On network two stages variable returns to scale Dea models

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Abstract

Multiplicative decomposition of stages indices is shown to be consistent with VRS network technologies. It is also shown why the primal dual correspondence breaks for serial network VRS models. Different VRS models can be associated with alternative transfer pricing systems, within the network. Multiplicative decomposition implies marginal cost pricing across stages. While other pricing systems (full cost) correspond to some of the known non-multiplicatively decomposable VRS models, proposed in the literature. Stages indices, therefore, respond not only to efficiency, but also to the network's distributive criteria across stages. The distributive contents of stage indices provide the key element for a solution to the problem of measuring scale efficiency in network systems. Multiplicative decomposable VRS models can be extended to more general network systems, containing both parallel and in series structures. The cost of this generalisation is that efficiency indices are referred to modified stages, that is to stages that include dummy processes. In perspective, these results contribute to show how organisational aspects, such as transfer pricing systems, could be modelled once network technologies are approached from the multiplier (ratio) side.

Keywords: Data envelopment analysis (DEA); Two stages network; Returns to scale; Scale efficiency; Duality

1 Introduction

There are some open issues in Data Envelopment Analysis (DEA) literature as to how convexity constraints should be embodied into network models in order to provide the Variable Return to Scale (VRS) technology. It is maintained that VRS technology is obtained by following the same procedure as in standard single stage model: i.e. by adding convexity constraints on intensity variables to the Constant Return to Scale

(CRS) model (Cook and Zhu 2014; Chen et al. 2016; Chen et al. 2009a; Chen et al. 2010a). The problem has firstly arisen in the attempt to provide a VRS network model with multiplicative decomposition of stage indices, as firstly proposed by Kao and Hwang (2008) and Liang et al. (2008) for the CRS model. In such a model, the inclusion of convexity constraints seems to be inconsistent with the requirements posed by multiplicative decomposition (Chen et al. 2009a; Kao and Hwang 2011; Sahoo et al. 2014; Chen and Zhu, 2018; Lim and Zhu 2019). Proposals to overcome the problem have been put forward by Wang and Chin (2010), who rely, however, on non convex technology, and by Sahoo et al. (2014), Lim and Zhu (2019), who have proposed a way to get multiplicative decomposition by a post optimality sharing rule.

However, it is still open the problem of identifying the reason of the apparent conflict between convexity and multiplicative decomposition. Even more so when it is observed that, in some cases CRS stage efficiency indices turn out to be higher than the corresponding VRS indices. This problem does not seems to be confined to models with multiplicative decomposition of stage indices. As shown in Chen et al. (2009b), Cook et al. (2010), Lim and Zhu (2019) by resorting to an additive objective function one can decompose the overall (network) efficiency index into the sum of its stages indices. However, even in these models, CRS stage indices turn out, in some case, to be higher than VRS indices. Casting, therefore, doubts as to what meaning should be attached to VRS technology in network systems. Quite inevitably, these results have also led to question the primal-dual correspondence, which holds for single stage models, between multiplier-based and envelopment-based network models, thus suggesting that they might constitute two different approaches in describing network technologies (Chen et al.2013; Sahoo et al. 2014; Lim and Zhu 2019).

There are, therefore, at least two problems linked to the contents of VRS network models: one is that multiplicative decomposition of divisional indices appears not to be consistent with VRS technology; the other is that, even abstaining from multiplicative decomposition, network VRS technology does not seem to hold the properties it has in a single stage technology, namely VRS efficiency indices being not lower than the CRS indices. This paper aims at providing an answer to these questions, and it does so by investigating the organizational underpinnings of multiplicative decomposition and their link with the primal-dual correspondence in the case of two stages serial network models.

The paper unfolds as follows. Sec. 2 shows the central role of the "multiplier linking constraint" in achieving multiplicative decomposition of stage indices. It also shows why the primal-dual correspondence breaks for serial network VRS models and, in this way, it provides a method to arrive at a multiplicatively decomposable network model. Sec. 3 shows how the multiplicity of VRS network specifications can be associated to different transfer pricing systems within the network. It is pointed out, therefore, how multiplicative decomposition implies marginal cost pricing within the network. Other pricing systems (full cost) correspond to some of the known non-multiplicatively decomposable VRS models proposed in the literature. Sec. 4. shows that multiplicatively and non-multiplicatively decomposable convex models are equivalent in as far as overall efficiency indices are concerned. What distinguishes them is the distributive criterion followed in allocating the overall index across stages. Therefore, multiplicative decomposition of indices comes to be one of the many possible criteria to allocate overall efficiency index. Non-multiplicatively decomposable models, associated with full cost pricing, are equally plausible, although they leave open the question of how to allocate, across stages, the overall index of efficiency. In Sec. 5 it is argued that stage scale efficiency cannot be computed, as in the single stage model, by the ratio of stage CRS to VRS indices, because these indices respond to distributive criteria and depend on assumptions about the network transfer pricing system. It is proposed that, in a network, scale efficiency should take up the nature of conditional measurement. That is, returns to scale assumptions have to be changed one stage at a time and the ”conditional” scale efficiency computed for the stage under assessment. This introduces some sort of “path dependence” type of indeterminacy in measuring scale efficiency, because, even in a two stages model, there are at least two alternative paths to go form a CRS to a VRS network model, one stage at a time. Sec. 6 contains an illustrative application of a VRS network model with multiplicative decomposable indices. Sec. 7 extends to more general network structures the approach developed in the previous sections. A final section contains some concluding comments and hints for further research.

2 The multiplier linking constraint

We consider a two-stages serial technology with sub-technologies connected in a system to form a network[1]. We assume there are n number of units, and each, in the first stage (A), uses m number of inputs to produce p number of intermediate outputs, which are then used as the only inputs by second stage (B), to produce s number of final outputs. For each unit k ($k = 1, ..., n$), the vectors $\mathbf{x}^k \in \mathcal{R}^m$; $\mathbf{z}^k \in \mathcal{R}^p$ and $\mathbf{y}^k \in \mathcal{R}^s$ are initial inputs, intermediate outputs and final outputs, respectively. The input matrix is defined by $X = [\mathbf{x}_1, ...\mathbf{x}_n] \in \mathcal{R}^{n\ x\ m}$; the intermediate output matrix by $Z = [\mathbf{z}_1, ...\mathbf{z}_n] \in \mathcal{R}^{n\ x\ p}$;and final output matrix by $Y = \left[\mathbf{y}_1, ...\mathbf{y}_n\right] \in \mathcal{R}^{n\ x\ s}$; It is also assumed that $X > 0; Z > 0; Y > 0$. The envelopment based network technology by Fare and Grosskopf (1996) can be taken as the generally accepted

[1]For "series" or "serial" network structure is meant a system where the whole of first stage output, and only that, is input to second stage. While "parallel" structures are made of processes operating independently.

reference for the definition of a network technology. Adding convexity constraints, we have, the envelopment input oriented VRS DEA model[2]:

$$E^I = \left\{ \min\ \theta \;\middle|\; \begin{array}{ll} X'\boldsymbol{\lambda}_A - \theta \mathbf{x}^k \leq 0 & -\mathbf{e}\boldsymbol{\lambda}_A = -1 \\ Z'\boldsymbol{\lambda}_B - Z'\boldsymbol{\lambda}_A \leq 0\ ; & -\mathbf{e}\boldsymbol{\lambda}_B = -1 \\ Y'\boldsymbol{\lambda}_B \geq \mathbf{y}^k & \boldsymbol{\lambda}_A, \boldsymbol{\lambda}_B \geq \mathbf{0} \end{array} \right\} \tag{2.1}$$

Where prime sign is for transpose and $\boldsymbol{\lambda}_A$, $\boldsymbol{\lambda}_B$ are the $(n\ x\ 1)$ vectors of intensity variables. Its dual provides the VRS network multiplier DEA model (Chen and Zhu 2004; Sahoo et al. 2014; Lim and Zhu 2019), which, in fractional form, is[3]:

$$E^I = \left\{ \max\ \frac{\mathbf{y}^k\mathbf{u} - h_A^I - h_B^I}{\mathbf{x}^k\mathbf{v}} \;\middle|\; \begin{array}{c} \frac{\mathbf{z}^j\mathbf{w} - h_A^I}{\mathbf{x}^j\mathbf{v}} \leq 1 \\ \frac{\mathbf{y}^j - h_B^I}{\mathbf{z}^j\mathbf{w}} \leq 1 \end{array} ;\ \begin{array}{c} \mathbf{v}, \mathbf{w}, \mathbf{u} \geq \mathbf{0} \\ h_A^I, h_B^I = \text{free} \\ j = 1, \ldots, n \end{array} \right\} \tag{2.2}$$

Overall (E^I) and stages efficiency indices (E_A^I, E_B^I) are defined as:

$$\begin{aligned} E^I &= \left(\mathbf{y}^k\mathbf{u} - h_A^I - h_B^I\right) / \, \mathbf{x}^k\mathbf{v} \\ E_A^I &= \left(\mathbf{z}^k\mathbf{w} - h_A^I\right) / \, \mathbf{x}^k\mathbf{v} \\ E_B^I &= \left(\mathbf{y}^k\mathbf{u} - h_B^I\right) / \, \mathbf{z}^k\mathbf{w} \end{aligned} \tag{2.3}$$

Therefore, overall efficiency cannot be decomposed into the product of stage efficiency indices ($E^I \neq E_A^I E_B^I$), due to the presence of scale variables (h_A^I, h_B^I).

The problem requires further examination, which can start by observing that in a multiplier model, a special linking constraint has to hold to get a multiplicative decomposition of indices. To show what this constraint is, let I_A, O_A; I_B, O_B, stand for stage A input and output (virtual) measures; and stage B input and output (virtual) measures, respectively. In an input oriented multiplier model, overall efficiency is given by the ratio of final output to initial input: O_B/I_A. If multiplicative decomposition has to hold, then it must be that:

$$\frac{O_B}{I_A} = \frac{O_A}{I_A}\frac{O_B}{I_B}$$

That is, multiplicative decomposition requires that:

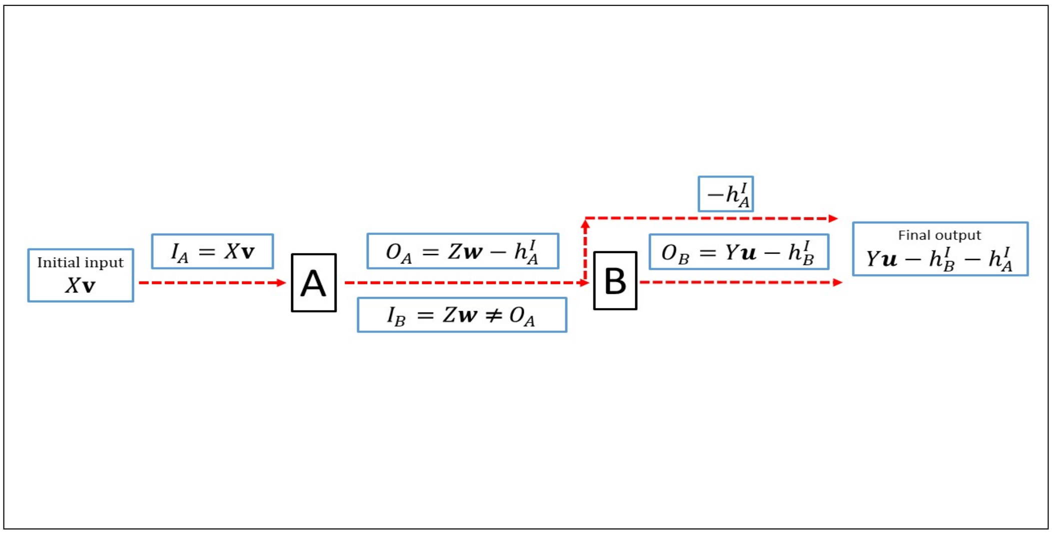
$$O_A = I_B \tag{2.4}$$

[2]The convexity constraints carry negative signs in order for the input oriented multiplier version to conform with the usage (Banker et al. 1984) of having scale variables (h) with negative signs.

[3]The fractional form is a convenient way to have at once the multiplier model and the implied definition of overall and stages indices for all the models following this first one.

Let us name it the "*multiplier linking constraint*". It is easily checked that it does not hold for model (2.2) where output from first stage is $O_A = Z\mathbf{w} - h_A^I$, while second stage input is $I_B = Z\mathbf{w}$. Fig 1 provides a flow chart representation of this point.

