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Theory of valuations on the space of quasi-concave functions

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Introduction

Valuation theory arose at the beginning of XX century in the context of the resolution of the Third Hilbert problem. It was strictly related, initially, to the idea of dissections and cutting: a valuation should have been a geometric set function, invariant w.r.t. dissections and *remodellying* of particular subsets of $\mathbb{R}^n$.

Let us recall the third of the problems posed by Hilbert, during the international conference of mathematicians in 1900 (see for instance [32]):

given two tetrahedra in $\mathbb{R}^3$ with equal base areas and heights, and hence volumes, is it possible to cut the first one in finitely many tetrahedral pieces which can be reassembled to yield the second one?

Max Dehn, one of Hilbert's students, proved that the answer is no, producing a counterexample and introducing a first prototype of valuation, Dehn's invariant, that he used to yield his counterexample. We refer to [27] and [28] for the problem details, as for the meaning of cutting and reassembling, and the proof of Dehn's answer.

Some years later, Hadwiger studied valuation theory rigorously, as functions defined on the set $\mathcal{K}^n$ of all convex and compact subsets of $\mathbb{R}^n$, that we will call **convex bodies**; see Chapters 1 and 2 of the present thesis. The definition of valuation on $\mathcal{K}^n$ is the following.

Definition. *A set function $\mu\colon \mathcal{K}^n \to \mathbb{R}$ is said to be a valuation if it holds*

$$\mu(K) + \mu(H) = \mu(K \cup H) + \mu(K \cap H),$$

for all $H, K \in K^n$ such that $H \cup K \in \mathcal{K}^n$.

Roughly speaking, a valuation is a set function that represents a generalization of the idea of measure. Indeed the Lebesgue measure on $\mathcal{K}^n$, V_n, is of course a valuation. We refer to Chapter 2 for more details and examples.

Hadwiger studied these set functions with particular interest in classification results. He established one of the most fundamental theorems in this theory.

Theorem. *[31] A function $\mu\colon \mathcal{K}^n \to \mathbb{R}$ is a continuous (w.r.t. Hausdorff metric) and rigid motion invariant valuation if and only if there exist $(n+1)$-coefficients $c_0, \cdots, c_n \in \mathbb{R}$ such that*

$$\mu(K) = \sum_{i=0}^{n} c_i V_i(K),$$

for every $K \in K^n$.

The functions $V_0, \cdots, V_n \colon \mathcal{K}^n \to \mathbb{R}$, appearing in the present statement, are the so-called **intrinsic volumes**, that play a fundamental role not only in valuation theory, but in convex geometry in general. We refer to Section 1.6 and Chapter 2 for details, results and more references concerning intrinsic volumes and valuations on convex bodies.

Hadwiger result gave a big impulse to this research area, i.e. classification of valuations. Indeed characterization results have been studied for valuations on $\mathcal{K}^n$ with different regularity, for instance upper/lower semi-continuity, and different kind of invariance, see for instance [44] and [49].
Another crucial contribution to valuations on $\mathcal{K}^n$ has been given by McMullen, see for instance [52]. We cite one of his most famous results, a homogeneous decomposition for continuous and translation invariant valuations.

We say that a valuation $\mu \colon \mathcal{K}^n \to \mathbb{R}$ is k-homogeneous, for $k \in \{0, \cdots, n\}$, if it holds

$$\mu(\lambda K) = \lambda^k \mu(K)$$

for all $K \in \mathcal{K}^n$ and $\lambda > 0$. The McMullen Decomposition Theorem is the following.

Theorem. *Let $\mu \colon \mathcal{K}^n \to \mathbb{R}$ be a continuous and translation invariant valuation. For every $k \in \{0, \cdots, n\}$ there exists a unique continuous, translation invariant and k-homogeneous valuation μ_k, such that it holds*

$$\mu = \sum_{k=0}^{n} \mu_k,$$

or equivalently

$$\mu(\lambda K) = \sum_{k=0}^{n} \lambda^k \mu_k(K),$$

for all $K \in \mathcal{K}^n$ and $\lambda > 0$.

Moreover we can consider not only real-valued valuations, but we can define them in a more general setting. Let us consider $(A, +)$ an Abelian semigroup; we say that a set function $\mu \colon \mathcal{K}^n \to (A, +)$ is a valuation if we have the same finite additivity condition $\mu(K) + \mu(H) = \mu(H \cup K) + \mu(H \cap K)$, for every $H, K \in \mathcal{K}^n$ such that $H \cup K \in \mathcal{K}^n$, where in this case $+$ is the operation defined on A.

An example of valuations of this type are the so-called Minkowski valuations, we refer to [43], where we have $A = \mathcal{K}^n$ and the operation $+$ is the Minkowski sum or vector sum (see Chapter 1). We refer to [47] and [68] for statements and results about this theory and other generalizations.

It has been considered also another kind of generalization of valuations notion, in the research area that is the topic of this thesis. Instead of $\mathcal{K}^n$, we change the domain of the valuation and we consider a function space X. We have two operations on X, replacing union and intersection: the pointwise maximum operator, $\vee$, and the pointwise minimum operator, $\wedge$. Hence we have the following definition.

Definition. *Let X be a function space. A functional $\mu \colon X \to \mathbb{R}$ is said to be a valuation if*

$$\mu(f) + \mu(g) = \mu(f \vee g) + \mu(f \wedge g)$$

for all $f, g \in X$ such that $f \vee g$ and $f \wedge g$ belong to X.

We consider, in this thesis, only real-valued valuations, anyway we observe that also in this case we can modify the definition and consider valuations μ from X to $\mathcal{K}^n$ or to some other function space or matrix-valued valuations. We refer to Chapter 4 for details and references about the related literature; in particular, we will explain a little bit more this research area, valuation theory on function spaces, presenting some past results and classification theorems for several spaces.

We studied a specific function space, the one formed by quasi-concave functions on $\mathbb{R}^n$. The main results we are going to present come from our papers [20] and [21]. We observe that results in Section 6.3 and in Chapter 7 do not belong to the previous papers and they are presented for the first time here.

Definition. *A function $f\colon \mathbb{R}^n \to \mathbb{R}_+$ is said to be quasi-concave if*

$$L_t(f) = \{x \in \mathbb{R}^n |\, f(x) \geq t\} \in \mathcal{K}^n \cup \{\varnothing\},$$

for every choice of $t > 0$.

We refer to Chapter 3 for all definitions and properties concerning these functions; now we are going to present a few remarks and examples. We denote by $\mathcal{QC}(\mathbb{R}^n)$ the space of quasi-concave functions. Some easy examples of quasi-concave functions are the Gaussian function and (positive multiples of) characteristic function of convex bodies,

$$s\chi_K(x) = \begin{cases} s & \text{if } x \in K, \\ 0 & \text{if } x \notin K, \end{cases}$$

with $K \in \mathcal{K}^n$ and $s > 0$.

In this thesis we are going to present various characterization results concerning valuations on $\mathcal{QC}(\mathbb{R}^n)$.

As we will see, there are many connections between this theory and valuations on $\mathcal{K}^n$. We will use these connections to obtain the main results of this thesis, for instance a Hadwiger type Theorem and a McMullen type Decomposition Theorem.
We define a notion of continuity for valuations on $\mathcal{QC}(\mathbb{R}^n)$ introducing a notion of convergence on quasi-concave function space: monotone and pointwise convergence.

Definition (Chapter 3, Section 3.2)**.** *We say that a sequence of quasi-concave functions f_i, $i \in \mathbb{N}$, converges to $f \in \mathcal{QC}(\mathbb{R}^n)$ in the monotone and pointwise convergence if:*

- *$f_i(x)$ is monotone either increasing or decreasing w.r.t i for every $x \in \mathbb{R}^n$ (with the same monotonicity).*
- *$\lim_{i \to +\infty} f_i(x) = f(x)$, for every $x \in \mathbb{R}^n$.*

In this thesis we will use mainly this convergence; in the final chapter we will introduce also another type of convergence, hypo-convergence. The main results will be presented and proved with the first convergence, but we observe that, as we will prove in Chapter 7, these results still hold with hypo-convergence.

A crucial property of valuations considered here is invariance. In the following we list the types of groups w.r.t. which we will define the invariance. We will consider always translation invariant valuation, in particular in Chapter 6 it will be the only invariance we will require. Moreover, to get Hadwiger type Theorem in Chapter 5, we will consider also the invariance of the valuation w.r.t elements of $O(n)$; composition of translation and elements of $O(n)$ will be denoted by rigid motion transformations. At the end, in the final chapter we will consider also $GL(n)$ and $SL(n)$-invariant valuations. In the following we present the definition of invariance for valuations defined on $\mathcal{QC}(\mathbb{R}^n)$.

Definition (Chapter 3, Section 3.2)**.** *Let μ be a real-valued valuation defined on $\mathcal{QC}(\mathbb{R}^n)$. We say that μ is invariant w.r.t. one of the transformations we have introduced above (translation, rigid motion, composition of translation and $GL(n)$ or $SL(n)$ elements), if for every transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ it holds*

$$\mu(f \circ T) = \mu(f),$$

for every $f \in \mathcal{QC}(\mathbb{R}^n)$.

We will consider in this thesis also different features that a valuation may have, like simplicity, monotonicity or homogeneity.

The main idea behind our characterization results is to find sufficient and necessary conditions such that we can represent the valuation μ in integral form. We will introduce in Chapter 5, Section 5.2, the notion of integral valuations, fundamental examples of valuations on $\mathcal{QC}(\mathbb{R}^n)$ that will replace the role of intrinsic volumes in these functional context.
Actually, these integral valuations come from intrinsic volumes. Let us fix $k \in \{0, \cdots, n\}$ and $f \in \mathcal{QC}(\mathbb{R}^n)$. For every $t \leq \max_{\mathbb{R}^n} f$ (we will see in Chapter 3 that the maximum always exists) we have that $L_t(f)$ is a convex body, so the function

$$\varphi_k\colon (0, \max_{\mathbb{R}^n} f] \to \mathbb{R}_+, \quad \varphi_k(t) = V_k(L_t(f))$$

is well-defined and we can extend φ_k to $(0, +\infty)$ just defining $\varphi_k(t) = 0$ for $t > \max_{\mathbb{R}^n} f$. The function φ_k is decreasing, so it has bounded variation. This means that it admits a distributional derivative which is a non-positive measure with support on $[0, +\infty)$. We denote this measure by $-S_k(f; \cdot)$.

We use these measures, $S_k(f; \cdot)$, depending on f, to define valuations on $\mathcal{QC}(\mathbb{R}^n)$, the so-called integral valuations. They are functionals of the form

$$\mu(f) = \int_0^{+\infty} \phi(t) dS_k(f; t), \quad \forall\, f \in \mathcal{QC}(\mathbb{R}^n). \tag{1}$$

In Chapter 5 we will study sufficient and necessary conditions for the finiteness of (1), because, since we are interested in valuations, we have to be sure that $\mu(f) \in \mathbb{R}$ for every choice of $f \in \mathcal{QC}(\mathbb{R}^n)$.

We present some of the main theorems of this thesis.

Theorem (Theorem 5.5.1; Theorem 1.1 in [20])**.** *A map $\mu\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ is an invariant, w.r.t. rigid motions, and continuous, w.r.t. pointwise and monotone convergence, valuation on $\mathcal{QC}(\mathbb{R}^n)$ if and only if there exist $n+1$ continuous functions ϕ_k, $k = 0, \ldots, n$ defined on $[0, +\infty)$, and $\delta > 0$ such that: $\phi_k \equiv 0$ in $[0, \delta]$ for every $k = 1, \ldots, n$, and*

$$\mu(f) = \sum_{k=0}^{n} \int_0^{+\infty} \phi_k(t) dS_k(f; t), \quad \forall\, f \in \mathcal{QC}(\mathbb{R}^n).$$

We observe that in this functional case the vector space of invariant and continuous valuations on $\mathcal{QC}(\mathbb{R}^n)$ has infinite dimension, differently from the case of valuations on $\mathcal{K}^n$.
Moreover in Chapter 5 we establish, in parallel, also characterization results for monotone valuations.

We consider also continuous valuations that are only invariant w.r.t. translations. We establish a functional counterpart of the McMullen Decomposition Theorem. First of all we introduce an operation for valuations μ defined on $\mathcal{QC}(\mathbb{R}^n)$. Let us fix $\lambda > 0$ and $f \in \mathcal{QC}(\mathbb{R}^n)$, we define the multiplication of f by the scalar factor λ as

$$\lambda \odot f(x) = f\left(\frac{x}{\lambda}\right),$$

for every $x \in \mathbb{R}^n$.

Hence a k-homogeneous valuation on quasi-concave functions, for $k \in \{0, \cdots, n\}$, is defined by

$$\mu(\lambda \odot f) = \lambda^k \mu(f),$$

for all $\lambda > 0$ and $f \in \mathcal{QC}(\mathbb{R}^n)$.
The homogeneous decomposition result is the following.

Theorem (Theorem 6.1.1; Theorem 1.1 in [21])**.** *Let $\mu\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ be a continuous, w.r.t. monotone and pointwise convergence, and translation invariant valuation. For all $k = 0, \dots, n$, there exists a unique μ_k continuous, w.r.t. monotone and pointwise convergence, translation invariant and k-homogeneous valuation, such that*

$$\mu = \sum_{k=0}^{n} \mu_k.$$

In Chapter 6 we will present also a characterization for 0- and n-homogeneous valuations and a polynomiality result again for continuous, w.r.t. monotone and pointwise convergence, and translation invariant valuations. To introduce polynomiality we need to define a notion of sum on $\mathcal{QC}(\mathbb{R}^n)$.

Definition. *Let f and $g \in \mathcal{QC}(\mathbb{R}^n)$. We define*

$$f \oplus g(x) = \sup_{y \in \mathbb{R}^n} \min\{f(x), g(y - x)\},$$

for $x \in \mathbb{R}^n$.

We refer to [56] for details about sum operation and the multiplication by non-negative scalars that we have already introduced for homogeneous valuations. We just observe that holds

$$L_t((\lambda \odot f) \oplus (\sigma \odot g)) = \lambda L_t(f) + \sigma L_t(g),$$

for every $t, \lambda, \sigma > 0$ and $f \in \mathcal{QC}(\mathbb{R}^n)$.

The result concerning polynomiality is the following.

Theorem (Theorem 6.3.5)**.** *Let μ be a continuous, w.r.t. monotone and pointwise convergence, translation invariant and k-homogeneous valuation on $\mathcal{QC}(\mathbb{R}^n)$. There exists a functional*

$$\overline{\mu}\colon (\mathcal{QC}(\mathbb{R}^n))^k \to \mathbb{R}$$

such that for $m \geq 1$, $\lambda_1, \cdots, \lambda_m > 0$ and $f_1, \cdots, f_m \in \mathcal{QC}(\mathbb{R}^n)$, one has

$$\mu((\lambda_1 \odot f_1) \oplus \cdots \oplus (\lambda_k \odot f_m)) = \sum_{r_1, \cdots, r_m = 0}^{k} \binom{k}{r_1 \cdots r_m} \lambda_1^{r_1} \cdots \lambda_k^{r_m} \, \overline{\mu}(f_1[r_1], \cdots, f_k[r_m]), \tag{2}$$

where $f_i[r_i]$ means that we count f_i in $\overline{\mu}$, r_i times. Moreover $\overline{\mu}$ is multilinear, translation invariant and symmetric.

If we fix $r \in \{0, \cdots, k\}$ and $g_1, \cdots, g_{k-r} \in \mathcal{QC}(\mathbb{R}^n)$ the map

$$\mathcal{QC}(\mathbb{R}^n) \ni f \mapsto \overline{\mu}(f[r], g_1, \cdots, g_{k-r})$$

is a continuous, w.r.t. monotone and pointwise convergence, translation invariant and r-homogeneous valuation on $\mathcal{QC}(\mathbb{R}^n)$

At the end, in Chapter 7 we will present characterization results for continuous, w.r.t. hypo-convergence, invariant, w.r.t. $SL(n)$ first and $GL(n)$ later, valuations on $\mathcal{QC}(\mathbb{R}^n)$. We conclude the final chapter with possible future developments.

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Notation

We work in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 1$, endowed with the usual inner product $(\cdot, \cdot)$ and norm $||\cdot||$. Given a subset A of $\mathbb{R}^n$, $\mathrm{int}(A)$, $\mathrm{cl}(A)$ and ∂A denote the interior, the closure and the topological boundary of A, respectively.

We will always denote by the same notation $\underline{0}$ the origin of all vector spaces we will consider like $\mathbb{R}^n$, function spaces and valuations.

For every $x \in \mathbb{R}^n$ and $r \geq 0$, let $\mathbb{B}(x, r)$ be the closed ball of radius r centered at x; for simplicity we will write $\mathbb{B}_r$ instead of $\mathbb{B}(\underline{0}, r)$ and $\mathbb{B}^n$ for the unit ball in $\mathbb{R}^n$. Moreover we will denote by $B(x, r)$, B_r and B^n the interior of $\mathbb{B}(x, r)$, $\mathbb{B}_r$ and $\mathbb{B}^n$ respectively, and by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$.

By a rotation of $\mathbb{R}^n$ we mean a linear transformation of $\mathbb{R}^n$ associated to a matrix in $O(n)$; we denote also by $SO(n)$ the special orthogonal group acting on $\mathbb{R}^n$. A **rigid motion** of $\mathbb{R}^n$ is the composition of a translation and a rotation of $\mathbb{R}^n$. We use also the notation $GL(n)$ for the general linear group acting on $\mathbb{R}^n$ and by $SL(n)$ the special linear group acting on $\mathbb{R}^n$.

The Lebesgue measure in $\mathbb{R}^n$ will be denoted by V_n, while by $\mathcal{H}^k$ we will denote the k-dimensional Hausdorff measure on $\mathbb{R}^n$, where $0 \leq k \leq n$, normalized so that $\mathcal{H}^n$ is equal to V_n, and we will use dx (instead of $d\mathcal{H}^n(x)$) to denote integration w.r.t. $\mathcal{H}^n$.

Chapter 1

Theory of Convex Bodies

In this chapter we will focus on some preliminar materials that will be crucial in this thesis.

We will start with definitions and examples of convex bodies and their properties. We will present also intrinsic and mixed volumes that will have a functional counterpart later.

We always refer to [64] for definitions and statements in this chapter, when not specified. See also [30], [37], [40] and [75] for introduction to convex geometry and [14], [15], [61], [63] and [65] for details about the theory of analytic aspects of convexity.

1.1 Basic definitions and properties

Definition 1.1.1. *A subset K of $\mathbb{R}^n$ is said to be a* ***convex body*** *if it is convex and compact. We denote by $\mathcal{K}^n$ the set of convex bodies in $\mathbb{R}^n$.*

Moreover, we define the dimension of K as the minimum natural number $0 \leq m \leq n$ such that there exists an affine subspace of $\mathbb{R}^n$, with such dimension, containing K. We will denote the dimension by $\dim(K)$.

A few basic and easy examples are the following:

- the closed Euclidean ball in $\mathbb{R}^n$ with any radius and center, is of course a convex body of dimension n.
- The closed segment with endpoints x and y in $\mathbb{R}^n$ belongs to $\mathcal{K}^n$ and it is of dimension 1. We denote that segment by $[x, y] = \{(1-\lambda)x + \lambda y \mid \lambda \in [0, 1]\}$.

Definition 1.1.2. *We define the convex hull of $A \subseteq \mathbb{R}^n$,* $\operatorname{conv}(A)$*, as the smallest convex set of $\mathbb{R}^n$ containing A:*

$$\operatorname{conv}(A) = \bigcap_{A \subseteq B \text{ convex } \subseteq \mathbb{R}^n} B.$$

Remark 1.1.3. We can see also that $\operatorname{conv}(A)$ is the set of all finite convex combinations of elements of A, that means

$$\operatorname{conv}(A) = \{\sum_{i=1}^{m} \alpha_i x_i : \mathbb{N} \ni m \geq 1, \ x_i \in A, \ \alpha_i \in [0, 1], \ \forall\, i \in \{1, \cdots, m\}, \ \sum_{i=1}^{m} \alpha_i = 1\}.$$

We can use the notion of convex hull to define some classes of convex bodies.

Definition 1.1.4. *We say that $K \subseteq \mathbb{R}^n$ is a simplex if it is the convex hull of affinely independent points in $\mathbb{R}^n$. Moreover we say that K is an m-simplex if it is the convex hull of $m+1$ affinely independent points.*

Remark 1.1.5. Any simplex in $\mathbb{R}^n$ is a convex body.

Simplices are important examples of a larger class of convex bodies: the family of **polytopes**.

Definition 1.1.6. *The convex hull of finitely many points in $\mathbb{R}^n$ is said to be a polytope. We denote by $\mathcal{P}^n$ the set of all polytopes.*

Remark 1.1.7. Clearly $\mathcal{P}^n \subseteq \mathcal{K}^n$ and also any simplex is a polytope.

Definition 1.1.8. *Let $C \subseteq \mathbb{R}^n$ be a convex set. A face of C is a subset $F \subset C$ such that if $x, y \in C$ and $\frac{x+y}{2} \in F$, then it holds $x, y \in F$.*

We just observe that if $K \in \mathcal{K}^n$ then every face of K is a convex body, and if $P \in \mathcal{P}^n$, then every face of P is a polytope.

We can define, moreover, some operations over $\mathcal{K}^n$. First, we notice that the intersection and the union are two operations on convex bodies, but while $\mathcal{K}^n$ is closed w.r.t. the first one, this is no longer true for the second, as the union of two convex bodies is not always convex.

We also have the following fundamental operations on $\mathcal{K}^n$.

Definition 1.1.9. *Let $K, H \in \mathcal{K}^n$ and $\alpha \geq 0$. Then we define:*

- *the Minkowski sum of K and L as*
$$K + H = \{x + y \mid x \in K,\ y \in H\}.$$
- *The scalar multiplication as*
$$\alpha K = \{\alpha x \mid x \in K\}.$$

It is easy to check the validity of the following proposition.

Proposition 1.1.10. *For any $H, K \in K^n$ and α, β non-negative real numbers, we have that $\alpha K + \beta H$ is still a convex body.*

We conclude this section with a connection between union/intersection and Minkowski sum.

Proposition 1.1.11 (Section 3.1 in [64])**.** *Let $K, H \in \mathcal{K}^n$ be convex bodies, such that $K \cup H$ is convex. Then we have*

- $$(K \cup H) + C = (K + C) \cup (H + C),$$
- $$(K \cap H) + C = (K + C) \cap (H + C),$$

for any $C \in \mathcal{K}^n$.

Moreover, it holds
$$(K \cup H) + (K \cap H) = K + L.$$

Remark 1.1.12. The first equality holds for any arbitrary subset of $\mathbb{R}^n$. The second one is an immediate consequence of the convexity of K.

1.2 Metric properties

Let us now focus on the metric onto the set of convex bodies. One of the most significant metrics on $\mathcal{K}^n$ is the one induced by **Hausdorff distance**.

Definition 1.2.1. *For any $K, H \in \mathcal{K}^n$ we define the Hausdorff distance as*

$$\delta(K,H) = \max\{\max_{x\in K} d(x,H), \max_{y\in H} d(K,y)\},$$

where for arbitrary $x \in \mathbb{R}^n$ and $I \subseteq \mathbb{R}^n$, $d(x,I) = \inf_{z\in I} ||x-z||$.

We have the following result.

Proposition 1.2.2. *$(\mathcal{K}^n, \delta)$ is a metric space.*

There is also another way to define Hausdorff distance, equivalent to the previous one, which uses the construction of the parallel set of a convex body.

Definition 1.2.3 (Parallel set)**.** *Let K be a convex body. For every $\epsilon > 0$, the parallel set K_ϵ is defined by*

$$K_\epsilon = K + \epsilon\mathbb{B}^n = \{x + y |\ x \in K,\ ||y|| \leq \epsilon\}.$$

As a consequence of Proposition 1.1.10, the parallel set is always a convex body, for every choice of $\epsilon > 0$.

Proposition 1.2.4. *The Hausdorff distance, $\delta(K,H)$, between $K, H \in \mathcal{K}^n$ can be expressed as follows*

$$\inf\{\epsilon \geq 0 |\ K \subseteq H_\epsilon,\ H \subseteq K_\epsilon\}.$$

Corollary 1.2.5. *For every $\epsilon > 0$ and convex bodies $K, H \in \mathcal{K}^n$, it holds*

$$\delta(K,H) \leq \epsilon \ \Leftrightarrow \begin{cases} K \subseteq H_\epsilon, \\ H \subseteq K_\epsilon. \end{cases}$$

We can consider now a sequence of convex bodies and we say that $\{K_i\}_{i\in\mathbb{N}} \subseteq \mathcal{K}^n$ converges to a convex body K if and only if

$$\lim_{i\to+\infty} \delta(K_i, K) = 0.$$

We have the following characterization result for a converging sequence in the Hausdorff metric.

Theorem 1.2.6 (Theorem 1.8.8 in [64])**.** *A sequence of convex bodies K_i converges to K if and only if the following two conditions hold together:*

- *for every $x \in K$ and for all $i \in \mathbb{N}$, there exists $x_i \in K_i$ such that $\lim_{i\to+\infty} x_i = x$.*
- *For any sequence of points, x_i, $i \in \mathbb{N}$, such that $x_i \in K_i$, $\forall\, i \in \mathbb{N}$, and $\exists\ \lim_{i\to+\infty} x_i = x$, then $x \in K$.*

The following are two fundamental results concerning the completeness and compactness of $\mathcal{K}^n$.

Theorem 1.2.7 (Completeness of $\mathcal{K}^n$)**.** *Every Cauchy sequence in $\mathcal{K}^n$ is convergent.*

Theorem 1.2.8 (Compactness of $\mathcal{K}^n$- **Blaschke Selection Theorem**). *Every bounded sequence of convex bodies (i.e. there exists $H \in \mathcal{K}^n$ such that any element of the sequence is contained in H) admits a convergent subsequence.*

We refer to [64], theorems from $1.8.3$ to $1.8.7$, for the proof. We just observe that these results are special cases (consequences) of corresponding statements valid in the class of compact subsets in $\mathbb{R}^n$.

We conclude this section with two results concerning δ. The first one concerns the Hausdorff limit of a decreasing sequence of convex bodies.

Lemma 1.2.9. *Let K_i be a decreasing sequence of convex bodies, i.e. $K_{i+1} \subseteq K_i$; then there exists the limit w.r.t. Hausdorff distance and it holds*

$$\lim_{i \to +\infty} K_i = \bigcap_{i=1}^{+\infty} K_i.$$

The last result is about the continuity of Lebesgue measure.

Theorem 1.2.10 (Theorem $1.8.20$ in [64]). *V_n, the Lebesgue measure in $\mathbb{R}^n$, is continuous on $\mathcal{K}^n$ w.r.t. Hausdorff distance.*

1.3 Regularity of the boundary

We consider now only convex bodies K with not empty interior, $\mathrm{int}(K) \neq \varnothing$. We present some classical results concerning the boundary of such bodies.

Proposition 1.3.1. *[64] Let K be a convex body in $\mathbb{R}^n$ with $\mathrm{int}(K) \neq \varnothing$. The boundary of K, ∂K, can be described locally as the graph of a convex function of $(n-1)$-variables.*

Proof. We fix $x_0 \in \partial K$. There exists $\overline{x} \in \mathrm{int}(K)$ such that $\overline{x} + \mathbb{B}_r \subseteq K$, for some $r > 0$.
We choose a coordinate system in $\mathbb{R}^n$ such that $-e_n = \frac{x_0 - \overline{x}}{||x_0 - \overline{x}||}$. Let Ω be the orthogonal projection of $\overline{x} + \mathbb{B}_r$ onto $e_n^{\perp} \sim \mathbb{R}^{n-1}$. Ω is a ball in $\mathbb{R}^{n-1}$ and hence it is convex.

For any $x \in \mathbb{R}^n$, we have $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. If we fix $x' \in \mathbb{R}^{n-1}$ and we consider a line s parallel to e_n through x', then we have

$$\varnothing \neq s \cap K = \{(x', t)|\ t \in [t_{min}, t_{max}]\},$$

for some $t_{min}, t_{max} \in \mathbb{R}$. Hence for every $x' \in \Omega$ we define

$$f(x') = \inf\{t \in \mathbb{R}|\ (x', t) \in K\}.$$

As it is easy to check, f is convex and it is the desired function. □

Corollary 1.3.2. *If $\mathrm{int}(K) \neq \varnothing$, then ∂K is the finite union of graphs of convex functions of $(n-1)$-variables.*

We list some properties that we can deduce from the previous facts. They can be deduced by standard regularity properties of convex function (see for instance [64]).

Proposition 1.3.3. *Let $K \in \mathcal{K}^n$ be a convex body with not empty interior.*

- *∂K is locally Lipschitz.*
- *∂K has finite $(n-1)$-Hausdorff dimension, $\mathcal{H}^{n-1}(\partial K) < +\infty$.*
- *∂K admits $\mathcal{H}^{n-1}$-a.e. tangent space.*

1.4 Support function

Definition 1.4.1. *Let K be a convex body, then we define the* ***support function*** *of K, $h_K \colon \mathbb{S}^{n-1} \to \mathbb{R}$ as*

$$h_K(u) = \sup\{(x, u) |\, x \in K\},$$

for any direction $u \in \mathbb{S}^{n-1}$.

It is possible to extend h_K to $\mathbb{R}^n$ as follows

$$h_K(x) = \begin{cases} \lambda h_K(u) & \text{if } x = \lambda u,\, \lambda > 0,\, u \in \mathbb{S}^{n-1}, \\ 0 & \text{if } x = \underline{0}. \end{cases}$$

In the following we will consider h_K as defined on $\mathbb{R}^n$, except when specified. It is also clear that h_K, just by definition, is 1-homogeneous. Moreover we have the following proposition.

Proposition 1.4.2. *h_K is a 1-homogeneous and sub-additive function on $\mathbb{R}^n$ (i.e. $h_K(x + y) \leq h_K(x) + h_K(y)$, for all x,y $\in \mathbb{R}^n$) for any choice of K.*

It is possible also to prove the vice versa.

Theorem 1.4.3. *Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a 1-homogeneous and sub-additive function, then there exists a unique convex body $K \in \mathcal{K}^n$ such that $f \equiv h_K$ on $\mathbb{R}^n$.*

We refer to [64] for the complete proof of the previous result. We mention that the convex body K can be described as follows, in terms of f,

$$\begin{aligned} K &= \{x \in \mathbb{R}^n |\, (x, y) \leq f(y),\ \forall\, y \in \mathbb{R}^n\} \\ &= \bigcap_{y \in \mathbb{R}^n} \{x \in \mathbb{R}^n |\, (x, y) \leq f(y)\}. \end{aligned}$$

It is possible to prove that $K \in K^n$ and the theorem statement by the conditions on f.

The previous theorem means that the support determines totally a convex body. Moreover we can find a convex body just with the definition of a function with the two properties mentioned above and this is a quite strong connection between geometric and analytic aspects of convexity.

We summarize the most important properties of the support function in the following remark, that show how crucial is this function in the Theory of Convex Bodies.

Remark 1.4.4. ***[Regularity of** h_K; see [40] and [64]]* Let K be a convex body in $\mathbb{R}^n$. By the 1-homogeneity and the sub-additivity of the support function we have that h_K is convex. Hence it is continuous on its effective domain and locally Lipschitz. By Rademacher's Theorem it follows that h_K is differentiable a.e. in $\mathbb{R}^n$.
Moreover, by Alexandroff's Theorem, see [9] and [64], it follows that h_K is a.e. twice differentiable in its domain in the sense that there exists a.e. a second-order Taylor expansion. This means that f is differentiable in x and there exists a symmetric linear map $Af(x) \colon \mathbb{R}^n \to \mathbb{R}^n$ such that it holds

$$f(y) = f(x) + (\nabla f(x), y - x) + \frac{1}{2}(Af(x)(y - x), y - x) + o(||y - x||^2),$$

for all $y \in \operatorname{dom}(f)$ and for a.e. $x \in \mathbb{R}^n$.

We refer for all the statements to [9], [60], [61], [64].

Lemma 1.4.5. *Let $K, L \in K^n$ and $\alpha, \beta \geq 0$, then we have $h_{\alpha K+\beta L} \equiv \alpha h_K + \beta h_L$.*

Proposition 1.4.6. *Let K and H be two convex bodies, then we have*

$$\delta(K, H) = ||h_K - h_L||_{L^\infty(\mathbb{S}^{n-1})}.$$

In particular for a sequence K_i in $\mathbb{R}^n$, $i \in \mathbb{N}$, and $K \in \mathcal{K}^n$, one has that $K_i \to K$ w.r.t. Hausdorff metric if and only if h_{K_i} converges uniformly to h_K in $\mathbb{S}^{n-1}$ and w.r.t. uniform convergence on compact subsets in $\mathbb{R}^n$.

At the end of this section, we present some examples of support functions.

1. Let $K = \mathbb{B}^n$ be the unit ball in $\mathbb{R}^n$. Then we have $h_K(x) = ||x||$, for every $x \in \mathbb{R}^n$.
2. Let $K = \{x_0\}$, then it follows easily that $h_K(x) = (x, x_0)$ and in particular $h_K \equiv 0$ if $K = \{\underline{0}\}$.
3. If we consider $K = [-x_0, x_0]$ a centered segment in $\mathbb{R}^n$, with $x_0 \neq \underline{0}$, then it holds

$$h_K(x) = \sup\{(x, y)|\, y \in K\} = \sup\{(x, tx_0)|\, t \in [-1, 1]\} = |(x_0, x)|.$$

In general, the support function of a convex body does not have a fixed sign. But we can claim that $h_K > 0$ if and only if the origin is a point that belongs to the interior of K and $h_K \geq 0$ if and only if the origin belongs to K.

1.5 Families of convex bodies and density results

There are many subfamilies of convex bodies that play an important role in this theory. We have already introduced the classes of simplices and polytopes, now we are going to show some other examples.

1.5.1 Zonotopes and zonoids

Definition 1.5.1. *We say that a convex body K is:*

- *a **zonotope** if it is the Minkowski sum of finitely many segments in $\mathbb{R}^n$;*
- *a **zonoyd** if it is the limit, in the Hausdorff sense, of a sequence of zonotopes.*

Remark 1.5.2. We can evaluate the support function of a zonotope K. According to the definition we can write $K = Y_1 + \cdots + Y_m$, where Y_i is a segment for every i. For simplicity we may assume that for every $j \in \{1, \cdots, m\}$, there exists $\underline{0} \neq y_j \in \mathbb{R}^n$ such that $Y_j = [-y_j, y_j]$. Indeed if we consider generic segments, in generic position in $\mathbb{R}^n$, then the zonotope that comes out it is just a translate of that one with every segments centered at the origin. Then we have

$$h_K(x) = \sum_{j=1}^{m} |(x, y_j)|,$$

for every $x \in \mathbb{R}^n$.

