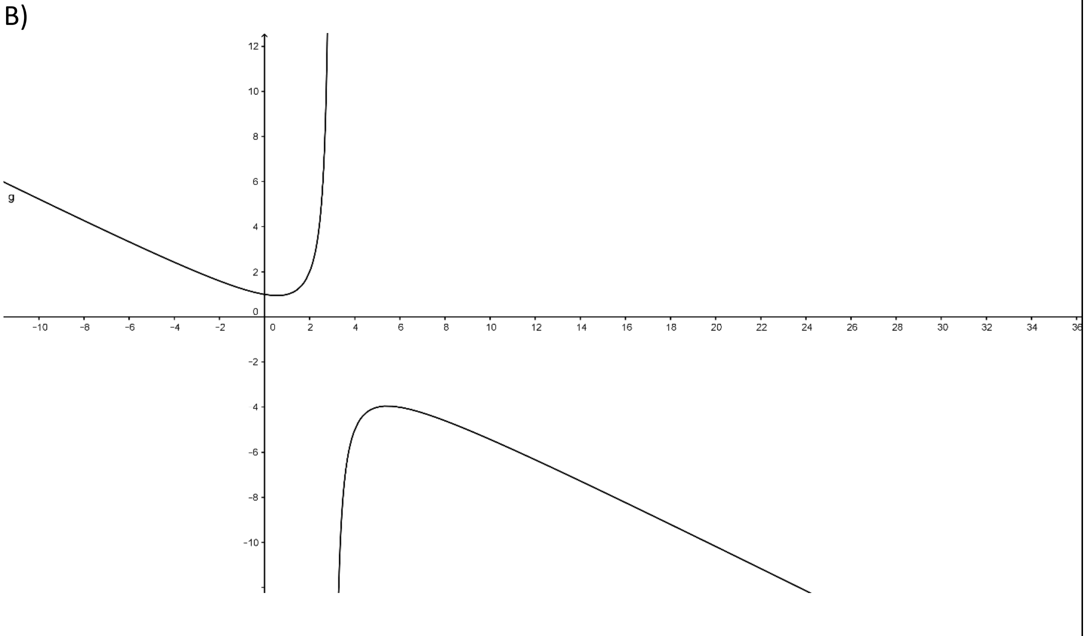
Activity7_2: realization DGc of the function $g(x) = \frac{x}{2} + \frac{3}{x-3}$. The task is the inverse of the previous one and it is on the following sheet of paper, which was given to each pair of students

Open the GeoGebra file Activity7_2.ggb and state which one of the following graphs represents the trajectory of the point $(x, g(x))$. Explain your own choice, describing on the sheet of paper how you chose your answer and why you rejected the others.

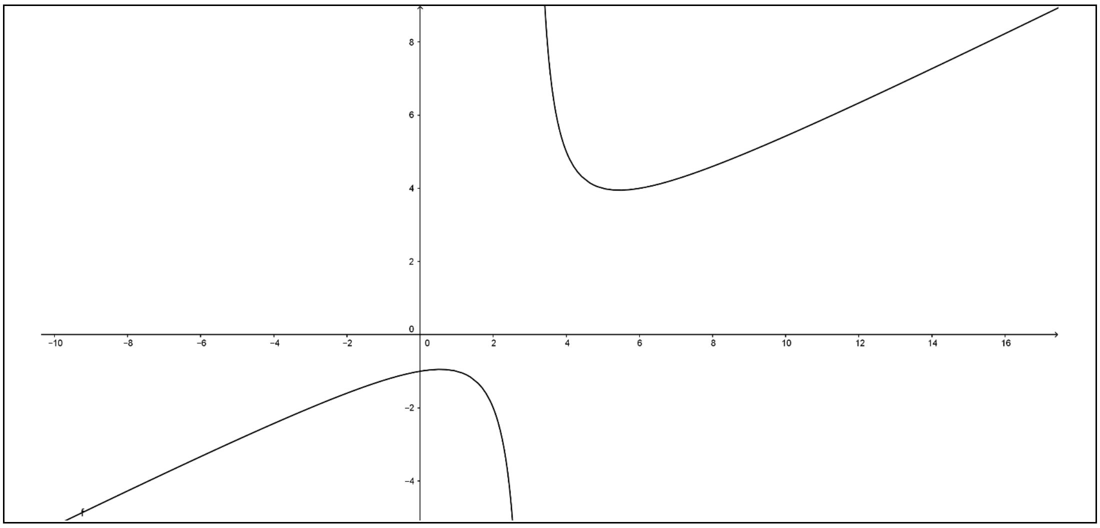
A)



B)



C)



Students dealt with this function in Activity3_1 with the realization DGpp so it would be interesting if someone recognized it and suggested some links with the one-dimensional realization. The two incorrect options are the graphs of the functions $\frac{1}{x-3}$ and $-\left(\frac{x}{2}+\frac{3}{x-3}\right)$, which were chosen because they both have a singularity for $x = 3$ but their behavior for x tending to negative and positive infinity is different.

After this activity, we used the IWB to show the construction of point $(x, g(x))$ and, by dragging x, we fellow its trajectory so students could check their answers. Then we asked them to do the same construction in the file Activity7_1 and to use it to verify their drawings. We expected that someone could suggest the activation of the Trace tool on the constructed point in order to better visualize its trajectory, otherwise we would do it.

We remark that the choice to show the construction on the IWB in order to give students the necessary tools is due to the fact that they were not used to working with this software. Indeed, it was not the main focus of the lesson, so we did not consider it worth to spend too much time on this part. At the same time, we think that by visualizing and reproducing the construction students may experience the importance of building two lines to obtain the desired point, and so the fact that a point in the Cartesian plane has two coordinates, not only one value. In particular, the lines have to pass through the independent and dependent variable respectively, so attention can be brought to the fact that the point $(x, g(x))$ exists thanks to both contributions.

The next activity supports the transition between SGc and DGpp, so the construction of a relation between two realizations of the same signifier.

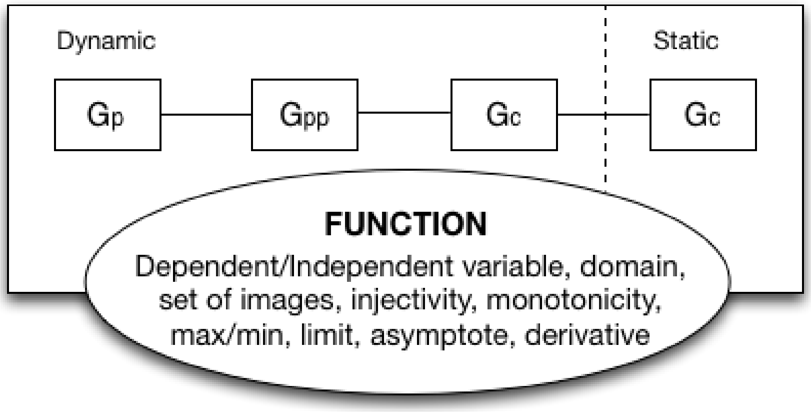
Activity7_3: realization DGpp of the function $h(x) = \frac{3}{2x} + 2$. The task is: *draw on a sheet of paper the graph of this function in the Cartesian plane.*

We performed certain transformations to $\frac{1}{x}$, in order to avoid immediately recognizable points that students could interpolate, for the same reason explained before. And we started from $\frac{1}{x}$ in order to have a function with vertical and horizontal asymptotes to foster students' discourse on these properties of the function, in particular during the transition from one dimension to two dimensions.

Design choices developed during the implementation of the sequence of activities:

- Activity7_3 was not proposed in class because students' construction of the point $(x, f(x))$ to verify their answers to Activity7_1 took up a lot of time.

4.4.3.8 Eighth lesson



Goal: to reinforce the links between the four different realizations DGp, DGpp, DGc and SGc and between these and the mathematical reference context. For this reason, the aim of the designed activities is to support the use of different symbolic artifacts simultaneously.

Students work in pairs both in the dynamic and static environments.

Activity8_1: realization DGpp of the function $f(x) = \begin{cases} \frac{3}{2x} + 2 & x > 0 \\ \frac{(x+1)^2(x+6)(x+3)}{x} - 1 & x \leq 0 \end{cases}$

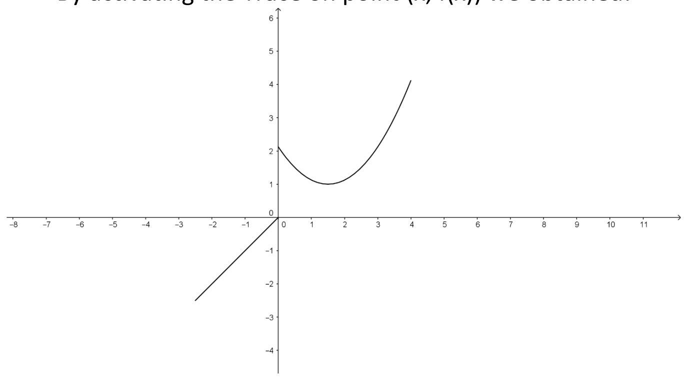
The task is*: draw on a sheet of paper the graph of this function on the Cartesian plane.*

We partially defined a function trying to obtain horizontal and vertical asymptotes, non-injectivity and coordinates that were not immediately recognizable. In this case it is not possible to build the point $(x, f(x))$ because the dynagraph is in one dimension.

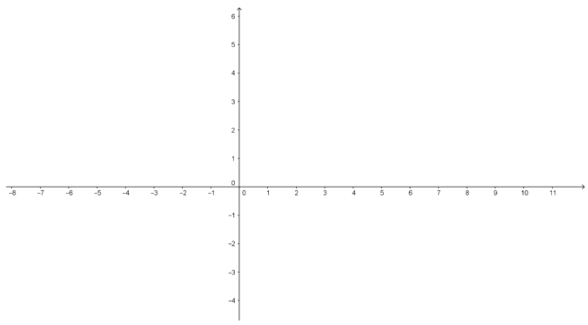
After this activity we showed on the IWB the same realization of the same function. However, it was possible for us to rotate the y-axis in order for it to become orthogonal with respect to the other one, and that, combining the origins, to obtain the Cartesian plane. Moreover, in the same file we built point $(x, f(x))$ and let students compare its trajectory with their drawings, potentially activating the trace tool on this point. In this way we let students work on the relation between three realizations of the same signifier (DGpp, SGc, DGc) and on the realization of the graph as the trajectory of the point $(x, f(x))$.

Activity8_2: the task is on a sheet of paper, given to each pair of students. It contains the following:

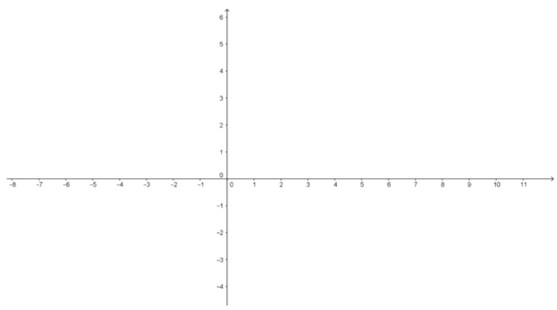
By activating the Trace on point (x, f(x)) we obtained:



Indicate what would colour by activating the Trace on x:



Indicate what would colour by activating the Trace on $f(x)$:



This activity focuses on the distinction between the independent variable, the dependent variable and the point $(x, f(x))$. The Trace tool has a twofold function: it transforms the (dynamic) trajectory into a (static) set of “crossed” values by eliminating the temporal nature of the dragging action that causes the movement of the variables and it helps to keep separate the notions of domain, set of images and graph because there are different parts of the same figure colored.

Activity8_3: the task is on a sheet of paper, given to each pairs of students. It contains the following:

> Draw on the Cartesian plane the graph of a function with the following properties:
>
> - Before x equals zero, if x increases $f(x)$ increases as well
> - When x is greater than six, they have opposite directions
> - As x moves forward, $f(x)$ moves more and more: for example, if x moves from -5 to -4, $f(x)$ moves only little, if x moves from 1 to 2 $f(x)$ moves much more
> - $f(x)$ can take on all the negative values and the positive values up to 10, because when it reaches 10 it goes back
> - They intersect approximately at 3,5
> - We can obtain some values of $f(x)$ in just one way, others in two, in three, or in four different ways

This task is very similar to Activity8_1 but this time the description of the dynamic graph with parallel axes is given and students have to draw it. So, this time, they were asked to translate the information expressed verbally into an iconic realization, but they also had to translate the description of a DGpp into a SGc passing from the dynamic to the static context and from one dimension to two dimensions.

Design choices developed during the implementation of the sequence of activities:

- The description of the properties that the function should have in Activity8_3 is made combining several expressions used by students during the sequence of activities.

4.4.4 A priori analysis

In the following table we made a list of all the activities designed for the experimental sequence; for each activity we specified which realization of the function is involved. Then, we colored the boxes referring to the mathematical objects, which are listed in the first line, depending on the goal of that activity. In particular:

- A black box indicates the mathematical object that has been mainly considered during the design process of the task. It means that the main goal of that activity was to foster students' discourse about this mathematical object.
- A grey box indicates the mathematical object that has been considered during the design process of the task, but with a secondary role. This means that we designed that activity in order to let students eventually also include this mathematical object in their discourse, because it represented like a sub-goal of the main goal indicated by the black box.
- A white box refers to a mathematical object that has not yet been included among the goals of the task during the design process.

As we can see in the table below, some mathematical objects were at the core of several activities, while others occurred only in some of them. For example, each activity focused on the dependence relation between the two variables, even if it was not the main goal of all of them, as suggested by the colors. At the same time, we can observe that the derivative was considered in the design of an activity for the first time in the last lesson.

This table refers to the analysis done for the task design and so it indicates the mathematical objects that each activity is designed to support. In particular, we highlighted our goals with respect to the mathematical context, by using three different priority levels determined by the use of white or grey or black. This allows the reader to have a general view of the whole experimental sequence that we designed, but it will also be useful in the *a posteriori* analysis, when, after the implementation of the sequence, we will create a similar table and then compare them.

Activity	Type	IN/DEP	DOM	RAN	INJ	MON	MAX/MIN	LIM	ASY	DER
1_1	DGp									
1_2	DGp									
1_3	DGpp									
2_1	DGpp									
2_2	DGpp									
2_3	DGpp									
3_1	DGpp									
3_2	DGpp									
4_1	DGpp									
4_2	DGpp									
5_1	DGc-DGpp									
5_2	DGc									
5_3	DGc									
6_1	DGc									
6_1bis	DGc									
7_1	DGc-SGc									
7_2	DGc-SGc									
8_1	DGpp-SGc									
8_2	SGc									
8_3	SGc									

Table 4.2. A priori analysis of the mathematical objects

4.5 DESIGN OF THE INTERVIEW

The design of the interview took place at the end of the implementation of the lessons.

The main goal was to make students deal with the static and the dynamic realizations of a function, so the problems that we designed were centred on the relation between different realizations; for example, students had to translate their discourse about a dynamic graph into discourse about a static realization of the same function. Moreover, we wanted to observe students' behaviours when asked to apply the formal definitions of some (supposedly new) mathematical objects, which had been discussed during the lessons but had never been introduced in formal mathematical terms. Indeed, we were interested in observing possible changes in their discourse.

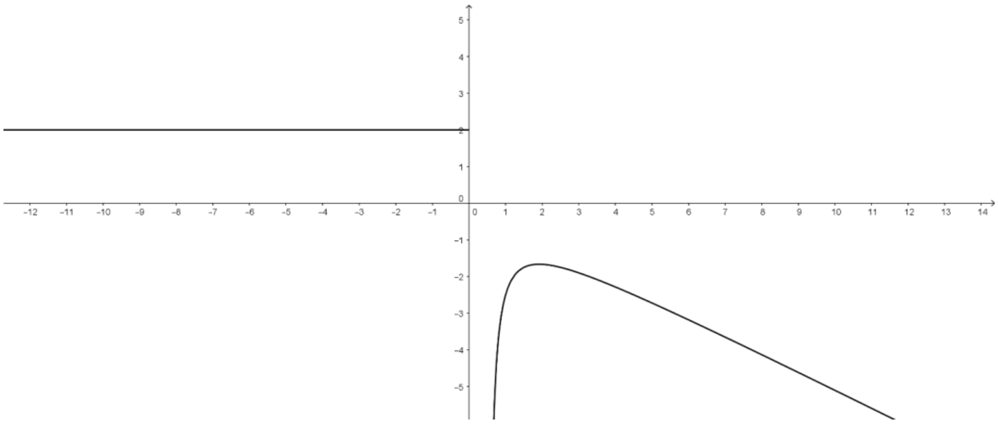
Now we are going to describe the tasks given to the pairs of students during the interview. In particular, we will see the transcript of the interview that we used with Alessio, which contains some of the activities designed for lessons 7 and 8, that took place when he was absent. Then, we will see the transcript of the interview that we used with Matilde and Nicco; for the interviews used with the pairs of other students we only changed the functions, since they took place within an interval of 3-4 days. In general, we defined specific functions in order to have examples of vertical or horizontal asymptotes, limited domains or limited sets of images, constant functions, maxima/minima and points of non-injectivity. These choices were made because we wanted the students to speak about the different mathematical properties of functions which they had encountered during the lessons.

4.5.1 The tasks for Alessio

1) Activity7_1
2) Activity7_2
3) Activity8_1
4) Activity8_2

4.5.2 The tasks for Matilde and Nicco

1) Imagine having this function represented with parallel axes in a GeoGebra file, how would you describe it?



2) One of you sits at the computer and the other one sits in front of him.
   The student at the computer opens the file named Intervista1.ggb and describes it to other student who, following the directions received, has to draw the graph of the function on the Cartesian plane on a sheet of paper.
   ATTENTION! The following actions are forbidden: to show any writing, to turn the computer screen, or to give information by using hand gestures.
3) Repeat the same activity, this time changing your spot and with the file Intervista2.ggb.
4) Carefully read the instruction on the sheet of paper that you received; it is the copy of a page belonging to a mathematical textbook for high school. Then answer these questions which refer to the graph below:

   a. *Colour the image of the interval [-1, 1]*
   b. *Colour the pre-image of the interval [-5, -3]*
   c. *Which is the domain of this function?*
   d. *Which is the set of images of this function?*
   e. *Is it injective? Why?*

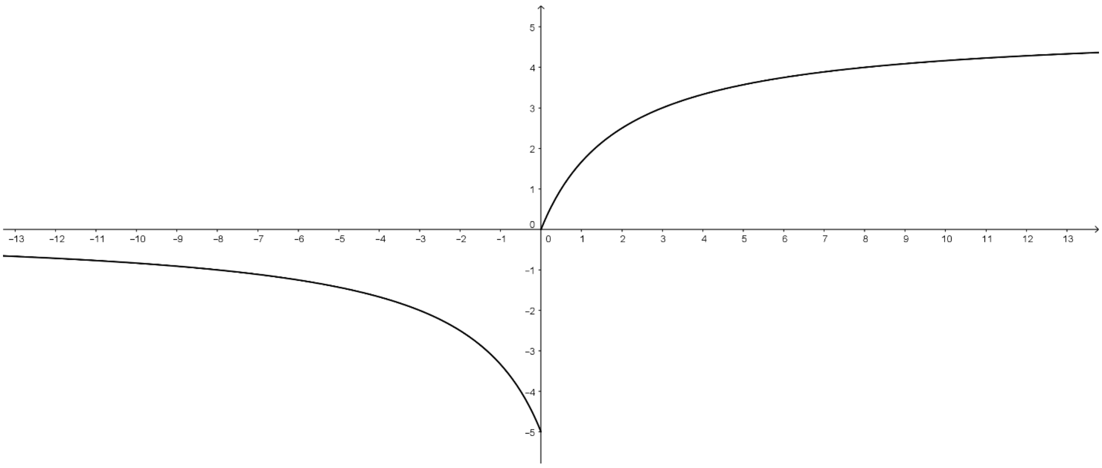
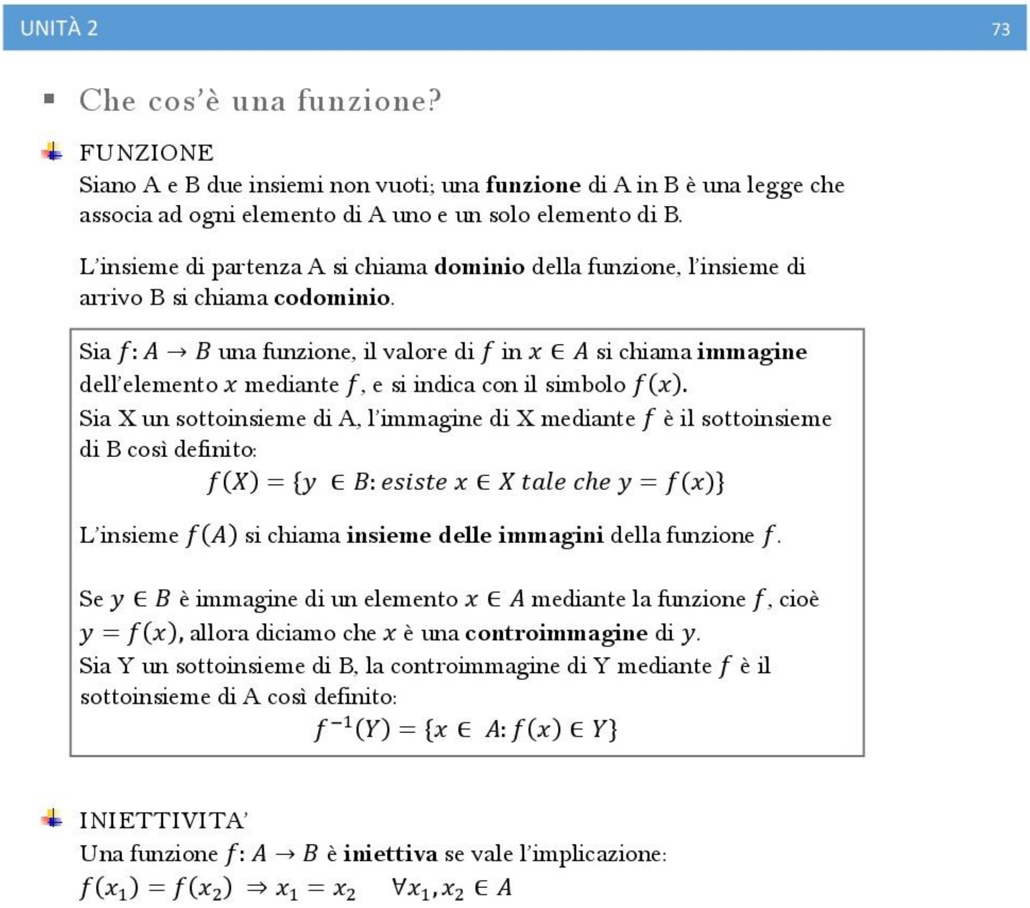


Figure 10 shows the sheet of paper that we gave to students in task 4. We wrote on the worksheet that it is the copy of a page from a mathematical textbook for high school, but we actually designed it by re-arranging on the same page the definitions taken from different sections of the book by Bergamini, Trifone & Barozzi (2005) and by giving to the page a realistic layout.



UNITÀ 2 73

▪ Che cos'è una funzione?

FUNZIONE

Siano A e B due insiemi non vuoti; una **funzione** di A in B è una legge che associa ad ogni elemento di A uno e un solo elemento di B.

L'insieme di partenza A si chiama **dominio** della funzione, l'insieme di arrivo B si chiama **codominio**.

Sia $f: A \rightarrow B$ una funzione, il valore di f in $x \in A$ si chiama **immagine** dell'elemento x mediante f, e si indica con il simbolo $f(x)$.
Sia X un sottoinsieme di A, l'immagine di X mediante f è il sottoinsieme di B così definito:

$$f(X) = \{y \in B: esiste\ x \in X\ tale\ che\ y = f(x)\}$$

L'insieme $f(A)$ si chiama **insieme delle immagini** della funzione f.

Se $y \in B$ è immagine di un elemento $x \in A$ mediante la funzione f, cioè $y = f(x)$, allora diciamo che x è una **controimmagine** di y.
Sia Y un sottoinsieme di B, la controimmagine di Y mediante f è il sottoinsieme di A così definito:

$$f^{-1}(Y) = \{x \in A: f(x) \in Y\}$$

INIETTIVITA'

Una funzione $f: A \rightarrow B$ è **iniettiva** se vale l'implicazione:

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad \forall x_1, x_2 \in A$

Figure 10. A book page used for the interview

4.6 DATA ANALYSIS

The data collected include video recordings, taken from the cameras and from the screen capturing software that was running in the background while students were working in GeoGebra, and students' worksheets produced during all the activities; both for the lessons and the interviews.

The data analysis process consisted of the transcription of the data and then of the analysis of the transcripts. In particular, we organized the transcripts in tables having the following structure:

#	When	Who	What is said	What is done

In fact, we transcribed the words spoken and made note of the dragging actions and gestures that took place in the videos; we specified the subject of each action and the researcher was marked as 'R'. We organized the transcripts in order to highlight the interplay between words, dragging actions and gestures within the student pairs' discourse, also including some screenshots of dragging or gesturing actions in the column called 'What is done'. The punctuation was added in this way:

- a comma is used for very short pauses,
- a period is also used for short pauses, in particular those pauses which seem to mark the end of a sentence;
- more consecutive periods denote longer pauses: the higher the number of periods is, the longer the pause lasted.

Moreover, transcripts contain the exact ways in which words were uttered by students, for instance, a student's utterance "B equals two when A is less than five and more than four" is written exactly that way and not as "B=2 when 4<A<5". During this phase, we also identified some critical events which could be significant with respect to our research questions.

Then we analyzed all the transcripts through different perspectives, depending on the research question we were addressing. In particular, we improved some tools offered by the theory to carry out the analyses. For example, we looked at how students used the dragging tool, both physically when manipulating the GeoGebra files and also by referring to it in their discourse, searching for recurring features. We will show these analyses in Chapter 5. Moreover, we created a coding scheme to identify instances of students' discourse mirroring potential expert discourse about specific mathematical objects. In the analyses, that we will show in Chapter 6, we will present excerpts from the transcripts to explain how we identified seeds of possible realizations of mathematical signifiers in students' discourse.

5 THE MEDIATION OF DRAGGING

During the transcription process of all the videos, we focused on the main features and on possible modification of students' discourse on covariation. The first thing we noticed was that it is rich in references to movement, time and space (Colacicco, Lisarelli & Antonini, 2017), as expected. Indeed, students were working on the activities that we designed and almost all of them involved the use of the software GeoGebra where, thanks to the dragging tool, activated through the mouse, they could experience the dependence relation that links a (dependent) tick to the one that is directly dragged. Thanks to the possibility of dragging they could also visualize the movements of the two ticks realizing the variables and the relation between these variations, that is the covariation. For this reason, one of our research questions concerns students' use of dragging when interacting with the DIMs that we designed. In particular, we were interested in investigating the role of dragging in the process of construction of a discourse on functions, in terms of its place for students and how they physically use it, how they communicate about it and through it.

The investigation of these aspects led us to identify different types of dragging, which students efficiently use for the exploration of the proposed DIMs, and different ways in which dragging mediates students' discourse, which seem to be supported by the designed activities in a DIE. In particular, we described the classification of the types of dragging in Chapter 2 and now we are going to show how it can be efficiently used to observe, describe and analyze students' discourse about functions in this specific context. The main goal behind our analyses is that of investigating if, and eventually how, the dragging mediated students' emergent discourse about functions. In particular, we are interested in studying how it is used by students as visual mediator in the communication with other students or with themselves; as expressed in our second research question.

In this chapter we will also show how the analysis of students' use of dragging allows us to investigate about the types of routines that seem to be supported by the activities that we designed. Moreover, we will describe a possible development of students' discourse towards that of an expert, that we observed along the sequence of lessons.

5.1 DRAGGING MEDIATED DISCOURSE

Gestures and dragging actions can be used both repeatedly to define a discursive pattern and as mediators to complement word use. From a preliminary analysis of the videos' transcripts we noticed that students' discourse is heavily mediated by dragging, where the mediation is intended according to the theory of commognition, as discussed in Chapter 2. This fact was not unexpected to us because we asked students to work in a DIE, where gestures and dragging actions play a central role in the communication. In particular, we have seen that characterizing features of the realizations of functions that we used are the dynamism. Indeed, the positions of the ticks realizing the variables can move under dragging. Moreover, we have seen that they are interactive, because they respond to students' dragging actions by maintaining the relationship between the variables. Therefore, we wanted to analyze in fine grain how students' discourse involved the mediation of dragging and gesturing actions and we carried out this analysis with tools that had previously been developed within the communicational approach to characterize mathematical discourse.

Drawing on the works of Morgan & Sfard (2016) and of Nachlieli & Tabach (2012), we chose to focus on specific aspects of students' discourse that are strictly related to the use of DIMs. Morgan and Sfard (2016) analyzed students' written discourse in a context of maths examinations, by developing an analytic scheme in which they expressed the aspects of the discourse that they wanted to investigate and the questions that guided the analysis of these aspects. Nachlieli and Tabach (2012) focused on students' discourse about functions in a context where they were introduced for the first time to this mathematical signifier; the analyses are led by the four-stage model (see section 2.1.1.2) for the evolution of the use of words that was described by Sfard (2008).

One of the aspects of students' discourse that we wanted to investigate was the use of dragging and gesturing actions as mediators in communication. We were also interested in studying a possible evolution in their use of these mediators, during the sequence of lessons. In order to capture key items for a fine-grained analysis of dragging mediated discourse, we formulated the following guiding questions:

- whether the dragging tool is physically used during the speaking;
- whether the dragging blends with a gesture becoming an act of dragsturing;
- whether the subject is a person or an object;
- whether mathematical objects are considered;
- which verb tense is used.

In other words, we set out to capture very specific features of the discourse, such as students' use of static or dynamic mediation, the type of mediator used, since it can be a symbolic artifact or a visible object employed in the communication, and whether it is physically manipulated, reproduced through a gesture or just seen. Moreover, we wanted to identify the subject and the object of the discourse, and the words used to address a mediator. Through these elements we also expected to be able to capture possible changes in dragging mediated discourse over time.

In the rest of this chapter we will show how the analysis of students' discourse leads to the identification of different ways in which dragging seems to mediate their discourse. In the next section we will give a detailed and operative description of these different ways.

5.1.1 Characterizing dragging mediated discourse

From the analysis of students' discourse, according to the research focuses explained above, we identified three different phases of what we called *dragging mediated discourse*: a passive phase, an active phase and a detached phase. We used the term 'phase' because it suggests a sort of temporal evolution from the first one to the third one. In fact, with respect to our sequence of lessons, we found most instances of the passive phase from the first to the third lesson; of the active phase from the second to the eighth lesson, and in the interviews; of the detached phase during the eighth lesson and the interviews.

Moreover, we are tempted to consider this temporal evolution of the phases also as a possible development of students' discourse towards a discourse closer to that of an expert mathematician, and we will focus on this aspect later in this chapter. The foundation of our idea can be placed in Sfard's work where she describes a four stage model of the development of word use (Section 2.1.1.2). In particular, she argues that students' word uses change during the individualization process of a discourse, going through a *passive use*; a *routine-driven use*; a *phrase-driven use* and, finally, an *object-driven use*. Starting from this idea, we described a model of the development of the role of dragging as a mediator in students' discourse, by identifying three main phases and giving them names that evoke those given by Sfard.

Now we are going to characterize each phase of dragging mediated discourse.

- **Passive phase**

  The dragging tool is physically used. The discourse is about dragging, it is a description of the direct action of the dragging tool on an object, while there are no mathematical objects considered. In this first phase of dragging mediated discourse the subject is a person, the student, and the object is the movement of the objects upon which she acts through dragging. An explorative dragging allows the student to answer to the question posed in the activity, or to a request of the teacher, that explicitly ask for dragging and for describing the effects of dragging. Indeed, the student uses types of dragging that we would not expect from an expert interacting with the DIM, as frequent attempts to drag the tick realizing the dependent variable that cannot be directly dragged. Moreover, sometimes the students show also to be surprised by what they see on the screen.

  The verb tense used within this discourse is the present simple and some typical expressions or words that characterize it are: "I can(not) move it", "you can(not) move it", "drag it", "move it" (in Italian: "(non) lo posso muovere", "(non) lo puoi muovere", "trascinalo", "spostalo").

- **Active phase**

  The dragging tool is physically used. The discourse is about the effects of dragging that are visible on the computer screen, it is a description of the perceived relations between the moving objects. The focus of this dragging mediated discourse is on the mathematical signifiers, while the action of dragging is not explicitly depicted and so the description looks as if it was independent from the person. The active phase sees a greater participation of the student in deciding what to move and how to move it and often her use of the different types of dragging mirrors that of a potential expert in the same task situation.

  The verb tense used within this discourse is the present simple and some typical expressions or words that characterize it are: "if x […] $f(x)$ […]", "when x […] $f(x)$ […]", "as x […] $f(x)$ […]" and other expressions like these that do not necessarily contain the labels 'x' and '$f(x)$' but that an expert can read in terms of the two variables (in Italian: "se x […] $f(x)$ […]", "quando x […] $f(x)$ […]", "man mano che x […] $f(x)$ […]").

- **Detached phase**

  The dragging tool is not physically used. The structure and the contents of the discourse are very similar to those in the active phase with the focus is on the relation between movements; but the dragging tool is not used to act upon any objects, so these movements are only imagined. In particular, the active use of dragging that characterizes the two other phases is here replaced by a realization of motion through gestures, also in a static context out of the DIE.

Within this discourse, verbs can be in the present tense, but also in the future tense or expressed in the "-ing" form. Typical expressions that characterize it are: "I imagine to drag", "by dragging x [...] $f(x)$ will move [...]" and other expressions like these that do not necessarily contain the labels 'x' and '$f(x)$' but that an expert can read in terms of the two variables (in Italian: "immagino di trascinare", "trascinando x [...] $f(x)$ si muoverà [...]").

The temporal evolution characterizing these three phases seems to also be related to the design of the activities. Indeed, generally speaking, initially the activites are implemented in a DIE, where students are asked to physically use dragging, and students' discourse is mostly in the passive phase; as the activities become paper-and-pencil based, correspondingly, students' discourse moves towards the detatched phase.

In the next three sections, we will analyze some excerpts, taken from the videos' transcripts, that we consider as representative examples of each phase of dragging mediated discourse. In particular, in the right column 'what is done' we also specify the type of dragging used by students by distinguishing between *continuous*, *discrete* and *impossible dragging* that can be objectively recognized; indeed, we identified them by looking at the movements happening on the computer screen. Moreover, we observe that in Italian there is a wide use of the impersonal form of the verbs and, in the following excerpts, we translate them by using the passive form; for example, "it is dragged".

5.1.2 Examples of dragging mediated discourse from the passive phase

In the following excerpt, which is taken from the first lesson, students are exploring the realization DGpp of the absolute value function and this is the first time that they are given this realization where the two axes can be separated out.

Excerpt 5.1 - Lesson 1

(realization DGpp of the function $f(x) = |x|$)

	When	Who	What is said	What is done
524	I1Mp3 12:33	R	These lines can be.. moved apart if you move the line	Activity1_3 She points to the screen
525		F	How? That is, what can I do?	
526		R	Take the line, not the tick	Initially he drags the tick to the right
527		F	Ehm the line	Now he drags the line downwards
528		R	You can move it now, if it is helpful for you, to understand what happens	She moves her hand up and down
529		F	That is, can I actually make a copy of it?	
530		R	Yes well done! …… Now maybe if you drag the tick you can see better	
531		F	This one? Or that one?	
532		R	Try!	

533		L	You cannot drag that one	Impossible dragging of the dependent variable
534		F	Ehm I cannot drag this one	
535		L	Try that one	
536	13:25	F	Let's try with this one	He drags the tick to the left

Franci and Lore's discourse in excerpt 5.1 is an example of dragging mediated discourse from the passive phase because it focuses on possible movements of the lines realizing the axes and the students' dragging actions are the objects of their discourse. There are no mathematical objects involved in the discourse because the students pay special attention to the construction. In particular, Franci and Lore use *wandering dragging* to explore the construction, as is also suggested by the researcher who explicitly tells them to drag one of the lines in order to discover what may happen (line 528). Then Franci expresses a movement that he observed, which consists in separating out the two lines, by making a copy of the line, at line 529, when he says "*can I actually make a copy of it*?". However, Franci's question at line 531 and the following *impossible dragging* of the tick realizing the dependent variable suggest that he is not keeping track of what the possible or impossible movements are.

We also notice that the subject of the discourse is always a person who acts on the file, some examples of this phenomenon are: "*what can I do?*" (line 525); "*you can move it* […]" (line 528) and "*you cannot drag that one*" (line 533).

The short episode in excerpt 5.2 happened at the end of the second lesson.

Excerpt 5.2 - Lesson 2

(Realization DGpp of the function $f(x) = \sqrt{x+3} - 2$)

	When	Who	What is said	What is done
148	I2C2 58:58	A	We did not see if B can be dragged	Activity2_2bis Impossible dragging of B
149		N	It cannot be dragged, it can never be dragged	
150		A	Why?!	

In this excerpt Alessio experiences an *impossible dragging* of B of which he is curious and for which he is in search for a reason at line 150. The students' discourse is about their impossibility of directly dragging one of the two ticks that according to Nicco "*can never be dragged*". Again, the subject used by the students is "we", so there is reference to be a person manipulating the file.

Excerpt 5.3 is taken from the beginning of the third lesson. Alessio and Nicco are exploring the realization DGpp of the function $f(x) = x + \frac{3}{x-3}$, with Nicco handling the mouse, and they are describing which movements are possible or impossible. The independent variable is labelled A, the dependent one is labelled B and the function is not defined at x=3, where it has a vertical asymptote.

Excerpt 5.3 - Lesson 3

(Realization DGpp of the function $f(x) = x + \frac{3}{x-3}$)

	When	Who	What is said	What is done
1	I3C2 06:50	A	This irritates me, move it [A] to three	Activity3_1 Nicco drags A in a neighborhood of $x = 3$
2		N	It does not go there	He uses the arrows of the keyboard to drag
3		A	Yes it does, you have to move it and then you can drag it. Look, now if you drag it, it goes one by one	
4		N		Continuous dragging of A forward and backward close to 3. Then impossible dragging of B
5		A	Try to overlap them, this is not possible, look at how it moves! But zoom in, like this it is too sensitive… a little bit more, oh zoom in…. Don't zoom out but make it bigger. Let's go and see if they meet. You have make it bigger, stretch a little bit more… but we do not see if they meet each other.	The two axes are overlapped. Then they look at large positive A-values
6		N		Impossible dragging of B
7	08:05	A	A is the only thing that can be dragged!	

In excerpt 5.3, Nicco uses the dragging tool in two different ways: through the mouse (line 1), through the arrows of the keyboard (line 2) and then through the mouse again (line 4 and line 6). So, he obtains two different qualities of motion of the independent variable, because the mouse induces *continuous dragging* (it depends on how the student moves the mouse but, in this case, the movement is quite uniform), while the arrows of the keyboard induce *discrete dragging* making the point jump by one. Alessio seems to notice this fact and he suggests to Nicco to use the mouse when he says "*you have to move it*" (line 3).

In general, the focus of the students' discourse is on what they can or cannot move on the screen. Indeed, the dialogue starts with Alessio asking to Nicco to drag A and move it (line 1), then the subject of the discourse becomes A for a while (line 2 and the beginning of line 3), until Alessio expresses what and how Nicco can move (line 3) and so the subject of his discourse is a person again.

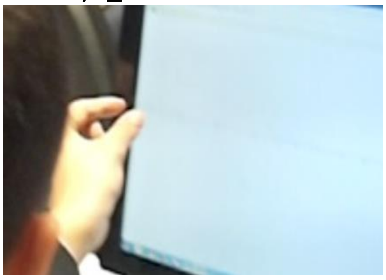
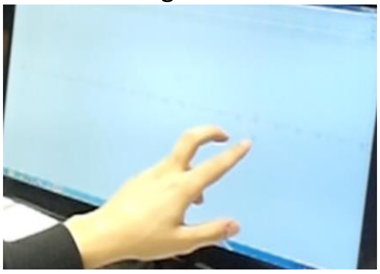
Dragging mediates their discourse so significantly that in two lines of this short excerpt Nicco substitutes the dragging for words (lines 4 and 6): he does not speak aloud, instead, through a dragging action he succeeds in communicating with Alessio, who replies. These are two examples of dragsturing, in which the dragging also fulfills a communicational purpose. The students do not seem to manipulate the file as an expert would do, especially Nicco who, at the end, experiences *impossible dragging* (line 6) because he tries to drag B which cannot be directly moved; but immediately Alessio, irritated, stresses that A is the only thing in the construction that they can drag (line 7).

5.1.3 Examples of dragging mediated discourse from the active phase

In the following excerpt, which is taken from the second lesson, Alessio is sharing with the researcher his findings about the realization DGpp of a function, after having explored it.

Excerpt 5.4 - Lesson 2

(Realization DGpp of the function $f(x) = \sqrt{x+3} - 2$)

	When	Who	What is said	What is done
284	I2Mp3 03:57	A	Now it is this one [A] that moves in an uniformly accelerated way because as this one goes forward this one [A] covers more space and this one [B] less and less and so it depends on the point of view because this one [A] moves by one and this one [B] moves a tiny bit (Fig. 5.4a) then this one [A] moves by one and this one [B] even by even less (Fig. 5.4b). Therefore it depends on… on the point of view	Activity2_2bis  Fig. 5.4a  Fig. 5.4b
285		R	But you decide how to move A	
286		A	Yes	
287		R	And so *you* describe how B moves	
288		A	Yes	
289		R	If I move A	
290	I2Mp3 05:00	A	And I move A by one..ehm but I do not know how to write it because over here [negative numbers] if I move A by one B moves by…I do not know about point five, while if I move A over here [large positive numbers] by one B moves by point two. As A increases…that is, I do not know how to say it	He is still dragging and he points to the negative and then the positive semi-axis on the screen

In excerpt 5.4 Alessio describes how the range of movement of the dependent variable depends on the position on the line of the interval of variation of the independent variable. In particular, he expresses the changes in range as "*depending on the point of view*" (line 284), because moving A by one for negative values he sees B moving by 0.5, while moving A by one for positive values he sees B moving by 0.2. So, he uses a combination of *discrete* and *continuous dragging*, both of which are *guided*, since he refers to moving A by one or by 0.2 (line 290) by showing two specific examples: an interval between two negative numbers and an interval between two large positive numbers.

His discourse is mediated by dragging actions and gestures (Fig. 5.4a and Fig. 5.4b), but, especially, it is about the relation between the movements resulting from the dragging of A. The subject is usually one of the two ticks, except in the last line where Alessio becomes the subject of the action, as can be seen from his expression "*if I move A* […]". Even if Alessio explicitly says "*I do not know how to say it*" (line 290), his discourse involves mathematical signifiers, indeed, it mirrors the discourse of an expert about the decreasing of the derivative function.

In the following excerpt Franci and Lore are exploring the behavior of a function in a neighborhood of the vertical asymptote and they are working with the realization DGpp.

Excerpt 5.5 - Lesson 3

(Realization DGpp of the function $f(x) = x + \frac{3}{x-3}$)

	When	Who	What is said	What is done
35	I3Mp1 04:35	L	If A goes that way [3+] B goes there [off the screen to the right] and it pops up there [off the screen to the left], while if A goes from the negative [3-] to the positive [3+]	Activity3_1 Lore holds the mouse and drags x in a neighborhood of 3
36		F	When B goes, when A goes..are you dragging A right?	
37		L	Yes only A can be dragged	
38	I3Mp1 04:55	F	Therefore if A goes from the positive to the negative… B keeps going in the positive, that is…	

The students' discourse in excerpt 5.5 is an example of dragging mediated discourse from the active phase because it is about the movements of the tick realizing the dependent variable when the other tick is dragged towards a specific direction. In particular, this dragging action is an example of *guided dragging* because Lore is exploring a specific configuration of the construction, that is with x moving in a certain interval of the x-axis, while Franci describes the movements observed on the screen. In particular, the two students express these movements in terms of "*if A goes* […] *then B goes* […]" (see lines 35 and 38); the subject of their discourse is not considered to be a person but the tick itself.

At line 36 Franci, who almost never holds the mouse, asks Lore which of the two ticks can be directly dragged and then he focuses again on the relation between the movement of A and that of B, that follow two different directions.

The same two students in the following excerpt are describing the behavior of a telephone plan depending on the time spent to call, which is expressed in hours; to do this they are looking at the realization DGpp of this function.

Excerpt 5.6 - Lesson 4

(Realization DGpp of the two functions $f(x) = \begin{cases} 7, \ x < 5 \\ 3 + floor(x), \ else \end{cases}$ and $g(x) = \begin{cases} \frac{5}{2}x, \ x < 6 \\ \frac{1}{2}x + 12, \ else \end{cases}$)

	When	Who	What is said	What is done
113	I4Mp1 12:30	L	A equals six and T_A [g(x)] is fifteen, A is seven.. look it is already diminished, before it had increased by a certain	Activity4_1 Lore holds the mouse. The grid is shown.

			amount of squares, now it increased just by half a square in one hour, just by half a square, more or less, see, at eight it continues to diminish	Discrete dragging
114		F	Yes	
115		L	Until	Continuous dragging to the right
116		F	they go and coincide at twenty-four	
117		L	Yes, so more or less the cost is not constant all the time, I mean the cost of the telephone cost	No dragging actions
118		F	No after which number?	
119		L	Seven	
120		F	After seven, so after seven hours it [the cost] is not constant anymore, while above [T_B]	
121	I4Mp2 13:20	L	No no no I mean that here [in a right neighborhood of 0] the cost increases, it increases more and more each hour, while after seven the cost is less than before that is it decreases with respect to the hours	He points to the screen in a right neighborhood of 0

Franci and Lore in excerpt 5.6 manipulate the dynamic file while speaking and the dragging mediates their discourse about the changing in ratio of the variations of f(x) to the variations of x, indeed they say "*after 7 hours the cost is not constant anymore*" (line 120) and "*for each hour the cost increases more and more, but after 7 the cost decreases with respect to the hours*" (line 121). As we can see, one of the two variables plays the role of subject in their discourse, which is about the relation linking the moving objects and it also involves mathematical signifiers.

Moreover, Lore seems to decide where and how to drag x: initially he uses *discrete dragging* by one, while counting the number of squares of the grid that $f(x)$ passes through, then *continuous dragging* to the right until he sees that "*they become matching at 24*" (line 116).

In the following excerpt Matilde is sharing with the researcher her findings about the function realized by the DGc in activity5_1.