Fig. 1: Model (2.2) network



The multiplier linking constraint does hold, instead, for a CRS network model (Kao and Hwang, 2008) where, by definition, scale variables are set to zero. Quite clearly the problem for VRS network models lies in the envelopment convexity constraints (as contained in 2.1) interfering with the multiplier linking constraint (2.4). In seeking a way to reconcile the two, it is useful to observe that algebraically equivalent forms of convexity constraints in the envelopment model give rise to different multiplier models. For instance, the system of linear equations that makes the convexity constraints in model (2.1) is in the form:

$$\begin{bmatrix} -\mathbf{e} & \mathbf{0} \\ \mathbf{0} & -\mathbf{e} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_A \\ \boldsymbol{\lambda}_B \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \tag{2.5}$$

However, by mean of an Elementary Row Operation (ERO), an equivalent system can be obtained. For instance, consider the following:

$$\begin{bmatrix} -\mathbf{e} & \mathbf{e} \\ \mathbf{0} & -\mathbf{e} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_A \\ \boldsymbol{\lambda}_B \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{2.6}$$

By replacing in model (2.1), the original convexity constraints (2.5) with (2.6), a new envelopment model is obtained, with the same solution set as that of the original model. However, the new envelopment model leads to a multiplier model different from (2.2), namely to:

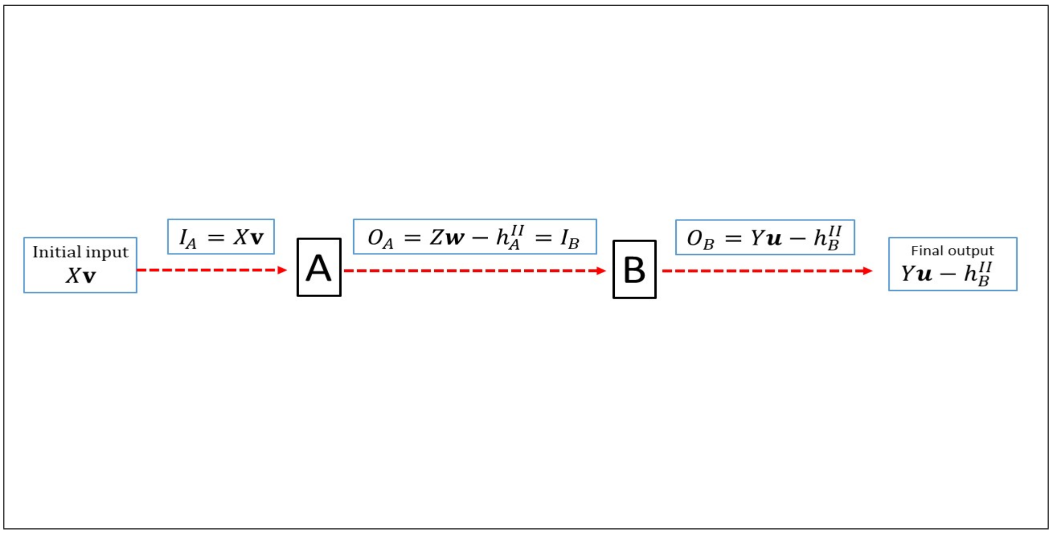
$$E^{II} = \left\{ \max \frac{\mathbf{y}^k\mathbf{u} - h_B^{II}}{\mathbf{x}^k\mathbf{v}} \;\middle|\; \begin{matrix} \frac{\mathbf{z}^j\mathbf{w} - h_A^{II}}{\mathbf{x}^j\mathbf{v}} \leq 1 \\ \frac{\mathbf{y}^j\mathbf{u} - h_B^{II}}{\mathbf{z}^j\mathbf{w} - h_A^{II}} \leq 1 \end{matrix} \; ; \; \begin{matrix} \mathbf{v}, \mathbf{w}, \mathbf{u} \geq \mathbf{0} \\ h_A^{II}, h_B^{II} = \text{free} \\ j = 1, \ldots, n \end{matrix} \right\} \tag{2.7}$$

This model does satisfy the multiplier linking constraint: $O_A = Z\mathbf{w} - h_A = I_B$ and therefore its efficiency indices are multiplicatively decomposable:

$$E^{II} = \frac{\left(\mathbf{y}^k\mathbf{u} - h_B^{II}\right)}{\mathbf{x}^k\mathbf{v}} = \frac{\mathbf{z}^k\mathbf{w} - h_A^{II}}{\mathbf{x}^k\mathbf{v}} \frac{\mathbf{y}^k\mathbf{u} - h_B^{II}}{\mathbf{z}^k\mathbf{w} - h_A^{II}} = E_A^{II} E_B^{II} \tag{2.8}$$

Fig 2 provides a flow chart representation of model (2.7) and comparison with Fig. 1 helps illustrating what changes occur in the definition of (virtual) input and (virtual) output in respect to model (2.2).

Fig. 2: Model (2.7) network



Although model (2.7) provides a VRS network model with multiplicatively decomposable indices, the procedure of how to get from the CRS envelopment model to the VRS multiplier with decomposable indices is still to be detailed. Although cumbersome to be put into words, the procedure is fairly straightforward once it is realized that the central issue is about selecting among alternative ERO on the convexity constraints (2.5), the one consistent with the multiplier linking constraint (2.4). From there, the multiplier model with decomposable indices would follow. A sketchy description is as follows:

1) Choose model's orientation and write down the relevant envelopment VRS model with convexity constraints in the standard form. That is, as in (2.5) for input oriented model, or in the form:

$$\begin{bmatrix} \mathbf{e} & \mathbf{0} \\ \mathbf{0} & \mathbf{e} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_A \\ \boldsymbol{\lambda}_B \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2.9}$$

for output oriented model[4];

2) By dual relations write down the multiplier model. At this stage the multiplier linking constraint (2.4) will not hold;

3) In the multiplier model, modify the allocation of scale variables (h) as required for the multiplier linking constraint (2.4) to hold. The desired allocation is obtained by moving the appropriate scale variable from the objective function to the relevant stage constraint. This move correspond to an ERO on the system of convexity constraints. The resulting model will satisfy the multiplier linking constraint (2.4) and, therefore, will have multiplicative decomposable indices.

As an example, the above procedure is used to get the output oriented VRS multiplier model. The starting point is the envelopment output oriented VRS model with convexity constraints in the standard form of (2.9). That is:

$$\begin{array}{l} Max\ \phi \\ \phi \mathbf{y}^k - Y'\boldsymbol{\lambda}_B \leq 0 \\ Z'\boldsymbol{\lambda}_B - Z'\boldsymbol{\lambda}_A \leq 0 \\ X'\boldsymbol{\lambda}_A \leq \mathbf{x}^k \\ e\boldsymbol{\lambda}_A = 1 \\ e\boldsymbol{\lambda}_B = 1 \end{array} \tag{2.10}$$

Duality relations lead to the multiplier model, in fractional form for ease of exposition:

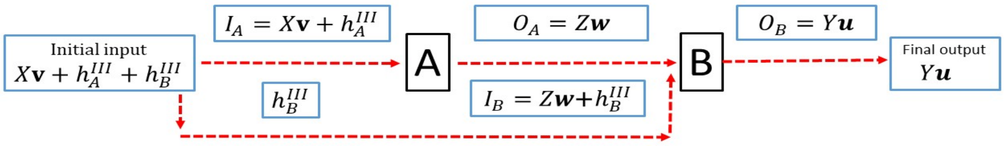
$$E^{III} = \left\{ \min \frac{\mathbf{x}^k\mathbf{v} + h_A^{III} + h_B^{III}}{\mathbf{y}^k\mathbf{u}} \;\middle|\; \begin{array}{l} \frac{\mathbf{x}^j\mathbf{v}+h_A^{III}}{\mathbf{z}^j\mathbf{w}} \geq 1 \\ \frac{\mathbf{z}^j\mathbf{w}+h_B^{III}}{\mathbf{y}^j\mathbf{u}} \geq 1 \end{array} ; \begin{array}{c} \mathbf{v}, \mathbf{w}, \mathbf{u} \geq \mathbf{0} \\ h_A^{III}, h_B^{III} = \text{free} \\ j = 1, ..., n \end{array} \right\} \tag{2.11}$$

With the help of Fig. 3, it is easily checked that, at this point, the multiplier linking constraint (2.4) does not hold because: $O_A = Z\mathbf{w} \neq I_B = Z\mathbf{w} + h_B^{III}$. To make the constraint hold, scale variable h_B^{III} has to become part of stage A output, instead of being part of initial input[5].

[4]This distinction being necessary to keep in line with the convention for scale variable (h) to take negative values in the case of increasing return to scale (Banker et al 1984).

[5]The alternative move of simply removing scale variable h_B^{III} from I_B (hence from initial inputs as well) would leave stage B without its scale variable. This would make stage B operate under costant return to scale.

Fig. 3: Model (2.11) network



All these moves correspond to an ERO that transforms the initial system of convexity constraints (2.9) into the following:

$$\begin{bmatrix} \mathbf{e} & \mathbf{0} \\ -\mathbf{e} & \mathbf{e} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_A \\ \boldsymbol{\lambda}_B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The initial VRS envelopment output oriented DEA model (2.10) becomes:

$$\begin{aligned} & Max\ \phi \\ & \phi \mathbf{y}^k - Y' \boldsymbol{\lambda}_B \leq 0 \\ & Z' \boldsymbol{\lambda}_B - Z' \boldsymbol{\lambda}_A \leq 0 \\ & X' \boldsymbol{\lambda}_A \leq \mathbf{x}^k \\ & e \boldsymbol{\lambda}_A = 1 \\ & -e \boldsymbol{\lambda}_A + e \boldsymbol{\lambda}_B = 0 \end{aligned}$$

and its dual, the multiplier model, in fractional form, is:

$$E^{IV} = \left\{ \min \ \frac{\mathbf{x}^k \mathbf{v} + h_A^{IV}}{\mathbf{y}^k \mathbf{u}} \ \middle| \ \begin{array}{l} \frac{\mathbf{x}^j \mathbf{v} + h_A^{IV}}{\mathbf{z}^j \mathbf{w} + h_B^{IV}} \geq 1 \\ \frac{\mathbf{z}^j \mathbf{w} + h_B^{IV}}{\mathbf{y}^j \mathbf{u}} \geq 1 \end{array} ; \ \begin{array}{c} \mathbf{v}, \mathbf{w}, \mathbf{u} \geq \mathbf{0} \\ h_A^{IV}, h_B^{IV} = \text{free} \\ j = 1, \ldots, n \end{array} \right\} \qquad (2.12)$$

Which shows that the multiplier linking constraint (2.4) does hold and efficiency indices are, therefore, multiplicatively decomposable:

$$E^{IV} = \frac{\mathbf{x}^k \mathbf{v} + h_A^{IV}}{\mathbf{y}^k \mathbf{u}} = \frac{\mathbf{x}^j \mathbf{v} + h_A^{IV}}{\mathbf{z}^j \mathbf{w} + h_B^{IV}} \frac{\mathbf{z}^j \mathbf{w} + h_B^{IV}}{\mathbf{y}^j \mathbf{u}} = E_A^{IV} E_B^{IV}$$

Of course, the procedure would be more cumbersome for network with more than two stages. However, the multiplier linking constraint (2.4) provides by itself the criterion to follow to get the multiplier model directly without having to go through the envelopment version. The central point, both for input and output oriented models, is that the primal (envelopment)-dual (multiplier) correspondence is no longer one-to-one as it happens for single stage models, and this is why defining technology from the multiplier (ratio) side opens the way to a wider set of technologies than that available under the definition from the envelopment side.

This result is, somehow, symmetrical to the one already observed in DEA network literature about the lack of frontier projections in network multiplier models (Chen et al. 2010b). There, alternative specifications of the envelopment linking constraint (Fare and Grosskopf, 1996) result in the same multiplier model. Sahoo et al. (2014) extend this approach to VRS models. Together these two observations suggest that primal and dual specifications of technology are ways to model different features of network processes.

3 Alternative pricing systems

The result from the previous section leads to a further question: what meaning to attach to different network convex models, such as models (2.2) and (2.7)[6]. For completeness, we observe that the set of two stages input oriented VRS models also includes a third and last model, which completes the set of all possible VRS multiplier models for a two stages network if elementary row operations (ERO) are limited to addition and/or subtraction and exclude the possibility of scaling. This statement can easily be shown to hold because all models have to come from an elementary row operation (ERO). If, starting from (2.5), as a first ERO one gets the system of constraints (2.6), then, in a two columns system, there is only one more possibility left, that is:

$$\begin{bmatrix} -\mathbf{e} & \mathbf{0} \\ \mathbf{e} & -\mathbf{e} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_A \\ \boldsymbol{\lambda}_B \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{3.1}$$

If the possibility of scaling is also admitted, then the set of alternative models becomes infinite even in a two stage model. Using (3.1) to replace (2.5) in model (2.1) leads to an algebraically equivalent model, whose multiplier version in fractional form is:

[6]Attention is being limited to input oriented models only for ease of exposition. The same arguments and answers apply to output oriented models.

$$E^V = \left\{ \max \frac{\mathbf{y}^k\mathbf{u} - h_A^V}{\mathbf{x}^k\mathbf{v}} \;\middle|\; \begin{array}{l} \frac{\mathbf{z}^j\mathbf{w} - h_A^V + h_B^V}{\mathbf{x}^j\mathbf{v}} \leq 1 \\ \frac{\mathbf{y}^j\mathbf{u}}{\mathbf{z}^j\mathbf{w} + h_B^V} \leq 1 \end{array} \;;\; \begin{array}{c} \mathbf{v}, \mathbf{w}, \mathbf{u} \geq \mathbf{0} \\ h_A^V, h_B^V = \text{free} \\ j = 1, ..., n \end{array} \right\} \tag{3.2}$$

It satisfies the convexity constraints in the envelopment formulation and, therefore, it is a valid alternative to models (2.2) and (2.7). Like model (2.2) it is not multiplicatively decomposable. We are going to show that these three models differ in the way scale variables are allocated among stages and that the reason of such a difference can be traced in the transfer pricing policy adopted within the network.

In order to compare the way scale variables are allocated, it is necessary to express each model in the same variables. We have already shown that models (2.2) and (2.7) come from imposing two different, though algebraically equivalent, systems of convexity constraints (2.5 and 2.6) to the basic CRS envelopment model. A similar link holds for model (2.2) and (3.2)[7]. Now, to compare models (2.2), (2.7) and (3.2), in their linearised forms (by mean of the Charnes and Cooper (1962) transformation), a starting observation is that their envelopment VRS versions are linked by an ERO; therefore, an *elementary column operation* (ECO) links the multiplier versions. It is straightforward to see that models (2.2) and (2.7) are linked by an ECO, in their linearised forms, and, therefore, they are equivalent up to the following change of variables:

$$\begin{aligned} h_B^I &= h_B^{II} - h_A^{II} \\ h_A^I &= h_A^{II} \end{aligned} \tag{3.3}$$

Where h_B^I, h_A^I are scale variables from model (2.2) and h_B^{II}, h_A^{II} are scale variables in model (2.7). By the same reasoning, we can compare models (2.7) and (3.2) to obtain that they also are equivalent, in their linearised forms, up to the following change of variable:

$$\begin{gathered} h_A^V = h_B^{II} \\ h_B^V = h_B^{II} - h_A^{II} \end{gathered} \tag{3.4}$$

Where h_A^V, h_B^V are scale variables in model (3.2) and, as before, h_B^{II}, h_A^{II} are scale variables in model (2.7). By applying these change of variables to the efficiency indices from model (2.2) and model (3.2), we are able to see how allocation of scale variables differs across models.