As we can see from Remark 1.5.2, a zonotope is, up to translation, centrally symmetric, i.e. $K = -K$, where $-K$ is the reflection w.r.t. the origin $\underline{0}$, i.e.

$$-K = \{-x |\ x \in K\}.$$

Taking the limit the same conclusion holds also for zonoids.

We can find also an explicit formula for the support function of a zonoid Y. Since it is the limit of a sequence of zonotopes Y_i, we have that h_Y is the uniform limit of h_{Y_i}. In the following, for this purpose, we will consider the support function as defined on $\mathbb{S}^{n-1}$. The idea is to reformulate the support function of the zonotope in the sense of measure theory.
We start with the following observation:

$$\sum_{j=1}^{m} |(u, y_j)| = \int_{\mathbb{S}^{n-1}} |(u, v)| d\mu(v),$$

where μ is the measure defined by

$$\mu = \frac{1}{2} \sum_{j=1}^{m} ||y_j||(\delta_{u_j} + \delta_{-u_j}),$$

where $u_j = \frac{y_j}{||y_j||}$ and δ_{u_j} is the Dirac mass concentrated in u_j.

This means that Y_i defines a sequence of even discrete measures. If Y_i converges to a zonotope Y, then it is possible to prove that this sequence converges weakly, up to subsequences, to an even Borel measure, depending on Y, $\mu(Y; .)$ on $\mathbb{S}^{n-1}$.

Hence we have the following result.

Theorem 1.5.3 (see [40] and [64])**.** *Let $Y \in \mathcal{K}^n$. Y is a zonoid if and only if there exists an even, Borel measure $\mu(Y; .)$ on $\mathbb{S}^{n-1}$ such that*

$$h_Y(u) = \int_{\mathbb{S}^{n-1}} |(u, v)| d\mu(Y; v), \quad \forall\, u \in \mathbb{S}^{n-1}.$$

Moreover $\mu(Y; .)$ is uniquely determined by Y.

We present the last result concerning zonotopes and zonoids, that was proved in [21] and will be used later.

Proposition 1.5.4. *Every zonoid $Y \in \mathcal{K}^n$ can be approximated by a decreasing sequence of zonotopes.*

Proof. Fix a natural number $i \geq 1$ and define

$$Y_i = Y + \frac{1}{2^i} \mathbb{B}^n.$$

Since Y_i is the Minkowski sum of zonoids, it is a zonoid itself and

$$h_{Y_i} = h_Y + \frac{1}{2^i}.$$

As every zonoid is the limit of zonotopes, for every Y_i there exists a zonotope Z_i such that

$$\delta(Z_i, Y_i) \leq \frac{1}{4}\frac{1}{2^i}.$$

We have

$$\delta(Y, Z_i) \to 0 \text{ as } i \to +\infty,$$

as a consequence of the triangle inequality and

$$h_Y + \frac{3}{2^{i+2}} = h_{Y_i} - \frac{1}{2^{i+2}} \leq h_{Z_i} \leq h_{Y_i} + \frac{1}{2^{i+2}}.$$

Then we obtain

$$Y + \frac{3}{2^{i+2}}\mathbb{B}^n \subseteq Z_i \subseteq Y + \frac{5}{2^{i+2}}\mathbb{B}^n.$$

So, we have the following conclusions:

- $Z_i \supseteq Y$;
- $Z_i \supseteq Z_{i+1}$.

Therefore, for every zonoid Y, we have created a decreasing sequence of zonotopes converging to Y. □

1.5.2 The classes of $C^{2,+}$ bodies and of polytopes. Density results

Definition 1.5.5. *We say that a convex body K is a $C^{2,+}$ body if:*

- *the boundary of K, ∂K, is C^2, i.e. it is locally the graph of a C^2 (convex) function.*
- *The Gaussian curvature of K at $x \in \partial K$, $G_K(x)$, is strictly positive for every $x \in \partial K$.*

In particular a $C^{2,+}$ body has non-empty interior. We have the following result, that we will use later and it is a basic result of the Theory of Convex Bodies.

Theorem 1.5.6. *The subsets of*

- *polytopes $\mathcal{P}^n$,*
- *$C^{2,+}$ bodies,*

are dense in K^n w.r.t. the Hausdorff distance.

According to Theorem 1.5.6, many problems concerning the Theory of Convex Bodies can be treated in one of these subfamilies of $\mathcal{K}^n$ and then extended to all the space.
We just observe, moreover, that for any convex body K we can find an approximating sequence of polytopes and of $C^{2,+}$ convex bodies that are inside the body and that one outside.

1.5.3 Density result for $C(\mathbb{S}^{n-1})$

We conclude this section with a density result about function spaces.

We know that the support function of a convex body is a continuous function on $\mathbb{R}^n$, ad in particular on the sphere $\mathbb{S}^{n-1}$. Now we focus on $C^2(\mathbb{S}^{n-1})$, that is the set of all twice differentiable functions over the sphere. Here twice differentiable means that the 1-homogeneous extension of $f \in C^2(\mathbb{S}^{n-1})$, defined by

$$\tilde{f}(x) = \begin{cases} ||x||f(\frac{x}{||x||}) & \text{if } x \in \mathbb{R}^n \setminus \{\underline{0}\}, \\ 0 & \text{if } x = \underline{0}, \end{cases}$$

is $C^2(\mathbb{R}^n \setminus \{\underline{0}\})$.
An easy computation shows that any function f like that is the difference of two support functions: if $f \in C^2(\mathbb{S}^{n-1})$, then there exist K, L convex bodies such that

$$f(u) = h_K(u) - h_L(u),$$

for every $u \in \mathbb{S}^{n-1}$. Hence we introduce the space

$$L^n = \{f \equiv h_K - h_L \text{ on } \mathbb{S}^{n-1} | \, K, L \in \mathcal{K}^n\},$$

and it holds $C^2(\mathbb{S}^{n-1}) \subseteq L^n \subseteq C(\mathbb{S}^{n-1})$.
The previous inclusions, with the density of $C^2(\mathbb{S}^{n-1})$ in $C(\mathbb{S}^{n-1})$ w.r.t. the usual convergence in $C(\mathbb{S}^{n-1})$, allow us to claim the following result.

Theorem 1.5.7. *L^n is dense in $C(\mathbb{S}^{n-1})$ w.r.t. the uniform convergence on the sphere.*

1.6 Intrinsic and Mixed volumes

1.6.1 Intrinsic volumes

There is a family of maps from $\mathcal{K}^n$ to $\mathbb{R}$ that are crucial in convex geometry and also in valuation theory. As we will see, they play an important role not only in $\mathcal{K}^n$, but also in many function spaces. We introduce this family, the intrinsic volumes, with a result concerning the Lebesgue measure of the parallel set. We recall that the parallel set of a convex body K, for an $\epsilon > 0$ is defined as

$$K_\epsilon = K + \epsilon\mathbb{B}^n = \{x + y | \, x \in K, \ ||y|| \leq \epsilon\},$$

and it is also a convex body.

There are many ways to define intrinsic volumes. We have chosen the one based on Steiner formula. In the following we will present the statement of Steiner's Theorem and we will introduce intrinsic volumes.

Theorem 1.6.1. ***[Steiner Formula]*** *There exist $(n+1)$ functions defined on $\mathcal{K}^n$,*

$$V_i \colon \mathcal{K}^n \to \mathbb{R},$$

for $i \in \{0, \cdots, n\}$, such that for every convex body K and $\epsilon > 0$, we have

$$V_n(K_\epsilon) = \sum_{i=0}^{n} V_i(K)\omega_{n-i}\epsilon^{n-i}, \tag{1.1}$$

where ω_{n-i} is the Lebesgue measure of the unit ball in $\mathbb{R}^{n-i}$. $V_0, \cdots, V_n$ are called ***intrinsic volumes****.*

The statement of the theorem claims that the Lebesgue measure of the parallel set is a polynomial w.r.t. ϵ and the coefficients, up to some normalization factor, depend only on K. These coefficients are the intrinsic volumes. The adjective "intrinsic" refers to the fact that the value of any $V_j(K)$ does not depend on the dimension of the ambient space; in particular if $\dim(K) = i < j$, then $V_j(K) = 0$ and $V_i(K)$ is exactly the i-Lebesgue measure of K.

Evaluating the Steiner formula at $\epsilon = 0$ we see that V_n is just the Lebesgue measure. We can make some other easy examples of such set functions.

Example 1.6.2.

- *V_0 is the Euler-Poincaré characteristic function, i.e. $V_0(K) = 1$, for every choice of K. Indeed, manipulating the Steiner formula we have*

$$V_n(K_\epsilon) = \epsilon^n \, V_n(\mathbb{B}^n + \frac{1}{\epsilon}K) = \epsilon^n \, \sum_{i=0}^{n} V_i(K)\frac{\omega_{n-i}}{\epsilon^i},$$

where the first equality holds because of the n-homogeneity of the Lebesgue measure. Hence we have

$$V_n(\mathbb{B}^n + \frac{1}{\epsilon}K) = \sum_{i=0}^{n} V_i(K)\frac{\omega_{n-i}}{\epsilon^i}$$

and taking the limit for ϵ tending to $+\infty$,

$$V_n(\mathbb{B}^n) = \omega_n V_0(K).$$

- *Let K be a convex body; again from Steiner formula it holds*

$$2V_{n-1}(K) + o(\epsilon) = \frac{V_n(K_\epsilon) - V_n(K)}{\epsilon}.$$

Hence, taking the limit for ϵ tending to 0^+, we have

$$2V_{n-1}(K) = \lim_{\epsilon \to 0^+} \frac{V_n(K_\epsilon) - V_n(K)}{\epsilon},$$

i.e. $V_{n-1}(K)$ is, up to a factor 2, the so-called ***Minkowski content*** *of K [see [8], Section 2.13].*

In particular, if K has non-empty interior, then the Minkowski content coincides with the $(n-1)$-Hausdorff measure of ∂K:

$$V_{n-1}(K) = \frac{1}{2}\,\mathcal{H}^{n-1}(\partial K).$$

In the general case we have

$$V_{n-1}(K) = \begin{cases} \frac{1}{2}\,\mathcal{H}^{n-1}(\partial K) & \textit{if } \operatorname{int}(K) \neq \varnothing, \\ \mathcal{H}^{n-1}(K) & \textit{otherwise.} \end{cases}$$

- $V_1(K)$ *is, up to some constant depending only on* n*, the mean width of* K*.*

  If we call $b_K(u) = h_K(u) - h_K(-u)$ *the width of* K *along the direction* $u \in \mathbb{S}^{n-1}$*, then we have*

$$V_1(K) = \frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} b_K(u) d\mathcal{H}^{n-1}(u).$$

For a generic $j \in \{2, \cdots, n-2\}$ it is not easy to find an explicit formula for V_j. In the following we show the computation for particular convex bodies, but usually we need tools from integral geometry to obtain a general formula for intrinsic volumes, see [37].

Example 1.6.3.

- *Let* $K = \mathbb{B}^n$ *be the unit ball. A simple calculation gives*

$$V_j(\mathbb{B}^n) = \frac{\omega_n}{\omega_{n-j}} \binom{n}{j},$$

  for every j*.*

- *Let* $P = I_1 \times \cdots \times I_n$ *be an interval in* $\mathbb{R}^n$*, with* $V_1(I_k) = x_k > 0$*. Then it holds*

$$V_j(P) = \sum_{1 \leq i_1 < \cdots < i_j \leq n} x_{i_1} \cdots x_{i_j},$$

  for every $j \in \{0, \cdots, n\}$*. In the special case of* $P = [0,1]^n$*, we have* $V_j(K) = \binom{n}{j}$*.*

1.6.2 Mixed volumes

The following result is a generalization of the Steiner formula.

Theorem 1.6.4. *[**Minkowski Theorem**] There exists a function* $V \colon (\mathcal{K}^n)^n \to \mathbb{R}$ *such that for every* $m \in \mathbb{N}$ *and for every choice of* $K_1, \cdots, K_m \in \mathcal{K}^n$ *and* $\lambda_1, \cdots, \lambda_m > 0$*, we have*

$$V_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{1 \leq i_1, \cdots, i_n \leq m} V(K_{i_1}, \cdots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}.$$

V *is called the* ***mixed volume****.*

As in the Steiner formula, we have that the Lebesgue measure of a linear combination of convex bodies can be written as a homogeneous polynomial of degree n w.r.t. $\lambda_1, \cdots, \lambda_m$.

We can link Minkowski result with Steiner formula and then mixed and intrinsic volumes.
If we take $m = 2$, $\lambda_1 = 1$, $\lambda_2 = \epsilon$ and $K_2 = \mathbb{B}^n$, then we have exactly the Steiner's Theorem and, by simple computations,

$$V_j(K) = \frac{\binom{n}{j}}{\omega_{n-j}} V(\underbrace{K, \cdots, K}_{j}, \underbrace{\mathbb{B}^n, \cdots, \mathbb{B}^n}_{n-j}),$$

for every $K \in \mathcal{K}^n$ and $j \in \{0, \cdots, n\}$.

Proposition 1.6.5. *For any* $K, L, K_1, \cdots, K_n \in \mathcal{K}^n$*,* $\alpha, \beta \geq 0$*,* $x_1, \cdots, x_n \in \mathbb{R}^n$*, we have the following properties:*

- *$V(K, \cdots, K) = V_n(K)$.*
- *V is symmetric.*
- *V is multilinear, i.e.*

$$V(\alpha K + \beta L, K_2, \cdots, K_n) = \alpha V(K, K_2, \cdots, K_n) + \beta V(L, K_2, \cdots, K_n),$$

  for all $K, L, K_2, \cdots, K_n \in \mathcal{K}^n$ and $\alpha, \beta \geq 0$.
- *V is translation invariant, i.e.*

$$V(K_1 + x_1, \cdots, K_n + x_n) = V(K_1, \cdots, K_n),$$

  for all $K_1, \cdots, K_n \in \mathcal{K}^n$ and $x_1, \cdots, x_n \in \mathbb{R}^n$.
- *V is invariant w.r.t. proper and improper rotations, i.e.*

$$V(gK_1, \cdots, gK_n) = V(K_1, \cdots, K_n), \quad \forall\, g \in O(n).$$

- *V is continuous w.r.t δ.*
- *V is positive and monotone in each argument.*

Corollary 1.6.6. *The intrinsic volumes are all δ-continuous, increasing, positive and rigid motion invariant.*

We observe also that intrinsic volumes satisfy the finite additivity condition, i.e. the valuation condition,

$$V_j(K) + V_j(H) = V_j(H \cup K) + V_j(H \cap K),$$

for all convex bodies K and H such that $H \cup K \in \mathcal{K}^n$. We will return exhaustively to this condition in Chapter 2, so right now, we focus on some other properties of intrinsic and mixed volumes.

Theorem 1.6.7 (Theorem 5.1.8 in [64])**.** *Let $K_1, \cdots, K_n$ be convex bodies. The following sentences are equivalent.*

- *$V(K_1, \cdots, K_n) > 0$,*
- *there exist segments $S_i \subseteq K_i$, $i \in \{1, \cdots, n\}$, with linearly independent directions.*

1.7 Surface area measure

The last tool of convex geometry we present in this chapter is the **surface area measure** of a convex body K, $S_{n-1}(K; \cdot)$. This is a Borel measure over the unit sphere $\mathbb{S}^{n-1}$ depending on K.

The main idea for the construction of this measure is the following: let $\omega \subseteq \mathbb{S}^{n-1}$ be a Borel subset of the sphere. We define $S_{n-1}(K; \omega)$ as follows

$$S_{n-1}(K; \omega) = \mathcal{H}^{n-1}(\{x \in \partial K |\ \nu_x \in \omega\}),$$

i.e. the $(n-1)$-Hausdorff measure of the set of all boundary points of K with unit outward normal vector belonging to ω (we recall that ν_x exists $\mathcal{H}^{n-1}$-a.e. on ∂K). We refer to [64], Chapters 4 and 5, for more details about the definition and well-posedness of $S_{n-1}(K; \cdot)$.

We can show some properties of surface area measure for particular convex bodies.

Lemma 1.7.1. *For every K, we have $S_{n-1}(K;\mathbb{S}^{n-1}) = 2V_{n-1}(K)$.*

If $\dim(K) \leq n-2$, then $S_{n-1}(K;\omega) = 0$, for every Borel set $\omega \subseteq \mathbb{S}^{n-1}$. Whereas if $\dim(K) = n-1$, let $u \in \mathbb{S}^{n-1}$ be a direction such that $K \subseteq u^{\perp}$, hence

$$S_{n-1}(K;\cdot) = V_{n-1}(K)(\delta_u(\cdot) + \delta_{-u}(\cdot)).$$

Example 1.7.2. *Let ω be a Borel subset of $\mathbb{S}^{n-1}$. We evaluate the surface area measure of ω for particular convex bodies.*

- $$S_{n-1}(\mathbb{B}^n;\omega) = \mathcal{H}^{n-1}(\omega).$$
- *Let $P \in \mathcal{P}^n$ be a polytope with $\dim(P) = n$, then we have*
  $$S_{n-1}(P;\omega) = \sum_{u\in\omega} V_{n-1}(F(P,u)),$$
  where $F(P,u)$ is the $(n-1)$-face of P such that the direction u is one of its outward unit normal vector and that belongs to ω.

Since $S_{n-1}(K;\cdot)$ is a measure, we can consider the integral over $\mathbb{S}^{n-1}$ w.r.t. it. In particular we have the following link between mixed volume and $S_{n-1}(K;\cdot)$.

Theorem 1.7.3 (Theorem 5.1.7 in [64])**.** *For every pair of convex bodies K and L, it holds*

$$V(L,K,\cdots,K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L(u)dS_{n-1}(K;u).$$

Remark 1.7.4. If $K = L$, then we have

$$V_n(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K(u)dS_{n-1}(K;u).$$

With this integral representation it is possible to prove the following properties.

Proposition 1.7.5. *Let $K \in \mathcal{K}^n$, then we have:*

- *for any $x \in \mathbb{R}^n$,*
  $$\int_{\mathbb{S}^{n-1}} (x,u)dS_{n-1}(K;u) = 0.$$
- *For any $x \in \mathbb{R}^n$,*
  $$S_{n-1}(K+x;\cdot) = S_{n-1}(K;\cdot).$$
- *For any rotation $g \in O(n)$ and Borel set $\omega \subseteq \mathbb{S}^{n-1}$,*
  $$S_{n-1}(gK;g\omega) = S_{n-1}(K;\omega).$$
- *For any sequence K_i of convex bodies converging to K, then $S_{n-1}(K_i;\cdot)$ converges weakly to $S_{n-1}(K;\cdot)$, i.e.*
  $$\int_{\mathbb{S}^{n-1}} f(u)dS_{n-1}(K_i;u) \to \int_{\mathbb{S}^{n-1}} f(u)dS_{n-1}(K;u),$$
  as i goes to $+\infty$, for any $f \in C(\mathbb{S}^{n-1})$.

The proof follows from the integral representation and properties of mixed volumes. The first statement of the previous proposition means that the integral of any linear function integrated w.r.t. any $S_{n-1}(K;\cdot)$ is null and we refer to this property saying that any surface area measure has centroid in 0.

We conclude with a fundamental result concerning surface area measure, the Minkowski's Existence Theorem, a characterization result of Borel measures, with certain additional hypothesis, in order to be the area measure of a convex body.

Theorem 1.7.6 (Theorem 8.2.2 in [64]). *Let μ be a Borel measure over the unit sphere, $\mathbb{S}^{n-1}$, such that*

$$\int_{\mathbb{S}^{n-1}} u d\mu(u) = 0$$

and $\mu(s) < \mu(\mathbb{S}^{n-1})$, for every great sub-sphere s of $\mathbb{S}^{n-1}$. Then there exists a unique (up to a translation) convex body K, with $\dim(K) = n$, such that $S_{n-1}(K;\cdot) = \mu(\cdot)$.

Chapter 2

Valuation Theory on $\mathcal{K}^n$

In this chapter we are going to present basic results concerning the theory of valuations on convex bodies.

We present the first definitions, examples and properties for valuations on $\mathcal{K}^n$. After that, we will focus on continuous and rigid motion invariant valuations, and present the Hadwiger Characterization Theorem. We will consider also continuous and only translation invariant valuations and we will describe the homogeneous decomposition of these valuations.

The last part is about the McMullen's conjecture and the Alesker's Density Theorem.

When not specified, we refer to [64] for the statements in this chapter. We refer also to [6], [7], [30], [31], [37], [52], [53] and [54] for some basic introduction to this theory.

2.1 Classes of valuations on convex bodies

Definition 2.1.1. *A set function $\mu\colon \mathcal{K}^n \to \mathbb{R}$ is a* ***valuation*** *if for every convex bodies K and H such that $K \cup H \in \mathcal{K}^n$, it holds*

$$\mu(K) + \mu(H) = \mu(K \cup H) + \mu(K \cap H). \tag{2.1}$$

We are interested also in some additional properties of the valuation, for instance we will frequently require that a valuation μ is continuous w.r.t. Hausdorff distance.
Moreover, we say that a valuation is invariant under the action of a group $G \leq GL(n)$ if and only if

$$\mu(gK) = \mu(K),$$

for every $g \in G$ and $K \in \mathcal{K}^n$ (where gK is the image of the action g through K).

In this thesis we will work with the following groups: Euclidean rigid motions, translations, $SL(n)$ and $GL(n)$.

The main goal of valuation theory is to establish characterization results for continuous (w.r.t. Hausdorff metric), invariant (w.r.t. translations and, possibly, a subgroup $G \leq GL(n)$) valuations on $\mathcal{K}^n$. We refer to [1], [2], [49], [52], [51] for some of the most relevant characterization results for valuations on $\mathcal{K}^n$.

Definition 2.1.2. *We denote by* $\mathrm{Val}(\mathcal{K}^n)$ *the set of all continuous and translation invariant valuations on $\mathcal{K}^n$.*

Another property that valuation may satisfy is monotonicity: we say that μ is monotone increasing (resp. decreasing) if

$$\forall\, K, H \in \mathcal{K}^n : \; K \subseteq H \;\Rightarrow\; \mu(K) \leq \mu(H) \text{ (resp. } \mu(K) \geq \mu(H)).$$

Theorem 2.1.3. *The intrinsic volumes $V_0, \cdots, V_n$ are all continuous, monotone increasing and rigid motion invariant valuations.*

Proof. From Corollary 1.6.6 we have all the properties except the valuation property.

This one is a consequence of Steiner formula (1.1) and Proposition 1.1.11. Indeed for H and K convex bodies such that $H \cup K \in \mathcal{K}^n$ one has

$$V_n((H \cup K)_\epsilon) = \sum_{i=0}^{n} V_i(H \cup K)\omega_{n-i}\epsilon^{n-i}.$$

By Proposition 1.1.11, it holds $V_n((H \cup K)_\epsilon) = V_n((H \cup K) + \epsilon\mathbb{B}^n) = V_n((H)_\epsilon + (K)_\epsilon)$ and since V_n is the Lebesgue measure, we have

$$V_n((H \cup K)_\epsilon) = V_n((H)_\epsilon) + V_n((K)_\epsilon) - V_n((H \cap K)_\epsilon).$$

Now we apply again Steiner formula to each terms of the right-hand side of the previous equation, and we obtain

$$\sum_{i=0}^{n} V_i(H \cup K)\omega_{n-i}\epsilon^{n-i} = \sum_{i=0}^{n} V_i(H)\omega_{n-i}\epsilon^{n-i} + \sum_{i=0}^{n} V_i(K)\omega_{n-i}\epsilon^{n-i} - \sum_{i=0}^{n} V_i(H \cap K)\omega_{n-i}\epsilon^{n-i}.$$

Comparing coefficients, we have the valuation property for every i,

$$V_i(H \cup K) = V_i(H) + V_i(K) - V_i(H \cap K).$$

□

Definition 2.1.4. *Let μ be a valuation on $\mathcal{K}^n$ and $i \in \{0, \cdots, n\}$. We say that μ is i-homogeneous if*

$$\mu(\lambda K) = \lambda^i \mu(K),$$

for every $\lambda > 0$ and $K \in \mathcal{K}^n$.

We denote by $\mathrm{Val}_i(\mathcal{K}^n)$ the subset of $\mathrm{Val}(\mathcal{K}^n)$ such that the valuation is also i-homogeneous.

Proposition 2.1.5. *The i-th intrinsic volume is i-homogeneous.*

Corollary 2.1.6. *For every $i \in \{0, \cdots, n\}$, it holds*

$$V_i \in \mathrm{Val}_i(\mathcal{K}^n).$$

We are able to show some examples of valuations with different types of properties.

Example 2.1.7.
- *We can define valuations through mixed volumes.*
  Let $C = (C_1, \cdots, C_i) \in (\mathcal{K}^n)^i$ *be a finite sequence of convex bodies, with* $i \in \{1, \cdots, n-1\}$. *Then we define*
  $$V_C \colon \mathcal{K}^n \to \mathbb{R},\ V_C(K) = V(\underbrace{K, \cdots, K}_{n-i \text{ times}}, C_1, \cdots, C_i).$$
  Then V_C *is a continuous, monotone increasing, translation invariant and* $(n-i)$*-homogeneous valuation, but in general not invariant w.r.t. all rigid motions.*

- *Let* μ *be any valuation on* $\mathcal{K}^n$ *and* C *a fixed convex body. Then the function* ϕ *defined as*
  $$\phi(K) = \mu(K + C),\ \forall\ K \in \mathcal{K}^n,$$
  is a valuation. Moreover if μ *is continuous, translation invariant and monotone, then also* ϕ *has the same properties.*

- *The* ***Affine area measure*** *defined by*
  $$\Omega(K) = \int_{\partial K} G_K(x)^{\frac{1}{n+1}} dx,$$
  where $G_K(x)$ *is the Gauss curvature at* x*, is a valuation, but not continuous, only upper semi-continuous w.r.t.* δ*-convergence.*

  Moreover Ω *is not only rigid motion, but also equi-affine invariant, i.e. invariant under the action of elements of the group* $SL(n)$*. See [40] and [44] for more details.*

- *Fix* $x_0 \in \mathbb{R}^n$*, then the Dirac mass on* x_0*,*
  $$\mu(K) = \delta_{x_0}(K) = \begin{cases} 1, & \textit{if } x_0 \in K, \\ 0, & \textit{otherwise}, \end{cases}$$
  is a valuation on $\mathcal{K}^n$*, but not invariant (either w.r.t. to rigid motions or translations). In general, any* σ*-finite Borel measure* m *on* $\mathbb{R}^n$ *defines a valuation on* $\mathcal{K}^n$*,* $\mu(K) = m(K)$*.*

If we focus now on $\mathrm{Val}(\mathcal{K}^n)$, we have the following decomposition result established by McMullen.

Theorem 2.1.8 (**McMullen's Decomposition Theorem**; [64], [52])**.** *Let* $\mu \colon \mathcal{P}^n \to \mathbb{R}$ *be a translation invariant valuation. For every* $i \in \{0, \cdots, n\}$*, there exists a unique valuation* μ_i *translation invariant and rational* i*-homogeneous, i.e.*
$$\mu_i(\lambda P) = \lambda^i \mu_i(P)$$
for every non-negative rational number $\mathbb{Q} \ni \lambda \geq 0$ *and* $P \in \mathcal{P}^n$*, such that*
$$\mu(P) = \sum_{i=0}^{n} \mu_i(P) \quad \forall\ P \in \mathcal{P}^n.$$

Moreover if μ *is defined on* $\mathcal{K}^n$ *and it is also continuous, i.e. it belongs to* $\mathrm{Val}(\mathcal{K}^n)$*, then for every*

$i \in \{0, \cdots, n\}$ *there exists a unique valuation* $\mu_i \in \mathrm{Val}_i(\mathcal{K}^n)$ *such that*

$$\mu(\lambda K) = \sum_{i=0}^{n} \lambda^i \mu_i(K),$$

for every $K \in \mathcal{K}^n$ *and* $\lambda \geq 0$. *This is also equivalent to the following decomposition for* μ*,*

$$\mu = \mu_0 + \cdots + \mu_n.$$

Remark 2.1.9. The previous statement is equivalent to

$$\mathrm{Val}(\mathcal{K}^n) = \bigoplus_{i=0}^{n} \mathrm{Val}_i(\mathcal{K}^n).$$

We have an immediate application of the McMullen's Theorem, concerning the behaviour of a homogeneous valuation on convex bodies with lower dimension.

Corollary 2.1.10. *Let* μ *be an* i*-homogeneous, translation invariant and continuous valuation. If* $K \in \mathcal{K}^n$ *has dimension strictly lower than i, then* $\mu(K) = 0$.

We present two other classes of valuations, even and odd.

Definition 2.1.11. *A valuation* $\mu \colon \mathcal{K}^n \to \mathbb{R}$ *is said to be even if*

$$\mu(K) = \mu(-K),$$

for every $K \in \mathcal{K}^n$. *Moreover,* μ *is odd if*

$$\mu(-K) = -\mu(K),$$

for every $K \in \mathcal{K}^n$.

We denote by $\mathrm{Val}^+(\mathcal{K}^n)$ (resp. $\mathrm{Val}^-(\mathcal{K}^n)$) the subset of $\mathrm{Val}(\mathcal{K}^n)$ which elements are all even (resp. odd). In the same way, we define $\mathrm{Val}_i^+(\mathcal{K}^n)$, $\mathrm{Val}_i^-(\mathcal{K}^n) \subseteq \mathrm{Val}_i(\mathcal{K}^n) \subseteq \mathrm{Val}(\mathcal{K}^n)$.

Theorem 2.1.12. *Every valuation* μ *can uniquely be written as the sum of an even and odd valuation. In particular we have*

$$\mathrm{Val}(\mathcal{K}^n) = \bigoplus_{i=0}^{n} \mathrm{Val}_i^+(\mathcal{K}^n) \oplus \mathrm{Val}_i^-(\mathcal{K}^n).$$

Note that for a valuation μ, we just take

$$\mu^+(K) = \frac{\mu(K) + \mu(-K)}{2}$$

and

$$\mu^-(K) = \frac{\mu(K) - \mu(-K)}{2}.$$

We present, at the end of this section, a polynomiality result for homogeneous valuation on $\mathcal{K}^n$. Let us fix $\mu \in \mathrm{Val}_m(\mathcal{K}^n)$, with $m \in \{1, \cdots, n\}$. The aim is to find a formula for the value of μ acting on any generic Minkowski combination of convex bodies, $\mu(\lambda_1 K_1 + \cdots \lambda_k K_k)$, where $\lambda_i \geq 0$, $K_i \in \mathcal{K}^n$, for $i \in \{1, \cdots, k\}$.

We have the following result.

Theorem 2.1.13 (Theorem 6.3.6 in [64])**.** *Let* $\mu \in \mathrm{Val}_m(\mathcal{K}^n)$*, with* $m \in \{1, \cdots, n\}$*. There exists a continuous, symmetric map* $\overline{\mu}\colon (\mathcal{K}^n)^m \to \mathbb{R}$*, which is translation invariant and Minkowski additive in each variable, such that*

$$\mu(\lambda_1 K_1 + \cdots + \lambda_k K_k) = \sum_{r_1, \cdots, r_k = 0}^{m} \binom{m}{r_1 \cdots r_k} \lambda_1^{r_1} \cdots \lambda_k^{r_k}\, \overline{\mu}(K_1[r_1], \cdots, K_k[r_k]), \tag{2.2}$$

holds for every $K_1, \cdots, K_k \in \mathcal{K}^n$ *and* $\lambda_1, \cdots, \lambda_k \geq 0$ *and where* $K_i[r_i]$ *means that we count* K_i *in* $\overline{\mu}$*,* r_i *times in* $\overline{\mu}$*. Moreover, if we fix* $r \in \{0, \cdots, m\}$ *and* $K_1, \cdots, K_{m-r} \in \mathcal{K}^n$*, then the map*

$$\mathcal{K}^n \ni K \mapsto \overline{\mu}(K[r], K_1, \cdots, K_{m-r})$$

is a valuation that belongs to $\mathrm{Val}_r(\mathcal{K}^n)$*.*

Remark 2.1.14. The previous statement tells us that $\mu(\lambda_1 K_1 + \cdots + \lambda_k K_k)$ is a homogeneous polynomial of degree m w.r.t. $\lambda_1, \cdots, \lambda_k$.

2.2 Characterization of continuous and rigid motion invariant valuations

The main result of this section is a celebrated result, proved by Hadwiger in [31] at the end of 50's; we notice that Klain in [33] at the end of XX century gave also a simpler proof.

Theorem 2.2.1 (Hadwiger's Theorem). *A function* $\mu\colon \mathcal{K}^n \to \mathbb{R}$ *is a continuous and rigid motion invariant valuation if and only if for all* $i \in \{0, \cdots, n\}$*, there exists a real constant* $c_i \in \mathbb{R}$ *such that*

$$\mu(K) = \sum_{i=0}^{n} c_i V_i(K),$$

for all convex body K*.*

Remark 2.2.2. The Hadwiger's Theorem says that all possible continuous and rigid motion invariant valuations are real combinations of intrinsic volumes, i.e. $V_0, \cdots, V_n$ form a basis for this vector space and its dimension is exactly $n + 1$.

There exists also a "monotone" counterpart of Hadwiger's Theorem, [31].

Proposition 2.2.3 (Hadwiger's Theorem for monotone valuations)**.** *A function* $\mu\colon \mathcal{K}^n \to \mathbb{R}$ *is a monotone increasing (resp. decreasing) and rigid motion invariant valuation if and only if for all* $i \in \{0, \cdots, n\}$*, there exists a positive (resp. negative) real constant* $c_i \geq 0$ (≤ 0) *such that*

$$\mu(K) = \sum_{i=0}^{n} c_i V_i(K),$$

for all convex body K*.*

We will not discuss the proof of Hadwiger's Theorem, we just want to introduce now another class of valuations, **simple valuations**, that it is fundamental for the proof given by Klain and also for our works.

Definition 2.2.4. *A valuation μ defined on $\mathcal{K}^n$ is said to be simple if*

$$\mu(K) = 0, \; \forall\, K \in \mathcal{K}^n : \; \dim(K) = 0.$$

An obvious example of simple valuation is the Lebesgue measure. Actually, Klain proved that V_n is the "only" continuous, translation invariant and simple valuation on $\mathcal{K}^n$.

Theorem 2.2.5 (Volume Theorem). *Let $\mu\colon \mathcal{K}^n \to \mathbb{R}$ be a continuous and rigid motion invariant valuation. μ is also simple if and only if there exists a real constant c_n such that*

$$\mu(K) = c_n V_n(K),$$

for all $K \in \mathcal{K}^n$.