Excerpt 5.7 - Lesson 5

(Realization DGc of the function $f(x) = -x + 5$)

	When	Who	What is said	What is done
73	I5Mp2 07:45	M	We could say a relation, B equals five minus A	Activity5_1
74		R	B equals five minus A	
75		M	Because we start from zero and B equals five, when A is on three, for example, five minus A, so five minus three equals two and it holds for all the values, also five minus ten that is minus five and so on	She starts dragging A from the origin to right. Discrete dragging

76		R	Okay, so you found that B is five minus A	
77		M	Yes and in any case in the end they are like inversely proportional in quotes because as one increases the other one decreases. Indeed, when A increases B starts moving, it also goes into negative [numbers] and also when A goes into negative [numbers] B [goes into] positive [numbers] and so it increases and decreases	When she says "indeed" she starts dragging A to the left. Continuous dragging
78		R	Okay, but what does inversely proportional mean?	
79	08:55	M	No, they are not inversely proportional, no, as one increases the other one decreases and vice versa	

In the first line of excerpt 5.7 Matilde proposes the algebraic realization of the function "*B equals five minus A*" and then, in order to explain how she obtained it, she uses the mediation of dragging actions. In particular, she initially uses *discrete test dragging* of x from 0 to 3 and then to 10 while showing that her formula holds for all these three values (line 75). Then she describes the relation between the two variables in terms of their opposite direction of movements, since "*as one increases the other one decreases*", while using *continuous test dragging* of x to left (line 77). This is an example of *test dragging* where the student expresses through words her idea before dragging and, after that, she tests it through dragging.

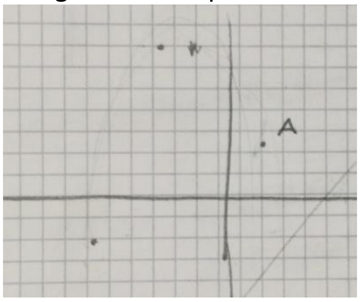
Moreover, the mediation of dragging for this communication can be also shown through the explicit references to dragging actions operated upon the ticks that can be found in Matilde's discourse, such as "*we start from zero*" (line 75) and "*B starts moving,* [it] *goes into negative* [numbers]" (line 77).

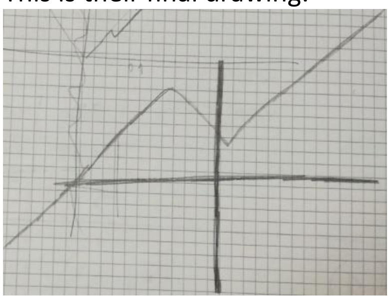
Below we analyze two excerpts taken from the seventh lesson. The first one is from the beginning of the lesson and Matilde and Nicco have to draw on paper the Cartesian graph of a function, by exploring it realized through the dynamic realization DGc. In particular, it is the first time during our sequence of lessons that students have to deal directly with a Cartesian graph.

Excerpt 5.8 - Lesson 7

(Realization DGc of the function $f(x) = \frac{1}{10}\left(\frac{x}{2}+4\right)(x+1)(x-2)+\frac{5}{2}$)

	When	Who	What is said	What is done
16	I7F1p1 11:00	R	What were you planning to do?	Activity7_1
17		N	We wanted to see where the intersection point between the axes passing through x and f(x) was	He moves the hand simulating a cross
18		R	And to trace?	
19		N	Yes, but	
20		M	But I do not know how to, we could start by marking some of	She makes continuous dragging of x to right

			the values and then, but it is not useful because there is something strange	
21	I7F1p1 12:15	N	Yes because, at the beginning it is like this more or less [Fig. 5.8a] but then it takes on the same value as it had here…	He simulates a curve passing through the three points:  Fig. 5.8a
			[…]	
40	I7F1p1 15:05	M	I'll try to do something, even if it is not useful… it does not work!	She activates the Trace on x and drags it, then she activates the Trace on $f(x)$ and drags x Continuous dragging
41		R	Do you see that it is slightly colored? What does it tell you? Even if you can't see the trace very well.	
42		N	For some specific x-values, f(x) stays still within a certain interval, after those values it goes to infinity	 Fig. 5.8b
43		R	Okay. You should try to put this information here	Pointing to the sheet of paper
		M	We have to identify where the point is for the first time, wait, here it is okay… so, therefore, it has to do like this if everything works well…no, then it does like this, until it reaches this point, well do you understand? I think, I hope, because otherwise I do not know how to start! Let's try again	Discrete dragging: she stops when x is at 0, then at -1, at -6 and then continuous dragging to the left. With the finger she traces a curve on the screen, from the second quadrant going down and a bit to the left.
44		R	What are you doing with your finger?	Matilde makes continuous dragging of x from 0 to left and again she uses a finger to trace the curve on the screen

45		M	I am marking the point, more or less now.. but I don't think it goes down again	 Fig. 5.8c
46		N	It is always going down, but it is not straight	
47		M	Wait, let me think, yes it is like this	
48		N	For me it does the same thing over there too [large positive x-values]	Continuous dragging of x from 0 to right
49	I7F1p1 17:40 C2 22:50	M	No because over there it does not go down, but not straight	
			[…]	
69	C2 24:30	M	wait, because it starts from here [(0, f(0))]	She drags x in a right neighborhood of 0
70		R	Is it from there that it starts going down?	
71		N	Yes, it goes down for a while and then	
72		M	A little bit down, so it does something, where is the point? The point is here	
73		R	And now? Good!	Matilde points to $(0, f(0))$ on the screen
74		M	Here! Is it here? Therefore it does like this…. no, it is here [(0, f(0))]	She drags x from 0 to 1 very slowly, then backward to 0
75		N	After one it starts going up again… there is something strange	He speaks before x reaches 1
76	C2 25:30	M	Well, and then it goes up again. Therefore it could be, give me the pencil.. it could be like this	Continuous dragging of x to right This is their final drawing:  Fig. 5.8d

In excerpt 5.8 Matilde and Nicco explore the realization DGc of a function, with Matilde holding the mouse, trying to find a way to keep track of the trajectory of $(x, f(x))$ that they have to draw on the paper.

As Nicco explains (line 17), they first search for "*the intersection point between the axes passing through x and f(x)*" and they mark some points of the Cartesian plane on the paper (Fig. 5.8a). However, Matilde notices that "*it is not useful because there is something strange*" (line 20) and so she decides to activate the trace on $f(x)$ and she uses *continuous wandering dragging* on x. By looking at the screen, Nicco observes that "*for some specific x-values, f(x) stays still within a certain interval, after that values it goes to infinity*" (line 42). The dragging and, especially, the trace left by $f(x)$ mediate his discourse, which is about the covariation of the two variables playing the role of subjects of the discourse.

At lines 43 and 45 Matilde follows with one finger the trajectory of the point $(x, f(x))$, although it is not visible on the screen, while she uses the other hand to carry out *continuous dragging* of x to the left. This is an example of *continuous handle dragging*: she is acting on x but her focus is on the movements of the imagined point $(x, f(x))$.

Moreover, we identify *continuous test dragging* at line 49, after Nicco makes a prediction about the behavior of $f(x)$ for large positive x -values (line 48).

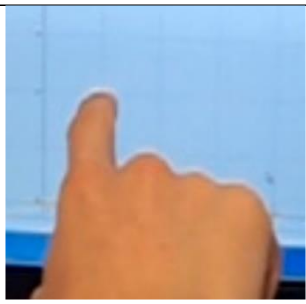
Finally, the two students explore the dynamic realization of the function for x in a right neighborhood of 0 and then also for positive large x -values, by using *continuous wandering dragging,* that leads them to complete their drawing (from line 69 to the end).

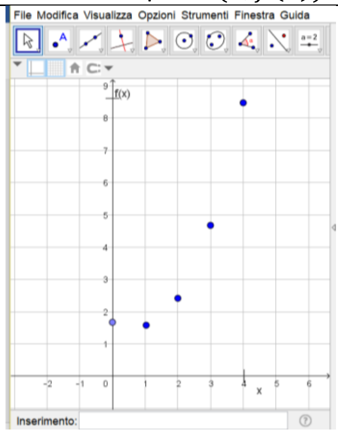
It is interesting to see how two other students in the class addressed the same activity. Their names are Bernardo and Carlo, we do not have videos recording their working along the whole sequence of lessons but the following excerpt is recorded by the mobile camera and it shows part of their discussion during the seventh lesson. As we are going to discuss, their dragging mediated discourse is at the active phase.

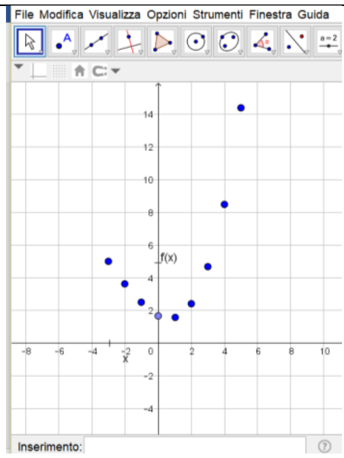
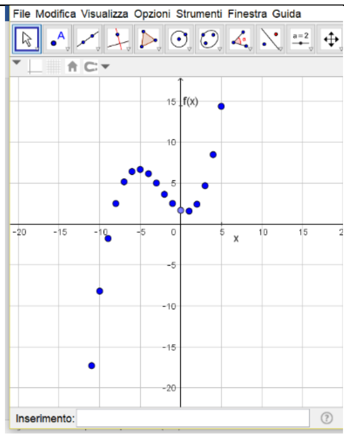
Excerpt 5.9 - Lesson 7

(Realization DGc of the function $f(x) = \frac{1}{10}\left(\frac{x}{2} + 4\right)(x+1)(x-2) + \frac{5}{2}$)

	When	Who	What is said	What is done
36	I7Mp1 04:35	B	Can you add color? Can you put color on f(x)?	Activity7_1 The grid is shown. Carlo holds the mouse
37		C	How do we do that?	
38		B	I do not know	
39		R	Now where should the point (x, f(x)) be?	
40		B	The point f(x) is a bit less than two	He points to the tick on the y -axis on the screen
41		C	x, f(x)	
42		B	but x f(x) is a segment	
43		R	The point of coordinates, that has two coordinates, one is x and	
44		B	Ah zero and a bit less than two	
45		R	And so where should it be?	

46		B	Where it was until now	Bernardo points to the tick on the y -axis while Carlo uses the arrow of the mouse to indicate $(x, f(x))$
47		R	The arrow?	
48		B	No, almost here where there is a light point	He points to the tick on the y -axis on the screen
49		R	Therefore it would be here	
50		B	The first point. Drag x on one… x on one is about here [Fig. 5.9a], go to x two	 Fig. 5.9a
51		C	Two and a half	Discrete dragging
52		B	If you go on three and a half, that is two and a half go on, go on three, no it dashes away, from two to three it dashes away… But we could, are there some tools to color where it is, more or less? That is, with the arrow	He points to other points $(x, f(x))$ in the first quadrant as he was doing in Fig. 5.9a
53		C	No	
54		B	Does it not exist? That is, if you have x at one and f(x) at ten you can't mark this point here on the plane [Fig.5.9b]?	 Fig. 5.9b
55		R	Ah you would like to draw them there?	
56	06:25	B	Yes, to draw the points on the graph	
			[…]	The researcher tells them how to use the point tool
86	I8Mp1 09:12	B	Move x on one	
87		C	Therefore I have to drag	
88		B	Yes, drag to one, well, now select the point tool and make it here	He points to $(1, f(1))$
89		C	I do not know if we can	
90		B	Yes we can, if we cannot	
91		C	Yes but it is not perfect	He builds the point $(1, f(1))$

92	09:40	B	What does it matter? More or less, put it on two now… so a bit at a time we get the staircase	
			[…]	Discrete dragging of x: Carlo drags it and stops at 2 to build the point $(2, f(2))$; at 3 to build the point $(3, f(3))$
102	10:20	B	Put it on four, it goes up up up, here… this looks like half a parabola	Discrete dragging to 4 He points to $(4, f(4))$
103		C	Can you take off your hand?	He builds the point $(4, f(4))$
104		B	It looks like half a parabola	 Fig. 5.9c
105		C	Now we should go to the negatives	
106		B	Go to the negatives and we see if it appears in same way…indeed this one [(1, f(1))] should be a bit more up, it is wrong… go to the negatives	
107		C	Exactly	He drags x to -1 and stops
108		B	It is not a parabola. Put it here!	He points to $(-1, f(-1))$
109		C	Ah	He builds the point $(-1, f(-1))$
110		B	It is not a parabola	
111		C	No it is not a parabola, it is something even stranger	Discrete dragging to -2 and he speaks after building the point $(-2, f(-2))$
112		B	I trust your precision	
113		C	Yes, I may get some millimeters wrong but… it does not look like	Discrete dragging to -3 and then he builds the point $(-3, f(-3))$
114	12:20	B	This is a very strange thing, can you mark where the lines cross each other? Zoom out, I want to see it for a moment	Carlo zooms out

				 Fig. 5.9d
			[...]	Discrete dragging of x to the left and Carlo builds other points
136	I7Mp2 01:00	B	Now can you zoom out? It's impressive how it dashes away, do you see it? Put the last point and then zoom out in order to let me see everything! Zoom out	Discrete dragging of x from -10 to -11 then Carlo builds the point $(-11, f(-11))$
137		C	No but it is too cool! Sorry, let's try to go on over there too [large positive x-values]	 Fig. 5.9e
138		B	Let's go on over there too	
139		C	Yes but it disappears	
140	01:55	B	Yes, they did it on purpose... why is it like this? Have we ever seen something similar?	

Bernardo and Carlo solve activity7_1 in a completely different way with respect to Matilde and Nicco. They use *discrete dragging* to move x by one along its axis and they build a set of points of the form $(x, f(x))$ directly in the interactive file, using the point tool.

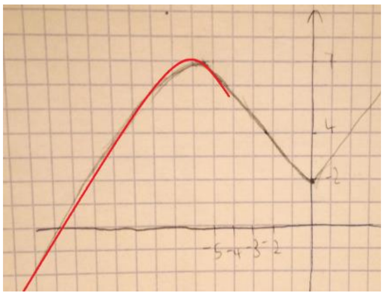
Initially, the two students seem to be looking at the movements of the tick '$f(x)$', for example at line 36 Bernardo asks "*can you put color on f(x)*?". Then the researcher brings their attention to the point $(x, f(x))$ (that is not visible on the screen) and at line 54 he asks again for the color but changing a little the question: "*if you have x at 1 and f(x) at 10 can you mark this point here on the plane*?".

After they have learned how to use the point tool, from line 86 on, they proceed by using *discrete guided dragging* until they obtain Figure 5.9e. In particular, Carlo drags x stopping at each whole number and, when he stops dragging, he constructs the point $(x, f(x))$. In this case, the students' discourse is mediated by dragging but their focus is on the results of their actions that are visible in the file. In particular, they discuss the shape of one of the possible curves that can be obtained by connecting the constructed points; they say "*it seems a half parabola*" (lines 102, 104). Moreover, they seem surprised by the behavior of $f(x)$ for large positive x -values: they remark "*it's impressive how it dashes*" (line 136) and "*it disappears*" (line 138).

In the following excerpt from Alessio's interview, the student is exploring the realization DGc of a function and he has to draw its Cartesian graph on a sheet of paper. He is describing the construction while he is manipulating it.

Excerpt 5.10 - Interview 1

(Realization DGc of the function $f(x) = \frac{1}{10}\left(\frac{x}{2} + 4\right)(x+1)(x-2) + \frac{5}{2}$)

	When	Who	What is said	What is done
13	C 15:15	A	From here, from minus one to minus two it moves more or less by one but from minus five to minus six…mm…….it moves, from minus five to minus six no let's do from minus four to minus five it moves by less than one, so as x decreases, that is, also the relationship existing between f(x) and x changes and so it cannot be like this [straight line] but it is a [curve]… that is to say it is not a broken line	Task1 Discrete-continuous-discrete dragging Then he draws the red part of the curve:  Fig. 5.10a

Alessio refers mainly to motion: he uses frequently the verb "to move", but the focus of his discourse is the search for a possible relation between movements, more than the movement itself. The subject is always a variable and not Alessio himself; initially it is unexpressed but then he says "*as x* [...]". This happens at the end of the excerpt where he describes changing in ratio of $f(x)$ to x in relation to the dragging of x. This is a very prototypical example of dragging mediated discourse from the active phase. Indeed, he acts on the file by dragging x to the left and at the same time his discourse focuses on the relationship between movements that he observes happening on the screen. So, the object of this discourse is not the action of dragging, it is only mediated by it: he uses a combination of *discrete* and *continuous dragging* of x while estimating the range of variation of $f(x)$, which suggests a good control over the dependency between the two variations.

At the very beginning of the excerpt, there is an example of dragsturing: when he says "*from here*" he uses the mediation of dragging as a gesture, because he indicates what he intends for "here" but he does not directly describe this action.

Finally, the excerpt below is taken from Alessio's interview when he is asked to draw the Cartesian graph of a function on a sheet of paper, given its realization DGpp.

Excerpt 5.11 - Interview 1

(Realization DGpp of the function $f(x) = \begin{cases} \frac{3}{2x} + 2, \ x > 0 \\ (x+1)^2(x+6)(x+3)\frac{1}{x} - 1, \ else \end{cases}$)

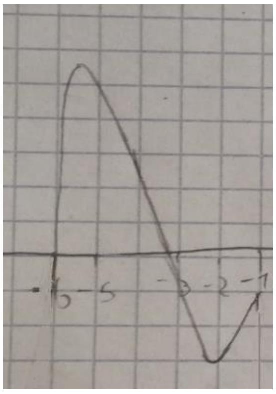
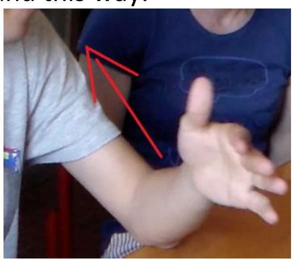
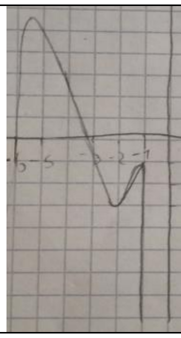
	When	Who	What is said	What is done
9	C 25:20	A	So, we start when f(x) is on the zero, x is on minus six then as it [x] goes on f(x) goes on but until….. minus five. Therefore, as x moves forward y makes, ehm.. from zero it reaches… so here [x=-6] it is zero, from zero it reaches, the top is five and a half…. And then does it come back down? Ehm no but it starts going down….. at minus five it is still going down but still it has no arrived at zero, it arrives at zero at minus, I do not know like minus three, a bit more than minus three. And then from minus three point two it keeps going down until.. minus three, when this [x] is at minus two point two…. And then it comes back up until minus one when it [x] is at minus one… then it comes back, then it goes away [Fig. 5.11b]	Task3 The grid is shown. Continuous dragging of x to the right and in the first pause he drags it back/forward in a neighborhood of -5. Again continuous dragging to the right. Then he draws:  Fig. 5.11a Finally, he moves his left hand this way:  Fig. 5.11b
10		R	Which one goes away?	
11		A	F(x) so this [the graph in a right neighborhood of -1] goes down	

				Fig. 5.11c
12		R	Why not up?	
13		A	Because from minus one it goes down towards..and then, then it arrives at, when this [x] is at one, so wait, at minus one it is over here…… Well, it arrives at, so from here [(0, 7)] it comes back for sure, but while coming back this [x] is still going on, this one goes on but as x goes on, y is going further and further down, that is, the ratio is always more [Fig. 5.11d] that is, as x goes on y changes but less and less in function of the movement of x, f(x) the function changes as x goes on	Continuous dragging of x from -1 to the right and he repeats this action several times. He points to large positive x -values on the paper and he makes this gesture:  Fig. 5.11d
14		R	The function change, but less and less?	
15	C 30:30	A	Yes, I mean like this [f(x) approaching 0], and then it is always flatter, that is, as x goes on the ratio of x to f(x) increases because, I do not know how, x goes on and f(x), as x goes on…	

In excerpt 5.11 Alessio explores and describes the realization DGpp of the function. So, he actually moves objects on the computer screen through the dragging tool but his discourse is about the variations of $f(x)$ as x moves on along its axis and it is not a description of the dragging action itself. Indeed, the role of subject in the discourse is played by one of the two variables, as if their movements were independent from the person. In particular, at lines 9, 10, 11 he focuses on the graph of the function for negative x -values and he describes all the movements of $f(x)$ in relation to the variations of x. In particular, at line 9 he says "*when f(x) is at 0, x is at -6* […] [f(x)] *decreases until -3, when this* [x] *is at* 2.2" while interacting with the file. We can notice that in both cases he first expresses the $f(x)$ -value and then the x -value, and indeed he drags x in order to see $f(x)$ take on a specific value and then he describes what happens "*when f(x) does a certain action*". However, in the same line he also drags x backward and forward in a neighborhood of the relative maximum point. In doing so he seems to be investigating a particular configuration of the two variables, combining *continuous handle dragging* with *continuous guided dragging.*

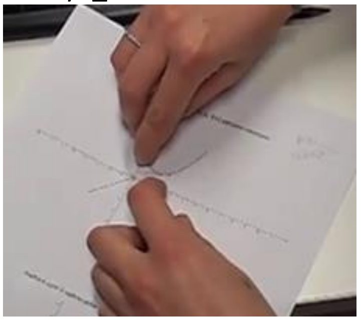
Then Alessio focuses on the ratio of $f(x)$ to x for large positive x -values. Indeed, at line 13, while using *continuous wandering dragging* of x, he says "*as x goes on, y is going further and* *further down, that is, the ratio is always more* […]" and he makes the gesture in Figure 5.11d; then at line 15 he adds "*as x goes on, the ratio of x to f(x) increases*".

5.1.4 Examples of dragging mediated discourse from the detached phase

In the following short excerpt from the eighth lesson, Matilde is working on a task using paper-and-pencil. By looking at the trace mark left by the point $(x, f(x))$ (that is, a bit of the Cartesian graph of a function) she has to mark on the same Cartesian plane, drawn on a sheet of paper, the trace mark that $f(x)$ would leave if trace were activated on it. Here she is exploring the graph only for positive abscissas.

Excerpt 5.12 - Lesson 8

(Realization SGc of the function $f(x) = \begin{cases} x, & -3 < x < 0 \\ \frac{1}{2}\left(x - \frac{3}{2}\right)^2 + 1, & 0 \le x \le 4 \end{cases}$)

	When	Who	What is said	What is done
158	I8Mp4 03:00	M	But wait, no no...x you have to move it inevitably towards here [to the right], because you do like this, you do like this because it has to stay perpendicular...while over here, here you do like that [Fig. 5.12a], it comes back up [Fig. 5.12b], yes...so there are some more marked parts in quotes, still, not here but here since it does...did you understand?	Activity8_2  Fig. 5.12a  Fig. 5.12b

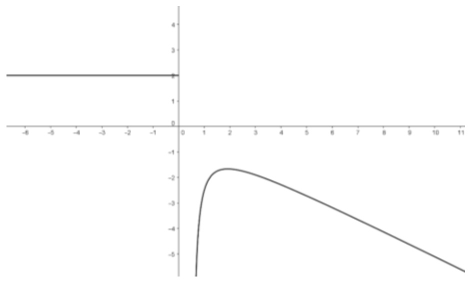
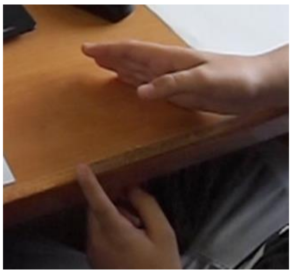
Analyzing the video, we saw how, while speaking aloud, Matilde moves her fingers on the sheet of paper as follows: her left hand moves from the origin to the right along the x -axis, her right hand from up to down and then from down to up along the y -axis (Fig. 5.12a and Fig. 5.12b). Moreover, her discourse is rich in references to space such as "towards here", "over here", "up", which need to be interpreted in association with her dragging actions, examples of dragsturing. At the end of the excerpt this dragging mediation even replaces words, when she doesn't explicitly say what "*it does*" through words but she moves her fingers on the graph to communicate it to her companion (during the pause indicated by the suspension dots).

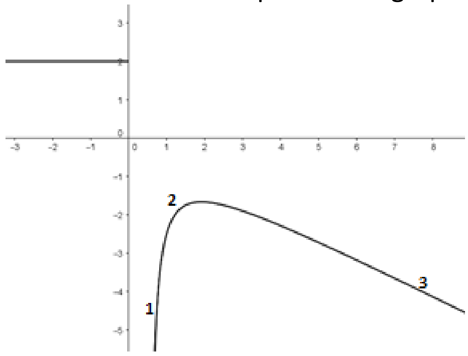
On the one hand, excerpt 5.12 could be seen as an example from the passive phase because the subject of Matilde's discourse is a person and she describes the direct action of dragging on the objects. On the other hand, we notice several differences with the excerpts illustrating the passive phase and we consider it to be an example from the detached phase. First of all, in this case the dragging tool is not physically used but a dragging action of the two variables re-created within the static context, and it plays the role of mediator in Matilde's discourse. Moreover, her discourse mirrors potential expert discourse on covariation, as we can see from the fact that she is able to evoke both ticks' movements with her body. Finally, she says "*there are some more marked parts*", suggesting that she is trying to find an answer to the question assigned in the activity, that is, discovering how to mark the trace of $f(x)$; so her discourse is about mathematical objects, as well. These elements all characterize the detached phase.

In the following excerpt Matilde is having a discussion with Nicco during the interview. They have been asked to describe a function, as if it was realized by a DGpp, given its Cartesian graph on a sheet of paper (Fig. 5.13a).

Excerpt 5.13 - Interview 2

(Realization SGc of the function $f(x) = \begin{cases} 2, \ x < 0 \\ -\frac{1}{3}(x + \frac{3}{x} + 2), \ else \end{cases}$)

	When	Who	What is said	What is done
1			[...]	Task1  Fig. 5.13a
2	MNm1 1:48	M	This is okay, I mean this is y equals two	She points to the graph of the function for negative x-values
3		R	And how would y equals two be visualized with parallel lines? With parallel lines I mean in a GeoGebra file like the ones that we have seen with parallel axes	
4		M	Yes yes ehm there is one of them that stays still at two and y that moves	
5		N	No x moves, x moves and y stays still at two	He keeps the right hand fixed while moving the left hand horizontally:  Fig. 5.13b

6	MNm1 02:50	M	While this one, you go forward with x and until it reaches one y goes down, I think, and then, going on, it comes up and then it goes down again…that is, you can drag x over here too [large positive x-values] and then y makes a sort of	After speaking she moves her hand down-up-down
			[…]	
12	MNm1 04:03	M	But here [positive x-semiaxis] is x or y faster? No y, no no no, if you drag in this direction [to the right] ……..x is faster	
13		N	Yes yes, x is faster while y is slower I mean in this case it seems that here [**1**] y would move faster, ehm because from here to arrive up here [**2**] it takes like this[4], then from up here to go back to this point [**3**] it takes like this[5] and…it is the same x, but faster, you drag one of them and so	He indicates different part of the graph:  Fig. 5.13c
14	MNm1 06:00	M	Exactly, no x is always faster I think…………. Wait, no but, are we sure that this is faster? Wait! Why faster?	During the pause she moves the hands as follows:  Fig. 5.13d  Fig. 5.13e

[4] In Italian he says "ci mette questo", and he opens his fingers as measuring the distance between the points 1 and 2 on the curve.

[5] In Italian he says "ci mette così", and he opens his fingers as measuring the distance between the points 2 and 3 on the curve.

As in the previous example, in excerpt 5.13 Matilde and Nicco's discourse is mediated by dragging, which is not physically used but dragging actions are referred to through the mediation of gestures. In particular, at line 5 Nicco moves his hands on the table describing the constant function $y = 2$ for which "*x moves and y stays still at 2*". He keeps one hand fixed while moving the other to the right and to the left (Fig. 5.13b). His gestures evoke *continuous guided dragging*, because he shows Matilde a specific configuration of the dependent variable, which stays still, while x can be moved to the right and to the left.

Then the students discuss the relationship between the variations of the two variables in terms of their different speeds for positive x -values. They search for "*which one between x and y moves faster*" (line 12) and Nicco expresses his observations by pointing to different parts of the graph on the paper (line 13). Matilde, instead, reproduces with her body the movements of the two variables along the Cartesian axes (line 14), using the dynamic visual mediation of dragsturing actions that are gestures on the paper but they blend with dragging actions previously made in an interactive file.

5.2 A SECOND LEVEL OF ANALYSIS: A DEVELOPING DISCOURSE

Provided with a model for the evolution of dragging mediated discourse and a description of different types of dragging used by students when interacting with DIMs, now we are going to investigate if, and eventually how, along the sequence of lessons students' discourse developed towards an expert's discourse on functions and on functional dependency in terms of covariation of two variables.

5.2.1 Students' use of precedents and their individualization of dragging

In the analyses of the excerpts above, that we presented to better characterize each phase of dragging mediated discourse, we also showed examples of students' use of different types of dragging and this analysis allowed us to better describe students' communicational actions. In this section we are going to focus on some recurring uses of these types of dragging that constitute possible precedents identified and performed by students within the specific task situations. We also describe how students' use of dragging seems to constantly change in relation to the different activities and to the context in which the activites are implemented. For example, we found episodes in which students use dragging actions in activities that ask to drag and explore the construction, but also in activities where it is not explicitely asked in the assignment; we even found drsagging to be employed, at times, in the paper-and-pencil environment. In relation to these different uses of dragging observed among different activities, we noticed a particular change occurring in various students' use of dragging, which we see as characterizing a process of individualization of dragging. Although this change does not necessarily happen for all the students, or in the same way in those for which it does happen, we were able to identify three main phases of dragging mediated discourse for various students. By *individualization* we mean a process that moves towards a use of dragging that is more similar to that expected from an expert. While speaking about the use of words in a mathematical discourse (see section 2.1.1.2), we have observed that in this study we are interested in knowing what specialised mathematical words are used by students; but also in identifying what informal words or constructs are used by students in certain ways that can be mirrored by experts' discourse. Simlilarly, we consider students' dragging and dragsturing actions and we investigate whether they can be mirrored by those used by experts. We describe experts' uses of dragging and dragsturing actions as having a more subjective dimension: there can be cases in which these actions are involved in experts' discourse but not prompted by specific requests to drag or gesture. It follows that we do not identify specifc types of dragging and dragsturing actions as characterizing experts' discourse, since they are necessarily related to the activity and to the context in which the activity is implemented.

The analyses of students' discourse illustrate a combination of *continuous* and *discrete dragging* during all the three phases, while *wandering* and *impossible dragging* seem to be predominant in the passive phase. Indeed, the examples from the active phase show students' use of *test, handle* and *guided dragging*. Then, the peculiarity of the detached phase is that the dragging tool is not physically used and so the different types of dragging are just evoked by students' gestures recreating dragging actions, that mainly consist of continuous movements of hands or fingers. We describe these changes in the use of dragging as possible steps in the process of individualization of dragging, which seems to move forward from the passive to the detached phase. Indeed, during the passive phase the visual mediation of dragging allows students to formulate a description of the construction involved in the specific task situation. Moreover, there are several examples of *impossible dragging* of the tick realizing the dependent variable, that we expect an expert would not do, at least after a first exploration of this kind of realization of functions. Then, the active phase sees a greater participation of students in deciding what and how to explore in the construction, that appears in their discourse which is about some effects of their dragging actions on the construction. With respect to the previous phase, there are no *impossible dragging* actions, suggesting that the impossibility of directly dragging one of the ticks has been identified by the students a precedent event. Moreover, c*ontinuous* and *discrete dragging* still intertwine in students' manipulations of the DIM but they also start combining different types of dragging, as *wandering* and *guided dragging* (excerpt 5.11), and *discrete* and *continuous dragging* (excerpt 5.4). We also found expressions of the form "when B […] A […]", that can be considered as examples of *handle dragging* where students move the tick A realizing the dependent variable in order to keep the other tick B in a specific position on the line, or to move it in a certain manner, and then they describe what happens on the screen when B behaves in this way. Since we consider this combination of different types of dragging as a potential routine that could be performed by an expert for exploring the specific realization of the function, we describe the change observed from the passive to the active phase in terms of students' individualization of the use of dragging.

Particularly interesting with respect to this process of individualization of dragging that we observed in students' discourse, are excerpts 5.8 and 5.9. Indeed, they show different procedures performed by the pair of students composed by Matilde and Nicco and that composed by Bernardo and Carlo for solving activity7_1, that asks them to draw on a paper the Cartesian graph of a function, given its dynamic realization DGc. The main difference that we noticed emerges from the types of dragging that are respectively used by the students: Matilde uses *continuous dragging* of x, while Carlo uses *discrete dragging* of x. In particular, Matilde traces with one finger the trajectory of the point $(x, f(x))$, without living any marks on the screen, while making a *continuous dragging* of x to the left. On the other hand, Carlo uses *discrete dragging* to move x by one along its axis, by stopping at each whole number, and he uses the point tool to build a set of points of the form $(x, f(x))$ directly in the DIM. So, at the end of these procedures, in Nicco and Matilde's discourse the curve realizing the function in the Cartesian plane is described as a result of the covariation of the two variables while Bernardo and Carlo's discourse is about the step-by-step process that they performed.

The analyses of these two excerpts show how the use of different types of dragging can be influenced and, at the same time, can influence the discourse. Moreover, the two students' different uses of dragging are both related to descriptions about the graph of the function that we could also expect from an expert and in this sense, we argue that an individualization of the use of dragging by students happened. Indeed, they mirror two different expert discourses about a curve in the Cartesian plane realizing the graph of function: a set of points of the form $(x, f(x))$ (discrete dragging) or the trajectory of the point $(x, f(x))$ (continuous dragging).

A further step in the development of this process of individualization of dragging can be identified in the detached phase of dragging mediated discourse, where the active use of the dragging tool is replaced by a realization of motion by using hands and fingers, also in a static context out of GeoGebra. Therefore, the different types of dragging are evoked by students' gestures that reproduce the movements of the variables along their respective axes within the realization SGc of a function. This shows how students have individualized the use of dragging for exploring the covariation of a function's variables, because it is precisely this aspect that constitutes the relationship between a dynamic graph DGc and a static graph SGc, and therefore that makes a Cartesian graph a possible realization of functional relationship. In fact, describing a curve in the Cartesian plane in terms of the movement of variables on the respective Cartesian axes is what an expert would do (even if without such an explicit reference to dragging actions). In particular, students seem to identify as possible *precedent* for the proposed task situation a procedure that they previously implemented in a DIE and so it cannot be performed exactly in the same way, because of the change of context, but they evoke it thanks to the mediation of dragsturing actions.

In particular, in the two excerpts 5.12 and 5.13 students make use of *continuous dragging*, because Matilde and Nicco move their hands reproducing a continuous movement of the tick realizing the independent variable. However, we interpreted differently the actions of the two students as *continuous wandering dragging* and *continuous guided dragging,* respectively. Indeed, Matilde describes the movements of her hands, that realize the movements of the two variables, for example she says "*here you do like that, it comes back up*" and so, the specific dragsturing actions are the objects of her discourse, as in the case of *wandering dragging*. While Nicco shows how the particular realization DGpp of a constant function would be, highlighting the difference between the behaviors of two ticks because one would move and the other would stay still and we called it *guided dragging* since he speaks about the behavior of the two variables before moving his hands for reproducing it. From these two examples it is possible to notice how many actions characterizing the experience within the DIE are used by the students as precedents requiring for replication in a paper-and-pencil context. In particular, their discourse heavily involves the visual mediation of different types of dragging, realized through acts of dragsturing, and this constitutes a phase in their process of individualizion.

In the next section we continue the analysis of dragging mediated discourse in this direction of research, which is aimed at characterizing the possible development of students' discourse along the sequence of lesson. For now, we described it by looking at their process of individualization of the use of dragging and at their identification of precedent events.

5.2.2 A possible turning point during the active phase

As we previously discussed, we used the word 'phase' for describing the three kinds of dragging mediated discourse because we observed a temporal evolution, from the passive to the detached, along the sequence of lessons, but we are tempted to consider this evolution also as a development of students' discourse towards a discourse closer to that of an expert mathematician. In particular, we identified a 'turning point' for this development happening during the active phase. In this section we are going to explain what we mean by 'turning point' and why we think that it can be associated to the second phase of dragging mediated discourse. To do this, we mainly focus on the routines performed by students along the three phases of dragging mediated discourse.

The passive phase is characterized by practical routines in the form of deeds and by discursive routines in the form of rituals. Indeed, in students' discourse in the excerpts 5.1, 5.2, 5.3 it is not possible to identify any references to a precedent event used by students because it required replication; they seem to start using the dragging tool to explore the DIM. For example, Nicco in excerpt 5.3 explores the construction trying to discover possible and impossible movements, which he does not seem to be aware of, as suggested by his *impossible dragging* actions. The desired outcome of students' procedures is a change in the objects, both in terms of physical changes in the construction and changes in their communication about the construction. Indeed, they have to give a description of what they see in the DIM but in several cases students substitute the dragging for words (e.g., excerpt 5.3). So, the dragging and dragsturing actions have a two-fold role because they allow students to communicate and, at the same time, they are the objects of students' discourse.

During the active phase of dragging mediated discourse students' performances are mainly communicational actions, which involve the physical use of dragging for manipulating the DIM. Therefore, a change in the objects is caused by the dragging actions, but the desired outcome of the routines seems to be the construction of a discourse about the mathematical properties of the function which is realized in the DIM. With respect to the previous phase, dragging is still used as a visual mediator that supports communication but there is a change in the object of students' discourse. Indeed, the description of the DIM is not accomplished with reference to the dragging actions but to mathematical signifiers realized by the DIM.

For example, we have seen in excerpt 5.5 that Franci uses the mediation of dragging to express the dependency of B. Indeed, he tells "w*hen B goes, when A goes.. are you moving A right*?" revealing that in order to choose how to build the sentence he would know which of the two ticks can be acted upon directly, because it may determine which one is independent.

Another example is excerpt 5.7 where Matilde's discourse contains explicit references to the dragging actions operated upon the ticks, but her focus is on finding a description for the specific function and on the relation between the two variables in terms of their opposite direction of movements. In particular, she uses the mediation of dragging to describe how she found the algebraic realization of the function and also to show that it holds for several values of the variables. Then, in excerpt 5.8, Matilde says that "*there is something strange*" which is interesting from a discursive point of view, because she identifies a 'strange' movement in the construction with 'something strange' and this can be seen as mirroring potential expert discourse on a specific property of the graph and, so, of the function realized.

Moreover, as we previously highlighted, in all these examples the subject of students' discourse is an object realized by the DIM, such as one of the variables or the function itself, and this makes the description independent from the person. This aspect usually characterizes experts' discourse; indeed the presentation of phenomena in an impersonal way is part of the process of objectification.

According to these observations, during the active phase of dragging mediated discourse, students perform discursive routines that can be described as standing at intermediate levels between rituals and explorations, that are the two extreme cases indicating "the process-oriented performance of the child" and "the outcome-oriented routine of the expert", respectively (Lavie, Steiner & Sfard, 2018). Moreover, together with a change in the routine, in the active phase also the mediation of dragging seems to change: it is used by students to better communicate, with other students or with themselves, about mathematical signifiers.

During the detached phase students seem to identify specific dragging actions as precedents for the proposed task situation. However, because of the design of some activities, that are implemented within a paper-and-pencil environment, they cannot physically use the dragging tool and so they re-create these actions with their body in the paper-and-pencil context. As in the previous two phases, the mediation of dragging is called into play but in a different way. In particular, students work with static realizations of mathematical signifiers and thanks to the visual mediation of dragging they evoke dynamic realizations of the same signifiers. In other words, dragging and dragsturing actions allow them to accomplish *saming* of a realization in paper-and-pencil context with a DIM.

Now we discuss about two examples of this process of saming. Matilde in both excerpts 5.12 and 5.13 evokes temporal and dynamic aspects characterizing the realization DGc of the two functions, in her discourse on the static realization SGc of the same functions. In particular, thanks to the dynamic visual mediation of dragsturing actions, she describes the covariation of the two variables bounded to the Cartesian axes. She seems to refer to a dynamic realization of the same functions, where the curve in the Cartesian plane is the result of the movements of two variables along their axes. Therefore, her process of saming, in both these cases, involves the realization of the functions SGc, given in the activities, with the realization DGc in a DIE; that she does not have at her disposal in that moment. In excerpt 5.13 also Nicco recreates a dragging action of the independent variable highlighting its dynamism with respect to the dependent one, because "*x moves and y stays still at 2*" (line 5). His dragsturing action is used as dynamic visual mediator in a discourse about the constant function $y = 2$ and it allows him to accomplish a saming of its realization SGc, given on the paper, with the algebraic expression of the function, which was found by Matilde few lines before in the excerpt and also with the realization DGpp of the same function, as suggested by the position and the movements of his hands.

According to these analyses, what we called 'turning point' for the development of students' discourse about covariation, towards the discourse of an expert, may be described as a change in discourse that we identified as sometimes happening during the active phase. In particular, during this phase the focus of the discourse changes: from being a description of possible and impossible movements in the DIM, it becomes a description of the relations existing between the movements observed on the screen, which means that students gradually involve mathematical signifiers into their discourse. Moreover, students start using the passive voice or they give variables the role of grammatical subject, instead of referring to themselves in the narration or to another person acting on the DIM. Together with these changes we observed a change in the routines performed by students that moved from rituals in the direction of explorations. Indeed, by analyzing how their uses of different types of dragging developed from the passive to the active phase, we observed that there were much more examples of *guided, handle* and *test dragging* than *impossible* and *wandering dragging*. Also the dynamic visual mediation of dragging seems to be used differently: during the passive phase dragging actions are the objects of students' discourse, in the active phase dragging and dragsturing actions are used by students to better communicate about mathematical signifiers, in the detached phase this kind of mediators is still used by students, even if they work in a static environment, especially through acts of dragsturing. Therefore, from the active phase the dynamic visual mediation of dragging and dragsturing actions becomes so intertwined with students' discourse that, at the same time, it subsumes the covariation.

5.3 CONCLUDING REMARKS

In this chapter, we studied students' use of dragging and, especially, we characterized its role as visual mediator in their communication about the DIMs that we designed. In order to this, we mainly used two tools of analysis, that we developed: the classification of different types of dragging described in Chapter 2 and a model for the evolution of dragging mediated discourse. Our analyses show that students performed different types of dragging to engage in mathematical discourse practices and their consistent use of dragsturing to complement their own speech, or as a response to their partner's, further contributes to the literature about dragging practices in DIEs.

Moreover, the analyses that we conducted through these tools, allowed us to describe the routines performed by students and to investigate how dragging mediates students' communication. In particular, we showed that along the sequence of lessons an individualization process of dragging takes place for students and that the role of dragging goes through the following changes:

- during the passive phase dragging allows students to communicate, and it is the object of their discourse;
- during the active phase dragging is used as a visual mediator which enlarges students' communicational actions about DIMs and it starts subsuming mathematical signifiers;
- during the detached phase dragging is used, blended with gestures, as visual mediator to evoke the dynamism of the DIMs and so it enlarges students' communicational actions and it subsumes the covariation even in a realization of function within a static environment. In particular, dragging and dragsturing actions allowed students to accomplish *saming* of a realization in paper-and-pencil context with a DIM.

According to these results, the temporal evolution characterizing the three phases of dragging mediated discourse along the sequence of lessons, may be also considered as a development of students' discourse about functions, towards a discourse closer to that of an expert mathematician. In particular, we identified a turning point for this developing process happening during the active phase, where several aspects of students' discourse change with respect to the previous phase and then they stay similar in the detached phase. Moreover, we also described a change in the routines performed by students moving from rituals in the direction of explorations. Indeed, they initially seem to perform practical routines in the form of deeds and discursive routines in the form of rituals; we did not identify any references to precedent events used by students that required replication. Then, their discourse becomes independent from the person, as is usually the case in experts' discourse, and students perform discursive routines that can be described as standing at intermediate levels between rituals and explorations.

Finally, our analyses suggest that dragsturing is an important mode for students of communicating dynamic and temporal aspects of functions. We think that the specific design of the DIMs facilitated the blending of dragging and gesturing actions, since the ticks were not labeled, and it led students to often refer to them through pointing gestures made with hands and fingers or through the mouse. However, an innovative and interesting result is that students used this type of non-linguistic communication also within a paper-and-pencil context. This finding seems to be in contrast with that of Ng (2016), according to which participants communicated about the fundamental calculus ideas differently within different types of environments. In particular, this is a defining feature of the detached phase of dragging mediated discourse where we showed that the visual mediation of different types of dragging, characterizing the experience within the DIE, is used by the students as precedent event requiring for replication and so it is realized through acts of dragsturing in paper-and-pencil context.

6 THE FORMATION OF A NEW MATHEMATICAL OBJECT

This chapter contains the analysis of the main features characterizing the students' emergent discourse about functions. We highlight the potential expert discourse mirrored and the seeds of possible realizations of mathematical signifiers, especially, the emergence of the covariational aspect of functions. This is made possible by the creation of a coding scheme, that we are going to describe, aimed at identifying different instances of specific patterns in discourse. Moreover, we present the *a posteriori* analysis of each task, comparing it with the *a priori* analysis, in order to have a feedback on the task design process.

6.1 SOME ASPECTS CHARACTERISING THE NEW MATHEMATICAL OBJECT

We are interested in analyzing the main features of the mathematical signifier 'function', which students are introduced to, as it is realized in their discourse; because we want to investigate what characterizes the specific discourse supported by the sequence of lessons and the DIMs that we designed.

In particular, we are going to analyze some excerpts taken from students' discourse during the lessons and the interviews, by focusing on:

- whether a formal vocabulary is used to communicate mathematical ideas;
- whether students' discourse is mirrored by potential expert discourse;
- whether visual mediators are involved in students' discourse;
- whether there are references to the dynamism of the realizations proposed, such as movement, time and space to describe functions and their properties;
- whether there is a shift in discourse where the DIM passes from being the object of students' discourse, eventually playing the role of mediator, to being the realization of the mathematical signifier 'function'.

6.1.1 Different expressions for $\boldsymbol{f(x) = y}$

In this section we describe different expressions used by the students to communicate about the correspondence between the values of the two variables, which is determined by the relation $f(\boldsymbol{x}) = \boldsymbol{y}$, for some $\boldsymbol{x}$ in the domain of the function. In particular, we selected excerpts that show a variety of expressions in students' discourse, and in which there is evident intertwining of static and dynamic aspects.

For example, in the following excerpt the researcher asks a question that prompts two different answers from Matilde and Anna: one of the students expresses a specific position of one variable, while the other student expresses the movement of one variable within an interval.