[7] See footnote above.

Table 1: Models (2.2) and (3.2) efficiency indices in terms of model (2.7)'s variables

Model (2.7) indices	Model (2.2) indices	Model (3.2) indices
$E^{II} = \frac{\mathbf{y}^k\mathbf{u}-h_B^{II}}{\mathbf{x}^k\mathbf{v}}$	$E^{I} = \frac{\mathbf{y}^k\mathbf{u}-h_B^{II}}{\mathbf{x}^k\mathbf{v}}$	$E^{V} = \frac{\mathbf{y}^k\mathbf{u}-h_B^{II}}{\mathbf{x}^k\mathbf{v}}$
$E_A^{II} = \frac{\mathbf{z}^k\mathbf{w}-h_A^{II}}{\mathbf{x}^k\mathbf{v}}$	$E_A^{I} = \frac{\mathbf{z}^k\mathbf{w}-h_A^{II}}{\mathbf{x}^k\mathbf{v}}$	$E_A^{V} = \frac{\mathbf{z}^k\mathbf{w}-h_A^{II}}{\mathbf{x}^k\mathbf{v}}$
$E_B^{II} = \frac{\mathbf{y}^k\mathbf{u}-h_B^{II}}{\mathbf{z}^k\mathbf{w}-h_A^{II}}$	$E_B^{I} = \frac{\mathbf{y}^k\mathbf{u}-h_B^{II}+h_A^{II}}{\mathbf{z}^k\mathbf{w}}$	$E_B^{V} = \frac{\mathbf{y}^k\mathbf{u}}{\mathbf{z}^k\mathbf{w}-h_A^{II}+h_B^{II}}$

Table 1 shows that the basic difference among the three models rests on the allocation of scale variables in stage B. Namely, scale variable h_A is part of input in models (2.7) and (3.2), while it is part output in model (2.2). As for scale variable h_B, it is part of output in models (2.7) and (2.2), while it is part of input in model (3.2). The different allocation of scale variables causes stage B efficiency indices to differ among models. Therefore, selecting one model over the others is not indifferent as far as second stage efficiency is concerned, while first stage and overall indices stay the same across models.

The issue, however, can conveniently be seen as the result of alternative transfer pricing policies within the network. The allocation of profits/losses to subsidiaries abroad is the core of transfer pricing models used by multinational companies. In allocating scale variables among stages, a network faces very much the same problem. Models (2.2), (2.7) and (3.2) have the unit prices of resource ($\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$) endogenously determined (actually they are the same across models) while they differ in the way they allocate scale variables (we shall refer to them also as: economic profits/losses, see Starrett 1977).

By looking at model (2.7) and Fig. 2, we can see that efficient units (k) in stage A are on the supporting hyperplane: $\mathbf{z}^k\mathbf{w} - \mathbf{x}^k\mathbf{v} - h_A^{II} = 0$, and price their output at: $\mathbf{z}^k\mathbf{w} - h_A^{II}$, that is to variable cost $\mathbf{x}^k\mathbf{v}$. Because variable cost is constant per unit, we are in front of a marginal cost pricing policy[8].

On the hyperplane: $\mathbf{y}^k\mathbf{u} - \left(h_B^{II} - h_A^{II}\right) - \mathbf{z}^k\mathbf{w} = 0$, efficient units in stage B take stage A's output ($\mathbf{z}^k\mathbf{w} - h_A^{II} = \mathbf{x}^k\mathbf{v}$) as input, and price their own output at: $\mathbf{y}^k\mathbf{u} - h_B^{II} = \mathbf{z}^k\mathbf{w} - h_A^{II}$. Therefore, they also follow marginal cost pricing[9] and bear the costs/benefits of their economic profits/losses, which means that their economic profits ($h_B^{II} \geq 0$) reduce gross revenue ($\mathbf{y}^k\mathbf{u}$) down to marginal cost ($\mathbf{z}^k\mathbf{w} + h_A^{II}$). In

[8]The quantity $\mathbf{x}^k\mathbf{v}$ stand for variable cost in accounting practice and, in general, differs from marginal cost. However, if we have a linear technology with fixed input to output ratios and given input prices (these latter are parameters in the hyperplane) then variable cost equals marginal cost. Limitedly to the rest of this section we shall use variable cost and marginal cost interchangeably.

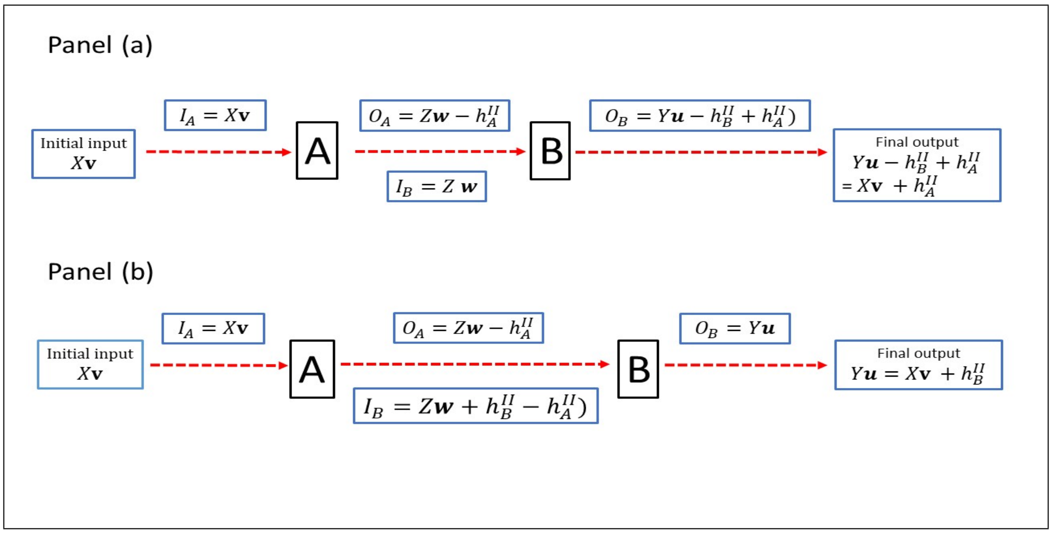
[9]For stage B the quantity: $\mathbf{z}^k\mathbf{w} + h_A^{II}$ represents marginal cost. No matter if it contains a profit/loss element. That is part of stage A pricing system.

case of economic losses ($h_B^{II} \leq 0$), instead, gross revenue would be increased up to marginal cost. Either way, efficiency requires zero accounting profits/losses.

From the network point of view (see Fig. 2) the combination of stage A and stage B marginal cost pricing leads the whole system to function accordingly, and final output to be priced at: $\mathbf{y}^k\mathbf{u}-h_B^{II} = \mathbf{z}^k\mathbf{w}-h_A^{II} = \mathbf{x}^k\mathbf{v}$, that is at the network's marginal cost ($\mathbf{x}^k\mathbf{v}$). It follows that the network as a whole bears its economic profits/losses by allocating them to the stage they arise from[10]. This goes to show that within the set of network convex models, multiplicative decomposition of stage indices implies marginal cost pricing within the network[11].

The same reasoning, once applied to model (2.2), allows to highlight the differences, with respect to model (2.7), in terms of transfer pricing policy. We have already shown that models (2.2) and (2.7) are equivalent up to a change of variable (3.3). Therefore, efficiency indices from model (2.2) can be re-written by using model (2.7) variables, as done in Table 1. Efficient units are on the same hyperplanes as those we have just been describing above for model (2.7). To illustrate the difference in terms of transfer pricing, however, a graphical representation of model (2.2), in terms of model (2.7) scale variables, is of better help.

Fig. 4: Model (2.3) (Panel a) and model (3.2) (Panel b) in terms of model (2.7)'s variables



Panel (a), in Fig. 4, shows that stage A pricing policy is the same as in model (2.7). That is, it operates under marginal cost pricing: economic profits/losses are born by the stage itself.

[10]In his seminal article, Hirshleifer (1956) derives optimal transfer prices that lead each unit to maximize the company profits. This result, however, requires some rather restrictive assumptions in the case of a network. Above all, that each unit operating costs be independent of the level of operations in other units (technological independence).

[11]This does not hold outside the set of convex network models, as model (4.4) in Sec.4 shows

The essential difference is in stage B, where input is priced at full cost $\mathbf{z}^k\mathbf{w} = \mathbf{x}^k\mathbf{v} + h_A^{II}$, (instead of stage A marginal cost: $\mathbf{z}^k\mathbf{w} - h_A^{II}$). This causes stage A economic profits/losses to be shifted forward onto final output at the benefit/cost of stage B revenue. Then, stage B prices its output at: $\mathbf{y}^k\mathbf{u} - h_B^{II} + h_A^{II} = \mathbf{z}^k\mathbf{w}$. That is, it bears the costs/benefits of its economic profits/losses (h_B^{II}). In short, first stage profits/losses h_A^{II} are shifted forward onto final output, while second stage profits/losses h_B^{II} are born by the stage. The network as a whole prices its output at: $\mathbf{y}^k\mathbf{u} - h_B^{II} + h_A^{II} = \mathbf{x}^k\mathbf{v} + h_A^{II}$, that is, at the network marginal cost ($\mathbf{x}^k\mathbf{v}$) plus first stage economic profits/losses (h_A^{II}). Therefore, differently from what happens in model (2.7), stage B does not follow marginal cost pricing, rather it follows a hybrid form of full cost pricing[12].

By the same type of reasoning, it can be shown (Panel (b), Fig. 4) that model (3.2) works according to a pricing system somehow symmetric to that of model (2.2). As in models (2.2), (2.7) and (3.2), first stage follows marginal cost pricing by setting its output at: $\mathbf{z}^k\mathbf{w} - h_A^{II} = \mathbf{x}^k\mathbf{v}$. Stage B, instead, prices its input at full cost: $\mathbf{z}^k\mathbf{w} - h_A^{II} + h_B^{II}$, that is, at marginal costs $\mathbf{z}^k\mathbf{w} - h_A^{II}$, plus its economic profits/losses h_B^{II}. Therefore, while stage A economic profits/losses are born by that stage, stage B economic profits/losses are shifted forward onto final output. Altogether, this pricing system leads to final output being priced at the network's marginal cost plus second stage economic profits/losses: $\mathbf{x}^k\mathbf{v} + h_B^{II}$.

To summarise, the three models can be distinguished by the pricing decision taken at stage B. While stage A follows marginal cost pricing in all the three models, stage B does the same in model (2.7); while it follows a sort of hybrid full cost rule in model (2.2); and follows the standard full cost pricing in model (3.2).

These results have a straightforward implication for the issue of multiplicative decomposition of network convex models. The pricing policies supported by models (2.2) and (3.2) are, from a general point of view, as plausible as the marginal cost pricing contained in model (2.7). Therefore, it is reasonable to maintain that the choice of which model to choose should be guided by what pricing policy is thought to be more appropriate for the case at hand and not simply by the multiplicative decomposition property.

Admittedly, if model (2.2) or (3.2) is chosen, then the question arises as to how allocate the overall index of efficiency across stages. This is because only model (2.7) provides an allocation, through its multiplicative decomposition, while models (2.2) and (3.2) do not contain a way to reconcile stages indices with the overall index. This issue is taken up in the following section.

[12] Full cost pricing by stage B would require its output being priced at marginal cost ($\mathbf{z}^k\mathbf{w} - h_A^{II}$) plus the stage's economic profits/losses (h_B^{II}).

4 The issue of multiplicative decomposition

Why should one seek multiplicative decomposition? Actually, what is sought is not multiplicative decomposition as such, rather a way to account for the sources of efficiency. Decomposition of overall index into stage efficiency indices conveys important bits of information similar to those provided by economic growth accounting models in determining factors' contribution to production. Multiplicative (Kao and Hwang 2008) and additive (Chen et al. 2009) decomposition of stage indices are the two basic methods used in the literature. While they have in common the definition of stage indices they differ largely in the objective function and in the manner indices are decomposed. Chen et al. (2009) firstly proposed the additive decomposition, one motivation being that of overcoming the difficulty posed by multiplicative decomposition in modelling Vrs technology. The overall efficiency (the objective function) is a weighted average of stages efficiency indices. Weights are endogenously determined and that facilitates the linearization of the two stages model. Such an objective function poses two different problems. The first is that being the weights endogenous they come to be different from units to units. Therefore the size of the stages comes to be part of the assessment which is meant to refer only to efficiency. Linked to this problem there is also the observation that endogenous weight might differ from the management's objectives. The second problem is that decomposition weight are non increasing along the sequence of stages. That is upstream stages receive higher weighs and will therefore effect overall efficiency more than downstream stages. In terms of efficiency indices this causes a bias in favour of downstream stages because indices decrease when the stage's weight increases. Ang and Chen (2016) have provided an alternative model with exogenous weights which overcomes the above shortcomings. However the model is non linear and subject to computing difficulties.