The continuity condition is necessary for the statement of the theorem. We have the following example for $n = 1$. (See [37], pag. 39)

Example 2.2.6. *We consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$, $\mathbb{R}_{\mathbb{Q}}$. Then we take $f\colon \mathbb{R}_{\mathbb{Q}} \to \mathbb{Q}$ an element of the dual, $\mathbb{R}^*_{\mathbb{Q}}$, of this vector space. Then we define*

$$\mu\colon \mathcal{K}^1 \to \mathbb{R}, \; \mu([a, b]) = f(b - a),$$

for every $-\infty < a \leq b < +\infty$.

μ is clearly a simple and rigid motion invariant (in this case it means invariant w.r.t. translations and central symmetries) valuation, but not continuous. Moreover it can not be a multiple of the Lebesgue measure, just because its image is a subset of $\mathbb{Q}$.

If we remove the rigid motion invariance assumption, taking only translations, then we have a modified version of Volume Theorem: there exists a constant $c \in \mathbb{R}$ such that

$$\mu(K) + \mu(-K) = c\mathrm{V}_n(K),$$

for all $K \in \mathcal{K}^n$. See [37], Theorem 8.1.5. An immediate consequence is the following result.

Theorem 2.2.7. *Let $\mu \in \mathrm{Val}^+(\mathcal{K}^n)$ be simple. Then there exists a constant $c_n \in \mathbb{R}$ such that*

$$\mu(K) = c_n \mathrm{V}_n(K), \quad \forall K \in \mathcal{K}^n.$$

The theorem says that we can omit the rotation invariance condition and have the same statement of Volume Theorem if we add the eveness condition.

We proved in [21] a slight variation of this result. We consider only the continuity w.r.t. decreasing sequence, nevertheless the statement is the same.

Theorem 2.2.8. *Let μ be a simple, even and translation invariant valuation on $\mathcal{K}^n$, continuous w.r.t. decreasing sequences. Then there exists a constant $c_n \in \mathbb{R}$ such that*

$$\mu(K) = c_n \mathrm{V}_n(K), \quad \forall K \in \mathcal{K}^n. \tag{2.3}$$

Proof. We follow the same idea of Theorems 8.1.4 in [37], 6.4.10 in [64] and in the works made by Klain in [33] and [36].
All the steps are the same, the only difference is the proof of (2.3) for zonoids. Since the valuation is continuous only w.r.t. decreasing sequences, we use Proposition 1.5.4 to pass from zonotopes to zonoids and then we have the thesis in the usual way. □

2.3 Characterization and density results concerning $\mathrm{Val}(\mathcal{K}^n)$

We focus now on some characterization theorems for continuous valuations, invariant w.r.t. translations. Note that, as we have the homogeneous decomposition result, Theorem 2.1.8, we may restrict our attention to homogeneous valuations.

We start with a 0-homogeneous valuation, μ_0. This means, by definition, that for every $\lambda \geq 0$ and convex body K, we have

$$\mu_0(\lambda K) = \mu_0(K).$$

In particular, if we choose $\lambda = 0$, then we get $\mu_0(\{0\}) = \mu_0(K)$, for every choice of K. Hence μ_0 is constant on $\mathcal{K}^n$ or, in other words, it is a multiple of the Euler-Poicaré characteristic function. We have the following proposition.

Proposition 2.3.1. $\mathrm{Val}_0(\mathcal{K}^n)$ *is a vector space of dimension* 1, *spanned by the Euler-Poincarè characteristic* V_0.

If we consider now the other limit case, i.e. n-homogeneous valuation, then we have a similar result to the Volume Theorem, proved by Hadwiger.

Theorem 2.3.2 (Hadwiger's Theorem for n-homogeneous valuations, [31])**.** *Let* $\mu \in \mathrm{Val}(\mathcal{K}^n)$*;* μ *is* n*-homogeneous if and only if it is, up to a multiplicative constant, the Lebesgue measure. This means that there exists* $c_n \in \mathbb{R}$ *such that*

$$\mu(K) = c_n V_n(K),$$

for every $K \in \mathcal{K}^n$.

The next theorem we show is the last one with a complete characterization of homogeneous valuations, this is the $(n-1)$-homogeneous case, due to McMullen.

Theorem 2.3.3 (McMullen's Theorem, [52])**.** *A valuation* μ *on* $\mathcal{K}^n$ *belongs to* $\mathrm{Val}_{n-1}(\mathcal{K}^n)$ *if and only if there exists a continuous function* $f : \mathbb{S}^{n-1} \to \mathbb{R}$ *such that*

$$\mu(K) = \int_{\mathbb{S}^{n-1}} f(u) dS_{n-1}(K; u),$$

for every $K \in \mathcal{K}^n$.

Moreover, f *is uniquely determined up to adding the restriction to* $\mathbb{S}^{n-1}$ *of a linear function.*

For the others homogeneous cases we do not have a characterization theorem. Moreover, also in the $(n-1)$-case we do not have anymore a vector space with finite dimension. Anyway we can say more. In [52] McMullen made a conjecture concerning a density result for $\mathrm{Val}(\mathcal{K}^n)$. We can summarize it in this way.

McMullen's conjecture. The mixed volume spans a dense subspace of $\mathrm{Val}(\mathcal{K}^n)$.

This means that the set of all linear and finite combinations of valuations of the form

$$K \mapsto V_C(K) = V(\underbrace{K, \cdots, K}_{i \text{ times}}, C_1, \cdots, C_{n-i}),$$

with all possible choices of $C = (C_1, \cdots, C_{n-i}) \in (\mathcal{K}^n)^{n-i}$ and $i \in \{0, \cdots, n\}$, is dense in $\mathrm{Val}(\mathcal{K}^n)$.

Remark 2.3.4.
- Density w.r.t. which convergence? We use the uniform convergence on compact sets of $\mathcal{K}^n$, i.e. we see $\mathrm{Val}(\mathcal{K}^n)$ as a Fréchet space with a countable family of seminorms. For $N \in \mathbb{N}$ we set
$$||\mu||_N = \sup\{|\mu(K)| \mid K \subseteq \mathbb{B}_N\},$$
where $\mathbb{B}_N$ is the ball centered in the origin and radius N.

- We can also fix the homogeneity index and prove the density for valuations of the form
$$K \mapsto V(\underbrace{K, \cdots, K}_{i \text{ times}}, C_1, \cdots, C_{n-i})$$
in $\mathrm{Val}_i(\mathcal{K}^n)$. Proving this statement for every i is equivalent to the conjecture for $\mathrm{Val}(\mathcal{K}^n)$ as a consequence of the McMullen's Decomposition Theorem.

- By Theorem 2.1.12 we can assume also that the valuations are all even (or resp. odd).

The conjecture was proved with a positive answer by Semyon Alesker in [5]. He used tecniques from abstract algebra, so we just present briefly the scheme of his proof.

The Alesker's main result about the McMullen's conjecture is called **Irreducibility Theorem** concerning valuations (continuous, i-homogeneous and even) that are invariant under the action of a generic Lie group G.

Alesker found a condition to establish the density of any G-invariant subspace in $\mathrm{Val}_i^+(\mathcal{K}^n)$ using representation theory.

Definition 2.3.5 (Representation of a group)**.** *A **representation** of a Lie group G on $\mathrm{Val}_i(\mathcal{K}^n)$ is a continuous and linear map*
$$G \times \mathrm{Val}_i^+(\mathcal{K}^n) \longrightarrow \mathrm{Val}_i^+(\mathcal{K}^n),\ (g, \mu) \longmapsto g \cdot \mu,$$
where $g \cdot \mu \in \mathrm{Val}_i^+(\mathcal{K}^n)$ is defined by
$$g \cdot \mu(K) = \mu(g^{-1}K),$$
for every $K \in \mathcal{K}^n$.

We say, moreover, that a linear subspace $Y \subseteq \mathrm{Val}_i^+(\mathcal{K}^n)$ is invariant under the action of G or G-invariant, if for every $g \in G$ and $\mu \in \mathrm{Val}_i^+(\mathcal{K}^n)$, we have $g \cdot \mu \in Y$.

Definition 2.3.6 (Irreducible representation)**.** *A representation is called **irreducible** if there are no other G-invariant and dense subspaces in $\mathrm{Val}_i^+(\mathcal{K}^n)$ except $\{\underline{0}\}$ and $\mathrm{Val}_i^+(\mathcal{K}^n)$.*

The Lie group that Alesker studied is $GL(n)$, the group of all invertible linear transformations of $\mathbb{R}^n$.

Theorem 2.3.7 (**Alekser's Irreducibility Theorem**)**.** *The representation of the group $GL(n)$ on $\mathrm{Val}_i^+(\mathcal{K}^n)$ is irreducible.*

A consequence of Alesker's Theorem is the positive answer of McMullen's conjecture.

Corollary 2.3.8. *The set spanned by the mixed volumes is $GL(n)$-invariant and not empty, hence it is dense in $\mathrm{Val}_i^+(\mathcal{K}^n)$.*

2.4 Klain map

We focus in this last section on the Klain map, a function defined on Val($\mathcal{K}^n$) that we can use to reconstruct a valuation. We start with some preliminaries.

Definition 2.4.1. ***(Grassmanian space)*** *Let $0 \le i \le n$, we define the i-Grassmanian as*

$$Gr(n,i) = \{F \; vector \; subspace \; of \; \mathbb{R}^n | \; \dim(F) = i\}.$$

Example 2.4.2.
- *If $i = n$, then $Gr(n,n) = \{\mathbb{R}^n\}$.*
- *If $i = 0$, then $Gr(n,0) = \{\underline{0}\}$.*
- *If $i = 1$, then $Gr(n,i) = \{$ (real) lines through the origin $\}$.*

Proposition 2.4.3. *$Gr(n,i)$ is a complete metric space w.r.t. the distance*

$$d(E,F) = \delta(E \cap \mathbb{B}^n, F \cap \mathbb{B}^n),$$

for all E and $F \in Gr(n,i)$.

To define the Klain map, we fix $i \in \{1, \cdots, n-1\}$ and we consider a valuation $\mu \in$ Val$_i(\mathcal{K}^n)$.

For any $E \in Gr(n,i)$, we set

$$\mathcal{K}(E) = \{K \in \mathcal{K}^n | \; K \subseteq E\}.$$

It is easy to prove that $\overline{\mu} \equiv \mu|_E$ is a valuation in Val$_i(\mathcal{K}(E))$.

By Hadwiger's Theorem 2.3.2, $\overline{\mu}$ is, up to a constant, the i-Lebesgue measure. There exists $c_E \in \mathbb{R}$ such that c_E depends only on E and

$$\overline{\mu}(K) = c_E V_i(K), \tag{2.4}$$

for every $K \subseteq E$ convex body. We can define the Klain function associated to μ as

$$KL_\mu \colon Gr(n,i) \to \mathbb{R}, \; KL_\mu(E) = c_E,$$

for every $E \in Gr(n,i)$.

Remark 2.4.4. We observe that KL_μ is well-defined since c_E in (2.4) is uniquely determined. Moreover it is continuous in $Gr(n,i)$ for every valuation μ.

The **Klain map** is defined as

$$KL \colon \text{Val}_i(\mathcal{K}^n) \to C(Gr(n,i), \mathbb{R})$$

such that for every valuation μ, we have $Kl_\mu \in C(Gr(n,i), \mathbb{R})$.

About the properties of the Klain map, we recall the following result that we established also for valuations on quasi-concave functions, as we will see in the following part of the thesis.

Theorem 2.4.5 ([64])**.** *Every valuation $\mu \in$ Val$_i^+(\mathcal{K}^n)$ is uniquely determined by its Klain function.*

The previuos statement tells us that the Klain map KL restricted on even valuation, Val$_i^+(\mathcal{K}^n)$, is injective and it determines totally μ.

Chapter 3

The space of quasi-concave functions

We start now to study the functional case, the subject of this thesis. We will study in this chapter quasi-concave functions defined on $\mathbb{R}^n$. In the research area of analytic aspects of convex geometry there are various works about this function space. We refer to [10], [13], [19], [56] and [57] for some literature about this theory. In these works the authors studied functional counterpart of convex geometry tools and statements like mixed and intrinsic volumes and functional version of Isoperimetric and Blaschke-Santaló Inequalities. We refer to the survey [55] for more details about functional extensions of the Theory of Convex Bodies.

Quasi-concave functions have applications not only on analytic aspects of convexity, but in many other theories of mathematics, see, for instance, [66] for applications in mathematical economics.

In this chapter we will prove some basic properties and present examples for this functional set and we will focus, in the last section, on the convergence, since we want also to study later the continuity of valuations.
We refer for this chapter, mainly, to the works [20] and [21].

3.1 Quasi-concave functions

Definition 3.1.1. *A function $f\colon \mathbb{R}^n \to \mathbb{R}_+$ is said to be quasi-concave if for every $t > 0$ the super-level set of f at t is either a convex body or the empty set,*

$$L_t(f) = \{x \in \mathbb{R}^n |\ f(x) \geq t\} \in \mathcal{K}^n \cup \{\varnothing\}.$$

We denote by $\mathcal{QC}(\mathbb{R}^n)$ the set of all quasi-concave functions. Note that here we adopt a notation similar to [56].

We have the following examples.

Example 3.1.2. • *The characteristic function of a convex body is quasi-concave, since we have for $s > 0$ and $K \in \mathcal{K}^n$*

$$s\chi_K(x) = \begin{cases} s & \text{if } x \in K, \\ 0 & \text{if } x \notin K, \end{cases}$$

and then

$$L_t(s\chi_K) = \begin{cases} K & \text{if } 0 < t \leq s, \\ \varnothing & \text{if } t > s. \end{cases}$$

- *The Gaussian function*
$$f(x) = a\exp(-\frac{||x-b||^2}{2c^2}),$$
defined on $\mathbb{R}^n$, for every choice of $a > 0$, $0 \neq c \in \mathbb{R}$ and $b \in \mathbb{R}^n$, is a quasi-concave function. Indeed the super-level sets are equal to
$$L_t(f) = \begin{cases} \mathbb{B}(b, -2c^2 \ln(\frac{t}{a})) & \text{if } 0 < t \leq a, \\ \varnothing & \text{if } t > a. \end{cases}$$

Proposition 3.1.3. *If $f \in \mathcal{QC}(\mathbb{R}^n)$, then the following statements hold:*

- $\lim_{||x|| \to +\infty} f(x) = 0.$
- *f is upper semi-continuous.*
- $\sup_{\mathbb{R}^n} f = \max_{\mathbb{R}^n} f < +\infty.$

Proof. To prove the first property, let $\epsilon > 0$. Hence $L_\epsilon(f)$ is compact, so there exists $R > 0$ such that $L_\epsilon(f) \subseteq \mathbb{B}_R$. This is exactly the definition of $\lim_{||x|| \to +\infty} f(x) = 0$.

The upper semi-continuity follows from the compactness of super-level sets.
Suppose now f is not identically equal to 0 and $\{x_i\}_{i\in\mathbb{N}} \subseteq \mathbb{R}^n$ is a maximizing sequence, i.e. $\lim_{i\to+\infty} f(x_i) = \sup_{\mathbb{R}^n} f$. Then since f decays to zero at infinity, the sequence x_i is compact. Hence we may assume that x_i converges to $x_0 \in \mathbb{R}^n$ and by upper semi-continuity it holds
$$f(x_0) \geq \limsup_{i\to+\infty} f(x_i) = \sup_{\mathbb{R}^n} f = \max_{\mathbb{R}^n} f.$$
□

Notation. We denote the maximum of $f \in \mathcal{QC}(\mathbb{R}^n)$ by $M(f)$. In the following when we will consider super-level sets $L_t(f)$ we will take $0 < t \leq M(f)$, because for $t > M(f)$ we have always $L_t(f) = \varnothing$.

Remark 3.1.4. For a quasi-concave function $f \in \mathcal{QC}(\mathbb{R}^n)$, the support of f is convex. We recall that
$$\text{supp}(f) = \text{cl}(\{x \in \mathbb{R}^n | \ f(x) > 0\}),$$
and we can rewrite it like
$$\text{supp}(f) = \text{cl}\Bigg(\bigcup_{k=1}^{+\infty} \Big\{x \in \mathbb{R}^n | \ f(x) \geq \frac{1}{k}\Big\}\Bigg).$$
The sets $\{x \in \mathbb{R}^n | \ f(x) \geq \frac{1}{k}\}$ form an increasing sequence of convex bodies and their union is convex. Of course this is not always a convex body, for instance take $f(x) = \exp(-\frac{||x||^2}{2})$. If f is of compact support, then $\text{supp}(f) \in \mathcal{K}^n$.

Proposition 3.1.5. *Let $f\colon \mathbb{R}^n \to \mathbb{R}_+$. Then f belongs to $\mathcal{QC}(\mathbb{R}^n)$ if and only if we have*

- $\lim_{||x|| \to +\infty} f(x) = 0.$
- *f is upper semi-continuous.*

- $f((1-\lambda)x+\lambda y) \geq \min\{f(x), f(y)\}$, *for every* $x, y \in \mathbb{R}^n$ *and* $\lambda \in (0,1)$.

Proof. If $f \in \mathcal{QC}(\mathbb{R}^n)$ we have already proved the first two properties. Now we take $x, y \in \mathbb{R}^n$ and $\lambda \in (0,1)$ and we suppose $f(x) > f(y) > 0$. It follows $x, y \in L_{f(y)}(f)$ convex set, hence $(1-\lambda)x+\lambda y \in L_{f(y)}(f)$. This means $f((1-\lambda)x+\lambda y) \geq f(y) = \min\{f(x), f(y)\}$.

Vice versa, let $t > 0$ such that $L_t(f)$ is not empty, we want to prove it is a convex body.

- If we suppose that $L_t(f)$ is not bounded, then for every $i \in \mathbb{N}$, there exists $x_i \in L_t(f)$ such that $||x_i|| \geq i$. This means that $\lim_{i\to+\infty} ||x_i|| = +\infty$. Then by the first property it holds

  $$\lim_{i\to+\infty} f(x_i) = 0.$$

  Hence, we have a contradiction because $f(x_i) \geq t$, for all $i \in \mathbb{N}$.

- By the upper semi-continuity $L_t(f)$ is closed.

- For any $x, y \in L_t(f)$ and $\lambda \in (0,1)$, we have $f((1-\lambda x)+\lambda y) \geq \min\{f(x), f(y)\} \geq t$, hence $L_t(f)$ is convex.

□

We have seen that a quasi-concave function is upper semi-continuous. Actually we can say more about its regularity. By Proposition 3.1.3, we have $\mathcal{QC}(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n)$ then $\mathcal{QC}(\mathbb{R}^n) \subseteq L^1_{loc}(\mathbb{R}^n)$.

In general, it is not true that a quasi-concave function f belongs to $L^1(\mathbb{R}^n)$. For instance the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } 0 \leq x \leq 1, \\ \frac{1}{x} & \text{if } x \geq 1. \end{cases}$$

is not an L^1-function, but it belongs to $\mathcal{QC}(\mathbb{R})$, in fact we have

$$L_t(f) = \left[0, \frac{1}{t}\right],$$

if $0 < t \leq 1$ and it is empty otherwise.

We conclude this section with a technical lemma that we will use later for the study of convergence of quasi-concave functions. If we fix a function $f \in \mathcal{QC}(\mathbb{R}^n)$ and we consider $L_t(f)$ for $0 < t \leq M(f)$, we want to know when it is possible to link this super-level set with $\{x \in \mathbb{R}^n |\ f(x) > t\}$. Usually it holds that $L_t(f)$ is the closure of such set, but it is not always true, not for all t. For instance, let $f = \chi_K$ be the characteristic function of $K \in \mathcal{K}^n$. For every $t \neq 1$ the statement is true, either when empty or K. But we have problems with $t = 1$, indeed $L_1(f) = K$, meanwhile

$$\{x \in \mathbb{R}^n |\ f(x) > 1\} = \varnothing.$$

We prove the following result. (Lemma 3.8 in [20])

Lemma 3.1.6. *Let $f \in \mathcal{QC}(\mathbb{R}^n)$ be a quasi-concave function. For all $t > 0$ except for at most countably many values we have*

$$L_t(f) = \mathrm{cl}(\{x \in \mathbb{R}^n |\ f(x) > t\}).$$

Proof. We fix $t > 0$ and we define

$$\Omega_t(f) = \{x \in \mathbb{R}^n : \ f(x) > t\} \text{ and } H_t(f) = \mathrm{cl}(\Omega_t(f)).$$

$\Omega_t(f)$ is convex for all t, since one has

$$\Omega_t(f) = \bigcup_{k\in\mathbb{N}} L_{t+\frac{1}{k}}(f).$$

Hence it follows that $H_t(f)$ is convex and $H_t(f) \subseteq L_t(f)$.
Now we define $D_t(f) = L_t(f) \setminus H_t(f)$. We want to prove that $D_t(f) \neq \varnothing$ for at most countably many values of t. We first note that if K and L are convex bodies with $\mathrm{int}(L) \neq \varnothing$, $K \subset L$ and $L \setminus K \neq \varnothing$, then $\mathrm{int}(L \setminus K) \neq \varnothing$, therefore

$$D_t(f) \neq \varnothing \ \Leftrightarrow \ \mathrm{V}_n(D_t(f)) > 0. \tag{3.1}$$

It comes from

$$D_t(f) = L_t(f) \setminus H_t(f) \subseteq L_t(f) \setminus \Omega_t(f) = \{x \in \mathbb{R}^n : \ f(x) = t\}$$

that

$$t_1 \neq t_2 \ \Rightarrow \ D_{t_1}(f) \cap D_{t_2}(f) = \varnothing. \tag{3.2}$$

For the rest of the proof we proceed by induction on the dimension, n.

For $n = 1$, we observe that if f is identically zero, then the lemma is trivially true.
If $\mathrm{supp}(f) = \{x_0\}$ and $f(x_0) = t_0$, then we have

$$L_t(f) = \{x_0\} = \mathrm{cl}(\Omega_t(f)), \ \forall\, 0 < t \neq t_0,$$

and the lemma is true.

We suppose next that $\mathrm{int}(\mathrm{supp}(f)) \neq \varnothing$; let $t_0 > 0$ be a number such that $\dim(L_t(f)) = 1$, for all $t \in (0, t_0)$ and $\dim(L_t(f)) = 0$, for all $t > t_0$. Moreover let $t_1 = \max_{\mathbb{R}} f \geq t_0$, we can observe that

$$L_t(f) = \mathrm{cl}(\Omega_t(f)) = \varnothing, \ \forall\, t > t_1 \text{ and } L_t(f) = \mathrm{cl}(\Omega_t(f)), \ \forall\, t \in (t_0, t_1).$$

Next we consider $t \in (0, t_0)$. Let us fix $\epsilon > 0$ and let K be a compact set in $\mathbb{R}$ such that $K \supseteq L_t(f)$ for every $t \geq \epsilon$. We define, for $i \in \mathbb{N}$,

$$T_i^\epsilon = \{\epsilon \leq t < t_0 : \ \mathrm{V}_1(D_t(f)) \geq \frac{1}{i}\}.$$

As $D_t(f) \subseteq K$ for all $t \geq \epsilon$ and taking (3.2) into account we obtain that T_i^ϵ is finite.

Hence the set

$$T^\epsilon = \bigcup_{i\in\mathbb{N}} T_i^\epsilon$$

is countable for every $\epsilon > 0$. By (3.1)

$$\{t \geq \epsilon : \ D_t(f) \neq \varnothing\}$$

is countable for every $\epsilon > 0$, so that

$$\{t > 0 : \ D_t \neq \varnothing\}$$

is also countable. That completes the proof for $n = 1$.

Assume now the claim of the lemma true up to dimension $n-1$, and let us prove it in dimension n. If $\dim(\mathrm{supp}(f))$ is strictly smaller than n, then, since $\mathrm{supp}(f)$ is convex, there exists an affine subspace H of $\mathbb{R}^n$, with $\dim(H) = n-1$ and containing the support of f.

In this case the assert of the lemma follows applying the induction assumption to the restriction of f to H. Next, we suppose that there exists $t_0 > 0$ such that

$$\dim(L_t(f)) = n,\ \forall\, t \in (0, t_0) \text{ and } \dim(L_t(f)) < n,\ \forall\, t > t_0.$$

By the same argument used in the one-dimensional case we can prove that

$$\{t \in (0, t_0) :\ D_t(f) \neq \varnothing\}$$

is countable. For $t > t_0$, there exists an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$ containing $L_t(f)$ for every $t > t_0$. To conclude the proof we apply the inductive hypothesis to the restriction of f to this hyperplane.

□

3.1.1 Operations on $\mathcal{QC}(\mathbb{R}^n)$

We start with some definitions of operations on quasi-concave functions that we will link them to the operations introduced on $\mathcal{K}^n$.

Definition 3.1.7. *Let f and g be two quasi-concave functions. We define*

$$(f \vee g)(x) = \max\{f(x), g(x)\},$$

the pointwise maximum operator on $\mathbb{R}^n$, and

$$(f \wedge g)(x) = \min\{f(x), g(x)\},$$

the pointwise minimum operator on $\mathbb{R}^n$.

We want to read these operations on super-level sets. Hence we have the following easy result.

Proposition 3.1.8. *[20] For any f and $g \in \mathcal{QC}(\mathbb{R}^n)$ and $t > 0$, we have*

$$L_t(f \vee g) = L_t(f) \cup L_t(g)$$

and

$$L_t(f \wedge g) = L_t(f) \cap L_t(g).$$

Remark 3.1.9. As a consequence of the previous proposition we have that for every quasi-concave functions f and g and $t > 0$, $L_t(f \wedge g) \in \mathcal{K}^n \cup \{\varnothing\}$ and so $f \wedge g \in \mathcal{QC}(\mathbb{R}^n)$. Meanwhile $f \vee g$ does not necessarily belong to $\mathcal{QC}(\mathbb{R}^n)$, because $L_t(f \vee g)$ is not always a convex body.

Example 3.1.10. *Let $K, H \in \mathcal{K}^n$ be two convex bodies and $s, r > 0$ two positive real numbers. If we take $f = s\chi_K$ and $g = r\chi_H$, then we have*

$$f \wedge g = \min\{r, s\}\chi_{K \cap H},\ f \vee g = \max\{r, s\}\chi_{K \cup H}.$$

In the following result we list some other properties of maximum and minimum operator on $\mathcal{QC}(\mathbb{R}^n)$.

Proposition 3.1.11. *Let $f, g \in \mathcal{QC}(\mathbb{R}^n)$ such that $f \vee g \in \mathcal{QC}(\mathbb{R}^n)$. One has that*

$$||f \vee g||_\infty = ||f||_\infty \vee ||g||_\infty$$

and

$$||f \wedge g||_\infty = ||f||_\infty \wedge ||g||_\infty.$$

Proof. 1. Let $x_1 \in \mathbb{R}^n$ such that $||f||_\infty = f(x_1)$, then it holds

$$f(x_1) \leq (f \vee g)(x_1) \leq ||f \vee g||_\infty.$$

We have also the same inequality for $||g||_\infty \leq ||f \vee g||_\infty$ and hence

$$||f||_\infty \vee ||g||_\infty \leq ||f \vee g||_\infty.$$

Vice versa let $\overline{x} \in \mathbb{R}^n$ such that $||f \vee g||_\infty = (f \vee g)(\overline{x})$. If $(f \vee g)(\overline{x}) = f(\overline{x})$, then we have

$$f(\overline{x}) \leq ||f||_\infty \leq ||f||_\infty \vee ||g||_\infty.$$

Similarly if $(f \vee g)(\overline{x}) = g(\overline{x})$, then we have $g(\overline{x}) \leq ||f||_\infty \vee ||g||_\infty$, and hence

$$||f \vee g||_\infty \leq ||f||_\infty \vee ||g||_\infty.$$

2. Let $\overline{x} \in \mathbb{R}^n$ such that $||f \wedge g||_\infty = (f \wedge g)(\overline{x})$, then we have

$$(f \wedge g)(\overline{x}) \leq f(\overline{x}) \leq ||f||_\infty \text{ and } (f \wedge g)(\overline{x}) \leq g(\overline{x}) \leq ||g||_\infty,$$

hence one has

$$||f \wedge g||_\infty \leq ||f||_\infty \wedge ||g||_\infty.$$

Vice versa we suppose there exists $||f \wedge g||_\infty < t < ||f||_\infty \wedge ||g||_\infty$, then we can read these two inequalities by super-level sets of f, g, $f \wedge g$ and $f \vee g$.

$||f \wedge g||_\infty < t$ means that $\varnothing = L_t(f \wedge g)$, while $t < ||f||_\infty \wedge ||g||_\infty$ means that $L_t(f)$ and $L_t(g)$ are not empty convex bodies. Hence we have $L_t(f \wedge g) = L_t(f) \cap L_t(g) = \varnothing$ and $L_t(f), L_t(g) \in \mathcal{K}^n$. By the assumption $f \vee g \in \mathcal{QC}(\mathbb{R}^n)$, we have

$$L_t(f \vee g) = L_t(f) \cup L_t(g) \in \mathcal{K}^n$$

and this is a contradiction with $L_t(f) \cap L_t(g) = \varnothing$.

□

Proposition 3.1.12.
- *If $f_1, \cdots, f_m$ are quasi-concave functions, then $\bigwedge_{i=1}^m f_i \in \mathcal{QC}(\mathbb{R}^n)$.*
- *Let $f_i \in \mathcal{QC}(\mathbb{R}^n)$, for $i \in \mathbb{N}$, then $\inf_i f_i$ is quasi-concave.*

Proof. The first statement follows easily by induction on m.

Now let $t > 0$, we claim

$$L_t(\inf_i f_i) = \bigcap_{i \in \mathbb{N}} L_t(f_i).$$

Indeed, if $x \in \bigcap_{i \in \mathbb{N}} L_t(f_i)$, then $f_i(x) \geq t$, for all i and then also for the infimum. Vice versa, if $x \in L_t(\inf_i f_i)$, then it holds $t \leq \inf_i f_i(x) \leq f_i(x)$ for every i, hence $x \in \bigcap_{i \in \mathbb{N}} L_t(f_i)$.
This means that $L_t(\inf_i f_i) \in K^n \cup \{\varnothing\}$. □

Since the maximum of two quasi-concave functions in general does not belong to $\mathcal{QC}(\mathbb{R}^n)$, then it does not always hold, clearly, that $\sup_i f_i$ is a quasi-concave function for $f_i \in \mathcal{QC}(\mathbb{R}^n)$, $i \in \mathbb{N}$. We can show the following example.

Example 3.1.13. *Let $N \in \mathbb{N}$ be a natural number strictly greater than 1. We define*

$$f_i(x) = \chi_{\mathbb{B}_N}(x) \vee 2\chi_{\mathbb{B}_{N-\frac{1}{i}}}(x), \quad i \in \mathbb{N},\ x \in \mathbb{R}^n.$$

Every f_i is a quasi-concave function and since the sequence is increasing w.r.t. i, then one has

$$\sup_{i\in\mathbb{N}} f_i(x) = \lim_{i\to+\infty} f_i(x) = f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}^n \setminus \mathbb{B}_N, \\ 1 & \text{if } x \in \partial\mathbb{B}_N, \\ 2 & \text{if } x \in \operatorname{int}(\mathbb{B}_N), \end{cases}$$

and f is not a quasi-concave function.

We can say something also about the support of maximum and minimum of two quasi-concave functions. Since $\operatorname{supp}(f)$ is the topological closure of the set of all x such that $f(x)$ is strictly positive, it holds

$$\{x \in \mathbb{R}^n |\ f \wedge g(x) > 0\} = \{x \in \mathbb{R}^n |\ f(x) > 0\} \cap \{x \in \mathbb{R}^n |\ g(x) > 0\}$$

and

$$\{x \in \mathbb{R}^n |\ f \vee g(x) > 0\} = \{x \in \mathbb{R}^n |\ f(x) > 0\} \cup \{x \in \mathbb{R}^n |\ g(x) > 0\}.$$

Taking the closure we conclude

$$\operatorname{supp}(f \wedge g) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(g) \text{ and } \operatorname{supp}(f \vee g) = \operatorname{supp}(f) \cup \operatorname{supp}(g), \tag{3.3}$$

for every f and g quasi-concave.

We consider now the composition of $f \in \mathcal{QC}(\mathbb{R}^n)$ with a rigid motion $T \colon \mathbb{R}^n \to \mathbb{R}^n$. We recall T is the composition of translations and elements of $O(n)$, i.e. the group of proper and improper rotations in $\mathbb{R}^n$.

Lemma 3.1.14. *[20] For all $f \in \mathcal{QC}(\mathbb{R}^n)$ and rigid motion T, $f \circ T \in \mathcal{QC}(\mathbb{R}^n)$. Indeed we have*

$$L_t(f \circ T) = T^{-1}(L_t(f)).$$

Example 3.1.15. *Let $K \in \mathcal{K}^n$ and $s > 0$. For the function $f = s\chi_K$ we have*

$$f \circ T = s\chi_{T^{-1}(K)},$$

with $T^{-1}(K) \in \mathcal{K}^n$, for every choice of $K \in \mathcal{K}^n$ and T rigid motion.

We define a notion of sum for quasi-concave functions and also a notion of multiplication by positive scalars.

Definition 3.1.16. *Let f and $g \in \mathcal{QC}(\mathbb{R}^n)$. We define for $x \in \mathbb{R}^n$,*

$$f \oplus g(x) = \sup_{y\in\mathbb{R}^n} \min\{f(x), g(y - x)\}.$$

Let us fix $\lambda > 0$, then we define

$$\lambda \odot f(x) = f\left(\frac{x}{\lambda}\right), \quad \forall\, x \in \mathbb{R}^n.$$

We refer to [56] for more details about these notions of operations, where they are used to establish functional Minkowski formula and functional Isoperimetric Inequality for functions in $\mathcal{QC}(\mathbb{R}^n)$. The following is a characterization of these operations in terms of super-level sets.

Proposition 3.1.17 (Proposition 4 in [56])**.** *Let f and g be two quasi-concave functions and $\lambda, \sigma > 0$. Then it holds*

$$L_t((\lambda \odot f) \oplus (\sigma \odot g)) = \lambda L_t(f) + \sigma L_t(g), \quad \forall\, t > 0.$$

Corollary 3.1.18. *$\mathcal{QC}(\mathbb{R}^n)$ is closed under the operations $\odot$ and $\oplus$.*

The definition of multiplication by scalars will be used also in Chapter 6 to define homogeneous valuation defined on $\mathcal{QC}(\mathbb{R}^n)$.