Excerpt 6.1 - Lesson 2

(Realization DGpp of the function $f(x) = \sqrt{x+3} - 2$)

	When	Who	What is said	What is done
225	I2Mp3 00:03	M	Put it [A] a bit forward…ah no, exactly on twenty-two, okay. So, B equals three when A is on twenty-two, but we have zero B, let's say that when B is on zero, when A is on one, and B on one when A is on six	Activity2_2 The grid is shown Anna holds the mouse

226		A	Yes	
227		M	And then.. it seems that B decelerated I don't know if	
228		A	In any cases, it [B] goes slower over there	
229	00:40	M	It goes slower, when A comes here B is a little bit more, it decelerates….if we can say to decelerate	She points to bigger and bigger values for A
			[…]	
239	01:15	R	Well! Now, how can I move B from zero to one?	
240		M	Ehm I have to put A on zero	
241		A	By moving A from one to six	
242		R	Let me see	Matilde takes the mouse
243		M	Because, when	
244		A	Put A on one	
245	01:35	M	If I put like this, it is at zero, if I put A at six it is on one	Guided dragging to have A=1 and then A=6

Excerpt 6.1 is taken from the second lesson, when Matilde worked with Anna, one of her classmates. In the excerpt, they mainly use the expression "the variable is on [a specific value]", which has a static nature, realizing the variable as an object that can be put on different numbers as Anna says at line 244 "*put A on one*". However, there are also several references to the dynamic aspects of the dynagraph, such as the description of B's velocity (see lines 227 and 228) and of A's movements (see line 229). In particular, the different answers of the two students to the question of the researcher (see line 239) show how both static and dynamic aspects coexist. Indeed, Matilde suggests to "*put A on zero*" while Anna to "*move A from one to six*". Moreover, in their attempt to find a correspondence between the values of the two variables, they also express the dependence relation referring to time, as indicated by their frequent use of the word 'when'.

Excerpt 6.2 - Lesson 4

(Realization DGpp of the two functions $f(x) = x^2 \; and \; g(x) = |x| + \frac{3}{2}$)

	When	Who	What is said	What is done
235	I4Mp3 00:18	E	When C is negative, D is positive	Activity4_2 He holds the mouse and she points to the screen
236		D	But look, it is much more up… when C is minus three, nine and a half	He writes on the paper, in a Cartesian plane that he drew
237		D	Minus two? Minus two will be here! Minus two, six and four… Minus one...almost one	She moves C by using the arrows on the keyboard He points to the screen while saying

				"here" and then he writes on the paper
238		E	However, they meet each other at zero	She is still using the arrows of the keyboard
239		D	Then put it at one	
240		E	They meet each other there, too… but wait, theoretically not exactly	
241		D	A little bit less	
242		E	At zero, too	
243		D	However, let's see, yes more or less. At two? At two is two, no four	
244		E	At three	
245	02:08	D	Nine	He draws a set of points in the paper

In excerpt 6.2, Elena and Davide's discourse is highly reified, characterized by the use of few verbs, short sentences and the focus seems to be on the static positions more then on the process to reach them. Students' exploration of the dynagraph consists in identifying pairs of values for the variables: for example, Davide's expression "*minus two, six and four*" (see line 237) describes the relation $f(-2) = 6{,}4$. It seems that the students were reading a table of values, without considering the temporal dimension of their dragging actions. Indeed, they use the arrows of the keyboard which shows discrete dragging of the independent variable which probably allows them to observe the product of the imposed movement. However, the students list the pairs of values according to the dependence relation, since they always express before the x-value and then the $f(x)$-value (e.g., lines 243, 244 and 245).

During the following lesson, Elena and Davide's discourse maintains the same features: again, they proceed by identifying in the DIM several pairs of values for the two variables and by plotting them on a Cartesian plane drawn on the sheet of paper. An example of this fact is shown in the excerpt below.

Excerpt 6.3 - Lesson 5

(Realization DGc of the function $f(x) = \sqrt{x+3} - 2$)

	When	**Who**	**What is said**	**What is done**
262	I5Mp6 00:04	D	At six it is one, put it……then at thirteen it is two……. and at twenty-two it is ok, so, let's do the negative values now, at minus one, or rather let's do at minus two it is minus one, at minus three it goes away, that is, a bit before, but at minus three it is a bit	Activity5_2 The grid is shown. He holds the mouse and she writes on the paper
263		E	It is almost minus three	
264		D	Then let's see some values in the middle, how much I took, you are at ten, at ten it is…one point six	Zoom in
265		E	At ten?	
266		D	Mm, but how is it possible?	He looks at the paper

267		E	Because I was wrong in doing this one and so	
268		D	Does it go up and then down?	
269		E	How is it possible? It is what you are telling to me	
270	01:40	D	No, absolutely not	

In the excerpt we can see Davide dragging the tick realizing the independent variable by using the mouse and he gives Elena a set of points of the form $(a, b = f(a))$ by telling her "at [a number] it is [a number]". Again, his discourse seems focused on static positions more than on dynamic relations between movements; however in the excerpt 6.3 there are also some expressions involving dynamic aspects. For example, at line 266 when Davide notices, by looking at the realization SGc, that Elena made a mistake because for him it is not possible that "*it goes up and then down*" (line 268). This observation is ambiguous since he does not explain what 'it' refers to, and it could be the dependent variable which moves up and down or the shape of the curve in their drawing.

Excerpt 6.4 is from the sixth lesson, and it captures Matilde working with Franci and Lore.

Excerpt 6.4 - Lesson 6

(Realization DGc of a function *ad hoc* defined, having [0, 12] as domain)

	When	Who	What is said	What is done
51	I6Mp1 08:35	F	Sorry, try to go at one for a moment, that is, we have to understand the meaning of this fact that it bounces	Activity6_1 She holds the mouse and Franci writes on the paper
52		M	Now it [x] is at one	
53		F	Eh, x equals one	
54		M	No, equals one it [f(x)] is one	
55		F	And when is f(x) at zero?	
56		L	It [f(x)] is zero, that is, when they are both zero, but also in each interval	Zoom in
57		M	No, on [x equals] one it [f(x)] gets to zero, so before one	After speaking she zooms out
58		F	Eh, on one, aaah before one does it [f(x)] get to one?	
59		L	I mean, after one month	
60		M	Yes	
61		F	And on [x equals] one it [f(x)] gets to zero	
62		L	During one month, zero cubic meters of water	
63		F	X equals one and zero cubic meters? While when x is close to one, but it is not at one, it is one cubic meter, in which sense?	He writes on the paper
64		M	It [f(x)] moves in the interval from zero to one, and at one it [f(x)] vanishes, then from one it will go up and from one to two it [f(x)] arrives at two, then it vanishes again	She moves her hands up and down

65		L	But it seems to me that it [f(x)] goes from zero to three	He moves one hand from one point in the air going up at another point
66		M	With [x equals] two? Not with two	She starts dragging again
67		L	Yes, with [x equals] two it [f(x)] arrives over there, while with three?	
68		M	With three, it arrives at	
69		L	At three	
70		F	With three it arrives at three, therefore between two and three	
71		M	At four it [f(x)] arrives at five and a half and with five also, at more than eight	
72	10:10	L	Yes, then go to six…ah it is back, however, they always go down in each interval	He points to the x-axis while saying "it is back"

Several times in this episode the students describe the variables as "staying on" or "being at" a certain value, which are expressions that Matilde already used in previous lessons. In excerpt 6.4 there are many examples of students' descriptions of the correspondence $\boldsymbol{f(x) = y}$ which is expressed statically (see lines 54, 56, 63) or with reference to the movement (see lines 57, 58, 61, 64, 67, 70, 71). The dynamism is involved in the description especially when Matilde moves her hands (see line 64) reproducing the movements of the dependent variable.

Excerpt 6.5 - Lesson 7

(Realization DGc of the function $g(x) = \frac{x}{2} + \frac{3}{x-3}$ and realization SGc of other four different functions)

	When	Who	What is said	What is done
212	I7Mp4 01:49-01:55	F	We have to find in which one among these four graphs it associates zero and one both with minus one	Activity7_2

Excerpt 6.5 is another example of description in terms of the process involved to reach a particular configuration. Indeed, Franci describes the correspondence between the variables' values as an association operated by the graph.

In conclusion, we have shown different excerpts from the sequence of lessons, containing students' discourse about the relation between the variables' values. If $a, b \in \mathbb{R}$, and f is a real function, we denote with A the independent variable and with B the dependent one, as most of the students did, all the following expressions are examples of students' descriptions of $f(a) = b$, that we found in the excerpts shown above:

1 “if I put A on *a*, it is on *b*”	7 “at *a*, it arrives at *b*”
2 “B is on *b* when A is on *a*”	8 “with *a* it arrives at *b*”
3 “B equals *b* when A is on *a*”	9 “equals *a*, it is *b*”
4 “it associates *a* with *b*”	10 “when A is *a*, *b* […]”
5 “when x is *a*, f(x) is *b*”	11 “at *a*, it is *b*”
6 “on *a* it gets to *b*”	12 “at *a*, *b* […]”

We listed them starting from expressions that explicitly mention the dependence relation or the dragging action of the user or the process needed to obtain that values. Then there are expressions that still contain some references to motion but in a less explicit way and finally there are the most reified realizations of the correspondence, even without any verbs. They actually do not appear in this order within students’ discourse, but scrambled up, revealing students’ attempt to build a discourse which takes into account (or which struggles with) a relationship between the dynamic action of dragging A, which brings it onto a certain position on the line, to the static condition “A equals a”.

6.1.2 The use of the word ‘function’

In order to investigate possible changes in students’ discourse, we selected all the excerpts where students use the word ‘function’, by using the Find Tool to identify its occurrences, and we analyze how it is used.

We noticed that very frequently students say “in function of”, which is an Italian expression used to describe a dependence relation, but it does not necessarily refer to the mathematical functional dependency, since it is commonly used in everyday situations. For this reason, now we are going to show all the excepts where the word ‘function’ is used not within this expression, highlighting possible patterns or developments in discourse.

During the first lesson, Matilde and Francesco already mentioned the word ‘function’ and in the following two excerpts we are going to describe in which context it happened.

Excerpt 6.6 - Lesson 1

(Realization DGp of the function $f(x) = -x + 5$)

	When	Who	What is said	What is done
425	I1Mp3 01:50-02:05	M	It could also be a function such that, when x is less than zero there is its image, when f(x) is like x that is, the absolute value, x becomes bigger than zero	Activity1_1 She tells the class her findings

Matilde uses the word ‘function’ when describing the DIM, hypothesizing that it was a realization of the absolute value function. In her description the dependence relation is initially expressed by the sentence “*when x is less than zero, there is its image*”, then her discourse about the relation between the two variables wanders from potential expert discourse mirrored. Moreover, she names them x and $f(x)$ giving other realizations of the signifiers ‘independent variable’ and ‘dependent variable’, as if the dynagraph was for her a possible realization of these mathematical signifiers.

Excerpt 6.7 - Lesson 1

(Realization DGpp of the function $f(x) = |x|$)

	When	Who	What is said	What is done
512	I1Mp3 11:08	F	I wanted to say that if it was a function, then there should be the inverse as well, so if I	Activity1_3
513		R	What does the inverse mean?	
514	11:50	F	It means that..wait, please do the movements…when we pass into positive numbers, then when the tick are separated out and one of them is in the positive [semiaxis] and the other one is in the negative [semiaxis] I should be able to move that one in the positive [semiaxis] also into the negative, because ehm..while I can move it only from the negative to the positive, so for me it is not a function because there should be an inverse as well	He gives the mouse to Lorenzo and tells R his observations pointing to the screen

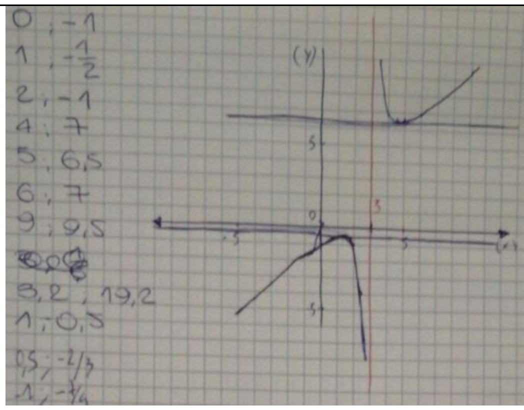
Francesco repeats twice the word 'function' and he also speaks about an inverse. In his discourse he does not refer to the dependence relation between the two variables, but he speaks about the ticks as two separated objects "*one of the ticks is in the positive* [semiaxis] *and the other one is in the negative* [semiaxis]" (see line 514). Moreover, it seems that he would recognize the DIM as a realization of a function if he had the possibility of dragging the dependent variable onto negative numbers, obtaining what he calls the 'inverse'. This observation seems an attempt to identify a *precedent* in his precedent-search-space involving similar task situations.

The main character of the episodes reported in the next excerpts is usually Davide, who seemed to employ quite often the word 'function' along the sequence of lessons.

Excerpt 6.8 - Lesson 3

(Realization DGpp of the function $f(x) = x + \frac{3}{x-3}$)

	When	Who	What is said	What is done
96	I3Mp2 00:05-00:15	D	Eh we have taken some points and we have put each of them here, then we have not taken them all so, more or less, I think that it comes in this way. But for three it does not exist as if it was a function that does not exist for three	Activity3_1 He shows to R their drawing:

				 Fig. 6.8a

Davide gives a description of the routine that he and his classmate usually perform for this type of task situation: it consists in identifying some pairs of values in the DIM and plotting them on a Cartesian plane on the paper, as shown in Figure 6.8a. Referring to the graph drawn, he treats it as a realization of "*a function that does not exist at three*" (see line 96). We interpret his discourse as related to a new branch, containing the realization DGpp, within his realization tree of the mathematical signifier "function not defined for $x = 3$". Indeed, from the realization DGpp he moves to SGc, which he seems quite familiar with, indeed, he also notices a gap in the domain of the function.

In the next lesson Davide was asked to describe his observations about this activity (Activity3_1) to the whole class and his explanation is reported in the following excerpt, where the student speaks without looking at the GeoGebra file.

Excerpt 6.9 - Lesson 4

	When	Who	What is said	What is done
3	I4Mp1 00:25-00:45	D	Eh we were able to understand that the function, a function x not exist for the value three and that all the other values it could not be less, it could not be more than minus a half and less than six and a half	Activity3_1 He tells the class his findings about the activity of the previous lesson, without opening the file

Davide uses the word 'function' again. He characterizes the domain, then he describes the set of images expressing the values that the function cannot be greater or lesser than. It is possible to see his discourse mirrored by potential expert discourse, if we identify the two extreme values that he expresses with constant functions. Otherwise, the dependent variable seems to be identified with the function itself in Davide's discourse.

In the next excerpt there are two episodes both belonging to the fifth lesson, in which two different students are involved.

Excerpt 6.10 - Lesson 5

(Realization DGc of the function $f(x) = -x + 5$)

	When	Who	What is said	What is done
4	I5C2 07:15-07:20	N	Eh but it will be something, it will be a function	Activity5_1 He explores the file, working with Alessio
			[…]	
197	I5Mp5 00:55-01:05	D	Yes, it is as the first one [DGc of activity5_1] but here [noises], but at the end the function is the same	Activity5_1bis He tells the class his idea

In the fifth lesson Nicco mentions for the first time the word 'function', identifying the DIM with a realization of a function. This excerpt is taken from the beginning of the lesson when students have just began exploring the interactive file of activity5_1 with the realization DGc. Since it is the first time that students see this type of realization, it is possible that because of the Cartesian plane Nicco identified a possible *precedent* in his precedent-search-space about task situations involving functions.

Then, Davide recognizes the DGpp of activity5_1bis and the DGc of activity5_1 as two realizations of the same function (see line 197) – this actually was the aim of that activities. This suggests that these are two different realizations both belong to his realization tree of the signifier $f(x) = x - 5$, which is the function considered.

Excerpt 6.11 - Lesson 6

(Realization DGc of a function defined ad hoc, having [0, 12] as domain)

	When	Who	What is said	What is done
85	I6Mp1 12:20-12:28	F	Yes f(x) sorry, it is a function……a function that vanishes for all whole numbers	Activity6_1 He talks with Lore and Matilde who manipulate the file, while he holds the pen to write on the paper

In excerpt 6.11, Franci describes the behaviour of the function that he sees realized by its DGc. From his words "*f(x), it is a function*" it seems that he identified the function with $f(x)$ itself, which is the label used for the dependent variable that moves bounded to the y-axis in the DIM. However, then he says "*a function that vanishes for all whole numbers*" which expresses the dependence on x.

In the following excerpt there are two episodes from the same lesson. In the first one, Davide is telling the entire class what he did to draw the Cartesian graph of the function in the activity7_1 and what he noticed about the behaviour of the function, by looking at the graph obtained. In the second one, Elena, who works paired with Davide, is speaking to the researcher.

Excerpt 6.12 - Lesson 7

(Activity7_1**:** realization DGc of the function $f(x) = \frac{1}{10}\left(\frac{x}{2} + 4\right)(x+1)(x-2) + \frac{5}{2}$;

Activity7_2: realization DGc of the function $g(x) = \frac{x}{2} + \frac{3}{x-3}$ and realization SGc of other four different functions)

	When	Who	What is said	What is done
263	I7Mp5 02:20	R	Davide and Elena how did you make it?	Activity7_1
264		D	Ehm so, we have done as usual, we have taken various x-values and we saw how the function behaved, we noticed that for .. when x assumed certain values the ... it created some curves that went up and down and then we managed to understand that for certain values of the function, that this function assumes, there are three solutions and therefore that for y varying between one and a half and about six or seven there are three x-values that satisfy such	He tells the class how they solved the activity
265		R	And did it help you to draw the trajectory?	
266		D	We noticed it later… to draw the graph we made as	
267		R	Using the values?	
268	03:30	D	Yes, as usual, and then we noticed this fact	
			[…]	
338	I7Mp6 06:07-06:13	E	And so here [in the Cartesian plane] we mark the two points and then we trace.. the function	Activity7_2 She talks with R

In excerpt 6.12 the researcher asks to Davide and Elena how they made the graph on the paper and Davide's answer focuses on the non-injectivity of the function, but this property is something that they noticed later, by looking at the Cartesian graph. Indeed, he explains the routine he and his classmate performed to solve the task which was the same "*as usual*" (line 264). In particular, Davide's use of the word 'solution' suggests that he refers to the equation $f(x) = k, k \in [1.5, 7]$, which is satisfied by different x-values. So in this case Davide employs the word 'function' as an expert.

Some moments later during the lesson, Elena says 'function' referring to the curve that realizes the graph of the function in the Cartesian plane.

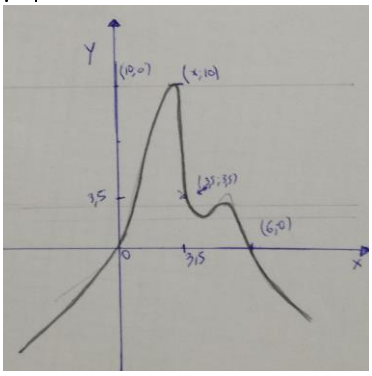
Finally, in the next excerpt there are two episodes happened during the last lesson.

Excerpt 6.13 - Lesson 8

(Activity8_1: realization DGpp of the function $f(x) = \begin{cases} \frac{3}{2x} + 2, \ x > 0 \\ \frac{(x+1)^2(x+6)(x+3)}{x} - 1, \ else \end{cases}$;

Activity8_3: description of different properties of a function to draw its realization SGc)

	When	Who	What is said	What is done
146	I8Mp3 03:04-03:05	N	But which function is it?	Activity8_1 He asks to R
			[…]	

277	I8Mp5 01:45-02:27	D	Here for three, for example, then also down here, and here for four values, and so we interpreted it in this way. Then the fact that they intersect at about three point five mans that the point three point five, three point five belongs to this function and so we also satisfied this request	Activity8_3 He tells R how they solved the activity, showing their drawing on the sheet of paper:  Fig. 6.13a
			[…]	
283	I8Mp5 03:12-04:06	D	However, before zero as x increases also f(x) increases, then the fact that going forward with x..the function moves very little so we have satisfied in this way the fact that between ten and six, between zero and six, that are the values on the x-axis, we have more movement, the function is more dense, I do not know how to say it, while when before zero, that is, when the values are less it is a bit more smooth, I do not know how to say it and then, I think that all the things are satisfied	

At this point, Nicco seems sure about the fact that he is working with the realization of a function, but he does not know which function, as his question to R, at line 146, suggests.

Then Davide repeats the word ‘function’ several times and he almost always uses it as an expert; except when he says that “*the function moves very little*” at line 283, which probably describes the movements of the dependent variable when the independent one is dragged in a certain interval.

The analyses highlight that not many students of the class say ‘function’ during the sequence of lessons; indeed all the excerpts are taken from the discourse of Matilde, Davide, Elena, Franci or Nicco. Moreover, we have observed that in several cases these students use the word ’function’ when talking about the independent variable and this is often done also by the experts, especially in oral communication.

In the case of Davide, since the first time the word ‘function’ appears in his discourse during the third lesson, till the last one during the eighth lesson, we see an evolution in the way that it is employed. In particular, his discourse is ever more mirrored by potential expert discourse in which the word ‘function’ is used. Moreover, during the sequence of lessons, we also see a development in students' use of the DIMs: from being the objects of the discourse, for many students theybecome possible realizations of the mathematical signifier 'function'. For example, in the excerpt 6.10 Nicco's discourse shows this transition which started during the fifth lesson, while we observed that for Franci part of the switch happened during the sixth lesson (see excerpt 6.11).

6.2 POSSIBLE REALIZATIONS OF FUNCTIONS' PROPERTIES

After showing an overview of the implications of the sequence of lessons on the whole class, aimed at identifying all the seeds of possible realizations of mathematical signifiers in students' discourse, in this section we are going to focus on certain selected pairs of students that we followed more closely. In particular, our purpose is to investigate if, and eventually how, these students' discourse is mirrored by a potential expert discourse (for a description of it, see section 2.1.1.2). For this reason, in the excerpts that we are going to show we added a column for the 'potential expert discourse' where we express what the student says as we expected from an expert. In some cases, this column is divided in two lines, because in the bottom line we add some further implications or other possible details that an expert would probably say. There could be also some notes in italic where we write possible corrections of students' mistakes. Moreover, we added another column for the 'code' that contains the mathematical signifiers realized by the corresponding discourse, including also the cases where the discourse contains seeds of realizations of these signifiers, expressed according to the coding scheme that we are going to explain.

6.2.1 A posteriori analysis

In this section we use the same table that we created for the *a priori* analysis (***Table 4.2***).

By looking at both **Table *6.1*** and ***Table 4.2***, for each activity it is possible to compare the *a priori* and *a posteriori* analyses in order to have an idea of the strength of each task in fostering a specific discourse. Moreover, analyzing all the selected instances for a given mathematical object, which are all the sentences with the same label, it is possible to have a general view of the development of students' discourse along the sequence of lessons, with respect to that object.

To create the table we follow a similar approach with respect to that used for ***Table 4.2***: a black box represents a mathematical object which students refer to in their discourse and a white box represents a mathematical object that does not appear in students' discourse. Moreover, the grey is used for an intermediate level, that is for the mathematical objects included in students' discourse but just a few number of times.

In order to choose which of the three colors should be used for each box and to have rigorous criteria for this choice, applicable to all the activities, we developed a coding scheme. We coded all the video transcripts by applying the labels: IN/DEP – DOM – RAN – INJ – MON – MAX/MIN – LIM – ASY – DER every time we identified a potential expert discourse mirrored about the corresponding mathematical signifier. Therefore, it is not necessary for a discourse to be expressed with a formal mathematical vocabulary in order to be coded through these labels. This process allowed us to assign a color based on the number of instances of a particular label within a specific activity: black indicates the most frequent labels, the grey for the less frequent and the white for absence of the label. Now, we provide a detailed description of each label, with a description of the coding; for example, we show some expressions or words considered representative for that mathematical signifier. In these expressions we always denote the two variables by x and $f(x)$, but in the excerpts we use the same labels for the mathematical signifiers even if in students' discourse they are denoted by A and B, or something similar.

- IN/DEP (independent/dependent variable): every time the asymmetric relation between the two variables is explicitly described, for example "$f(x)$ moves in function of x"; or it is expressed by using temporal references, for example "as x increases, $f(x)$…"; or it is used the hypothetical form which conveys causality "if x…, $f(x)$…".
  Moreover, the same label is used to mark seeds of realizations of the relation $f(\boldsymbol{x}) = \boldsymbol{y}$, for some $\boldsymbol{x}$ in the domain of the function; that we described in 6.1.1.

- DOM (domain): marks discourse involving references to impossible movements, or values, of $f(x)$, expressed with respect to the corresponding movements, or values, of x; for example "$f(x)$ disappears after… [corresponding x-value]" or "$f(x)$ does not exist between… [corresponding interval of x]".
- RAN (set of images): every discourse which involves references to possible or impossible values for $f(x)$, that can also be expressed in dynamic terms as possible or impossible movements of $f(x)$; for example "$f(x)$ does/does not go from… to…" or "$f(x)$ does not go past…" or "$f(x)$ does not exceed…" or "from here on $f(x)$ always exists".

- INJ (injectivity): marks discourse on non-injectivity of the function, in particular, in each case where the fact that a particular value of $f(x)$ can be obtained for, at least, two different values of x. For example, "$y = \boldsymbol{y}$ can be obtained in two, three,… different ways".

- MON (intervals of monotonicity): every time the direction of movements of the two variables is explicated to emphasize that they have the same direction, in the case of an increasing function, or two opposite directions, in the case of a decreasing function. For example, "$f(x)$ and x move in the same direction" or "$f(x)$ and x are symmetrical" or "$f(x)$ comes back".

- MAX/MIN (relative or absolute maximum/minimum): it is used every time the word 'maximum' or 'minimum' is used, but also for discourse referring to a point where $f(x)$ changes its direction or stops or bounces. For example, "$f(x)$ arrives at … and then it moves down again" or "$f(x)$ bounces" or "$f(x)$ stops and it doesn't pass over…".

- LIM (finite or infinite limit): marks discourse involving references to the behavior of the dependent variable at the extremities of the domain, for example "$f(x)$ does not stop anymore" or "$f(x)$ goes to inifinity".

- ASY (vertical or horizontal asymptote): marks the descriptions of the movement of $f(x)$ when it moves so much slowly that it seems fixed and it is the case of a horizontal asymptote, for example "$f(x)$ stays still here" or "$f(x)$ approaches… but it never touch it"; or the descriptions of the path followed by $f(x)$ as it was a circle around the computer screen and it is the case of a vertical asymptote, for example "$f(x)$ arrives at an infinite point here and then it comes back from an infinite point there" or "$f(x)$ reappears over there".

- DER (derivative): marks discourse about changes in the speed of $f(x)$ and its possible acceleration/deceleration, for example "x moves a little and $f(x)$ dashes, then x moves a little and also $f(x)$ moves a little"; or to mark all the descriptions of changes in the movements of $f(x)$ that are expressed with respect to the movements of x, for example "$f(x)$ moves less and less in relation to x" or "$f(x)$ passes over x".

Activity	Type	IN/DEP	DOM	RAN	INJ	MON	MAX/ MIN	LIM	ASY	DER
1_1	DGp									
1_2	DGp									
1_3	DGpp									
2_1	DGpp									
2_2	DGpp									
2_3	DGpp									
3_1	DGpp									
3_2	DGpp									
4_1	DGpp									
4_2	DGpp									
5_1	DGc-DGpp									
5_2	DGc									
5_3	DGc									
6_1	DGc									
6_1bis	DGc									
7_1	DGc-SGc									
7_2	DGc-SGc									
8_1	DGpp-SGc									
8_2	SGc									
8_3	SGc									

Table 6.1. Mathematical objects in all students' discourse

Table 6.1 shows that several mathematical objects intertwine in students' discourse along the whole sequence of lessons. So the designed activities seem to support the formation of discourse about functions and the description of a specific property of a function involves the observation of several other aspects characterizing the same function.

The first notable difference that we can observe between **Table 6.1** and ***Table 4.2*** concerns the derivative. Indeed, it is realized in students' discourse almost during the whole sequence of lessons, especially in the activities 4_1 and 5_3, even if this mathematical object was not included in the *a priori* analysis until the last lesson. It seems that the dynagraphs, where movement is essential to explore the functional relation, promote discourse about the speed and changes in speed of the ticks realizing the variable, this discourse is mirrored by potential expert discourse about the derivative of the function. Moreover, activity3_1, which was designed to support discourse about the domain and the set of images of a function, as indicated by the black boxes in ***Table 4.2***, seems actually to promote discourse about the set of images and the monotonicity properties, but not about the domain. This observation could suggest a review of the task or the choice of the particular function defined for the activity, in order to foster further discourse on possible values for the independent variable, according to the goals of the lesson. For example, the question "For which x values does $f(x)$ exist?" could be added in activity3_1bis.

As we expected, in some activities the choice of asking the questions to students after giving them time to explore the file seems to have fostered richness in their discourse describing their explorations – more so than when answering the questions. Differently from what we expected, when solving activity7_2 no one in the class mentions that the same function was also used in activity3_1.

6.2.2 Nicco and Alessio

Table 6.2 shows the list of the mathematical objects that can be found in Nicco and Alessio's discourse, who worked together for six lessons.

As previously discussed for the whole class, seeds of realizations of several mathematical signifiers characterizing functions can be found in Nicco and Alessio's discourse.

Activity	Type	IN/DEP	DOM	RAN	INJ	MON	MAX/MIN	LIM	ASY	DER
1_1	DGp									
1_2	DGp									
1_3	DGpp									
2_1	DGpp									
2_2	DGpp									
2_3	DGpp									
3_1	DGpp									
3_2	DGpp									
4_1	DGpp									
4_2	DGpp									
5_1	DGc-DGpp									
5_2	DGc									
5_3	DGc									
6_1	DGc									
6_1bis	DGc									

Table 6.2. Mathematical objects in Nicco and Alessio's discourse

If we consider **Table 6.2** and ***Table 4.2*** we can notice some differences: for example activity 2_3 mainly proposes realizations of the domain, and not of the set of images of the function, as was indicated in the *a priori* analysis. This could suggest reviewing the task or the choice of the particular function. However, ***Table 6.1*** shows that this difference is not observed for all the students and so it might be due to the particular focus of Nicco and Alessio' discourse during the explorations of the DIMs.

There are also some differences between **Table 6.1** and **Table 6.2**, for example, activity4_1 has a black box for the set of images if we consider the whole class's discourse, while Nicco and Alessio's discourse seems not to be mirrored by potential expert discourse about this mathematical object, as indicated by the white box in **Table 6.2**. The same remark holds for activity5_2 with respect to the domain of the function and also in this case the question "For which x values does $f(x)$ exist?" could be added to bring students' attention on the independent variable. On the contrary, in activity4_2 the only possible realizations of the intervals of monotonicity that have been observed in the class's discourse come from Nicco and Alessio, while all the other students in the class are mainly focused on the set of images of the function.

Now we are going to analyze Nicco and Alessio's discourse and how it develops during the sequence of lessons, showing some possible realizations of mathematical objects that we have identified, coded and listed in **Table 6.2**. Since the mathematical objects are intertwined in students' discourse, an expression is rarely coded with only one label. Therefore, we are going to present the excerpts following the chronological order of their happening and not dividing them according to the codes that characterize them.

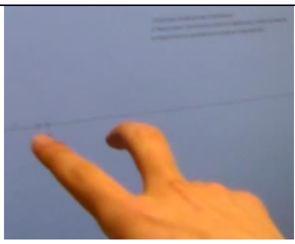
Excerpt 6.14, from the first lesson, documents students' attempts to establish which tick moves and, possibly, how it moves. This type of exploration supports discourse about the asymmetric relation between the two variables and about the behaviour of the function, that decreases.

Excerpt 6.14 - Lesson 1

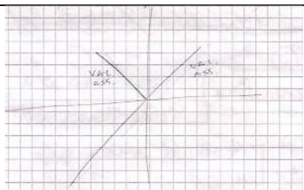
(Activity1_1: realization DGp of the function $f(x) = -x + 5$;

Activity1_2: realization DGp of the function $f(x) = |x|$;

Activity1_3: realization DGpp of the function $f(x) = |x|$)

	When	Who	What is said	What is done	Potential expert discourse	Code
70	I1Mp1 06:42	R	What moves?	Activity1_1		
71		A	They move symmetrically and in opposite directions, that is, if this one [x] moves a bit this one [f(x)] moves the same but in the opposite direction	 Fig. 6.14a	The function is strictly decreasing	MON
72		R	But do they move both?			
73		A	yes..absolutely!			
74		R	Do you agree?			

75		N	Eh they move both, we move just one of them			
76	I1Mp1 07:08	A	No they move both			
			[…]			
209	I1Mp2 02:15-02:20	A	In my opinion they move in a symmetrical way because if one goes away, the other goes away, if one approaches the other, the other approaches too	He tells the class their findings and he moves his hands to the right and to the left, without manipulating the file	The function is strictly decreasing	IN/ DEP MON
			[…]			
221	I1Mp2 03:00-03:18	A	yes! That is, we said that they move both because, that is, with respect to the two fixed points that are zero and one, by moving maybe B to the right, A moves to the left and then it goes below zero and by moving B to the left A goes to the right	He tells the class their findings and he moves his hands, without manipulating the file	A depends on B. The function is strictly decreasing.	IN/ DEP MON
			[…]			
233	I1Mp2 03:47	N	That one does not move	Alessio takes the mouse to drag again		
234		R	Let me see, do they move both or does it move just one of them?			
235		A	Ah only one of them moves, it's true			
236		R	But what does it mean that just one of them moves?	She asks this question to the class		
237		A	That is, they move both but one of them	He is not dragging anymore		
238		R	I see them moving!			
239	04:08	A	But one of them moves in function of the other one, of A and B, that is, B can		A moves depending on B	IN/ DEP

			move while A moves in function of B			
			[…]			
327	I1Mp2 11:28	N	When minus x arrives at zero	Activity1_2		
328	I1Mp2 11:38	A	Yes, that is, we have the intersection point between minus x and plus x	He points to the screen and closes together his second and third finger	$-x$ intersects $+x$ $f(0) = 0$	
			[…]			
360	I1Mp2 15:30	N	They are x and absolute value…and if x is positive then only the positive it exists	He drags x along positive real numbers	It is the absolute value function and for positive x-values $f(x) = x$	
361		R	Therefore, over here [negative x-axis] there is one if them			
362		A	Yes, so both x and minus x			
363		N	The fact that the graph of the absolute value… they are three	 Fig. 6.14b		
364		A	That is, if x is negative, and also if x is positive that is, it is always positive so it is absolute value function	He points to the screen, moving his hand to the right and to the left a number of times	For all positive and negative x-values the function is positive, so it is the absolute value function	
365		R	Therefore, which one is x? X is that variable that can go into positive and negative numbers?			
366		A	Yes			
367		N	And so, minus x, absolute value of x,	He points to the screen firstly x which is negative		

			but if it was positive the absolute value is positive	and secondly f(x) which is positive. R nods		
368		A	Therefore, this one is the absolute value of x, while these two are x without absolute value	When he sees just one tick; when he sees the two ticks	$f(x) = \|x\|$	
369	I1Mp2 16:38	N	Absolute value of minus x… this is minus x and this is the absolute value of minus x			
			[…]			
500	I1Mp3 10:07	R	So which one are you dragging now?	Activity1_3		
501		A	This one [f(x)]	He holds the mouse and points to the screen with the other hand		
502		R	The one above			
503		A	And this one [f(x)] moves in function of this one [x]		$f(x)$ depends on x	IN/ DEP
504		R	And is it the same of what you said before?			
505		A	Yes..because no one of them disappears	He stops dragging before speaking	The function is always defined	DOM
506		N	For negative [values of x] there exist negative and absolute value	He takes the mouse to drag x		
507		A	It does not disappear, because this one [f(x)] is the absolute value of minus x, and this one [x] of x, and so they are two different segments	The class listen to him and he moves his hands without dragging		
508		R	Is there never a point that disappears?			
509		A	No			
510		R	While according to you, it should disappear, that is, does it seem strange to you?			

511	I1Mp3 11:00	A	No no no			
			[…]			
546	I1Mp3 14:14	A	This one [x] moves in both the negative and positive [numbers]…this one [f(x)] is the absolute value of x and this one [x] is x	He holds the mouse and drags, while pointing and following the movements of the ticks on the screen with the other hand	$f(x) = \|x\|$	
547	I1Mp3 14:26	N	Yes, that one above is this [graph of the absolute value in Fig. 6.14b] and that one below is only this part [the line y=x in Fig. 6.14b]			

The dependence relation is expressed by students thanks to the difference between direct and indirect motion, as shown by Nicco explicitly referring to his possibility of dragging just one of the two ticks (see line 75). This type of observations is also prompted by some questions of the researcher such as "*but do they move both*?" or "*but what does it mean that just one of them moves*?" (lines 72 and 236). At a certain point in the excerpt Alessio says that the movements of one variable depend on the other one (see line 503) and we can figure out which variable he is talking about by watching at his pointing gestures, since he does not distinguish them by words. Also, Alessio explains that by moving one tick the other one moves (see lines 221, 239) and he denotes with B the independent variable and with A the dependent one. In line 221, as in 71 and in 209, his discourse contains references to the opposite directions of movement of the variables and so, it is mirrored by potential expert discourse about the interval of monotonicity of the function, where it is decreasing. Alessio mainly uses the mediation of gestures, together with words, to communicate his observations to the whole class and to the researcher.

However, Nicco and Alessio do not always express the dependence relation between the two variables; indeed there are several examples in which they speak about them as two separate entities (e.g., lines 328, 368, 507, 546, 547). In particular, when exploring the dynagraph of the absolute value function, Nicco and Alessio do not immediately identify the independent variable and the dependent one. They assume that there are three different objects: $x, f(x), -x$; which they treat as separate entities, not linked to one another. This means, for example, that they describe $f(x)$ independently from x, which might even not exist (see lines 368, 546). Also, their drawing on the paper (Fig. 6.14b) seems consistent with this description involving three different elements. However, the students are building discourse on the dependence relation, trying to put together different objects which they still see as distinct. There are some instances of this discourse which is forming, for example, when Alessio at line 364 describes the absolute value function, probably identifying a *precedent* from his precedent-search-space, he describes the behavior of $f(x)$ as depending on the sign of x.

During the second lesson the focus is mainly on the domain and the set of images of the functions. In particular, students' discourse is rich in references to values that the variables can or cannot assume and to unexpected movements or direction of movements. Moreover, in the excerpt 6.15 we will see that Nicco and Alessio's discourse on the dependence relation is still evolving.

Excerpts 6.15 - Lesson 2

(Activity2_1: Realization DGpp of the function $f(x) = e^{x-1} + \frac{1}{25}$;

Activity2_2: realization DGpp of the function $f(x) = \sqrt{x+3} - 2$;

Activity2_3: realization DGpp of the function $f(x) = \sqrt{(x^2-1)(x^2-4)}$)

	When	Who	What is said	What is done	Potential expert discourse	Code
45	I2C2 37:00	A	Over here as you move, it seems that it becomes more sensitive, like B to the movements of A, because here if [x] moves by one, I do not know, one millimeter, this one [f(x)] moves maybe by two millimeters, instead here if this [x] moves by one millimeter, here this one [f(x)] moves by two centimeters so I do not know...	Activity2_1	The derivative is not constant	IN/DEP DER
			[…]			
66	I2C2 28:28	A	There is a bug, there is a bug!	Activity2_2 He drags A forward and backward in a neighborhood of -3 and B disappears from the screen		
67		N	Where is [f(x)] from [x equals] minus three? There is not [f(x)] from minus three		In a left neighborhood of -3 the function is not defined	DOM
68		A	Try a second to go behind minus three?	He asks to other two students		
69		N	After minus three does it [f(x)] disappear also to you?			

70		A	Professor, to us B disappears after minus three			
71		N	Yes, it disappears also to them, no, so it is not a bug in the program!			
72	I2C2 28:52	A	Does it disappear? It's a magic, magic!			
			[…]			
109	I2C2 40:55-41:22	A	But in my opinion, now it is this [f(x)] that moves with uniformly accelerated motion, because as this one [x] goes forward this [x] moves in more space and this one [f(x)] always less, so it depends on the point of view anyway, because this [x] moves by one and this one [f(x)] moves very little, then this [x] moves by one and this one [f(x)] always less, therefore it depends on, I do not know, on the point of view, for me	He talks to R and moves his hands pointing to the screen	As x grows, the ratio of Δf(x) to Δx decreases The derivative is decreasing	IN/DEP DER
			[…]			
113	I2C2 44:17	A	Oh my god! if it [x] is minus one, B disappears	Activity2_3 Alessio makes an impossible dragging on B	The function is not defined at -1	IN/ DEP DOM
114		N	Yes but go further, it reappears, look			
115		A	It's true! Probably also before, because we did not go beyond minus three			
116	I2C2 44:54	N	B does not exist in the interval between [x equals] minus two		The function is not defined on the interval [-2; ?)	DOM
			[…]	They stay in silence while Nicco continues dragging		

117	I2C2 45:30	A	What? No! Keep going forward, go go go	They see B coming back when dragging A from 0 to 1		
118		N	Also between [x equals] one and two it [f(x)] does not exist, between one and two it does not exist!		The function is not defined on the interval [1; 2]	DOM
119	I2C2 46:27	A	It is as if it bounced………for me, it seems a bouncing ball	Fast dragging where B moves “well” and slow dragging around the “critical points” such as x=1 and x=2	In a left neighbourhood of 1 the function decreases, then it does not exist and in a right neighbourhood of 2 it increases	MON
			[…]	They are quite		
120	I2C2 48:50	A	We have seen that in the interval between minus two	He tells R their findings		
121		N	Between minus two and minus one and between one and two it [f(x)] does not exist, B does not exist		The function is not defined on the intervals [-2; -1] and [1; 2] The domain is R except for these two intervals	DOM
122		R	And these numbers are…that is, do they refer to A or B?			
123	I2C2 49:13	A	It would be in the intervals of A between one and two and between minus one and minus two, that B does not exist		The function is not defined on the intervals [1; 2] and [-2; -1]	DOM
			[…]	Activity2_3bis		
140	I2C2 54:54	N	Nooo, B does not go on zero	For several times he drags A from -2 towards the left, in a small interval, and he brings it back	It does not exist $\boldsymbol{x}$ such that $f(\boldsymbol{x}) = 0$	RAN
141		A	And B does not either go on four		It does not exist $\boldsymbol{x}$ such that $f(\boldsymbol{x}) = 4$	RAN
142		N	But how not? But do you see that from		$f(x)$ takes on all positive values	RAN

			here on it always exists? How can it not go there?!	He still drags A from -2 towards the left	(0, +∞) is a subset of the set of images	
143		A	It does that thing, how is it called that thing like this? It is symmetrical, it is a…?		The function is even	INJ
144		N	Parabola			
145	I2C2 55:45	A	Yes, a parabola but it had a particular name, symmetry, no, how is it called?			
146		N	from zero point two…what do you do? No bring it back here	Nicco reads the third question and explores the file by dragging		
147		A	Nicco, for me it is a parabola…… But for me, we do not have to focus on the numbers, but more on the characteristics		The graph is a parabola	
148		A	We did not see if we could drag B	Impossible dragging		
149		N	We cannot drag it, we can never drag it		$f(x)$ is the dependent variable	IN/ DEP
150	I2C2 59:00	A	Why?!			

In the first episode reported in the excerpt (line 45), the students are working on Activity2_1 and Alessio gives a detailed description of the range of variations of the dependent variable, given a fixed interval of variation of the independent one. Also later in this excerpt he makes some other observations about the speed of the two variables and, in particular, about the changes in speed of the dependent variable with respect to the independent one (e.g. line 109). These types of discourse are mirrored by potential expert discourse about the derivative of the function, and so about the slope of its graph restricted to a fixed interval.

Similarly to what is described in the studies by Healy & Sinclair (2007), Alessio and Nicco are initially surprised by the vanishing of the dependent variable from the screen. At lines 68, 69, 70 they even ask other students and the researcher if their file had some problems, since "*from minus three it vanishes*" and it seems strange to them. They refer to the disappearance of the dependent variable and they express it with respect to the value of the independent one, which is -3, mirroring potential expert discourse about the domain of the function. As already discussed in the *a priori* analysis, in the realization DGpp of a function the domain has to be identified by observing the movements of the dependent variable. In a similar way they realize the domain of the function of activity2_3 by expressing for which x-value $f(x)$ disappears from the screen (see lines 113-116), and we can infer what variable they refer to by looking at their *dragsturing* actions. Then, for example at line 123, Alessio expresses the distinction between the two variables by using the labels A and B.

Moreover, Alessio associates the movements of the dependent variable to the movements of a bouncing ball (see line 119) mirroring potential expert discourse about the behavior of the function in a neighbourhood of the interval not belonging to its domain. Indeed, in a left neighborhood of 1 and in a right neighbourhood of 2, that are the two extremes of this interval, he slowly drags the independent variable as focusing the attention on the changing in direction of movement of the dependent variable. In the last excerpt the students speak about symmetry and about a parabola: it may be that the bouncing movements and the existence of a minimum lead them to evoke the image of a parabola, recalling a *precedent*.

From the last three lines, we can see that Nicco and Alessio's discourse on the dependence is still forming: they are not able to justify the fact that B cannot be directly dragged and they even try impossible dragging on it.

The aim of activity2_3bis was to mediate a transition from variation in terms of movement to variation within a set. In particular, when talking about the set of images of a function, mathematically we intend the set of possible values for the dependent variable; this definition is independent from any specific value taken on by the independent variable. So, the activation of the trace tool on the dependent variable could support the transition from the description of all possible movements to the description of the static set of values which are taken on. Indeed, the trace tool might allow to overcome the temporal dimension, since it leaves a static mark on the screen. From the episodes analyzed it seems that Nicco and Alessio realize the domain and the set of images of the function dynamically in terms of possible or impossible movements (see lines 68, 140); but also statically referring to possible or impossible positions (see lines 116, 121, 123).

The discourse on injectivity, which starts to arise when students talk about the parabola and the bouncing ball, during the third lesson comes to light, according to the goals of the lesson. For example, in the excerpt 6.16 we will see that during the activity3_1 Nicco distinguishes between the extreme points which the dependent variable takes on for only one x-value and all the other values which have at least two pre-images.