While in what follows we shall concentrate on multiplicative decomposition, it cannot be left unsaid that it also faces some indeterminacy problem because of the non-uniqueness of its efficiency indices. That is because intermediate outputs do not appear neither in the objective function nor in the normalisation constraint. This implies that their value can change and the overall efficiency index remain the same. That is, the same overall efficiency index is consistent with many values of stage indices. Kao and Hwang (2008) recognised the problem and, in order to limit the indeterminacy, proposed to solve a couple (in a two stages mode) of post optimality models aiming at finding the highest efficiency score for one stage (the leader) while keeping the overall efficiency at optimal level. The rationale being that one might want to give priority to either stage in selecting among alternative optima. Though this procedure has become common practice in applied work it still remains a matter of concern especially in cases where non priority among stages is sought or no reliable information is available to assign priority.

In proposing model (2.2), Sahoo et al. (2014) provide an *ex post* multiplicative decomposition of the overall index of efficiency. Though slightly different, the decomposition proposed by Lim and Zhu (2019) is based on the same ex-post logic [13]. Using the symbols from model (2.2), Sahoo et al. (2014) proposed decomposition is:

$$\begin{aligned} & E^I = E^I_A * E^I_B * K^I \\ \Rightarrow \quad & \frac{\mathbf{y}^k\mathbf{u} - h^I_A - h^I_B}{\mathbf{x}^k\mathbf{v}} = \frac{\mathbf{z}^k\mathbf{w} - h^I_A}{\mathbf{x}^k\mathbf{v}} * \frac{\mathbf{y}^k\mathbf{u} - h^I_B}{\mathbf{z}^k\mathbf{w}} * \left(\frac{\mathbf{y}^k\mathbf{u} - h^I_A - h^I_B}{\mathbf{y}^k\mathbf{u} - h^I_B} \frac{\mathbf{z}^k\mathbf{w}}{\mathbf{z}^k\mathbf{w} - h^I_A} \right) \end{aligned} \tag{4.1}$$

Where E^I, E^I_A and E^I_B are as from (2.3) and K^I is identically defined by:

$$K^I \equiv \frac{E^I}{E^I_A * E^I_B} \tag{4.2}$$

That is K^I is a measure of the "gap" between the product of stages' indices and the overall index of efficiency. It has the role of a “residual” in the process of allocating overall productivity to stages of production. As such, K^I is not a measure of efficiency independent from stages' indices, even if its constitutive elements can be traced back to parts of efficiency indices, as Sahoo et al. (2014, Remarks 1-3, p. 587) and (4.1) show.

This point deserves, however, some further investigation which can start by recalling the equivalence among models (2.2), (2.7) and (3.2) as summarised by Table 1. In short, there it is shown that overall and first stage indices are the same across models, while second stage indices are different. If *ex post* multiplicative decomposition, such as (4.1), is also applied to model (3.2), thus determining K^V factor (computed as in (4.2), but with indices from model (3.2)), then because of the equality across models of first stage and overall efficiency indices (see Table 1), and because multiplicative decomposition holds for model (2.7), that is: $E^{II} = E^{II}_A E^{II}_B$, it must follow that efficiency indices from models (2.2), (2.7) and (3.2) are linked in the following manner:

$$\begin{aligned} E^{II} &= E^I = E^V \\ E^{II}_A &= E^I_A = E^V_A \\ E^{II}_B &= E^I_B * K^I = E^V_B * K^V \end{aligned} \tag{4.3}$$

[13] In terms of the symbols we have been using, and taking the case of intermediate measures being the same between stages (i.e. the Kao and Hwang, 2008 assumption), Lim and Zhu (2019) decomposition amounts to define second stage efficiency as: $E^I_B = \left(\mathbf{y}^k\mathbf{u} - h^I_A - h^I_B\right) / \left(\mathbf{z}^k\mathbf{w} - h^I_A\right)$, which is different from the definition used in their model (D3), which, in turn, is the same as model (2.2). This is why we call this decomposition "ex-post". If the change of variable (3.3) is applied to this definition of second stage index, one gets the same index as that in model (2.7) which is multiplicatively decomposable. Therefore, the definition proposed by Lim and Zhu (2019) is consistent with multiplicative decomposition of indices, though not with the model they adopt.

Where superscript I, II and V stand for models (2.2), (2.7) and (3.2) indices, respectively. The first two lines simply repeat the fact that first stage and overall indices are the same across models. The last line shows that if in models (2.2) and (3.2) factors K^I and K^V are allocated to stage B index, then the two models yield the same index as model (2.7). Therefore, the three models not only represent the same network technology, and that is known because they all come from the same envelopment model, but they also have the same distribution of efficiency across stages.

Of course, this conclusion can be reached only because model (2.7) shows how to allocate factors K^I and K^V to that end. There is, however, nothing compelling in allocating factors K^I and K^V in this way. That is, models (2.2) and (3.2) can be used to support any other different allocation of the K factors across stages. Actually one would expect a different allocation across stages from that provided by model (2.7), right because models (2.2) and (3.2) imply different transfer pricing systems within the network.

However, models (2.2) and (3.2) require additional assumptions to arrive at a full distribution of overall efficiency index across stages[14]. This causes some indeterminacy as how to interpret the results for benchmarking purposes[15] when using model (2.2) or (3.2), and suggests that further research is needed in order to link the internal pricing policies to a consistent way of sharing the overall index of efficiency across stages.

One possibility could be that of resorting to a Nash type of cooperative bargaining game, by extending to VRS technology the model proposed by Du et al. (2011) for CRS technology. Stages' disagreement constraints could be provided by stages' indices resulting from models (2.2) or (3.2). However the network's overall index of efficiency might turn out to be lower than that from model (2.2) or (3.2). That is because Du et al. (2011) model gives the optimal value for the network attainable through cooperation. That is because Du et al. (2011) model gives the optimal value for the network attainable under the constraint of the solution having to be cooperative. That is, by non-cooperation (as in models (2.2), (2.7) and (3.2)) higher overall efficiency could be achieved. It is of interest to observe, that for zero disagreement point, Du et al. (2011) model reduces to the multiplicative decomposable CRS model by Liang et al. (2008). It is straightforward matter to shown that such a property would continue to hold for VRS technology provided model (2.7) is used, right because it is multiplicatively decomposable.

In closing this section, a final remark is in line with the section's main contents:

[14] It is worth observing that by resorting to an additive (ex post) decomposition, instead of the multiplicative one, would not help because then the three models would all have a residual attached.

[15] Being K different across units, it can happen that, at stage level, efficiency ranking comes to depend on the assumed distribution of K across stages.

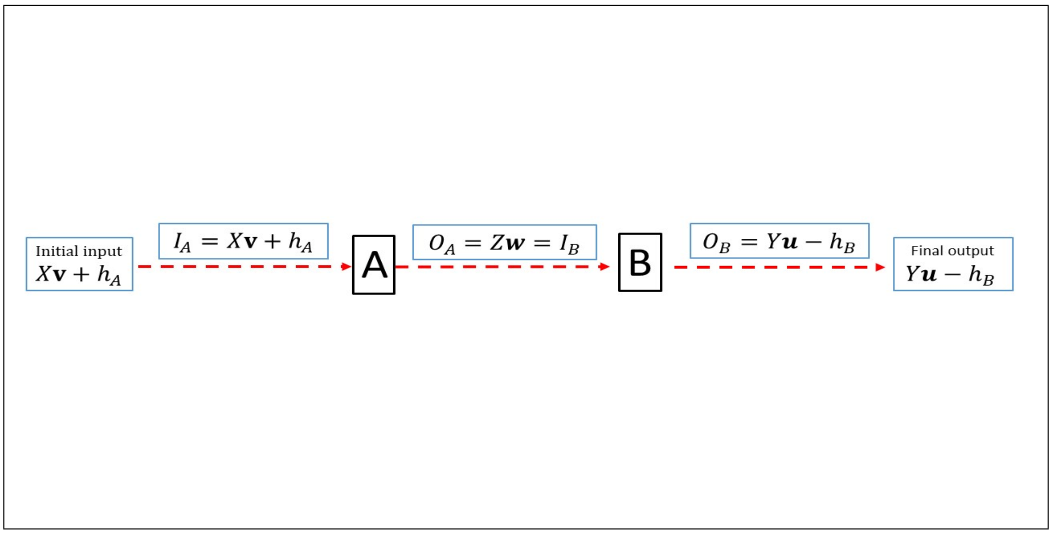
the role of the network internal pricing system. The possibility arises that in building a VRS network model one might end up in a trade-off between making the model consistent either with a given transfer pricing rule, or with convexity. For instance the model by Wang and Chin (2010):

$$E^{VI} = \left\{ \max \frac{\mathbf{y}^k\mathbf{u} - h_B^{VI}}{\mathbf{x}^k\mathbf{v}+h_A^{VI}} \;\middle|\; \begin{array}{l} \frac{\mathbf{z}^j\mathbf{w}}{\mathbf{x}^j\mathbf{v}+h_A^{VI}} \leq 1 \\ \frac{\mathbf{y}^j\mathbf{u}-h_B^{VI}}{\mathbf{z}^j\mathbf{w}} \leq 1 \end{array} \quad ; \quad \begin{array}{c} \mathbf{v}, \mathbf{w}, \mathbf{u} \geq \mathbf{0} \\ h_A^{VI}, h_B^{VI} = \text{free} \\ j = 1, ..., n \end{array} \right\} \tag{4.4}$$

has the property of multiplicative decomposition of indices but is not convex, neither in input, nor in output orientation. Non convexity is easily seen by working out its dual, the envelopment input oriented model, whose constraints on intensity vectors turn out to be: $\mathbf{e}\boldsymbol{\lambda}_A = \theta$; and $\mathbf{e}\boldsymbol{\lambda}_B = 1$. By a suitable change of variables: $\boldsymbol{\lambda}_A/\theta = \widehat{\boldsymbol{\lambda}}_A$; $\boldsymbol{\lambda}_B/\theta = \widehat{\boldsymbol{\lambda}}_B$ and $1/\theta = \phi$, one gets the equivalent output oriented model, with intensity constraints: $\mathbf{e}\widehat{\boldsymbol{\lambda}}_A = 1$; and $\mathbf{e}\widehat{\boldsymbol{\lambda}}_B = \phi$.

Fig. 5 shows that model (4.4) is a combination of first stage full cost pricing with second stage marginal cost pricing. It follows, with changes, the original idea put forward by Kao and Hwang (2011), by joining together an output oriented model, for the first stage, and an input oriented, in the second stage[16]. We do not, however, further investigate this non-convex class of models because out of our present scope. We observe, though, that models such as (4.4), when supplemented by organisational aspects, like the network internal pricing system, further widen the interest in a line of research based on approaching the network technology from the multiplier (fractional) side.

Fig. 5: Model (4.4) network



[16]Actually, Kao and Hwang (2011)'s was not a network model because made up of two separate stages: an input oriented first stage and an output oriented second stage. Combining these two stages into a network would result in a model, which is neither convex, nor multiplicatively decomposable.

5 Scale efficiency in network models

The peculiarity of CRS efficiency indices being not smaller than VRS indices, thus casting doubts on the meaning of scale efficiency measures in network systems, has so far been observed in network models with additive decomposition (Chen et al. 2009b, Cook et al. 2010 , Lim and Zhu, 2019) and also in models with ex-post multiplicative decomposition Lim and Zhu (2019).

The problem, however, persists in our multiplicatively decomposable model (2.7). Applying this model to the Kao and Hwang (2008) data reveals that, in 6 out of 24 observations, stage A CRS index is higher than the VRS index. Stage B and the overall VRS indices, instead, obey the usual rule of being not lower than CRS indices[17]. Inspection of model (2.7) shows one obvious though important fact, that is, first stage output is not part of the objective function, which contains only second stage output. Therefore, in going from CRS to VRS technology, optimal values for first stage variables will be selected so as to increase (relatively to CRS solution) final output, i.e. second stage output. First stage output is no more than an intermediate variable. This is the simple explanation of why in a network model it can happen that stages' CRS efficiency indices are higher than VRS indices. This explanation holds also for models with additive decomposition, where to be maximised is the weighted average of stages efficiency indices[18].

What is in question is not the overall (i.e. network) scale efficiency index, which shows the usual properties as in a single stage model (the same is observed in Lim and Zhu 2019). Inded, it can easily be argued that in a two stages network two additional (scale) variables are added, relatively to the CRS model, therefore, network VRS optimal solution cannot be lower than the CRS's. However, network scale efficiency index can be multiplicatively decomposed, as a result of its components (CRS &VRS indices) being themselves decomposable, and it is open to question what meaning to assign to stage indices arising from such a decomposition.

The answer to this question comes from the analysis carried out in Sec.3. The decomposition by stages of the overall index follows the network transfer pricing system. And models (2.2), (2.7), (3.2) show how alternative assumptions about the pricing system lead to different allocation, across stages, of the same overall index of efficiency. In other words, stage decomposition responds to a distributive criterion and it would be meaningless to use it to describe a technology feature such as scale efficiency.

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[17]The same happens for models (2.2) and (3.2). Interestingly, in model (3.2) it also happens that, for one unit, CRS index is higher than VRS index even for second stage.

[18]In these models one would expect that also second stage CRS indices could turn out to be not smaller than VRS indices, right because, differently from model (2.7), the model's objective function is not made only of second stage output.

However a reasonable way to get a scale efficiency index for each stage could be that of introducing the change in technology (from CRS to VRS) one stage at a time, in a step-by-step procedure. This would provide the increase in network efficiency due to changing the technology assumption in one stage, while keeping the other stage's technology unchanged, and would provide the same sort of information about stage scale efficiency as in a single stage model. It would have, though, the nature of "conditional" scale efficiency. Conditional on the other stage's returns to scale assumption.