We conclude this subsection with the last operation we want to consider over quasi-concave functions, that one we will use to define even valuation. Let $R\colon \mathbb{R}^n \to \mathbb{R}^n$ be the function defined by $R(x) = -x$, the reflection w.r.t the origin. It holds that $L_t(f \circ R) = -L_t(f)$, for every $f \in \mathcal{QC}(\mathbb{R}^n)$ and $t > 0$ and this means that the operation $\mathcal{QC}(\mathbb{R}^n) \ni f \mapsto f \circ R$ is closed for quasi-concave functions, i.e. $f \circ R \in \mathcal{QC}(\mathbb{R}^n)$, for any $f \in \mathcal{QC}(\mathbb{R}^n)$.

3.1.2 Families of quasi-concave functions

Log-concave functions.
First of all we define the function space, $Conv(\mathbb{R}^n)$, as the space of all $u\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ convex, lower semi-continuous, proper and coercive. By proper we mean that

$$\mathrm{dom}(u) = \{x \in \mathbb{R}^n |\, u(x) < +\infty\} \neq \varnothing,$$

and by coercive

$$\lim_{||x|| \to +\infty} u(x) = +\infty.$$

With these properties we have that there exists $\min_{\mathbb{R}^n} u > -\infty$ and the sub-level set $\{x \in \mathbb{R}^n |\, u(x) \leq t\}$ is a convex body, for every $t \geq \min_{\mathbb{R}^n} u$.

Definition 3.1.19. *A function f is said to be log-concave if it is of the form*

$$f(x) = \exp(-u(x)),$$

with $u \in Conv(\mathbb{R}^n)$.

We denote by $\mathcal{LC}(\mathbb{R}^n)$ the set of log-concave functions.

We refer to [23], [22], [24] and [58] for more details and results about $Conv(\mathbb{R}^n)$ and $\mathcal{LC}(\mathbb{R}^n)$. The space of log-concave functions is a fundamental family of function space, we refer to [17], [19], [38] for some literature about this family related to convex analysis.

Remark 3.1.20. There are some easy connections between $\mathcal{LC}(\mathbb{R}^n)$ and $Conv(\mathbb{R}^n)$. Let $u \in Conv(\mathbb{R}^n)$ and $f = \exp(-u)$. Then

- $\mathrm{supp}(f) = \mathrm{dom}(u)$.
- $L_t(f) = \{x \in \mathbb{R}^n |\, \exp(-u(x)) \geq t\} = \{x \in \mathbb{R}^n |\, u(x) \leq -\ln(t)\}$, for every $t > 0$. Hence f is quasi-concave.

- $M(f) = \max_{\mathbb{R}^n} f = \exp(-\min_{\mathbb{R}^n} u)$.

Gaussian function is of course log-concave and also characteristic function of convex bodies. Indeed, it holds

$$\chi_K \equiv \exp(-I_K),$$

where I_K is the indicatrix function of K, i.e.

$$I_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{if } x \notin K. \end{cases}$$

Not all quasi-concave functions are log-concave, indeed if we consider again the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } 0 \leq x \leq 1, \\ \frac{1}{x} & \text{if } x \geq 1, \end{cases}$$

then f does not belong to $\mathcal{LC}(\mathbb{R}^n)$.

Simple functions.

Definition 3.1.21. *A quasi-concave function f is said to be **simple** if it is of the form*

$$f \equiv \bigvee_{i=0}^{m} t_i \chi_{K_i}, \tag{3.4}$$

where $\mathbb{N} \ni m \geq 1$, $K_i \in \mathcal{K}^n$, $t_i > 0$ and one has:

1. $K_m \subseteq \cdots \subseteq K_1$.
2. $0 < t_1 < \cdots < t_m$.

We denote by $\mathcal{SQC}(\mathbb{R}^n)$ the space of simple functions.

Simple functions play a fundamental role in valuation theory on quasi-concave functions because, as we will see later, they are helpful to find connections between valuations on $\mathcal{K}^n$ and their functional counterpart.
A simple calculation gives us the following proposition (see [20]).

Proposition 3.1.22. *Let f be a simple function of the form* (3.4)*, then for every $0 < t \leq t_m$ it holds*

$$L_t(f) = K_i,$$

if $t_{i-1} < t \leq t_i$, where we set $t_0 = 0$.

In particular the previous result proves that $\mathcal{SQC}(\mathbb{R}^n) \subseteq \mathcal{QC}(\mathbb{R}^n)$.
We can also observe that every simple function is compactly supported, in fact $\text{supp}(f) = K_1$. Moreover they are continuous almost everywhere, except for the boundaries of $K_1, \cdots, K_m$.

It is possible also to construct quasi-concave functions with a similar form to simple functions, but with infinitely many "steps".

Example 3.1.23. *Let $K_i = \mathbb{B}_i$ be the ball centered at the origin and with radius i. Then the function*

$$f(x) = \bigvee_{i=1}^{+\infty} \frac{1}{i}\chi_{\mathbb{B}_i}(x),$$

is quasi-concave, in fact $L_t(f) = \mathbb{B}_i$ if $\frac{1}{i} \leq t < \frac{1}{i-1}$. In this case f is not with compact support anymore.

The space $\mathcal{QC}_N(\mathbb{R}^n)$.

For a fixed natural number N, we define

$$\mathcal{QC}_N(\mathbb{R}^n) = \{f \in \mathcal{QC}(\mathbb{R}^n) | \, ||f||_\infty \leq N, \; \mathrm{supp}(f) \subseteq \mathbb{B}_N\}.$$

The space $\mathcal{QC}_N(\mathbb{R}^n)$ can be useful to introduce a metric on the space of valuations on $\mathcal{QC}(\mathbb{R}^n)$, the idea is to replace the convergence on compact subset of $\mathcal{K}^n$ with something that we can use for function space, but we can anyway link with the convex bodies counterpart. Indeed we will see that if we consider only functions that are characteristic functions of convex bodies, then it is possible to return to the $\mathcal{K}^n$ case.

We can observe that $\mathcal{QC}_N(\mathbb{R}^n)$ and the space of simple functions have non-empty intersection, just take $f = t\chi_K$ for $K \subseteq \mathbb{B}_N$ and $0 < t \leq N$.

In the following we present a few properties of $\mathcal{QC}_N(\mathbb{R}^n)$.

Proposition 3.1.24. *Let f_i, $i \in \mathbb{N}$, be a sequence of quasi-concave functions such that $f_i \in \mathcal{QC}_N(\mathbb{R}^n)$. Then the function $\inf_i f_i$ belongs to $\mathcal{QC}_N(\mathbb{R}^n)$. In particular $\mathcal{QC}_N(\mathbb{R}^n)$ is closed under the minimum operator of finite number of functions.*

Proof. First we prove the closedness in $\mathcal{QC}_N(\mathbb{R}^n)$ of $\wedge$ under finite number of functions and then we pass to infimum. We want to prove that for every $m \geq 2$, one has

$$\bigwedge_{i=1}^{m} f_i \in \mathcal{QC}_N(\mathbb{R}^n).$$

At the beginning let $m = 2$, so if we have f_1 and f_2 in $\mathcal{QC}_N(\mathbb{R}^n)$, then of course the minimum is a quasi-concave function. Moreover we have

$$||f \wedge g||_\infty = (f \wedge g)(\overline{x}) \leq N,$$

for some $\overline{x}$ maximum point and the inequality holds because $f(x), g(x) \leq N$ for all $x \in \mathbb{R}^n$. As a consequence of (3.3), we claim that $\mathrm{supp}(f \wedge g) \subseteq \mathbb{B}_N$ and then $f \wedge g \in \mathcal{QC}_N(\mathbb{R}^n)$.
By induction on m, we have immediately that $\bigwedge_{i=1}^{m} f_i \in \mathcal{QC}_N(\mathbb{R}^n)$.

Now we consider the infimum, $\inf_i f_i$. We have already proved that

$$L_t(\inf_i f_i) = \bigcap_i L_t(f_i),$$

so $\inf_i f_i$ is a quasi-concave function. Moreover we know that $||f_i||_\infty \leq N$ and $\text{supp}(f_i) \subseteq \mathbb{B}_N$, for all i, hence $\inf_i f_i \in \mathcal{QC}_N(\mathbb{R}^n)$.

If we have $x \in \mathbb{R}^n \setminus \mathbb{B}_N$, then there exists an open ball $B(x,r) \subseteq \mathbb{R}^n \setminus \mathbb{B}_N$ such that for every $y \in B(x,r)$ we have $f_i(y) = 0$ and then $\inf_i f_i(y) = 0$. This means that $B(x,r) \subseteq \mathbb{R}^n \setminus \text{supp}(\inf_i f_i)$. At the end, since $f_i(x) \leq N$ for every i, then $\inf f_i(x) \leq N$ for every $x \in \mathbb{R}^n$, so it holds

$$|| \inf_i f_i||_\infty \leq N,$$

and that means $\inf_i f_i \in \mathcal{QC}_N(\mathbb{R}^n)$. □

Strictly related to $\mathcal{QC}_N(\mathbb{R}^n)$ is the space of quasi-concave functions with compact support that we will denote by $\mathcal{QC}_c(\mathbb{R}^n)$. Indeed we are able to prove the following.

Lemma 3.1.25.

$$\mathcal{QC}_c(\mathbb{R}^n) = \bigcup_{N \geq 1} \mathcal{QC}_N(\mathbb{R}^n).$$

Proof. First we observe $\mathcal{QC}_N(\mathbb{R}^n) \subseteq \mathcal{QC}_c(\mathbb{R}^n)$ for every $N \geq 1$, just by definition of $\mathcal{QC}_N(\mathbb{R}^n)$. Vice versa, for a compactly supported function there exists $N_1 \in \mathbb{N}$ such that $\text{supp}(f) \subseteq \mathbb{B}_{N_1}$. Moreover there exists $N_2 \geq 0$ such that $||f||_\infty \leq N_2$. Hence taking $\bar{N} = \max\{N_1, N_2\}$, we have $f \in \mathcal{QC}_{\bar{N}}(\mathbb{R}^n)$. □

3.2 Monotone and pointwise convergence on $\mathcal{QC}(\mathbb{R}^n)$

We focus in this section on the convergence we want to consider in $\mathcal{QC}(\mathbb{R}^n)$.

Definition 3.2.1. *(Monotone and pointwise convergence) We say that a sequence of quasi-concave functions f_i, $i \in \mathbb{N}$, converges to $f \in \mathcal{QC}(\mathbb{R}^n)$ in the monotone and pointwise convergence if:*

- *$f_i(x)$ is monotone either increasing or decreasing w.r.t i for every $x \in \mathbb{R}^n$ (with the same monotonicity).*
- *$\lim_{i \to +\infty} f_i(x) = f(x)$, for every $x \in \mathbb{R}^n$.*

We want to read this convergence also in terms of super-level sets, to link the functional convergence to Hausdorff one for convex bodies.

If we consider a sequence $f_i \to f$ converging pointwise and monotone, we want to establish some results for the sequence $L_t(f_i)$.

Lemma 3.2.2 (Lemma 3.9 in [20]). *Let $\{f_i\}_{i \in \mathbb{N}} \subseteq \mathcal{QC}(\mathbb{R}^n)$ and $f \in \mathcal{QC}(\mathbb{R}^n)$. Assume that $f_i \nearrow f$ pointwise in $\mathbb{R}^n$. Then for all $t > 0$, except at most for countably many values, one has*

$$\lim_{i \to +\infty} L_t(f_i) = L_t(f).$$

Proof. For every $t > 0$, the sequence of convex bodies $L_t(f_i)$ is increasing in i and $L_t(f_i) \subseteq L_t(f)$ for every i. In particular this sequence admits a limit $L_t \subset L_t(f)$. We choose $t > 0$ such that

$$L_t(f) = \text{cl}(\{x \in \mathbb{R}^n | \ f(x) > t\}).$$

By Lemma 3.1.6 we know that this condition holds for every t except at most countably many values.

It is clear that for every x such that $f(x) > t$ we have $x \in L_t$, hence $L_t \supset \{x \in \mathbb{R}^n : f(x) > t\}$; on the other hand, as L_t is closed, we have that $L_t \supset L_t(f)$. Hence $L_t = L_t(f)$ and proof is complete. □

Lemma 3.2.3 (Lemma 3.10 in [20]). *Let $\{f_i\}_{i\in\mathbb{N}} \subseteq \mathcal{QC}(\mathbb{R}^n)$ and $f \in \mathcal{QC}(\mathbb{R}^n)$. Assume that $f_i \searrow f$ pointwise in $\mathbb{R}^n$. Then for all $t > 0$*

$$\lim_{i\to+\infty} L_t(f_i) = L_t(f).$$

Proof. The sequence $L_t(f_i)$ is decreasing and its limit, denoted by L_t, contains $L_t(f)$. On the other hand

$$L_t = \bigcap_{i\in\mathbb{N}} L_t(f_i)$$

(Lemma 1.8.1 in [64]), if $x \in L_t$ then $f_i(x) \geq t$ for every i, so $f(x) \geq t$, i.e. $x \in L_t(f)$. □

In conclusion we can garantee the convergence of $L_t(f_i)$ for every t only in the decreasing case.

This is not the only difference between increasing and decreasing convergence. We focus now on the closedness of $\mathcal{QC}(\mathbb{R}^n)$.

Lemma 3.2.4. *$\mathcal{QC}(\mathbb{R}^n)$ is closed under decreasing and pointwise convergence.*

Proof. From the proof of Lemma 3.2.3 we have

$$L_t(f) = \bigcap_{i\in\mathbb{N}} L_t(f_i) \in \mathcal{K}^n \cup \{\varnothing\},\ \forall\, t > 0$$

and so $f \in \mathcal{QC}(\mathbb{R}^n)$. □

We do not have the same result for increasing sequences. For instance let

$$f_i(x) = \chi_{[-1+\frac{1}{i},1-\frac{1}{i}]}(x)$$

be a sequence of functions in $\mathcal{QC}(\mathbb{R})$. f_i is increasing w.r.t. i, but

$$f_i(x) \to f(x) = \chi_{(-1,1)}(x)$$

pointwise and $f \notin \mathcal{QC}(\mathbb{R})$.
Moreover the sequence of quasi-concave functions $f_i(x) = (1 - \frac{1}{i})\chi_{[0,i]}(x)$ is another example of increasing sequence that converges pointwise to a non quasi-concave function $f = \chi_{[0,+\infty)}(x)$.

We have seen in the first example that an increasing sequence does not preserve closedness of super-level sets, meanwhile the second one does not preserve the boundness of $L_t(f)$.

Nevertheless we claim that convexity is preserved by pointwise convergence.
Indeed let $f_i \in \mathcal{QC}(\mathbb{R}^n)$ be a sequence of functions converging pointwise to f. We fix $t > 0$ such that $L_t(f) \neq \varnothing$. If x and $y \in L_t(f)$ and $\lambda \in (0, 1)$, then we have

$$f_i((1-\lambda)x + \lambda y) \geq \min\{f_i(x), f_i(y)\},\ \forall\, i \in \mathbb{N},$$

by quasi-concavity of f_i.

Since $f_i(x) \to f(x)$ and $f_i(y) \to f(y)$, by the continuity of minimum operator, it holds

$$f((1-\lambda)x + \lambda y) \geq \lim_{i\to+\infty} \min\{f_i(x), f_i(y)\} = \min\{f(x), f(y)\} \geq t.$$

Hence $(1-\lambda)x + \lambda y \in L_t(f)$ and $L_t(f)$ is convex.

Another consequence of monotone and pointwise convergence is the following.

Lemma 3.2.5. *Let $\{f_i\}_{i\in\mathbb{N}} \subseteq \mathcal{QC}(\mathbb{R}^n)$, one has*

$$f_i \nearrow f \;\Rightarrow\; M(f_i) \nearrow M(f)$$

and

$$f_i \searrow f \;\Rightarrow\; M(f_i) \searrow M(f).$$

Proof. Let $f_i \nearrow f$ pointwise. We denote by x_i one of the points (possibly unique) in $\mathbb{R}^n$ such that

$$M(f_i) = f_i(x_i)$$

and by $\overline{x}$ the one for $M(f)$.

Hence we have the following inequalities

$$f_i(x_i) \leq f_{i+1}(x_i) \leq f_{i+1}(x_{i+1})$$

where the first one holds because f_i is increasing with x_i fixed, and the second one holds because $f_{i+1}(x_{i+1})$ is the maximum for f_{i+1}. Hence the sequence $\{f_i(x_i)\}_{i\in\mathbb{N}}$ is increasing, so we have that there exists

$$\lim_{i\to+\infty} f_i(x_i) = \sup_{i\in\mathbb{N}} f_i(x_i) \leq f(\overline{x}),$$

where the inequality holds because $f_i \leq f$ for every $x \in \mathbb{R}^n$.

Vice versa we have

$$f(\overline{x}) = \lim_{i\to+\infty} f_i(\overline{x}) \leq \lim_{i\to+\infty} f_i(x_i).$$

We suppose $f_i \searrow f$ pointwise. As above, we denote by x_i one of the points (possibly unique) in $\mathbb{R}^n$ such that

$$M(f_i) = f_i(x_i)$$

and by $\overline{x}$ that one for $M(f)$. By the monotone assumption of f_i we have

$$f_{i+1}(x_{i+1}) \leq f_i(x_{i+1}) \leq f_i(x_i),$$

then $f_i(x_i)$ is decreasing in $i \in \mathbb{N}$ and

$$\exists \lim_{i\to+\infty} f_i(x_i) = \inf_{i\geq 1} f_i(x_i) \geq f(\overline{x}).$$

Vice versa we suppose that there exists $t > 0$ such that $f(\overline{x}) < t < \inf_{i\geq 1} f_i(x_i) \leq f_i(x_i)$ for every $i \in \mathbb{N}$. Hence we have $L_t(f) = \varnothing$, $\mathcal{K}^n \ni L_t(f_i)$, for all $i \in \mathbb{N}$. Moreover since $f_i \searrow f$, it holds also

$$\mathcal{K}^n \ni L_t(f_i) \to L_t(f) = \varnothing,$$

then we have a contradiction. □

The last results that we want to show concerning monotone and pointwise convergence are density results.

Proposition 3.2.6. *For every quasi-concave function $f \in \mathcal{QC}_c(\mathbb{R}^n)$ with compact support, there exists a sequence of simple functions $f_i \in \mathcal{SQC}(\mathbb{R}^n)$ such that $f_i \nearrow f$.*

Proof. We know that there exists K convex body such that

$$L_t(f) \subseteq K$$

for every t.

Fix $i \in \mathbb{N}$, we consider the dyadic partition $\mathcal{P}_i$ of $[0, M(f)]$:

$$\mathcal{P}_i = \{t_j = j\frac{M(f)}{2^i} \ : \ j = 0, \cdots, 2^i\}.$$

Set

$$K_j = L_{t_j}(f), \ f_i = \bigvee_{j=1}^{2^i} t_j \chi_{K_j}.$$

f_i is a simple function and by $t_j\chi_{K_j} \leq f$ we have $f_i \leq f$ in $\mathbb{R}^n$.

The sequence f_i is increasing since $\mathcal{P}_i \subseteq \mathcal{P}_{i+1}$ and by $f_i \leq f$ it holds

$$\lim_{i\to+\infty} f_i(x) \leq f(x), \ \forall\, x \in \mathbb{R}^n$$

(in particular the support of f_i is contained in K, for every i). We establish the reverse inequality.

Let $x \in \mathbb{R}^n$, if $f(x) = 0$ then trivially

$$f_i(x) = 0 \text{ for all } i \in \mathbb{N}, \text{ hence } \lim_{i\to+\infty} f_i(x) = f(x).$$

Assume that $f(x) > 0$ and fix $\epsilon > 0$. Let $i_0 \in \mathbb{N}$ be such that $2^{-i_0}M(f) < \epsilon$. Let $j \in \{1, \cdots, 2^{i_0} - 1\}$ be such that

$$f(x) \in \Big(j\frac{M(f)}{2^{i_0}}, (j+1)\frac{M(f)}{2^{i_0}}\Big].$$

Then

$$f(x) \leq j\frac{M(f)}{2^{i_0}} + \frac{M(f)}{2^{i_0}} \leq f_{i_0} + \epsilon \leq \lim_{i\to+\infty} f_i(x) + \epsilon.$$

Hence the sequence f_i converges pointwise to f in $\mathbb{R}^n$. □

Proposition 3.2.7. *The space of quasi-concave functions with compact support is dense in $\mathcal{QC}(\mathbb{R}^n)$ under monotone and pointwise convergence.*

Proof. For every $f \in \mathcal{QC}(\mathbb{R}^n)$, we take

$$f_i = f \wedge M(f)\chi_{\mathbb{B}_i}.$$

It holds that $f_i \in \mathcal{QC}_c(\mathbb{R}^n)$ for every $i \in \mathbb{N}$, the sequence is increasing w.r.t i and it converges pointwise to f. □

Chapter 4

Valuations on function spaces

We are going to present briefly an introduction to the theory of valuations on function spaces, focusing on characterization results.

We recall first the notion of valuation on a generic function space X, that we have already seen in the introduction.

Definition 4.0.1. *Let X be a function space. A real-valued valuation on X is a functional $\mu\colon X \to \mathbb{R}$ such that it holds*

$$\mu(f) + \mu(g) = \mu(f \vee g) + \mu(f \wedge g),$$

for every $f, g \in X$ such that $f \vee g$ and $f \wedge g \in X$.

Remark 4.0.2. We recall that $f \vee g$ is the pointwise maximum operator and $f \wedge g$ is the pointwise minimum operator, as we have seen in Chapter 3.

We refer to [46] for a survey about valuations on function spaces. We will consider continuous valuations w.r.t. different notions of convergence, depending on the space X. Moreover we will study also the invariance property of a valuation; also this condition will depend on the space, but we can say that a valuation μ is invariant if

$$\mu(f \circ T) = \mu(f),$$

for any $f \in X$ and transformation T that will belong to $SO(n)$, to translation transformations or to $SL(n)$ acting on $\mathbb{R}^n$ or $\mathbb{S}^{n-1}$.

In many cases the idea is to find an integral representation of the valuation μ like

$$\mu(f) = \int_Y K(f(x))dx,$$

or linear and finite combinations of that, where K is a real-valued function, usually continuous, defined on $\mathbb{R}$ and Y might be $\mathbb{R}^n$ or some of its subsets, like for instance $\mathbb{S}^{n-1}$. The integral representation, on space of functions which have derivatives of some type, may involve the gradient of f as well, or higher order derivatives, when they are available.

We start our investigation with the papers written by Klain, [34] and [35], that we can consider the first papers in this area. Actually he established characterization results for valuations on star-shaped sets, but we can interpret them as results for valuations on a particular function space.

Definition 4.0.3. *A set $K \subseteq \mathbb{R}^n$ is a star-shaped set w.r.t. the origin if K contains the origin, and if for every line l passing through $\underline{0} \in \mathbb{R}^n$, the set $K \cap l$ is a closed interval. In the following when we will refer to a star-shaped set K, we will mean w.r.t. the origin.*

Remark 4.0.4. Every convex body, containing the origin, is a star-shaped set w.r.t. the origin.

We can associate to every star-shaped set, K, a function called radial function , $\rho_K \colon \mathbb{S}^{n-1} \to \mathbb{R}_+$, defined by

$$\rho_K(u) = \max\{\lambda \geq 0 : \ \lambda u \in K\}, \quad \forall\, u \in \mathbb{S}^{n-1}.$$

Definition 4.0.5. *Let $p > 0$. A star-shaped set $K \subseteq \mathbb{R}^n$ is an L^p-star if the radial function $\rho_K \in L^p(\mathbb{S}^{n-1})$.*

Remark 4.0.6. Since the radial function of a star-shaped set is non-negative, we introduce the space $L^p_+(\mathbb{S}^{n-1})$ of all non-negative functions in $L^p_+(\mathbb{S}^{n-1})$, and of course $\rho_K \in L^p_+(\mathbb{S}^{n-1})$, for every K L^p-star.

We denote by S^n the set of all L^n-star in $\mathbb{R}^n$. We observe that every convex body, containing the origin, is contained in S^n and moreover S^n is closed under union and intersection.

Klain studied valuations defined on S^n. The reason of the choice of this particular space depends on the following fact: let K be a star-shaped set, then we can evaluate the Lebesgue measure of K and it holds

$$V_n(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K^n(u)\, d\mathcal{H}^{n-1}(u).$$

Hence we require that $K \in S^n$, i.e. $\rho_K \in L^n(\mathbb{S}^{n-1})$, because we want to consider star-shaped sets K with finite volume.

So Klain established characterization results for particular class of star-shaped sets, but with the notion of radial function we can link functions on $L^n_+(\mathbb{S}^{n-1})$ to S^n. Indeed not only K is determined uniquely by ρ_K, but also any non-negative function on $L^n(\mathbb{S}^{n-1})$ determines uniquely a star-shaped set.

Definition 4.0.7 (Valuation on S^n; [34] and [35])**.** *A set function $\tilde{\mu} \colon S^n \to \mathbb{R}$ is a valuation if*

$$\tilde{\mu}(K \cup H) + \tilde{\mu}(K \cap H) = \tilde{\mu}(K) + \tilde{\mu}(H),$$

for all K and $H \in S^n$.

Klain studied continuous and invariant valuations on S^n. Continuity is w.r.t. L^n-convergence on radial functions:

$$K_i \to K \text{ if and only if } \lim_{i \to +\infty} \int_{\mathbb{S}^{n-1}} |\rho_{K_i}(u) - \rho_K(u)|^n \, d\mathcal{H}^{n-1}(u) = 0,$$

for every K_i and $K \in S^n$, $\forall\, i \in \mathbb{N}$. Invariance is w.r.t. $SO(n)$, i.e. the special orthogonal group on $\mathbb{R}^n$, the subgroup of $O(n)$ which elements have determinant equal to 1.

The most significant example of continuous and invariant valuation on S^n is of course the Lebesgue measure, V_n.

We have the following result, where Klain proved sufficient and necessary conditions to represent $\tilde{\mu}$ in integral form like $V_n(K)$.

Theorem 4.0.8 (Theorem 2.8 in [35]). *A set function $\tilde{\mu}\colon S^n \to \mathbb{R}$ is a continuous and invariant valuation if and only if there exists a continuous function $G\colon [0,+\infty) \to \mathbb{R}$ such that $|G(x)| \leq ax^n + b$, for some $a, b \geq 0$ and for all $x \geq 0$, and we have the following integral representation*

$$\tilde{\mu}(K) = \int_{\mathbb{S}^{n-1}} G \circ \rho_K(u)\, d\mathcal{H}^{n-1}(u), \quad \forall\, K \in S^n.$$

As we have already noticed, there is a correspondence between S^n and $L^n_+(\mathbb{S}^{n-1})$, so we can use this connection to define valuations on the function space $L^n_+(\mathbb{S}^{n-1})$. Indeed we observe first that the following properties hold:

1. for every $K_1, K_2 \in S^n$,

$$\rho_{K_1 \cup K_2}(u) = \rho_{K_1} \vee \rho_{K_2}(u), \quad \forall\, u \in \mathbb{S}^{n-1}.$$

2. For every $K_1, K_2 \in S^n$,

$$\rho_{K_1 \cap K_2}(u) = \rho_{K_1} \wedge \rho_{K_2}(u), \quad \forall\, u \in \mathbb{S}^{n-1}.$$

These properties allow us to define valuations on $L^n_+(\mathbb{S}^{n-1})$ starting with $\tilde{\mu}\colon S^n \to \mathbb{R}$, a valuation on star-shaped sets. Indeed, if $f \in L^n_+(\mathbb{S}^{n-1})$, then we know there exists a unique $K \in S^n$, such that

$$\rho_K(u) = f(u), \quad \forall\, u \in \mathbb{S}^{n-1}.$$

Hence we define the functional

$$\mu\colon L^n_+(\mathbb{S}^{n-1}) \to \mathbb{R}, \quad \mu(f) = \tilde{\mu}(K) \quad \forall\, f \in L^n_+(\mathbb{S}^{n-1}),$$

where $f \equiv \rho_K$ and $K \in S^n$. Consequently we can rewrite Klain's Characterization Theorem in the following way.

Theorem 4.0.9. *A functional $\mu\colon L^n_+(\mathbb{S}^{n-1}) \to \mathbb{R}$ is a continuous and invariant valuation if and only if there exists a continuous function $G\colon [0,+\infty) \to \mathbb{R}$ such that $|G(x)| \leq ax^n + b$, for some $a, b \geq 0$ and for all $x \geq 0$, and we have the following integral representation*

$$\mu(f) = \int_{\mathbb{S}^{n-1}} G \circ f(u)\, d\mathcal{H}^{n-1}(u), \quad \forall\, f \in L^n_+(\mathbb{S}^{n-1}).$$

Summarizing, we can consider Klain's results like the first results concerning valuations on function spaces. After that we have two different directions of development. The first one, historically, was established by Tsang, [67], and it is a characterization results for valuations on Lebesgue space.

Actually he obtained classification results in a more general setting. He considered valuations on $L^p(X)$, the Lebesgue space defined on a generic measure space $(X, \mathcal{F}, \sigma)$ with suitable conditions on it. So he established characterization theorems for a pure abstract function space, independently of the geometric setting of convex bodies or star-shaped sets.

We recall briefly definitions and notions concerning $L^p(X)$. Let $(X, \mathcal{F}, \sigma)$ be a measure space, we say that a σ-measurable function $f\colon X \to [-\infty, +\infty]$ is p-summable, with $1 \leq p < +\infty$, if

$$\int_X |f(x)|^p \, d\sigma(x) < +\infty.$$

The functional $||f||_p = \left(\int_X |f(x)|^p \, d\sigma(x)\right)^{\frac{1}{p}}$ is a semi-norm over all p-summable functions. We identify all p-summable functions that are equal σ-a.e. and we define $L^p(X)$ the space of all these functions with this identification.

Hence, $||\cdot||_p \colon L^p(X) \to [-\infty, +\infty]$ is a norm and $(L^p(X), ||\cdot||_p)$ is a Banach space.

Definition 4.0.10. *A measure space $(X, \mathcal{F}, \sigma)$ is called non-atomic if for all $E \in \mathcal{F}$, there exists $F \in \mathcal{F}$ such that $F \subsetneq E$ and $0 < \sigma(F) < \sigma(E)$.*

Remark 4.0.11. We will use in Chapter 5 the notion of non-atomic measure space. We will consider Radon measure ν over $[0, +\infty)$ and in particular we will use the equivalent form of non-atomic property, i.e. $\nu(\{t\}) = 0$, for all $t > 0$.

The main result due to Tsang is the following statement.

Theorem 4.0.12 (Theorem 1.2 in [67]). *Let $(X, \mathcal{F}, \sigma)$ be a non-atomic measure space and let*

$$\mu \colon L^p(X) \to \mathbb{R}$$

be a continuous, w.r.t. $L^p(X)$-convergence, valuation. Let us suppose that there exists a continuous function $h \colon \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ such that

$$\mu(\alpha\chi_E) = h(\alpha)\sigma(E), \quad \forall\, \alpha \in \mathbb{R}, \forall\, E \in \mathcal{F},$$

with $\sigma(E) < +\infty$ and where χ_E is the characteristic function of E. Then there exist $\gamma, \delta \geq 0$ such that $|h(\alpha)| \leq \gamma|\alpha|^p + \delta$, for all $\alpha \in \mathbb{R}$ and

$$\mu(f) = \int_X h \circ f(x) \, d\sigma(x), \quad \forall\, f \in L^p(X).$$

In addition if $\sigma(X) = \infty$, then $\delta = 0$.

If we take $X = \mathbb{S}^{n-1}$ and $\sigma = \mathcal{H}^{n-1}$, then, applying Theorem 4.0.12, we have the following statement.

Theorem 4.0.13 (Theorem 1.4 in [67]). *A functional $\mu \colon L^p(\mathbb{S}^{n-1}) \to \mathbb{R}$ is a continuous and $SO(n)$-invariant valuation if and only if there exists a continuous function $h \colon \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ and there exist real numbers $\gamma, \delta \geq 0$ such that $|h(\alpha)| \leq \gamma|\alpha|^p + \delta$, for all $\alpha \in \mathbb{R}$ and*

$$\mu(f) = \int_{\mathbb{S}^{n-1}} h \circ f(u) \, d\mathcal{H}^{n-1}(u),$$

for all $f \in L^p(\mathbb{S}^{n-1})$.

If we take $L^p_+(\mathbb{S}^{n-1})$, i.e. the subspace of $L^p(\mathbb{S}^{n-1})$ with non-negative functions, then the previous result still holds, and we can read it in terms of valuations on star-shaped sets with p-summable radial functions.

Meanwhile, if we take $X = \mathbb{R}^n$, endowed with the n-Lebesgue measure, we have a complete characterization result for valuations on $L^p(\mathbb{R}^n)$, for every $1 \leq p < +\infty$.

Theorem 4.0.14 (Theorem 1.3 in [67])**.** *A functional $\mu\colon L^p(\mathbb{R}^n) \to \mathbb{R}$ is a continuous and translation invariant valuation if and only if there exists a continuous function $h\colon \mathbb{R} \to \mathbb{R}$ such that $\exists\, \gamma \geq 0$ with the property $|h(\alpha)| \leq \gamma|\alpha|^p$, for all $\alpha \in \mathbb{R}$, and*

$$\mu(f) = \int_{\mathbb{R}^n} h \circ f(x)\, dx,$$

for all $f \in L^p(\mathbb{R}^n)$.

The second generalization, more recent, of Klain's works, was established by Tradacete and Villanueva in [73] and [72] and they obtained classification results for rotationally invariant and continuous valuations on $C_+(\mathbb{S}^{n-1})$, i.e. the space of all non-negative and continuous functions defined on $\mathbb{S}^{n-1}$. Here continuous means w.r.t. uniform convergence on $\mathbb{S}^{n-1}$.

Their results are connected to the work of Klain because every function in $C_+(\mathbb{S}^{n-1})$ determines uniquely a star-shaped set with continuous radial function, and the vice versa holds as well. A star-shaped set with continuous radial function is called star body.
We split in two thoerems the classification results of [73] and [72]. The first one was established by Villanueva.

Theorem 4.0.15. *[73] A functional $\mu\colon C_+(\mathbb{S}^{n-1}) \to \mathbb{R}_+$ is a rotation invariant and continuous valuation verifying $\mu(\underline{0}) = 0$, where $\underline{0}$ is the zero function, if and only if there exists a continuous function $h\colon [0, +\infty) \to \mathbb{R}_+$ such that $h(0) = 0$ and*

$$\mu(f) = \int_{\mathbb{S}^{n-1}} h \circ f(u)\, d\mathcal{H}^{n-1}(u),$$

for every $f \in C_+(\mathbb{S}^{n-1})$.