Excerpt 6.16 - Lesson 3

(Realization DGpp of the function $f(x) = x + \frac{3}{x-3}$)

	When	Who	What is said	What is done	Potential expert discourse	Code
34	I3C2 11:14	A	But we must see how B moves in function of A. Then... then... then, after five... After that A passed over five, B moves in the same direction... in the same direction, because if you move	Activity3_1 Nicco drags A to the right and then brings it back fast	$f(x)$ depends on x. In a right neighborhood of 5 the function is strictly increasing while in the interval [0; 4]* the function is decreasing (**[1.5; 3) and (3; 5]*)	IN/ DEP MON MAX/ MIN

			it [x] in here... do you see that it [f(x)] does move in the same direction? Instead try from zero to four, go to zero. Go! See you are an inept! Do you see that it moves in the opposite direction? Instead, when it [x] arrives at five, go to five, it [f(x)] starts moving in the same direction		The derivative changes the sign at $x = 5$, which is a relative minimum point	
35		N	After five?			
36	I3C2 12:25	A	No, yes…..after that A passed over five	He laughs and he stresses A		
			[...]			
54	I3C2 22:53	N	All the values can be obtained in two different ways	He drags A forward and backwards	All the $f(x)$-values have two distinct pre-images The function is non injective	INJ
55		A	What?			
56	I3C2 23:14	N	It is possible to have all the values in two different ways….all the values except for the minimum…		All the $f(x)$-values, except for the minimum, have two distinct pre-images The function is non injective	INJ
			[…]			
81	I3C2 27:55	A	yes, that is, in the interval where A moves from one to five, B moves from minus zero point five to six point five, but I do not know	Nicco holds the mouse	$f(1) = -0.5; f(5) = 6.5$ The function has a vertical asymptote at $x = 3$	ASY
82		N	No, while A moves from one to five, B moves… that is, how can be			
83		R	But it does not move			
84		N	Ehm, exactly! It is as if it arrived at an infinite point over there and then it started from infinite point here		$\lim_{x\to 3-} f(x) = -\infty$ $\lim_{x\to 3+} f(x) = +\infty$	LIM ASY

85	I3C2 28:20	A	As if it moved around and came back			
			[…]			
127	I3C2 38:55	R	So, B is leaving this trace, how can this fact help us?	She suggests them to activate the trace on B		
128	I3C2 39:35	A	To see where B passes throw. Then, as we said, B first passes everywhere, it stops at six point five and at minus zero point five. And then it follows, that is, in the time interval in which A is between zero and five, B moves in the opposite direction to A, while when A passes over five B moves in the same direction. We have seen this.	Alessio holds the mouse and he answers to R. He drags A from 5 to the right and to the left, then he drags A from 1 to -1 and vice versa for several times.	The set of images is all R except for the interval (-0.5; 6.5). For x∈[0; 5]* the function is decreasing, for $x >$ 5 the function is increasing (**In the intervals [1.5; 3) and (3; 5] the function is decreasing*) The function has a relative minimum point at 5	RAN MON MAX/ MIN

In the excerpt 6.16 we can see that Alessio describes the DIM focusing on the behavior of the function, expressing its intervals of monotonicity in terms of relative movements of the two variables and his discourse is rich in references to time and to the position of the ticks on the line. His discourse is mirrored by potential expert discourse about the dependence relation between A and B. Indeed, he directly explicates this relation ("*B moves in function of A",* line 34), or he refers to it by mentioning the role of time ("*when A…B…",* line 36, 81, 128*;* "*while A…B…",* line 82).

At line 85 of the excerpt there is an example of possible realization of the vertical asymptote, which is the dependent variable "*moving behind the screen"*. Furthermore Nicco, in the line before, speaks about the same variable going to negative infinity and then to positive infinity mirroring potential expert discourse about the mathematical object of limit, which in this case is $\lim_{x \to 3\pm} f(x)$.

In the last line of the excerpt, after that the researcher suggested to activate the trace on $f(x)$ and asked them what it could be used for, Alessio's discourse has some seeds of possible realizations of all the main properties of the function. For example, his discourse is mirrored by potential expert discourse about the set of images of the function, about the maximum and minimum values and about the intervals of monotonicity; in fact he distinguishes between concord or discord movements of the variables.

While the students' discourse on the dependence relation between the two variables is almost the same as potential expert discourse, in the excerpt below they are mainly involved in describing the behavior of the function and its derivative.

Excerpts 6.17 - Lesson 4

(Activity4_1: realization DGpp of the functions $f(x) = \begin{cases} 7, \ x < 5 \\ 3 + floor(x), \ else \end{cases}$

and $g(x) = \begin{cases} \frac{5}{2}x, \ x < 6 \\ \frac{1}{2}x + 12, \ else \end{cases}$;

Activity4_2: realization DGpp of the functions $f(x) = x^2$ and $g(x) = |x| + \frac{3}{2}$)

	When	Who	What is said	What is done	Potential expert discourse	Code
23	I4C2 16:00	A	This is half an hour	Activity4_1 The grid is shown. Nicco holds the mouse		
24		N	Yes but it is always increasing, look		The function is strictly increasing	MON
25		A	It increases but look, it increases and here it [f(x)] slows down and it [f(x)] reaches the other [x]		The function is increasing but the derivative is not constant. There exist a x-value such that $f(x) = x$	MON
26		N	Twenty-four hours, ah, in a day			
27		A	In a day he pays, in a day he pays twenty-four, he pays twenty-four euros per day It is enough…….Sorry but..he pays, pays eighteen euros twenty-one hours	The ticks are fixed: A = T_A = 24 Then Nicco drags A to 0	$f(24) = 24; f(21) = 18$	
28		N	Yes, it is okay but he pays twenty-four euros in twenty-four hours			
29		A	Yes, but we must describe the motion of the line, we do not have to... do you see that here it		The difference quotient increases and then it decreases.	DER

			goes faster and here it goes slower? You don't understand... see here, he pays, see here, see here [0] they are equal, it surpasses it and then it reaches it, then, no, we don't write like that		$f(0) = 0$ The derivative is not constant	
30		N	Anyway, he pays one euro per hour		The difference quotient is 1	DER
31	I4C2 17:25	A	On average, but do we have to average? Is there the word media written? No, there is written to describe and comment all the information that it is possible to gain		The average rate of change is 1	DER
			[…]			
106	I4C2 37:37	A	Ah I understood, but we must speak about sets, not a point, points where B and D are not	Activity4_2 He has just read the task	We look at the set of images and not at the set of points $(x, f(x))$	RAN
107		N	We can put B and D together, and C and A together, no probably only these two [B and D]	They drag the B-line overlapping it to the D-line		
108		A	Then, not here because D goes… no, actually it depends	He drags C obtaining B and D overlapped and says "not here" but then he drags A in order to have A=C and he seems confused		
109		N	These two [B and D] must have the same value and these [A and C]?!			
110		A	That is, possibly, from one to less		For example, I have to find	MON RAN

			than two this one [B] goes backward and, this [D] from one to three goes forward and so they don't have points in common, is it like this that you mean?		an interval [a; b] where the function f is decreasing while the other function g is increasing and $f(a) < g(a)$	
111		R	Yes, but did you always look at D or one time at B and the other time at D?			
112		A	Ehm, once B and once D, do we have to do like this, no? So... from zero on B goes forward, but also from... so he, A always goes forward, that is, B always goes forward so we have to see when, in which range of C, D goes backward, because if B always goes forward	He drags A from zero to the right and then to the left	In a right neighborhood of 0 the function f is increasing, actually, it is always increasing	MON
113		R	For example, if there is			
114		A	But B does not reach zero, B arrives at a half, therefore we have to see which is in C, the time interval when D goes from a half to zero	He drags C in a neighborhood of 0 while A=0, B=3/2	$f(x)$ never vanishes: $f(x) > 0$; so we have to find the pre-images of [0; 0.5] with respect to the function g	RAN
115		R	Okay			
116	I4C2 39:34	A	Ehm but how do I write it?	Then they write what follows:		

				 Fig. 6.17a		

Activity4_1 involves an always increasing function, as mirrored by Nicco's discourse at line 24. However, they do not just observe this propriety, indeed in the following line Alessio puts the focus on the different speeds of the two ticks, which move in the same direction with $f(x)$ in general bigger then x, but there exists a value such that $x = f(x)$. His discourse is mirrored by potential expert discourse about the changes in slope of the graph, even if the function keeps growing. Moreover, Alessio's detailed description of relative motions of the two variables and of possible changes in speed of the dependent one with respect to the independent one is a possible realization of the derivative of the function as the limit of the difference quotient. Probably, it is thanks to his attempt to describe as much as possible information about the DIM, which is the aim expressed by Alessio at line 31, that his discourse is so rich of realizations of mathematical objects.

Activity4_2 has been designed to let students deal with the difference between the set of images and the set of points belonging to the graph. At line 106, Alessio's discourse is mirrored by potential expert discourse about the set of image of the function which is different from the set of points $(x, f(x))$. Then, at line 110, he explains to Nicco what they have to do to find the solution, by making an example of possible movements of the two dependent variables, and at line 112 he tries to adapt his example to this particular case where "*B goes always ahead*". Finally, at line 114 Alessio finds the solution which he expresses dynamically and referring to the time interval when D moves.

In the last line of the excerpt Alessio admits that he does not know how to write down their findings, highlighting how much the mediation of the DIM and of gestures facilitates the communication.

Excerpts 6.18 - Lesson 5

(Activity5_1bis: Realization DGp of the function $f(x) = -x + 5$;

Activity5_3: Realization DGc of the function *ad hoc* defined $f(x) = -\frac{x^2}{25} + x + 1$)

	When	Who	What is said	What is done	Potential expert discourse	Code
55	I5C2 21:30	N	But it is the same	Activity5_1bis He refers to Activity5_1		
56		A	It is the same, but there is just one line… we are smart! […]			

57	27:00	R	Did you compare the two files together?			
58		N	Yes, according to us they are the same but just on one line		They are two realizations of the same function	
59		A	That is, the second is just on one line			
60		N	Also here they intersect at two point five		$f(2.5) = 2.5$	
61	I5C2 27:30	A	Because we made the grid and the numbers too, and they intersect on two point five and before we found that A plus B equals five, in fact		In both cases the algebraic expression of the function is $x + y = 5$	
			[…]			
109	I5C2 51:00	A	Look, look, look, when it [x] arrives at fifteen, that is, there an increasing of production, then, wait	Activity5_3 He drags A and stops on 1, on 2, then continuous dragging until B goes down	The function has a relative maximum point at 15* (**x=13*)	MAX/ MIN
110		N	No it goes down even before		The function is decreasing	MON
111		A	No it goes down at fifteen		In a right neighborhood of 15* the function is decreasing (**x=13*)	MON
112		N	it goes down even before	He takes the mouse	The function is decreasing	MON
113		A	It's true, it's true, at thirteen it goes down, go, go, no, zoom in a little		In a right neighborhood of 13 the function is decreasing	MON
114		N	It goes down after thirteen		In a right neighborhood of 13 the function is decreasing	MON

115		A	For me it goes down after twelve and a half… so, it goes up, up, up, from when is it still? So, it goes up until…from twelve to thirteen it is stationary at seven point twenty-five	He takes the mouse and zooms in, he starts dragging from A=11	The function increases until reaching the value $f(x) = 7.25$ for a certain x around 12 and in a right neighborhood of 12.5 the function is decreasing	MON
116		R	Then, describe and comment all the information	She reads the task		
117		N	The maximum to reach it is seven point twenty-five tons, because then it comes back	He talks to R	The maximum value is 7.25	MAX/ MIN
118		A	That is, it is always increasing, then from twelve to thirteen it is stationary	He continues dragging	The function is strictly increasing and it is constant for $x\epsilon[12; 13]$ The derivative is positive and then it vanishes	MON DER
119		N	It arrives at a maximum point and then		The function has a relative maximum point	MAX/ MIN
120	I5C2 52:55	A	From twelve to thirteen it is still seven point twenty-five, and then the production starts decreasing		For $x\epsilon[12; 13]$ $f(x) = 7.25$ In a right neighborhood of 13 the function is decreasing	MON

The first episode in the excerpt shows that Nicco and Alessio look at DGc in activity5_1 and DGpp in activity5_1bis as two different realizations of the same function. In particular, their *saming* among the two types of graphs seems to arise from their observations that in both cases $f(2.5) = 2.5$ and also that the algebraic expression of the function $x + y = 5$ holds.

Then, during the activity5_3 Nicco and Alessio describe several features of the function. In particular, in the excerpt 6.18 we can see that they focus on the maximum value that the dependent variable can take on and they express it with respect to the value of the independent variable. Moreover, their description concerns the behavior of the function in a neighborhood of that value and Alessio's discourse is especially about the dynamic relations between the two variables. He expresses the movements of the dependent variable at lines 113, 115 in terms of "*going up/down*", then at line 118 as "*increasing*" and, finally, at line 120 he translates the information in relation to the task saying that "*the production* *starts decreasing*". In this way he succeeds in realizing the intervals of monotonicity of the function.

In lines 115, 118, Nicco and Alessio observe that the function "*increases and from twelve to thirteen it becomes stable*", mirroring potential expert discourse about the derivative of the function which is positive and at x=12 it vanishes.

6.2.3 Nicco and Matilde

During the last two lessons and the interview Nicco worked with Matilde and from the video transcripts we can see that, in general, Matilde speaks more than him and, usually, she also brings new ideas in their discussions.

In their discourse it is possible to identify many seeds of possible realizations of mathematical signifiers, as described in **Table 6.3**.

Differently from other students of the class, Nicco and Matilde's discourse is not mirrored by any potential expert discourse about the domain of the function, when solving activity7_2 and activity8_1. Indeed, they seem to focus on some other properties of the functions, as their behavior in terms of intervals of monotonicity and asymptotic behavior. It also seems to be a good choice of focus for them, because dealing with these properties may be a key element in order to complete the activities. In particular, the main goal of these two activities is to support the passage from a dynamic realization to a static one and it could be done by looking at different aspects of the function, but the dynamic realizations DGpp and DGc seem to support Nicco and Matilde's discourse on these specific aspects (monotonicity properties, derivative and asymptotic behavior of the function) contributing in this way to their success in the activities.

Activity	Type	IN/DEP	DOM	RAN	INJ	MON	MAX/ MIN	LIM	ASY	DER
7_1	DGc-SGc									
7_2	DGc-SGc									
8_1	DGpp-SGc									
8_2	SGc									
8_3	SGc									

Table 6.3. Mathematical objects in Nicco and Matilde's discourse

The first excerpt contains two episodes characterizing the first time that these two students worked together.

Excerpt 6.19 - Lesson 7

(Activity7_1: realization DGc of the function $f(x) = \frac{1}{10}\left(\frac{x}{2}+4\right)(x+1)(x-2)+\frac{5}{2}$;

Activity7_2: realization DGc of the function $g(x) = \frac{x}{2}+\frac{3}{x-3}$ and realization SGc of other four different functions)

	When	Who	What is said	What is done	Potential expert discourse	Code
41	I7C2 20:55 Fp1 12:40	N	So, for some x-values, f(x) stays within a certain interval, after that values it [f(x)] goes to infinity	Activity7_1  Fig. 6.19a	For a certain interval in the domain the function is limited. $\lim_{x\to+\infty} f(x) = +\infty$	RAN LIM
42		R	Okay. You should try to write this here	Pointing to the sheet of paper		
43		M	You have to see where is the point the first time, wait, here it is okay [f (0)]... so, therefore, it must do like this, if everything is fine... no, please don't do it, then it does so, so until getting here, [*a point in the bottom-left part of the screen*], well did you understand how it does?! I think so, I hope so, because otherwise I don't know from where to start! Let's try again, wait	Discrete dragging: she stops when x is at 0, then at -1, at -6 and then continuous dragging to the left. With the finger she traces a curve on the screen, from the second quadrant going down and a bit to the left		
44		R	What are you doing with your finger?	Matilde drags from $x = 0$ to the left		
45		M	I am keeping the point/place, more or less now.. but for me, it does not go down again	 Fig. 6.19b		
46		N	It ever goes down, but not straightly		$\lim_{x\to-\infty} f(x) = -\infty$	LIM

47		M	Wait, let me think, yes, like this			
48		N	For me, it does the same also over there	She drags from $x = 0$ to the right		
49	I7C2 22:45	M	No, because over there it doesn't go down, so it goes up		The function increases for positive values of x	MON
			[…]			
102	I7C2 35:00 Fp2 09:45 I7C2 35:56	M	Because then, sincerely, the values of... of... because in this case [A] it is as if the g(x)-values were constant because it moves in this way, so it is as if only x changes and g(x) remained still, and instead in this case [in the dynagraph] also g(x) moves in relation to x, therefore going, going down or, anyway, since the result decreases, that is, being in relation and g(x) remaining still, here it is the trace of the point while on the other part [for positive x-values] it cannot be this [A] because, mmm... are we sure?	Activity7_2 She explains to R why she thinks that the answer is not A but C, without dragging, only looking at the paper	In A the function has an horizontal asymptote: $\lim_{x \to -\infty} g(x) = 0$	ASY LIM DER

In activity7_1, which presents the realization DGc of a function, students are asked to draw the trajectory of the point $(x, f(x))$ on a sheet of paper. Matilde uses her finger "*to keep the point/place*", as she says at line 45 of the excerpt 6.19, while dragging x along its axis in the DIM. We observe that in Italian, the word 'point' can be used to indicate the point $(x, f(x))$ but also to indicate a specific place on the screen, and since it is not clear what Matilde refers to, we translated it into point/place. Actually, what happens seems a combination of the two possibilities, indeed she is keeping trace of the position on the screen of the point $(x, f(x))$. In particular, she moves her finger on the computer screen following the trajectory of $(x, f(x))$, even if this point is not visible on the screen. Meanwhile she also holds the mouse and drags x from right to left, combining continuous and discrete dragging, proving to have a high control over what moves in the DIM. In particular, in order to transition from a realization DGc to a realization SGc of the function, she uses the mediation of this dragsturing action with a twofold role: communicating with her partner (line 43), and discovering what shape the curve should have.

Moreover, we observe that at line 41 Matilde's discourse is quite the same of potential expert discourse. Indeed, her descriptions of the movements of the dependent variable, such as "*it goes to infinity*" and "*it ever goes down*" (see lines 41 and 46), are mirrored by potential expert discourse about the behavior of the function at the extremities of the domain and in this case it is an unlimited function.

In the second part of the excerpt (see line 102) she works on activity7_2 and her discourse is mirrored by potential expert discourse about the asymptotic behavior of the function $\frac{1}{x-3}$ for large x-values, since she draws attention to the small variations of the dependent variable with respect to the variations of the independent one. In particular, she says "*as if g(x)-values were constant*" and "*as if only x changes*" referring to one of the Cartesian graphs on the sheet of paper and then she contrasts this particular property with what happens in the realization of another function in the DGc where "*g(x) moves in relation to x*". So Matilde seems to focus on the different range of variations of the two ticks, mirroring potential expert discourse about the derivative of the function seen as the limit of the difference quotient.

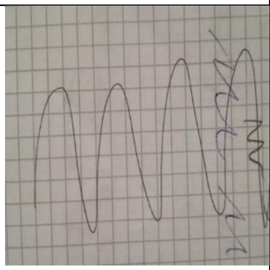
Excerpt 6.20 - Lesson 8

(It is given a description of different properties of a function to draw its realization SGc)

	When	Who	What is said	What is done	Potential expert discourse	Code
121	I8C2 40:50	N	So, for me, here before zero if x increases, also f(x) increases, so it should be a line	Activity8_3 He reads the first description of the function in the task: "Before zero, if x increases, also f(x) increases"	For $x < 0$ the function increases, so it should be a line* (**the graph is not necessarily a line*)	MON
			[...]			
138	I8C2 43:55	M	They intersect at three point five and some f(x)-values can be obtained in just one way, other in two ways, other in three ways and other in four different ways..so, at a certain point there is a situation of equality, in a certain sense	She reads the description of the function in the task: "They intersect at 3.5 and some f(x)-values can be obtained in just one way, other in two ways, other in three ways and other in four different ways"	$f(3.5) = 3.5$ The function is non injective, so an horizontal line may intersect the Cartesian graph in more than one point	INJ

				Then she moves one hand horizontally		
139		N	Or something like this, that is, for f(x) equals minus two, it [x] can be zero or it [x] can be three		-2 has two pre-images that are 0 and 3: $f(0) = -2 = f(3)$	INJ
140		M	Yes yes, but this one I don't know how, when x is bigger than six….they have opposite directions, what does it mean?	She reads the description in the task: "when x is bigger than six, they have opposite directions"		
141		N	For me, it means that this one [x] is positive and this one [f(x)] is negative			
142		M	If they intersect at three point five			
143		N	Yes but it could be that			
144		M	Wait, when x is bigger than six, they have opposite directions, no, it means that it [f(x)] comes back		For $x > 6$ the function decreases	MON
145		N	Yes			
146		M	I got it, I got it, but something happens somewhere, some f(x)-values, here [Fig. 6.20a] we have the f(x)-values obtained in one, two, and three ways because here is the same as here, here is the same as here, here is the same as here… and then, before zero if x increases, also f(x) increases, so it is in this way, when x is bigger than six they have different directions, so I had to	She points to some x-values with the same image:  Fig. 6.20a She reads the description in the task: "*man mano che x va avanti si muove sempre di più ad esempio se* va da cinque"	There are different x-values having the same image. $f(x)$ takes all negative and positive values* (**smaller than 10)* There are two different pre-images of 3.5: $f(3.5) = 3.5$ and $f(7) = 3.5$	INJ RAN MAX / MIN

			do it bigger, anyway, if x goes this way [to the right] f(x) goes to negative numbers, as x goes on, it moves more and more, for example if it goes from five, here we are, we have to do this one, f(x) takes on all negative and positive values…ah we should get there, wait… Then we draw it better! They intersect at three point five, indeed…this [(3.5, 3.5)] is the intersection point at three point five, some values can be obtained in just one way, other in two ways, indeed here [(3.5, 3.5)], for example, we have three point five and we have it also here [(7, 3.5)], but here [on the x-axis at 3.5] it is three point five and here [on the x-axis at 7] it is a bit more than six, it will be about seven	She extends the curve in order to obtain the maximum value for $f(x)$ at 10	The function is non-injective	
147		N	But other values in three or four ways, this graph is not good, there must be some strange turns somewhere	He moves one hand making circles on the sheet of paper:  Fig. 6.20b	There should be more different intervals of monotonicit y And more relative extreme points	MON MAX / MIN
148		M	No, also here [(0, -5)], if f(x) is about five here, it is the same here [(9, -5)], wait, this is minus five and	Her gestures indicate the horizontal line $y = -5$	The pre-images of -5 are 0 and 9	INJ

149	I8C2 47:30	N	But there are two, there are only two ways in our graph			
			[…]			
154	I8C2 48:40	M	Ehm, there could be something like this [Fig. 6.20c]….but the fact that it moves faster and slower	 Fig. 6.20c During the pause she reads the description in the task: "Man mano che x va avanti $f(x)$ si muove sempre di più"		INJ DER
155	I8C2 49:15	N	Eh, they are these lines [Fig. 6.20d], they could be one a little more steep and the other less steep… the problem is that I don't know which one should be more, and which one less, steep	 Fig. 6.20d	The curve in the Cartesian plane should be more and less steep The faster the y grows the steeper the curve is, because the derivative increases	DER

We find very interesting to analyze Nicco and Matilde' discourse when working on the last activity, since it contains the seeds of many mathematical properties of functions. In the excerpt 6.20 students are asked to draw on the Cartesian plane the graph of a function with some given properties, so they have to read and interpret all the properties described on the paper and translate them in the realization SGc of a function. In some parts of the excerpt the students read the description given in the activity, but there are also many other parts where they rephrase it with their words or they use it to gain information about the curve. For example, Nicco explains the increasing of both variables simultaneously as if the graph in the Cartesian plane was a line and he expresses it at line 121.

At line 138, Matilde, after reading the description in the task, talks about a "*situation of equality*", moving her hand horizontally as referring to a horizontal line intersecting the curve in the Cartesian plane in at least two points. The potential expert discourse mirrored in this case is about the non injectivity of the function, as suggested by her dragsturing action involving a horizontal line intersecting the graph of the function. Nicco answers her by showing an example, indeed he fixes a f(x)-value, "*f(x) equals minus two*", and he expresses two different x-values for which f(x) takes on that fixed value. Since the subject of his discourse is f(x), the potential expert discourse mirrored is about the pre-images of that specific f(x)-value. Then he supposes that if the variables are oppositely directed then they have different signs, while Matilde's discourse is mirrored by the potential expert discourse about the interval of monotonicity of the function where it decreases. In particular, when, she says that the dependent variable "*goes back*" (see line 144). At line 146 she checks if their drawing fits the descriptions in the task, by worrying in particular about the properties that in the potential expert discourse are described as the non-injectivity and the maximum. Nicco supposes the existence of "*some strange turns*" in order to satisfy the condition which increases the number of pre-images to three and four, because in their graph *"there are only two ways*" (see line 149).

Another realization of the non-injectivity of the function is proposed at line 154 by Matilde, but her drawing in Figure 6.20c does not fit the other conditions that are given in the task, for example "*the fact that it moves faster and slower*". Then, in line with our *a priori* analysis, Nicco seems to succeed in *saming* between changes in the speed of the dependent variable and changes in the slope of the curve. In particular, he explains it to Matilde at line 155 with the visual mediation of gestures (see Fig. 6.20d) and his discourse is mirrored by potential expert discourse about the derivative of the function that defines the slope of the curve in the Cartesian plane.

The following excerpts are taken from the interview with Nicco and Matilde.

Excerpt 6.21 - Interview 2

(Realization DGpp of the function $f(x) = \begin{cases} \left|\frac{(x-1)(x-4)(x+8)}{8}\right| - 6, & x < 4.94 \\ \frac{x-5}{5}, & else \end{cases}$)

	When	Who	What is said	What is done	Potential expert discourse	Code
1	MNm 2 00:07 C 15:15	N	The zero for f(x) is the value minus seven point five of x, so f(x) equals zero when x equals minus seven point five	Task2 with Nicco at the pc and Matilde drawing.	$f(-7.5) = 0$	
2		M	Okay, no wait.....when x is minus seven point five and not f(x)?	She drew the Cartesian plane on the paper and puts a point in it		
3		N	Yes! Then, f(x) never goes under minus six, it is always greater than or equals to minus six, f(x) greater than or equals to minus six.... f(x) is between... because it also never goes over an interval...more or less it will be fourteen point two or fourteen point twenty-five	She writes this information in a small space in the bottom part of the paper	$f(x) \geq -6$ $f(x) \leq 14.2$	RAN
4		M	So, between minus six and?			

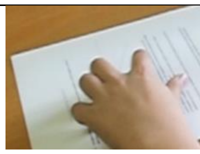
5		N	Between minus six and fourteen point twenty-five…. And when x, no, fourteen point five, and when x is in the interval	Zoom in and out	$-6 \leq f(x) \leq 14.5$	RAN
6		M	But also equals to fourteen point five?		But can $f(x)$ takes on 14.5?	
7		N	Also equals, less than and equals to, when f(x) is at, at, that is, fourteen point five, which is the maximum, x equals minus four point five….. while, when it is at minus six, so the end of the other interval, x equals minus eight	She draws other two points	$f(x) \leq 14.5$ The pre-image of 14.5, which is the maximum, is -4.5 While the pre-image of -6 is -8	RAN MAX/MIN
8		M	Can you tell me some points?!			
9	MNm 2 03:15 C 18:25	N	No, wait, because this graph is strange…it is very strange! No, f(x) exists also after fourteen point five			
10		M	That is, does it break for a while?	She moves the pe in the air as drawing a short segment		
11		N	Then, by going to the positive [numbers], no wait… that is, because x arrives at minus four point five, then f(x) arrives at fourteen point five, then if I keep dragging it backward		$f(-4.5) = 14.5$	
12		M	If you keep dragging backward, x?			
13		N	If I still move x backward, f(x) comes back	He moves his left hand to the left on the table for two times	The function is increasing in a left neighborhood of -4.5	MON
14		M	It goes down because you told me that, when x is minus eight, it [f(x)] is at minus six		$f(-8) = -6$	
15		N	Exactly, it comes back at minus six and then it goes up again, it goes up until, it doesn't stop anymore	Now he moves the same hand to the right while dragging x to the left	In a left neighbourhood of -8 (because f(-8)=-6) the function decreases	MON LIM

					while in a right neighbourhood it increases $\lim_{x \to -\infty} f(x) = +\infty$	
					So the derivative changes the sign and -6 is a relative minimum value for f	
15 bis		M	Can you tell me some points? I have three of them!			
16		N	f(x) goes to one, so, when f(x) equals one.. it [x] is equal to minus eight point five	Zoom in and out	The pre-image of 1 is -8.5	
17		M	x?			
18		N	When x equals minus eight point five, f(x) is one…. Then, when x is about minus six, f(x) equals twelve… but f(x) equals twelve also when x equals minus three point twenty-five	She draws other two points	$f(-8.5) = 1$ $f(-6) = 12$ but there is another pre-image of 12 that is -3.25	INJ
19		M	Is it [f(x)] always equal to twelve? I mean, f(x) equals twelve on [x equals] minus six			
20		N	f(x) equals twelve			
21		M	on x that is on minus six			
22		N	Yes, about minus six, and on minus three point twenty-five… so it twill be a… Ah, f(x) equals zero also when x equals five	She draws other points and he moves his hand like making a small hill in the air after saying “it will be a”	Another zero of the function is at $x = 5$	INJ
23		M	Five or minus five?			
24		N	Five!			
25		M	Okay, can you tell me some points over there, like for positive x-values?			

26	C 21:50 MNm3 00:00	N	When x equals one, f(x) is minus six ... .then, ah that's why here [in a neighborhood of x = 3] it goes back, when x is minus two point five, when x is two point five, f(x) is minus three. I have no idea about what a strange graph it is!	He drags x from 1 to 4 rapidly and then slowly to 2.5	$f(1) = -6$ The function is increasing, $f(2.5) = -3$ and then it is decreasing The derivative changes the sign and the function has a relative maximum	MON MAX/MIN
27		M	Go on in this way			
28		N	Then, x arrives at four, probably I already told it to you, f(x) equals minus six		$f(4) = -6$	
29		M	No! Yes! Go even more far, like about seven or eight			
30		N	I put it on one, when x equals ten, f(x) equals one, it goes slower, passed over zero, then since x passes over five f(x) passes over zero, f(x) goes slower, maybe it's slowing down and...		I search for the pre-image of 1, $f(10) = 1$ Then for $x > 5, f(x) > 0$ and the function increases more slowly The difference quotient decreases	MON DER
31		M	When x is ten			
32		N	When x is ten, f(x) equals one		$f(10) = 1$	
33		M	One or minus one?			
34		N	one.. while minus one, when x….. then f(x) is minus one when x equals four point seventy-five	Discrete dragging	The pre-image of -1 is 4.75	
35		M	f(x) is minus one, when is x four point seventy-five?			
36		N	When it is four point seventy-five, yes, but it [f(x)] is minus one also when.. it's better if I say that, when x equals zero, f(x) is minus two… there is another minus one	She starts tracing the curve, without marking the paper	But there is another pre-image of -1 $f(0) = -2$	INJ
37		M	Go to f(x) about five, six			

38		N	Mmmm….. Eh, I don’t know if you have enough space, f(x) is five when x is thirty, it should be something like this, then	He points to the graph in Task1 on the sheet of paper	The pre-image of 5 is 30	
39		M	But is f(x) one when x is ten?		Is 10 the pre-image of 1?	
40		N	Yes, it [f(x)] is one when it [x] is ten and when it [x] is, because there will be another one, when it [x] is ten		Yes, but there are two distinct pre-images of 1	INJ
41		M	When it [x] is minus eight point five			
42		N	Also when it is minus, yes, but then there is also another one too, when it [x] is minus zero point seventy-five	Zoom in	Another pre-image of 1 is -0.75	INJ
43		M	When is x minus zero point seventy-five?			
44		N	f(x) is one			
45		M	Go to the negative numbers, with both x and f(x)			
46		N	So, f(x) doesn’t go under minus six		$f(x) \geq -6$	RAN
47		M	Okay, before. You only gave me one point [of the form (x, f(x))] x at minus eight and f(x) at minus six			
48		N	Eh because then it dashes!	He drags x to the left	The function decreases quickly The difference quotient takes on an high values –in absolute value- for $x < 8.5$	MON DER
49		M	What does it mean that then it dashes? Is there nothing in the middle? Like at minus two, minus three			
50		N	X minus two? when x equals minus two, f(x) is seven point five	Zoom in	$f(-2) = 7.5$	
51		M	When x is minus two?			

52		N	f(x) is seven point five			
53		M	But is there nothing before minus eight and minus six?			
54		N	Before minus eight			
55		M	Before that x is minus eight, I mean			
56		N	Ah before that x is..?			
57		M	Yes, that is, about seven point five or minus six			
58		N	Minus six, yes, ah no, negative x? No when x equals minus six, f(x) is eleven		$f(-6) = 11$	
59		M	No, what did you say? Repeat, repeat, when f(x) is minus six			
60		N	No, when x is minus six, f(x) is eleven point five	She points (-6,0) and then moves the hand up and to the right to intersect the y-axis at about 12	$f(-6) = 11.5$	
61		M	And sorry, when x is minus three point twenty-five?		$f(-3.25)$?	
62		N	f(x) is twelve point twenty-five…. Did I tell you about minus eight, right? minus eight for x			
63		M	Minus six for f(x)?			
64	C 29:15 MNm3 07:25	N	Yes then, I can give you f(x) equals zero again.. when x is minus seven point five, f(x) equals zero	She moves the pen on the paper as drawing the curve for negative x-values but still without leaving any trace	I can tell you another pre-image of 0 $f(-7.5) = 0$	INJ
			[…]			
72	MNm3 08:35	N	For example, there is x equals twenty-five, f(x) is four, it moves slowly, I told	He points to the graph for $x > 0$ in Task1:	$f(25) = 4$ and the function increases slowly	MONDER

			you that it is like this here [Fig. 6.21a]	 Fig. 6.21a	The difference quotient takes on a small value in a neighborhood of 25	
73		M	Repeat it! f(x) is four?			
74		N	f(x) is four when x is twenty-five.. I also have f(x) equals three if you want, such that x equals twenty…. Ah it moves five by five, basically, every five x, f(x) makes one.. so, when f(x) equals two, for example, f(x) equals two it [x] is fifteen, when f(x) equals three, it [x] is twenty, yes, when x equals twenty, f(x) equals three, when x equals twenty-five f(x) equals four, then five, six	Discrete dragging	The pre-image of 4 is 25; the pre-image of 3 is 20. The ratio between the variations of $f(x)$ and x is 1 to 5. The pre-image of 2 is 15; the pre-image of 3 is 20, $f(20) = 3$ and $f(25) = 4$	DER
			[…]			
78	C 33:00	N	Ah yes yes, when x equals minus five, f(x) arrives at fourteen point twenty-five, about fourteen point five, and then, as x keeps going to negative numbers, f(x) goes down again, it comes back at minus six when x is at minus eight and then, passed over minus eight, so when x passes over minus eight, f(x) goes up to positive numbers again, eh, here it [f(x)] goes fast…it [f(x)] goes very fast		$f(-5) = 14.25$ The function is increasing for $-8 < x < -5$, while it is decreasing for $x < -8$ and $f(-8) = -6$ $\lim_{x \to -\infty} f(x) = +\infty$ and the function decreases rapidly The derivative changes the sign at $x = -8$ where the function has a relative minimum and the difference quotient takes on high values at x in a left	MON LIM DER MAX / MIN

					neighborhood of -8	
79		M	But is it possible that for f(x) between minus one and minus six, when x is positive, it goes like back and forth?	She moves her right hand up/down and then to the right/to the left	Is it possible that the behaviour of the function changes for positive x-values such that $-6 < f(x) < -1$?	MON
80		N	Yes, it is possible because when x equals zero, I move it, f(x) goes backward, then, passed over x equals one f(x) goes on again		In a neighborhood of $x = 0$ the function decreasing and in a right neighborhood of $x = 1$ it is increasing The derivative changes the sign at 1 where the function has a relative minimum	MON MAX/MIN
81		M	Then, it comes back again and it goes forth		Then the function decreases and increases again	MON
82		N	Then, it comes back again and it goes forth, it goes on slowly		Then the function decreases and it increases slowly The derivative changes the sign and the function has a relative minimum.	MON DER MAX/MIN

					Then the difference quotient takes on small values	
83		M	From there it [f(x)] goes on to infinity…slowly? Doesn't it go five by five?!		$\lim_{x \to +\infty} f(x) = +\infty$	LIM DER
84		N	Five by five is slow! Here it goes twenty-three by twenty-three		Here the ratio between the variations of $f(x)$ and x is 1 to 23	DER
85	MNm 3 13:00	M	There is a little problem, this [Fig. 6.21b] could be the going back and forth, this is the rising, the fact that from zero point twenty-five, it creates me many problems, I really don't know how to fit it because in this way it seems a bit strange to me... something like this, and where do I put these [two points that she drew in the third quadrant]? Since x passes over minus one, does f(x) get to minus six?	She points to the curve for $0 < x < 5$:  Fig. 6.21b		
86		N	Since x passes over minus one?	He drags x to the right		
87		M	It [f(x)] goes down, down and then it goes up again	She moves her right hand down and up	The function is decreasing and then it is increasing	MON
88		N	f(x) goes down		The function is decreasing	MON
89		M	That is, it [f(x)] goes forward, it [f(x)] goes backward	She moves her left hand to the right and to the left	The function is increasing, it is descreasing	MON
90		N	f(x) goes backward since x arrives at one, more or less, when x passes over one		The function is decreasing in a left neighborhood of x= 1	MON
91		M	yes, and f(x) is minus six, then it [f(x)] goes forward			
92		N	when x passes over minus one, it [f(x)] goes on	She moves her left hand up	The function is increasing in a right	MON

					neighborhood of x= -1* (**x=1*)	
93	MNm3 14:20	M	At [x equals] minus three, it [f(x)] comes back		-3 is a relative maximum for the function	MAX/MIN
94		N	f(x)… yes, then it comes back at minus six			
95		M	And it [f(x)] goes forward again, to infinity, five by five		$\lim_{x \to +\infty} f(x) = +\infty$ and the ratio between the variations of $f(x)$ and x is 1 to 5	LIM DER

In the excerpt 6.21 Nicco is exploring the realization DGpp of a function and he is describing it to Matilde who has to draw the Cartesian graph of the same function on a sheet of paper. It is possible to observe that several times she asks him for some points $(x, f(x))$ (see lines 8, 15bis, 25, 37, 47) and that sometimes he expresses them starting from the $f(x)$-value and then explaining the corresponding x-value. This example of students' discourse is mirrored by potential expert discourse about the pre-image of a point belonging to the codomain of the function (see lines 16, 18, 22, 30, 34, 36, 38, 40, 64, 74). Moreover, in some of these examples (see lines 18, 22, 36, 40, 64) students focus on the number of pre-images existing for a given $f(x)$-value mirroring potential expert discourse about the injectivity of the function. Indeed, if there exist more than one pre-image the function is non injective.

There are several examples where Nicco and Matilde's discourse is mirrored by potential expert discourse about the derivative of the function. In particular, they describe changings in speed of the tick realizing the dependent variable (see lines 30, 48, 72, 78) mirroring the derivative in terms of limit of the difference quotient, but also changings in direction of the tick realizing the dependent variable (see lines 15, 26, 78, 80, 82) mirroring the changes of sign of the derivative. Consequently, in this second case their discourse is also mirrored by potential expert discourse about relative maximum/minimum points of the function. We observe that in the excerpt 6.21 the word 'maximum' is used only one time by Nicco at line 7 ("*when f(x) is at 14.5, which is the maximum, x equals -4.5*").

Moreover, students' discourse about the direction of movement of $f(x)$ with respect to the direction of x, and its possible changings, is mirrored by potential expert discourse about different intervals of monotonicity of the function. In general, the potential expert discourse can be obtained by replacing in students' discourse "*f(x) goes forward*" with "the function increases" and "*f(x) goes backward*" with "the function decreases" (see lines 13, 15, 26, 30, 48, 72, 78-82, 89-93). However, there are also some examples of students saying "f*(x) goes down/up*" (see lines 14, 85, 87, 88) and this description of the direction of movement of $f(x)$ along the y-axis refers to the two dimensional realization of the function where the y-axis is vertical. In fact, Nicco who works with the realization DGpp of the function uses the verb 'to go down' just one time, at line 88, while the other three examples are taken from Matilde's descriptions, who has to draw the Cartesian graph of the function. In all the other cases she uses 'to go backward/forward' too, as if she was translating for him her discourse about the realization SGc. In particular, at line 92 she says "*when x passes -1 [f(x)] goes forward"* while moving her hand up, so her gestures refer to the realization in the Cartesian plane and her words to the dynagraph. Moreover, it is interesting to notice that sometimes Nicco drags the tick to the left, as he explicitly says for example at lines 11 and 13 "*moving x backward"* or at line 78 *"as x goes to the negative numbers".* In these cases the description of $f(x)$ going backward (forward) is mirrored by potential expert discourse about the increasing (decreasing) behavior of the function.

Finally, Nicco and Matilde's discourse in the excerpt 6.21 is mirrored by potential expert discourse about the limits of the function for x tending to infinity. For example, students describe the behavior of the tick realizing the dependent variable through expressions like "*it doesn't stop anymore*" (line 15), "*then it goes up to positive values*" (line 78), and Matilde uses explicitly the word 'infinity' at line 83 "*from there it goes towards to infinity*" and at line 95 "*and again it goes towards to infinity*".

We also notice that Nicco's discourse at lines 38 and 72 may be considered as an attempt to identifying *precedents* within his *precedent-search-space,* in particular he seems to take task1 of the interview for a similar task situation, because he points to the graph on the sheet of paper which they dealt with at the beginning of the interview.

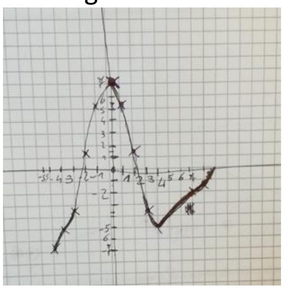
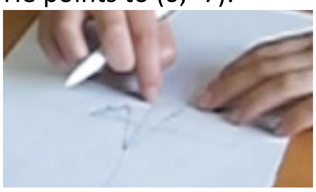
Excerpt 6.22 - Interview 2

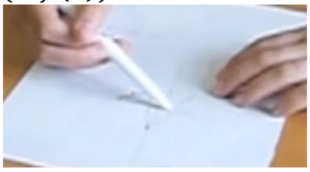
(Realization DGpp of the function $f(x) = \begin{cases} -5.18, & x < -4.6 \\ 10\frac{sinx}{x} - 3, & -4.6 \leq x < 6.2 \\ \frac{3}{2}x - 12.4, & 6.2 \leq x < 8.3 \end{cases}$)

	When	Who	What is said	What is done	Potential expert discourse	Code
11	MNm5 03:15	M	It goes up, it goes up, but in the moment that f(x) arrives at zero and x goes up more than eight and a half, f(x) disappears	Task3 with Matilde at the pc and Nicco drawing.	The function increases until vanishing and it is not defined for all $x > 8.5$	MON DOM
12		N	Wait, when f(x) equals zero, when x is equal to?		What is the pre-image of 0?	
13		M	wait… Okay, okay it [f(x)] disappeared. Did you understand this or I should repeat it for you?	She zooms out and drags x to the right, she stops at $x = 40$		

14		N	But is there a point where f(x), that is, the point x when f(x) equals zero?		Does it exist a value $\boldsymbol{x}$ such that $f(\boldsymbol{x}) = 0$?	DOM
15		M	F(x) equals zero when x is two point twenty-five		The pre-image of 0 is 2.25	
16		N	Two point twenty-five?			
17		M	yes… when x is two point twenty-five, f(x) equals zero		$f(2.25) = 0$	
18		N	So it goes up here, no, it is two point twenty-five but.. because, then, here it goes down, then it goes up, then give me another point after that it goes up again	With the pen he follows the trajectory of $(x, f(x))$:  Fig. 6.22a	The function is decreasing then it is increasing The derivative changes the sign and the function has a relative minimum point	MON MAX/ MIN
19		M	After that it goes up, where? Tell me exactly because I don't have..			
20		N	When x equals seven, f(x) should be, you told me about		$f(7)$?	
21		M	Minus two point twenty-five, minus two			
22		N	Ah.. so, when x equals seven, is f(x) minus two? Okay, and another point that goes up?	He draws the point (7, -2)	$f(7) = -2$	
23		M	Going up I have it at eight, that is like.... it [f(x)] is minus one point five and x equals eight		$f'(8) > 0; f(8) = -1.5^*$ ($^*f(8) = -0.5$)	DER

24		N	For example, x equals twelve or thirteen?			
25		M	It [f(x)] disappeares! Yes, because f(x) arrives at zero and it corresponds to eight and a half for x, so, from eight and a half on there isn't f(x) anymore… yes, there isn't	Zoom out during the suspension dots	It does not exist. The function is not defined for all $x > 8.5$	DOM
26		N	Ah okay, it disappears! Okay, tell me some negative values for x		Ah, it does not exist	DOM
27		M	X minus one corresponds to five point twenty-five… minus two, to one and a half, I'm saying minus two referring to x, I always tell you first the value of x, minus three to minus three and a half…	Discrete dragging He draws the points in the paper	$f(-1) = 5.25; f(-2) = 1.5; f(-3) = -3.5^*$ ($^*f(-3) = -2.5$)	
28		N	minus three to minus three and a half?			
29		M	Yes, of f(x)... minus four to minus five, more or less		$f(-4) = -5$	
30		N	As that in the other side, minus four of x, yes, then does it go up again?			
31	MNm5 07:25	M	And then, from, oooh, wait, keep calm…. eh it [f(x)] stops.. when x arrives	She drags x in a neighbourhood of -5.5 and then stops dragging for a while	For $x < -5.5^*$ the function becomes constant (*for $x < -4.5$)	RAN

			at about minus five and a half, f(x) stops, it doesn't go further, so it stays between			
32		N	Minus five and a half, yes, so how much is f(x)?			
33		M	when f(x) is a little bit less than minus five, it is minus six point eighty, something like that, but it doesn't exist, that is, no, it is not right that it [f(x)] doesn't exist, it stops		The function is constant: $f(x) = -6.80$ * (*$f(x) = -5.20$)	RAN
34		N	It stops and then it doesn't go on anymore? That is, does f(x) move when x goes from minus five point five to eight and a half? Then doesn't f(x) move anymore?	Continuous dragging to the right His final drawing is the following one:  Fig. 6.22b		
35		M	No! Tell me what movements it does, according to you, so, tell me when it goes up and when it goes down, please!		Tell me about the possible monotonicity of the function	MON
36		N	So, f(x) goes from minus seven	He points to (0, -7):  Fig. 6.22c	The set of images is from minus seven	RAN

37		M	No, tell me going up, down because otherwise		Tell me about the intervals of monotonicity	MON
38		N	so, when x equals minus five point five, f(x), I mean the graph, starts going up, it goes up, up, up	With the pen he follows the trajectory of $(x, f(x))$:  Fig. 6.22d	From $x = -5.5$ the graph starts going up The function is increasing	MON
39		M	Until? quickly, quickly	Continuous dragging to the right		
40		N	yes, then a bit more slowly but it's ok, until seven, and then from seven it goes down again		In a left neighbourhood of 7 the function increases while in a right neighbourhood it decreases A relative maximum of the function is 7	MAX/ MIN
41		M	Yes, until?			
42		N	Until minus five, then after minus five it goes up		A relative minimum of the function is -5	MAX/ MIN
43		M	After minus five of f(x)?!			
44		N	yes, it goes up…then you told me that it stops			
45	MNm5 09:55	M	Until disappearing at eight		$f(8)$ does not exist	DOM

Oppositely from the previous excerpt, in the excerpt 6.22 Matilde is exploring the realization DGpp of a function and he is describing it to Nicco.

Also in this excerpt there are several examples in which, especially Nicco, identifies pairs of coordinate $(x, f(x))$ by fixing a $f(x)$-value and then expressing one of its pre-image (see lines 12, 14, 15). Moreover, at line 30, again he seems to identify a *precedent* when he says "*it is like that on the other side*", probably referring to the piece of curve for $x < 0$.