To describe the procedure in some detail, let $E_{(\mathrm{B=Crs;\ A=Crs})}$ stand for the overall efficiency index of CRS network model, where the CRS assumption holds for both stages, as made clear by text within brackets. This index is obtained from model (2.1) by removing the convexity constraints and, by dual relations, set in its multiplier form. From Kao and Hwang (2008), it is known that this index is multiplicatively decomposable as:

$$E_{(\mathrm{B=Crs;\ A=Crs})} = E_{A(\mathrm{B=Crs;\ A=Crs})} * E_{B(\mathrm{B=Crs;\ A=Crs})}$$

Then, let $E_{(\mathrm{B=Vrs;\ A=Crs})}$ be the overall efficiency index of a model obtained from model (2.1) with stage A convexity constraints removed while stage B convexity constraint is maintained and, again, expressed in its multiplier form. This index is also multiplicatively decomposable because it complies with the multiplier linking constraint (2.4):

$$E_{(\mathrm{B=Vrs;\ A=Crs})} = E_{A(\mathrm{B=Vrs;\ A=Crs})} * E_{B(\mathrm{B=Vrs;\ A=Crs})}$$

Therefore, scale efficiency index for stage B, conditional on stage A being on CRS, is:

$$SE_{B(\mathrm{A=Crs})} = \frac{E_{(\mathrm{B=Crs;\ A=Crs})}}{E_{(\mathrm{B=Vrs;\ A=Crs})}} \tag{5.1}$$

Which can be decomposed as:

$$\begin{aligned} SE_{B(\mathrm{A=Crs})} &= \frac{E_{A(\mathrm{B=Crs;\ A=Crs})}}{E_{A(\mathrm{B=Vrs;\ A=Crs})}} * \frac{E_{B(\mathrm{B=Crs;\ A=Crs})}}{E_{B(\mathrm{B=Vrs;\ A=Crs})}} \\ &= \Delta E_{A(\mathrm{A=Crs})} * \Delta E_{B(\mathrm{A=Crs})} \end{aligned} \tag{5.2}$$

Now let $E_{(\mathrm{B=Vrs;\ A=Vrs})}$ be the overall efficiency index of model (2.7), where both stages are under convexity constraint. Then, stage A scale efficiency index, conditional on stage B being on VRS, can be computed as:

$$SE_{A(\mathrm{B=Vrs})} = \frac{E_{(\mathrm{B=Vrs;\ A=Crs})}}{E_{(\mathrm{B=Vrs;\ A=Vrs})}} \tag{5.3}$$

and decomposed as:

$$\begin{aligned} SE_{A(\text{B=Vrs})} &= \frac{E_{A(\text{B=Vrs; A=Crs})}}{E_{A(\text{B=Vrs; A=Vrs})}} * \frac{E_{B(\text{B=Vrs; A=Crs})}}{E_{B(\text{B=Vrs; A=Vrs})}} \\ &= \Delta E_{A(\text{B=Vrs})} * \Delta E_{B(\text{B=Vrs})} \end{aligned} \tag{5.4}$$

Indices (5.1) and (5.3) provide a measure of scale efficiency for the stage under assessment (B and A, respectively) with the usual properties as in a single stage model, with the provision of being conditional on the other stage's returns to scale assumption.

Stage decomposition of these indices, according to (5.2) and (5.4), only provides a measure of the way the overall increase in efficiency is shared across stages. In this following the results from Sec. 3, which has shown how multiplicative decomposition has a distributive meaning. If, for example, in (5.4) stage B index for a given unit happens to be greater than one ($\Delta E_{B(\text{B=Vrs})} \geq 1$), that has only the meaning of a decrease in stage B efficiency when stage A moves from CRS to VRS.

Of course, to provide a measure of scale efficiency for a given stage, one could follow a path different from that followed to get to (5.1) and (5.3). By taking $E_{(\text{B=Crs; A=Crs})}$ as a starting point, $E_{(\text{B=Crs; A=Vrs})}$ could then be computed and their ratio would provide scale efficiency index for stage A : $SE_{A(\text{B=Crs})}$, conditional on stage B being on CRS[19]. This index would in general be different from (5.3. The same would hold for scale efficiency index $SE_{B(\text{A=Vrs})}$ as the ratio of $E_{(\text{B=Crs; A=Vrs})}$ to $E_{(\text{B=Vrs; A=Vrs})}$, which would in general be different from (5.1). Therefore, this step-by-step type of assessment introduces a sort of "path dependency" issue because, even in the simplest network, made of only two stages, there are at least two different paths one could follow. The conditional feature, of this way in computing scale efficiency for network systems, contains, therefore, some inherent indeterminacy.

6 An illustrative application

For the 24 Taiwanese non-life insurance companies of Kao and Wang (2008), Table 2 shows efficiency indices under CRS and VRS assumptions. CRS indices are the same as in Kao and Hwang (2008) and can be obtained from any of the VRS models presented in the text (i.e. models (2.2), (2.7) and (3.2)), by dropping all scale variables. The same models, with all the scale variables included, provide the VRS indices in Table 2.

[19]It is of some interest to observe that such a combination of assumptions about stages returns to scale leads to a multiplier network model that is not multiplicatively decomposable because does not conform to the multiplier linking constraint (2.4). Thus providing an indirect support to the necessity of keeping the information contents of multiplicative decomposition of scale efficiency distinct from the content of conditional scale efficiency index.

Multiplicative decomposition holds only for model (2.7). Confirming the results obtained in Sec. 3, indices from the three VRS models are the same, as far as overall and stage A are concerned, while they differ for stage B.

However, if stage B indices from model (2.2) are scaled by the K^I factor (as computed in (4.2)), then one gets the same indices as from model (2.7). The same equivalence would hold between stage B indices from model (2.7) and those from model (3.2) provided the latter are scaled by the K^V factor described in Sec. 4.

Of course, there is no compelling reason to apply K^I or K^V factor entirely on stage B indices. Any other distribution of the K factor across stages would be consistent with models' assumptions. That is why, models (2.2) and (3.2) need to be supplemented by additional assumptions in order to arrive at a complete distribution of the overall index across stages. Short of these assumptions, the use of models (2.2) and (3.2) for benchmarking purposes is necessarily limited, for efficiency ranking would depend on how one decides to allocate the K factor.

Table 2: CRS and VRS efficiency indices for two stages input oriented models

	CRS			VRS mod (2.2)				VRS mod (2.7)			VRS mod (3.2)			
Units	Overall	Stage A	Stage B	Overall	Stage A	Stage B	K^I	Overall	Stage A	Stage B	Overall	Stage A	Stage B	K^V
1	0.6992	0.9926	0.7045	0.7366	0.9886	0.7434	1.0023	0.7366	0.9886	0.7451	0.7366	0.9886	0.7448	1.0004
2	0.6248	0.9985	0.6257	0.7109	1	0.7111	0.9997	0.7109	1	0.7109	0.7109	1	0.7192	0.9885
3	0.6900	0.6900	1	0.6903	0.6903	1	1	0.6903	0.6903	1	0.6903	0.6903	1	1
4	0.3042	0.7243	0.4200	0.3123	0.7172	0.4238	1.0275	0.3123	0.7172	0.4355	0.3123	0.7172	0.4255	1.0234
5	0.7670	0.8307	0.9233	1	1	1	1	1	1	1	1	1	1	1
6	0.3897	0.9606	0.4057	0.4904	0.9118	0.5323	1.0104	0.4904	0.9118	0.5379	0.4904	0.9118	0.5561	0.9672
7	0.2766	0.6706	0.4124	0.4479	0.7511	0.5938	1.0042	0.4479	0.7511	0.5963	0.4479	0.7511	0.6131	0.9726
8	0.2752	0.6630	0.4150	0.5292	0.7834	0.7216	0.9361	0.5292	0.7834	0.6756	0.5292	0.7834	0.8365	0.8076
9	0.2233	1	0.2233	0.2790	1	0.2756	1.1234	0.2790	1	0.2790	0.2790	1	0.2952	0.9451
10	0.4660	0.8615	0.5408	0.6286	0.8619	0.7272	1.0029	0.6286	0.8619	0.7294	0.6286	0.8619	0.7438	0.9805
11	0.1639	0.6468	0.2534	0.3329	0.7184	0.4580	1.118	0.3329	0.7184	0.4633	0.3329	0.7184	0.5011	0.9247
12	0.7596	1	0.7596	0.7827	0.9064	0.8618	1.0020	0.7827	0.9064	0.8635	0.7827	0.9064	0.8662	0.9969
13	0.2078	0.6720	0.3093	0.6133	0.8021	0.7634	1.0016	0.6133	0.8021	0.7646	0.6133	0.8021	0.8736	0.8752
14	0.2886	0.6699	0.4309	0.4055	0.7254	0.5550	1.0072	0.4055	0.7254	0.5590	0.4055	0.7254	0.5850	0.9556
15	0.6138	1	0.6138	0.8809	1	0.8795	1.0016	0.8809	1	0.8809	0.8809	1	0.8853	0.9950
16	0.3202	0.8856	0.3615	0.3846	0.9107	0.4169	1.0130	0.3846	0.9107	0.4223	0.3846	0.9107	0.4473	0.9441
17	0.3600	0.6276	0.5736	0.7242	0.7242	1	1	0.7242	0.7242	1	0.7242	0.7242	1	1
18	0.2588	0.7935	0.3262	0.3261	0.6589	0.4857	1.0190	0.3261	0.6589	0.4949	0.3261	0.6589	0.5504	0.8992
19	0.4112	1	0.4112	0.6763	1	0.6566	1.0300	0.6763	1	0.6763	0.6763	1	0.7644	0.8847
20	0.5465	0.9332	0.5857	0.9021	0.9021	1	1	0.9021	0.9021	1	0.9021	0.9021	1	1
21	0.2008	0.7321	0.2743	0.4437	0.8821	0.3376	1.4899	0.4437	0.8821	0.5030	0.4437	0.8821	0.2170	2.3180
22	0.5895	0.5895	1	1	1	1	1	1	1	1	1	1	1	1
23	0.4203	0.8426	0.4989	0.6610	0.9757	0.6202	1.0923	0.6610	0.9757	0.6774	0.6610	0.9757	0.5675	1.1938
24	0.1348	0.4287	0.3145	0.1899	0.4399	0.3489	1.2373	0.1899.9492	0.4399	0.4318	0.1899	0.4399	0.3616	1.1938

Tab. 3 contains in the second column the index of scale efficiency for the entire network computed as the ratio of overall CRS index to overall VRS index. Then Table 3 has two sets of conditional scale efficiency indices for stages A and B. The first set is computed according to eq. (5.1, for stage B, and eq. (5.3, for stage A, and is reported in the 3rd and 6th column of Tab. 3. In the case of this first path in changing returns to scale assumptions, scale efficiency indices can be decomposed, as shown in columns with the "Delta" headings. The second set of scale efficiency indices is computed, by following a different path in changing the assumptions on returns to scale, and is shown in 9th and last columns of Tab. 3. For these latter measures of scale efficiency, no "Delta" headed column is added because they cannot be decomposed multiplicatively.

Interpretation of the results is slightly different from conventional, one stage, measure of scale efficiency. For instance, the second column shows that unit 3 has stage B scale efficient ($SE_B = 1$), conditional on stage A operating under CRS. For stage A (SE_A), instead, with second stage operating under VRS (6th column), no unit is scale efficient. The conditional element affects the measure rather significantly, although not so much as to alter the ranking, as it can be seen by comparing scale efficiency for the same stage under different assumption about returns to scale on the other stage. For instance, stage A scale efficiency, conditional on stage B being under VRS (6th column), is different from the same measure with stage B under CRS (10th column). Although the two measure seem to follow a similar tendency. The same holds for stage B scale efficiency under the two alternative assumptions (3rd and last columns).

Scale efficiency measures from eq. (5.1) and (5.3), columns 3rd and 6th, can be decomposed multiplicatively according to eq. (5.2) and eq. (5.4), respectively. Decomposition for stage B scale efficiency (3rd column) is reported in columns 4th& 5th under the headings of "Delta" to signify that these indices are to be interpreted as a measure of the variation of stage efficiency. If the index is greater than one, there is a reduction in efficiency, while an increase in efficiency shows by an index lower than one. Take, for instance, Unit 1 and the case of returns to scale assumption being changed so as to move stage B from CRS to VRS, while keeping stage A under CRS, (3rd column). Unit 1 would, then, have stage B conditional scale efficiency at SE_B=0.9508 (i.e. 0.0492 scale inefficient) and this would cause stage A to remain at the same efficiency level as before the move ($Delta_A$=1 , 4th column), while stage B would experience an increase in efficiency ($Delta_B$=0.9598, 5th column).