In the second work, [72], Tradacete and Villanueva removed the non-negativity condition for the valuation μ defined on $C_+(\mathbb{S}^{n-1})$, proving a Jordan-like Decomposition Theorem for valuations.

Theorem 4.0.16 (Theorem 1.1 in [72])**.** *Let $\mu\colon C_+(\mathbb{S}^{n-1}) \to \mathbb{R}$ be a rotation invariant and continuous valuation such that $\mu(\underline{0}) = 0$. Then there exist two rotation invariant and continuous valuations $\mu_+, \mu_-\colon C_+(\mathbb{S}^{n-1}) \to \mathbb{R}_+$ such that $\mu_+(\underline{0}) = \mu_-(\underline{0}) = 0$ and*

$$\mu(f) = \mu_+(f) - \mu_-(f),$$

for all $f \in C_+(\mathbb{S}^{n-1})$.

Corollary 4.0.17 (Corollary 1.2 in [72])**.** *A functional $\mu\colon C_+(\mathbb{S}^{n-1}) \to \mathbb{R}$ is a rotation invariant and continuous valuation if and only if there exists a continuous function $h\colon [0, +\infty) \to \mathbb{R}_+$ such that*

$$\mu(f) = \int_{\mathbb{S}^{n-1}} h \circ f(u)\, d\tilde{\mathcal{H}}^{n-1}(u), \quad f \in C_+(\mathbb{S}^{n-1}),$$

where $\tilde{\mathcal{H}}^{n-1}$ is the Hausdorff measure over the sphere normalized such that $\tilde{\mathcal{H}}^{n-1}(\mathbb{S}^{n-1}) = 1$.

Tradacete and Villanueva established also results concerning valuation theory in more general settings, for instance they removed the rotationally invariant condition or they considered valuations on Banach lattices. We refer to [69], [70] and [71] for their works on this theory.

The next step is a classification result for Sobolev spaces, $W^{1,p}(\mathbb{R}^n)$. We present briefly the work established by Ma in [50].
As usual we denote by $W^{1,p}(\mathbb{R}^n)$ the space of all functions $f \in L^p(\mathbb{R}^n)$ such that the weak gradient ∇f belongs to $L^p(\mathbb{R}^n)$. For each $f \in W^{1,p}(\mathbb{R}^n)$ we define the Sobolev norm as

$$||f||_{W^{1,p}(\mathbb{R}^n)} = \Big(||f||_p^p + ||\nabla f||_p^p\Big)^{\frac{1}{p}}.$$

Equipped with this norm, $W^{1,p}(\mathbb{R}^n)$ is a Banach space.

Remark 4.0.18. For every $f, g \in W^{1,p}(\mathbb{R}^n)$, we have that $f \vee g$ and $f \wedge g$ belong to $W^{1,p}(\mathbb{R}^n)$.

Hence a real-valued valuations on $W^{1,p}(\mathbb{R}^n)$ is a functional

$$\mu\colon W^{1,p}(\mathbb{R}^n) \to \mathbb{R},$$

such that

$$\mu(f) + \mu(g) = \mu(f \vee g) + \mu(f \wedge g), \quad \forall\, f, g \in W^{1,p}(\mathbb{R}^n)$$

and we also set $\mu(\underline{0}) = 0$.
In the following we will consider valuations on $W^{1,p}(\mathbb{R}^n)$ that are continuous w.r.t. $W^{1,p}(\mathbb{R}^n)$-norm and invariant w.r.t. composition of translations and $SL(n)$-transformations. We are going to present the result established by Ma with the additional condition of homogeneity. She proved also a characterization theorem for valuations that are not homogeneous, for which we refer to Theorem 5 in [50].

Definition 4.0.19. *We say that $\mu\colon W^{1,p}(\mathbb{R}^n) \to \mathbb{R}$ is q-homogeneous, with $q \in \mathbb{R}$, if $\mu(\lambda f) = |\lambda|^q \mu(f)$, for every $\lambda \in \mathbb{R}$ and $f \in W^{1,p}(\mathbb{R}^n)$.*

The statement is the following.

Theorem 4.0.20 (Theorem 4 in [50])**.** *Let $1 \leq p < n$. A functional $\mu\colon W^{1,p}(\mathbb{R}^n) \to \mathbb{R}$ is a continuous, $SL(n)$ and translation invariant and q-homogeneous valuation if and only if $p \leq q \leq \frac{np}{n-p}$ and $\exists\, c \in \mathbb{R}$ such that*

$$\mu(f) = c||f||_q^q,$$

for every $q \in W^{1,p}(\mathbb{R}^n)$.

We can find some other characterization results in [12], [39] and [74], where the authors studied valuations defined, respectively, in the space of definable functions, in Orlicz space and in $BV(\mathbb{R}^n)$. We observe that Ma's Theorem 4.0.20 is not the only result concerning valuations on Sobolev space. We refer to [47], [59] and [68] for results about Minkowski (and L^p-Minkowski) valuations defined not only on Sobolev space but also in other function spaces. We recall, as we have seen in the introduction, that a Minkowski valuation is a valuation with values on $(\mathcal{K}^n, +)$ where $+$ is the Minkowski sum (and with L^p-Minkowski we mean that we consider the operation $+_p$, the L^p-Minkowski sum, see [64]).

We mention that there are also results concerning matrix-valued valuations defined on Sobolev space, for instance in [45] and in [48].

The last function space we want to consider is $Conv(\mathbb{R}^n)$. We have already defined this space in Chapter 3, we recall that it is the space of all convex functions $u\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ that are lower semi-continuous, proper and coercive.

Colesanti, Ludwig and Mussnig are studying valuations on $Conv(\mathbb{R}^n)$ and they have already established some results that we are going to present.
In this case we have that only the maximum pointwise operator is closed in $Conv(\mathbb{R}^n)$, not in general the minimum one. Hence for the definition of valuation on $Conv(\mathbb{R}^n)$ we have to require that

$$\mu(u) + \mu(v) = \mu(u \vee v) + \mu(u \wedge v),$$

for every $u, v \in Conv(\mathbb{R}^n)$ such that $u \wedge v \in Conv(\mathbb{R}^n)$.

We consider continuous valuations w.r.t epi-convergence in $Conv(\mathbb{R}^n)$ (see Chapter 7, Section 1) and $SL(n)$ and translation invariant valuations.
We present two basic examples of valuations in $Conv(\mathbb{R}^n)$ that are also fundamental in classification theorems.

Example 4.0.21. *[23]*

1. *Let $K_0\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, then*
$$\mu\colon Conv(\mathbb{R}^n) \to \mathbb{R}, \quad \mu(u) = K_0(\min_{\mathbb{R}^n} u)$$
is a continuous, translation and $SL(n)$-invariant valuation.

2. *Let us fix $K_1\colon \mathbb{R} \to \mathbb{R}$ continuous such that*
$$\int_0^{+\infty} t^{n-1} K_1(t) dt < \infty. \tag{4.1}$$
Hence the functional $\mu\colon Conv(\mathbb{R}^n) \to \mathbb{R}$ defined by
$$\mu(u) = \int_{\mathrm{dom}(u)} K_1 \circ u(x)\, dx, \quad \forall\, u \in Conv(\mathbb{R}^n)$$
is a continuous, translation and $SL(n)$-invariant valuation.
We observe that condition (4.1) *is equivalent to $\mu(u) < +\infty$, for every choice of $u \in Conv(\mathbb{R}^n)$.*

One of the main results is the following statement.

Theorem 4.0.22. *[23] A functional $\mu\colon Conv(\mathbb{R}^n) \to \mathbb{R}$ is a continuous, translation and $SL(n)$-invariant valuation if and only if there exist two continuous functions K_0 and K_1 such that*

$$\int_0^{+\infty} t^{n-1} K_1(t) dt < +\infty$$

and

$$\mu(u) = K_0(\min_{\mathbb{R}^n} u) + \int_{\mathrm{dom}(u)} K_1 \circ u(x)\, dx,$$

for every $u \in Conv(\mathbb{R}^n)$.

We refer to [23], [24] for the proof and also to [3] and [18] for more informations concerning this theory. Moreover we refer to [22] and [58] for results about Minkowski valuations on $Conv(\mathbb{R}^n)$ and $\mathcal{LC}(\mathbb{R}^n)$, i.e. the space of log-concave function space that as we have seen, is related to $Conv(\mathbb{R}^n)$.

Chapter 5

Valuations on $\mathcal{QC}(\mathbb{R}^n)$, continuous and invariant

This chapter contains one of the main results of this thesis.

We will present the definition of valuations for quasi-concave functions, i.e. functionals with finite additivity property. We will equip these valuations with continuity notion and we will require translation and $O(n)$ invariance, this means invariance w.r.t. composition of translations and $O(n)$ transformations, that are proper rotations and reflections.

We will see how to link such functionals with valuations on $\mathcal{K}^n$ and how to use this connection to obtain the main theorem of this chapter (a Hadwiger type Theorem), a complete characterization of continuous, $O(n)$ and translation invariant valuations on $\mathcal{QC}(\mathbb{R}^n)$.

Roughly speaking for a valuation μ on quasi-concave functions, we define a family of valuations $\tilde{\mu}_t$ defined on $\mathcal{K}^n$ depending by a positive real parameter $t > 0$. Applying results for $\mathcal{K}^n$, like Hadwiger and Volume Theorem, we will come back to $\mathcal{QC}(\mathbb{R}^n)$ and obtain our results.

At the end of the chapter we will consider also the monotone case, showing a characterization result also for them.

5.1 Definitions and examples of valuations on $\mathcal{QC}(\mathbb{R}^n)$

Definition 5.1.1. *A functional $\mu\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ is said to be a valuation if*

1. $\mu(\underline{0}) = 0$ *where here $\underline{0}$ is the zero constant function.*

2. *For all quasi-concave functions f and g, such that $f \vee g \in \mathcal{QC}(\mathbb{R}^n)$, one has*

$$\mu(f \wedge g) + \mu(f \vee g) = \mu(f) + \mu(g).$$

We list immediately other hypothesis we will consider on μ.

Definition 5.1.2.

- *Continuity:*

$$\lim_{i \to +\infty} \mu(f_i) = \mu(f),$$

whenever f_i goes to f w.r.t. monotone and pointwise convergence.

- *Invariance:*
$$\mu(f \circ T) = \mu(f),$$
for every $f \in \mathcal{QC}(\mathbb{R}^n)$ and T rigid motion on $\mathbb{R}^n$.

- *Monotonicity: increasing if*
$$f \leq g \; pointwise \;\Rightarrow\; \mu(f) \leq \mu(g),$$
decreasing if
$$f \leq g \; pointwise \;\Rightarrow\; \mu(f) \geq \mu(g).$$

- *Simplicity:*
$$\mu(f) = 0,$$
whenever $f \in \mathcal{QC}(\mathbb{R}^n)$ with $\dim(\mathrm{supp}(f)) < n$.

In the following, when we will assume that μ is *invariant*, it has to be intended w.r.t. rigid motion if not otherwise specified.

We have introduced the notion of continuity w.r.t. pointwise and monotone convergence, the following remark clarifies a little bit more why we have chosen this convergence.

Remark 5.1.3. A brief discussion on the choice of the convergence in $\mathcal{QC}(\mathbb{R}^n)$
A natural choice of a topology in $\mathcal{QC}(\mathbb{R}^n)$ would be the one induced by pointwise convergence. Let us observe that this choice would be too restrictive, with respect to the theory of continuous and rigid motion invariant (but translations are enough) valuations. Indeed, any translation invariant valuation μ on $\mathcal{QC}(\mathbb{R}^n)$ such that
$$\lim_{i\to\infty} \mu(f_i) = \mu(f)$$
for every sequence f_i in $\mathcal{QC}(\mathbb{R}^n)$, converging to some $f \in \mathcal{QC}(\mathbb{R}^n)$ pointwise, must be the valuation constantly equal to 0. To prove this claim, let f be a quasi-concave function with compact support, let e_1 be the first vector of the canonical basis of $\mathbb{R}^n$ and set
$$f_i(x) = f(x - i\, e_1) \quad \text{for all } x \in \mathbb{R}^n, \quad \text{for all } i \in \mathbb{N}.$$
The sequence f_i converges pointwise to the function $f_0 \equiv 0$ in $\mathbb{R}^n$, so that, by translation invariance, and as $\mu(f_0) = 0$, we have $\mu(f) = 0$. Hence μ vanishes on each function f with compact support. By Theorem 3.2.7 we know also that every element of $\mathcal{QC}(\mathbb{R}^n)$ is the pointwise limit of a sequence of functions in $\mathcal{QC}_c(\mathbb{R}^n)$. Hence $\mu \equiv 0$.

Example 5.1.4. *1. Let us fix $K \subseteq \mathbb{R}^n$ convex body, with $\dim(K) = n$. We define the following map*
$$\mu_K \colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}, \; \mu_K(f) = \int_K f(x)dx.$$
We know μ is well-defined since any quasi-concave function belongs to $L^1_{loc}(\mathbb{R}^n)$.

Consider now f and $g \in \mathcal{QC}(\mathbb{R}^n)$ with $f \vee g \in \mathcal{QC}(\mathbb{R}^n)$ and define $K_1 = \{x \in K : \; f(x) \geq g(x)\}$ and $K_2 = \{x \in K : \; f(x) < g(x)\}$, then it holds
$$\int_K (f \wedge g)(x)dx = \int_{K_1} (f \wedge g)(x)dx + \int_{K_2} (f \wedge g)(x)dx = \int_{K_1} g(x)dx + \int_{K_2} f(x)dx.$$

In a similar way we have $\int_K (f \vee g)(x)dx = \int_{K_1} f(x)dx + \int_{K_2} g(x)dx$ and then if we add each term, we have exactly the valuation property.

μ_K is not rigid motion invariant: by a change of variables we have

$$\int_K f(T(x))dx \underbrace{=}_{y=T(x)} \int_{T^{-1}(K)} f(y)dy,$$

and the last term is not equal to $\mu_K(f)$ in general.

μ_K is continuous: if $f_i \nearrow f$ pointwise, by Monotone Convergence Theorem we have

$$\lim_{i\to+\infty} \mu_K(f_i) = \lim_{i\to+\infty} \int_{\mathbb{R}^n} f_i(x)\chi_K(x)dx = \int_{\mathbb{R}^n} f(x)\chi_K(x)dx = \mu_K(f).$$

Applying Lebesgue Convergence Theorem we have the same converging result also for decreasing sequences, since it holds $f_i\chi_K \leq f_1\chi_K$ for all i and $f_1\chi_K$ is summable in $\mathbb{R}^n$.

Moreover μ_K is increasing and simple.

2. *We fix $h \in \mathbb{R}^n$ and define*

$$\mu_h \colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R},\ \mu_h(f) = f(h).$$

Clearly μ_h is a valuation, also continuous because of the pointwise convergence, but it is not invariant.

Moreover μ_h is increasing, but in general not simple.

We have seen some examples of positive valuations, i.e. $\mu(f) \geq 0$, for all $f \in \mathcal{QC}(\mathbb{R}^n)$, but this sign condition is not necessary. Indeed we can make examples of non-positive valuations, for instance $\mu_K(f) = -\int_K f(x)dx$, for every $f \in \mathcal{QC}(\mathbb{R}^n)$, with a fixed convex body $K \subseteq \mathbb{R}^n$ with $\dim(K) = n$.

In general we may define:

$$\mu(f) = \int_{K_1} f(x)dx - \int_{K_2} f(x)dx = \int_{\mathbb{R}^n} f(x)(\chi_{K_1}(x) - \chi_{K_2}(x))dx,$$

for every $f \in \mathcal{QC}(\mathbb{R}^n)$, fixed K_1 and K_2 convex bodies of $\mathbb{R}^n$ with dimensions equal to n.

Now we want to focus on the first connection between valuations on quasi-concave functions and valuations on convex bodies. We will see how to define a valuation on quasi-concave functions starting with the intrinsic volumes.

Let us fix $f \in \mathcal{QC}(\mathbb{R}^n)$, we already know that

$$L_t(f) \in \mathcal{K}^n, \quad \forall\, t \in (0, M(f)].$$

Let us fix also $k \in \{0, \cdots, n\}$ and now we consider the k-th intrinsic volume, $V_k \colon \mathcal{K}^n \to \mathbb{R}$, acting on super-level set, $V_k(L_t(f))$. In this way, we have just defined a function

$$t \in (0, M(f)] \longmapsto V_k(L_t(f))$$

that we can extend to $\mathbb{R}_+$ putting $V_k(L_t(f)) = 0$, for $t > M(f)$.
On the other hand, if we fix $t > 0$, these functionals act on f as valuations (for every choice of k), indeed one has

$$V_k(L_t(f \vee g)) + V_k(L_t(f \wedge g)) = V_k(L_t(f) \cup L_t(g)) + V_k(L_t(f) \cap L_t(g)) \quad (5.1)$$
$$= V_k(L_t(f)) + V_k(L_t(g)), \quad (5.2)$$

for every choice of quasi-concave functions f and g such that $f \vee g \in \mathcal{QC}(\mathbb{R}^n)$, by properties of maximum and minimum operators and by valuation condition of k-th intrinsic volume on $\mathcal{K}^n$.
This valuation is invariant, but not continuous. Indeed, for a fixed $t > 0$, we consider

$$f_i(x) = (t - \frac{1}{i})\chi_K(x),$$

for $i \in \mathbb{N}$ and $K \in \mathcal{K}^n$ fixed. We have $f_i(x) \nearrow f(x) = t\chi_K(x)$, pointwise, but $L_t(f_i) = \varnothing$, for every i, while $L_t(f) = K$.

Summarizing, for a fixed $k \in \{0, \cdots, n\}$, we have a family of valuations on $\mathcal{QC}(\mathbb{R}^n)$, defined through V_k, depending on $t > 0$.

Is it possible to define a valuation, starting from this family, that is also continuous? This is what we call **integral valuation**.

5.2 Integral valuations

5.2.1 Continuous integral valuations

For a fixed $f \in \mathcal{QC}(\mathbb{R}^n)$ and $0 \leq k \leq n$, we have already defined the function

$$\varphi_k \colon (0, +\infty) \to \mathbb{R}_+ \quad t \longmapsto V_k(L_t(f)).$$

The idea is to consider the average of this function:

$$\int_0^{+\infty} \varphi_k(t)dt = \int_0^{M(f)} V_k(L_t(f))dt.$$

We study now the measurability of φ_k: by Lemma 3.1.6 it follows that φ_k is continuous for a.e. $t > 0$ and then it is measurable. In fact, let us fix $t_0 > 0$ such that

$$L_{t_0}(f) = \mathrm{cl}(\{x \in \mathbb{R}^n : \ f(x) > t_0\}) \neq \varnothing.$$

Let us consider now a sequence t_i converging to t_0; we want to show that $L_{t_i}(f) \to L_{t_0}(f)$ w.r.t Hausdorff metric.

- If there exists $x_i \in L_{t_i}(f)$, for every $i \in \mathbb{N}$, such that $x_i \to x \in \mathbb{R}^n$, then we have $f(x_i) \geq t_i$ and by upper semi-continuity of f,

$$f(x) \geq \limsup_{i \to +\infty} f(x_i) \geq \lim_{i \to +\infty} t_i = t_0.$$

  Hence it holds $x \in L_{t_0}(f)$.

- Let $x \in L_{t_0}(f)$. If $f(x) > t_0$, then there exists $\bar{i} \in \mathbb{N}$ such that $f(x) > t_i$ for all $i \geq \bar{i}$, and so we have $x \in L_{t_i}(f)$, for $i \geq \bar{i}$.

  If $f(x) = t_0$, then $\exists\, x_i \to x$ such that $f(x_i) > t_0$. Let us fix $\overline{x}_i$ such that $f(\overline{x}_i) > t_0$, then there exists $\bar{i} \in \mathbb{N}$ such that
$$f(\overline{x}_i) > t_i,\ \forall\, i \geq \bar{i} \text{ and } \overline{x}_i \in L_{t_i}(f).$$

The continuity of V_k implies the continuity of $V_k(L_t(f))$ for a.e. $t > 0$.

We have proved the following result.

Lemma 5.2.1. *For every choice of* $k \in \{0, \cdots, n\}$, φ_k *is continuous for a.e.* $t > 0$ *and then it is measurable w.r.t. Lebesgue measure on* $(0, +\infty)$.

We focus now on the finiteness of the integral. We are interested in valuations on $\mathcal{QC}(\mathbb{R}^n)$, so we need a functional that is finite for every choice of f.

Easy examples can tell us that this is not the case of $\int_0^{+\infty} \varphi_k(t)dt$. For instance, we take $n = k = 1$ and
$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } 0 \leq x \leq 1, \\ \frac{1}{x} & \text{if } x \geq 1. \end{cases}$$
Then we have $V_1(L_t(f)) = V_1([0, \frac{1}{t}]) = \frac{1}{t}$, for $0 \leq t \leq 1$, so one has
$$\int_0^{+\infty} \varphi_1(t)dt = \int_0^1 \frac{1}{t}dt = +\infty.$$

Therefore we need something more to control the super-level set of a quasi-concave function; the problem is that $V_k(L_t(f))$ can decay with arbitrarily slow speed as $t \to 0^+$. Hence we need a weight and then a weighted average. We can do the following.

We observe that φ_k is a decreasing function, which vanishes for $t > M(f)$. In particular φ_k has bounded variation in $[\delta, M(f)]$ for every $\delta > 0$, hence there exists a Radon measure defined in $(0, +\infty)$, that we will denote by $S_k(f; \cdot)$, such that
$$-S_k(f; \cdot) \text{ is the distributional derivative of } \varphi_k$$
(see, for instance, [8]). Note that, as φ_k is decreasing, we put a minus sign in this definition to have a non-negative measure. The support of $S_k(f; \cdot)$ is contained in $[0, M(f)]$.

Let ϕ be a continuous function defined on $[0, +\infty)$, such that $\phi(0) = 0$. We consider the functional on $\mathcal{QC}(\mathbb{R}^n)$ defined by
$$\mu(f) = \int_0^{+\infty} \phi(t)dS_k(f; t), \quad f \in \mathcal{QC}(\mathbb{R}^n). \tag{5.3}$$
The aim of this subsection is to prove that this is a continuous and invariant valuation on $\mathcal{QC}(\mathbb{R}^n)$. As a first step, we need to find some condition on the function ϕ which guarantee that the above integral is well-defined for every f.

Assume that
$$\exists\, \delta > 0 \text{ s.t. } \phi(t) = 0 \text{ for every } t \in [0, \delta]. \tag{5.4}$$

In the following we usually refer to a function ϕ with this property saying that it satisfies the δ-condition. Then

$$\int_0^{+\infty} \phi_+(t)dS_k(f;t) = \int_\delta^{M(f)} \phi_+(t)dS_k(f;t)$$
$$\leq M\,\left(V_k(L_\delta(f)) - V_k(L_{M(f)}(f))\right) < +\infty,$$

where $M = \max_{[\delta,M(f)]}\phi_+$ and ϕ_+ is the positive part of ϕ. Analogously we can prove that the integral of the negative part of ϕ, denoted by ϕ_-, is finite, so that μ is well-defined.

We will prove that, for $k \geq 1$, condition (5.4) is necessary as well. Clearly, if $\mu(f)$ is well-defined (i.e. is a real number) for every $f \in \mathcal{QC}(\mathbb{R}^n)$, then

$$\int_0^{+\infty} \phi_+(t)dS_k(f;t) < +\infty \quad \text{and} \quad \int_0^{+\infty} \phi_-(t)dS_k(f;t) < +\infty \qquad \text{for all } f \in \mathcal{QC}(\mathbb{R}^n).$$

Assume that ϕ_+ does not vanish identically in any right neighborhood of the origin. Then we have

$$\psi(t) := \int_0^t \phi_+(\tau)\,d\tau > 0 \quad \text{for all } t > 0.$$

Consequently we can define the following function

$$t \to h(t) = \int_t^1 \frac{1}{\psi(s)}ds, \quad t \in (0,1],$$

that is strictly decreasing. As $k \geq 1$, we can construct a function $f \in \mathcal{QC}(\mathbb{R}^n)$ such that

$$V_k(L_t(f)) = h(t) \quad \text{for every } t > 0. \tag{5.5}$$

Indeed, consider a function of the form

$$f(x) = w(\|x\|), \quad x \in \mathbb{R}^n,$$

where $w \in C^1([0,+\infty))$ is positive and strictly decreasing. Then $f \in \mathcal{QC}(\mathbb{R}^n)$ and $L_t(f) = \mathbb{B}_{r(t)}$, where

$$r(t) = w^{-1}(t)$$

for every $t \in (0, f(0)]$ (note that $f(0) = M(f)$). Therefore

$$V_k(L_t(f)) = c\,(w^{-1}(t))^k$$

where c is a positive constant depending on k and n. Hence if we choose

$$w = \left[\left(\frac{1}{c}\,h\right)^{1/k}\right]^{-1},$$

(5.5) is verified. Hence the distributional derivative of $S_k(f;\cdot)$ is the measure that has a density, w.r.t. Lebesgue measure dt, equals to $\frac{1}{\psi(t)}$, and

$$\int_0^{+\infty} \phi_+(t)dS_k(f;t) = \int_0^{M(f)} \frac{\psi'(t)}{\psi(t)}dt = +\infty.$$

In the same way we can prove that ϕ_- must vanish in a right neighborhood of the origin. We have proved the following result.

Lemma 5.2.2. *Let $\phi \in C([0,+\infty))$ and $k \in \{1,\dots,n\}$. Then ϕ has finite integral w.r.t. the measure $S_k(f;\cdot)$ for every $f \in \mathcal{QC}(\mathbb{R}^n)$ if and only if ϕ verifies* (5.4).

Next we show that (5.3) defines a continuous and invariant valuation.

Proposition 5.2.3. *Let $k \in \{1,\dots,n\}$ and $\phi \in C([0,+\infty))$ be such that* (5.4) *is verified. Then* (5.3) *defines an invariant and continuous valuation on $\mathcal{QC}(\mathbb{R}^n)$.*

Proof. For every $f \in \mathcal{QC}(\mathbb{R}^n)$ we have already defined the function $\varphi_k \colon (0, M(f)] \to \mathbb{R}$ as

$$\varphi_k(t) = V_k(L_t(f)).$$

As already remarked, this is a decreasing function. In particular it has bounded variation in $[\delta, M(f)]$. Let ϕ_i, $i \in \mathbb{N}$, be a sequence of functions in $C^\infty([0,\infty))$, with compact support, converging uniformly to ϕ on compact sets. As $\phi \equiv 0$ in $[0,\delta]$, we may assume that the same holds for every ϕ_i. Then we have

$$\mu(f) = \lim_{i\to\infty} \mu_i(f),$$

where

$$\mu_i(f) = \int_0^{+\infty} \phi_i(t) dS_k(f;t) \quad \forall\, f \in \mathcal{QC}(\mathbb{R}^n).$$

By the definition of distributional derivative of a measure, we have, for every f and for every i:

$$\int_0^{+\infty} \phi_i(t) dS_k(f;t) = \int_0^{+\infty} \varphi_k(t)\phi_i'(t)dt = \int_0^{M(f)} V_k(L_t(f))\phi_i'(t)dt.$$

As we have already seen, if $f, g \in \mathcal{QC}(\mathbb{R}^n)$ are such that $f \vee g \in \mathcal{QC}(\mathbb{R}^n)$, for every $t > 0$, one has

$$V_k(L_t(f \vee g)) + V_k(L_t(f \wedge g)) = V_k(L_t(f)) + V_k(L_t(g)).$$

Multiplying both sides times $\phi_i'(t)$ and integrating on $[0,+\infty)$ we obtain

$$\mu_i(f \vee g) + \mu_i(f \wedge g) = \mu_i(f) + \mu_i(g).$$

Letting $i \to +\infty$ we deduce the valuation property for μ.

In order to prove the continuity of μ, let $f_i, f \in \mathcal{QC}(\mathbb{R}^n)$, $i \in \mathbb{N}$, and assume that the sequence f_i is either increasing or decreasing w.r.t. i, and it converges pointwise to f in $\mathbb{R}^n$. Note that in each case there exists a constant $M > 0$ such that $M(f_i), M(f) \le M$ for every i. Consider now the sequence of functions $\varphi_{k,i}(t) = V_k(L_t(f_i))$. By the monotonicity of the sequence f_i, and that of intrinsic volumes, this is a monotone sequence, w.r.t. i, of decreasing functions, and it converges a.e. to φ_k in $(0,+\infty)$, by Lemmas 3.2.2 and 3.2.3. In particular the sequence $\varphi_{k,i}$ has uniformly bounded total variation in $[\delta, M]$. Consequently, the sequence of measures $S_k(f_i;\cdot)$, converges weakly to the measure $S_k(f;\cdot)$ as $i \to +\infty$ (see for instance [8, Proposition 3.13]). Hence, as ϕ is continuous

$$\lim_{i\to+\infty} \mu(f_i) = \lim_{i\to\infty} \int_\delta^M \phi(t)\, dS_k(f_i;t) = \int_0^M \phi(t)\, dS_k(f;t) = \mu(f).$$

Finally, the invariance of μ follows directly from the invariance of intrinsic volumes w.r.t. rigid motions. □

In the statement of Proposition 5.2.3 we did not consider the case $k = 0$. Since it is quite different from the others k, for instance we do not need to require the δ-condition, we will present that at the end of this section, together with some other observations concerning the other extremal case, $k = n$.

5.2.2 Monotone (and continuous) integral valuations

Now we introduce a slightly different type of integral valuations, which will be needed to characterize all possible continuous and monotone valuations on $\mathcal{QC}(\mathbb{R}^n)$. Note that, as it will be clear in the sequel, when the involved functions are smooth enough, the two types can be reduced one to another by an integration by parts.

Let $k \in \{0, \dots, n\}$ and let ν be a Radon measure on $[0, +\infty)$; assume that

$$\int_0^{+\infty} V_k(L_t(f))d\nu(t) < +\infty, \quad \forall f \in \mathcal{QC}(\mathbb{R}^n). \tag{5.6}$$

We will return later on explicit condition on ν such that (5.6) holds. Then we define the functional $\mu\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ by

$$\mu(f) = \int_0^{+\infty} V_k(L_t(f))d\nu(t), \quad \forall\, f \in \mathcal{QC}(\mathbb{R}^n). \tag{5.7}$$

Proposition 5.2.4. *Let ν be a Radon measure on $[0, +\infty)$ which verifies* (5.6)*; then the functional defined by* (5.7) *is a rigid motion invariant and monotone increasing valuation.*

Proof. The proof that μ is a valuation follows, as usual, by properties of maximum and minimum operators and valuation condition for intrinsic volumes, as in the proof of Proposition 5.2.3.

The same can be done for invariance and monotonicity. □

If we do not impose any further assumption the valuation μ needs not to be continuous. Indeed, for example, if we fix $t = t_0 > 0$ and let $\nu = \delta_{t_0}$ be the delta Dirac measure at t_0, then the valuation

$$\mu(f) = V_n(L_{t_0}(f)), \quad \forall f \in \mathcal{QC}(\mathbb{R}^n),$$

is not continuous. To see it, let $f = t_0\chi_{\mathbb{B}^n}$ (recall that $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$) and let

$$f_i = t_0(1 - \frac{1}{i})\chi_{\mathbb{B}^n}, \quad \forall\, i \in \mathbb{N}.$$

Then f_i is a monotone sequence of elements of $\mathcal{QC}(\mathbb{R}^n)$ converging pointwise to f in $\mathbb{R}^n$. On the other hand

$$\mu(f_i) = 0 \quad \forall\, i \in \mathbb{N},$$

while $\mu(f) = V_n(\mathbb{B}^n) > 0$. The next result asserts that the presence of atoms is the only possible cause of discontinuity for μ. We recall that a measure ν defined on $[0, +\infty)$ is said non-atomic if $\nu(\{t\}) = 0$ for every $t \geq 0$.

Proposition 5.2.5. *Let ν be a Radon measure on $[0, +\infty)$ such that* (5.6) *holds and let μ be the valuation defined by* (5.7)*. Then the two following conditions are equivalent:*

i) ν is non-atomic,

ii) μ is continuous.

Proof. Suppose that *i)* does not hold, than there exists t_0 such that $\nu(\{t_0\}) = \alpha > 0$.

Define $\varphi\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$\varphi(t) = \int_0^t d\nu(s).$$

φ is an increasing function with a jump discontinuity at t_0 of amplitude α. Now let $f = t_0\chi_{\mathbb{B}^n}$ and $f_i = t_0(1-\frac{1}{i})\chi_{\mathbb{B}^n}$, for $i \in \mathbb{N}$. Then f_i is an increasing sequence in $\mathcal{QC}(\mathbb{R}^n)$, converging pointwise to f in $\mathbb{R}^n$. On the other hand

$$\mu(f) = \int_0^{t_0} V_k(\mathbb{B}^n) d\nu(s) = V_k(\mathbb{B}^n)\nu((0,t_0]) = V_k(\mathbb{B}^n)\,\varphi(t_0)$$

and similarly

$$\mu(f_i) = V_k(\mathbb{B}^n)\,\varphi\left(t_0 - \frac{1}{i}\right).$$

Consequently

$$\lim_{i\to+\infty} \mu(f_i) < \mu(f).$$

Vice versa, suppose that *i)* holds. We observe that, as ν is non-atomic, every countable subset has measure zero w.r.t. ν. Let $f_i \in \mathcal{QC}(\mathbb{R}^n)$, be a sequence such that either $f_i \nearrow f$ or $f_i \searrow f$, pointwise in $\mathbb{R}^n$, for some $f \in \mathcal{QC}(\mathbb{R}^n)$. Set

$$\varphi_{k,i}(t) = V_k(L_t(f_i)), \quad \varphi_k(t) = V_k(L_t(f)) \quad \forall\, t \geq 0, \quad \forall\, k \in \mathbb{N}.$$

The sequence $\varphi_{k,i}$ is monotone w.r.t i and, by Lemmas 3.2.2 and 3.2.3, converges to φ_k ν-a.e. Hence, by the continuity of intrinsic volumes and the Monotone Convergence Theorem, we obtain

$$\lim_{i\to\infty} \mu(f_i) = \lim_{i\to\infty} \int_0^{+\infty} \varphi_{k,i}(t)\, d\nu(t) = \int_0^{+\infty} \varphi_k(t)\, d\nu(t) = \mu(f).$$

□

Now we are going to find a more explicit form of condition (5.6). We need the following lemma.