First of all we observe that Nicco and Matilde's discourse about the direction of movement of the tick realizing the dependent variable is mirrored by potential expert discourse about the behavior of the function which is increasing or decreasing (see lines 11, 18, 35, 37, 38). In particular, we notice that Matilde uses the verb 'to go up/down' even if this time she is manipulating the DIM where the function is realized through a DGpp and so she sees the ticks 'moving right/left'. As we previously observed, it seems as if she was translating for Nicco her discourse about the realization DGpp into a discourse about the two dimensional realization of the same function. Moreover, there are two examples of mirrored potential expert discourse about relative maximum and minimum points (see lines 40, 42) where students focus on the existence of different intervals of monotonicity of the function.

As discussed for the excerpt 6.21 students' discourse about changes in the direction of movement of $f(x)$ is mirrored by potential expert discourse about the derivative of the function, that changes the sign (see lines 18, 22). Then, at line 23 of the excerpt 6.22, Matilde says "*at 8 there is one* [point $(x, f(x))$] *where it* [the graph] *goes up, which is -1.5"* mirroring potential expert discourse about the positive value of the derivative at $x = 8$.

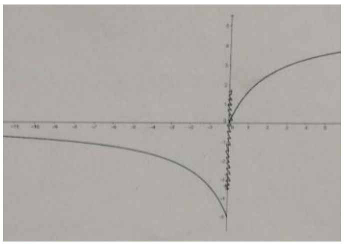
At lines 11, 25, 26, 34, 44, 45 students' discourse is mirrored by potential expert discourse about the domain of the function, which is not defined for all x bigger than 8.5. In particular, they describe the behavior of the tick that realizes the dependent variable which "*disappears*" for a certain x-value. This description of the domain is supported by the dynagraph proposed, where one of the two ticks is always visible and draggable while the other one depends on the choice of the function. In this case the function is defined as constant for all $x < 5.5$ while it is not defined for all $x > 8.5$ and so in the dynagraph, which Matilde is manipulating, for $x < 5.5$ the tick realizing the dependent variable is visible but it does not move while it is not defined for $x > 8.5$ (and so it is not visible on the screen). In the excerpt we can see Matilde's attempt to explain this difference by looking at her choice of the verbs: 'to stop' and 'to disappear'. For example she says "*when x arrives at 5.5 f(x) stops*" (see line 31) or "*when f(x) is more or less -6.80 it does not exist, no it is not right that f(x) does not exist, it stops"* (see line 33) or "*until disappearing at 8*" (see line 45). However, Nicco's discourse seems focused on the movements of f(x) which actually "*moves when x goes from -5.5 to 8.5*" as he says at line 34, but this description does to not highlight the different behavior of the function in a right neighborhood of 8.5 and in a left neighborhood of -5.5. Also in the graph that he draws (Fig. 6.22b) the function is defined only for $-5.5 < x < 8.5$, so he does not trace the curve where the function is defined as constant.

Excerpt 6.22bis - Interview 2

(Realization SGc of the function $f(x) = \begin{cases} \sqrt{-x} - 5, & x < 0 \\ \sqrt{x}, & x \geq 0 \end{cases}$)

	When	Who	What is said	What is done	Potential expert discourse	Code
11	MNm6 01:39 C 51:40	M	It is as I told you before, if x is eight, f(x) is zero, f(x) is the image of x, we never say like this	Task4	$f(8) = 0; f(x)$ is the image of x	RAN

			but I think that it works			
12		N	Ah, yes okay			
			[…]	They read the definition again		
13	MNm6 03:30	M	I'm not sure about what I told you			
14		R	For the image, you mean?			
15		M	Yes			
16		N	The value taken by f in x			
17		R	Let me know what it doesn't work for you			
18		N	The image of the interval minus one, one… that is		The image of the interval [-1; 1]	RAN
19		R	The image is what you said			
20		M	F(x), it is the corresponding one, okay, well, and we understood it			
21		R	But you said the image of a point			
22		M	Yes, for an interval it is different, I think			
23		R	That is, the image of all the points that stays within this interval minus one, one			
24		M	Anyway, so, minus one is about here… it i sas if it was the trace left by f(x)?	 Fig. 6.22e Then she moves the pen along the y -axis		RAN
25		R	When you drag x, where?			
26		M	When I drag x from minus one to one, and the trace left by f(x) is the image		The image of the interval [-1; 1] is	IN/ DEP RAN

					the set of values which $f(x)$ takes on	
27	MNm6 05:20	R	Yes	Matilde colors as follows:  Fig. 6.22f		

In the last task of the interview, the students have to express the image of an interval, by looking at its Cartesian graph on a paper and having at their disposal the formal mathematical definition of 'image of a subset of the domain'. Initially Matilde explains to Nicco what the image of a point is, through an example taken from the previous activity, then the researcher helps them to generalizing their idea and adapting it to an interval instead of a point. At line 24 Matilde proposes a realization of the mathematical signifier 'image' that evokes the dynamic environment, in fact she moves her hand up/down on the y -axis on the paper and she says "*as if it was the trace left by f(x)*". Then, at line 26, Matilde's discourse mirrors potential expert discourse about the image of an interval as the set of values which $f(x)$ takes on, and she actually marks the right interval on the paper.

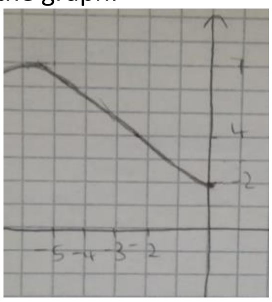
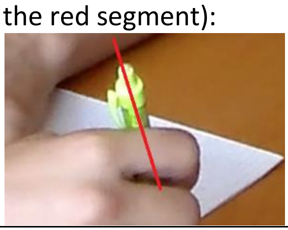
6.2.4 Excerpts from Alessio's interview

The following two excepts are taken from Alessio's interview. As discussed in Chapter 4, he was interviewed alone and the activities were not the same that the other students had. For example, in the following excerpt Alessio is working on the first task that we gave to him, that involves the realization DGc of a function and he has to explore the dynamic interactive file in GeoGebra in order to draw the Cartesian graph of same function on the paper.

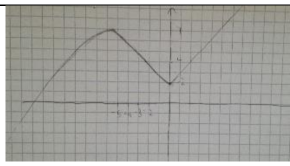
Excerpt 6.23 - Interview 1

(Realization DGc of the function $f(x) = \frac{1}{10}\left(\frac{x}{2} + 4\right)(x+1)(x-2) + \frac{5}{2}$)

	When	Who	What is said	What is done	Potential expert discourse	Code
1	C 09:45	A	so, I have to draw the trajectory of (x, f(x)), that is, f(x) moves in function of x, by moving x also f(x) moves, but can I see?	Task1 He still does not drag anything	$f(x)$ depends on x	IN/ DEP
2		R	Obviously!			
3		A	When x is zero, f(x) is one point eight more or less, then if x goes to minus	In the pause he drags x to -12 and then in a neighborhood of -5.5	$f(0) = 1.8; f(-2) = 3.8$ $f(x) \leq 6.5$	RAN MAX/ MIN

			two, f(x) goes to three point eight, can we go on? Yes, then…… f(x) doesn't go more than six point five and it reaches the maximum at minus five point seven, eight, and now I have to draw the trajectory! So.. but can I take two poits and then tracing from the two points?		The function has a relative maximum point at -5.7	
4		R	Yes	Slow continuous dragging within negative numbers		
5		A	The maximum is seven		A relative maximum is 7	MAX/ MIN
6		R	And you said that it reaches it?			
7		A	It reaches it when it is at five, more or less, so it is here [(-5,0)], I was wrong, however it is good. Then it comes back, but… I don't know but I think that a straight line is not correct	He draws on the paper the following part of the graph:  Fig. 6.23a Then he drags x to the right	In a right neighborhood of $x = -5$ the function decreases. It is not a line.	MON
8		R	Why do you think that a straight line is not correct?			
9		A	Eeeeeh then, because….. when from one point five….. so, I think that here [from negative infinity to -5] it is a straight line to… [Fig. 6.23b] But they are two different trajectories or do	In the first pause he drags x from -3 to -5 and then to 2. He moves the right hand in this way (along the red segment): 	For x in (-∞, -5] the graph is a line	

			they have to be linked together [the graph for positive and negative x-values]?	Fig. 6.23b		
10		R	Eh let's see, how can we understand it?			
11		A	linked! Eh they are linked together, since f(x) moves in function of x, there are not two x, there is just one so, yes, then when this [x] is at zero, it [f(x)] is about two… and then it goes up again, but this is to infinity, while this one……. From here, from minus one to minus two it [f(x)] moves by? About one	He drags x in a neighborhood of 0, in the first pause he draws the point (0, 1.8) and then he drags x slowly to 5. In the second pause he drags x to the left, within a neighborhood of -5 and backward/forward between -1 and -2	$f(x)$ depends on x $f(0) = 2$ The function is increasing in a right neighborhood of 0 $\lim_{x \to +\infty} f(x) = +\infty$ $\lvert f(-1) - f(-2) \rvert = 1$	IN/ DEP MON LIM
12		R	f(x)?			
13	C 15:15	A	Yes, f(x) moves by one, more or less, while from minus five to minus six…….it moves, from minus five to minus six, no, let's do from minus four to minus five, it moves by less than one, so as x decreases, that is, also the ratio of f(x) to x changes, so it cannot be like this [straight line] but it is a [curve]… that is, it's not a broken line	In the first pause he zooms in and out	The ratio of Δf(x) to Δx is 1 more or less; then $\lvert f(-4) - f(-5) \rvert < 1$ The ratio of Δf(x) to Δx is not constant, so the graph is not a line In a left neighborhood of -5 the derivative is increasing	DER
14		R	Okay, so is for this reason that you said that it cannot be a broken line? I mean, it cannot be a line	He finally draws the following graph:		

				 Fig. 6.23c		
15		A	Yes, more or less! So, also this one [the graph for positive x-values] I think… because, from one to zero…about one, no a bit less, while from one to two eh, also here, and so this is a broken line for me, but I don't know, it is a bit strange		$\lvert f(1) - f(0)\rvert$ is the same of $\lvert f(2) - f(1)\rvert$ so the graph is a line The derivative is constant	DER
16		R	Which one?			
17		A	This here	He points to the graph for positive x -values		
18		R	Ah for positive x-values, and here [Fig. 6.23d] does it go down to infinity?	 Fig. 6.23d	$\lim_{x\to-\infty} f(x) = -\infty$?	LIM
19	C 18:05	A	Yes, it seems like that, yes, at least by looking at what we can see, yes, it goes down to infinity	He drags x to the left, zooming out, to -15	$\lim_{x\to-\infty} f(x) = -\infty$	LIM

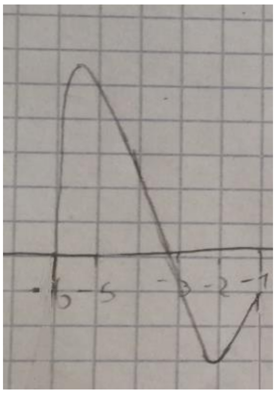
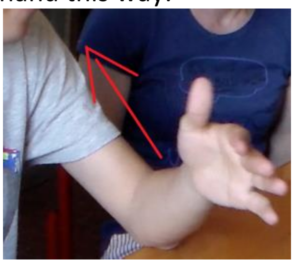
The dependence relation between the two variables is realized by Alessio in the first line of the excerpt through the expression "*f(x) moves in function of x*", that he says even before starting to manipulate the DGc.

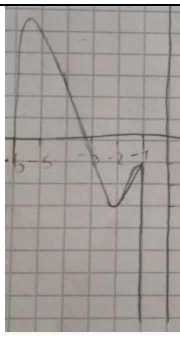
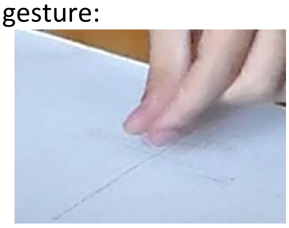
We think that the most interesting feature of the excerpt 6.23 is the presence of seeds of possible realizations of the derivative of the function. In particular, at lines 11 and 13 Alessio looks at the relation between the variations of the two variables for negative x -values and since he finds out that "*the ratio of f(x) to x changes*" he states that in that interval the graph is not a line. Then, from an expert point of view, at line 15 Alessio computes the ranges of variation of $f(x)$ over the intervals [0; 1] and [1; 2] and he compares the two values discovering that they are the same. For this reason, he concludes that for positive x -values the graph has to be a line. His discourse mirrors potential expert discourse about the behavior of the derivative function which in this case is increasing for $x < -5$, and it is not constant as would be for a linear function.

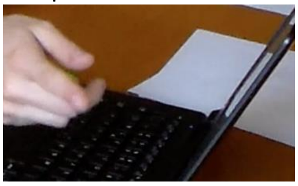
In lines 11 and 19 Alessio's discourse is mirrored by potential expert discourse about the limit of the function for x approaching positive and negative infinity respectively. In both cases the function is unlimited, as he expresses by “*it goes up again, but this is to infinity*” and “*it goes down to infinity”.*

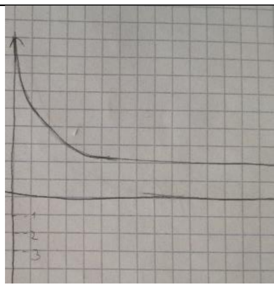
Excerpt 6.24 - Interview 1

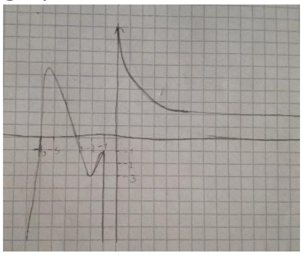
(Realization DGpp of the function $f(x) = \begin{cases} \frac{3}{2x} + 2, \ x > 0 \\ (x+1)^2(x+6)(x+3)\frac{1}{x} - 1, \ else \end{cases}$)

	When	Who	What is said	What is done	Potential expert discourse	Code
9	C 25:20	A	Then, let’s start from when f(x) is on zero, x is on minus six, then as it [x] goes on, f(x) goes on but until….. minus five. So, as x goes on, y makes, eh.. from zero it arrives… so here [x=-6] is zero, from zero it arrives, the peak is five and a half…. And then does it go down again? Eh no, but it starts going down….. at minus five it keeps going down but it is not still arrived at zero, it arrives at zero at minus, like minus three, a bit more than minus three. And then, from minus three point two it is still going down until.. minus three, when this [x] is at minus two point two…. And then it goes up until minus one, when it is at minus one… the nit comes	Task3 The grid is activated He drags x to the right and in the first pause he drags back/forward in a neighborhood of -5. Then he draws:  Fig. 6.24a Finally, he moves his left hand this way:  Fig. 6.24b	A pre-image of 0 is -6, in a right neighborhood of -6 the function is increasing and it has a relative maximum point at -5. $f(-6) = 0$, the relative maximum is 5.5 and in a right neighborhood of -5 the function decreases. A pre-image of 0 is -3.2 where the function still decreases and -3 is a relative minimum such that $f(-2.2) = -3$ The function is increasing for x in [-2.2, -1], $f(-1) = -1$ and in a right neighborhood of -1 $f(x)$ disappears	MON MAX/ MIN DOM DER

			back, the nit goes away [Fig. 6.24b]!		The derivative changes the sign at -5; $f'(-3.2) < 0$; the derivative changes the sign at -2.2; in a right neighborhood of -1 the function is not defined	
10		R	Who goes away?			
11		A	F(x), so this one [the graph in a right neighborhood of -1] goes down	 Fig. 6.24c	$\lim_{x \to 0-} f(x) = -\infty$	LIM ASY
12		R	Why not up?			
13		A	Because from minus one it gown down, it goes to.. and then, the nit arrives, when this [x] is at one, so wait, at minus one is here…… Then, it arrives at, so from here [(0, 7)] it comes back for sure, but while coming back this [x] keeps going on, this one goes on but as x goes on, y gets more and more down, but the ratio is still [Fig. 6.24d], that is, as x goes on, y changes less and less, in function of the movement of x, f(x), the function changes as x goes on	He drags x from $x = -1$ to the right and he repeats this action. He points to large positive x -values on the paper and he makes this gesture:  Fig. 6.24d	In a right neighborhood of -1 the function decreases. For $x \geq 0$ the function is again defined and it decreases; then as x grows the ratio of Δf(x) to Δx decreases The derivative is negative and it decreases as x approaches positive infinity	MON DOM DER ASY

14		R	The function changes, but less?			
15		A	Yes, like this [f(x) approaching 0], and then it is always more flat, as x goes on the ratio of x to f(x) is bigger and bigger because, I don't know how, x goes on and f(x), as x goes on.. that is…		The slope of the graph tends to 0. As x grows also the ratio of Δx to Δf(x) increases The derivative decreases for positive large x -values	DER ASY
16		R	So, there [Fig. 6.24e] as x goes on, f(x) comes back	She points to the screen:  Fig. 6.24e	The function decreases for positive large x -values	MON
17		A	Yes, here for positive numbers, as x goes on f(x) moves back getting very close to zero, that is, f(x) decreases more and more as x goes on, but as x goes on f(x) is always decreasing [Fig. 6.24f], but less than before, as x goes on	 Fig. 6.24f	The function decreases and $\lim_{x\to+\infty} f(x) = 0^*$ The ratio of Δf(x) to Δx decreases as x grows (* $\lim_{x\to+\infty} f(x) = 2$) The function has a horizontal asymptote	MON LIM ASY DER
18		R	Is it slower?			
19	C 31:35	A	Yes, it is slower in decreasing, so, it is like this, it doesn't touch zero, so… it arrives at two, I don't know….. that is, I don't know where it start, where f(x) comes from	He points to the positive y -axis on the paper	The function decreases slowly and $f(x) > 0$; $f(x)$, approaches 2. I do not know where the function is defined The difference quotient decreases	MON DER DOM
20		R	When does it appear?			

21		A	When it appears, if it [x] is one or two			
22		R	When do you not see it for sure?			
23		A	So…. I have to see when f(x) is at nine because here I did it of height nine….. more or less zero	In the first pause he looks at the numbers on the y -axis on the paper, then he drags x in a neighborhood of zero	A pre-image of 9 is about 0	
24		R	More or less zero, so does it appear again?			
25		A	here! A little more further, no, here it's good, on zero more or less		0 belongs to the domain of the function	DOM
26		R	Yes, according to what you can see			
27		A	on zero, and then it arrives at… at [x equals] two, a little more than two, then it starts falling, that is, less and less so, it is this [Fig. 6.24g], for me the line is like this, these two are not broken, but, they are	 Fig. 6.24g	At x around 2 the derivative decreases and the function is decreasing	DER MON
28		R	Are all curves?			
29		A	yes!			
30		R	So, at minus six it is zero and then, for smaller x-values? Let's try to continue a bit here	She points to the negative x-axis on the paper	$f(-6) = 0$	
31	C 33:50	A	At minus six it is zero, then, as it [x] moves backward, it is always more.. here [x=-4] I don't know, wait, here from minus six to seven eeeh goodbye! It becomes more and more sensible, here [negative infinity] it is the opposite, here		$f(-6) = 0$ For x tending to negative infinity the derivative tends to infinity	DER LIM

			[positive infinity], that is, as x moves backward this [the graph] becomes more and more…			
32		R	So, how can we do it?			
33		A	Like this! No no no, yes like this, always more open, no wait, but I'm not sure about it… yes, for me it's good, for me yes, as it moves on this [the graph] becomes more and more… that is, for each small x-variation, f(x) moves more and more, that is here, I don't know, to go from minus six to minus seven, f(x) moves from minus one to minus fifteen and, from minus seven to minus eight, f(x) moves from minus fifteen to minus forty or minus fifty.. According to me it is like this.	He draws the curve for $x < -6$ and he erases it, repeating these actions several times. Finally, he draws the following graph:  Fig. 6.24h	The graph becomes steep because the ratio of Δf(x) to Δx is very high, for example: $\lvert f(-7) - f(-6) \rvert = \lvert -40 - (-15) \rvert$ $\lim_{x \to -\infty} f(x) = -\infty$	DER LIM
34	C 35:45	R	Well, so, from here we can see that there are some values that f(x) takes on more than one time, for example?	She looks at the drawing on the paper	There are some points of non injectivity	INJ
35		A	Such as, from minus one, eeeh because this one does never come back here, so from minus one to infinity	He points to the y -axis going down from -1		
36		R	To minus infinity			

37		A	Yes, to minus infinity, from minus one to minus infinity and.. from five point five, actually, the values that f(x) takes on, from minus three to minus infinity, twice, from minus one to minus three, four times, from five point five to minus one, only twice, ah no, and from five point five to two, three times, yes three times	He looks at the drawing on the paper	For all $f(x)$ ϵ(-∞; -3] there are two pre-images For all $f(x)$ ϵ(-3; -1) there are four pre-images For all $f(x)$ ϵ(-1; 2) there are two pre-images For all $f(x)$ ϵ(2; 5.5] there are three pre-images	INJ
38		R	And over five point five?			
39	C 37:05	A	One			

Alessio's discourse in the first line of the excerpt is very similar to potential expert discourse about the possible monotonicity properties, the intersection points with the x -axis, the relative extreme points and the domain of the same function. Moreover, at line 11 Alessio realizes in the Cartesian plane the point outside the domain of the function by drawing an almost vertical line as part of the graph, which evokes the existence of a vertical asymptote.

In line 13 Alessio starts describing the function for positive x -values. In particular, his discourse is mirrored by potential expert discourse about the behavior of the function that is decreasing but "*as x goes on, y changes less and less*". This observation, that Alessio repeats in other words in lines 15 and 17, is mirrored by potential expert discourse about the derivative as limit of the difference quotient, which expresses the average rate of change of the function. It is interesting to look at his gestures in Figure 6.24d and Figure 6.24f because they are very similar and they are used as visual mediators during the description of the getting smaller variations of the dependent variable. In line 17 Alessio's discourse is mirrored by potential expert discourse about the horizontal asymptote $y = 0$, which actually is $y = 2$, when he explains that "*as x goes on f(x) moves back getting very close to zero*". Moreover, in line 27 he identifies x around 2 as the point where the function "*starts decreasing slower*" that, from an expert point of view, is a changing in the behavior of the derivative function which still remains negative. Finally, we analyse Alessio's discourse about the behaviour of the function for x tending to negative and positive infinity respectively. In line 31 he compares the two situations and he says that for x tending to negative infinity "*it becomes more and more sensible*" while for x tending to positive infinity *"it is the opposite*". His discourse is mirrored by potential expert discourse about the study of the behaviour of the function in the extremities of the domain, since it has finite limit for x tending to positive infinity and infinite limit for x tending to negative infinity. Then, before realizing it in the

Cartesian plane (Fig. 6.24h), in line 33 he looks at the relation between the variations of the two variables for large negative x -values, gaining information about the slope of the graph.

From line 34 to the end of the excerpt Alessio's discourse about the non-injectivity of the function is the same as potential expert discourse. In fact, he expresses the number of pre-images of each value belonging to the set of images of the function.

6.2.5 Franci and Lore

The two students worked together during all the lessons and they were also interviewed together. As we will see in the excerpts below, Franci and Lore have two different roles along the whole sequence of activities: Lore holds the mouse and Franci writes their observations on the sheet of paper.

Table 6.4 shows the list of the mathematical objects that can be found in Franci and Lore's discourse, who worked together during the whole sequence of lessons and then they were also interviewed together. There are two blank lines in ***Table 6.4***, because we do not have sufficient recordings of their discussions during activity7_1 and they did not have enough time to complete Activity8_3. However, by looking at the table we can notice that Franci and Lore's discourse, developing during all the activities, is mirrored by potential expert discourse about several mathematical objects.

Activity	Type	IN/DEP	DOM	RAN	INJ	MON	MAX/ MIN	LIM	ASY	DER
1_1	DGp									
1_2	DGp									
1_3	DGpp									
2_1	DGpp									
2_2	DGpp									
2_3	DGpp									
3_1	DGpp									
3_2	DGpp									
4_1	DGpp									
4_2	DGpp									
5_1	DGc-DGpp									
5_2	DGc									
5_3	DGc									
6_1	DGc									
6_1bis	DGc									
7_1	DGc-SGc									
7_2	DGc-SGc									
8_1	DGpp-SGc									
8_2	SGc									
8_3	SGc									

Table 6.4. Mathematical objects in Franci and Lore's discourse

In line with what we expected and what we have seen with Nicco and Alessio, during the first lesson the students are mainly involved in the construction of a discourse on the dependence relation between the two ticks.

Excerpt 6.25 - Lesson 1

(Realization DGpp of the function $f(x) = |x|$)

	When	Who	What is said	What is done	Potential expert discourse	Code
542	I1Mp3 13:45	F	So this one [dependent tick] cannot have, doesn't have the negative and so this one [dependent tick] is the absolute value and this one [independent tick] not, or this [dependent tick] is the absolute value of this [independent tick]	Activity1_3 Lore holds the mouse and Franci points to the screen	One of the two ticks is x and the other is abs(x) $f(x) = \|x\|$	IN/ DEP
543		L	Something like that, yes			
544		R	Something like that			
545	I1Mp3 14:05	F	The first is the absolute value of the second		$f(x) = \|x\|$	IN/ DEP

As we can see from the excerpt 6.25, when exploring the function in activity1_3 Franci tries to combine the two ticks together, by relating them through a dependence relation. However, finding this relation does not seem immediate, in fact at line 542 he initially describes them as two separated entities ("*this one is absolute value and this one not*") and then he supposes that "*this is the absolute value of this*". Finally, his discourse at line 545 "*the first [tick] is the absolute value of the second [tick]*" is the same as potential expert discourse mirrored, where *the first tick* is intended to be $f(x)$ and *the second tick* to be x.

Excerpt 6.26 - Lesson 2

(Realization DGpp of the function $f(x) = e^{x-1} + \frac{1}{25}$)

	When	Who	What is said	What is done	Potential expert discourse	Code
31	I2Mp1 06:13	F	It [B] is a little bit greater, so B is greater than A, if A equals one, B is greater than A	Activity2_1 Lore holds the mouse	In a neighborhood of $x = 1$: $f(x) > x$; $f(1) > 1$	IN/ DEP
32		L	But in all the cases, because if A equals zero, B is greater than zero		For all x in the domain $f(x) > x$; $f(0) > 0$	IN/ DEP
33		F	Then, after one, we will check later, anyway, so B is always greater than A?	He writes down		

34		L	Yes, basically yes, A is zero and B is about zero point three		$f(0) = 0.3$	
35		F	Yes, but the important thing is that it's bigger, then, after one, so, if A is greater than one			
36		L	Yes			
37		F	If A is greater than one	He writes down		
38	I2Mp1 07:00	L	It [f(x)] goes faster and faster, they are never aligned		For all $x > 1, f(x) \neq x$, and the derivative increases	DER
			[…]			
62	I2Mp1 09:15	F	In the interval between zero and one, if you move A to one, that is, considering that B started further, A is faster, so in the interval between zero and one A is faster than B…….. so, A is faster than B then, passed over one, then, if A arrives at one, B is a little bit further… however, A is faster	Lore holds the mouse and Franci writes down their description (see Fig. 6.26)	for $x \in [0; 1]$ the ratio of Δf(x) to Δx is smaller than 1 $f(1) > 1$	IN/ DEP DER
63		L	After that, look at B…. it increases because if A is constant	Continuous dragging to the right with an almost constant speed	In a right neighborhood of 1 the function is increasing	MON
64		F	I know, but as A increases, B increases more and more, that is, it increases more and more, look, try to drag slowly		In a right neighborhood of 1 the function is increasing and also the derivative is increasing	MON DER
65		L	We start with a distance of.. the difference would be			
66		F	As zero point one		$f(1) - 1 = 0.1$	
67		L	It's minimum, and it increases more and more		As x grows $f(x) - x$ increases For all $x \geq 1, f'(x) > 1$	DER
68	I2Mp1 11:00	F	Yes, but a lot, so it doesn't increase in a			

			directly proportional way, it increases of…			

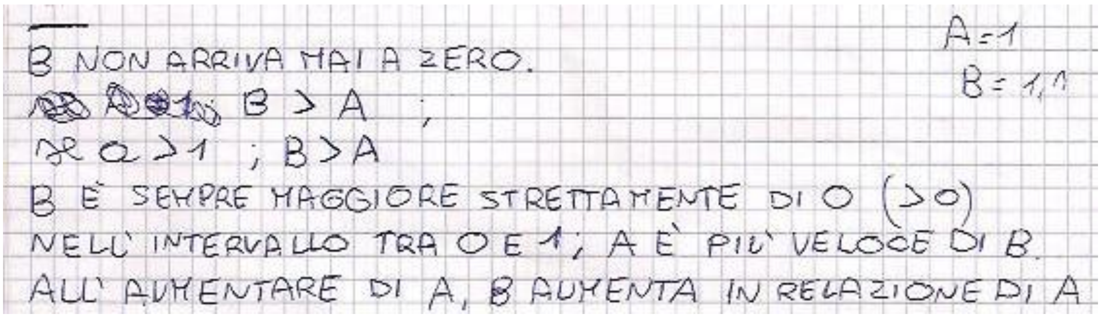


Fig. 6.26. Excerpt taken from students' worksheet

In the excerpt 6.26 Franci and Lore describe the movements of the two ticks in terms of "if A…B…", but their discourse is not still properly mirrored by potential expert discourse about dependency, for example at line 34 Lore considers A and B separately.

Meanwhile we identified the seeds of possible realizations of other mathematical signifiers. In order to find a value for which the two ticks are overlapped Franci and Lore observe not only the values that are taken on but also the speeds of the ticks. They discover that B is always greater than A (e.g. line 32 and Figure 6.26) and that in a right neighborhood of 1 it moves faster than A (e. g. line 64); their discourse is mirrored by potential expert discourse about the existence of different intervals of monotonicity and the derivative of the function. In particular, from line 65 to the end the students describe the changes in distance between the two ticks that "*from the minimum value, it ever increases*", mirroring potential expert discourse about the derivative function. The value of the derivative gives information about the slope of the graph, that in this case for $x \geq 1$ is steeper than the graph of $y = x$.

Excerpt 6.27 - Lesson 3

(Realization DGpp of the function $f(x) = x + \frac{3}{x-3}$)

	When	Who	What is said	What is done	Potential expert discourse	Code
27	I3Mp1 03:45	F	Where does B arrive?	Activity3_1 Axes overlapped Lore holds the mouse		
28		L	B moves on and then it appears from that side [negative values]	He drags A from right to left in a neighborhood of $x = 3$	In a right neighborhood of 3 the function is decreasing; in a left neighborhood of 3 the function is negative $x = 3$ is a vertical asymptote	MON ASY

29		F	Eh, it appears from that side, what does it mean? That it was positive and it starts from negative [numbers]		$\lim_{x\to 3+} f(x) = +\infty$ $\lim_{x\to 3-} f(x) = -\infty$	LIM ASY
30		L	Yes, but why, then			
31		F	We don't know why, we have to look at what we see….here [large positive x-values] they move together	He points to the screen	For large positive x -values the function is increasing	MON
32		F	Wait, after six point six, six point four, what happens? B appears from negative [numbers]?		6.4 is a relative minimum	MAX/ MIN
33		L	No no, it depends because it is strange			
34		F	Go, go			
35		L	If A goes that way [to 3 from the right] B goes there [positive infinity] and it comes from that side [negative infinity]. While, if A moves from negative to positive	He drags A and he also points to the screen	$\lim_{x\to 3+} f(x) = +\infty$	LIM
36		F	When B goes, when A goes..are you dragging A right?			IN/ DEP
37		L	Yes, we can drag only A		A is the independent variable	IN/ DEP
38		F	Therefore, if A moves from positive [3^+] to negative [3^-]..B keeps going to positive, that is		In a right neighborhood of 3 the function is positive and strictly decreasing	IN/ DEP MON
39		L	Buti t is not always true, because here [x=9] B follows A, and, once arrived at six point four, point five, it comes back and it reappears when A passes over, it reappears when A is two, no, it reappears when A is here [x=2.5], more or less	He drags A from right to left	For large positive x -values the function is increasing, 6.5 is a relative minimum. In a neighborhood of 3 the function is not defined, there is a vertical asymptote	MON MAX/ MIN DOM ASY

40	I3Mp1 06:05	F	It will be two point six……. This thing is strange because…how can a number going to positive and suddenly becoming negative? That is, if it goes on to positive, it goes on to positive, that is, suddenly it should change its sign because.. if it go on to positive, it would go on to infinity, while at a certain point it comes back and it doesn't make sense			

The function in activity3_1 has been defined in order to support students' emerging discourse about the domain, indeed it is not defined at $x = 3$ where there is a vertical asymptote. In line with the expectations, at lines 33 and 40 the two students say "*it is strange*" and at line 40 "*it has no sense*" suggesting that they are surprised by the vanishing of the dependent variable. In particular, Franci seems surprised by the possibility for a number "*to go toward the positivity and to suddenly become negative*" after taking on positive values (see line 40), and his discourse is mirrored by potential expert discourse about the existence of a vertical asymptote. Moreover, Franci at line 29 and Lore at line 30 of the excerpt 6.27 say that they do not give meaning to the movements of B.

Students' discourse in the excerpt 6.27 is also mirrored by potential expert discourse about the monotonicity properties of the function (see lines 28, 31, 38 and 19). However, when looking at most of these examples we have to pay attention to Lore's dragging action because he usually drags A from right to the left and this is important because an analysis of the monotonicity properties involves the relation between the directions the two ticks. For example, at line 39 Lore says "*B goes forward and then it appears again over there [negative values]*" for describing B moving to the right as A moves to the left and he uses 'then' referring to smaller A-values; in fact, it is mirrored by potential expert discourse "in a right neighborhood of 3 the function is decreasing; in a left neighborhood of 3 the function is negative".

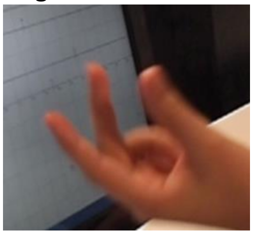
Students' discourse at lines 36, 37, 38 is mirrored by potential expert discourse about the dependence relation between the two ticks, which is experienced by the students in terms of direct/indirect motion within the DIM. In fact, Franci, who never holds the mouse to manipulate the file, has to ask Lore which tick is directly draggable to formulate the sentence "*if A goes from negative to positive [values] then B keeps going to positive [values]*" (see line 38).

At lines 28, 31, 35, 39 there are examples of students' use of 'this' for indicating both the ticks and 'here' to describe particular positions of the ticks on the lines; these descriptions are understandable by looking at their pointing gestures to the screen or their dragging actions which are, actually, dragsturing actions.

Excerpts 6.28 - Lesson 4

(Realization DGpp of the two functions $f(x) = \begin{cases} 7, & x < 5 \\ 3 + floor(x), & else \end{cases}$

and $g(x) = \begin{cases} \frac{5}{2}x, & x < 6 \\ \frac{1}{2}x + 12, & else \end{cases}$)

	When	Who	What is said	What is done	Potential expert discourse	Code
44	I4Mp1 06:00	F	So, one third of the day.. after eight hours, [T_A equals] sixteen, eh after eight hours, it [A] seems moving faster	Activity4_1 The grid is shown	For $x > 8, f'(x)$ decreases	DER
45		L	Yes, that is, it seems that they…	While dragging he moves the other hand closing two fingers:  Fig. 6.28a	As x grows $f(x) - x$ becomes smaller The derivative is decreasing	DER
46		F	That is, A moves always in the same way, but B takes same spaces in less time… then, coming back, this thing is normal	At the end it is not clear what he refers to	As x grows the ratio of Δf(x) to Δx increases* (**the derivative is decreasing)*	DER
47	I4Mp1 06:37	L	T_A is never less than zero and also for twenty-four		$f(x)$ is always positive	RAN
			[…]			
107	I4Mp1 12:20	L	At seven it is still decreasing because			
108		F	What is decreasing?			
109		L	The distance, I mean		For $x \geq 7, f(x) - x$ decreases For $x \geq 7$ the derivative is decreasing	DER
110		F	And so, in other words?			

111		L	The speed of T_A		For $x \geq 7$ the speed of $f(x)$ decreases For $x \geq 7$ the derivative is decreasing	DER
112		F	Is it decreasing?			
113	I4Mp1 12:52	L	A equals six and T_A is fifteen, A is seven, look, it is already lesser, before it was increased by a certain number of squares, now it is increased just by a half [square] in one hour, just half a square, at eight it is always decreasing	Discrete dragging	$f(6) = 15$; $f(7) - f(6) < f(6) - f(5)$: $f(8) - f(7) < f(7) - f(6)$ For $x \geq 7$ the derivative is decreasing	DER

During the fourth lesson Franci and Lore's discourse is mainly mirrored by potential expert discourse about the set of images and the derivative of the function. For example, at lines 44 and 46 of the excerpt 6.28 Franci describes the relative speed of the two ticks, while Lore mainly focuses on the distance between the two ticks, which becomes smaller as x grows, as he expresses through the gesture in Figure 6.28a and through words at lines 109 and 113. In particular, he computes $f(x) - x$ for some specific x-values and then he compares the resulting values, mirroring potential expert discourse about the behavior of the derivative function that has different intervals of monotonicity. Indeed, this comparison between different values of $f(x) - x$ gives him information about the speed of T_A, as he explains to Franci at line 111.

At line 113 Lore says "*at eight it is always decreasing*" by using an instantaneous reference, because eight is a point, but he describes the behavior of the function, which is decreasing, and the monotonicity property of a function concerns an interval of its domain. Therefore, we identify in Lore's discourse a seed of realization of the passage to the limit, mirroring potential expert discourse about the derivative of the function, especially about $f'(8)$.

Moreover, Lore's discourse at line 47 is mirrored by potential expert discourse about the set of images of the function, which takes on only positive values.

Franci and Lore's discourse seems now to be properly mirrored by potential expert discourse on dependency, also when they describe the function in activity5_1, which involves for the first time the realization DGc. The following excerpt shows a short part of their discussion during this activity.

Excerpt 6.29 - Lesson 5

(Realization DGc of the function $f(x) = -x + 5$)

	When	Who	What is said	What is done	Potential expert discourse	Code
25	I5Mp1 02:02	F	A is four times B, but it doesn't work	Activity5_1	$x = 4f(x)^*$ ($^*f(x)=-x+5$)	
26		L	No, because if A is two, B is three, if A is three, B is two		$f(2) = 3; f(3) = 2$	IN/ DEP
27		F	So, as the numbers on the x-axis increase			
28		L	It increases more and more	He does not move anything		
29	I5Mp1 02:30	F	But it [B] goes down, look, go to eight, nine, ten, it goes down, try to go and see if something changes at the bottom [large positive x-values]…no, nothing changes, then	Lore drags A to the right, then quickly drags A onto 8 and he continues dragging A to the right	The function is strictly decreasing	MON

In line with our observations above, the two students express the dependency through the expressions "*if A… B…*" (see line 26) and "*if going on along the x-axis then…*" (see line 27).

At the beginning of the excerpt, which almost corresponds to the beginning of the lesson, Franci searches for a rule to express the relation between the two variables. Then they focus on possible changes in direction of the dependent variable, which is actually what an expert would do when studying the monotonicity properties of the function, At line 29, Franci's discourse is mirrored by potential expert discourse "as x grows, the function is strictly decreasing". Moreover, we notice that instead of using "to go back" as in the previous lessons Franci for the first time says "*[B] goes down*" (see line 29). It is interesting because it happens when he is working for the first time with the realization DGc of the function, which involves the second dimension and so he sees the tick realizing the independent variable moving vertically.

Excerpt 6.30 - Lesson 7

(Realization DGc of the function $g(x) = \frac{x}{2} + \frac{3}{x-3}$ and realization SGc of other four different functions)

	When	Who	What is said	What is done	Potential expert discourse	Code
229	I7Mp4 03:10	L	What can we do to see the positive numbers? well…for example, from three, no! … three….sorry but, three….. it reappears from the top, there is something wrong, three can be both up and down	Activity7_2 He drags x maintaining $(x, f(x))$ in the first quadrant	$\lim_{x\to 3-} f(x) = -\infty$ $\lim_{x\to 3+} f(x) = +\infty$	ASY

230		F	What does it mean that three can be both up and down?			
231		L	Look…that is, we were here, less than three, and it was minus ninety-nine, do you remember?	He zooms in at 2.9 on the x -axis and then he zooms out at -99 on the y -axis	For x in a left neighborhood of 3 $f(x) = -99$ $\lim_{x \to 3-} f(x) = -\infty$	LIM
232		F	Yes			
233		L	Then, if I go on, look at f(x), then it appears again	He zooms in at 3.1 on the x -axis	$\lim_{x \to 3+} f(x) = +\infty$	LIM
234		F	It doesn't appear anymore…did it appear before?			
235		L	Yes, wait, I have to zoom out…….look!	He zooms out at a big number on the y -axis		
236		F	Oh where did it reappear?			
237	I7Mp4 04:55	L	Three, where could it also be? It dashes away!		$f(3)$ does not exist	DOM ASY

When exploring the DIM in activity7_2, Franci and Lore do not seem to identify activity3_1 (where there was the same function) as a *precedent*, but their discourse is very similar in both cases (see Excerpt 6.27).

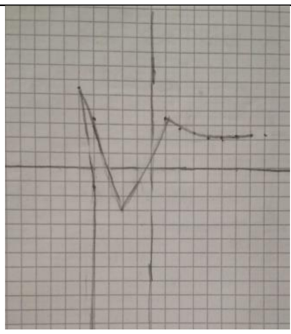
In particular, their discourse in the excerpt 6.30 is mirrored by potential expert discourse about the vertical asymptote of the function which is not defined at $x = 3$; for example when they say that "*the three can be both up and down*" (see lines 229, 230) or "*it dashes away*" (see line 237) or "*then f(x) appears again*" (see line 233). This activity involves the realization DGc of the function and it might be guessed also from the students' choice of words. Indeed, they describe the tick realizing the dependent variable going down and appearing from the top, instead of making a circle around the screen, as they did in the excerpt 6.27 where they were exploring the realization DGpp of this function.

However, also in this case they seem surprised by the movements of the dependent variable which they probably did not expect to see, for example Lore says "*there is something wrong*" at line 229.

Excerpt 6.31 - Lesson 8

(Realization DGpp of the function $f(x) = \begin{cases} \frac{3}{2x} + 2, \ x > 0 \\ \frac{(x+1)^2(x+6)(x+3)}{x} - 1, \ else \end{cases}$)

	When	Who	What is said	What is done	Potential expert discourse	Code
78	I8Mp2 03:00	L	What a beautiful graph!	Activty8_1		

				 Fig. 6.31a		
79		F	Eh, this [Fig. 6.31a] should be the graph, I cannot understand, but if you notice it, it never belongs to this [the fourth] quadrant, it intersects all the other and it never goes there [fourth quadrant], zero there isn't, and it doesn't pass through the origin		$f(0) \neq 0$ the function is not defined at $x = 0$	DOM
80		L	No, but sorry, so here in the graph that we did, if x is zero			
81		F	Zero doesn't disappear, zero disappeared at all		The function is not defined at $x = 0$	DOM
82		L	No, here if x is zero, it intersects the y-axis, anyway			
83		F	Yes, I know, but it should be a kind of parabola, but we didn't do the parabola			
84		L	It passes through one, more or less			
85		F	I think that it is			
86		L	And here, that is, here zero	He takes the mouse and zooms out with $x = 0$		
87		F	It's not visible, there isn't, that is, I don't know how to explain it		$f(0)$ does not exist; the function is not defined at $x = 0$	DOM

88		L	Here it is, here it is			
89		F	F(x) about minus four hundred and seventy, however, the important thing is to know that it is down		$f(0)$ is around -470 in a left neighborhood of 0 $f(x)$ is around -470	LIM
90		L	So wait, if x is zero?			
91	I8Mp2 04:10	F	Minus four hundred and seventy, it means that you have to take a point here	He points to a large negative number on the x -axis in the paper		
			[…]			
102	I8Mp2 04:50	L	Then wait, if it is here [1] it will be like here [2] but it is like here [2] randomly, so it decreases, then, when it [x] is one it moves up again and it does this [3], so the two graphs are not linked, I don't know why I liked them, that is	 Fig. 6.31b	In a left neighborhood of 0 the function is decreasing $\lim_{x \to 0-} f(x) = -\infty$	MON LIM
103		F	Are they not linked? Then, if they are not linked, how can we know that they are not linked?			
104		L	Because if x is zero, y is minus four hundred, something like that		$f(0)$ is around -400 in a left neighborhood of 0 $f(x)$ is around -400	LIM
105		F	And so, they are separated, but			
106		L	And so, that is, it starts from zero, it goes down and comes up, like this [Fig. 6.31c] more or less	He points to this piece of curve:	For $x \in$ [-3.5; -2] the function is decreasing and for $x \in$ [-2; -1] it is increasing	MON

				 Fig. 6.31c		
107		F	What does it? It starts like this and then goes			
108		L	That is, if x equals zero			
109	I8Mp2 06:10	F	X equals zero, they have two different values		$f(0^-)$ is different from $f(0^+)$ $\lim_{x\to 0-} f(x) = -\infty$ $\lim_{x\to 0+} f(x) = +\infty$	LIM

The excerpt 6.31 is from the last lesson and during the activity8_1 students are asked to pass from the realization DGpp to the realization SGc of a function. The function is not defined for $x = 0$, where there is a vertical asymptote; again Franci and Lore express uncertainty doing this, for example at line 87 Franci explicitly says "*I do not know how to explain it*".

However, their discourse is mirrored by potential expert discourse about the domain and the different limits for x tending to 0 from right or left. Indeed, at lines 79 and 81 they notice that in their drawing (Fig. 6.31a) $f(0)$ exists while in the DIM "*zero disappeared*", since $x = 0$ is not in the domain of the function. But then, at a certain point during their explorations, they say to have found a value for $f(0)$, which is around -470 (see line 89), and at line 102 their description is about different limits of the function for x in left and right neighborhoods of zero. At lines 104 and 109 they also say, as happened in the previous lesson, that "*x equals zero has two different values*". They do not nevertheless modify their drawing.

Then at line 83 Franci mentions a parabola, as *saming* their drawing on the paper with a possible realization of a parabola in the Cartesian plane, even if he says that they "*did not study it yet*". Moreover, at lines 102 and 106 Lore identifies possible intervals of monotonicity of the function, when looking at the realization SGc on the paper after exploring the DIM again to check their drawing.

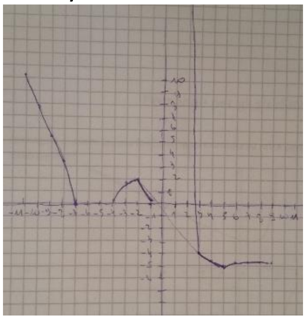
The following excerpt is taken from the interview with Franci and Lore.