Table 3: Network, conditional scale efficiency (SE) and changes in stages efficiency indices (Delta)

	Network	conditional on A=CRS			conditional on B=VRS			conditional on B=CRS	conditional on A=VRS
Units	SE	SE_B	Delta E_A	Delta E_B	SE_A	Delta E_A	Delta E_B	SE_A	SE_B
1	0.9492	0.9508	1	0.9508	0.9984	1.0040	0.9944	0.9952	0.9538
2	0.8789	0.8789	1	0.8789	0.9999	0.9985	1.0014	0.9994	0.8793
3	0.9996	1	1	1	0.9996	0.9996	1	0.9996	1
4	0.9740	0.9893	1	0.9893	0.9845	1.0099	0.9748	0.9830	0.9908
5	0.7670	0.9158	0.9918	0.9233	0.8375	0.8375	1	0.7670	1
6	0.7947	0.8035	1.0544	0.7621	0.9888	0.9992	0.9896	0.9878	0.8043
7	0.6175	0.6199	0.8917	0.6952	0.9961	1.0013	0.9948	0.9727	0.6349
8	0.5200	0.5351	0.9303	0.5752	0.9717	0.9098	1.0681	0.9504	0.5471
9	0.8004	0.8078	1	0.8078	0.9908	1	0.9908	0.9934	0.8057
10	0.7413	0.7439	1	0.7439	0.9965	0.9996	0.9969	0.9996	0.7416
11	0.4923	0.4998	0.9032	0.5534	0.9852	0.9968	0.9884	0.9792	0.5029
12	0.9705	0.9735	1.1045	0.8814	0.9969	0.9988	0.9980	0.9973	0.9731
13	0.3388	0.3390	0.8368	0.4051	0.9996	1.0012	0.9984	0.9454	0.3584
14	0.7117	0.7173	0.9245	0.7759	0.9923	0.9989	0.9934	0.9984	0.7130
15	0.6968	0.6979	1	0.6979	0.9985	1	0.9985	0.9980	0.6982
16	0.8326	0.8449	0.9762	0.8655	0.9853	0.9962	0.9891	0.9802	0.8493
17	0.4971	0.4977	0.8677	0.5736	0.9988	0.9988	1	0.9574	0.5193
18	0.7936	0.8106	1.1958	0.6779	0.9791	1.0071	0.9722	0.9820	0.8083
19	0.6080	0.6345	1	0.6345	0.9582	1	0.9582	0.9152	0.6643
20	0.6058	0.6545	1.1176	0.5857	0.9256	0.9256	1	0.8780	0.6900
21	0.4526	0.7375	0.9755	0.7561	0.6136	0.8508	0.7212	0.4780	0.9467
22	0.5895	1	1	1	0.5895	0.5895	1	0.5895	1
23	0.6359	0.8063	1	0.8063	0.7888	0.8635	0.9134	0.7269	0.8749
24	0.7098	0.9565	1	0.9565	0.7420	0.9745	0.7614	0.7301	0.9721

7 Extensions

The three basic models (2.2), (2.7) and (3.2) we have been dealing with so far are characterised by: 1) containing only two stages; 2) each stage is made of one and only one process, hence there is equivalence between stages and processes; 3) stages (processes) are organised in series, while parallel systems are not accounted for. In this section we show how to extend the basic idea of multiplicative decomposition of VRS systems to these more general networks.

1) Let us start from the most straightforward extension, that of more than two stages, while keeping the structure in series with one process per stage. Convexity constraints (2.5); (2.6), and (3.1), in a three stages case become, respectively:

$$\begin{bmatrix} -\mathbf{e} & 0 & 0 \\ 0 & -\mathbf{e} & 0 \\ 0 & 0 & -\mathbf{e} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_A \\ \boldsymbol{\lambda}_B \\ \boldsymbol{\lambda}_C \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \tag{7.1a}$$

$$\begin{bmatrix} -\mathbf{e} & \mathbf{e} & 0 \\ 0 & -\mathbf{e} & \mathbf{e} \\ 0 & 0 & -\mathbf{e} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_A \\ \boldsymbol{\lambda}_B \\ \boldsymbol{\lambda}_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \tag{7.1b}$$

$$\begin{bmatrix} -\mathbf{e} & 0 & 0 \\ \mathbf{e} & -\mathbf{e} & 0 \\ 0 & \mathbf{e} & -\mathbf{e} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_A \\ \boldsymbol{\lambda}_B \\ \boldsymbol{\lambda}_C \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \tag{7.1c}$$

The corresponding models to (2.2), (2.7) and (3.2) for the three stages case are, respectively:

$$E^{I} = \left\{ \max \frac{\mathbf{y}^k\mathbf{u} - h_A^I - h_B^I - h_C^I}{\mathbf{x}^k\mathbf{v}} \;\middle|\; \begin{array}{l} \frac{\mathbf{z}_A^j\mathbf{w}_A - h_A^I}{\mathbf{x}^j\mathbf{v}} \leq 1 \\ \frac{\mathbf{z}_B^j\mathbf{w}_B - h_B^I}{\mathbf{z}_A^j\mathbf{w}_A} \leq 1 \\ \frac{\mathbf{y}^j\mathbf{u} - h_C^I}{\mathbf{z}_B^j\mathbf{w}_B} \leq 1 \end{array} ; \begin{array}{c} \mathbf{v}, \mathbf{w}_A, \mathbf{w}_B, \mathbf{u} \geq \mathbf{0} \\ h_A^I, h_B^I, h_C^I = \text{free} \\ j = 1, ..., n \end{array} \right\} \tag{7.2a}$$

$$E^{II} = \left\{ \max \frac{\mathbf{y}^k\mathbf{u} - h_C^{II}}{\mathbf{x}^k\mathbf{v}} \;\middle|\; \begin{array}{l} \frac{\mathbf{z}_A^j\mathbf{w}_A - h_A^{II}}{\mathbf{x}^j\mathbf{v}} \leq 1 \\ \frac{\mathbf{z}_B^j\mathbf{w}_B - h_B^{II}}{\mathbf{z}_A^j\mathbf{w}_A - h_A^{II}} \leq 1 \\ \frac{\mathbf{y}^j\mathbf{u} - h_C^{II}}{\mathbf{z}_B^j\mathbf{w}_B - h_B^{II}} \leq 1 \end{array} ; \begin{array}{c} \mathbf{v}, \mathbf{w}_A, \mathbf{w}_B, \mathbf{u} \geq \mathbf{0} \\ h_A^{II}, h_B^{II}, h_C^{II} = \text{free} \\ j = 1, ..., n \end{array} \right\} \tag{7.2b}$$

$$E^{V} = \left\{ \max \frac{\mathbf{y}^k\mathbf{u} - h_A^V}{\mathbf{x}^k\mathbf{v}} \;\middle|\; \begin{array}{l} \frac{\mathbf{z}_A^j\mathbf{w}_A - h_A^V + -h_B^V}{\mathbf{x}^j\mathbf{v}} \leq 1 \\ \frac{\mathbf{z}_B^j\mathbf{w}_B - h_C^V}{\mathbf{z}_A^j\mathbf{w}_A + -h_B^V} \leq 1 \\ \frac{\mathbf{y}^j\mathbf{u}}{\mathbf{z}_B^j\mathbf{w}_B + h_C^V} \leq 1 \end{array} ; \begin{array}{c} \mathbf{v}, \mathbf{w}_A, \mathbf{w}_B, \mathbf{u} \geq \mathbf{0} \\ h_A^V, h_B^V, h_C^V = \text{free} \\ j = 1, ..., n \end{array} \right\} \tag{7.2c}$$

Where, as compared to the two stages models, an intermediate stage(process) has ben added to produce output z_B by use of intermediate output z_A from stage A. To save on notation, the same symbols as in models (2.2), (2.7) and (3.2) have been used for scale variables, although the network contains three stages. It is of some interest to observe how efficiency indices from the three basic models (2.2, 2.7 and 3.2) are linked in a three stages case. Scale variables equivalence for two stages models are now extended to include the third stage. Therefore, equivalence (3.3) will contain an additional row given by: $h_C^I = h_C^{II} - h_B^{II}$, and equivalence (3.4) will also have: $h_C^V = h_C^{II} - h_B^{II}$. For ease of further reference we merge these equivalences in Table 4.

Table 4: Scale variables equivalences for three stages models

	model (7.2a)	model (7.2c)
Stage A	$h_A^I = h_A^{II}$	$h_A^V = h_B^{II}$
Stage B	$h_B^I = h_B^{II} - h_A^{II}$	$h_B^V = h_B^{II} - h_A^{II}$
Stage C	$h_C^I = h_C^{II} - h_B^{II}$	$h_C^V = h_C^{II} - h_B^{II}$

By means of these equivalences, efficiency indices can be written using the same scale variable. Taking scale variable from model (7.2b) as reference, we have:

Table 5 : Models (7.2a) and (7.2c) efficiency indices in terms of model (7.2b)'s scale variables

Model (7.2b) indices	Model (7.2a) indices	Model (7.2c) indices
$E^{II} = \frac{\mathbf{y}^k\mathbf{u}-h_C^{II}}{\mathbf{x}^k\mathbf{v}}$	$E^{I} = \frac{\mathbf{y}^k\mathbf{u}-h_C^{II}}{\mathbf{x}^k\mathbf{v}}$	$E^{V} = \frac{\mathbf{y}^k\mathbf{u}-h_C^{II}}{\mathbf{x}^k\mathbf{v}}$
$E_A^{II} = \frac{\mathbf{z}_A^k\mathbf{w}_A-h_A^{II}}{\mathbf{x}^k\mathbf{v}}$	$E_A^{I} = \frac{\mathbf{z}_A^k\mathbf{w}_A-h_A^{II}}{\mathbf{x}^k\mathbf{v}}$	$E_A^{V} = \frac{\mathbf{z}_A^k\mathbf{w}_A-h_A^{II}}{\mathbf{x}^k\mathbf{v}}$
$E_B^{II} = \frac{\mathbf{z}_B^k\mathbf{w}_B-h_B^{II}}{\mathbf{z}_A^k\mathbf{w}_A-h_A^{II}}$	$E_B^{I} = \frac{\mathbf{z}_B^k\mathbf{w}_B-h_B^{II}+h_A^{II}}{\mathbf{z}_A^k\mathbf{w}_A}$	$E_B^{V} = \frac{\mathbf{z}_B^k\mathbf{w}_B-h_B^{II}}{\mathbf{z}_A^k\mathbf{w}_A-h_A^{II}}$
$E_B^{II} = \frac{\mathbf{y}^k\mathbf{u}-h_C^{II}}{\mathbf{z}_B^k\mathbf{w}_B-h_B^{II}}$	$E_C^{I} = \frac{\mathbf{y}^k\mathbf{u}-h_C^{II}+h_B^{II}}{\mathbf{z}_B^k\mathbf{w}_B}$	$E_C^{V} = \frac{\mathbf{y}^k\mathbf{u}-h_C^{II}}{\mathbf{z}_B^k\mathbf{w}_B-h_B^{II}}$

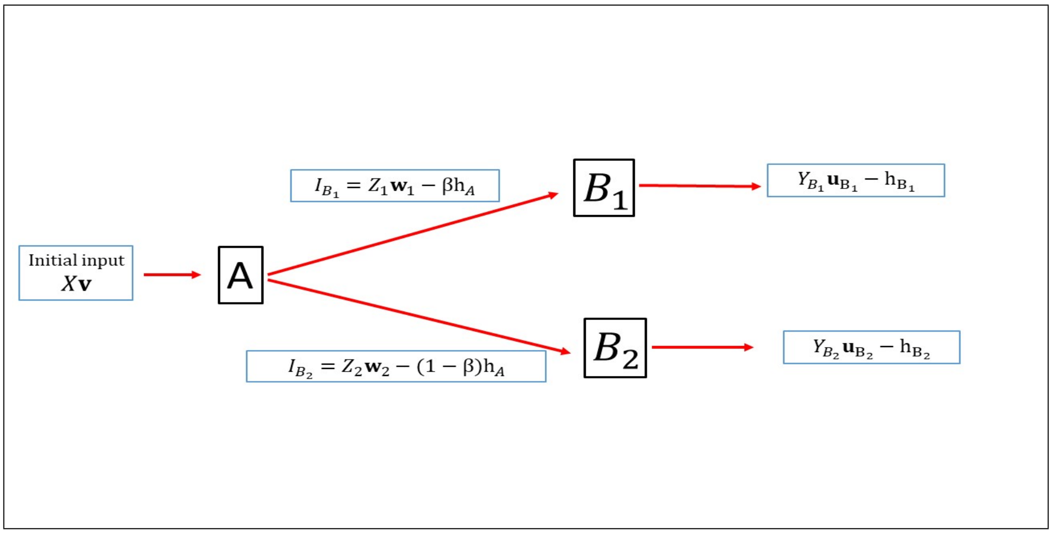
Showing that as more stages are added to the basic models, only first stage and overall indices remain the same across models. The remaining stages have different efficiency indices, giving an even greater role to the choice of models for benchmarking purposes.

2) A different type of extension is that of increasing the number of processes within a stage, hence breaking the so far held correspondence between stages and processes. It corresponds to a move from the basic series system, so far adopted, towards a parallel systems. As far as the series system continues to constitute the backbone of the overall system, that is the organization in series holds at the stages level, although no longer holds at the process level, the basic requirements for multiplicative decomposition of (stage) indices stay the same. That is, the multiplier linking constraint (2.4) has to hold between stages, though not necessary between processes. We show this point with reference to an example contained in Kao (2014) which is also going to be addressed, with some modifications, in the final part of this section in discussing an even more general extension. In that example the case is made of a stage A which by means of external input x produces (with a single process) two intermediate outputs, z_1 and z_2. These are then inputs to two distinct processes (B_1 and B_2) in stage B leading to final products y_{B1} and y_{B2}. Modification to model (2.7) as required by the network structure gives the following fractional model:

$$E = \left\{ \max \frac{\mathbf{y}_{B_1}^k\mathbf{u}_{B_1} - h_{B_1} + \mathbf{y}_{B_2}^k\mathbf{u}_{B_2} - h_{B_2}}{\mathbf{x}^k\mathbf{v}} \;\middle|\; \begin{array}{l} \frac{\mathbf{z}_1^j\mathbf{w}_1+\mathbf{z}_2^j\mathbf{w}_2-h_A}{\mathbf{x}^j\mathbf{v}} \leq 1 \\ \frac{\mathbf{y}_{B_1}^j\mathbf{u}_{B_1}-h_{B_1}}{\mathbf{z}_1^j\mathbf{w}_1+\beta h_A} \leq 1 \\ \frac{\mathbf{y}_{B_2}^j\mathbf{u}_{B_2}-h_{B_2}}{\mathbf{z}_2^j\mathbf{w}_2+(1-\beta)h_A} \leq 1 \end{array} \;;\; \begin{array}{c} \mathbf{v},\mathbf{w}_1,\mathbf{w}_2,\mathbf{u}_{B_1},\mathbf{u}_{B_2} \geq \mathbf{0} \\ h_A, h_{B_1}, h_{B_2} = \text{free} \\ j = 1,...,n \end{array} \right\}$$

A chart flow of this structure is in Fig 6. The interesting feature is that because we are dealing with a VRS model, not only the two outputs from stage A are to be allocated to processes in stage B, and here it is assumed that the whole of each intermediate output goes into a single process in stage B (other more general allocation could be considered, for instance proportional), but also the scale variable from first stage has to be allocated. We have assumed that stage A scale variable (h_A) is proportionally ($0 \leq \beta \leq 1$) allocated to processes in stage B. The proportion does not matter for multiplicative decomposition, what matters is that the whole of the scale variable has to be input to stage B, otherwise the multiplier linking constraint (2.4) would not hold. Indeed, use of model (2.2) or model (3.2) instead of model (2.7) easily shows that multiplicative decomposition would not hold.