Lemma 5.2.6. *Let $\phi : [0,+\infty) \to \mathbb{R}$ be an increasing, non-negative and continuous function with $\phi(0) = 0$ and $\phi(t) > 0$, for all $t > 0$. Let ν be a Radon measure such that $\phi(t) = \nu([0,t])$, for all $t \geq 0$. Then*

$$\int_0^1 \frac{1}{\phi^k(t)} d\nu(t) = +\infty, \ \forall k \geq 1.$$

Proof. Fix $\alpha \in [0,1]$. The function $\psi : [\alpha, 1] \to \mathbb{R}$ defined by

$$\psi(t) = \begin{cases} \dfrac{1}{k-1}\phi^{1-k}(t) & \text{if } k > 1, \\[2ex] \ln(\phi(t)) & \text{if } k = 1, \end{cases}$$

is continuous and of bounded variation in $[\alpha, 1]$. Its distributional derivative is the measure that has a density, w.r.t. ν, equals to $\frac{1}{\phi^k}$.

Hence, for $k > 1$,

$$\frac{1}{k-1}[\phi^{1-k}(\alpha) - \phi^{1-k}(1)] = \psi(1) - \psi(\alpha) = \int_\alpha^1 \frac{d\nu}{\phi^k(t)}.$$

The claim of the lemma follows letting $\alpha \to 0^+$. A similar argument can be applied to the case $k = 1$. □

Proposition 5.2.7. *Let ν be a non-atomic Radon measure on $[0, +\infty)$ and let $k \in \{1, \ldots, n\}$. Then* (5.6) *holds if and only if*

$$\exists\, \delta > 0 \text{ such that } \nu([0, \delta]) = 0. \tag{5.8}$$

Proof. We suppose that there exists $\delta > 0$ such that $[0, \delta] \cap \mathrm{supp}(\nu) = \varnothing$. Then we have, for every $f \in \mathcal{QC}(\mathbb{R}^n)$,

$$\mu(f) = \int_\delta^{M(f)} V_k(L_t(f))d\nu(t) \le V_k(L_\delta(f)) \int_\delta^{M(f)} d\nu(t) \tag{5.9}$$

$$= V_k(L_\delta(f))(\nu([0, M(f)]) - \nu([0, \delta])) < +\infty. \tag{5.10}$$

Vice versa, assume that (5.6) holds. By contradiction we suppose that for all $\delta > 0$ we have $\nu([0, \delta]) > 0$. We define

$$\phi(t) = \nu([0, t]), \quad t \in [0, 1]$$

then ϕ is continuous (as ν is non-atomic) and increasing; moreover $\phi(0) = 0$ and $\phi(t) > 0$, for all $t > 0$. The function

$$\psi(t) = \frac{1}{t\phi(t)}, \quad t \in (0, 1],$$

is continuous and strictly decreasing. Its inverse ψ^{-1} is defined in $[\psi(1), \infty)$; we extend it to $[0, \psi(1))$ setting

$$\psi^{-1}(r) = 1 \quad \forall\, r \in [0, \psi(1)).$$

Then

$$V_1(\{r \in [0, +\infty) :\ \psi^{-1}(r) \ge t\}) = \begin{cases} \psi(t), & \forall\, t \in (0, 1] \\ 0 & \forall\, t > 1. \end{cases}$$

We define now the function $f\colon \mathbb{R}^n \to \mathbb{R}$ as

$$f(x) = \psi^{-1}(||x||), \quad \forall\, x \in \mathbb{R}^n.$$

Then

$$L_t(f) = \{x \in \mathbb{R}^n :\ \psi(||x||) \ge t\} = \mathbb{B}_{\frac{1}{t\phi(t)}},$$

and

$$V_k(L_t(f)) = c\, \frac{1}{t^k \phi^k(t)} \quad \forall\, t \in (0, 1],$$

where $c > 0$ depends on n and k. Hence, by Lemma 5.2.6, it holds

$$\int_0^{+\infty} V_k(L_t(f))d\nu(t) = \int_0^1 V_k(L_t(f))d\nu(t) \ge c \int_0^{+\infty} \frac{d\nu(t)}{\phi^k(t)} = +\infty.$$

□

The following proposition summarizes some of the results we have found so far.

Proposition 5.2.8. *Let $k \in \{0, \dots, n\}$ and let ν be a Radon measure on $[0, +\infty)$ which is non-atomic and, if $k \geq 1$, verifies condition* (5.8). *Then the map $\mu\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ defined by* (5.7) *is an invariant, continuous and increasing valuation.*

5.2.3 The connection between the two types of integral valuations

When the regularity of the involved functions permits, the two types of integral valuations that we have seen can be obtained one from each other by a simple integration by parts.

Let $k \in \{0, \dots, n\}$ and $\phi \in C^1([0, \infty))$ be such that $\phi(0) = 0$. For simplicity, we may assume also that ϕ has compact support. Let $f \in \mathcal{QC}(\mathbb{R}^n)$. By the definition of distributional derivative of an increasing function we have:

$$\int_0^{+\infty} \phi(t)\, dS_k(f;t) = \int_0^{+\infty} \phi'(t) V_k(L_t(f)) dt.$$

If we further decompose $-\phi'$ as the difference of two non-negative functions, and we denote by ν_1 and ν_2 the Radon measures having those functions as densities, we get

$$\int_0^{+\infty} \phi(t)\, dS_k(f;t) = \int_0^{+\infty} V_k(L_t(f)) d\nu_1(t) - \int_0^{+\infty} V_k(L_t(f)) d\nu_2(t).$$

The assumption that ϕ has compact support can be removed by a standard approximation argument. In this way we have seen that each valuation of the form (5.3), if ϕ is regular, is the difference of two monotone integral valuations of type (5.7).

Vice versa, let ν be a Radon measure (with support contained in $[\delta, +\infty)$, for some $\delta > 0$), and assume that it has a smooth density w.r.t. the Lebesgue measure:

$$d\nu(t) = \phi'(t) dt$$

where $\phi \in C^1([0, +\infty))$, and it has compact support. Then

$$\int_0^{+\infty} V_k(L_t(f))\, d\nu(t) = \int_0^{+\infty} \phi(t)\, dS_k(f;t).$$

Also in this case the assumption that the support of ν is compact can be removed. In other words each integral monotone valuation, with sufficiently smooth density, can be written in the form (5.3).

5.2.4 The extremal cases

We conclude this section with some remarks concerning the extremal cases, i.e. $k = 0$ and $k = n$.

If $k = 0$, then we recall that V_0 is the Euler-Poincaré characteristic, so it holds $V_0(L_t(f)) = 1$, for all $0 < t \leq M(f)$ and for all $f \in \mathcal{QC}(\mathbb{R}^n)$. Hence the integral valuation is written as

$$\mu_0(f) = \phi(M(f)),$$

for some continuous function $\phi\colon [0, +\infty) \to \mathbb{R}$.

We do not need the δ-condition for the finiteness of μ_0, moreover we can prove the following statement.

Proposition 5.2.9. *The functional $\mu_0(f) = \phi(M(f))$ is a continuous, invariant valuation on $\mathcal{QC}(\mathbb{R}^n)$, for any fixed function $\phi\colon [0,+\infty) \to \mathbb{R}$ continuous.*

Proof. The continuity of μ_0 follows from Proposition 3.2.5 and from the continuity condition of ϕ. Since T is a rigid motion on $\mathbb{R}^n$, then $M(f) = M(T(f))$, for every $f \in \mathcal{QC}(\mathbb{R}^n)$ and hence μ_0 is invariant.

Moreover the valuation property comes from the finite additivity property of $M(f)$ as we have established in Proposition 3.1.11. □

For $k = n$ we have another simple representation of integral valuation due to the Layer Cake Principle (see for instance [8] or [42]).

Proposition 5.2.10. *Let ϕ be a continuous function on $[0,+\infty)$ verifying* (5.8)*. Then for every $f \in \mathcal{QC}(\mathbb{R}^n)$ we have*

$$\int_0^{+\infty} \phi(t)\, dS_n(f;t) = \int_{\mathbb{R}^n} \phi(f(x))dx. \tag{5.11}$$

Proof. As ϕ can be written as the difference of two non-negative continuous function, and (5.11) is linear w.r.t. ϕ, there is no restriction if we assume that $\phi \geq 0$. In addition we suppose initially that $\phi \in C^1([0,\infty))$ and it has compact support. Fix $f \in \mathcal{QC}(\mathbb{R}^n)$; by the definition of distributional derivative, we have

$$\int_0^{+\infty} \phi(t)\, dS_n(f;t) = \int_0^{+\infty} V_n(L_t(f))\phi'(t)dt.$$

There exist $\phi_1, \phi_2 \in C^1([0,+\infty))$, strictly increasing, such that $\phi = \phi_1 - \phi_2$. Now one has

$$\int_0^{+\infty} V_n(L_t(f))\phi_1'(t)dt = \int_0^{+\infty} V_n(\{x \in \mathbb{R}^N \,:\, \phi_1(f(x)) \geq s\})ds = \int_{\mathbb{R}^n} \phi_1(f(x))dx,$$

where in the last equality we have used the Layer Cake Principle [Proposition 1.78 in [8]]. Applying the same argument to ϕ_2 we obtain (5.11) when ϕ is smooth and compactly supported. For the general case, we apply the result obtained in the previous part of the proof to a sequence ϕ_i, $i \in \mathbb{N}$, of functions in $C^1([0,+\infty))$, with compact support, which converges uniformly to ϕ on compact subsets of $(0,+\infty)$. The conclusion follows from a direct application of the Dominated Convergence Theorem. □

5.3 Connection with $\mathcal{K}^n$

In this section we will focus on connections between valuations on $\mathcal{QC}(\mathbb{R}^n)$ and $\mathcal{K}^n$. We have already seen how to define valuations on quasi-concave functions with intrinsic volumes, now we present the way to define valuations on convex bodies starting with $\mu\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$.

If μ is a valuation on $\mathcal{QC}(\mathbb{R}^n)$, then we fix $t > 0$ and we define

$$\tilde{\mu}_t\colon \mathcal{K}^n \to \mathbb{R} \quad \text{as} \quad \tilde{\mu}_t(K) = \mu(t\chi_K).$$

By properties of maximum and minimum operators related to union and intersection between convex bodies we can see that $\tilde{\mu}_t$ is a valuation on $\mathcal{K}^n$. Indeed for every K and $H \in \mathcal{K}^n$ such that $H \cup K \in \mathcal{K}^n$

one has

$$\begin{aligned}\tilde{\mu}_t(K \cup H) + \tilde{\mu}_t(K \cap H) &= \mu(t\chi_{K\cup H}) + \mu(t\chi_{K\cap H}) \\ &= \mu(t\chi_K \vee t\chi_H) + \mu(t\chi_K \wedge t\chi_H) = \mu(t\chi_K) + \mu(t\chi_H) \\ &= \tilde{\mu}_t(K) + \tilde{\mu}_t(H).\end{aligned}$$

The valuation $\tilde{\mu}_t$ is also invariant, indeed let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a rigid motion, then

$$\tilde{\mu}_t(T(K)) = \mu(t\chi_{T(K)}) = \mu(t\chi_K \circ T^{-1}) = \mu(t\chi_K) = \tilde{\mu}_t(K).$$

Meanwhile $\tilde{\mu}_t$ is only continuous w.r.t. decreasing sequences of convex bodies. If $K_i \searrow K$, then $t\chi_{K_i} \searrow t\chi_K$, because $K = \bigcap_{i\geq 1} K_i$, hence if $x \in K$, then $x \in K_i$, for all i, and

$$t\chi_{K_i}(x) = t = t\chi_K(x)$$

for all $i \in \mathbb{N}$. Moreover if $x \notin K$, there exists $\bar{i}$ such that $x \notin K \subseteq K_i$, for all $i \geq \bar{i}$ and then

$$t\chi_{K_i}(x) = 0 = t\chi_K(x)$$

for all $i \in \mathbb{N}$.

About increasing sequences, let us consider $K_i = \mathbb{B}_{1-\frac{1}{i}}$ and $K = \mathbb{B}^n$. Clearly $K_i \nearrow K$, but $\chi_{K_i} \nearrow \chi_K$ for a.e. x (not for $x \in \partial K$), then we can not say that $\tilde{\mu}_t(K_i) \to \tilde{\mu}_t(K)$.
Anyway, even with this asymmetry between decreasing and increasing sequences, we will be able to obtain Hadwiger type result and a characterization theorem like Volume Theorem.

Lemma 5.3.1. *Let $\mu\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ be a continuous valuation. Then, for a fixed convex body $K \in \mathcal{K}^n$, the function*

$$(0, +\infty) \ni t \mapsto \tilde{\mu}_t(K) = \mu(t\chi_K)$$

is continuous.

Proof. Let us fix $t_0 \in (0, +\infty)$. Suppose $t_i \searrow t_0$, then it holds that $t_i\chi_K \searrow t_0\chi_K$ pointwise and by continuity of μ we have

$$\tilde{\mu}_{t_i}(K) \to \tilde{\mu}_{t_0}(K).$$

In a similar way we have the convergence of $\tilde{\mu}_{t_i}(K)$ to $\tilde{\mu}_{t_0}(K)$ for an increasing sequence $t_i \nearrow t_0$. □

Proposition 5.3.2. *Let $\mu\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ be a continuous valuation. If*

$$\mu(t\chi_P) = 0,\ \forall t > 0,\ \forall P \in \mathcal{P}^n,$$

then $\mu \equiv 0$.

The proof of this result follows closely the lines of that one of Theorem 1.2 in [20], we refer also to Proposition 4.3 in [21].

Proof. **First step.** We know, by [64], that for every $K \in \mathcal{K}^n$ there exists a decreasing sequence of polytopes P_i, such that P_i converges to K. Since the sequence is decreasing, we have

$$t\chi_{P_i} \searrow t\chi_K, \quad \text{as} \quad i \to +\infty.$$

By continuity of the valuation we have $\mu(t\chi_{P_i}) \to \mu(t\chi_K)$. Since $\mu(t\chi_{P_i}) = 0$ for all $i \in \mathbb{N}$, we obtain $\mu(t\chi_K) = 0$ for all $t > 0$ and $K \in \mathcal{K}^n$.

Second step. We prove that μ vanishes on simple functions g of the form

$$g = \bigvee_{i=1}^{m} t_i \chi_{K_i},$$

where $m \in \mathbb{N}$ is fixed, $0 < t_1 < \cdots < t_m$, and $K_1, \ldots, K_m$ are convex bodies such that $K_1 \supset K_2 \supset \cdots \supset K_m$. We proceed by induction on m.

If $m = 1$, the assertion follows from the previous step. Assume, now, that the claim has been proved up to some $m \geq 1$. Let

$$g = \bigvee_{i=1}^{m+1} t_i \chi_{K_i}$$

be a simple function. Using the valuation property of μ, we can write

$$\mu(g) = \mu\left(\bigvee_{i=1}^{m} t_i \chi_{K_i}\right) + \mu(t_{m+1}\chi_{K_{m+1}}) - \mu\left(\bigvee_{i=1}^{m} t_i \chi_{K_i} \wedge t_{m+1}\chi_{K_{m+1}}\right) \tag{5.12}$$

$$= -\mu\left(\bigvee_{i=1}^{m} t_i \chi_{K_i} \wedge t_{m+1}\chi_{K_{m+1}}\right). \tag{5.13}$$

The last equality holds as $\mu(t_{m+1}\chi_{K_{m+1}}) = \mu(\bigvee_{i=1}^{m} t_i \chi_{K_i}) = 0$ by the induction assumption. Since

$$\bigvee_{i=1}^{m} t_i \chi_{K_i} \wedge t_{m+1}\chi_{K_{m+1}} = t_1 \chi_{K_{m+1}},$$

we obtain $\mu(g) = 0$ again by the induction assumption.

Third step. We use Theorem 3.2.6 and the continuity of valuation to extend the result from simple functions to quasi-concave ones with compact support. Then we apply Theorem 3.2.7 to obtain $\mu \equiv 0$ for all quasi-concave function.

□

5.4 Simple valuations

We have already defined simple valuations on $\mathcal{QC}(\mathbb{R}^n)$; we recall that means $\mu(f) = 0$, for every quasi-concave function f with $\dim(\mathrm{supp}(f)) < n$.

In this section we present a version of the Volume Theorem for quasi-concave functions. The statement is the following.

Theorem 5.4.1. *A map $\mu\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ is an invariant, continuous (w.r.t. pointwise and monotone convergence) and simple valuation if and only if there exists $\phi_n\colon [0, +\infty) \to \mathbb{R}$ continuous, with $\phi_n \equiv 0$ in $[0.\delta]$, for some $\delta > 0$, such that*

$$\mu(f) = \int_0^{+\infty} \phi_n(t) dS_n(f;t) = \int_{\mathbb{R}^n} \phi_n(f(x)) dx, \tag{5.14}$$

for every $f \in \mathcal{QC}(\mathbb{R}^n)$.

We will focus now on the proof of this result.
We already know, by the theory of integral valuations, that a functional like $\int_0^{+\infty} \phi_n(t) dS_n(f;t)$, with the δ-condition for ϕ_n, is an invariant and continuous valuation. Moreover by the second integral representation, it is easy to see that it is also simple.

Vice versa, we have to prove now that every valuation μ invariant, continuous and simple is of the form (5.14).

We start with the following observation: if μ on $\mathcal{QC}(\mathbb{R}^n)$ is simple, then $\tilde{\mu}_t$ is simple on $\mathcal{K}^n$, for every $t > 0$, since we have $\mathrm{supp}(t\chi_K) = K$.

By Theorem $2.2.5$, there exists a constant, depending on $t > 0$, $\phi_n(t) \in \mathbb{R}$, such that

$$\tilde{\mu}(K) = \phi_n(t) V_n(K),$$

for every $K \in \mathcal{K}^n$. ϕ_n is our function. We summarize the previous considerations in the following proposition.

Proposition 5.4.2. *Let μ be an invariant, continuous and simple valuation on $\mathcal{QC}(\mathbb{R}^n)$. Then there exists a continuous function ϕ_n on $[0, +\infty)$, such that*

$$\mu(t\chi_K) = \phi_n(t)\, V_n(K)$$

for every $t \geq 0$ and for every $K \in \mathcal{K}^n$.

We have to prove that we can extend this result from characteristic functions $t\chi_K$ to all quasi-concave functions $f \in \mathcal{QC}(\mathbb{R}^n)$, and that ϕ_n satisfies the δ-condition. For this goal we pass from characteristic functions to simple ones first and then we will apply density results.

First of all we recall that by $-S_n(f;\cdot)$ we denote the distributional derivative of $\varphi_n(t) = V_n(L_t(f))$.

In the case of $f = s\chi_K$, where $s > 0$ and $K \in \mathcal{K}^n$, we can compute explicitly the measure and it becomes

$$S_n(s\chi_K;\cdot) = V_n(K)\delta_s(\cdot),$$

where δ_s denotes the Dirac measure on $s > 0$.
Now we want to extend this formula from $f = s\chi_K$ to $f = \bigvee_{j=1}^m t_j\chi_{K_j} \in \mathcal{SQC}(\mathbb{R}^n)$. We can see that in this case

$$S_n\Big(\bigvee_{j=1}^{m} t_j\chi_{K_j};\cdot\Big) = \sum_{j=1}^{m-1}(V_n(K_j) - V_n(K_{j+1}))\delta_{t_j}(\cdot) + V_n(K_m)\delta_{t_m}(\cdot). \tag{5.15}$$

Remark 5.4.3. We also observe that the same formula holds for V_k, with $k < n$.

Lemma 5.4.4. *Let μ be an invariant, continuous and simple valuation on $\mathcal{QC}(\mathbb{R}^n)$. Then, for every simple function $f \in \mathcal{SQC}(\mathbb{R}^n)$ we have*

$$\mu(f) = \int_0^{+\infty} \phi_n(t)\, dS_n(f;t),$$

with $\phi_n\colon [0, +\infty) \to \mathbb{R}$ the function defined by Proposition $5.4.2$.

Proof. Let $f = \bigvee_{j=1}^m t_j\chi_{K_j} \in \mathcal{SQC}(\mathbb{R}^n)$ be a simple function. We prove the following formula

$$\mu(f) = \sum_{j=1}^{m-1} \phi_n(t_j)(V_n(K_j) - V_n(K_{j+1})) + \phi_n(t_m)V_n(K_m); \tag{5.16}$$

by (5.15), this is equivalent to the statement of the lemma. Equality (5.16) will be proved by induction on m. For $m = 1$ its validity follows from Proposition 5.4.2. Assume that it has been proved up to $m - 1$. Set

$$g = t_1\chi_{K_1} \vee \cdots \vee t_{m-1}\chi_{K_{m-1}}, \quad h = t_m\chi_{K_m}.$$

We have that $g, h \in \mathcal{SQC}(\mathbb{R}^n)$ and

$$g \vee h = f \in \mathcal{SQC}(\mathbb{R}^n), \quad g \wedge h = t_{m-1}\chi_{K_m}.$$

Using the valuation property of μ and Proposition 5.4.2 we get

$$\mu(f) = \mu(g \vee h) = \mu(g) + \mu(h) - \mu(g \wedge h) \tag{5.17}$$

$$= \mu(g) + \phi(t_m)V_N(K_m) - \phi(t_{m-1})V_N(K_m). \tag{5.18}$$

On the other hand, by induction

$$\mu(g) = \sum_{j=1}^{m-2} \phi_n(t_j)(V_n(K_j) - V_n(K_{j+1})) + \phi_n(t_{m-1})V_n(K_{m-1}).$$

The last two equalities complete the proof. □

Proof of Theorem 5.4.1.

Now we are ready to extend the result to all quasi-concave functions.

Step 1. Let us fix $f \in \mathcal{QC}_c(\mathbb{R}^n)$, we know by Theorem 3.2.6 that there exists an increasing sequence, f_i, of simple functions converging pointwise to f. By continuity we have

$$\mu(f) = \lim_{i\to+\infty} \mu(f_i) = \lim_{i\to+\infty} \int_0^{+\infty} \phi_n(t)dS_n(f_i;t).$$

By Lemma 3.2.2, a further consequence is that

$$\lim_{i\to+\infty} \varphi_{n,i}(t) = \varphi_n(t) \quad \text{for a.e. } t \in (0,\infty),$$

where

$$\varphi_{n,i}(t) = V_n(L_t(f_i)), \quad i \in \mathbb{N}, \quad \varphi_n(t) = V_n(L_t(f))$$

for $t > 0$. We consider now the sequence of measures $S_n(f_i;\cdot)$, $i \in \mathbb{N}$; the total variation of these measures in $(0,\infty)$ is uniformly bounded by $V_n(K)$, moreover they are all supported in $(0, M(f))$. As they are the distributional derivatives of the functions $\varphi_{n,i}$, which converges a.e. to φ_n, we have that (see for instance [8, Proposition 3.13]) the sequence $S_n(f_i;\cdot)$ converges weakly in the sense of measures to $S_n(f;\cdot)$. This implies that

$$\lim_{i\to+\infty} \int_0^{+\infty} \bar{\phi}(t)\, dS_n(f_i;t) = \int_0^{+\infty} \bar{\phi}(t)\, dS_n(f;t) \tag{5.19}$$

for every function $\bar{\phi}$ continuous in $(0, \infty)$, such that $\bar{\phi}(0) = 0$ and $\bar{\phi}(t)$ is identically zero for t sufficiently large. In particular (recalling that $\phi(0) = 0$), we can take $\bar{\phi}$ such that it equals ϕ in $[0, M(f)]$. Hence, as the support of the measures $S_n(f_i; \cdot)$ is contained in this interval, we have that (5.19) holds for ϕ as well. This proves the validity of (5.14) for functions with bounded support.

Step 2. This is the most technical part of the proof. The main goal here is to prove that ϕ is identically zero in some right neighborhood of the origin. Let $f \in \mathcal{QC}(\mathbb{R}^n)$. For $i \in \mathbb{N}$, let

$$f_i = f \wedge (M(f)\chi_{\mathbb{B}_i}).$$

The function f_i coincides with f in $\mathbb{B}_i$ and vanishes in $\mathbb{R}^n \setminus \mathbb{B}_i$; in particular it has bounded support. Moreover, the sequence f_i, $i \in \mathbb{N}$, is increasing and converges pointwise to f in $\mathbb{R}^n$. Hence

$$\mu(f) = \lim_{i \to +\infty} \mu(f_i) = \lim_{i \to +\infty} \int_0^{+\infty} \phi_n(t)\, dS_n(f_i; t).$$

Let $\phi_{n,+}$ and $\phi_{n,-}$ be the positive and negative parts of ϕ_n, respectively. We have that

$$\lim_{i \to +\infty} \left[\int_0^{+\infty} \phi_{n,+}(t)\, dS_n(f_i; t) + \int_0^{+\infty} \phi_{n,-}(t)\, dS_n(f_i; t) \right]$$

exists and it is finite. We want to prove that this implies that $\phi_{n,+}$ and $\phi_{n,-}$ vanishes identically in $[0, \delta]$ for some $\delta > 0$.

By contradiction, assume that this is not true for $\phi_{n,+}$. Then there exist three sequences t_i, r_i and ϵ_i, $i \in \mathbb{N}$, with the following properties: t_i tends decreasing to zero; $r_i > 0$ is such that the intervals $C_i = [t_i - r_i, t_i + r_i]$ are contained in $(0, 1]$ and pairwise disjoint; $\phi_{n,+}(t) \geq \epsilon_i > 0$ for $t \in C_i$. Let

$$C = \bigcup_{i \in \mathbb{N}} C_i\,, \quad \Omega = (0, 1] \setminus C.$$

Next we define a function $\gamma : (0, 1] \to [0, +\infty)$ as follows. $\gamma(t) = 0$ for every $t \in \Omega$ while, for every $i \in \mathbb{N}$, γ is continuous in C_i and

$$\gamma(t_i \pm r_i) = 0, \quad \int_{C_i} \gamma(t) dt = \frac{1}{\epsilon\, i}.$$

Note in particular that γ vanishes on the support of $\phi_{n,-}$ intersected with $(0, 1]$. We also set

$$g(t) = \gamma(t) + 1 \quad \forall\, t > 0.$$

Observe that

$$\int_0^1 \phi_{n,-}(t) g(t) dt = \int_0^1 \phi_{n,-}(t) dt < +\infty.$$

On the other hand

$$\int_0^1 \phi_{n,+}(t) g(t) dt \geq \int_0^1 \phi_n(t) \gamma(t) dt = \sum_{i=1}^{\infty} \int_{C_i} \phi_{n,+}(t) \gamma(t) dt \tag{5.20}$$

$$\geq \sum_{i=1}^{+\infty} \epsilon_i \int_{C_i} \gamma(t) dt = +\infty. \tag{5.21}$$

Let

$$G(t) = \int_t^1 g(s)ds \quad \text{and} \quad \rho(t) = [G(t)]^{1/n}, \quad 0 < t \leq 1.$$

As γ is non-negative, g is strictly positive, and continuous in $(0,1)$. Hence G is strictly decreasing and continuous, and the same holds for ρ. Let

$$S = \sup_{(0,1]} \rho = \lim_{t \to 0^+} \rho(t),$$

and let $\rho^{-1} : [0,S) \to \mathbb{R}$ be the inverse function of ρ. If $S < \infty$, we extend ρ^{-1} to be zero in $[S,+\infty)$. In this way, ρ^{-1} is continuous in $[0,+\infty)$, and $C^1([0,S))$. Let

$$f(x) = \rho^{-1}(\|x\|), \quad \forall\, x \in \mathbb{R}^n.$$

For $t > 0$ we have

$$L_t(f) = \begin{cases} \{x \in \mathbb{R}^n \,:\, \|x\| \leq \rho(t)\} & \text{if } t \leq 1, \\ \varnothing & \text{if } t > 1. \end{cases}$$

In particular $f \in \mathcal{QC}(\mathbb{R}^n)$. Consequently,

$$V_n(L_t(f)) = c\,\rho^n(t) = c\,G(t) \quad \forall\, t \in (0,1],$$

where $c > 0$ is a dimensional constant, and $S_n(f;\cdot)$ has density, w.r.t. dt, equals to $c\,g(t)$.

By the previous considerations

$$\int_0^{+\infty} \phi_{n,+}(t)dS_n(f;t) = c\int_0^{+\infty} \phi_{n,+}(t)g(t)dt = +\infty, \quad \int_0^{+\infty} \phi_{n,+}(t)dS_n(f;t) < +\infty.$$

Clearly we also have that

$$\int_0^{+\infty} \phi_{n,+}(t)dS_n(f;t) = \lim_{i\to\infty} \int_0^{+\infty} \phi_{n,+}(t)dS_n(f_i;t),$$

and the same holds for $\phi_{n,-}$; here f_i is the sequence approximating f defined before. We reached a contradiction.

Step 3. The conclusion of the proof proceeds as follows. Let $\bar{\mu} : \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ be defined by

$$\bar{\mu}(f) = \int_0^{+\infty} \phi_n(t)\, dS_n(f;t).$$

By the previous step, and by the results of integral valuations, this is well-defined, and it is an invariant and continuous valuation. Hence the same properties are shared by $\tilde{\mu} = \mu - \bar{\mu}$; on the other hand, by Step 1 and the definition of $\bar{\mu}$, this vanishes on functions with bounded support and then on $f = t\chi_P$, for every $t > 0$ and $P \in \mathcal{P}^n$. By Proposition 5.3.2 we have

$$\tilde{\mu}(f) = 0, \ \forall\, f \in \mathcal{QC}(\mathbb{R}^n),$$

and then

$$\mu(f) = \int_0^{+\infty} \phi_n(t)dS_n(f;t),$$

for all $f \in \mathcal{QC}(\mathbb{R}^n)$.

□

5.5 Characterization Theorem

We are now at the main result of this chapter, a complete characterization result for continuous and rigid motion invariant valuations.

Theorem 5.5.1. *A map $\mu\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ is an invariant and continuous, w.r.t. pointwise and monotone convergence, valuation if and only if there exist $(n+1)$ continuous functions ϕ_k, $k = 0, \dots, n$ defined on $[0, +\infty)$, and $\delta > 0$ such that: $\phi_k \equiv 0$ in $[0, \delta]$ for every $k = 1, \dots, n$, and*

$$\mu(f) = \sum_{k=0}^{n} \int_0^{+\infty} \phi_k(t) dS_k(f; t) \quad \forall\, f \in \mathcal{QC}(\mathbb{R}^n).$$

Proof. We proceed by induction on n. For the first step of induction, let μ be an invariant and continuous valuation on $\mathcal{QC}(\mathbb{R})$. For $t > 0$ let

$$\phi_0(t) = \mu(t\chi_{\{0\}}).$$

This is a continuous function in $\mathbb{R}$, with $\phi_0(0) = 0$. We consider the map $\mu_0\colon \mathcal{QC}(\mathbb{R}) \to \mathbb{R}$,

$$\mu_0(f) = \phi_0(M(f))$$

where as usual $M(f) = \max_{\mathbb{R}} f$. By what we have seen about integral valuations, this is an invariant and continuous valuation. Note that it can be written in the form

$$\mu_0(f) = \int_0^{+\infty} \phi_0(t)\, dS_0(f; t).$$

Next we set $\bar{\mu} = \mu - \mu_0$; this is still an invariant and continuous valuation, and it is also simple. Indeed, if $f \in \mathcal{QC}(\mathbb{R})$ is such that $\dim(\mathrm{supp}(f)) = 0$, this is equivalent to say that

$$f = t\chi_{\{x_0\}}$$

for some $t \geq 0$ and $x_0 \in \mathbb{R}$. Hence

$$\mu(f) = \mu(t\chi_{\{0\}}) = \phi_0(t) = \mu_0(f).$$

Therefore we may apply Theorem 5.4.1 to μ_1 and deduce that there exists a function $\phi_1 \in C([0, +\infty))$, which vanishes identically in $[0, \delta]$ for some $\delta > 0$, and such that

$$\bar{\mu}(f) = \int_0^{+\infty} \phi_1(t)\, dS_1(f; t) \quad \forall\, f \in \mathcal{QC}(\mathbb{R}).$$

The proof in the one-dimensional case is complete.

We suppose that the Theorem holds up to dimension $n-1$. Let H be an hyperplane of $\mathbb{R}^n$ and define $\mathcal{QC}_H(\mathbb{R}^n) = \{f \in \mathcal{QC}(\mathbb{R}^n) : \mathrm{supp}(f) \subseteq H\}$. $\mathcal{QC}_H(\mathbb{R}^n)$ can be identified as $\mathcal{QC}(\mathbb{R}^{n-1})$; moreover μ restricted to $\mathcal{QC}_H(\mathbb{R}^n)$ is trivially still an invariant and continuous valuation. By the induction assumption, there exists $\phi_k \in C([0, +\infty))$, $k = 0, \dots, n-1$, such that

$$\mu(f) = \sum_{k=0}^{n-1} \int_0^{+\infty} \phi_k(t)\, dS_k(f; t) \quad \forall\, f \in \mathcal{QC}_H(\mathbb{R}^n).$$

In addition, there exists $\delta > 0$ such that $\phi_1, \dots, \phi_{n-1}$ vanish in $[0, \delta]$. Let $\bar{\mu}\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ as

$$\bar{\mu}(f) = \sum_{k=0}^{n-1} \int_0^{+\infty} \phi_k(t)\, dS_k(f;t).$$

This is well-defined for $f \in \mathcal{QC}(\mathbb{R}^n)$ and it is an invariant and continuous valuation. The difference $\mu - \bar{\mu}$ is simple; applying Theorem 5.4.1 to it, as in the one-dimensional case, we complete the proof. □

Remark 5.5.2. If we apply Theorem 5.5.1 to $f = s\chi_K$ with $s > 0$ and $K \in \mathcal{K}^n$, we obtain a Hadwiger type formula for convex bodies

$$\tilde{\mu}_s(K) = \sum_{i=0}^{n} \phi_i(s) V_i(K).$$

5.6 Monotone valuations

We focus now on monotone, continuous ad invariant valuations, as usual continuous w.r.t. pointwise and monotone convergence and invariant w.r.t. rigid motions.

We will show a counterpart of Theorem 5.5.1 and we will use integral monotone valuations defined by Radon measure in Section 5.2.2.
We consider only the case of increasing valuations, and that allows us to say that $\mu(f) \geq 0$, for every quasi-concave function f on $\mathbb{R}^n$.

Theorem 5.6.1. *A map μ is an invariant, continuous and monotone increasing valuation on $\mathcal{QC}(\mathbb{R}^n)$ if and only if there exists $(n+1)$ Radon measures on $[0, +\infty)$, ν_k, $k = 0, \dots, n$, such that each ν_k is non-negative, non-atomic and, for $k \geq 1$, the support of ν_k is contained in $[\delta, +\infty)$ for a suitable $\delta > 0$, and*

$$\mu(f) = \sum_{k=0}^{n} \int_0^{+\infty} V_k(L_t(f))\, d\nu_k(t), \quad \forall\, f \in \mathcal{QC}(\mathbb{R}^n).$$

The proof is divided into three parts.