Excerpt 6.32 - Interview 3

(Realization DGpp of the function $f(x) = \begin{cases} \frac{1}{x-2} - 5, \ x > 2 \\ \frac{1}{4}\sqrt{-(3x+21)(2x+2)(x+4)}, \ else \end{cases}$)

	When	Who	What is said	What is done	Potential expert discourse	Code
22	C 20:45 FLm2 03:50	F	Well six is minus five, a bit less than minus five, then do I go on?	Task2 with Franci at the pc and Lore drawing the points on the paper	$f(6)$ is around -5	
23		L	Yes	Discrete dragging		
24		F	Seven is minus four and eight more close to minus five		$f(7) = -4.8$	
25		L	That is, like seven?			
26		F	Eh no, I'm at seven now			
27		L	Like six, sorry			
28		F	Wait, now I look, because I don't remember it! Yes, yes, like six, exactly, that is, it [f(x)] doesn't move but the strange thing, what I can tell to you is that between six and seven, that is, it moves very little, so much little that I think it is even impossible to notice it	Zoom out and he drags x within the interval [6, 7]	$f(7) - f(6)$ is about 0 because $f(x)$ takes on the same value For $x \in$ [6; 7] the derivative is about 0	DER MON
29		L	eight?			
30		F	While eight is always less than minus five, so now there are different values between six and eight, there are different values approaching minus five, then nine, we are always a little less than minus five, so I would go on until the trend doesn't change	discrete dragging zoom out discrete dragging again	$f(8)$ is around -5 There are several pre-images of about -5, also $f(9)$ is around -5. I would search for a change in the behavior	INJ MON
31		L	Yes, for example look at eleven, twelve, to see if			

			something changes			
32		F	No, ten doesn't change, neither eleven, twelve, thirteen, fourteen, no, it doesn't seem to me… I would go back to negative again		The function is constant for all $x > 5$	MON
33		L	Minus five, because then I don't know, minus five			
34		F	Minus five, there isn't, the correspondence doesn't exist, or even in these intervals, do you want to know minus six? Neither minus six, so for sure it will be a function that… well, it appears again at minus seven and it is zero	Zoom out and in Discrete dragging	The function is not defined on [-6; -5] $f(-7) = 0$	DOM
35		L	Try minus eight, probably			
36		F	Minus eight is three and a half, more or less, but we have also to precise that it is positive		$f(-8) = 3.5$	
37		L	Sorry but, does it go up?		Is the function decreasing?	MON
38		F	Minus eight, three and a half yes		$f(-8) = 3.5$	
39		L	Minus nine, let's go till we have some values			
40		F	Well, minus nine, five and a half, minus ten, almost eight, well, and minus eleven.. ah, yes minus eleven is	Discrete dragging	$f(-9) = 5.5; f(-10) = 8; f(-11)$ is around 10	

			a little more than ten			
41	C 24:25	L	Then I think that it can be enough	Finally he draws:  Fig. 6.32a		

The function for this activity has been chosen such that it is not always defined and the value of its derivative is very close to zero for large positive x —values, in order to investigate students' possible description of these two properties. In the excerpt 6.32 we can see that Franci and Lore's discourse is mirrored by potential expert discourse about the derivative, the monotonicity properties, the non injectivity and the domain of the function.

First of all, they observe that *"[f(x)] does not move*" or *"[f(x)] moves very little, so much little that I think it is even impossible to notice it*", mirroring potential expert discourse about the derivative function, which takes on values very close to zero from a certain positive x —value on; and Franci proposes to search for a possible change in the "*trend*" (see lines from 28 to 32), that might be what an expert would call 'behavior' when looking at the intervals of monotonicity of the function. They also speak about the non injectivity of the constant function $f(x) = -5$, since they observe that there are several pre-images of -5 (see line 30), and this leads Lore to draw the horizontal line $y = -5 \; for \; x \geq 5$. Concerning the intervals of monotonicity, at lines 34 and 36 Lore is dragging x to the left and so his discourse "*does it goes up again?"* mirrors potential expert discourse "does the function decrease?" even if he is talking about a growing of the $f(x)$ -value.

Moreover, at line 34 Franci identifies an interval where "*the correspondence does not exist*" and he finds out that "*it will be a function such that it exists for x=-7*" properly mirroring potential expert discourse about the domain.

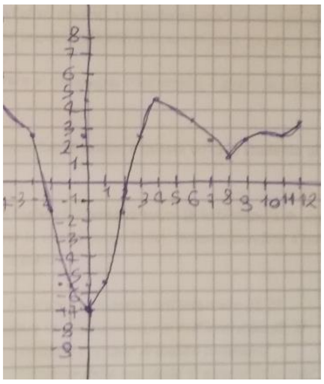
Excerpt 6.33 - Interview 3

(Realization DGpp of the function $f(x) = \begin{cases} 5.18, \;\; x \leq -4.6 \\ -10\frac{sinx}{x} + 3, \;\; -4.6 < x < 8.2 \\ \frac{2}{5}x - \frac{3}{2}, \;\; else \end{cases}$)

	When	Who	What is said	What is done	Potential expert discourse	Code
21	C 35:40 FLm5 02:15	L	However, for now by moving x within the positive [numbers], so by	Task3 with Lore at the pc and Franci drawing Discrete dragging Zoom out and in	The function is increasing in a left neighborhood	MON

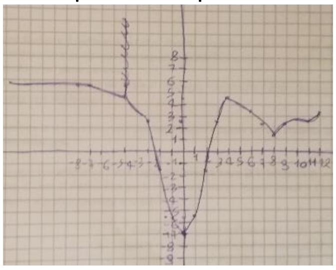
			increasing x, also f(x) increases, for now… x equals four, f(x) about four point nine… five is always a little more than four point nine		of 5; $f(5) = 4.9$	
22		F	Five is a little more than four point nine, then, I don't even know if marking it, since the trend, at the end we are interested in the trend, if it is what I think… six?	He does not write anything	We are interested in the behavior of this function	MON
23		L	Six is a little less than three and a half, about three point four, it is going a little down	Discrete dragging	$f(6) = 3.5$; in a right neighborhood of 5 the function is decreasing	MON
24		F	It is going down		It is decreasing	MON
25		L	Yes, arrived at, well, a little more than five and then it went down		In a right neighborhood of 5 the function is decreasing $x = 5$ is a relative maximum point	MON MAX /MIN
26		F	What a strange thing! Seven, a little more than two? We hope well! Eight, one point eight, it is following more or less, then stop at nine, that is, go to the other, nine, a little more than two?	He repeats what Lore says At the end he uses the pen to move on the curve that he drew	$f(9)$ is around 2; there are several pre-images of that value around 2. -7 is a relative minimum: in a right neighborhood of -7 the function is	MAX/ MIN MON INJ

			There are many, a little more than two! So it seems that it goes up… so, wait a second, here I see that minus seven is the point, as we did at math, like the vertex of a parabola?! It could be something like that, but I don't know, because here I started from minus seven then I went up, then I went down and now, with ten, sorry?		increasing and then it decreases There is also a relative maximum point where the derivative changes the sign	
27		L	Ten, we are at two and a half		$f(10) = 2.5$	
28	C 38:30	F	Eh, so it went up again		The function is increasing	MON
			[…]	They find that $\lim_{x \to +\infty} f(x) = +\infty$ and that for$-4.5 < x < -6, f(x)$ has always the same value, that is around 5		
35	C 42:35 FLm6	L	I look if it is constant, no, from now on it is constant		For $x < -4.5$ the function is constant	MON
36		F	That is, do they have the same value, more or less?		all $x < -4.5$ have the same value as image	INJ
37		L	Yes, now it is constant, because it doesn't move, now I'm going on but it doesn't move, it is still there, minus sixteen, still there	Continuous dragging to the left	The function is constant, the derivative is 0 $f(-16)$ is still around 5	MON DER

38		F	So, wait a second, because			
39		L	No Franci, it [f(x)] is always here [around 5]			
40		F	That is, is it still five point four, five point six? A little more than five… so, does it go up?	He moves his hand up as making a vertical line in the air		
41		L	Not now, it is, yes, it is always constant its value		The function is constant	MON
42		F	So it goes straight		Is it a line?	
43		L	Now, I don't know the shape of your graph		I do not know the shape of the graph	
44		F	Yes, but it doesn't have a shape! That is, there will be a succession of points one on another one		It might be a vertical* line (**horizontal*)	
45		L	No no, look also at mine… a straight line and I don't know, in both my and your graph there is a succession of points where the value is always constant, the f(x)-value	In the same paper there the Cartesian graph drawn by Lore in the previous activity He does not drag anything	A horizontal line; there are several x -values such that $f(x)$ takes on the same value	
46		F	Then, does it tell you something about the function? Some information, some details? What I can tell you is that its shape, this one seems a parabola, but we have not studied them in details,	He draws this curve  Fig. 6.33a	The graph seems a parabola	

			yet, and it has a bit strange shape			
47		R	How would you go on here, for negative values?			
48		F	How I would go on, eh, sorry, go to minus seven for a second			
49		L	Minus seven is always a little greater than five, from minus six on…	Zoom out	$f(-7)$ is around 5, for x in a left neighborhood of -6 it is a constant function	MON
50		F	So they are still in the same point, so it will go… eh!			
51		L	That is, x equals minus six for example, and f(x) is always greater than five	He moves his hand up	$f(-6)$ is around 5	
52		F	Yes, minus eight			
53		L	Minus seven, minus nine, they are all the same, sixteen, wait, I can go on but I don't think that there could be a turning point… it is still there also at fifty	He drags to the left and he zooms in at (0,5)	-7; -9 and 16* have the same image; $f(50)$ is* around 5 *(*-16 and $f(-50)$)* $f(x) = 5$ for all $x \leq -6$	MON
54		F	Wait a second, because if the ordinate moves and the abscissa is always in the same point, or anyway it doesn't go up, that is, for me we are stuck here, but I don't know, because y can, how I can say it, we can		The derivative is 0	DER

			move x as we want			
55		L	which? what?			
56		F	x, minus one, minus two, that is, minus five, minus six			
57		L	You will obtain a continuous line, I mean, straight line, like the one that I obtained before, even if I didn't do the entire line	He moves both his hands horizontally	The graph is an horizontal line	
58		F	a line following the trend			
59		L	Yes, a straight line			
60		F	It should be like this, parallel to the ordinates	He draws the vertical line x=-5	A vertical line, parallel to the y -axis	
61		L	No! x moves, x			
62		F	not y			
63		L	Eh but, it's not true that it doesn't move, but it is	He moves one hand horizontally as before		
64		F	Yes, but it corresponds to the same point, in other words if x has a value and y has always the same	He looks at his drawing on the paper		
65		L	However, you do all the points, minus seven put five, a little more than five, then minus eight put the same value again, minus nine the same value	Franci is still looking at his drawing	You can plot the points (-7, 5); (-8, 5); (-9, 5)	

66	C 48:25	F	Yes, yes, it's true, I was wrong, so minus seven should be here, then minus eight……….. so it is parallel to the axis of abscissas!	He starts drawing the points that Lore told him to put in the plane  Fig. 6.33b	The graph is a horizontal line, parallel to the x -axis	

The function that we used for this activity is always defined on the real number set and it is constant on a specific interval of the domain. In the excerpt 6.33 Lore is manipulating the realization DGpp of the function and he is telling to Franci the coordinates of some points $(x, f(x))$; he also describes the function in terms of movements of the two ticks. In fact, there are several examples of the two students' discourse mirroring potential expert discourse about the monotonicity properties of the function (see lines 21, 22, 23, 24, 25, 26, 28, 35, 37, 41, 49). This is actually what Franci says to be interested in to drawing the graph: 'the trend' (see lines 22 and 58). Moreover, the students' description of changes in direction of the tick realizing the dependent variables, at lines 25 and 26, is mirrored by potential expert discourse about the existence of relative maximum or minimum points of the function.

From line 35 to the end of the excerpt students' discourse focuses on the interval of the domain where the function is constant. Franci seems to have some difficulties with the realization of this property graphically in the Cartesian plane; at line 46 he even asks Lore if there are some more information about the function written in the DIM. At line 45 Lore suggests him to look at the graph in the previous activity (Fig. 6.32a) which also had "*a set of points where the f(x)-value is constant*", and at line 65 he invites him to apply the following routine: plotting in the Cartesian plane a set of points having different abscissas and the same value of the ordinate. Nonetheless, at lines 40, 44 and 60 Franci describes a vertical line in the Cartesian plane as a possible realization of the constant function and he also draws it. Then he changes his idea, explaining that "[*the line] has to be parallel to the x-axis*" (see line 66) and he finally modifies his drawing by delating the vertical line and tracing a horizontal line (Fig. 6.33b).

As happened in the excerpt 6.31, we observe that at lines 26 and 46 Franci describes the shape of his graph by speaking about a parabola, as *saming* his drawing in the Cartesian plane with a possible realization of parabola.

6.2.6 Davide and Elena

Davide and Elena worked together during the whole sequence of lessons, except for the second because they were not at school when it took place. It is interesting to analyze this pair of students' discourse, because they did not follow the intended path that we designed, with respect to the type of realization of the function being involved. Indeed, their routine for each activity consisted in identifying a set of points $(x, f(x))$ and plotting them in a Cartesian plane which they drew on a sheet of paper. This means that ever since the first lesson they actually worked with the realization SGc of the function, instead of getting there after a series of activities in the dynamic environment, as we designed.

By looking at **Table 6.5** we notice that it is quite empty compared to **Table 6.1**. There also appear to be some completely white lines; we conjecture that it could be due to these students' alternative approach to the activities which does not allow for as much mirroring of experts' discourse.

In general, these students use some mathematical technical words but not always in a formally correct way; especially Davide seems to evoke routines from his precedent-search-space on task situations involving functions. He also employs visual mediators both symbolic and iconic, proposing different realizations of the mathematical objects of his discourse during the activities.

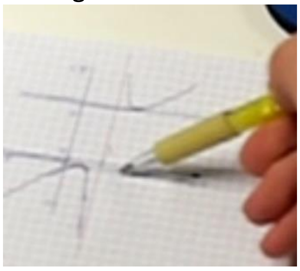
Activity	Type	IN/DEP	DOM	RAN	INJ	MON	MAX/ MIN	LIM	ASY	DER
1_1	DGp									
1_2	DGp									
1_3	DGpp									
3_1	DGpp									
3_2	DGpp									
4_1	DGpp									
4_2	DGpp									
5_1	DGc-DGpp									
5_2	DGc									
5_3	DGc									
6_1	DGc									
6_1bis	DGc									
7_1	DGc-SGc									
7_2	DGc-SGc									
8_1	DGpp-SGc									
8_2	SGc									
8_3	SGc									

Table 6.5. Mathematical objects in Davide and Elena's discourse

As we will see in some of the following excerpts, Davide and Elena prefer working with the realization SGc of the function instead of with the dynamic realizations proposed in the activities. However, they are not always able to interpret this realization, for example, in many cases Davide refers to a point $(x, f(x))$ belonging to the curve, but his discourse is mirrored by potential expert discourse about the behavior of the dependent variable.

Excerpt 6.34 - Lesson 3

(Realization DGpp of the function $f(x) = x + \frac{3}{x-3}$)

	When	Who	What is said	What is done	Potential expert discourse	Code
181	I3Mp2 08:14	R	Is this [realization SGc] coherent with this one [realization DGpp]?	Activity3_1		
182		D	Yes, because here all the values that B can take on, I don't know, because it cannot be greater than this point and less than this point. Then B is minus one for A equals zero and then, which one among these values is it possible to obtain at zero, one, this, I didn't understand the question	He uses the pen to indicate the relative maximum point and then the relative minimum point in the Cartesian graph on the paper. He reads the question in the task.	The set of images is limited. A pre-image of -1 is 0	RAN
183		R	You told me that the values that B can take on are all except for	She points to an interval in between the relative extremes		RAN
184		D	Except for three			DOM
185		E	Except for that part, yes			RAN
186		R	This part here, while these values that it can take on, that is, in how many ways can you obtain these down and these up?		Is the function injective?	INJ
187		D	One			
188		R	Just in one way? how?			
189		D	Yes, because according to me, there is only a number that	He moves the pen as tracing vertical lines:  Fig. 6.34a		INJ

190		R	Or there is only one A-value, by recalling this A			
191		D	There is only one A-value that is six and a half, or there is only one value that is, however, that is zero or that is minus a half		There exists only a pre-image of 6.5, of 0 and of $-\frac{1}{2}$* (*0 is not in the set of images and there are two pre-images of -1/2*)	INJ
192		R	There is also just one value that makes it, if it was seven, seven and a half?	She points to (0, 7.5) on the paper	Is there only one pre-image of 7.5?	
193		D	Yes, yes			INJ
194		R	Is it always just one that of A?			
195	I3Mp2 09:30	D	Yes			INJ

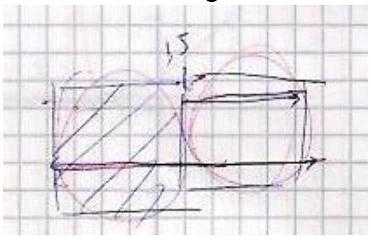
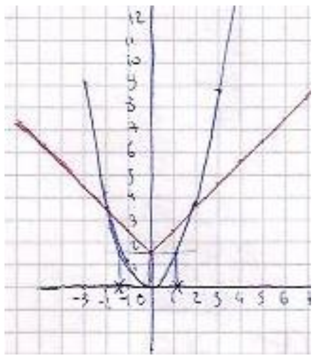
At line 184 of the excerpt 6.34 Davide says that three is the only value that the dependent variable B does not take on but, actually, this is the value which the independent variables A cannot take on. Since at line 182 he refers to an interval of values which B cannot take on, it is not clear which one of the two variables he thinks that does not exist at three. This difference is important from the point of view of potential expert discourse mirrored because it would be about the domain or about the set of images of the function. Moreover, at line 182 Davide's discourse is about "*the values that B can take on*" and so he looks at the extreme values of the set of images but he shows them by pointing to the relative maximum point $(x_1, f(x_1))$ and the relative minimum point $(x_2, f(x_2))$ on the paper, instead of pointing to $f(x_1)$ and $f(x_2)$ on the y -axis.

In the last part of the excerpt, we observe that Davide's discourse is not mirrored by potential expert discourse about the non-injectivity of the function. In particular, when the researcher asks him to find the number of pre-images of a given $f(x)$ -value, he looks at his drawing on the paper and he traces vertical lines (Fig. 6.34a), which intersect the curve at only one point. The mediation of these lines could be used to realize the being well-defined of the function, but not to investigate its injectivity (in this case the lines should be horizontal). However, Davide does not change his idea (see lines 193 and 195) despite the challenges of the researcher at lines 192 and 194.

In the following part of the lesson, which is not reported in the excerpt, thanks to the request of the researcher to check their answer in the realization DGpp of the same function, Davide and Elena recognize that there are some points of non-injectivity. It is interesting that, even if they usually prefer working with the realization SGc, they see the realization of more properties of the function when manipulating its dynagraph.

Excerpt 6.35 - Lesson 4

(Realization DGpp of the two functions $f(x) = x^2$ *and* $g(x) = |x| + \frac{3}{2}$)

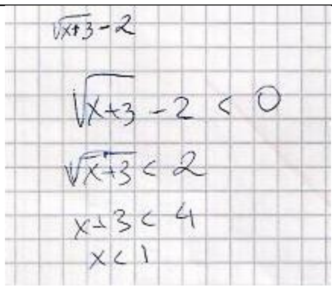
	When	Who	What is said	What is done	Potential expert discourse	Code
280	I4Mp3 07:25	D	Basically, this one [g(x)] greater than one point five and the other [f(x)] everywhere… disjoint? So it is the opposite, this one, this part here [the interval [0, 1.5) in Fig. 6.35a]... unless this one [g(x)] then goes down again, but we don't know	Activity4_2 He draws the diagram:  Fig. 6.35a And then he marks the interval [0,3/2] on the y -axis	The set of images of $g(x)$ is [1.5, +∞) while the set of images of $f(x)$ is [0, +∞). The interval (0, 1.5) is not in common.	RAN
			[…]	They are quite looking at the diagram		
281	I4Mp3 08:20	D	Wait, but from one and a half downward…….so, from these points, no?	He traces the horizontal line $y = \frac{3}{2}$ and he finds the x -values such that $f(x) = \frac{3}{2}$:  Fig. 6.35b		RAN
282	I4Mp3 08:50	E	yes, so from minus one to one			

To solve activity4_2, first of all Davide and Elena build the Cartesian graph of the function on the paper, as for all the other activities. However, in the excerpt 6.35 we can see that Davide uses also the visual mediation of a diagram, shown in the Fig. 6.35a, which actually reminds of the realization DGpp of the two functions, with the trace activated on the dependent variables. By looking at the diagram he describes the set of images of the two functions and he also identifies an interval that they do not have in common. In fact, from the diagram it is possible to see that one of the two ticks realizing a dependent variable does not mark the interval [0, 3/2] and that both of them touch all the values greater than $\frac{3}{2}$. Initially Davide expresses the interval on the y -axis but then, after a short pause of reflection indicated by the suspension markers, at line 281 he passes to the x -axis. In order to do this, he traces the line $y = \frac{3}{2}$ on the paper and he marks the values on the x -axis that are the abscissas of the intersection points with the curve (Fig. 6.35b).

The following excerpt shows another example where Davide involves some visual mediators in his discourse, not presented in the activity or suggested by the researcher, in particular he writes an inequality on the sheet of paper as rephrasing the question posed by the researcher.

Excerpt 6.36 - Lesson 5

(Realization DGc of the function $f(x) = \sqrt{x+3} - 2$)

	When	Who	What is said	What is done	Potential expert discourse	Code
328	I5Mp7 03:30	R	What are the A-values such that B takes on values less than zero	Activity5_2 They have drawn the Cartesian graph on the paper (Fig. 6.36)	For which x the function is negative?	IN/ DEP
329		D	However, we can just do a inequality, for me, no?		We should solve an inequality	
330		R	Which inequality would it be?			
331		D	Square root of x plus three minus two less than zero	He writes the inequality on paper	$\sqrt{x+3} - 2 < 0$	
332		R	And what should you find out?			
333		D	So, it would be like this (Fig. 6.36a)… then, I don't remember how to do it, wait… less than one? Yes, for all x less than one	 Fig. 6.36a		
334		R	So, for x less than one you find that	She points to $x = 1$ in the graph on the paper		
335		D	Indeed, one, it's true, for x equals one it is zero, because one plus three, four, so square root two, and it results zero, so for all [x] less		$f(1) = 0$ because 1+3=4 and $\sqrt{4} = 2$; for all $x < 1$	

			than one it results less than zero		then $f(x) < 0$	
336		R	And here, how can we see it?	She points to the DIM		
337		D	Here we see it, where is A? A is here, B is here, going down… that is, obviously for the values less than one we always go… they are negative	He drags A from 1 to the left and brings it back at 1		
338		R	What are negative?			
339		D	B-values			
340	I5Mp7 04:50	R	Okay			

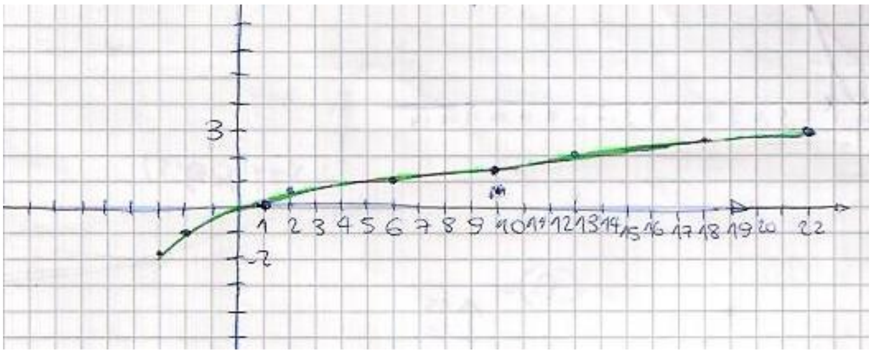


Fig. 6.36. Excerpt taken from students' worksheet

Analyzing the types of mediators used by students, in the excerpt 6.36 Davide employs symbolic visual mediation to answer the question at line 328. Indeed, at line 329 he proposes a realization of the correspondence expressed by the researcher in terms of dependence relation between the two variables, involving an analytic expression of the function. Then he writes the inequality on the paper (Fig. 6.36a) and solves it. In particular, at line 333 he says "*I do not remember how to do this*" and this expression seems an attempt to identify a possible ritual from his *precedent-search-space* for a task situation involving inequalities.

Then the researcher asks him to interpret the solution that he found, by looking at the dynamic realization of the function. At line 337 he succeeds in doing this, since he drags the tick realizing the independent variable from 1 to the left, even if he seems to have some troubles in finding the words to describe what he sees on the screen, as suggested by his discourse "*obviously for the values smaller than one it is, it goes… that is, they are negative*", where the subject is unexpressed.

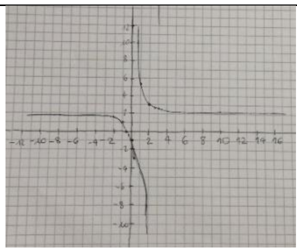
The following excerpt is taken from the interview with Davide and Elena.

Excerpt 6.37 - Interview 4

(Realization DGpp of the function $f(x) = \frac{3}{2x-1} + 2$)

	When	Who	What is said	What is done	Potential expert discourse	Code
1	C 27:45	D	then, for x equals zero, f(x) equals minus one…for x equals one, f(x) is five point one, a little bit more than five	Task3 with Davide at the pc and Elena drawing. The grid is activated. Wandering dragging	$f(0) = -1$; $f(1) = 5.1$	
2		E	A little bit more than five?			
3		D	Yes…then for x equals two, f(x) is three, for x equals three, f(x) is a little bit greater than two point five, for x equals four, f(x) is two point four, somehow it slows, slows, indeed…f(x) doesn't arrive, wait	After speaking he zooms out	$f(2) = 3$ $f(3)$ is around 2.5; $f(4) = 2.4$ The difference quotient decreases	DER
4		E	Does it arrive at two?		Is there $\boldsymbol{x}$ such that $f(\boldsymbol{x}) = 2$?	
5		D	Probably f(x) is two, but not exactly, for thirty		A pre-image of 2 is 30	
6		E	For example, for x equals ten, how much is f(x)?		$f(10)$?	
7		D	A little bit more than two…then, when x equals one, did I tell you that it is about five?	He drags x and then he answers	$f(1)$ is around 5	
8		E	When x equals one, five, yes!			
9		D	when x, f(x) is about ten, for zero point six, that is, when x is zero point six, f(x) is ten… until for about a half, for x zero point five, f(x) disappears	He drags x in a neighborhood of 0.5	$f(0.6)$ is around 10 $f(0.5)$ does not exist	DOM
10		E	Ah! So, at a half, when x is at a half,	She does not write anything	$f(0.5)$ does not exist	DOM

			does f(x) disappear?			
11		D	Yes, for coming back again… it could come back beforee, but for x equals zero point two, f(x) is minus threee……..f(x) is zero for x equals minus one third…….for x equals minus one, f(x) is one		$f(0.2) = -3$; a pre-image of 0 is $-\frac{1}{3}$; $f(-1) = 1$	
12		E	X minus one, is f(x) one?		$f(-1) = 1$	
13		D	Minus one, f(x) is one… for minus two, let's say that it slows, more or less similar to what we saw before, it slows again close to two, indeed, when x is at minus two, f(x) is one point six and f(x) doesn't arrive at two, never here, more or less		In a left neighborhood of -2 the function decreases slowly $f(-2) = 1.6$; $f(x) < 2$ In a left neighborhood of -2 the derivative tends to 0 and $\lim_{x \to -\infty} f(x) = 2$	ASY DER
14		E	For example, x minus eight?			
15		D	When x is minus eight, f(x) is one point eight, it's difficult for you to reach f(x) equals two, if not impossible, so the values that f(x) doesn't ever reach are two… while f(x) doesn't exist for a half…do you need something more?	He drags x backward and forward along the x -axis	$f(-8) = 1.8$ $f(x) < 2$ The function is not defined at $x = \frac{1}{2}$	ASY RAN DOM
16	C 34:15	E	I don't think so!	Before speaking, she moves the pen on the paper but without tracing the curve		
			[…]	Elena's final drawing is:		

				 Fig. 6.37a		
25	C 37:20	D	Now, I don't know if it is relevant or not, but for me, this is a function, it is a fraction having x at the numerator and, probably, at the denominator a half minus x?	He speaks to R	The algebraic expression of this function should be like: $\frac{x}{\frac{1}{2}-x}$	DOM
26		R	At the denominator a half minus x, why?			
27		D	Because for a half it disappears, so when the denominator is zero, if it was a half minus x, when x is a half it would be zero		For $x = \frac{1}{2}, f(x)$ does not exist, so the denominator might be $\frac{1}{2} - x$	DOM
28		R	And why did you say x at the numerator?			
29		D	Eh, I don't know why I do say x at the numerator, no no, I don't know it			
30		R	For large x-values, how much is it?		$\lim_{x \to +\infty} f(x)$?	LIM
31	C 38:40	D	For large x-values it is always, about two, and so here it could be this part plus two, but now I don't know what there is at the numerator		$\lim_{x \to +\infty} f(x) = 2$ So the algebraic expression could be: $\frac{x}{\frac{1}{2}-x} + 2$	LIM ASY

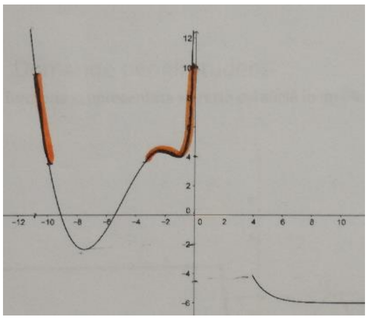
Davide and Elena's discourse in the excerpt 6.37 is mainly focused on some static positions and not on the movements of the two ticks, in fact Davide tells her a set of points $(x, f(x))$ which she plots in the Cartesian plane on the paper and then she traces a curve passing through all these points. This is the same routine performed by the two students during the whole sequence of lessons.

In some lines, though, their discourse is mirrored by potential expert discourse about mathematical objects as the derivative, for example at lines 3 and 13 where Davide describes a change in speed of the dependent variable; or the domain of the function which is not defined at $x = 0.5$ and they express it at lines 9, 10 and 15 by "*f(x) disappears*". Moreover, Davide proposes a symbolic realization of the domain of the function, at line 25, when he tells to the researcher a possible algebraic expression of the function. In particular, he seems to identify a *precedent* about the relation between a point of non-definition for a function and its algebraic expression and he explains it at line 27 "*because for [x equals] a half [f(x)] disappears, so when the denominator is zero, if it [the denominator] was a half minus x then for x equals a half it would be zero".*

Excerpt 6.38 - Interview 4

(Realization SGc of a function, that is showed in Fig. 6.38a)

	When	Who	What is said	What is done	Potential expert discourse	Code
5	DMm6 01:40	E	Color the image of the interval	Task4 She reads the task		RAN
6		D	Four, ten. Is the interval in x or in y? I mean, the interval on the axis			
7		R	The image			
8		E	The image is…	She reads the definition again		
9		D	Ah, color the image, the image.. ehm, this here, from four to ten here	He points to the interval [4, 10] on the y -axis on the sheet of paper		
10		E	No, wait			
11		D	yes…the image of x is on y, while the pre-image of y is on x			IN/ DEP
12		E	Yes, yes so the image of the interval four, ten			
13		D	From four to ten, do we color it here on the axis?			
14		R	Is that the image of the interval four, ten?			
15		E	Yes, yes			
16		D	No, but it's not just one, there are more than one, the image… this is	He points to the interval [4, 10] on the y -axis on the sheet of paper and then he		

			the interval four, ten, the image is all this part and then till ten	marks some parts of the curve (see Fig. 6.38)		
17		E	Yes, it is that part			
18		D	Right?			
19		R	Mm, I don't tell you if it is right or wrong			
20		D	How can we color it? Do we trace it?			
21	DMm6 03:15	R	Yes!	He colors as follows:  Fig. 6.38a		

In the last activity of the interview, after reading the formal mathematical definition of image of a subset of the domain through a function f, students are asked to express the image of the interval [4; 10], given the Cartesian graph of the function. Davide and Elena answer this task by identifying the interval [4; 10] on the y -axis and then Davide marks the points belonging to the curve whose ordinate belong to this interval. We notice that the students seem not to doubt their idea, even if the researcher at line 14 challenges them by asking "*is that the image of the interval [4, 10]*?".

Their discourse seems not to be mirrored by potential expert discourse, because they should consider the interval [4; 10] on the x -axis and searching for the corresponding values on the y -axis, not on the curve. Moreover, they do not correctly answer the question in the activity, even if they are given the realization of the function SGc that seemed to be the realization preferred by them.

6.3 DISCOURSE ON COVARIATION

The analyses of the excerpts, taken from the lessons and the interviews, show several instances of covariation in students' discourse. In particular, we are interested in the covariation of quantities that vary following a dependence relation, so we focus on students' discourse about the relation between variations when they describe it according to the functional dependency.

We notice that in the excerpts above it is possible to identify three different levels of mirrored expert discourse on covariation: covariation of space and time, covariation of the two variables and covariation of ratios. Now we are going to describe them, by bringing some examples.

1. Covariation of space and time: description of the behavior of a variable increasing or decreasing over time, or having the same amount of change with respect to time. In

general, the reference to time is explicitly expressed by students, while the space is intended as the range of variation of a tick on the line.
For example, in excerpt 6.16 Alessio describes the movement of B with respect to "*the interval of time that A moves […]*" (see line 128) or in excerpt 6.28 Franci's discourse is about the dependent variable "*taking the same space in less time*" (see line 46).

2. Covariation of the two variables: description of the variation of the dependent variable with respect to the variation of the independent one. It is usually expressed in terms of "if A moves […] then B […]" or "when A […], B moves […]" and this type of covariation is the most frequently described by students, as we can see from the analyses above.
For example, in excerpt 6.15 at line 109 Alessio relates the variation of $f(x)$ to the variation of x by comparing their ranges of movement. Similarly, Nicco in excerpts 6.19 (see line 41) and 6.21 (see line 78) describes the movement of $f(x)$ for x varying within fixed intervals.

3. Covariation of ratios: description of changes in ratio of Δf(x) to Δx, which is also a description of the speed of a variable with respect to the speed of the other one. Indeed, we have seen that in some cases students describe possible changing in ratio of the variation of $f(x)$ to the variation of x, calculated on different intervals of the real number line.
Examples of this type of covariation can be found in the excerpts above where students' discourse mirrors potential expert discourse about the derivative of the function. For example, in excerpts 6.23 (see line 13) and 6.24 (see line 13) Alessio focuses on changes in ratio of the variation of $f(x)$ to the variation of x, observing that "*f(x) changes less and less in function of the movement of x*".

In the light of these observations, we argue that the activities with the dynamic and static realizations of functions that we proposed supports the emergence of a discourse mirroring expert discourse on covariation. In particular, students expressed covariation by using words and verbs that refer to movement and to changing over time, and with the help of dynamic visual mediators such as gestures, dragging and dragsturing actions. Indeed, we have seen many examples of students describing the dependence relation between the two variables through the description of the range of variations (or of possible changing in speed) of the dependent variable, in relation to the range of variations (or of possible changing in speed) of the independent one. We noticed that Nicco and Alessio's discourse is mirrored by potential expert discourse about covariation more than anyone else in the class. Moreover, it is possible to see that the examples used to explain the three types of covariation are taken from excerpts happened throughout the entire sequence of the lessons. Therefore, although the three different levels of discourse on covariation that we identified do not seem to be related to certain activities, they are probably supported by the specific context used in the activity and by the realizations of functions proposed to students.

6.4 CONCLUDING REMARKS

In this chapter we have analyzed the main features characterizing students' emerging discourse about functions and their properties, by looking at its similarities and differences with potential expert discourse mirrored. We found that all the students show difficulties in finding the words to describe their explorations, for example, they say "*how can I write it down*?" (excerpt 6.17, line 116) or "*otherwise I would not know how to begin with it*" (excerpt 6.19, line 43) or "*I do not know to explain it*" (excerpt 6.31, line 87). However, they heavily use dynamic visual mediators such as gestures and dragging actions to better communicate their observations. There are also some examples of their attempts to identify *precedents,* as possible routines being performed in a specific task situation (see excerpts 6.7; 6.10; 6.14; 6.15; 6.21; 6.22; 6.36; 6.37).

During the analyses we highlighted the various seeds of possible realization of many mathematical signifiers that we identified in students' discourse. In general, the designed activities seem to support students' emergent discourse about many mathematical objects, that is in line with our expectations expressed in the *a priori* analysis. Indeed, from the excerpts presented in this chapter it is possible to see that in some occasions the researcher participates in the pairs of students' discussions but the explorations carried on by the pairs of students are mainly guided by the activities. However, there are also some episodes where the questions of the researcher prompt the students' answers and, especially, we found them during the first lessons, while there is almost any trace of them during the interviews. We reported examples in the excerpts involving Nicco and Alessio, Davide and Elena and during Alessio's interview.

By comparing *a priori* analysis with *a posteriori* analysis of the sequence of lessons we conclude that ***Table 6.1*** shows the number of realizations of functions properties contained in the potential expert discourse mirrored and that most of them are the same with respect to that considered in ***Table 4.2***. However, we have seen how at the end of the sequence of lessons students' discourse on functions is not still objectified. In fact, students express significant calculus ideas in multimodal ways incorporating language, gestures and dragsturing actions during the activities, without using a formal vocabulary. For example, they share a mutual understanding of what "this" or "it" or "there" refers to, even without stating what they mean explicitly.

One central focus of our analyses was students' discourse about the functional dependency. First of all, we investigated how students express the relation '$\boldsymbol{f(x) = y}$', identifying different possible ways that show how dynamic aspects related to dragging actions in the DIMs and more formal and reified expressions intertwine in students' discourse. In particular, we found instances that explicitly mention the process needed to obtain that specific $f(x)$ −value, or that contain some references to motion, but in a less explicit way; and also some reified expressions, even without any verbs. Moreover, we highlighted that in most of the excerpts shown above, students express the dependence relation between the ticks realizing the two variables in terms of covariation of quantities and, in particular, we identified three different levels of covariation depending on the two covarying quantities.

Another aim of our analyses was investigating students' use of the DIMs in their discourse and the analyses suggest that, along the sequence of lessons, there is a sort of development of the DIM: from mediator used by students' in their communication it gradually seems to become a realization of the mathematical signifier 'function'. In Section 6.1.2 we showed this passage in relation to students' use of the word 'function'. Moreover, during the analyses of the excerpts we observed that some students propose other possible realizations of functions as *saming* the DIM with these realizations. For example, Davide introduces the algebraic expression of two functions, when interacting with the realization DGc (excerpt 6.36) and with the realization DGpp (excerpt 6.37), while Franci describes the Cartesian graph of a specific function, that he drew on the paper, as a possible realization of a ‘parabola’ (excerpt 6.33).

7 DISCUSSION AND CONCLUSIONS

In this concluding chapter we explain how the study that we conducted about the use of a dynamic approach to introduce students to functions and their graphs, led to significant findings with respect to the research questions we had set out to investigate. In particular, this study provides an adequate description of students' learning process when they are introduced to the mathematical discourse on functions through a specific dynamic approach. The analyses of students' use of dragging when interacting with the DIMs proposed brought us to identify different types of dragging and to describe the significant role played by dragging in students' discourse. In particular, we developed a model for the evolution of "dragging mediated discourse" that shed light on the development of students' discourse and on their use of dynamic mediators, as gestures and dragsturing actions, both in the dynamic and static context. Moreover, the activities with the DIMs proposed seemed to support the emergence of a discourse on functions and their properties that started as a description of the relation between the movements of two quantities, one depending on the other. This result is in line with our *a priori* analysis, since our dynamic approach to functions and their graphs is designed to highlight this dynamic aspect of functional dependency, that is covariation. Then, the description of a "potential expert discourse mirrored" provided a lens through which it was possible to analyze students' discourse by focusing on the seeds of possible realizations of mathematical signifiers involved. This analysis allowed us to describe the richness of their discourse with respect to the properties of functions and their graphs. Indeed, we found many examples of students' discourse about mathematical signifiers characterizing significant properties of functions, that they expressed by using a non-formal mathematical vocabulary and through a discourse that, in most of the cases, was not objectified. However, we were able to point out these examples by matching students' discourse with a potential expert discourse that they mirrored, especially focusing on the mathematical signifiers that were realized.

After answering the four research questions, we contextualize our findings within the existing literature, highlighting the theoretical contributions that this study offers and then we describe possible implications and directions for further research.

Our findings have no statistical ambitions because of the limited number of cases analyzed. However, the fine grain qualitative analysis that was carried out provided a richness in detail and depth which would not have otherwise been possible. Furthermore, many commonalities emerged during the analyses and so, in a search for more general results, a quantitative research could be fruitfully grounded upon our findings.

7.1 ANSWERS TO THE RESEARCH QUESTIONS

The research questions we proposed to investigate were:

1. Does students' discourse emerging during the proposed activity sequence involve covariation? If so, in what ways?
2. What is the role of dragging, if it has one, as dynamic interactivemediator in students' discourse? Is it used to express covariation? If so, how?
3. What recurrent features is it possible to identify in students' discourse about the different realizations of functions that we design within dynamic and static environments? And in students' attempt to relate them?

4. How does students' discourse compare to potential expert discourse about functions and their properties? In particular, from an expert's point of view, what seeds of realizations of mathematical objects is it possible to identify in students' discourse?

Now we discuss about the results of the analyses that we conducted, highlighting in which sense they provide meaningful insights with respects to each of these research questions.

7.1.1 Research question 1

In this study we analyzed the emergence and evolution of a discourse on functions, working with a group of students who were introduced to functions for the first time, at least concerning their math classrooms at high school. In particular, we focused on some features of their discourse, as suggested by the theory of commognition to investigate about a learning process, and we found several instances where they expressed dependency between the two variables in terms of covariation. These istances have been identified in Chapter 6 by looking at how their discourse possibly mirrored potential expert discourse and this analysis allowed us to distinguish between three different levels of covariation expressed by students, depending on the two covarying quantities considered.

In other words, we showed that the activities with the DIMs that we designed to realize functions supported the emergence of students' discourse on the movements of the two ticks realizing the variables and on the relation between these movements, that is a discourse on covariation. In particular, we identified the following three different kinds of covariation expressed by students, depending on the two quantities considered: covariation of space and time, covariation of the two variables and covariation of ratios.

Covariation of space and time

In some cases, student described the behavior of a variable increasing or decreasing over time, or having the same amount of change with respect to time. In particular, in order to express the range of variation of a variable they referred to the space walked by the tick on the line in the DIM and they discussed its evolution over time.

For example, in the following excerpt from the third lesson (see the analysis in chapter 6) Alessio and Nicco were exploring the realization DGpp of a function and Alessio described the direction of movement of the dependent variable B with respect to the time interval within which the other variable A moved from 5 to 0, and then when A moved in a right neighborhood of 5. The covariation of space and time is expressed here for the tick A realizing the independent variable. Indeed, he described the range of variation of A identifying it with the space walked by the tick on the line, with respect to the interval of time that this variation lasted.

In Figure 7.1, below the excerpt, there is the written description given by Nicco and Alessio about the same aspect of the DIM, where it is possible to see that they referred to the "*in the time interval from 1 to 5*" to discuss the covariation.

Excerpt 7.1 - Lesson 3

(It is taken from excerpt 6.16)

	When	Who	What is said	What is done
127	I3C2 37:00	R	So, B is leaving this trace, how can this fact help us?	Activity3_1

				The trace tool is activated on B
128		A	To see where B passes throw. Then, as we said, B first passes everywhere, it stops at six point five and at minus zero point five. And then it follows, that is, in the time interval in which A is between zero and five, B moves in the opposite direction to A, while when A passes over five B moves in the same direction.	Continuous dragging of A from 5 towards 0 and then to the right

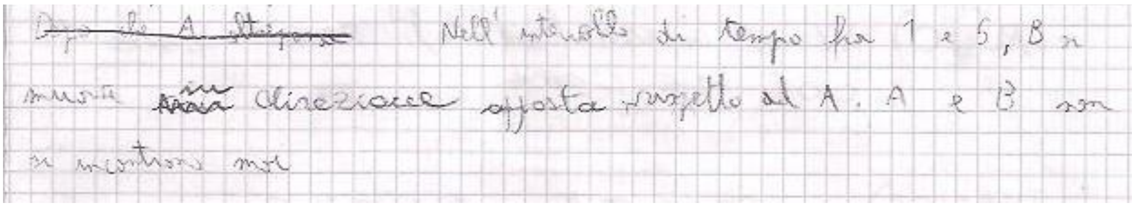


Fig. 7.1. Excerpt taken from students' worksheet

Covariation of the two variables

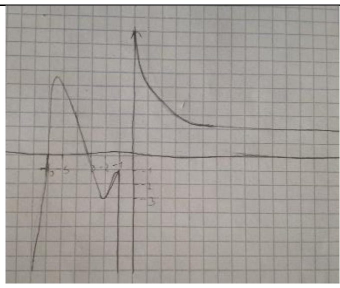
In other cases, students described the variation of the dependent variable with respect to the variation of the independent one, that is a discourse on the covariation between two variables, one depending on the other. We reported several examples of this type of discourse on covariation in Chapter 6, because it was the one most frequently addressed by students, and now we just recall one of them.

For example, in the following excerpt Alessio was exploring the realization DGpp of a function and he had to draw on a paper the realization SGc of the same function. From this short excerpt we can see that before realizing it in the Cartesian plane (Figure 7.2), he looked at the relation between the variations of the two variables for large negative x-values, gaining information about the slope of the graph. In particular, he related the small variations of $f(x)$ to the wider variations of x, by comparing the range of their movements along the lines. The resulting discourse is on the covariation between the two variables.

Excerpt 7.2 - Interview 1

(It is taken from excerpt 6.24)

	When	Who	What is said	What is done
33	C 34:15	A	As this [x] goes on, it [pointing to the graph] becomes more and more ... that is, for a very small change of x, f(x) moves more and more, that is, here I do not know but to get from minus six to minus seven, f(x) goes from minus one to minus fifteen and from minus seven to minus eight, f(x) goes from minus fifteen to minus forty or	Task3 He draws the curve for $x < -6$ and he erases it, repeating these actions several times. Finally, he draws the following graph:

			minus fifty.. In my opinion it is like this.	 Fig. 7.2

Covariation of ratios

There were also some cases in which students focused on the changes in ratio of Δf(x) to Δx, which is a description of the speed of a variable with respect to the speed of the other one. In particular, they looked at the ratio of a variation of $f(x)$ to a fixed variation of x (for example, within a segment of length 1), then they looked at the ratio of a variation of $f(x)$ to the same fixed variation of x but within another interval of the real number line. Then, they described the relation between these two ratios and so we called it covariation of ratios.

For example, during the second lesson Alessio gave the description reported in excerpt 7.3 (see the analysis in chapter 6), while interacting with the realization DGpp of a function. In particular, his discourse was about the different ranges of variation of the dependent variable given a fixed amount of change of the independent variable, but in correspondence of different parts of the real number line. He made this description in terms of the "sensitivity" of the dependent variable to the movements of the other variable.