Fig. 6: Network with two processes in second stage



Following Kao (2009a, 2014) efficiency index for stage B is the weighted average of processes B_1 and B_2 indices, with the weights (ω) given by the share of resources used by each process. That is by:

$$\begin{aligned} E_B &= \omega E_{B_1} + (1-\omega)\, E_{B_2} \\ &= \frac{\mathbf{z}_1^k\mathbf{w}_1 - \beta h_A}{\mathbf{z}_1^k\mathbf{w}_1 + \mathbf{z}_2^k\mathbf{w}_2 - h_A} E_{B_1} + \frac{\mathbf{z}_2^k\mathbf{w}_2 - (1-\beta)\, h_A}{\mathbf{z}_1^k\mathbf{w}_1 + \mathbf{z}_2^k\mathbf{w}_2 - h_A} E_{B_2} \\ &= \frac{\mathbf{y}_{B_1}^k\mathbf{u}_{B_1} - h_{B_1} + \mathbf{y}_{B_2}^k\mathbf{u}_{B_2} - h_{B_2}}{\mathbf{z}_1^k\mathbf{w}_1 + \mathbf{z}_2^k\mathbf{w}_2 - h_A} \end{aligned}$$

Hence multiplicative decomposition follows from:

$$\begin{aligned} E &= E_A * E_B \\ &= \frac{\mathbf{z}_1^k\mathbf{w}_1 + \mathbf{z}_2^k\mathbf{w}_2 - h_A}{\mathbf{x}^k\mathbf{v}} * \frac{\mathbf{y}_{B_1}^k\mathbf{u}_{B_1} - h_{B_1} + \mathbf{y}_{B_2}^k\mathbf{u}_{B_2} - h_{B_2}}{\mathbf{z}_1^k\mathbf{w}_1 + \mathbf{z}_2^k\mathbf{w}_2 - h_A} \\ & \frac{\mathbf{y}_{B_1}^k\mathbf{u}_{B_1} - h_{B_1} + \mathbf{y}_{B_2}^k\mathbf{u}_{B_2} - h_{B_2}}{\mathbf{x}^k\mathbf{v}} \end{aligned}$$

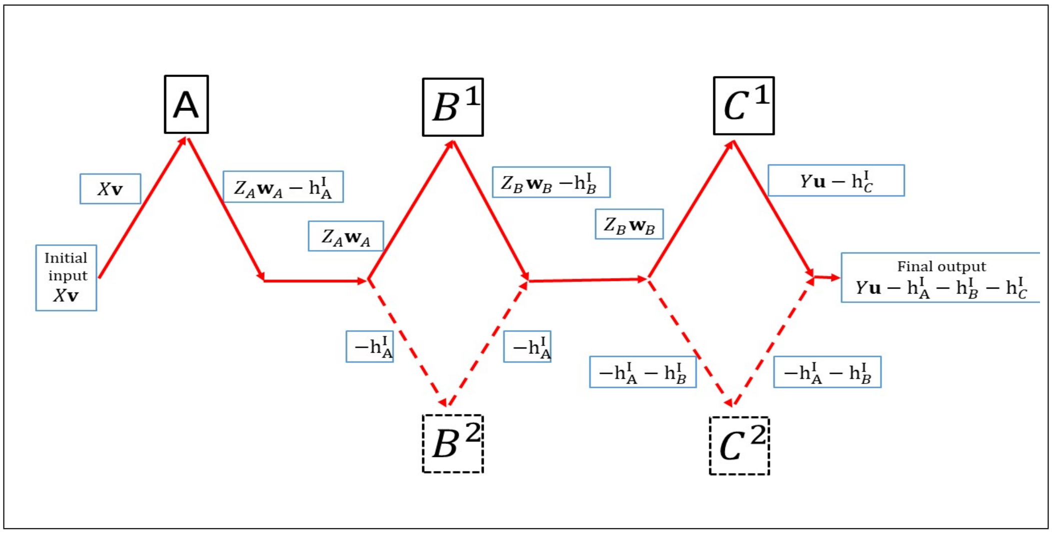
I short, extension of VRS mutiplicative decomposable model (2.7) to the case of parallel systems is also straightforward matter as long as the backbone of the whole system remains that of a series. The only change is that stages instead of processes are linked in series.

3) A more general type of extension concerns the case of process specific inputs and outputs. That is, when one or more processes, in addition to intermediate outputs produced by upstream stages (processes), also use as inputs resources coming from outside the network. Equivalently, the same problem is posed by stages (processes) whose outputs are, at least partially, destined to outside the network, instead of being fully absorbed by downstream stages (processes). Following Kao (2014), we shall refer to these as exogenous inputs and outputs, respectively. For a useful graphical representation of a general network system (one made up of parallel and series structures) we refer to Kao (2014, p. 118.). Quite obviously, the case is a hard one for multiplicative decomposition of efficiency indices because due to exogenous inputs and outputs there is no way the multiplier linking constraint (2.4) can hold, not even at the stage level.

One possible way out is that provided in Kao (2009b, 2014) by means of the "dummy" processes. A general network system can be transformed into a series of parallel systems by using dummy processes whose role is to carry forward inputs to be used in downstream stages and outputs produced by upstream stages. A dummy process has the same inputs and outputs (hence is efficient by definition) and its use is that of helping to represent a general network system as a series of sub systems. These subsystems may contain both actual and dummy processes. Kao (2009b, 2014) has already shown that by means of dummy processes multiplicative decomposition of stage indices can be obtained in general network under the CRS assumption. We shall now provide the extension to the case of VRS assumption and by the same token to the wider set of non constant returns to scale.

The extension relies on two rather self evident observations. The first is that all basic network VRS models, such as two stages models (2.2), (3.2) and (2.7), or three stage models (7.2a), (7.2c) and (7.2b), and in general models with any number of stages, can be reconciled with multiplicative decomposition of efficiency indices by means of dummy processes on scale variables. Let us take for instance the case of model (7.2a) which is known not to be multiplicatively decomposable. By means of dummy processes we modify the network so as to make the modified stages consistent with the multiplier linking constraint (2.4), hence with multiplicative decomposition. Fig. 7 shows the modified network, where processes with continuous borders are actual processes and those with dotted borders are dummy processes. Stages B and C are modified stages and are made up of actual and dummy processes, while stage A only contains one actual process.

Fig. 7: A three stages network modified for scale variables



It is easily seen that efficiency indices for the overall network and for the modified stages are:

$$
\begin{aligned}
E^I &= \frac{\mathbf{y}^k\mathbf{u} - h^I_A - h^I_B - h^I_C}{\mathbf{x}^k\mathbf{v}} \\
E^I_A &= \frac{\mathbf{z}^k_A\mathbf{w}_A - h^I_A}{\mathbf{x}^k\mathbf{v}} \\
E^I_B &= \frac{\mathbf{z}^k_B\mathbf{w}_B - h^I_A - h^I_B}{\mathbf{z}^k_A\mathbf{w}_A - h^I_A} \\
E^I_C &= \frac{y^k\mathbf{u} - h^I_A - h^I_B - h^I_C}{\mathbf{z}^k_B\mathbf{w}_B - h^I_A - h^I_B}
\end{aligned}
\tag{7.3}
$$

And multiplicative decomposition holds: $E^I = E^I_A * E^I_B * E^I_C$. For brevity we omit to show the same results for model (7.2c). The key point is that dummy processes are organised in such a way as to make the multiplier linking constraint (2.4) hold for the modified stages. And because the lack of linking constraint is what prevents the model to be multipicatively decomposable, the role of dummy processes is right that of imposing the multiplier linking constraint on the modified stages.

A further interesting aspect of this result is that efficiency indices for the modified stages turn out to be the same as those one could obtain by directly applying model (7.2b) which is known to be multiplicatively decomposable and does not need to go through the dummy processes procedure. This point is easily shown by replacing scale variables in (7.3) with scale variables from model (7.2b) as in Table 5. We have that:

$$
\begin{aligned}
E^I &= \frac{\mathbf{y}^k\mathbf{u} - h_A^I - h_B^I - h_C^I}{\mathbf{x}^k\mathbf{v}} = \frac{\mathbf{y}^k\mathbf{u} - h_C^{II}}{\mathbf{x}^k\mathbf{v}} = E^{II} \\
E_A^I &= \frac{\mathbf{z}_A^k\mathbf{w}_A - h_A^I}{\mathbf{x}^k\mathbf{v}} = \frac{\mathbf{z}_A^k\mathbf{w}_A - h_A^{II}}{\mathbf{x}^k\mathbf{v}} = E_A^{II} \\
E_B^I &= \frac{\mathbf{z}_B^k\mathbf{w}_B - h_A^I - h_B^I}{\mathbf{z}_A^k\mathbf{w}_A - h_A^I} = \frac{\mathbf{z}_B^k\mathbf{w}_B - h_B^{II}}{\mathbf{z}_A^k\mathbf{w}_A - h_A^{II}} = E_B^{II} \\
E_C^I &= \frac{y^k\mathbf{u} - h_A^I - h_B^I - h_C^I}{\mathbf{z}_B^k\mathbf{w}_B - h_A^I - h_B^I} = \frac{y^k\mathbf{u} - h_C^{II}}{\mathbf{z}_B^k\mathbf{w}_B - h_B^{II}} = E_C^{II}
\end{aligned}
\tag{7.4}
$$

That is model (7.2b) indices are the same as those in (7.3) obtained from model (7.2a). In the light of the previous discussion, the result is not surprising because model (7.2b) as it stands satisfies the multiplier linking constraint, while model (7.2a), which in its original form does not satisfy the constraint, is made to comply by means of the dummy processes. Of course, the same result also holds for model (7.2c). In other words, the result says that when dummy processes are used to modify stages' organisation then the distinction between models with and without multiplicatively decomposable indices vanishes because they all provide the same multiplicatively decomposable indices for the modified stages.

Now the question arises as to whether this result, which clearly holds for network systems organised in series in their original stages, still holds for more general systems which, although not organised in series in their original stages, can be transformed into a series of modified stages by means of dummy processes. The answer must be yes, because the rule guiding the allocation of dummy processes in a network (the multiplier linking constraint) holds true irrespective of the original structure of the network and of the presence of scale variables. We therefore expect the above result of equivalence among alternative VRS models, when applied to network modified by dummy processes, to hold true even for more complex network.

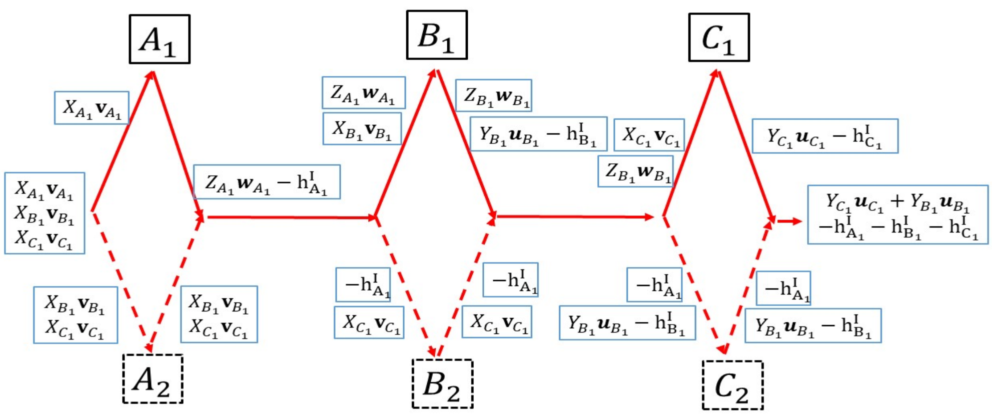
Take for instance the example contained in Tone and Tsutsui (2009) as modified in Kao (2014) about the electricity service where electric power (z_{A_1}) is generated in stage A from initial inputs x_{A_1} and is then sent to the Distribution Division (stage B) which uses some additional input (skilled workers, x_{B_1}), sells some of the electricity (y_{B_1}) to large consumers and sends he rest (z_{B_1}) to the Transmission Division which uses some external input (skilled workers z_{C_1}) to provide electricity (y_{C_1}) to small consumers. The presence of external inputs and outputs makes this network a rather good example for a general multistage system because some parts of it are organised in parallel and some in series.