5.6.1 Identification of the measures ν_k, $k = 0, \dots, n$.

We proceed as in the proof of Proposition 5.4.2. Fix $t > 0$ and consider the map $\tilde{\mu}_t\colon \mathcal{K}^n \to \mathbb{R}$,

$$\tilde{\mu}_t(K) = \mu(t\chi_K), \quad K \in \mathcal{K}^n.$$

This is a rigid motion invariant valuation on $\mathcal{K}^n$ and, as μ is increasing, $\tilde{\mu}_t$ has the same property. Hence, by Proposition 2.2.3, there exist $(n+1)$ coefficients, depending on t, that we denote by $\psi_k(t)$, $k = 0, \dots, n$, such that

$$\tilde{\mu}_t(K) = \sum_{k=0}^{n} \psi_k(t) V_k(K) \quad \forall\, K \in \mathcal{K}^n. \tag{5.22}$$

We prove that each ψ_k is continuous and monotone in $(0,+\infty)$. Let us fix the index $k \in \{0,\dots,n\}$, and let Δ_k be a closed k-dimensional ball in $\mathbb{R}^n$ of radius 1. We have

$$V_j(\Delta_k) = 0 \quad \forall\, j = k+1,\dots,n,$$

and

$$V_k(\Delta_k) =: c(k) > 0.$$

Fix $r \geq 0$; for every j, V_j is positively homogeneous of degree j, hence, for $t > 0$,

$$\mu(t\chi_{r\Delta_k}) = \sum_{j=0}^{k} r^j V_j(\Delta_k)\psi_j(t).$$

Consequently

$$\psi_k(t) = \frac{1}{V_k(\Delta_k)} \cdot \lim_{r\to+\infty} \frac{\mu(t\chi_{r\Delta_k})}{r^k}.$$

By the properties of μ, the function $t \to \mu(t\chi_{r\Delta_k})$ is non-negative, increasing and vanishes for $t = 0$, for every $r \geq 0$; these properties are inherited by ψ_k.

As for continuity, we proceed in a similar way. To prove that ψ_0 is continuous we observe that the function

$$t \;\to\; \mu(t\chi_{\Delta_0}) = \psi_0(t)$$

is continuous, by the continuity of μ. Assume that we have proved that $\psi_0,\dots,\psi_{k-1}$ are continuous. Then by the equality

$$\mu(t\chi_{\Delta_k}) = \sum_{j=1}^{k} V_j(\Delta_k)\psi_j(t),$$

it follows that ψ_k is continuous.

We have proved the following fact.

Proposition 5.6.2. *Let μ be an invariant, continuous and increasing valuation on $\mathcal{QC}(\mathbb{R}^n)$. Then there exists $(n+1)$ functions $\psi_0,\dots,\psi_n$ defined in $[0,+\infty)$, such that* (5.22) *holds for every $t \geq 0$ and for every K. In particular each ψ_k is continuous, increasing, and vanishes at $t = 0$.*

For every $k \in \{0,\dots,n\}$ we denote by ν_k the distributional derivative of ψ_k. In particular as ψ_k is continuous, ν_k is non-atomic and

$$\psi_k(t) = \nu_k([0,t)), \quad \forall\, t \geq 0.$$

Since ψ_k are non-negative functions, by Proposition 2.2.3, ν_k are non-negative measures.

5.6.2 The case of simple functions

Let f be a simple function, i.e. is of the form

$$f = t_1\chi_{K_1} \vee \cdots \vee t_m\chi_{K_m}$$

with $0 < t_1 < \cdots < t_m$, $K_1 \supset \cdots \supset K_m$ and $K_i \in \mathcal{K}^n$ for every i. The following formula can be proved with the same method used for (5.16)

$$\mu(f) = \sum_{k=0}^{n} \sum_{i=1}^{m} (\psi_k(t_i) - \psi_k(t_{i-1})) V_k(L_{t_i}(f)), \tag{5.23}$$

where we have set $t_0 = 0$. As

$$\psi_k(t_i) - \psi_k(t_{i-1}) = \nu_k((t_{i-1}, t_i])$$

and $L_t(f) = K_i$ for every $t \in (t_{i-1}, t_i]$, we have

$$\mu(f) = \sum_{k=0}^{n} \int_0^{+\infty} V_k(L_t(f))\, d\nu_k(t). \tag{5.24}$$

In other words, we have proved Theorem 5.6.1 for simple functions.

5.6.3 Proof of Theorem 5.6.1

Let $f \in \mathcal{QC}(\mathbb{R}^n)$ and let f_i, $i \in \mathbb{N}$, be the sequence of functions built in the proof of Theorem 5.4.1, Step 2. We have seen that f_i is increasing and converges pointwise to f in $\mathbb{R}^n$. In particular, for every $k = 0, \ldots, n$, the sequence of functions $\varphi_{k,i}(t) = V_k(L_t(f_i))$, $t \geq 0$, $i \in \mathbb{N}$, is monotone increasing and it converges a.e. to $\varphi_k(t) = V_k(L_t(f))$ in $(0, +\infty)$. By Monotone Convergence Theorem, we have

$$\lim_{i \to +\infty} \int_0^{+\infty} V_k(L_t(f_i))\, d\nu_k(t) = \int_0^{+\infty} V_k(L_t(f))\, d\nu_k(t)$$

for every k. Using (5.24) and the continuity of μ we have that the representation formula (5.24) can be extended to every $f \in \mathcal{QC}(\mathbb{R}^n)$.

Note that for a simple function each term of the sum in the right hand-side is non-negative, hence we have that

$$\int_0^{+\infty} V_k(L_t(f))\, d\nu_k(t) < \infty \quad \forall\, f \in \mathcal{QC}(\mathbb{R}^n).$$

Applying Proposition 5.2.7 we obtain that, if $k \geq 1$, there exists $\delta > 0$ such that the support of ν_k is contained in $[\delta, +\infty)$. The proof is complete.

□

Chapter 6

Continuous and translation invariant valuations

We are now going to study valuations μ defined on $\mathcal{QC}(\mathbb{R}^n)$ that are continuous, as usual w.r.t. pointwise and monotone convergence, and invariant w.r.t. translations only, instead of rigid motion transformations.

Definition 6.0.1. *We denote by* $\mathrm{Val}(\mathcal{QC}(\mathbb{R}^n))$ *the set of all continuous and translation invariant valuations on quasi-concave functions.*

We will present a Decomposition Theorem for $\mathrm{Val}(\mathcal{QC}(\mathbb{R}^n))$ so we have to introduce the notion of homogeneous valuations. To define that, we recall the operation $\odot$ we have introduced in Chapter 3. For a fixed $\lambda > 0$, we define

$$\lambda \odot f(x) = f\left(\frac{x}{\lambda}\right) \quad \forall\, x \in \mathbb{R}^n.$$

We have already seen that for any $f \in \mathcal{QC}(\mathbb{R}^n)$, $\lambda \odot f$ is still a quasi-concave function, indeed it holds $L_t(\lambda \odot f) = \lambda L_t(f)$, for every $t > 0$.

Definition 6.0.2. *A valuation* $\mu \in \mathrm{Val}(\mathcal{QC}(\mathbb{R}^n))$ *is said to be* k*-homogeneous, with* $k \in \{0, \cdots, n\}$*, if*

$$\mu(\lambda \odot f) = \lambda^k \mu(f)$$

for all $f \in \mathcal{QC}(\mathbb{R}^n)$ *and* $\lambda > 0$*. The class of* k*-homogeneous valuations in* $\mathrm{Val}(\mathcal{QC}(\mathbb{R}^n))$ *will be denoted by* $\mathrm{Val}_k(\mathcal{QC}(\mathbb{R}^n))$.

Remark 6.0.3 ([21])**.** Let $\lambda > 0$ and $\mu \in \mathrm{Val}(\mathcal{QC}(\mathbb{R}^n))$. The functional $\overline{\mu}\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$\overline{\mu}(f) = \mu(\lambda \odot f), \quad \forall\, f \in \mathcal{QC}(\mathbb{R}^n),$$

belongs to $\mathrm{Val}(\mathcal{QC}(\mathbb{R}^n))$.

We recall also the reflection function (w.r.t. the origin),

$$R\colon \mathbb{R}^n \to \mathbb{R}^n, \text{ defined by } R(x) = -x.$$

Hence also in this case we have

$$f \circ R \in \mathcal{QC}(\mathbb{R}^n), \quad \forall\, f \in \mathcal{QC}(\mathbb{R}^n),$$

and it holds that $L_t(f \circ R) = -L_t(f)$, for all $t > 0$.

Definition 6.0.4. *We say that a valuation* $\mu \in \mathrm{Val}(\mathcal{QC}(\mathbb{R}^n))$ *is even if*

$$\mu(f \circ R) = \mu(f),$$

for all $f \in \mathcal{QC}(\mathbb{R}^n)$*. The class of even (resp.* k*-homogeneous and even) valuations in* $\mathrm{Val}(\mathcal{QC}(\mathbb{R}^n))$ *will be denoted by* $\mathrm{Val}^+(\mathcal{QC}(\mathbb{R}^n))$ *(resp.* $\mathrm{Val}_k^+(\mathcal{QC}(\mathbb{R}^n))$*).*

Proposition 6.0.5. *If* μ *is* k*-homogeneous on* $\mathcal{QC}(\mathbb{R}^n)$*, then for every choice of the parameter* $t > 0$*, the valuation* $\tilde{\mu}_t$ *is* k*-homogeneous on* $\mathcal{K}^n$*.*

Proof. The statement follows from the following facts:

$$\tilde{\mu}_t(\lambda K) = \mu(t\chi_{\lambda K}),$$

and

$$x \in \lambda K \ \Leftrightarrow\ \frac{x}{\lambda} \in K.$$

Hence we have $\tilde{\mu}_t(\lambda K) = \mu(\lambda \odot t\chi_K) = \lambda^k \mu_t(K)$. □

Remark 6.0.6. In general $\mu \in \mathrm{Val}(\mathcal{QC}(\mathbb{R}^n))$ does not imply that $\tilde{\mu}_t \in \mathrm{Val}(\mathcal{K}^n)$ because as we have seen in the previous chapter, $\tilde{\mu}_t$ is only continuous w.r.t decreasing sequences.

6.1 Decomposition Theorem for $\mathrm{Val}(\mathcal{QC}(\mathbb{R}^n))$

We are going to prove a McMullen type Theorem for valuations in $\mathrm{Val}(\mathcal{QC}(\mathbb{R}^n))$.

Theorem 6.1.1. *Let* $\mu\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ *be a continuous and translation invariant valuation. For all* $k = 0, \dots, n$ *there exists a unique* $\mu_k \in \mathrm{Val}_k(\mathcal{QC}(\mathbb{R}^n))$*, such that*

$$\mu = \sum_{k=0}^{n} \mu_k.$$

Proof. If we fix $t > 0$, then we can define $\widetilde{\mu}_t\colon \mathcal{P}^n \to \mathbb{R}$ as

$$\widetilde{\mu}_t(P) = \mu(t\chi_P) \quad \forall\, P \in \mathcal{P}^n,$$

which is a translation invariant valuation on $\mathcal{P}^n$. By Theorem 2.1.8, there exist $\widetilde{\mu}_{t,k}$, $k = 0, ..., n$, translation invariant and rational k-homogeneous valuations on $\mathcal{P}^n$, such that

$$\widetilde{\mu}_t(\lambda P) = \sum_{k=0}^{n} \lambda^k \widetilde{\mu}_{t,k}(P) \tag{6.1}$$

for every $P \in \mathcal{P}^n$ and for all rational $\lambda > 0$.

If we write (6.1) for $\lambda = 1$, we have

$$\mu(t\chi_P) = \widetilde{\mu}_{t,0}(P) + ... + \widetilde{\mu}_{t,n}(P).$$

Similarly, for $\lambda = 2$, we get

$$\mu(t\chi_{2P}) = \widetilde{\mu}_{t,0}(P) + ... + 2^n \widetilde{\mu}_{t,n}(P).$$

We can repeat this argument for $\lambda = 3, \dots, n+1$ to obtain a system of $n+1$ linear equations and a matrix of Vandermonde type associated to it:

$$M := \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & 2^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & n+1 & \dots. & (n+1)^n \end{pmatrix} \tag{6.2}$$

Since M is invertible, we have

$$\widetilde{\mu}_{t,k}(P) = \sum_{j=1}^{n+1} c_{k,j}\mu(t\chi_{jP}), \tag{6.3}$$

where $c_{k,j}$ are the coefficients of the inverse matrix of M, which are independent by t and P. Moreover, we observe that by Lemma 5.3.1, the continuity of μ implies that $\widetilde{\mu}_t$ is dilation continuous, i.e. $\lambda \mapsto \widetilde{\mu}_t(\lambda P)$ is continuous for every P.
Then equation (6.3) implies that all $\widetilde{\mu}_{t,k}$ are dilation continuous, for all $t > 0$. Since $\widetilde{\mu}_{t,k}$ are also rational k-homogeneous, we can conclude that they are k-homogeneous for all real positive number λ and for all $t > 0$.

Now, we want to determine a set of valuations μ_k on $\mathcal{QC}(\mathbb{R}^n)$, $k = 0, \dots, n$, such that $\mu_k(t\chi_P) = \widetilde{\mu}_{t,k}(P)$ for every polytope P and every $t > 0$. So we define $\mu_k \colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ by

$$\mu_k(f) = \sum_{j=1}^{n+1} c_{k,j}\mu(j \odot f), \quad \forall\, f \in \mathcal{QC}(\mathbb{R}^n). \tag{6.4}$$

For every k, μ_k is a continuous translation invariant valuation on $\mathcal{QC}(\mathbb{R}^n)$, because it is finite linear combination of functionals of the form $\mu(j \odot f)$ that belong to $\mathrm{Val}(\mathcal{QC}(\mathbb{R}^n))$ by Remark 6.0.3.

As next step, for $\lambda > 0$ and $k = 0, \dots, n$, we define $\overline{\mu}_k \colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ by

$$\overline{\mu}_k(f) = \mu_k(\lambda \odot f) - \lambda^k \mu_k(f),$$

which turns out to be a continuous valuation. Furthermore, for $P \in \mathcal{P}^n$,

$$\overline{\mu}_k(t\chi_P) = \mu_k(t\chi_{\lambda P}) - \lambda^k \mu_k(t\chi_P) = \widetilde{\mu}_{t,k}(\lambda P) - \lambda^k \widetilde{\mu}_{t,k}(P).$$

Since $\widetilde{\mu}_{t,k}$ is a homogeneous valuation of degree k on $\mathcal{P}^n$, by McMullen's Theorem 2.1.8, we have $\overline{\mu}_k(t\chi_P) = 0$. By Proposition $5.3.2$, we obtain $\overline{\mu}_k(f) = 0$ for all $f \in \mathcal{QC}(\mathbb{R}^n)$ and this means that μ_k is homogeneous of degree k.

Let us define $\widetilde{\mu} \colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$, as

$$\widetilde{\mu}(f) = \mu(f) - \sum_{k=0}^{n} \mu_k(f),$$

which results a translation invariant, continuous valuation such that, for every $t > 0$ and $P \in \mathcal{P}^n$, $\widetilde{\mu}(t\chi_P) = 0$. Then $\widetilde{\mu}(f) = 0$ for every $f \in \mathcal{QC}(\mathbb{R}^n)$ by Proposition $5.3.2$ and we have the conclusion

$$\mu(f) = \sum_{k=0}^{n} \mu_k(f)$$

for every $f \in \mathcal{QC}(\mathbb{R}^n)$.
Finally, we conclude the proof with a remark about uniqueness. If we have

$$\mu = \sum_{k=0}^{n} \mu_k = \sum_{k=0}^{n} \sigma_k,$$

then we are able to write

$$0 = \sum_{k=0}^{n} \mu_k - \sigma_k.$$

By the homogeneity of μ_k and σ_k, we obtain the uniqueness.

□

We consider now the two following results concerning simple and n-homogeneous valuations on $\mathcal{QC}(\mathbb{R}^n)$.

Theorem 6.1.2. *An even functional μ belongs to $\mathrm{Val}^+(\mathcal{QC}(\mathbb{R}^n))$ and it is simple if and only if there exist a (necessarily unique) function ϕ_n from $[0,+\infty)$ to $\mathbb{R}$ and $\delta > 0$, with the following properties:*

- *ϕ_n is continuous in $[0,\infty)$,*
- *$\phi_n(t) = 0$ for all $t \in [0,\delta]$,*

such that

$$\mu(f) = \int_0^{+\infty} \phi_n(t) dS_n(f;t)$$

for every $f \in \mathcal{QC}(\mathbb{R}^n)$.

Proof. First of all, we observe that the "if" part is a simple consequence of the final part of Lemma 5.2.2 (note that simplicity follows directly from basic properties of integrals).

For the other implication, we fix $t > 0$ and we consider the map $\widetilde{\mu}_t \colon \mathcal{K}^n \to \mathbb{R}$ defined by

$$\widetilde{\mu}_t(K) = \mu(t\,\chi_K), \quad \forall\, K \in \mathcal{K}^n.$$

This is a translation invariant, even, valuation, which is additionally simple (as μ is simple) and continuous w.r.t. decreasing sequences. Therefore, by Theorem 2.2.8, there exists a real constant, depending on t, $\phi_n(t)$, such that

$$\widetilde{\mu}_t(K) = \phi_n(t)\mathrm{V}_n(K), \quad \forall\, K \in \mathcal{K}^n. \tag{6.5}$$

The function ϕ_n is continuous by the continuity of μ and it is univocally determined by (6.5). The integral form of μ and the additional condition on ϕ_n can be obtained using the same argument of the proof of Theorem $5.4.1$ or Lemma 6.5 and Theorem 1.2 in [20]. □

Theorem 6.1.3. *Let μ be a functional on $\mathcal{QC}(\mathbb{R}^n)$. μ belongs to $\mathrm{Val}_n(\mathcal{QC}(\mathbb{R}^n))$ if and only if there exist a (necessarily unique) function $\phi_n \colon [0,+\infty) \to \mathbb{R}$ and a number $\delta > 0$, with the following properties:*

- *ϕ_n is continuous,*

- $\phi_n(t) = 0$, *for all* $t \in [0, \delta]$,

such that

$$\mu(f) = \int_0^{+\infty} \phi_n(t) dS_n(f;t)$$

for all $f \in \mathcal{QC}(\mathbb{R}^n)$.

Proof. The same argument as in the proof of Theorem 6.1.2 can be used, where Theorem 2.2.8 has to be replaced by Theorem 2.3.2. Moreover, even for this case, the "if" part is a consequence of the final part of Lemma 5.2.2 and the n-homogeneous property follows from the integral properties. □

We have seen in these two last results that we can obtain a similar Volume Theorem for $\mathcal{QC}(\mathbb{R}^n)$ valuations, like Theorem 5.4.1, without rotation invariance, but just adding the eveness hypothesis in Theorem 6.1.2. Moreover in Theorem 6.1.3 we do not need even the simplicity condition, we can replace it by n-homogeneity hypothesis.

We conclude this section with a remark about 0-homogeneous valuations $\mu \in \mathrm{Val}_0(\mathcal{QC}(\mathbb{R}^n))$, which we are able to characterize. In this case, we have

$$\mu(\lambda \odot f) = \mu(f)$$

for all $f \in \mathcal{QC}(\mathbb{R}^n)$ and $\lambda > 0$. As in the n-homogeneous case, we first look at the valuation defined on $\mathcal{P}^n$. We set, for $t > 0$,

$$\tilde{\mu}_t \colon \mathcal{P}^n \to \mathbb{R},\ \tilde{\mu}_t(P) = \mu(t\chi_P).$$

We observe that $\tilde{\mu}_t$ is a 0-homogeneous, translation invariant valuation, then (by [64, p. 353]) we are able to say that $\tilde{\mu}_t$ is constant on $\mathcal{P}^n$,

$$\tilde{\mu}_t(P) \equiv \tilde{\mu}_t(\{0\})$$

for all $P \in \mathcal{P}^n$. So we obtain $\mu(t\chi_P) = \mu(t\chi_{\{0\}})$, i.e. it is equal to a function depending only on t:

$$\mu(t\chi_P) = \phi_0(t)$$

for all $t > 0$ and $P \in \mathcal{P}^n$. In particular, we have that $\phi_0 \colon [0, +\infty) \to \mathbb{R}$ is continuous, and

$$\mu(t\chi_P) = \phi_0(M(t\chi_P))$$

where $M(t\chi_P) = \max_{\mathbb{R}^n} t\chi_P$.

We define now $\overline{\mu}(f) = \phi_0(M(f))$ for every $f \in \mathcal{QC}(\mathbb{R}^n)$. This is a continuous, translation invariant valuation on quasi-concave functions. Applying Proposition 5.3.2 to $\tilde{\mu} := \mu - \overline{\mu}$, we have $\tilde{\mu} \equiv 0$, hence $\mu(f) = \phi_0(M(f))$.

6.2 The Klain function on $\mathrm{Val}(\mathcal{QC}(\mathbb{R}^n))$

Let $\mu \in \mathrm{Val}_k(\mathcal{QC}(\mathbb{R}^n))$ with $k \in \{1, \dots, n-1\}$. We fix $E \in \mathrm{Gr}(n,k)$, we recall this means that E is a vector subspace of dimension k in $\mathbb{R}^n$, and we define

$$\mathcal{QC}_E(\mathbb{R}^n) = \{f \in \mathcal{QC}(\mathbb{R}^n) : \mathrm{supp}(f) \subset E\}.$$

After fixing a coordinate system on E we can identify $\mathcal{QC}_E(\mathbb{R}^n)$ as $\mathcal{QC}(\mathbb{R}^k)$, so we have $\mu|E \in \mathrm{Val}_k(\mathcal{QC}(\mathbb{R}^k))$.

Applying Theorem 6.1.3 we deduce that there exists a function $\phi_E\colon [0,+\infty) \to \mathbb{R}$, depending on E, such that

$$\mu(f) = \int_0^{+\infty} \phi_E(t) dS_k(f;t)$$

for every $f \in \mathcal{QC}_E(\mathbb{R}^n)$. Moreover ϕ_E is continuous and there exists $\delta > 0$ such that $\phi_E(t) = 0$ for all $t \in [0,\delta]$.

Hence we can define a function, Kl_μ, that we will call the **Klain function of** μ, as follows:

$$\mathrm{Kl}_\mu\colon [0,+\infty) \times \mathrm{Gr}(n,k) \to \mathbb{R}, \quad \mathrm{Kl}_\mu(t,E) = \phi_E(t).$$

This is equivalent to the identity:

$$\mu(f) = \int_0^{+\infty} \mathrm{Kl}_\mu(t,E) dS_k(f;t) \tag{6.6}$$

for all $f \in \mathcal{QC}(\mathbb{R}^n)$ such that $\mathrm{supp}(f) \subset E$.

We choose now $f = t\chi_K \in \mathcal{QC}(\mathbb{R}^n)$, where $t > 0$ and K is a convex body contained in E. We obtain exactly

$$\mu(t\chi_K) = \mathrm{Kl}_\mu(t,E)\, V_k(K).$$

We have proved the following proposition.

Proposition 6.2.1. *If $\mu \in \mathrm{Val}_k(\mathcal{QC}(\mathbb{R}^n))$, then for all $E \in \mathrm{Gr}(n,k)$ and $t > 0$, there exists a unique real number $\phi_E(t)$ such that*

$$\widetilde{\mu}_t|E = \phi_E(t)V_k.$$

In particular, $\phi_E(t)$ is the Klain function of μ evaluated at (t,E).

Proposition 6.2.2. *The map $\mathrm{Kl}\colon \mu \mapsto \mathrm{Kl}_\mu$ is injective on $\mathrm{Val}_k^+(\mathcal{QC}(\mathbb{R}^n))$.*

Proof. Let $\mu, \sigma \in \mathrm{Val}_k^+(\mathcal{QC}(\mathbb{R}^n))$ such that $\mathrm{Kl}_\mu = \mathrm{Kl}_\sigma$ in $[0,+\infty) \times \mathrm{Gr}(n,k)$. We fix $t > 0$ and we consider $\widetilde{\mu}_t$ and $\widetilde{\sigma}_t$ on $\mathcal{P}^n$. By Theorem 2.4.5, we obtain $\widetilde{\mu}_t = \widetilde{\sigma}_t$ for all $t > 0$.

We define now $\overline{\mu}\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ as $\overline{\mu} = \mu - \sigma$, and we observe that $\overline{\mu} \in \mathrm{Val}_k^+(\mathcal{QC}(\mathbb{R}^n))$. Furthermore, for $f = t\chi_P$, we have $\overline{\mu}(t\chi_P) = \mu(t\chi_P) - \sigma(\chi_P) = \widetilde{\mu}_t(P) - \widetilde{\sigma}_t(P) = 0$. Then, by Proposition 5.3.2, we have $\overline{\mu} = 0$ in $\mathcal{QC}(\mathbb{R}^n)$. □

Let $\mu \in \mathrm{Val}_k(\mathcal{QC}(\mathbb{R}^n))$ be also rotation invariant, so $\mathrm{Kl}_\mu(t,E)$ does not depend by E. This follows immediately from the definition of the Klain function and the fact that for every $E, F \in \mathrm{Gr}(n,k)$ there exists a rotation $g \in O(n)$ such that $E = gF$. In this case, we get the Klain function as a function of t, hence

$$\mathrm{Kl}_\mu(t,E) = \phi(t)$$

for $t \in [0,+\infty)$ and all $E \in \mathrm{Gr}(n,k)$. We have the following characterization theorem.

Theorem 6.2.3. *If $\mu \in \mathrm{Val}_k(\mathcal{QC}(\mathbb{R}^n))$ and it is also rotation invariant, then*

$$\mu(f) = \int_0^{+\infty} \phi(t) dS_k(f;t), \quad \forall\, f \in \mathcal{QC}(\mathbb{R}^n),$$

where $\phi(t) = \mathrm{Kl}_\mu(t,E)$.

Proof. We observe that, by the property of the Klain function, the quantity

$$\int_0^{+\infty} \phi(t) dS_k(f;t)$$

is a continuous, rigid motion and k-homogeneous valuation on $\mathcal{QC}(\mathbb{R}^n)$. We conclude the proof applying Proposition 5.2.5 to $\mu(\cdot) - \int_0^{+\infty} \phi(t) dS_k(\cdot\,;t)$. □

6.3 Polynomiality

We present now an application of the homogeneous decomposition contained in Theorem 6.1.1 for Val$(\mathcal{QC}(\mathbb{R}^n))$, i.e. a polynomiality result like Theorem 2.1.13.

We first recall the sum and multiplication by scalars we have defined on $\mathcal{QC}(\mathbb{R}^n)$: for f and $g \in \mathcal{QC}(\mathbb{R}^n)$ we have

$$f \oplus g(x) = \sup_{y \in \mathbb{R}^n} \min\{f(x), g(y-x)\} \quad \forall\, x \in \mathbb{R}^n.$$

We have also a scalar multiplication operation $\odot$ that we have studied for homogeneous valuations. We want to consider linear combinations of quasi-concave functions of the form

$$(\lambda_1 \odot f_1) \oplus \cdots \oplus (\lambda_k \odot f_k),$$

with $\lambda_i > 0$, $f_i \in \mathcal{QC}(\mathbb{R}^n)$ and $i \in \{1, \cdots, k\}$. We know, by Corollary 3.1.18, that $\mathcal{QC}(\mathbb{R}^n)$ is closed w.r.t. $\oplus$ and $\odot$.

Remark 6.3.1. The operation $\oplus$ is commutative and it comes from the commutativity property of Minkowski sum for super-level sets.

Let us fix $\mu \in \mathrm{Val}_m(\mathcal{QC}(\mathbb{R}^n))$, with $m \in \{1, \cdots, n\}$. Following the same steps for valuations on $\mathrm{Val}_m(\mathcal{K}^n)$ showed in [64], Section 6.3, we start with a function $g \in \mathcal{QC}(\mathbb{R}^n)$ and we define the following functional

$$\mu_g \colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}, \quad \mu_g(f) = \mu(f \oplus g),\ \forall\, f \in \mathcal{QC}(\mathbb{R}^n).$$

Proposition 6.3.2. *For any $g \in \mathcal{QC}(\mathbb{R}^n)$ fixed, we have $\mu_g \in$ Val$(\mathcal{QC}(\mathbb{R}^n))$.*

Proof.

- **Valuation property.** Let $f_1, f_2 \in \mathcal{QC}(\mathbb{R}^n)$ such that $f_1 \vee f_2 \in \mathcal{QC}(\mathbb{R}^n)$. We prove that

$$(f_1 \vee f_2) \oplus g = (f_1 \oplus g) \vee (f_2 \oplus g). \tag{6.7}$$

and

$$(f_1 \wedge f_2) \oplus g = (f_1 \oplus g) \wedge (f_2 \oplus g). \tag{6.8}$$

We use the characterization of $\oplus$ with super-level sets and we have

$$\begin{aligned} L_t((f_1 \vee f_2) \oplus g) =& L_t(f_1 \vee f_2) + L_t(g) = (L_t(f_1) \cup L_t(f_2)) + L_t(g) = \\ & (L_t(f_1) + L_t(g)) \cup (L_t(f_2) + L_t(g)) = L_t(f_1 \oplus g) \cup L_t(f_2 \oplus g) = \\ & L_t((f_1 \oplus g) \vee (f_2 \oplus g)), \end{aligned}$$

for every $t > 0$, where for the previous equalities we used Proposition 1.1.11. In a similar way we have also the proof of (6.8). By the valuation property of μ, we deduce that also μ_g is a valuation.

- **Translation invariance.** Let $x \in \mathbb{R}^n$. We denote by T_x the translation $T_x(y) = y - x$, for every $y \in \mathbb{R}^n$. Then we have

$$L_t((f \circ T_x) \oplus g) = L_t(f \circ T_x) + L_t(g) = L_t(f) + x + L_t(g) = L_t(f \oplus g) + x,$$

  for every $t > 0$, and then one has $\mu_g(f \circ T_x) = \mu_g(f)$ for every choice of $f \in \mathcal{QC}(\mathbb{R}^n)$.

- **Continuity w.r.t. pointwise and monotone convergence.** It comes from the fact that $f_i \nearrow f$ implies $f_i \oplus g \nearrow f \oplus g$ pointwise in $\mathbb{R}^n$ and the same for decreasing sequences.

Hence we have $\mu_g \in \text{Val}(\mathcal{QC}(\mathbb{R}^n))$. □

We apply now Theorem 6.1.1 and then there exist $(n+1)$ valuations, depending on g,

$$\mu_i(\cdot, g)\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R},$$

such that $\mu_i(\cdot, g) \in \text{Val}_i(\mathcal{QC}(\mathbb{R}^n))$ and

$$\mu_g(f) = \sum_{i=0}^{n} \mu_i(f, g)$$

holds for every $f \in \mathcal{QC}(\mathbb{R}^n)$. Equivalently we have this first polynomial expression

$$\mu((\lambda \odot f) \oplus g) = \sum_{i=0}^{n} \lambda^i \mu_i(f, g), \tag{6.9}$$

for every $f \in \mathcal{QC}(\mathbb{R}^n)$ and $\lambda > 0$.

Proposition 6.3.3. *For any fixed $f \in \mathcal{QC}(\mathbb{R}^n)$ and $i \in \{0, \cdots n\}$, the functional*

$$\mu_i(f, \cdot)\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}, \quad g \mapsto \mu_i(f, g)$$

is a continuous (w.r.t. monotone and pointwise convergence) and translation invariant valuation.

The proof comes easily from equation (6.9): we use equalities (6.7) and (6.8). By the commutativity property of $\oplus$, Remark 6.3.1, and by valuation property of μ, we have the thesis.

Remark 6.3.4. For any $i \in \{0, \cdots, n\}$ the map

$$\mu_i\colon (\mathcal{QC}(\mathbb{R}^n))^2 \to \mathbb{R}$$

is symmetric, by the commutativity property, Remark 6.3.1, and translation invariant.

We want now to extend this result to a generic combination of quasi-concave functions. Following [64], Remark 6.3.3 and Theorem 6.3.4, and the previous proposition, we repeat the same argument and we get inductively

$$\mu((\lambda_1 \odot f_1) \oplus \cdots \oplus (\lambda_k \odot f_k)) = \sum_{r_1, \cdots, r_k = 0}^{m} \binom{m}{r_1 \cdots r_k} \lambda_1^{r_1} \cdots \lambda_k^{r_k} \, \mu_{r_1, \cdots, r_k}(f_1, \cdots, f_k),$$

for every $f_i \in \mathcal{QC}(\mathbb{R}^n)$ and $\lambda_i > 0$, $i \in \{1, \cdots, k\}$. If we choose $k = m$, we define

$$\overline{\mu}(f_1, \cdots, f_m) = \mu_{1,\cdots,1}(f_1, \cdots, f_m)$$

and then

$$\mu_{r_1,\cdots,r_k}(f_1, \cdots, f_k) = \overline{\mu}(\underbrace{f_1, \cdots, f_1}_{r_1}, \cdots, \underbrace{f_k, \cdots, f_k}_{r_k}) = \overline{\mu}(f_1[r_1], \cdots, f_k[r_k]).$$

Summarizing, we have the following result.

Theorem 6.3.5. *Let $\mu \in \mathrm{Val}_m(\mathcal{QC}(\mathbb{R}^n))$. There exists a functional $\overline{\mu}\colon (\mathcal{QC}(\mathbb{R}^n))^m \to \mathbb{R}$ such that for $k \geq 1$, $\lambda_1, \cdots, \lambda_k > 0$ and $f_1, \cdots, f_k \in \mathcal{QC}(\mathbb{R}^n)$, one has*

$$\mu((\lambda_1 \odot f_1) \oplus \cdots \oplus (\lambda_k \odot f_k)) = \sum_{r_1,\cdots,r_k=0}^{m} \binom{m}{r_1 \cdots r_k} \lambda_1^{r_1} \cdots \lambda_k^{r_k}\, \overline{\mu}(f_1[r_1], \cdots, f_k[r_k]), \tag{6.10}$$

Moreover $\overline{\mu}$ is multilinear, translation invariant and symmetric.

Moreover if we fix $r \in \{0, \cdots, m\}$ and $g_1, \cdots, g_{m-r} \in \mathcal{QC}(\mathbb{R}^n)$ the map

$$\mathcal{QC}(\mathbb{R}^n) \ni f \mapsto \overline{\mu}(f[r], g_1, \cdots, g_{m-r})$$

is a valuation that belongs to $\mathrm{Val}_r(\mathcal{QC}(\mathbb{R}^n))$.