Excerpt 7.3 - Lesson 2

(It is taken from excerpt 6.15)

	When	Who	What is said	What is done
45	I2C2 37:00	A	Over here as you move, it seems that it becomes more sensitive, like B to the movements of A, because here if [x] moves by one, I do not know, one millimeter, this one [f(x)] moves maybe by two millimeters, instead here if this [x] moves by one millimeter, here this one [f(x)] moves by two centimeters so I do not know...	Activity2_1

In all the three kinds of covariation expressed by students, we observed an use of words and verbs that refer to movement and to changing over time, and of dynamic visual mediators such as gestures, dragging and dragsturing actions. Therefore, the activities designed with the DIMs to realize functions seemed to support discourses on functions and their properties that draw attention to the covariational aspects, which was our first goal behind the choice of a dynamic approach to functions and graphs of functions.

7.1.2 Research question 2

The dynamic realizations of functions that we designed are made possible by the use of a DIE and, especially, by the dragging tool and for this reason we were interested in analyzing students' use of dragging as dynamic visual mediator in their communication about the DIM. In line with what we expected, the analyses reported in Chapter 5 show that students performed different types of dragging to engage in mathematical discourse practices and, in particular, dragging had a two-fold role for students: it *allowed* them to speak and it even *became necessary* to speak about covariation.

We developed a classification of different types of dragging, that we used as tool of analysis and it allowed us to describe the routines performed by students and to investigate about the mediation of dragging in students' communication. In line with what we expected, we found that students performed mainly discursive routines and by analyzing the interplay between the types of dragging used and the descriptions related to the dragging actions we observed a progressive modification from rituals in the direction of explorations.

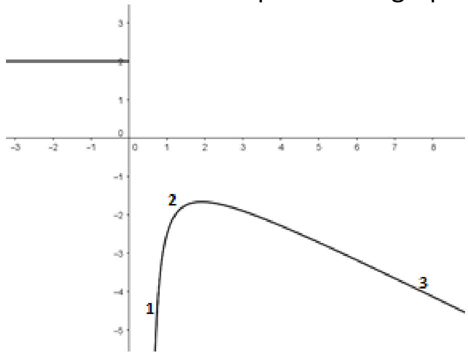
Moreover, we described a model for the evolution of dragging mediated discourse, that shows how during the sequence of lessons an individualization process of dragging took place for students. In particular, during the passive phase of dragging mediated discourse, dragging was the object of students' discourse, then during the active phase dragging enlarged students' communicational actions about DIMs and in some cases it subsumed mathematical signifiers. In this phase we identified a possible turning point for the development of students' discourse about functions, towards the discourse of an expert mathematician. Indeed, during the active phase students' performances were mainly communicational actions involving the physical use of dragging for manipulating the DIMs and so a change in the objects of the construction is caused by the dragging actions. In particular, the tick realizing the independent variable moves under the direct action of dragging and it causes the indirect motion of the other tick, realizing the dependent variable. However, the desired outcome of the routines seemed to become a discourse about the mathematical properties of the function realized by the DIM. This is why we spoke about turning point, intended as a change of focus in the discourse: from being a description of possible and impossible movements in the DIM, it progressively became a description of possible relations existing between the movements of the ticks, that are observed on the computer screen. Moreover, students focused their attention on the speed of the ticks, the direction of their movements, the range of their variations on the real number line and it suggested us that they gradually involved mathematical signifiers into their discourse. They also started using the passive voice or they gave variables the role of grammatical subject, instead of referring to themselves in the narration or to another person acting on the DIM.

As we were previously discussing about, together with these changes we observed a change in the routines performed by students, moving from rituals in the direction of explorations. Indeed, by analyzing how their uses of different types of dragging developed from the passive to the active phase, we observed that there were much more examples of *guided, handle* and *test dragging* than *impossible* and *wandering dragging*. We considered this aspect to be relevant because, according to our description of these types of dragging in Chapter 2, we expect that an expert mathematician would use *guided, handle* and *test dragging*, both *continuous* and *discrete*, but we do not expect to find many examples of *impossible dragging* in a potential expert discourse about the DIMs.

Finally, we described the detached phase of dragging mediated discourse in which dragging was used by students as visual mediator to communicate about the dynamism of the DIMs in a context out of the DIE. For example, in the following excerpt (see the analysis in chapter 5), Matilde and Nicco were working with the realization SGc of a function and, in particular, they were comparing the speeds of the two variables for positive x-values. In order to do this, they referred to the movements of the ticks along their axis and to the students' dragging actions in a realization of the same function in a DIE, even if it was not present at that moment. Indeed, at line 12 Matilde says "*if you drag*" and at line 14 she even moves her hands along the lines, using the dynamic visual mediation of dragsturing.

Excerpt 7.4 - Interview 2

(It is taken from excerpt 5.13)

	When	Who	What is said	What is done
12	MNm1 04:03	M	But here [positive x-semiaxis] is x or y faster? No y, no no no, if you drag in this direction [to the right] ……..x is faster	Task1
13		N	Yes yes, x is faster while y is slower I mean in this case it seems that here [**1**] y would move faster, ehm because from here to arrive up here [**2**] it takes like this[6], then from up here to go back to this point [**3**] it takes like this[7] and…it is the same x, but faster, you drag one of them and so	He indicates different part of the graph:  Fig. 7.4a
14	MNm1 06:00	M	Exactly, no x is always faster I think…………. Wait, no but, are we sure that this is faster? Wait! Why faster?	During the pause she moves the hands as follows:  Fig. 7.4b

[6] In Italian he says "ci mette questo", and he opens his fingers as measuring the distance between the points 1 and 2 on the curve.
[7] In Italian he says "ci mette così", and he opens his fingers as measuring the distance between the points 2 and 3 on the curve.

				 Fig. 7.4c

The identification of a detached phase of dragging mediated discourse was an innovative and interesting result because it showed that the close connection between dragging and the temporal and dynamic aspects of functions entered students' discourse within a static context as well. Indeed, we found that dragging and dragsturing actions allowed them to accomplish *saming* of a realization in paper-and-pencil context (the Cartesian graph) with a DIM.

7.1.3 Research question 3

From the analyses in both Chapter 5 and Chapter 6 we did not find evidence of significant changes in students' discourse that can be related to the passage from one realization of functions to another one. In particular, we observed that the characteristics of Davide and Elena's discourse were quite the same during the whole sequence of lessons. We think that it was mainly because they have always worked with the Cartesian graph drawn on a paper as realization of functions, yet from the first lesson. Indeed, they used the DIMs proposed to identify pairs of values in order to plot some points in a Cartesian plane and to trace a curve passing through them. Even if the task situation almost always involved the exploration of a DIM and the description of the observed movements, also in a written discourse, their main *task* during all the activities was that of drawing the Cartesian graph of the function. In particular, we did not find any examples of their attempts to *same* the realization SGc of a function with the other realizations that we proposed to them. However, in some cases, we noticed that Davide accomplished a saming of the Cartesian graph with the analytic expression of the same function, because he expressed it by looking at the Cartesian graph and then he also used it to describe properties of the function.

According to these observations, we can conclude that Davide and Elena were a bit apart from the other pairs of students, who were not introduced to the realization SGc of a function until the seventh lesson. However, we found that also the other students' discourse did not present substantially different features when they were asked to pass from the dynamic realization in one dimension (DGpp) to that in two dimensions (DGc) and not even when going from a DIE to a paper-and-pencil environment. This fact has been also highlighted in the previous section when we discussed about students' use of dragging and dragsturing actions to mediate the communication with other students within the static context as well. At this regard, we think that the specific design of the construction in the DIMs facilitated the blending of dragging and gesturing actions, for example the ticks realizing the variables were not labeled and it brought students to often refer to them through pointing gestures made with hands and fingers or through the mouse. The analyses suggested that this type of non-linguistic communication, that is an act of dragsturing, was a recurrent feature in students' discourse about the different realizations of functions that we designed. Dragsturing actions represented an important mode for students of communicating dynamic and temporal aspects of functions both in static and dynamic environments.

Moreover, we noticed that very often students had difficulties in communicating about the features of the realizations of functions proposed, that were properties of functions, and, in particular, it seemed to be difficult for them to put their observations into a written discourse. For example, they showed difficulties in finding the words to describe their explorations and they expressed them explicitly through expressions like "*how can I write it down*?" (excerpt 6.17, line 116) or "*otherwise I would not know how to begin with it*" (excerpt 6.19, line 43) or "*I do not know to explain it*" (excerpt 6.31, line 87). We also considered students' acts of substituting dragsturing actions for words in the verbal communication as showing possible difficulties that they encountered.

Another aspect that we investigated during the analyses was students' search for *precedents,* as possible routines being performed in a specific task situation. We found several examples of students' attempts to identify *precedents,* that we described in Chapter 6, and we noticed that this happened especially when they were introduced to a new (for them) realization of function. For example, when we asked them to pass from the realization DGpp of a function to the realization DGc of the same function, they tried to perform similar routines. It also happened in the passage from the realization DGc to the realization SGc, when they still tried to perform discursive routines involving dragging. Moreover, we showed several excerpts where students described the function as being "a parabola". For example, it happened in some episodes where they found out that the function had a maximum or a minimum point, as if they identified this property to be typical of a parabola, from their precedent-search-space.

Concerning the second part of our research question, that is about the relationship between the different realizations of functions proposed to students along the sequence of lessons (DGp, DGpp, DGc and SGc), we mainly identified students' attempts to build some of these relations during the interview. Indeed, we designed the interview with some activities where we explicitly asked them to pass from one realization to another one. We noticed that students' discourse involved within this type of task situations was still characterized by a richness in references to movement and time, as it was when they had to interact with just one of these realizations. In particular, the analyses of their discourse showed that along the whole sequence of lessons, and also during the interviews, they focused on the "processes" more than on the "objects". Indeed, the focus of their discourses was mainly on the dynamic and temporal relationships between the ticks realizing the variables. This attention to the processes brought them to non-objectified discourses, both when interacting with dynamic and static realizations of functions. It can be seen in their descriptions of possible movements of $f(x)$ (e.g. "*B goes everywhere, it stops at six point five and at minus zero point five*", excerpt 6.16; "f*(x) does not pass over six point five*", excerpt 6.23) and in their descriptions of the reciprocal movements of the two variables (e.g. "*they move symmetrically and in opposite directions*", excerpt 6.14; "*after [x equals] thirteen it [f(x)] goes down",* excerpt 6.18); and in many other examples that can be found in the previous chapter.

Moreover, even if students did not make use of a formal mathematical vocabulary, a sort of development in their discourse occurred from the first lesson to the last one. For example, the labels A and B that were used by students to name the ticks realizing the variables at the beginning of the sequence of lessons, were then replaced by 'x' and '$f(x)$'. Even more significant is that the analyses that we conducted in both Chapter 5 and Chapter 6 highlighted how students' use of the DIMs in their discourse developed during the sequence of lessons: from mediator involved to better communicate, it gradually seemed to become a realization of the mathematical signifier ‘function’. In particular, we showed this development in relation to students’ use of the word ‘function’, but we also observed that some students made a process of *saming* between the different realizations of function proposed. For example, when we asked students to interact with both the realizations DGp and DGc of the function $f(x) = -x + 5$, almost all of them immediately observed that they were the same thing; through discourses similar to that of Nicco and Alessio in the following excerpt (see the analysis in chapter 6).

Excerpt 7.5 - Lesson 5

(It is taken from excerpt 6.18)

	When	Who	What is said	What is done
55	I5C2 21:30	N	But it is the same	Activity5_1bis
56		A	It is the same, but there is just one line… we are smart! […]	
57	I5C2 27:00	R	Did you compare the two files together?	
58		N	Yes, according to us they are the same but just on one line	
59		A	That is, the second is just on one line	
60		N	Also here they intersect at two point five	
61	I5C2 27:30	A	Because we made the grid and the numbers too, and they intersect on two point five and before we found that A plus B equals five, in fact	

Similarly, as we previously discussed, some students accomplished saming also between a graphical realization of a function and an analytic realization of the same function. For example, it happened in excerpt 6.36 (Davide at line 333) and in excerpt 5.7 (Matilde at line 73).

7.1.4 Research question 4

Besides shedding light on the interplay between the use of words, gestures and dragging actions, that we addressed through the other research questions, the analyses conducted in this study also provided a deeper understanding of pairs of students’ mathematical discourse about functions, when interacting with a DIE.

In Chapter 6 we investigated about a possible relation between students’ discourse and that of an expert, in terms of similarities and differences, and we showed how by using a non-formal mathematical vocabulary students’ discourse was mirrored by potential expert discourse about many properties of functions and their graphs. In particular, dynamic and temporal aspects related to dragging actions and more formal and reified expressions intertwined in students’ discourse about their interaction with the dynamic and static realizations of functions proposed. However, during the whole sequence of lessons students’ discourse was not objectified. Indeed, they engaged in the development of a mathematical discourse about the dynamic and temporal nature of functional dependency and of the other properties of functions, through everyday language. We also highlighted that, for doing this, they seemed to be helped by multimodal ways of communicating that incorporated language, gestures and dragsturing actions.

Moreover, through ***Table 4.2*** and ***Table 6.1*** we pointed out the close relationship between *a priori* and *a posteriori* analyses of the whole sequence of lessons. Indeed, the designed activities proved to support students' emergent discourse about many mathematical signifiers, in line with our expectations expressed in the *a priori* analysis. In particular, we showed that the potential expert discourse mirrored contained possible realizations of many properties of functions and we observed that most of them were also considered in ***Table 4.2***. Now we are going to discuss what seeds of possible realizations of mathematical signifiers it was possible to identify in students' discourses, from an expert point of view.

- Dependency: in order to realize the dependence relation between the values taken on by the two ticks, students used different expressions that we have listed in Section 6.1.1. Some of these realizations explicitly refer to motion and to the dragging actions made in the DIM, other still contain these references to the dynamism but in a less explicit way. Moreover, we found some very reified expressions, even without any verbs, used to realize the dependency.
- Domain and set of images: they are realized in students' discourse in terms of possible movements of the tick realizing the independent and the dependent variable, respectively.
- Injectivity: we noticed that the injectivity was not a property described by students, while we identified some discourses about the non-injectivity of a function, that seemed to be observed by students. These discourses realized the non-injectivity as a characterization of several x-values which make $f(x)$ taking the same value.
- Monotonicity: the monotonicity properties of a function, or the intervals of monotonicity, were realized by discourses about the relation between the directions of movements of the two variables. An expert would describe a function as increasing if students described both variables having the same direction of movement, while an expert would describe a function as decreasing if students described the variables moving in opposite directions.
- Relative or absolute maximum/minimum: in a similar way with respect to an interval of monotonicity, a possible realization of an extremum point was the identification of a change in direction of the dependent variable, with the independent one always following the same direction. Moreover, we found that students realized a maximum point also through expressions like "the independent variable stops and goes back again" or through the help of gestures reproducing in the air this behavior of the ticks or drawings on the sheet of paper.
- Limits and asymptotes: vertical asymptotes with the left and right limits having different signs were realized in students' discourse by expressions of surprise. In some cases, they even asked to other students or to the researcher if their DIM was broken because one of the ticks suddenly disappeared out of their screen and then it quickly shot back from the other side of the screen. This sudden disappearing and reappearing of the tick has been also realized by students through the image of a small object moving around the screen, like a satellite that is orbiting.

While horizontal asymptotes were realized in students' discourse through descriptions of the speed of the tick realizing the dependent variable which drastically slows down and then it approaches a specific number.

- Derivative: in many cases students' discourse contained references to the rate of change of the ticks along the axes. For example, students distinguished between a constant growth and an accelerated growth by moving the tick realizing the independent variable at constant speed along the x-axis and describing the movement of the tick realizing the dependent variable. In particular, they realized differently $f(x)$ always having the same increment or whisking off the screen. Otherwise, some students focused on the range of variation of $f(x)$ with respect to the range of variation of x, by looking at different intervals of variation on the real number line and comparing them.

  Finally, some of the students also realized the observed changes in the derivative of a function by changing the slope of the curve that they were drawing in the Cartesian plane.

We would also observe that each line in ***Table 6.1*** corresponds to a specific activity and so to a realization of one function. The presence of many colored squares for only one activity shows how the description of some properties of a function involved students' observation of several aspects characterizing the same function, and these aspects intertwined in their discourse. The following two excerpts represent two short examples of this feature of students' discourse. We took them from the analyses in Chapter 6 in order to show the plurality of mathematical signifiers involved in students' discourse, also in very short excerpts like these, and we expressed the specific signifiers that we identified through some codes in the last right column.

Excerpt 7.6 - Lesson 2

(It is taken from excerpt 6.15)

	When	Who	***What is said***	**What is done**	**Potential expert discourse**	**Code**
118		N	Also between [x equals] one and two it does not exist, between one and two it does not exist		The function is not defined on the interval [1; 2]	DOM
119	I2C2 46:27	A	It seems that it bounced………for me it seems a bouncing ball	Fast dragging where B moves "well" and slow dragging around the "critical points"	There is a relative extreme point	MAX/ MIN MON

Excerpt 7.7 - Lesson 3

(It is taken from excerpt 6.27)

	When	Who	*What is said*	What is done	Potential expert discourse	Code
39		L	But not always, because here [x = 9] B still follows A and when it is arrived at six point four or point five it takes and goes back and it reappears when A passes over, it reappears when A is two, no, it reappears when A is here [x = 2.5] more or less	Activity3_1 He drags A from right to left	For large positive x-values the function is increasing, 6.5 is a relative minimum. In a neighborhood of 3 the function is not defined, there is a vertical asymptote	MON MAX/ MIN DOM ASY

A notable difference that we pointed out between ***Table 4.2*** and ***Table 6.1*** is about the derivative, which was realized by students' discourse even when this mathematical signifier was not included in the *a priori* analysis. This was probably due to the nature of the DIMs proposed, where the movement is essential to explore the construction and, thus, the functional relation. Therefore, the exploration of the DIMs seemed to promote discourses about the speed and changes in speed of the two ticks, and these types of discourse mirrored potential expert discourse about the derivative of the function.

Finally, we already discussed in Chapter 6 possible modifications in some of the tasks given in the activities, that were designed to support students' discourse about specific mathematical objects but they actually seemed to put the focus on other aspects of the functions involved.

7.2 Contextualization of findings within the literature and main research contributions of this study

In this section we situate our results within the field of mathematics education. In particular, we discuss how they can be considered with respect to the literature on the teaching and learning of functions and about the use of DIEs in classroom practices. Moreover, we present the study's theoretical contributions to research employing the commognitive framework and, especially, to studying the use of DIEs in teaching and learning processes.

7.2.1 Contextualization of the study's findings with respect to the literature

We chose the theory of commognition as theoretical framework for this study because we shared with it the main assumptions about the learning of mathematics. In particular, seeing mathematics as a special discourse, whose objects are discursive objects, allows us to look at "understanding mathematical objects" in terms of communicating about them. In order to do this, students engage in a process of individualization of a discourse, which corresponds to how learning occurs. Indeed, communication and cognition are seen as two manifestations of a same phenomenon and we were completely in line with this view. Especially, we considered worth to underline that communication has to be intended to include all forms of communication and not just the verbal one. Indeed, we were also interested in students' use of gestures and dragging actions.

Moreover, it seemed to us that commognition could provide operative tools of analysis that allow to capture very fine details and to avoid as much as possible the role of the interpretation in the analyses. Indeed, Sfard described four main features characterizing the mathematical discourse, that are words, visual mediators, narratives and routines, and it can be investigated how they are used by students in their discourse. We thought that this type of analysis could be insightful with respect to the problem of studying how students, who were introduced to functions through a specific dynamic approach, were learning functions. This is because we had the tools to describe this learning process in terms of what and how students communicated about functions. In particular, it seemed to us that analyzing the main features of students' discourse on functions we could describe their learning process without making any kind of unfounded inferences but focusing on what and how they communicated, which characterize their thinking. Indeed, one of our concern was that of objectivity, which is always a delicate point in educational research where it is impossible to keep the subjectivity of the researcher aside, because she inevitably has to make some choices influencing the study.

However, conducting the analyses was not always straightforward given the tools offered by the theory. In particular, we identified our main problem to be the choice of using a DIE, because we noticed that several aspects related to this specific context could be found in students' discourse and it was not easy for us to address them in the light of commognition. Students' discourse was rich in references to the dynamic and temporal aspects characterizing their interaction with the DIMs. This was especially pointed out in chapter 6, when we analyzed Nicco and Alessio's discourse and also Nicco and Matilde's discourse. Moreover, they did not always use words, visual mediators and narratives that were specific of a mathematical discourse. For example, in chapter 5, we observed that dragging was heavily involved in their communication with themselves, with other students and with the teacher, and it seemed to not be used merely as a mediator to better communicate. Indeed, at times, it subsumed the dependence relation, as discussed in Section 5.2.1.

We concluded that our main difficulties emerged from the use of a particular realization of function that is implemented in a DIE. The theory of commognition has indeed been developed in the context of written discourses and not specifically addressing the use of digital artifacts in the teaching and learning process.

As we have pointed out, we were interested in investigating about students' consistent use of dragging, especially in relation to their learning of functions and their description of covariation between quantities. Dragging seemed to play a central role in their interaction with the DIMs used to realize functions and so, in their discourse but, according to the theory of commognition, it could be described as a visual mediator. Therefore, we looked for a more detailed characterization of visual mediation and we based on the studies conducted by Ng (2014, 2016). She worked with bilingual learners on derivative function by using a DIE and her analyses provided strong evidence that there was an interplay between modes of communication, mathematical thinking and use of visual mediators. In particular, the participants in her study communicated about fundamental calculus ideas differently when prompted by different types of visual mediators. Indeed, she designed activities on the derivative function involving both dynamic and static environments and observed bilingual students working with them. What she found was that there were several changes in their discourse in relation to changing in the environment. Differently from her findings, we did not find significant differences in students' discourse in the passage from dynamic to static environment, as we largely discussed above, they even communicated with the help of the dynamic visual mediation of dragging without physically using the tool. In line with this direction of research, we integrated her description of visual mediation and we will better illustrate it in the next section.

Indeed, an important part of this study consisted in refining some results taken from the literature that were useful and interesting for our research problem but we had to describe them through different theoretical lenses. Indeed, several studies that we referred to, about functions and also about the use of DIEs, were grounded on theories having different backgrounds and different underlying assumptions about the learning process, with respect to ours. For example, we found that students heavily used the dynamic visual mediation of the dragging, and sometimes also the trace tool, to communicate about functions in terms of dynamic and temporal relations between the variations and the movements visible on the screen. A similar investigation about the use of DIEs and about the role played by the specific design of the activities in supporting students' discourse about the dynamic features of functions and graphs, has been carried on by Falcade, Laborde & Mariotti (2007) who described very similar results but through a different perspective, because their study was based on the theory of semiotic mediation.

On the other hand, the literature that we took into consideration that draws on Sfard's theory have often a different focus of research. Indeed, we noticed that some studies grounded on commognition (Nachlieli & Tabach, 2012; Ng, 2018), investigate students' individualization process of the discourse of an expert and so, what happens during the hypothetical sequence of lessons is the introduction of a formal vocabulary by the teacher, then the researcher analyzes students' progressively objectification of their discourse on a specific mathematical object.

Differently, our goal behind the design of the sequence of lessons was that of supporting the emergence of a students' discourse about function that involved the dynamic and temporal aspects characterizing this mathematical object. Indeed, we think that they are usually left implied when students are introduced to functions starting from the formal definition or the static realization of a function in the Cartesian plane. For this reason, we preferred to stress these aspects of the discourse, and so of students' learning, that are related to the covariation of two quantities, one depending on the other; and we did not give to students the definitions or the formal mathematical terms that can be found written in textbooks. As Caspi & Sfard (2012) pointed out, even if they were discussing about the teaching and learning of arithmetic, an approach like this

> "minimizes the danger of purely ritualized learning, as a result of which the student would only be able to see algebraic discourse as a 'discourse for others'." (p. 65).

Indeed, a significant result of this study is that during the sequence of lessons students communicated about functions and their properties through discourses rich in reference to movement and time, focusing on the processes rather than on the objects; but we also found remarkable similarities between the students' discourse and a more formal reified discourse on functions. We have especially highlighted this aspect in the analyses, by the description of expert discourse mirrored.

Another important problem of research concerns the possible effects of a formal introduction of the word 'function' and its properties, in the form of possible changes in students' discourse. In particular, it would be interesting to provide students with the formal mathematical terms used by expert mathematicians to indicate the properties of functions and their graphs, after a sequence of lessons similar to that one designed in this study. In this way, it could be investigated if the realizations of the mathematical signifiers, that already emerged in students' discourse during the sequence of lessons, change; and eventually how they change in relation to the objectified discourse of an expert. For example, Ayalon, Watson & Lerman (2017) tried to force an object view of functions (Sfard, 1991) for studying possible implications in students' understanding of functions and, in particular, they wanted to explore the role of the word 'function' in students' development of their concept image. What they found was that

> "evidence for an object view of functions is mostly implicit, in that students were responding to the word as if it was a noun describing something, which could have been a set of actions, a process, or an object. […] Students have different 'object views' and the treatment of the idea of function as a noun is not enough to guarantee a full range of meanings – becoming an object and being fully understood could be separate lines of development" (p. 16).

This is completely in line with our findings about students' discourse. Indeed, on one hand we found that they did not developed an objectified discourse on functions, and they even used very few times the word 'function'. But, on the other hand, we highlighted the richness of seeds of possible realizations of mathematical signifiers, related to functions and their properties, in students' discourse.

7.2.2 Theoretical contributions of this study

This study accounts for three main contributions from a theoretical point of view.

1. First of all, we contribute to the refinement of Sfard's notion of *visual mediator* by introducing a notion of DIM (Dynamic Interactive Mediator) that characterizes a specific kind of mediator, which takes into account the distinctive features of a DIE. This notion is grounded on the distinction between two kinds of visual mediation, dynamic and static, that has been suggested by Ng (2014, 2016). We further developed her line of research on dynamic mediation since we consider this distinction between static and dynamic to be very important, especially when functions and graphs are the objects of the discourse. Moreover, we decided to move the focus away from "visual" and we introduced the term "interactive". Indeed, gestures, dragging and dragsturing actions, as the constructions within a DIEs, play a significant role in this study and they are used by students as mediators that are not only visual. In particular, the design of specific DIMs allowed us to realize mathematical relations through dynamic and temporal changes (Ng & Sinclair 2013; Núnez, 2003). This is particularly significant in a study involving functions, because DIMs can be used to communicate about the asymmetric relation between the two variables in terms of covariation of quantities. In this way, also the various properties of functions can be realized as different relationships between the movements of the variables within their sets of definition. On the contrary, a discourse on functions involving static visual mediators does not necessarily express the properties of functions with respect to the covariation of the variables, but it may refer to other realizations as an analytic expression or the drawing of a graph.

In particular, we observe that a feature characterizing the notion of DIM, that makes it more significant as theoretical contribution, is the fact that in experts' discourse it is used as realization of a mathematical signifier. For example, in this study we asked students to interact with particular graphs implemented within the software GeoGebra, that in a potential expert discourse are used as realizations of functions while for the students, at least initially, they were the objects of their discourse.

2. Moreover, we proposed a possible integration to the notion of *dragsturing*, that was originally described by Ng (2014). In particular, according to our analyses, the description of an act of dragsturing as subsuming both gesturing and dragging actions can be extended in order to include also that cases in which there is not a physical use of the dragging tool. This integration is important because the use of DIEs in the process of learning has long-term effects on mathematical thinking, which can be observed in the characteristic discursive patterns produced through their use. Therefore, it allows a more complete description of students' actions, and so of their discourse, when they are engaged in activities implemented within both dynamic and static environments.
In this study we found that students used dragsturing actions to communicate about functions, also out of the context of the DIE. This fact has been highlighted by the identification of a detached phase of dragging mediated discourse, characterized by students' use of the dynamic visual mediation of dragging in their discourse about functions realized through the Cartesian graph on a paper. Therefore, we described this kind of actions, not involving a physical use of the dragging tool but that are actually gestures realizing dragging actions, as examples of dragsturing.
For example, we have shown in excerpt 5.12 that Matilde said "*x you have to move it inevitably towards here [to right], because you do like this, you do like this because it has to stay perpendicular*", while moving her hands on a sheet of paper containing the drawing of the Cartesian graph of a function. In particular, she was moving her fingers along the axes, as reproducing dragging actions in a dynamic realization of the same function.

3. Finally, we further contribute to the current literature about dragging practices in DIEs. First of all, through the description of the consistent use of dragging and dragsturing actions to complement the use of words that, in some cases, even replace the use of words. Moreover, we developed a classification of different types of dragging, that we summarized in ***Table 2.1*** and it can be used to investigate about the routines performed by students within particular task situations implemented in a DIE. Indeed, the identification of which types of dragging are used by the students, *a posteriori* gives information about possible patterns in discourse. In particular, our description was inspired by Arzarello et al.'s (2002) study where they identified different dragging modalities used by students in the process of generating conjectures about geometric open problems. The classification that we developed is more general, because it does not specifically address the exploration of geometric problems, and we also obtained the distinction between the types of dragging in a different way. Indeed, they adopted a different theoretical lens that allowed them to operate the distinction among the different modalities depending on students' goals behind a specific dragging action. According to our theoretical framework, it is not possible to identify these goals, because they are not considered as part of the

discourse and the focus of the analyses is the discourse itself. Therefore, for us, the only way to identify students' goals is that they communicated about them, that is, students expressed their goals explicitly in their discourse.

Our analyses show that students performed different types of dragging to engage in mathematical communication about the different realizations of functions and we succeeded in characterizing them through two levels of analysis: one refers to the quality of the movements that are visible on the computer screen, while the other takes into account the interplay between a dragging action and students' discourse, that involves verbal descriptions and gestures, in the moment of dragging.

In the analyses we showed that this analytical tool allows to analyze students' discourse on dynamic graphs and, in general, on functions, and we think it has value at the cognitive, didactical and epistemological levels. From a cognitive point of view, the tool is important because the classification proposed was identified *a posteriori*, through an empirical analysis of the data collected. Epistemologically, the use of different types of dragging suggests what types of routines are performed by students and they can be put in relation with the routines that we expected from an expert mathematician within the same discourse. It follows that, from a didactical point of view, the teacher could promote specific types of dragging, through appropriate tasks, knowing what to expect and how to gradually foster the transition of routines in the form of rituals to explorations.

7.3 DIDACTICAL IMPLICATIONS

In terms of implications for the teaching and learning of functions and graphs of functions in classroom, this study suggests a specific design of activities that can be employed in order to exploit the use of dynamic realizations of functions within a DIE to support the emergence of students' discourse on functions in terms of covariation of two quantities.

In relation to the use of a DIE, we gave insight into possible uses of dragging as dynamic mediators in students' discourse. In particular, we described a model for the evolution of dragging mediated discourse that show some characterizing features of the discourse that seem to be supported by the activities within the DIE. These are significant information to take into account when implementing such a dynamic approach to functions in the classroom.

Our analyses show how the discourse developed by students interacting with the realizations of functions proposed can be mirrored by potential expert discourse on functions and many functions' properties. By considering the fact that students have always had several difficulties in learning functions, as reported by the literature, we can conclude that this didactical implication is important. Indeed, the activities can be used to have the students engage in discussions with other students and with the teacher, that is how a learning process occurs according to Sfard's theoretical framework. At this regard, this study suggests that opportunities for students to work in pairs or small groups allow them to engage actively in the development of their mathematical discourse; a contribution in this direction is given also by the task situations designed by the teacher who, for example, can use open questions that ask for discussion and for a written description.

Moreover, we expect that in a teaching and learning process the role of the teacher is that of promoting discourses on mathematical signifiers that can be found in students' discourse but that should evolve towards potential discourses of an expert. In particular, we expect that during the activities with the dynamic realizations of functions significant aspects characterizing students' discourse are promoted by the teacher. It may happen when their discourse mirrors potential expert discourse on functions, or some properties of functions, and it is also expected to evolve. We found many properties of functions that were realized in students' discourse, by mirroring potential expert discourse. For example, we found that the property of 'injectivity' was not described by students, but we identified some cases of students' discourse mirroring expert discourse on the non-injectivity of a function. This is an interesting and important observation in terms of didactical implications, because we expect that it has to be considered at the moment in which these notions are introduced to students in a formal way, for example when giving them a definition or a word to indicate them.

During the analyses, we highlighted how the discourse of our students was rich in references to the dynamic and temporal aspects of functions and graphs of functions and also their frequent use of non-formal mathematical words. On the other hand, we used a coding scheme and we defined a potential expert discourse mirrored to investigate about the large number of realizations of mathematical signifiers, related to functions and their graphs, that can be found in students' discourse.

Now we show some examples, they are very short excerpts taken from the analyses in Chapter 6 and for each of them we specify the mathematical signifier with the potential expert discourse mirrored.

- "*They move symmetrically and towards opposite directions.*" (Excerpt 6.14, line 71)
  Monotonicity properties: the function is decreasing.
- *"Between minus two and minus one and between one and two it does not exist, B does not exist." (Excerpt 6.15, line 121)*
  Domain of the function: it is not defined on the intervals [-2; -1] and [1; 2].
- "*It is as it came from an infinite point over there and it started moving infinite point over here." (Excerpt 6.16, line 84)*
  Asymptote and limits: $\lim_{x \to 3-} f(x) = -\infty$; $\lim_{x \to 3+} f(x) = +\infty$
- "*There are some f(x) values that can be obtained in just one way, other values in two ways, other values in three ways and others in four.*" (Excerpt 6.20, line 138)
  Injectivity: the function is not injective.

We have reported here only some examples, but in the previous chapters we gave a detailed description of the type of students' discourse that the designed activities seem to support and we think that this may be a very important issue for a teacher, in order to be aware of what to expect and in which direction to work.

Moreover, we discussed the importance for students of developing and individualizing a discourse, of working with different realizations of a mathematical signifier and of accomplishing a process of saming among these realizations, beyond the particular DIE used. This is an important didactical implication of this study, which is in line with other research studies, even based on completely different theoretical backgrounds. In particular, Duval (2006) expressed the notion of *register of semiotic representation* of a mathematical object, and he highlighted the importance of *conversions* in and *treatments* between different representations.

However, our findings suggest different hypotheses to be refined and investigated in future research and we are going to discuss about some of them in the next section.

7.4 LIMITATIONS AND FURTHER RESEARCH

We would like to conclude this chapter by expressing some limitations that we identified about our study, introducing some general questions that arise from it and by outlining possible directions for future research that might be carried on from this study.

In the first chapter we introduced the importance within the field of mathematics education of improving the teaching and learning of functions and we discussed about how focusing on the covariational aspects of functions can play a significant role for this purpose. Our results specifically address questions related to the use of DIEs to realize functions and to the possible role of dragging with regard to the issue of teaching and learning functions when such a dynamic approach is adopted. Given our findings, a discussion should be opened about whether, as a mathematics education community, we are interested in fostering a discourse on functions as described by our analyses, which is rich in references to the dynamic and temporal components. As we have seen, the use of a formal mathematical vocabulary is not the main goal of the activities, while we have discussed in the introduction that, traditionally, it is considered at the core of many teaching experiments. In particular, it means to debate about the possibility of considering a dynamic approach to introduce functions, which involves the specific one dimensional realization that we designed within a DIE, as part of the mathematics curriculum in high schools. In that case we should take into account issues related to the dragging tool, as the possibility of using the mediation of different types of dragging and the emergence, and eventually a development, of a dragging mediated discourse. For example, how to deal with students' use of the dynamic visual mediation of dragging in their communication about functions and graphs of functions also out of the DIE. It should be discussed about the possibly choice, of a hypothetical teacher, of avoiding its use or supporting it.

At this regard, an aspect that has not been considered in this study but that would be worth to do is a focus on the role of the teacher, especially during the classroom discussions. Indeed, we intentionally chose to keep the researcher as much as possible out of the students' discussions, asking them to work in pairs during the whole sequence of classroom activities, in order to let them construct their own discourse on functions, by interacting only with the DIMs that we designed and with other students. In particular, we chose to do like this because we wanted to focus on students' learning process, by looking at their choice of words, of visual mediators and of routines being performed, and we did not want to influence these choices, and so their construction of a discourse. However, a further research could focus more on the teaching process and so on the use of a similar dynamic approach to introduce functions and their graphs, by analyzing the role of the teacher.

Moreover, we think that an appropriate methodology could be a longer term teaching experiment to allow the introduction of the formal mathematical definitions of functions and their properties, but this is something that need for further investigations. We have this impression, because from a theoretical point of view gaining some insights about the transition of the DIM from being the object of students' discourse to be used in the same discourse as realization of function, as we hypothesized in Chapter 6, would be a very significant result for the research.

Furthermore, it would be also interesting to deepen the analysis of gestures used by students to communicate about functions and graphs. Indeed, we mainly looked at dragsturing actions but there were also some examples of different gestures used by the students that we did not analyzed. For example, in a discourse about the maximum point of a function, students may use the visual mediation of a gesture obtained by moving the hand to the right and suddenly to the left, as referring to the one dimensional realization of function; or by tracing with the hand a sort of semicircle, as referring to a curve in the Cartesian plane.

We also suggest that it would be beneficial to investigate whether certain students benefit more, or less, from being introduced to functions through such a dynamic approach. For example, activities that make use of the DIMs that we designed may reveal to support especially students who have been labelled as carriers of specific mathematical learning disabilities. Indeed, these activities foster the development of informal discourse that is multimodal (it includes dynamic interactions, gestures, dynamic interactive mediation,…) and has the potential of developing into formal mathematical discourse, thus giving a larger range of students the possibility to participate to mathematical discourse.

We conclude with an important issue that very often arises in the field of math education: the problem of designing tasks that are in line with to the educational goals behind a specific activity, that is, the problem of generating "good problems" aimed at achieving certain educational goals. Contextualized within this greater research problem, further research could investigate the design of activities with DIMs such as those we designed, in order to find significant examples and problems for the implementation of the specific dynamic realization of function.

Appendix A

Here there is the original Italian version of the activities implemented during the sequence of lessons and the interviews.

LEZIONE 1

Attività1_1: DGp della funzione $f(x) = -x + 5$.

Esplorare la situazione, individuare e descrivere i movimenti possibili attraverso il trascinamento e trascrivere sul quaderno le proprie osservazioni.

Attività1_2: DGp della funzione $f(x) = |x|$.

Esplorare la situazione, individuare e descrivere i movimenti possibili attraverso il trascinamento e trascrivere sul quaderno le proprie osservazioni.

Attività1_3: DGpp della funzione $f(x) = |x|$.

Esplorare la situazione, individuare e descrivere i movimenti possibili attraverso il trascinamento e trascrivere sul quaderno le proprie osservazioni.

LEZIONE 2

Attività2_1: DGpp della funzione $f(x) = e^{x-1} + \frac{1}{25}$.

Esplorare la situazione, individuare e descrivere i movimenti possibili attraverso il trascinamento e trascrivere sul quaderno le proprie osservazioni.

Attività2_2: DGpp della funzione $f(x) = \sqrt{x+3} - 2$.

Esplorare la situazione, individuare e descrivere i movimenti possibili attraverso il trascinamento e trascrivere sul quaderno le proprie osservazioni.

1) *È possibile avere B=3? Come?*
2) *È possibile avere B= -3? Come?*
3) *Come faccio a muovere B da 0 a 1?*
   Giustificare sul foglio le risposte.

Attività2_3: DGpp della funzione $f(x) = \sqrt{(x^2-1)(x^2-4)}$.

Esplorare la situazione, individuare e descrivere i movimenti possibili attraverso il trascinamento e trascrivere sul quaderno le proprie osservazioni.

1) *È possibile avere B=4? Come?*
2) *È possibile avere B= -4? Come?*
3) *Trascinando A da -2 a 2, quali sono tutti i valori che può assumere B?*
   Giustificare sul foglio le risposte.

LEZIONE 3

Attività3_1: DGpp della funzione $f(x) = x + \frac{3}{x-3}$.

Esplorare la situazione, individuare e descrivere i movimenti possibili attraverso il trascinamento e trascrivere sul quaderno le proprie osservazioni.

1) *È possibile avere B= -1? Come?*
   È possibile avere B=1? Come?
2) *Quali sono tutti i valori che può assumere B?*
3) *Quali di questi valori è possibile ottenerli in 0; 1; 2; 3...modi diversi?*

Attività3_2: DGpp di una funzione definita *ad hoc,* con dominio ristretto all'intervallo [0, 48].

Descrivere e commentare tutte le informazioni che è possibile ricavare sulla percentuale di carica della batteria di un cellulare al variare del tempo, nell'arco di 48 ore.

LEZIONE 4

Attività4_1: DGpp delle due funzioni $f(x) = \begin{cases} 7, \ x < 5 \\ 3 + floor(x), \ altrimenti \end{cases}$ e $g(x) = \begin{cases} \frac{5}{2}x, \ x < 6 \\ \frac{1}{2}x + 12, \ altrimenti \end{cases}$

Descrivere e commentare tutte le informazioni che è possibile ricavare sulle tariffe telefoniche di Aldo (T_A) e di Bianca (T_B), espresse in euro, al variare delle ore di chiamate effettuate. Confrontare poi le due tariffe.

Attività4_2: DGpp delle due funzioni $f(x) = x^2$ *e* $g(x) = |x| + \frac{3}{2}$.

Scegliere, se possibile, un intervallo in cui far variare A e C in modo che l'insieme dei valori assunti da B e l'insieme dei valori assunti da D siano disgiunti.

LEZIONE 5

Attività5_1: DGc della funzione $f(x) = -x + 5$.

Esplorare la situazione, individuare e descrivere i movimenti possibili attraverso il trascinamento e trascrivere sul quaderno le proprie osservazioni.

Attività5_1bis: DGpp della funzione $f(x) = -x + 5$.

Esplorare la situazione e confrontare con l'attività precedente.

Attività5_2: DGc della funzione $f(x) = \sqrt{x+3} - 2$.

1) *È possibile avere B=3? Se sì, come?*
   È possibile avere B= -3? Se sì, come?
2) *Quali sono tutti i valori che può assumere B?*
3) *Quali sono i valori di A per cui B assume valori minori di 0?*

Giustificare sul foglio le risposte.

Attività5_3: DGc della funzione $f(x) = -\frac{x^2}{25} + x + 1$, l'asse x è etichettato con "litri" e l'asse y con "tonnellate".

Descrivere e commentare tutte le informazioni che è possibile ricavare sulla produzione stagionale di agrumi di un'azienda agricola in funzione della quantità di fertilizzante utilizzato.

LEZIONE 6

Attività6_1: DGc di una funzione definita *ad hoc*, l'asse x è etichettato con "mesi" e l'asse y con "m^3".

Descrivere e commentare tutte le informazioni che è possibile ricavare sulla quantità di acqua necessaria per irrigare un giardino, con 100 metri quadrati di prato, per ogni mese dell'anno.

Attività6_1bis: DGc di una funzione definita *ad hoc*, l'asse x è etichettato con "m^3" e l'asse y con "euro".

Descrivere e commentare tutte le informazioni che è possibile ricavare sul costo dell'acqua in funzione del consumo.

Attività da consegnare dopo aver lasciato un po' di tempo per Attività6_1 e Attività6_1bis:

Il proprietario di un giardino è preoccupato per le spese che deve sostenere per irrigare i suoi 100 metri quadrati di prato, e si pone i seguenti quesiti:

c) Mi capiterà mai di dover pagare meno di 5 euro per l'irrigazione del prato? Se sì, quando? Se no, perché?
d) Mi capiterà mai di dover pagare più di 50 euro per l'irrigazione del prato? Se sì, quando? Se no, perché?

Tenendo aperti i file Attività6_1 e Attività6_1bis, rispondere sul foglio ai due quesiti.

Descrivere poi come varia nel tempo, nel corso di ogni mese, la spesa per l'acqua che il proprietario deve sostenere per irrigare il suo giardino (non sono considerati i costi di canoni fissi mensili).

LEZIONE 7

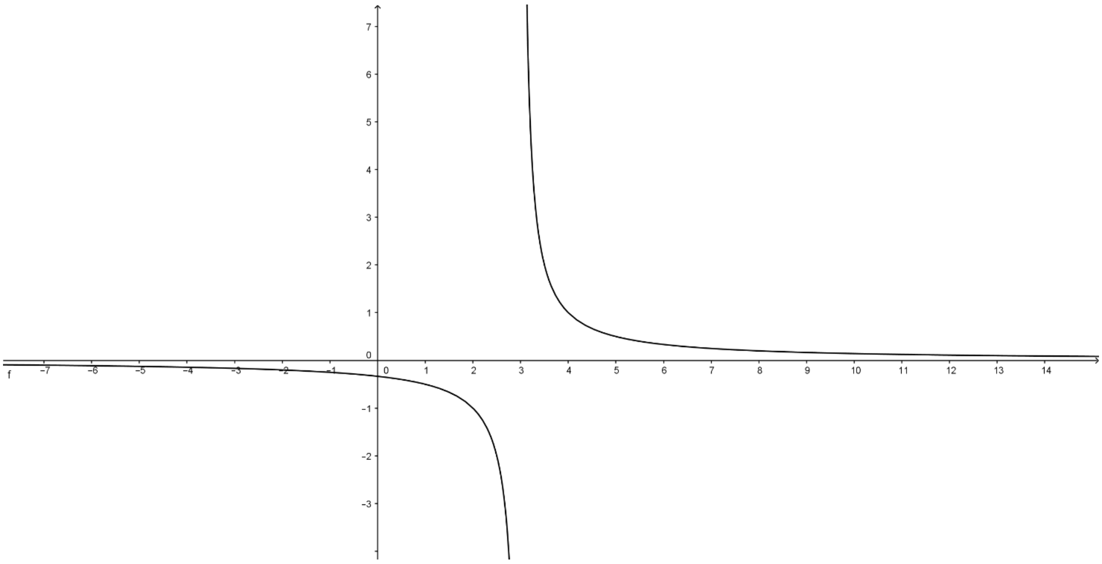
Attività7_1: DGc della funzione $f(x) = \frac{1}{10}\left(\frac{x}{2} + 4\right)(x+1)(x-2) + \frac{5}{2}$.

Disegnare sul foglio la traiettoria del punto (x, f(x)).