As for the VRS model let us, for instance, use model (7.2a) which when applied to the case at hand gives the following:

$$
\begin{aligned}
&Max\ \mathbf{y}^k_{B_1}\mathbf{u}_{B_1} + \mathbf{y}^k_{C_1}\mathbf{u}_{C_1} - h^I_{A_1} - h^I_{B_1} - h^I_{C_1} \\
&\mathbf{x}^k_{A_1}\mathbf{v}_{A_1} + \mathbf{x}^k_{B_1}\mathbf{v}_{B_1} + \mathbf{x}^k_{C_1}\mathbf{v}_{C_1} = 1 \\
&Z_{A_1}\mathbf{w}_{A_1} - X_{A_1}\mathbf{v}_{A_1} - \mathbf{h}^I_{A_1} + \mathbf{s}_{A_1} = 0 \\
&Y_{B_1}\mathbf{u}_{B_1} + Z_{B_1}\mathbf{w}_{B_1} - X_{B_1}\mathbf{v}_{B_1} - Z_{A_1}\mathbf{w}_{A_1} - \mathbf{h}^I_{B_1} + \mathbf{s}_{B_1} = 0 \\
&Y_{C_1}\mathbf{u}_{C_1} - Z_{B_1}\mathbf{w}_{B_1} - X_{C_1}\mathbf{v}_{C_1} - \mathbf{h}^I_{C_1} + \mathbf{s}_{C_1} = 0
\end{aligned}
\tag{7.5}
$$

As it stands we know the model is not multiplicatively decomposable. However by means of dummy processes we can transform the network as in Fig. 8. The network is now made of a series of modified stages, it conforms to the multiplier linking constraint, as a simple visual inspection confirms, and therefore its efficiency indices are multiplicatively decomposable. Of course, as observed by Kao (2014), decomposable indices refer to the modified stages, not to the original stages(processes), therefore they contain a bias due to the presence of dummy processes whose efficiency is one by definition.

Fig. 8: Model (7.5) modified system

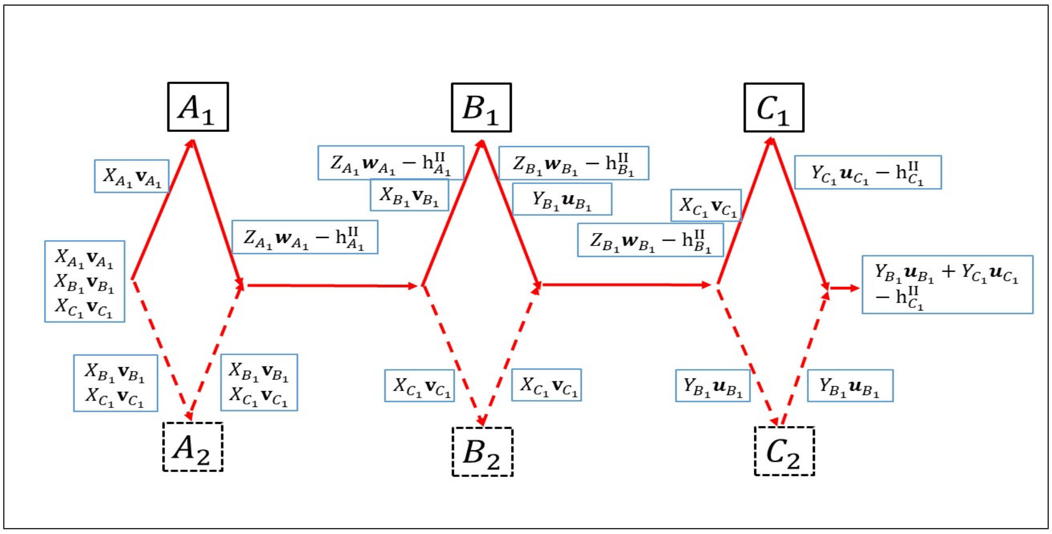


As a last point it remains to show that indices from model (7.5) in addition of being multiplicatively decomposable are the same as the indices obtained from any other VRS model For this purpose let us take the case of model (7.2b) which when applied to the Kao (2014) data is:

$$
\begin{aligned}
&Max\ \mathbf{y}^k_{B_1}\mathbf{u}_{B_1} + \mathbf{y}^k_{C_1}\mathbf{u}_{C_1} - h^{II}_{C_1} \\
&\mathbf{x}^k_{A_1}\mathbf{v}_{A_1} + \mathbf{x}^k_{B_1}\mathbf{v}_{B_1} + \mathbf{x}^k_{C_1}\mathbf{v}_{C_1} = 1 \\
&Z_{A_1}\mathbf{w}_{A_1} - X_{A_1}\mathbf{v}_{A_1} - \mathbf{h}^{II}_{A_1} + \mathbf{s}_{A_1} = 0 \\
&Y_{B_1}\mathbf{u}_{B_1} + Z_{B_1}\mathbf{w}_{B_1} - X_{B_1}\mathbf{v}_{B_1} - Z_{A_1}\mathbf{w}_{A_1} + \mathbf{h}^{II}_{A_1} - \mathbf{h}^{II}_{B_1} + \mathbf{s}_{B_1} = 0 \\
&Y_{C_1}\mathbf{u}_{C_1} - Z_{B_1}\mathbf{w}_{B_1} - X_{C_1}\mathbf{v}_{C_1} + \mathbf{h}^{II}_{B_1} - \mathbf{h}^{II}_{C_1} + \mathbf{s}_{C_1} = 0
\end{aligned}
\tag{7.6}
$$

Because of the presence of exogenous inputs and outputs the model would not yield multiplicatively decomposable indices. To this end, use of dummy processes leads to a modified system whose representation is as in Fig. 9.

Fig. 9: Model (7.6) modified system



By computing the modified stage indices for model (7.5) and model (7.6) and using equivalences from Table 5 to express them in the same scale variables (those of model (7.6) provides the following equivalences:

$$
\begin{aligned}
E_A^I &= \frac{\mathbf{z}^k_{A_1}\mathbf{w}_{A_1} + \mathbf{x}^k_{B_1}\mathbf{v}_{B_1} + \mathbf{x}^k_{C_1}\mathbf{v}_{C_1} - h^I_{A_1}}{\mathbf{x}^k_{A_1}\mathbf{v}_{A_1} + \mathbf{x}^k_{B_1}\mathbf{v}_{B_1} + \mathbf{x}^k_{C_1}\mathbf{v}_{C_1}} = \frac{\mathbf{z}^k_{A_1}\mathbf{w}_{A_1} + \mathbf{x}^k_{B_1}\mathbf{v}_{B_1} + \mathbf{x}^k_{C_1}\mathbf{v}_{C_1} - h^{II}_{A}}{\mathbf{x}^k_{A_1}\mathbf{v}_{A_1} + \mathbf{x}^k_{B_1}\mathbf{v}_{B_1} + \mathbf{x}^k_{C_1}\mathbf{v}_{C_1}} = E_A^{II} \\
E_B^I &= \frac{y^k{}_{B_1}\mathbf{u}_{B_1} + z^k{}_{B_1}\mathbf{w}_{B_1} + \mathbf{x}^k_{C_1}\mathbf{v}_{C_1} - h^I_{A_1} - h^I_{B_1}}{\mathbf{z}^k_{A_1}\mathbf{w}_{A_1} - h_{A_1} + \mathbf{x}^k_{B_1}\mathbf{v}_{B_1} + \mathbf{x}^k_{C_1}\mathbf{v}_{C_1}} = \frac{y^k{}_{B_1}\mathbf{u}_{B_1} + z^k{}_{B_1}\mathbf{w}_{B_1} + \mathbf{x}^k_{C_1}\mathbf{v}_{C_1} - h^{II}_{B}}{\mathbf{z}^k_{A_1}\mathbf{w}_{A_1} - h^{II}_{A} + \mathbf{x}^k_{B_1}\mathbf{v}_{B_1} + \mathbf{x}^k_{C_1}\mathbf{v}_{C_1}} = E_B^{II} \\
E_C^I &= \frac{\mathbf{y}^k_{C_1}\mathbf{u}_{C_1} + \mathbf{y}^k_{B_1}\mathbf{u}_{B_1} - h^I_{A_1} - h^I_{B_1} - h^I_{C_1}}{y^k{}_{B_1}\mathbf{u}_{B_1} + z^k{}_{B_1}\mathbf{w}_{B_1} + \mathbf{x}^k_{C_1}\mathbf{v}_{C_1} - h^I_{A_1} - h^I_{B_1}} = \frac{\mathbf{y}^k_{C_1}\mathbf{u}_{C_1} + \mathbf{y}^k_{B_1}\mathbf{u}_{B_1} - h^{II}_{C}}{y^k{}_{B_1}\mathbf{u}_{B_1} + z^k{}_{B_1}\mathbf{w}_{B_1} + \mathbf{x}^k_{C_1}\mathbf{v}_{C_1} - h^{II}_{B}} = E_C^{II} \\
E^I &= \frac{\mathbf{y}^k_{C_1}\mathbf{u}_{C_1} + \mathbf{y}^k_{B_1}\mathbf{u}_{B_1} - h^I_{A_1} - h^I_{B_1} - h^I_{C_1}}{\mathbf{x}^k_{A_1}\mathbf{v}_{A_1} + \mathbf{x}^k_{B_1}\mathbf{v}_{B_1} + \mathbf{x}^k_{C_1}\mathbf{v}_{C_1}} = E_A^I * E_B^I * E_C^I = E_A^{II} * E_B^{II} * E_C^{II} = \frac{\mathbf{y}^k_{C_1}\mathbf{u}_{C_1} + \mathbf{y}^k_{B_1}\mathbf{u}_{B_1} - h^{II}_{C}}{\mathbf{x}^k_{A_1}\mathbf{v}_{A_1} + \mathbf{x}^k_{B_1}\mathbf{v}_{B_1} + \mathbf{x}^k_{C_1}\mathbf{v}_{C_1}} = E^{II}
\end{aligned}
$$

That is indices computed by model (7.5) and model (7.6) are the same and they are multiplicatively decomposable. Therefore, the concluding observation to this section is that multiplicative decomposable VRS models can be extended to more general network systems than the pure system organised in series, provided that allocation of scale variables by means of dummy processes obeys the multiplicative linking constraint (2.4). In these more general networks, the difference among alternative VRS models vanishes, that is because dummy processes force multiplicative decomposition even on models that do not possess it in their original, non modified version. The cost of this generalisation, common to VRS as well as to CRS models, is that efficiency indices are referred to modified stages, that is to stages that in addition to actual processes also include dummy processes. In this way introducing a bias, which can be isolated, as done in Kao (2014), but at the cost of losing the multiplicative decomposition.

Table 6 contains the results of applying model (7.6) under CRS and VRS specification to the data in the example contained in Kao (2014). The addition with respect to the results shown in Kao (2014) concerns the VRS indices. We observe that, as expected, indices are multiplicatively decomposable. We also observe that VRS indices for modified stages are at cases lower than the corresponding CRS indices. This does not come as a surprise in light of the discussion in sec.5. However, what could be the meaning of scale efficiency indices in the presence of dummy processes is a matter that deserves further investigation.

Table 6: Kao (2014)'s example under CRS and VRS assumptions

	Network system		Stage A		Stage B		Stage C	
Units	CRS	VRS	CRS	VRS	CRS	VRS	CRS	VRS
A	0.3825	0.3888	0.6000	1	0.6468	1	0.9856	0.3887
B	0.3865	0.6301	1	1	1	1	0.3865	0.6301
C	1	1	1	1	1	1	1	1
D	1	1	1	1	1	1	1	1
E	0.3297	1	1	1	1	1	0.3297	1
F	0.4416	0.5931	1	1	1	1	0.4416	0.5931
G	0.6263	0.8633	1	1	1	1	0.6263	0.8633
H	0.5200	1	0.7462	1	0.6968	1	1	1
I	0.9959	1	1	1	1	1	0.9959	1
J	0.5024	0.8247	1	1	1	1	0.5024	0.8247

8 Concluding remarks

Multiplicative decomposition of stages indices is shown to be consistent with VRS network series technologies. This result comes to depend on the model satisfying a constraint, named the multiplier linking constraint. Not all convex network technologies satisfy such a constraint. It is shown how VRS network multiplier models can be amended to make them consistent with the multiplier linking constraint, hence with multiplicative decomposition of stage indices.

The multiplier linking constraint does not have a counterpart on the envelopment model, in the sense that alternative formulations of the multiplier linking constraint are consistent with the same envelopment model. This leads to a break in the primal-dual correspondence, which holds, instead, for one stage DEA models. Thus opening the way to two separate approaches in modelling network structures: one from the envelopment side and the other from the multiplier (ratio) side.

It is shown that multiplicative decomposition can be associated with transfer pricing rules, within the network, based on marginal cost pricing. Therefore, in choosing a multiplicatively decomposable model one should have some evidence in support of such a pricing strategy. Non multiplicatively decomposable models can be associated to full cost pricing within the network. Adoption of these models requires additional assumptions in order to arrive at a complete distribution, across stages, of the overall efficiency index. That is because these models, by themselves, do not provide such a distribution.

Associating each network convex technology to a different transfer pricing system has made it possible to show that VRS network models differ in their distributive criterion used to allocate the same overall index of efficiency across stages. Thus making clear that stages indices respond not only to efficiency but also to distributive criteria. This has provided the key element to propose a solution to the vexed question of how to measure scale efficiency in network system. The proposed measure operates through a step-by-step procedure. Returns to scale assumptions are changed, from CRS to VRS, one stage at a time and the "conditional" scale efficiency computed for the stage under assessment. The meaning of such a "conditional" index is, therefore, slightly different from the conventional, one stage, measure. This step-by-step type of assessment introduces, in addition, a sort of "path dependency" issue because, even in the simplest two stages network, there are at least two different paths one could follow. Therefore some indeterminacy is inherent to this way in computing scale efficiency.

Multiplicative decomposable VRS models can be extended to more general network systems, containing both parallel and in series structures, provided that allocation of scale variables by means of dummy processes obeys the multiplicative linking constraint. In these more general networks, the difference among VRS models vanishes, because dummy processes force multiplicative decomposition even on models that do not possess it in their original form. The cost of this generalisation is that efficiency indices are referred to modified stages, that is to stages that include dummy processes.

Seen in perspective, these results contribute to show how organisational aspects of network processes, such as transfer pricing systems, turn out to be relevant in modelling network technologies once these are approached from the multiplier (ratio) side. This seems to be a promising line of research. One which could provide more general results on how to model network organisational arrangements by means of new definitions of convex technologies.

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