In other words Theorem 6.3.5 tells us that $\mu((\lambda_1 \odot f_1) \oplus \cdots \oplus (\lambda_k \odot f_k))$ can be expressed as an homogeneous polynomial of degree m w.r.t. $\lambda_1, \cdots, \lambda_k$.

Chapter 7

Final remarks and future developments

In this final chapter we are going to present final remarks, results and some possible future problems to study.

In Section 1 we will describe a different type of convergence on $\mathcal{QC}(\mathbb{R}^n)$, *hypo-convergence*, that we will use in this chapter. We will see that every result we have established with monotone and pointwise convergence holds also with hypo-convergence. In Section 2 we will present results concerning valuations on $\mathcal{QC}(\mathbb{R}^n)$ that are $GL(n)$ and $SL(n)$-invariant.
We conclude this final chapter with a few words about future developments and questions still open.

7.1 Hypo-convergence

We introduce now another convergence for quasi-concave functions sequences, the **hypo-convergence**. Roughly speaking it is a set convergence of their hypo-graphs.

In this section we will focus first on the Painlevé-Kuratowski set convergence and then we will use it to define hypo-convergence. We refer to [11] and to [62] for details and proof of the following statements.

We consider first of all the two following collections of subsets of $\mathbb{N}$:

$$N_\infty = \{N \subseteq \mathbb{N} |\ \mathbb{N} \setminus N \text{ finite}\}.$$

$$N_\infty^\# = \{N \subseteq \mathbb{N} |\ N \text{ infinite}\}.$$

The following definition is the **Painlevé-Kuratowski** convergence for a sequence of subsets of $\mathbb{R}^n$.

Definition 7.1.1. *Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{R}^n$.*

- *The inner limit is the set defined by*

$$\liminf_{i\to+\infty} C_i = \{x|\ \exists\ N \in N_\infty,\ \exists\ x_i \in C_i\ (i \in N):\ x_i \to x\}.$$

- *The outer limit is the set defined by*

$$\limsup_{i\to+\infty} C_i = \{x|\ \exists\ N \in N_\infty^\#,\ \exists\ x_i \in C_i\ (i \in N):\ x_i \to x\}.$$

- *If the outer and inner limits are equal, then this set is called the limit, in the sense of Painlevé-Kuratowski, of the sequence C_i.*

This convergence is defined for generic subsets of $\mathbb{R}^n$, not necessarily for convex bodies. We have also the following properties and examples.

Lemma 7.1.2 (Characterizations of set limits)**.** *For a sequence of subsets $C_i \subseteq \mathbb{R}^n$ we have*

$$\liminf_{i\to+\infty} C_i = \{x| \limsup_{i\to+\infty} d(x, C_i) = 0\} = \bigcap_{N\in N_\infty^\#} \mathrm{cl}\Big(\bigcup_{i\in N} C_i\Big),$$

and

$$\limsup_{i\to+\infty} C_i = \{x| \liminf_{i\to+\infty} d(x, C_i) = 0\} = \bigcap_{N\in N_\infty} \mathrm{cl}\Big(\bigcup_{i\in N} C_i\Big).$$

Example 7.1.3.

1. *A sequence of balls $\mathbb{B}(x_i, \rho_i)$ converges to $\mathbb{B}(x, \rho)$ if and only if $x_i \to x$ and $\rho_i \to \rho$.*
2. *Let D_1 and D_2 be two different closed subsets of $\mathbb{R}^n$. We define*

$$C_i = \begin{cases} D_1 & \text{if } i \text{ is odd,} \\ D_2 & \text{if } i \text{ is even.} \end{cases}$$

It is easy to see that $\limsup_{i\to+\infty} C_i = D_1 \cup D_2$, whereas $\liminf_{i\to+\infty} C_i = D_1 \cap D_2$, hence there is no convergence for C_i, unless $D_1 = D_2$.

Proposition 7.1.4.

- *An increasing sequence of sets C_i has always a limit, in the sense of Painlevé-Kuratowski, and*

$$\lim_{i\to+\infty} C_i = cl(\bigcup_{i\in\mathbb{N}} C_i).$$

- *A decreasing sequence C_i has always a limit, in the sense of Painlevé-Kuratowski, and*

$$\lim_{i\to+\infty} C_i = \bigcap_{i\in\mathbb{N}} cl(C_i).$$

We introduce now hypo-graph and in parallel epi-graph, we will see that the set convergences that will come out from these two sets are strictly related.

Definition 7.1.5 (Hypo-graph/Epi-graph)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be any function, then its hypo-graph is defined by*

$$\mathrm{hypo}(f) = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R}| \ f(x) \geq \alpha\}.$$

Similarly the epi-graph of f is

$$\mathrm{epi}(f) = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R}| \ f(x) \leq \alpha\}.$$

We have immediately some easy remarks to do.

Remark 7.1.6.

- f is convex if and only if $\mathrm{epi}(f)$ is convex in $\mathbb{R}^{n+1}$.

- f is concave if and only if $\text{hypo}(f)$ is convex in $\mathbb{R}^{n+1}$.
- f is upper semi-continuous if and only if its hypo-graph is closed in $\mathbb{R}^{n+1}$. In particular if $f \in \mathcal{QC}(\mathbb{R}^n)$, then $\text{hypo}(f)$ is closed in $\mathbb{R}^{n+1}$.

Definition 7.1.7 (Hypo/Epi-convergence)**.** *A sequence of functions f_i, $i \in \mathbb{N}$, defined in $\mathbb{R}^n$, is said to hypo-converge to f if*

$$\lim_{i \to +\infty} \text{hypo}(f_i) = \text{hypo}(f),$$

in the sense of Painlevé-Kuratowski.

Moreover, f_i epi-converges to f if

$$\lim_{i \to +\infty} \text{epi}(f_i) = \text{epi}(f),$$

w.r.t. Painlevé-Kuratowski convergence.

Remark 7.1.8. f_i hypo-converges to f if and only if $-f_i$ epi-converges to $-f$.

We equip now $\mathcal{QC}(\mathbb{R}^n)$ with hypo-convergence. We have the following statements.

Proposition 7.1.9.

- *If $f \leq g$, then* $\text{hypo}(f) \subseteq \text{hypo}(g)$.
- *If $f_i \in \mathcal{QC}(\mathbb{R}^n)$ converges to $f \in \mathcal{QC}(\mathbb{R}^n)$ pointwise and monotone increasing w.r.t. i, then*

$$\lim_{i \to +\infty} \text{hypo}(f_i) = \text{hypo}(f).$$

- *If $f_i \in \mathcal{QC}(\mathbb{R}^n)$ converges to $f \in \mathcal{QC}(\mathbb{R}^n)$ pointwise and monotone decreasing w.r.t. i, then*

$$\lim_{i \to +\infty} \text{hypo}(f_i) = \text{hypo}(f).$$

Proof. The first statement is a merely consequence of definition of the hypo-graph.

- Since f_i is increasing, we have that also $\text{hypo}(f_i)$ is increasing w.r.t. i. This means that there exists

$$\lim_{i \to +\infty} \text{hypo}(f_i) = cl(\bigcup_{i \in \mathbb{N}} \text{hypo}(f_i)).$$

  Moreover $f_i \leq f$, so we obtain $\text{hypo}(f_i) \subseteq \text{hypo}(f)$ and we can say that

$$cl(\bigcup_{i \in \mathbb{N}} \text{hypo}(f_i)) \subseteq \text{hypo}(f).$$

  Now, let (x, α) be an element of $\text{hypo}(f)$. Then we have $\alpha \leq f(x) = \sup_{i \in \mathbb{N}} f_i(x)$. We can say that there exists $\bar{i} \in \mathbb{N}$, such that $\alpha \leq f_i(x)$, for all $i > \bar{i}$ and we can conclude that $(x, \alpha) \in cl(\bigcup_{i \in \mathbb{N}} \text{hypo}(f_i))$. So we obtain $\lim_{i \to +\infty} \text{hypo}(f_i) = \text{hypo}(f)$.

- In a similar way as before, we have that $\text{hypo}(f_i)$ is a decreasing sequence and $\text{hypo}(f) \subseteq \text{hypo}(f_i)$, so we can say that

$$\text{hypo}(f) \subseteq \bigcap_{i \in \mathbb{N}} \text{hypo}(f_i) = \bigcap_{i \in \mathbb{N}} cl(\text{hypo}(f_i)) = \lim_{i \to +\infty} \text{hypo}(f_i).$$

  Let (x, α) be an element of $\bigcap_{i \in \mathbb{N}} \text{hypo}(f_i)$, then we have $\alpha \leq f_i(x)$, for all i and $\alpha \leq \lim_{i \to +\infty} f_i(x) = f(x)$. We can conclude with $(x, \alpha) \in \text{hypo}(f)$.

□

The last two results tell us that pointwise and monotone convergence implies hypo-convergence. In particular we have same density results we have established in the previous section, i.e. $\mathcal{SQC}(\mathbb{R}^n)$ is dense in $\mathcal{QC}_c(\mathbb{R}^n)$ and $\mathcal{QC}_c(\mathbb{R}^n)$ is dense in $\mathcal{QC}(\mathbb{R}^n)$ w.r.t. hypo-convergence and also the space of quasi-concave functions is not closed under hypo-convergence.

We focus now on a different way, but equivalent, to define epi/hypo-convergence.

Definition 7.1.10 (Γ-convergence)**.** *Let $f_i\colon \mathbb{R}^n \to \mathbb{R}$ be a sequence of functions. We say that f_i Γ-converges to f ($f = \Gamma - \lim_i f_i$) if for every $x \in \mathbb{R}^n$, one has:*

- *for every sequence $x_i \to x$, it holds*

$$\liminf_{i\to+\infty} f_i(x_i) \geq f(x).$$

- *There exists $x_i \to x$ such that*

$$\limsup_{i\to+\infty} f_i(x_i) \leq f(x) \; if\ and\ only\ if\ there\ exists \; \lim_{i\to+\infty} f_i(x_i) = f(x).$$

This definition is functional, not geometric like epi-convergence. It was introduced by DeGiorgi (see for instance [26] and [25]) to study problems related to Calculus of Variations. We refer also to [16] and [29] for more details and properties of this type of convergence. We observe that they define Γ-convergence for sequences of functions defined in a generic metric space (X, d). This is because they usually consider not functions, but functionals defined on some function space equipped with a suitable topology.
We will focus on $X = \mathbb{R}^n$ with the standard Euclidean metric structure.

Remark 7.1.11. Γ-convergence is equivalent to epi-convergence. Moreover f_i hypo-converges to f if and only if $-f_i$ Γ-converges to $-f$. Hence f_i hypo-converges to f if and only if for every $x \in \mathbb{R}^n$ one has:

- for every sequence $x_i \to x$, it holds

$$\limsup_{i\to+\infty} f_i(x_i) \leq f(x).$$

- There exists a sequence $x_i \to x$, such that

$$f(x) \leq \liminf_{i\to+\infty} f_i(x_i) \text{ if and only if there exists } \lim_{i\to+\infty} f_i(x_i) = f(x).$$

Theorem 7.1.12 (Compactness principle for Γ-convergence; Proposition 2.16 in [29])**.** *Any sequence of functions $f_i\colon \mathbb{R}^n \to \mathbb{R}$ admits a subsequence f_{i_k} Γ-converging to some function f.*

We list now a few properties for quasi-concave functions with this convergence.

Remark 7.1.13. If f_i is a sequence of log-concave functions in $\mathcal{LC}(\mathbb{R}^n)$, then there exists a sequence of convex functions $u_i \in Conv(\mathbb{R}^n)$ such that $f_i = \exp(-u_i)$. Let $f = \exp(-u)$ be another log-concave function. Hence it holds that f_i hypo-converges to f if and only if u_i epi-converges to u.

Theorem 7.1.14. *If $K_i \to K$ w.r.t. δ, then χ_{K_i} hypo-converges to χ_K.*

Proof. Suppose $x \in K$. We have:

- $\forall x_i \to x$, if $\exists \{x_{i_k}\} \subseteq \{x_i\}$ such that $x_{i_k} \in K_{i_k}$, then $\chi_{K_{i_k}}(x_{i_k}) = 1$ and

$$\chi_K(x) = 1 = \limsup_{i \to +\infty} \chi_{K_i}(x_i).$$

  Otherwise if $x_i \notin K_i$ for all i, or there exists $\bar{i}$ such that $x_i \notin K_i$ for all $i \geq \bar{i}$, then $\limsup_{i \to +\infty} \chi_{K_i}(x_i) = 0 < \chi_K(x)$.

- From the δ convergence we have that $\exists\, x_i \in K_i$ such that $x_i \to x$ and so

$$\liminf_{i \to +\infty} \chi_{K_i}(x_i) = 1 = \chi_K(x).$$

Suppose now $x \notin K$:

- $\forall x_i \to x$, if $\exists \{x_{i_k}\} \subseteq \{x_i\}$ such that $x_{i_k} \in K_{i_k}$, then $x_{i_k} \to x$ and by the properties of δ we have $x \in K$ and this is a contradiction. So it holds $x_i \notin K_i$ for all i, or there exists $\bar{i}$ such that $x_i \notin K_i$ for all $i \geq \bar{i}$, then $\limsup_{i \to +\infty} \chi_{K_i}(x_i) = 0 = \chi_K(x)$.

- Any sequence that tends to x is alright, since we have

$$0 \leq \liminf_{i \to +\infty} \chi_{K_i}(x) \leq \limsup_{i \to +\infty} \chi_{K_i}(x_i) = 0 = \chi_K(x).$$

□

Lemma 7.1.15. *For all $K \in \mathcal{K}^n$, $\{a_i\}_{i \in \mathbb{N}} \subseteq \mathbb{R}_+$ such that $a_i \to a \geq 0$, one has $f_i(x) = a_i \chi_K(x)$ hypo-converges to $f(x) = a\chi_K(x)$.*

Proof. Suppose for simplicity that $\dim(K) = n$.

- Let us consider first $x \in K$, we want to prove the two statements that defined hypo-convergence. Suppose $x_i \to x$, we consider two cases:

  1. If $x \in \mathrm{int}(K)$, then there exists $\bar{i}$, such that $x_i \in K$ for all $i \geq \bar{i}$, and that means

  $$\limsup_{i \to +\infty} f_i(x_i) = \lim_{i \to +\infty} a_i = a = f(x).$$

     The same result if we consider $x \in \partial K$ and there exists x_{i_k} a subsequence of x_i such that $x_{i_k} \in \mathrm{int}(K)$.

  2. If $x \in \partial K$ and $x_i \in \mathbb{R}^n \setminus K$, then it holds

  $$\limsup_{i \to +\infty} f_i(x_i) = 0 \leq f(x) = a.$$

  For the $\liminf$ inequality just take $x_i \equiv x$.

- For the case $x \notin K$, if $x_i \to x$, then $x_i \notin K$ for $i \geq \bar{i}$ and hence

$$\limsup_{i\to+\infty} f_i(x_i) = 0 = f(x).$$

And for the other inequality again take $x_i \equiv x$.

□

We focus also on the convergence of sequences of super-level sets w.r.t. hypo-convergence.

Proposition 7.1.16. *Let $f_i\ f \in \mathcal{QC}(\mathbb{R}^n)$ be quasi-concave functions. If f_i hypo-converges to f, then*

$$\lim_{i\to+\infty} L_t(f_i) = L_t(f),$$

in the Hausdorff metric, for a.e. $t > 0$.

Proof. We use the characterization property for Hausdorff convergence, Theorem 1.2.6.

1. If for every $i \in \mathbb{N}$, there exists $x_i \in L_t(f_i)$ such that $x_i \to x \in \mathbb{R}^n$, then by hypo-convergence we have
$$\limsup_{i\to+\infty} f_i(x_i) \leq f(x).$$
Since $x_i \in L_t(f_i)$, it holds $f_i(x_i) \geq t$ for every $i \in \mathbb{N}$, then it holds $f(x) \geq t$, i.e. $x \in L_t(f)$.

2. Let $x \in L_t(f)$. For a.e. $t > 0$ we have, by Lemma 3.1.6,
$$L_t(f) = \mathrm{cl}(\{x : \ f(x) > t\}).$$
Let now suppose $f(x) > t$. Since there exists a converging sequence $x_i \to x$, such that
$$\lim_{i\to+\infty} f_i(x_i) = f(x),$$
one has that there exists $\bar{i} \in \mathbb{N}$ such that $f_i(x_i) \geq t$, for all $i \geq \bar{i}$ and hence $x_i \in L_t(f_i), \forall\, i \geq \bar{i}$.

   If $f(x) = t$, then there exists $x_i \to x$ such that $f_i(x) > t$, for all $i \in \mathbb{N}$.
   Let us fix i, then $\exists\, \overline{x}_i$ such that, by hypo-convergence, $||\overline{x}_i - x_i|| < \frac{1}{i}$ and $f_i(\overline{x}_i) \geq t$.
   Then we have
$$||\overline{x}_i - x|| \leq ||\overline{x}_i - x_i|| + ||x_i - x|| \to 0 \text{ for } i \to +\infty.$$
Hence $\overline{x}_i \to x$ and $\overline{x}_i \in L_t(f_i)$.

□

At the end we show that we can obtain the same results of Chapters 5 and 6 with this new notion of convergence on $\mathcal{QC}(\mathbb{R}^n)$.

First of all we make the following remark.

Remark 7.1.17. By Theorem 7.1.14 we have that for any continuous, w.r.t. hypo-convergence, valuation on $\mathcal{QC}(\mathbb{R}^n)$ and for any $t > 0$, the valuation

$$\tilde{\mu}_t \colon \mathcal{K}^n \to \mathbb{R}, \quad \tilde{\mu}_t(K) = \mu(t\chi_K)$$

is continuous in $\mathcal{K}^n$.

We observe this is one of the main differences with the continuity of valuation w.r.t. monotone and pointwise convergence.

We prove that the continuous integral valuations introduced in Section 5.2 are also continuous w.r.t. hypo-convergence. We need the following proposition.

Proposition 7.1.18. *Let $f_i, f \in \mathcal{QC}(\mathbb{R}^n)$, for all $i \in \mathbb{N}$. If f_i hypo-converges to f, then $M(f_i)$ converges to $M(f)$.*

Proof. Let us fix $x_i, \overline{x} \in \mathbb{R}^n$ such that

$$M(f_i) = \max_{x\in\mathbb{R}^n} f_i(x) = f_i(x_i)$$

and

$$M(f) = \max_{x\in\mathbb{R}^n} f(x) = f(\overline{x}).$$

By hypo-convergence we have that there exists $y_i \to \overline{x}$ such that $\lim_{i\to+\infty} f_i(y_i) = f(\overline{x})$. Hence we have

$$f(\overline{x}) = \liminf_{i\to+\infty} f_i(y_i) \leq \liminf_{i\to+\infty} f_i(x_i).$$

Vice versa we suppose there exists $t > 0$ such that $f(\overline{x}) < t < \limsup_{i\to+\infty} f_i(x_i)$. First of all we observe that we can choose t such that $L_t(f_i) \to L_t(f)$ in Hausdorff metric.
By $f(\overline{x}) < t$ we have $L_t(f) = \varnothing$ and by $t < \limsup_{i\to+\infty} f_i(x_i)$ we have, up to subsequences, $L_t(f_i) \in \mathcal{K}^n$, for i greater than some $\overline{i}$. So we have

$$\mathcal{K}^n \ni L_t(f_i) \to \varnothing$$

and we have a contradiction.
Hence we obtain

$$\limsup_{i\to+\infty} f_i(x_i) \leq f(\overline{x})$$

and then the thesis.

□

We recall the notion of integral valuations, i.e. a functional defined as

$$\mu(f) = \int_0^{+\infty} \phi(t) dS_k(f;t), \quad \forall\, f \in \mathcal{QC}(\mathbb{R}^n),$$

where $-S_k(f;\cdot)$ is the distributional derivative of $\varphi_k(t) = V_k(L_t(f))$, with $k \in \{0, \cdots, n\}$ and $\phi\colon [0, +\infty) \to \mathbb{R}$ is continuous, $\phi(0) = 0$ and, if $k \geq 1$, $\phi(t) = 0$, for all $t \in [0, \delta]$, for some $\delta > 0$.
We observe that the δ-property is a sufficient and necessary condition for the finiteness also in this case, indeed Proposition 5.2.3 still holds because we do not need continuity assumption for the proof.

Proposition 7.1.19. *Let $f_i, f \in \mathcal{QC}(\mathbb{R}^n)$, $i \in \mathbb{N}$, such that f_i hypo-converges to f. Then we have*

$$\int_0^{+\infty} \phi(t) dS_k(f_i;t) \to \int_0^{+\infty} \phi(t) dS_k(f;t),$$

for every $\phi \in C([0, +\infty))$ with $\phi(t) = 0$, for all $t \in [0, \delta]$, for some $\delta > 0$ and for $k \geq 1$.

Proof. By the previous proposition, we may suppose that $\exists M > 0$ such that $M(f_i), M(f) \leq M$. Moreover we can choose $\delta > 0$ such that $L_\delta(f_i) \to L_\delta(f)$ and we observe that $L_M(f_i) = L_M(f) = \varnothing$, for every $i \in \mathbb{N}$.
We already know that $\varphi_{k,i}(t) = V_k(L_t(f_i))$ and $\varphi_k(t) = V_k(L_t(f))$ are decreasing functions in t, and hence of bounded variation in $[\delta, M]$, for every f_i and f, $i \in \mathbb{N}$. We prove that $V_k(L_t(f_i))$ is of uniform bounded variation in $[\delta, M]$.

For every $i \in \mathbb{N}$, the total variation of $\varphi_{k,i}(t)$ is $V_k(L_\delta(f_i)) - V_k(L_M(f_i)) = V_k(L_\delta(f_i))$ and we know that it converges to $V_k(L_\delta(f))$, so the sequence is bounded in i, and then $\varphi_{k,i}$ is of uniform bounded variation in $[\delta, M]$. Hence $S_k(f_i;\cdot)$ weakly converges to the measure $S_k(f;\cdot)$, so we have

$$\int_0^{+\infty} \phi(t) dS_k(f_i;t) \to \int_0^{+\infty} \phi(t) dS_k(f;t).$$

□

With the previous result we have the following statement.

Theorem 7.1.20. *Let $k \in \{1, \cdots, n\}$, $\phi \in C([0, +\infty))$ with $\phi(t) = 0$, for all $t \in [0, \delta]$, for some $\delta > 0$. Then the functional*

$$\mu(f) = \int_0^{+\infty} \phi(t) dS_k(f;t)$$

is an invariant (w.r.t. rigid motions) continuous (w.r.t. hypo-convergence) valuation.

Moreover if $\phi_0 \in C([0, +\infty))$, then we have that

$$\mu(f) = \phi_0(\max_{\mathbb{R}^n} f)$$

is an invariant, continuous (w.r.t. hypo-convergence) valuation.

Since a sequence of quasi-concave functions that converges monotone and pointwise to some $f \in \mathcal{QC}(\mathbb{R}^n)$, also hypo-converges to f, we have now the same results of Chapters 5 and 6. We list briefly some results that we will use in Section 2.

Theorem 7.1.21. *(Homogeneous Decomposition Theorem) If μ is a continuous (w.r.t. hypo-convergence) and translation invariant valuation, then for all $i \in \{0, \cdots, n\}$, there exists a continuous (w.r.t. hypo-convergence), translation invariant and i-homogeneous valuation such that*

$$\mu = \sum_{i=0}^n \mu_i.$$

Theorem 7.1.22. *(Characterization Theorem for 0 and n-homogeneous valuations) Let $k = 0$, then a continuous (w.r.t. hypo-convergence) and translation invariant valuation is 0-homogeneous if and only if there exists $\phi_0 \colon [0, +\infty) \to \mathbb{R}$ continuous such that*

$$\mu_0(f) = \phi_0(\max_{\mathbb{R}^n} f),$$

for all $f \in \mathcal{QC}(\mathbb{R}^n)$.

Moreover if $k = n$, then a continuous (w.r.t. hypo-convergence) and translation invariant valuation is n-homogeneous if and only if there exist $\phi_n\colon [0, +\infty] \to \mathbb{R}$ continuous and $\delta > 0$ such that $\phi_n(t) = 0$, for all $t \in [0, \delta]$, and it holds

$$\mu_n(f) = \int_0^{+\infty} \phi_n(t) dS_n(f; t),$$

for all $f \in \mathcal{QC}(\mathbb{R}^n)$.

Proposition 7.1.23. *Let $\mu\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ be a continuous (w.r.t. hypo-convergence) valuation such that*

$$\mu(t\chi_P) = 0$$

for all $t > 0$ and $P \in \mathcal{P}^n$ polytope. Then it holds $\mu(f) = 0$ for all $f \in \mathcal{QC}(\mathbb{R}^n)$.

7.2 $GL(n)$ and $SL(n)$-invariant valuations

In this section we are going to prove characterization results concerning continuous, translation invariant valuations on $\mathcal{QC}(\mathbb{R}^n)$ that are also first $SL(n)$ and then $GL(n)$-invariant. See [49] for similar results in valuation theory on convex bodies.

First of all we are going to prove some easy remarks.

Lemma 7.2.1. *Let $f \in \mathcal{QC}(\mathbb{R}^n)$ and $T \in GL(n)$, then it holds $f \circ T \in GL(n)$. Moreover for every $t > 0$, we have that $\tilde{\mu}_t$ is $GL(n)$-invariant on $\mathcal{K}^n$.*

Proof. Let $t > 0$, such that $L_t(f) \in \mathcal{K}^n$, then we have

$$L_t(f \circ T) = T^{-1}(L_t(f)) \in \mathcal{K}^n.$$

Now let us consider $K \in \mathcal{K}^n$ and $T \in GL(n)$. It holds

$$\tilde{\mu}_t(T(K)) = \mu(t\chi_{T(K)}) = \mu(t\chi_K \circ T^{-1}) = \mu(t\chi_K).$$

□

Lemma 7.2.2. *Let $\phi_0, \phi_n \in C([0, +\infty))$ such that $\phi_0(0) = 0$ and $\phi_n(t) = 0$, for all $t \in [0, \delta]$, for some $\delta > 0$. Then we have*

$$\mu_0(f) = \phi_0(\max_{\mathbb{R}^n} f)$$

is $GL(n)$-invariant and

$$\mu_n(f) = \int_0^{+\infty} \phi_n(t) dS_n(f; t)$$

is $SL(n)$-invariant.

Proof. Let us observe that for all $T \in GL(n)$ and $f \in \mathcal{QC}(\mathbb{R}^n)$, we have

$$M(f \circ T) = M(f).$$

Indeed if $M(f \circ T) = f(T(\tilde{x}))$ and $M(f) = f(\overline{x})$ with $\tilde{x}, \overline{x} \in \mathbb{R}^n$, then it holds

$$f(T(\tilde{x})) \leq f(\overline{x}).$$

Since there exists $x_1 \in \mathbb{R}^n$ such that $\overline{x} = T(x_1)$, then it holds also

$$f(\overline{x}) = f(T(x_1)) \leq f(T(\tilde{x})).$$

Hence μ_0 is $GL(n)$-invariant.
The invariance of μ_n comes from the invariance w.r.t. $SL(n)$ of the Lebesgue measure and the same steps of the proof of Proposition 5.2.3. □

The characterization results for $SL(n)$-invariant valuations is the following.

Theorem 7.2.3. *A functional $\mu \colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ is a continuous (w.r.t. hypo-convergence), $SL(n)$ and translation invariant valuation if and only if there exist $\phi_0, \phi_n \colon [0, +\infty) \to \mathbb{R}$ continuous such that $\phi_n(t) = 0$, for all $t \in [0, \delta]$, for some $\delta > 0$, and*

$$\mu(f) = \phi_0(\max_{\mathbb{R}^n} f) + \int_0^{+\infty} \phi_n(t) dS_n(f; t),$$

for all $f \in \mathcal{QC}(\mathbb{R}^n)$.

Proof. We suppose that μ is a continuous and invariant (w.r.t. translations and $SL(n)$) valuation. By Theorem 7.1.21, for all $i \in \{0, \cdots, n\}$ there exists μ_i continuous, invariant (w.r.t. translations) and i-homogeneous valuation such that

$$\mu = \sum_{i=0}^{n} \mu_i \quad \text{on } \mathcal{QC}(\mathbb{R}^n).$$

We observe that μ_i is also $SL(n)$-invariant for all $i \in \{0, \cdots, n\}$. Hence for every $t > 0$, $\tilde{\mu}_{i,t}$ is $SL(n)$-invariant. Since, if $i \in \{1, \cdots, n-1\}$, $\tilde{\mu}_{i,t}$ is both i-homogeneous and $SL(n)$-invariant on $\mathcal{K}^n$, we have that $\tilde{\mu}_{i,t}(K) = 0$ for every $t > 0$ and for every $K \in \mathcal{K}^n$. By Proposition 7.1.23, $\mu_i(f) = 0$ for all $f \in \mathcal{QC}(\mathbb{R}^n)$ and then

$$\mu = \mu_0 + \mu_n.$$

By Theorem 7.1.22, we have the characterization result, i.e.

$$\mu(f) = \phi_0(\max_{\mathbb{R}^n} f) + \int_0^{+\infty} \phi_n(t) dS_n(f; t)$$

with the appropriate properties for ϕ_0 and ϕ_n.
We have the vice versa by Theorem 7.1.22. □

We want to prove now characterization result for continuous (w.r.t. hypo-convergence), $GL(n)$ and translation invariant valuations. We need the following proposition.

Proposition 7.2.4. *Let μ be a continuous (w.r.t. hypo-convergence), $GL(n)$ and translation invariant and i-homogeneous valuation on $\mathcal{QC}(\mathbb{R}^n)$, with $i \in \{1, \cdots, n\}$. Then we have that $\mu(f) = 0$ for all $f \in \mathcal{QC}(\mathbb{R}^n)$.*

Proof. Since dilations are $GL(n)$-transformations and the operation $\lambda\odot$ is a dilation with $\frac{1}{\lambda}$, $\lambda > 0$, as the dilation coefficient, we have

$$\mu(f) = \mu(\lambda \odot f) = \lambda^i \mu(f), \quad \forall\, \lambda > 0,\, \forall\, f \in \mathcal{QC}(\mathbb{R}^n).$$

Since it holds for every $\lambda > 0$, we have $\mu(f) = 0$ for every $f \in \mathcal{QC}(\mathbb{R}^n)$. □

Theorem 7.2.5. *A functional $\mu\colon \mathcal{QC}(\mathbb{R}^n) \to \mathbb{R}$ is a continuous (w.r.t. hypo-convergence), $GL(n)$ and translation invariant valuation if and only if there exists $\phi_0\colon [0, +\infty) \to \mathbb{R}$ continuous and $\phi_0(0) = 0$ such that*

$$\mu(f) = \phi_0(\max_{\mathbb{R}^n} f),$$

for all $f \in \mathcal{QC}(\mathbb{R}^n)$.

Proof. We already know that $\mu(f) = \phi_0(\max_{\mathbb{R}^n} f)$ is a continuous (w.r.t. hypo-convergence), $GL(n)$ and translation invariant valuation on $\mathcal{QC}(\mathbb{R}^n)$. Vice versa by Theorem 7.2.3 we have

$$\mu(f) = \phi_0(\max_{\mathbb{R}^n} f) + \int_0^{+\infty} \phi_n(t)dS_n(f;t),$$

for all $f \in \mathcal{QC}(\mathbb{R}^n)$. Since $\int_0^{+\infty} \phi_n(t)dS_n(f;t)$ is n-homogeneous, by previous proposition we have

$$\int_0^{+\infty} \phi_n(t)dS_n(f;t) = 0, \quad \forall\, f \in \mathcal{QC}(\mathbb{R}^n),$$

and then

$$\mu(f) = \phi_0(\max_{\mathbb{R}^n} f),$$

for every $f \in \mathcal{QC}(\mathbb{R}^n)$. □

7.3 Future developments

The problem we want to study consists to find a right definition of family of continuous (w.r.t. hypo-convergence) and invariant (w.r.t. translations) valuations, through polynomiality result, such that it would be dense in valuation space, w.r.t. a suitable topology (a possible idea for this topology is to use the family of functions $\mathcal{QC}_N(\mathbb{R}^n)$ introduced in Chapter 3). To justify this problem, we present briefly the work made in [56] by V. Milman and L. Rotem.

They studied quasi-concave function space to obtain functional counterpart of Minkowski Theorem and Brunn-Minkowski and Alexandrov-Fenchel Inequalities. They introduced a notion of "volume" for function $f \in \mathcal{QC}(\mathbb{R}^n)$,

$$V(f) = \int_{\mathbb{R}^n} f(x)dx$$

and they proved the following statement.

Theorem 7.3.1 (Theorem 6 in [56])**.** *Let us fix $f_1, \cdots, f_m \in \mathcal{QC}(\mathbb{R}^n)$. Then the function*

$$F\colon (\mathbb{R}_+)^m \to [0, +\infty],$$

defined by

$$F(\lambda_1, \cdots, \lambda_m) = \int_{\mathbb{R}^n} [(\lambda_1 \odot f_1) \oplus \cdots \oplus (\lambda_m \odot f_m)](x)dx$$

is a homogeneous polynomial of degree n, with non-negative coefficients. If we write

$$F(\lambda_1, \cdots, \lambda_m) = \sum_{i_1, \cdots, i_n = 1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(f_{i_1}, \cdots, f_{i_n})$$

for a symmetric function V, then

$$V(f_1, \cdots, f_n) = \int_0^{+\infty} V(L_t(f_1), \cdots, L_t(f_n))dt.$$

Hence they define a functional counterpart of mixed volumes, that they call **mixed integrals**, for quasi-concave functions. Since we are interested in valuation properties, we can not use directly this notion because we want

$$V(f_1, \cdots, f_n) < +\infty$$

for every choice of $f_1, \cdots, f_n \in \mathcal{QC}(\mathbb{R}^n)$.

The idea is to use Theorem 6.3.5, the polynomiality result for a generic m-homogeneous valuation. We have to choose carefully this valuation, because we do not have a unique notion of "volume" for quasi-concave functions. If we consider a n-homogeneous valuation (continuous w.r.t. hypo-convergence and translation invariant), then we know that there exist a function $\phi_n\colon [0, +\infty) \to \mathbb{R}$ and $\delta > 0$ such that $\phi_n(t) = 0$, for all $t \in [0, \delta]$ and

$$\mu(f) = \int_0^{+\infty} \phi_n(t) dS_n(f; t)$$

for every $f \in \mathcal{QC}(\mathbb{R}^n)$.
Therefore the notion of volume for f depends on ϕ_n and the notion of "mixed integrals" we would like to introduce will depend by the functional $\overline{\mu}$ that appear in polynomiality result.

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