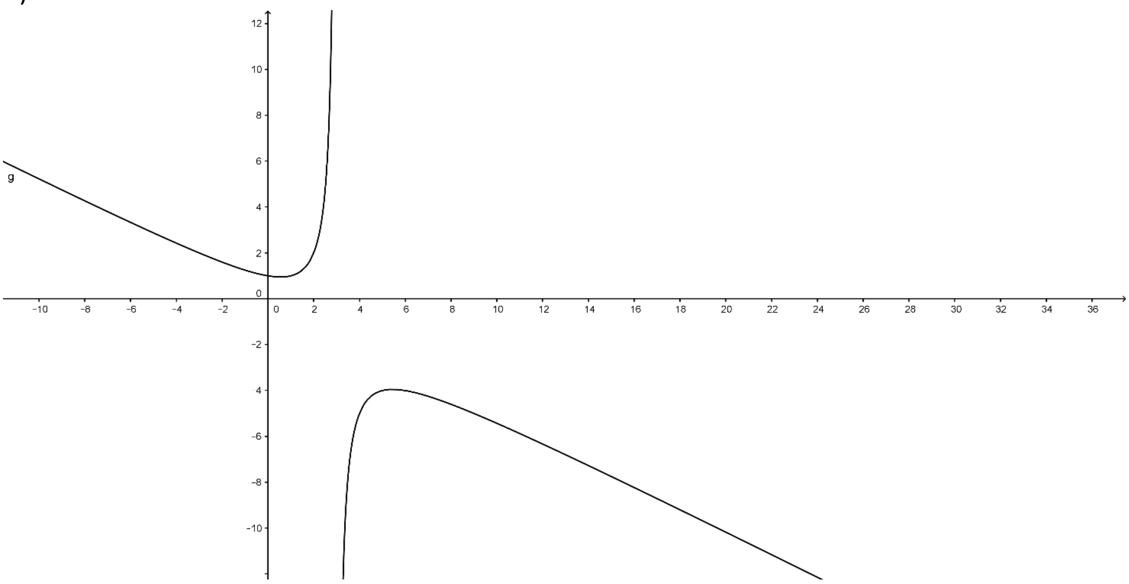
Attività7_2: DGc della funzione $g(x) = \frac{x}{2} + \frac{3}{x-3}$.

Aprire il file Geogebra Attività7_2.ggb e indicare quale tra i grafici sotto riportati rappresenta la traiettoria del punto $(x, g(x))$. Argomentare per scritto la propria scelta, descrivendo quale opzione avete indicato e perché, quali invece avete scartato e perché.

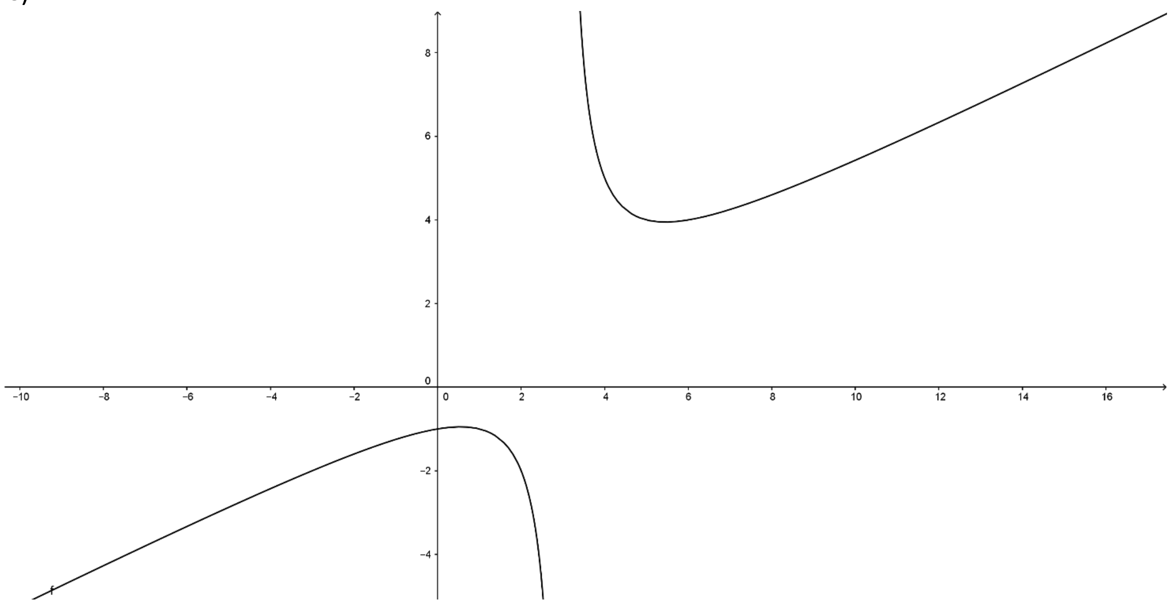
A)



B)



C)

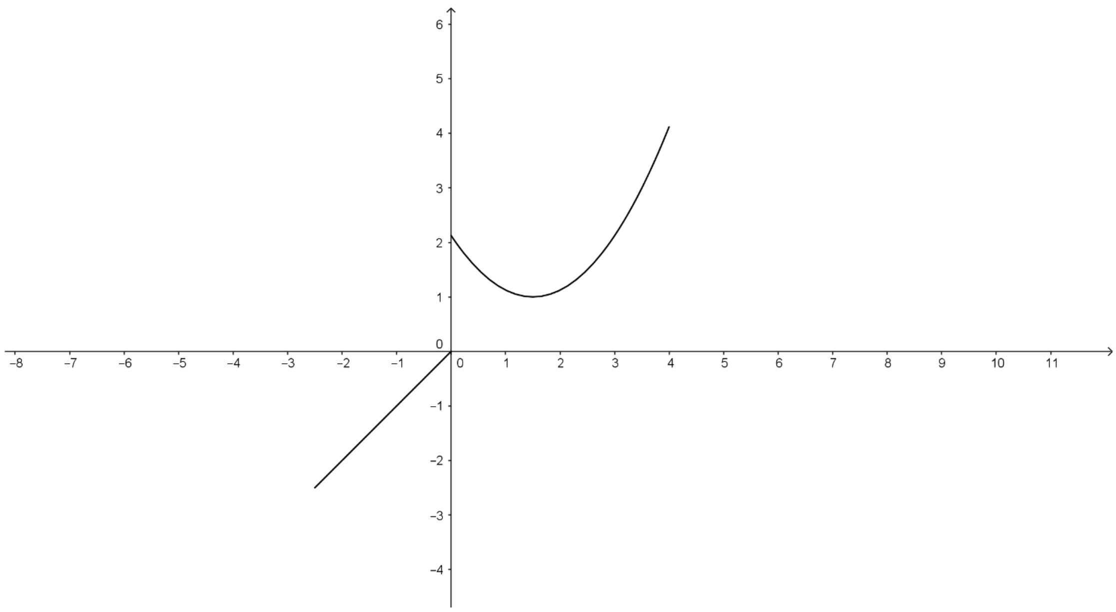


LEZIONE 8

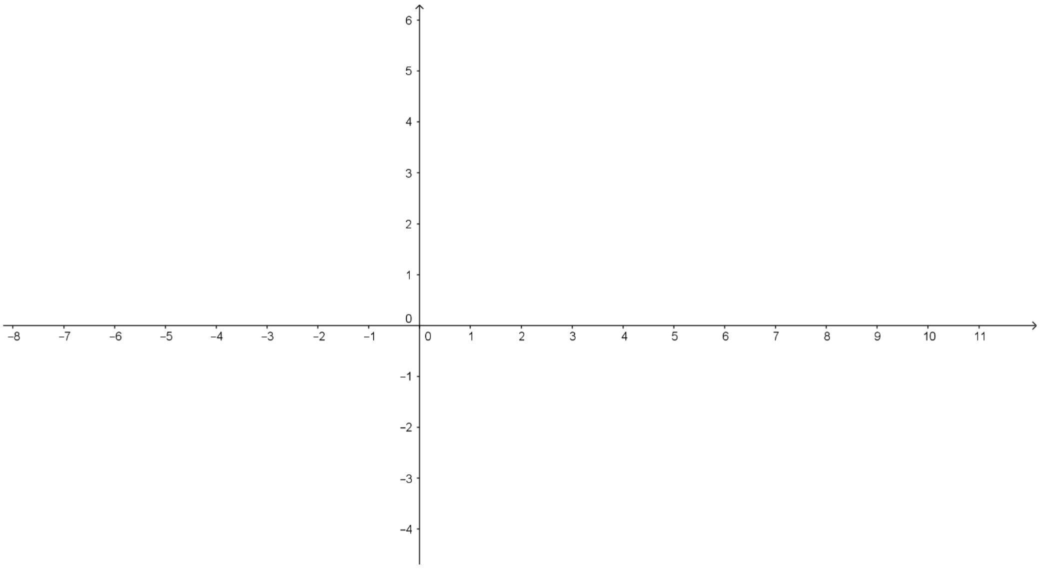
Attività8_1: DGpp della funzione $f(x) = \begin{cases} \dfrac{3}{2x} + 2 & x > 0 \\ \dfrac{(x+1)^2(x+6)(x+3)}{x} - 1 & x \leq 0 \end{cases}$

Disegnare sul foglio il grafico di questa funzione nel piano cartesiano.

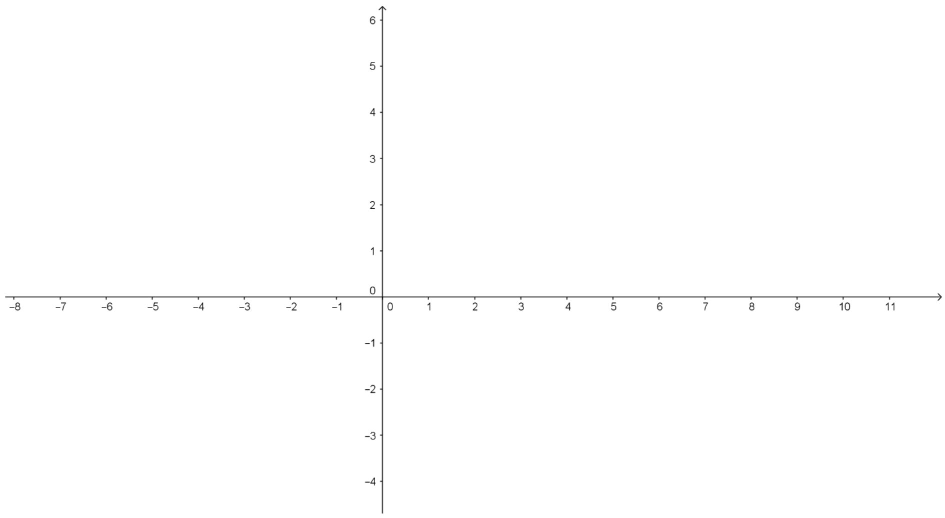
Attività8_2: *Attivando la traccia sul punto* $(x, f(x))$ *abbiamo ottenuto:*



Indicare cosa si colorerebbe attivando la traccia su x:



Indicare cosa si colorerebbe attivando la traccia su $f(x)$:



Attività8_3: *Disegnare il grafico, nel piano cartesiano, di una funzione che abbia le seguenti proprietà:*

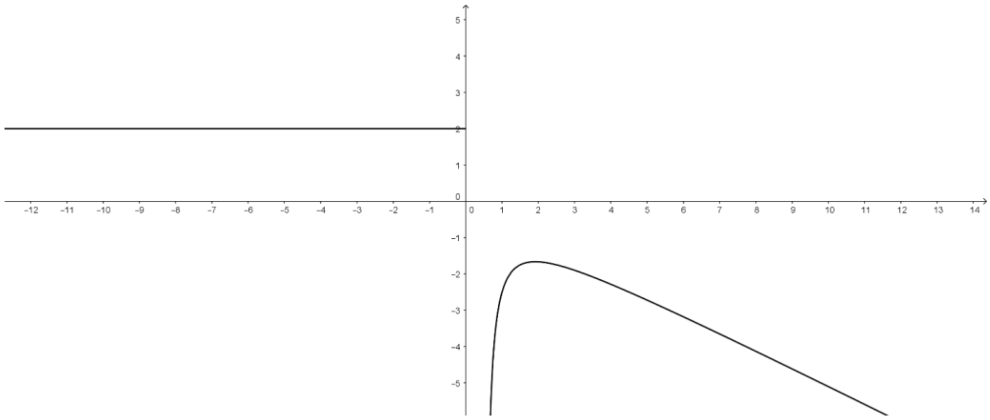
- *Prima di zero se aumenta x aumenta anche f(x)*
- *Quando x è maggiore di 6 hanno versi opposti*
- *Man mano che x va avanti f(x) si muove sempre di più: ad esempio, se x va da -5 a -4 f(x) si muove di pochissimo, se x va da 1 a 2 f(x) si muove di più spazio*
- *f(x) può assumere tutti i valori negativi e quelli positivi minori di 10, perché quando arriva a 10 poi torna indietro*
- *Si intersecano a 3,5 circa*
- *Alcuni valori di f(x) si possono ottenere in un solo modo, altri in due, altri in tre e altri in quattro modi diversi*

INTERVISTA 1: Alessio

5) Attività7_1
6) Attività7_2
7) Attività8_1
8) Attività8_2

INTERVISTA 2: Matilde e Nicco

5) Immagina di avere questa funzione rappresentata su rette parallele in un file GeoGebra, come la descriveresti?



6) Uno studente si mette al pc e l'altro davanti. Lo studente che lavora al pc apre il file Intervista1.ggb[8] e lo descrive al compagno, che seguendo le sue indicazioni deve disegnare sul foglio il grafico della funzione nel piano cartesiano.
   ATTENZIONE: è vietato mostrare al compagno cose scritte, o girare lo schermo del pc, o dargli indicazioni con le mani.

7) Ripetere la stessa cosa, scambiandovi di posto, e questa volta lo studente al pc deve aprire il file Intervista2.ggb[9].

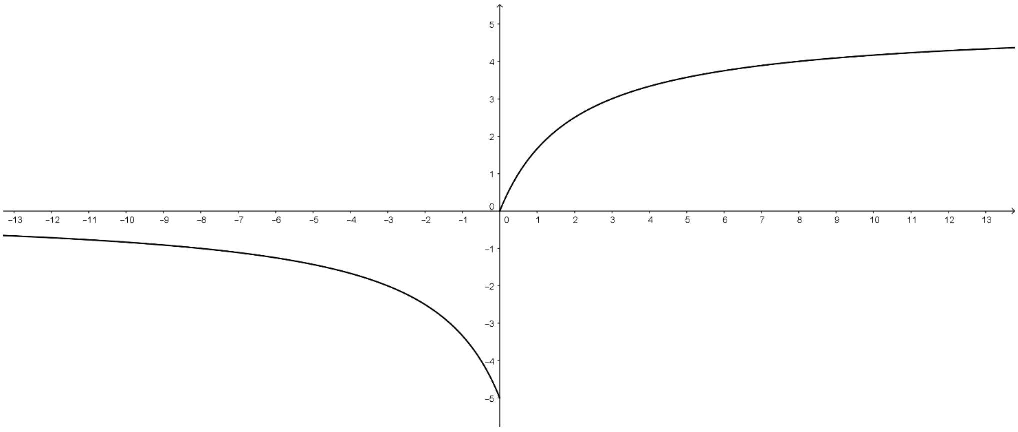
8) Leggere con attenzione l'altro foglio che avete ricevuto (è la fotocopia di una pagina presa da un libro di testo di matematica per la scuola secondaria). Dopo rispondere a questi cinque quesiti relativi al grafico riportato di seguito:

   f. *Colorare l'immagine dell'intervallo [-1, 1]*
   g. *Colorare la controimmagine dell'intervallo [-5, -3]*
   h. *Qual è il dominio di questa funzione?*
   i. *Qual è l'insieme delle immagini di questa funzione?*
   j. *È una funzione iniettiva? Perché?*

---

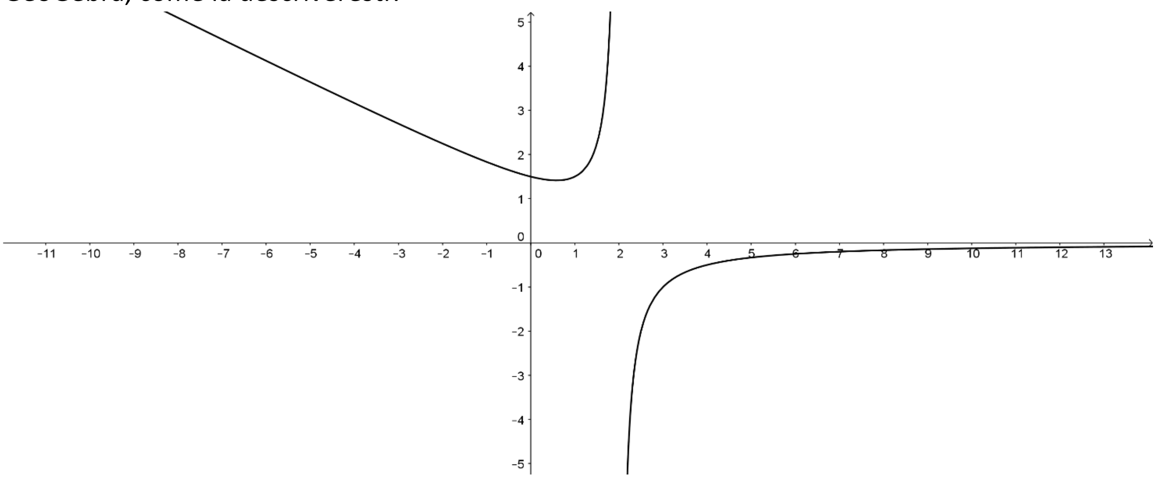
[8] DGpp della funzione $f(x) = \begin{cases} \left|\frac{(x-1)}{2}(x-4)\frac{(x+8)}{4}\right| - 6, \; x < 4.94 \\ \frac{x-5}{5}, \; altrimenti \end{cases}$

[9] DGpp della funzione $f(x) = \begin{cases} -5.18, \; se\ x < -4.6 \\ 10\frac{sinx}{x} - 3, \; se - 4.6 \le x < 6.2 \\ \frac{3}{2}x - 12.4, \; se\ 6.2 \le x < 8.3 \end{cases}$



INTERVISTA 3: Franci e Lore

1) Immagina di avere questa funzione rappresentata su rette parallele in un file GeoGebra, come la descriveresti?



2) Uno studente si mette al pc e l'altro davanti. Lo studente che lavora al pc apre il file Intervista1.ggb[10] e lo descrive al compagno, che seguendo le sue indicazioni deve disegnare sul foglio il grafico della funzione nel piano cartesiano.
ATTENZIONE: è vietato mostrare al compagno cose scritte, o girare lo schermo del pc, o dargli indicazioni con le mani.

3) Ripetere la stessa cosa, scambiandovi di posto, e questa volta lo studente al pc deve aprire il file Intervista2.ggb[11].

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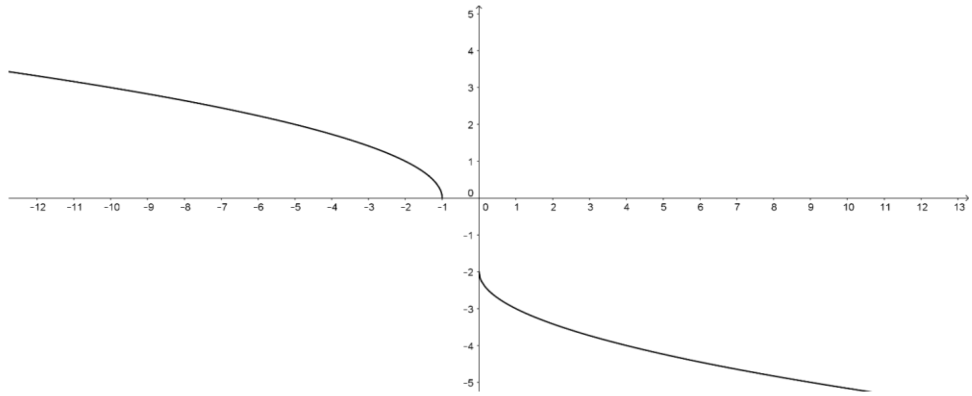
[10] DGpp della funzione $f(x) = \begin{cases} \frac{1}{x-2} - 5, \ x > 2 \\ \frac{1}{4}\sqrt{-(3x+21)(2x+2)(x+4)}, \ altrimenti \end{cases}$

[11] DGpp della funzione $f(x) = \begin{cases} 5.18, \ se\ x < -4.6 \\ -10\frac{sinx}{x} + 3, \ se - 4.6 \leq x < 8.2 \\ \frac{2}{5}x - \frac{3}{2}, \ altrimenti \end{cases}$

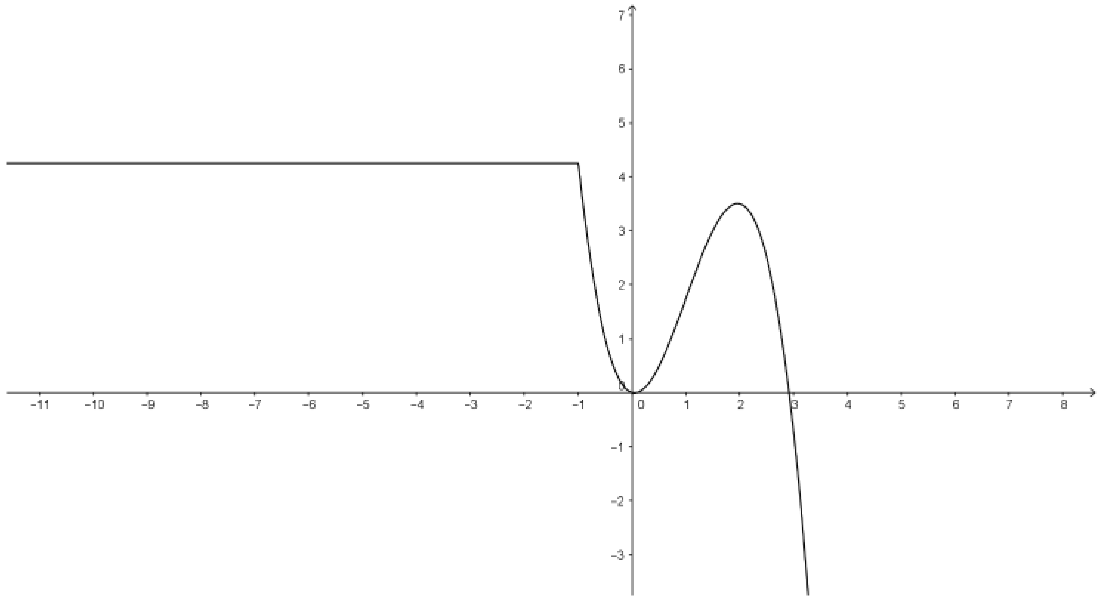
4) Leggere con attenzione l'altro foglio che avete ricevuto (è la fotocopia di una pagina presa da un libro di testo di matematica per la scuola secondaria). Dopo rispondere a questi cinque quesiti relativi al grafico riportato di seguito:

   a. *Colorare l'immagine dell'intervallo [0, 2]*
   b. *Colorare la controimmagine dell'intervallo [0, 2]*
   c. *Qual è il dominio di questa funzione?*
   d. *Qual è l'insieme delle immagini di questa funzione?*
   e. *È una funzione iniettiva? Perché?*



INTERVISTA 4: Davide e Elena

1) Immagina di avere questa funzione rappresentata su rette parallele in un file GeoGebra, come la descriveresti?

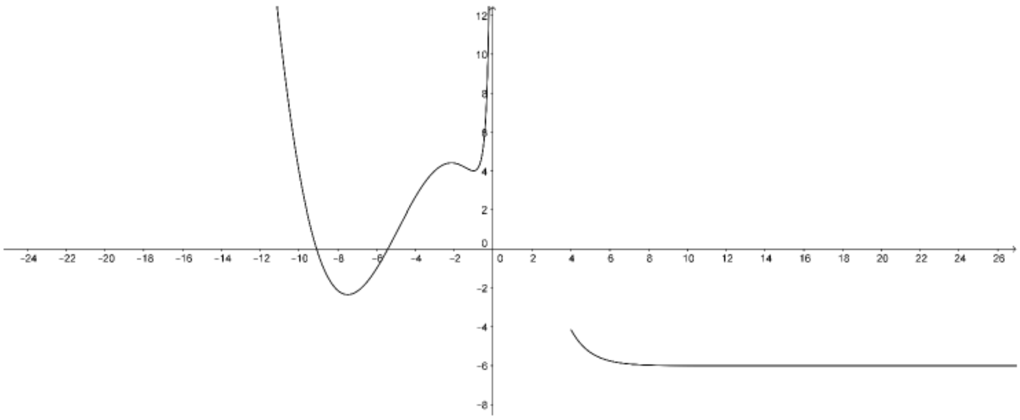


2) Uno studente si mette al pc e l'altro davanti. Lo studente che lavora al pc apre il file Intervista1.ggb[12] e lo descrive al compagno, che seguendo le sue indicazioni deve disegnare sul foglio il grafico della funzione nel piano cartesiano.
ATTENZIONE: è vietato mostrare al compagno cose scritte, o girare lo schermo del pc, o dargli indicazioni con le mani.

3) Ripetere la stessa cosa, scambiandovi di posto, e questa volta lo studente al pc deve aprire il file Intervista2.ggb[13].

4) Leggere con attenzione l'altro foglio che avete ricevuto (è la fotocopia di una pagina presa da un libro di testo di matematica per la scuola secondaria). Dopo rispondere a questi cinque quesiti relativi al grafico riportato di seguito:

   a. *Colorare l'immagine dell'intervallo [4, 10]*
   b. *Colorare la controimmagine dell'intervallo [0, 2]*
   c. *Qual è il dominio di questa funzione?*
   d. *Qual è l'insieme delle immagini di questa funzione?*
   e. *È una funzione iniettiva? Perché?*



[12] DGpp della funzione $f(x) = \frac{1}{4}\sqrt{\frac{(x-8)}{2}(x+1)(x-6)(x-1)}$

[13] DGpp della funzione $f(x) = \frac{3}{2x-1} + 2$

Appendix B

The original Italian version of all the excerpts that have been used in the thesis is available here:

https://drive.google.com/file/d/1HVHedzI88afVv4-fr5uHlIOfqLtV36Um/view?usp=sharing

Bibliography

Antonini, S., & Martignone, F. (2009). Students' utilization schemes of pantographs for geometrical transformations: a first classification. In: *Proceedings of the Sixth Conference of European Research of Mathematics Education*, (pp. 1250–1259).

Antonini, S., Baccaglini-Frank, A. & Lisarelli, G. (under review). From experiences in a dynamic environment to written discourse on functions: snapshots, live-photos and scenes. *Digital Experience in Mathematics Education*.

Arcavi, A., & Nachmias, R. (1993). What is your family name, Ms. Function? - Exploring families of functions with a non-conventional representation. *Journal of Computers in Mathematics and Science Teaching, 12*(3/4), 315-329.

Arcavi, A., & Schoenfeld, A. H. (1988). On the Meaning of Variable. *Mathematics Teacher, 81*, 420-427.

Arzarello, F., Micheletti, C., Olivero, F., Paola, D., & Robutti, O. (1998). A model for analysing the transition to formal proofs in geometry. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, 2, (pp. 32–39). Stellenbosh: PME.

Arzarello, F., Olivero, F., Paola, D., & Robutti, O., (2002). A cognitive analysis of dragging practises in Cabri environments. *ZDM*, *34*(3), 66-72.

Aspinwall, L., Haciomeroglu, E. S., & Presmeg, N. (2008). Students' verbal descriptions that support visual and analytic thinking in Calculus. *Proceedings of PME 32 and PME-NA 30, 2*, 97–104.

Ayalon, M., Watson, A., & Lerman, S. (2016). Reasoning about variables in 11 to 18 year olds: informal, schooled and formal expression in learning about functions. *Mathematics Education Research Journal, 28*(3), 379-404.

Ayalon, M., Watson, A., & Lerman, S. (2017). Students' conceptualisations of function revealed through definitions and examples. *Research in Mathematics Education Research Journal, 19*(1), 1–19.

Baccaglini-Frank, A., & Mariotti, M. A., (2010). Generating Conjectures in Dynamic Geometry: the Maintaining Dragging Model. *International Journal of Computers for Mathematical Learning, 15*(3), 225-253.

Baccaglini-Frank, A., Mariotti, M. A., & Antonini, S. (2009). Different Perceptions of Invariants and Generality of Proof in Dynamic Geometry. In Tzekaki, M., & Sakonidis, H. (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education, Vol. 2*, pp. 89-96. Thessaloniki, Greece: PME.

Barnes, M. (1988). Understanding the function concept: some results of interviews with secondary and tertiary students. *Research on Mathematics Education in Australia*, 24-33.

Bartolini Bussi, M. G., & Mariotti, M. A., (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English et al. (Eds.), *Handbook of International Research in Mathematics Education,* second edition, (pp. 746-783). New York and London: Routledge.

Bell, A., & Janvier, C. (1981). The interpretation of graphs representing situations. *For the Learning of Mathematics, 2,* 34–42.

Bergamini, M., Trifone, A., & Barozzi, G. (2005). Corso base blu di matematica, Volume 5. Zanichelli Editore.

Boyer, C. (1946). Proportion, equation, function: three steps in the development of a concept. *Scripta Mathematica, 1*(16), 5-13.

Bruner, J. S. (1996). *The culture of education*. Cambridge, MA: Harvard University Press.

Caponi, M., & Lisarelli, G. (2018). Realization trees: a commognitive lens. In Bergqvist, E., Osterholm, M., Granberg, C., & Sumpter, L. (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education, 5*, (pp. 212). Umea, Sweden: PME.

Carlson, M. P. (1998). A cross sectional investigation of the development of the function concept. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in Collegiate Mathematics Education. III. CBMS Issues in Mathematics Education* (pp. 114-162). Providence, RI: American Mathematical Society.

Carlson, M., & Oehrtman, M. (2005). Key aspects of knowing and learning the concept of function. Research Sampler 9. MAA Notes.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378.

Caspi, S., & Sfard, A. (2012). Spontaneous meta-arithmetic as a first step toward school algebra. *International Journal of Educational Research, 51-52*, 45–65.

Chinn, C. A, & Sherin, B. L. (2014). Microgenetic methods. In *The Cambridge Handbook of the Learning Sciences*, Second Edition, (pp. 171-190). Cambridge University Press.

Colacicco, G., Lisarelli, G., & Antonini, S., (2017). Funzioni e grafici in ambienti digitali dinamici. *Didattica della Matematica: dalla ricerca alle pratiche d'aula*, *2*, 7-25.

Confrey J., & Smith, E. (1995). Splitting, covariation and their role in the development of exponential function. *Journal for Research in Mathematics Education, 26,* 66–86.

Confrey, J. (1991). The concept of exponential functions: A student's perspective. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 124–159). New York, NY: Springer.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. *The development of multiplicative reasoning in the learning of mathematics,* 293-330. State University of New York Press.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26,* 135–164.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process schema. *Journal of Mathematical Behavior, 15*(2), 167-192.

Cuoco, A. (1995). Computational Media to Support the Learning and Use of Functions. In: diSessa A.A., Hoyles C., Noss R., Edwards L.D. (Eds.) *Computers and Exploratory Learning*. NATO ASI Series (Series F: Computer and Systems Sciences), *146*. Springer, Berlin, Heidelberg.

Dubinsky E., & Harel G., (1992). The nature of the process conception of function. In G. Harel and E. Dubinsky (Eds.), *The concept of function: aspects of epistemology and pedagogy*, 85–106. MAA Notes.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking, In D. Tall (Ed.). *Advanced Mathematical Thinking* (pp. 95-126). Boston: Kluwer.

Duval, R. (2006). The cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*(1-2), pp. 103-131.

Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior, 39,* 135–155.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics*, *21*, 521-544.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education*, *24*(2), 94-11.

Falcade, R. (2001). L'environment Cabri-géomètre outil de médiation sémiotique pour la notion de graphe d'une function. Memoire de DEA, UJF, Grenoble.

Falcade, R. (2003). Instruments of semiotic mediation in Cabri for the notion of function. *Proceedings of the 3rd Conference of the European society for Research in Mathematics Education,* Bellaria: Italy.

Falcade, R., Laborde, C., & Mariotti, M. A. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics*, *66*, 317-333.

Ferrara, F., & Ferrari, G. (2018). Thinking in movement and mathematics: a case study. In Bergqvist, E., Osterholm, M., Granberg, C., & Sumpter, L. (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education*, *2*, (pp. 419-426). Umea, Sweden: PME.

Freudenthal, H. (1983). Didactical phenomenology of mathematical structures. Dordrecht: D. Reidel.

Goldenberg, E. P., (1995). Ruminations about dynamic imagery (and a Strong Plea for Research). In R. Sutherland and J. Mason (Eds.), *Exploiting Mental Imagery with Computers in Mathematics Education* (pp. 202-224). Berlin: Springer.

Goldenberg, E. P., Lewis, P., & O'Keefe, J. (1992). Dynamic representation and the development of an understanding of functions. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy, 25*. MAA Notes.

Grugnetti, L., Marchini, C., & Maffini, A. (1999). Le concept de fonction dans l'école italienne; usage de l'épistémologie et de l'histoire des mathématiques pour en clarifier le sens. *Histoire et épistémologie dans l'education mathématique*, (pp. 421-444). Louvain.

Hazzan, O., & Goldenberg, E. P. (1997). Student's understanding of the notion of function. *International Journal of Computers for Mathematical Learning, 1*(3), 263–290.

Healy, L., & Hoyles, C. (2001). Software tools for geometrical problem solving: potentials and pitfalls. *International Journals of Computers for the Learning of Mathematics, 6*, 235-256.

Healy, L., & Sinclair, N. (2007). If this is our mathematics, what are our stories? *International Journal of Computers for Mathematical Learning, 12*, 3 – 21.

Hershkowitz, R., & Vinner, S. (1983). The role of critical and non-critical attributes in the concept image of geometrical concepts. In R. Hershkowitz (Ed.) *Proceedings of the 7th International Conference for the Psychology of Mathematics Education* (pp. 223-228). Rehovot, Israel: Weizmann Institute of Science.

Hitt, F., & González-Martín, A. S. (2015). Covariation between variables in a modelling process: The ACODESA (collaborative learning, scientific debate and self-reflection) method. *Educational Studies in Mathematics*, *88*(2), 201–219.

Hitt, F., & González-Martin, A. S. (2016). Generalization, Covariation, Functions and Calculus. In A. Gutierrez, G. Leder, & P. Boero (Eds.), *The Second Handbook of Research on the Psychology of Mathematics Education,* (pp. 3-38). Sense Publishers.

HoIzI, R. (1996). How does 'dragging' affect the learning of geometry. *International Journal of Computers for Mathematical Learning, 1*, 169-187.

Jayakody, G. (2015). Commognitive conflicts in the discourse of continuous functions. In (Eds.) T. Fukawa-Connelly, N. Infante, K. Keene, and M. Zandieh, *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 611-619). Pittsburgh, Pennsylvania.

Johnson, H. L., & McClintock, E. (2018). A link between students' discernment of variation in unidirectional change and their use of quantitative variational reasoning. *Educational Studies in Mathematics, 97*(3), 299-316.

Kafetzopoulos, G.I., Psycharis, G. (2016). Conceptualising function as covariation through the use of a digital system integrating cas and dynamic geometry. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.), *Proceedings of the 40th Conference of IGPME*, 3, (pp. 67–74).

Kaput, J. J. (1992). Patterns in students' formalization of quantitative patterns. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 290–318). Washington, DC: Mathematical Association of America.

Kleiner, I. (1989). Evolution of the function concept: a brief survey. *College Mathematics Journal, 20*(4), 282–300.

Laborde, C. (1999). Core geometrical knowledge for using the modelling power of Geometry with Cabrigeometry. *Teaching Mathematics and its Applications, 18*(4), 166–171.

Laborde, C. (2001). Integration of Technology in the Design of Geometry Tasks with Cabri-Geometry. *International Journal of Computers for Mathematical Learning, 6*, 283-317. Netherlands: Kluwer Academic Publishers.

Laborde, C. (2003). Technology used as a tool for mediating knowledge in the teaching if mathematics: the case of Cabri-geometry. Plenary speech delivered at the Asian Technology Conference in Mathematics.

Laborde, C. (2005). Robust and Soft Constructions: two sides of the use of dynamic Geometry environments. *Proceedings of the 10th Asian Technology Conference in Mathematics*, (pp. 22-35). Korea National University of Education, Cheong-Ju, South Korea.

Laborde, C., & Mariotti, M. A. (2001). Grounding the notion of function in a DGS. *Cabri World 2001.*

Lagrange, J. B. (2010). Teaching and learning about functions at upper secondary level: designing and experimenting the software environment Casyopée. *International Journal of Mathematical Education in Science and Technology, 41*(2), 243-255.

Lakoff, G., & Núnez, R. (2000). Where mathematics comes from: How the embodied mind brings mathematics into being. New York: Basic Books.

Lavie, I., Steiner, A., & Sfard, A. (2018). Routines we live by: from ritual to exploration. *Educational Studies in Mathematics*, 1-24.

Leung, A. (2008). Dragging in a Dynamic Geometry Environment through the Lens of Variation. *International Journal of Computers for Mathematical Learning, 13*(2), 135-157.

Leung, A., Baccaglini-Frank, A., & Mariotti, M.A. (2013). Discernment in dynamic geometry environments. *Educational Studies in Mathematics, 84*(3), 439–460.

Lewis, K. E. (2017). Designing a Bridging Discourse: Re-Mediation of a Mathematical Learning Disability. *Journal of the Learning Sciences, 26*(2), 320-365.

Lopez-Real, F., & Leung, A. (2006). Dragging as a conceptual tool in dynamic geometry. *International Journal of Mathematical Education in Science and Technology, 37*(6), 665-679.

Malik, M. A. (1980). Historical and pedagogical aspects of the definition of a function. *International Journal of Mathematics Education in Science and Technology*, *11*(4), 489-492.

Mariotti, M. A. (2002). Influence of technologies advances on students' math learning, In English, L. et al. *Handbook of International Research in Mathematics Education,* (pp. 695-723). Lawrence Erbaum Associates.

Mariotti, M.A. (2006). Proof and proving in mathematics education. A. Gutiérrez & P. Boero (Eds.) *Handbook of Research on the Psychology of Mathematics Education*, (pp. 173-2014). Sense Publishers, Rotterdam, The Netherlands.

Mariotti, M. A. (2010). Riflessioni sulla dinamicità delle figure. In G. Accascina and E. Rogora (Eds.) *Seminari di Geometria Dinamica* (pp. 271-296). Roma, Italy: Edizioni Nuova Cultura.

Mariotti, M. A. (2015). Transforming Images in a DGS: The Semiotic Potential of the Dragging Tool for Introducing the Notion of Conditional Statement. In *Transformation - A Fundamental Idea of Mathematics Education* (pp. 155–172). New York, NY: Springer New York.

Mariotti, M. A., Laborde, C., & Falcade, R. (2003). Function and graph in a DGS environment. In N. A. Pateman, B. J. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PMENA*, 3, (pp. 237–244). University of Hawai'i, Honolulu, HI, USA: CRDG, College of Education.

Markovits, Z., Eylon, B., & Bruckheimer, M. (1986). Functions today and yesterday. *For the Leaming of Mathematics,* 6(2), 18-24.

Markovits, Z., Eylon, B., & Bruckheimer, M. (1988). Difficulties students have with the function concept. In S. Wagner & C. Kieran (Eds.), *The Ideas of algebra*, K-12 (pp. 43-60). Reston (VA): National Council of Teachers of Mathematics.

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago: University of Chicago Press.

Menz, P. M. (2015). *Unfolding of Diagramming and Gesturing between Mathematics Graduate Student and Supervisor during Research Meetings*. Doctoral Dissertation, Simon Fraser University, CA.

MIUR (2010). Schema di regolamento recante "Indicazioni nazionali riguardanti gli obiettivi specifici di apprendimento concernenti le attività e gli insegnamenti compresi nei piani degli studi previsti per i percorsi liceali". http://www.indire.it/lucabas/lkmw_file/licei2010/indicazioni_nuovo_impaginato/_decreto_indicazioni_nazionali.pdf

Monk, S., & Nemirovsky, R. (1994). The case of Dan: Student construction of a functional situation through visual attributes. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education. I. CBMS Issues in Mathematics Education* (pp. 139-168). Providence, RI: American Mathematical Society.

Morgan, C., & Sfard, A. (2016). Investigating changes in high-stakes mathematics examinations: a discursive approach, *Research in Mathematics Education*, *18*(2), 92-119.

Moschkovich, J. (2007). Using two languages when learning mathematics. *Educational Studies in Mathematics, 64*(2), 121-144.

Nachlieli, T., & Tabach, M. (2012). Growing mathematical objects in the classroom—the case of function. *International Journal of Educational Research*, *51-52*, 10–27.

Nagle, C., Tracy, T., Adams, G., & Scutella, D. (2017). The notion of motion: covariational reasoning and the limit concept, *International Journal of Mathematical Education in Science and Technology, 48*(4), 573-586.

Ng, O. (2014). The interplay between language, gestures, and dragging and diagrams in bilingual learners' mathematical communications. In P. Liljedahl, S. Oesterle, C. Nicol, D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36, 4,* (pp. 289–296). Vancouver: PME.

Ng, O. (2016). Comparing calculus communication across static and dynamic environments using a multimodal approach. *Digital Experiences in Mathematics Education, 2*(2), 115-141.

Ng, O. (2016a). *Language, gestures and touchscreen dragging in school calculus: bilinguals' linguistic and non linguistic communication*. Doctoral Dissertation, Simon Fraser University, CA.

Ng, O., & Sinclair, N. (2013). Gestures and temporality: Children's use of gestures on spatial transformation tasks. In Lindmeier, A. M. & Heinze, A. (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education, 3*, (pp. 361-368). Kiel, Germany: PME.

Ng, O. (2018). Examining Technology-Mediated Communication Using a Commognitive Lens: the Case of Touchscreen-Dragging in Dynamic Geometry Environments. International Journal of Science and Mathematics Education. Advanced online publication.

Niss, M. A. (2014). Functions Learning and Teaching. In S. Lerman, *Encyclopedia of Mathematics Education* (pp. 238-241). Springer Reference.

Núnez, R. (2003). Do real numbers really move? Language, thought, and gesture: The embodied cognitive foundations of mathematics. In R. Hersh (Ed.), 18 *Unconventional essays on the nature of mathematics* (pp. 160–181). New York: Springer.

Olivero, F. (2002). The Proving Process within a Dynamic Geometry Environment. PhD Thesis, Bristol, UK: University of Bristol.

Sahin-Gur, D., & Prediger, S. (2018). "Growth goes down, but of what?" A case study on language demands in qualitative calculus. In Bergqvist, E., Osterholm, M., Granberg, C., & Sumpter, L. (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education*, *4*, (pp. 99-106). Umea, Sweden: PME.

Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking covariation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berenson & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education–North America, 1*, (pp. 298–304). Raleigh: North Carolina State University.

Sasso, L. (2015). LA matematica a colori, Edizione Blu per il secondo biennio. Petrini Editore.

Schoenfeld, A. H., Smith, J., & Arcavi, A. (1993). Learning: The Microgenetic Analysis of One Student's Evolving Understanding of a Complex Subject Matter Domain. In R. Glaser (Ed.) Advances in Instructional Psychology, 4, 55-175. Erlbaum, NJ.

Schwartz, B., & Dreyfus, T. (1995). New actions upon old objects: A new ontological perspective on functions. *Educational Studies in Mathematics*, *29*, 259–291.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22(1)*, 1-36.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification - the case of function. In G. Harel, & E. Dubinsky, *The concept of function: Aspects of epistemology and pedagogy* (pp. 59-84). Washington, DC: MAA.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing.* Cambridge: Cambridge University Press.

Sfard, A. (2009). What's all the fuss about gestures? A commentary. *Educational Studies in Mathematics, 70*, 191–200.

Sfard, A., & Lavie, I. (2005). Why Cannot Children See as the Same What Grown-ups Cannot See as Different? Early Numerical Thinking Revisited. *Cognition and Instruction*, *23*(2), 237-309.

Siegler, R. S. (2006). Microgenetic analyses of learning. In D. Kuhn & R. Siegler (Eds.), Handbook of child psychology (pp. 464–510). Hoboken, NJ: John Wiley & Sons.

Sierpinska, A. (1988). Epistemological remarks on functions. *Proceedings of the 12th International Conference on the Psychology of Mathematics Education*, *3,* 568-575.

Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 25–58). Washington DC: MAA.

Sinclair, N., & Gol Tabaghi, S. (2010). Drawing space: Mathematicians' kinetic conceptions of eigenvectors. *Education Studies in Mathematics, 74*(3), 223-240.

Sinclair, N., & Yurita, V. (2008). To be or to become: how dynamic geometry changes discourse. *Research in Mathematics Education, 10*, 135-150.

Sinclair, N., & Robutti, O. (2013). Technology and the Role of Proof: The Case of Dynamic Geometry. In A. J. Bishop, M. A. Clements, C. Keitel & F. Leung (Eds.), *Third international handbook of mathematics education*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Sinclair, N., Healy, L., & Reis Sales, C. (2009). Time for telling stories: Narrative thinking with Dynamic Geometry. *ZDM*, *41*, 441-452.

Sinclair, N., & Moss, J. (2012). The more it changes, the more it becomes the same: the development of the routine of shape identification in dynamic geometry environment. *International Journal of Educational Research*, *51-52*, 28-44.

Solomon, Y., & O'Neill, J. (1998). Mathematics and narrative. *Language and Education*, *12*(3), 210–221.

Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity and proof. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 495-511). New York: Macmillan.

Tall, D. (1996). Function and Calculus, In: A.J. Bishop et al. (Eds.) *International Handbook of Mathematics Education,* 289–325. The Netherlands: Kluwer Academic Publishers.

Tall, D. (2009). Dynamic mathematics and the blending of knowledge structures in the calculus. *ZDM*, *41*(4), 481–492.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151-169.

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, *26*, 229–274.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education,* WISDOMe Monographs (Vol. 1, pp. 33–57). Laramie: University of Wyoming.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation and functions: Foundational ways of mathematical thinking. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). Reston, VA: National Council of Teachers of Mathematics.

UMI, MIUR, SIS e MATHESIS (2003). *Matematica 2003*. Lucca, Italia: Liceo Scientifico "A. Vallisneri".

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology*, *14*, 293-305.

Vinner, S. (2002). The Role of definitions in the Teaching and Learning of Mathematics. In Tall D. (Eds.) *Advanced mathematical Thinking*, Mathematics Education Library, *11*. Springer, Dordrecht.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, *20*, 356–366.

Vinner, S., & Hershkowitz, R. (1980). Concept images and common cognitive paths in the development of some simple geometrical concepts. In R. Karplus (Ed.), *Proceedings of the Fourth International Conference for the Psychology of Mathematics Education* (pp. 177-184). Berkeley: University of California, Lawrence Hall of Science.

Vygotsky, L. S. (1987). Thinking and speech. In R. W. Rieber, & A. C. Carton (Eds.), *The collected works of L. S. Vygotsky*. New York: Plenum Press.

Weingarden, M., & Heyd-Metzuyanim, E. (2018). Examining explorative instruction according to the realization tree assessment tool. In Bergqvist, E., Osterholm, M., Granberg, C., & Sumpter, L. (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education*, *4*, (pp. 427-434). Umea, Sweden: PME.

Wittgenstein, L. (1953/2003). *Philosophical investigations: The German text, with a revised English translation* (3rd ed., G. E. M. Anscombe, Trans.). Malden, MA: Blackwell.

Yerushalmy, M. (1991). Students perceptions of aspects of algebraic function using multiple representation software. *Journal of Computer Assisted Learning, 7*, 42–57.