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Asymptotic analysis of solutions related to the game-theoretic p-laplacian

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Introduction

The *game-theoretic* (or *normalized* or *one-homogeneous*) *p-laplacian* is defined as

$$\Delta_p^G u = \frac{1}{p}|\nabla u|^{2-p}\Delta_p u, \tag{1}$$

for $p \in (1, \infty)$, where $\Delta_p u$ is the usual p-laplacian

$$\Delta_p u = \operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big),$$

and as

$$\Delta_\infty^G u = \frac{\langle \nabla^2 u \nabla u, \nabla u\rangle}{|\nabla u|^2},$$

in the extremal case $p = \infty$. With these notations, by computing formally the divergence in (1), Δ_p^G can be written for any $p \in [1, \infty]$ as

$$\Delta_p^G u = \frac{1}{p}\Delta u + \left(1 - \frac{2}{p}\right)\Delta_\infty^G u, \tag{2}$$

where Δ is the classical Laplacian.

Works by Peres, Schramm, Scheffield and Wilson (see [50, 51]) have emphasized the role of Δ_p^G in stochastic differential equations in the context of *game theory*. Indeed, equations for Δ_p^G appear when one considers the limiting value for vanishing length of steps of certain two-players games called *tug-of-war (or TOW) games with noise* in the case $p \in (1, \infty)$ and TOW games in the case $p = \infty$. In this context, it is possible to consider a stochastic game between two opponents, one who wants to maximize and the other who wants to minimize the payoff. Heuristically, the game consists of a combination of random moves (which correspond to noise and are dictated by Δ) and moves that are orthogonal to the gradient (which correspond to TOW and depend on the operator Δ_∞^G). In this context, we also mention works by Manfredi, Parviainen and Rossi, in [43, 45, 44, 46].

The relation between TOW games and differential equations similar to those involving the game-theoretic p-laplacian are considered by Nyström and Parviainen in the context of *market manipulation* and *option pricing* (see [47]).

There is also a growing interest for equations involving the game-theoretic p-laplacian in connection to numerical methods for *image enhancement or restoration* (see [20] and

[10]). Typically, for a possibly corrupted image represented by a function u_0, it is considered an evolutionary process based on Δ_p^G with initial data u_0 and homogeneous Neumann boundary conditions. As explained in [20], the different choice of p affects the direction in which the brightness evolves; the 1-homogeneity of Δ_p^G ensures that such an evolution does not depend on the brightness of the image. The relation between solutions of parabolic equations and a corresponding parametrized elliptic equation is examined in [10] for the classical p-laplacian, and can be extended to the case of Δ_p^G in hand.

Besides the cited applications, problems for Δ_p^G have been recently studied by Attouchi and Parviainen [5], Attouchi, Parviainen and Ruosteenoja [6], Parviainen and Ruosteenoja [48], A. Björn, J. Björn and Parviainen [15], Parviainen and Vázquez [49], Banerjee and Garofalo [7, 8], Does [20], Juutinen and Kawohl [29], Kawohl, Krömer and Kurtz [30] as well as Banerjee and Kawohl [9] and Jin and Silvestre [27].

Observe that, having in mind the formal decomposition (2), when $p = 2$ we simply obtain $\Delta_2^G = \Delta/2$ and that, for $p \neq 2$, Δ_p^G can be seen as a proper singular perturbation of Δ/p. Indeed, one notices that $\Delta_\infty^G u$ has discontinuous coefficients when $\nabla u = 0$. Nevertheless, Δ_p^G is uniformly elliptic (in case $p \in (1, \infty)$) and (degenerate) elliptic in the case $p = \infty$.

It is evident that, for $p \neq 2$, Δ_p^G is nonlinear. However, Δ_p^G is somewhat reminiscent of the lost linearity of the Laplace operator, since it is 1-homogeneous, that is

$$\Delta_p^G(\lambda u) = \lambda \Delta_p^G u \quad \text{for any} \quad \lambda \in \mathbb{R},$$

differently from Δ_p, which instead is $(p-1)$-homogeneous. The nonlinearity of Δ_p^G is indeed due to its non-additivity. Nevertheless, Δ_p^G acts additively if one of the relevant summands is constant and, more importantly, on functions of one variable and on radially symmetric functions. We shall see in the sequel that these last properties are decisive for the purposes of this thesis.

Also, when $p \neq 2$, differently from Δ_p, Δ_p^G is not in divergence form. This fact implies that we cannot apply the standard theory of distributional weak solutions. We need to consider the theory of *viscosity solutions*. The main tool of this theory we will use is the *comparison principle*, that we recall and adapt to our purposes in Chapter 2, together with some versions of the *strong maximum principle* and the *Hopf-Oleinik lemma*, which will be used in Chapter 5.

In this thesis, we focus on the connection between asymptotic formulas for solutions of certain game-theoretic p-laplacian problems and some geometrical features of the relevant domain.

We will generally consider a domain Ω in $\mathbb{R}^N$, with $N \geq 2$, not necessarily bounded, with boundary $\Gamma \neq \varnothing$. We shall consider viscosity solutions $u = u(x,t)$ of the following initial-boundary value problem:

$$u_t = \Delta_p^G u \quad \text{in} \;\; \Omega \times (0, \infty), \tag{3}$$
$$u = 0 \quad \text{on} \;\; \Omega \times \{0\}, \tag{4}$$
$$u = 1 \quad \text{on} \;\; \Gamma \times (0, \infty). \tag{5}$$

Also, we shall consider viscosity solutions u^ε of the one-parameter family of boundary value problems

$$u^\varepsilon - \varepsilon^2 \Delta_p^G u^\varepsilon = 0 \quad \text{in} \;\; \Omega, \tag{6}$$
$$u^\varepsilon = 1 \quad \text{on} \;\; \Gamma. \tag{7}$$

Our attention will focus on the asymptotic analysis for small positive values of parameters t and ε.

When $p = 2$, the two cases are strongly connected. Indeed, by taking advantage of the linearity of (3), one can use a modified *Laplace transform*, to obtain that,

$$u^\varepsilon(x) = \frac{1}{\varepsilon^2} \int_0^\infty u(x,t) e^{-\frac{t}{\varepsilon^2}} \, dt \quad \text{for} \;\; x \in \overline{\Omega}, \; \varepsilon > 0. \tag{8}$$

In the case $p \neq 2$ the two problems are no more equivalent. Nevertheless, if Ω has spherical or one-dimensional symmetry, due to the fact that for radial or one-dimensional functions the operator Δ_p^G acts linearly, (8) still holds true. This observation will be crucial. A proper manipulation of (8) will give suitable barriers to estimate the parabolic solution (see Lemma 2.15).

In what follows, we will describe the main results of this thesis, which are mainly contained in the two papers [14, 13]. The case of problem (3)-(5) is considered in [14], whereas [13] addresses the case of problem (6)-(7). Under the assumption that Ω merely satisfies the topological assumption $\Gamma = \partial \left(\mathbb{R}^N \setminus \overline{\Omega} \right)$, in Theorem 3.5 we establish for $p \in (1, \infty]$ the asymptotic profile of the solution of (3)-(5) for small values of time:

$$\lim_{t \to 0^+} 4t \log u(x,t) = -p' \, d_\Gamma(x)^2, \; x \in \overline{\Omega}. \tag{9}$$

Here, by p' we mean the conjugate exponent of p, that is $1/p + 1/p' = 1$, for $p \in (1, \infty)$, and $p' = 1$, when $p = \infty$. Also, by $d_\Gamma(x)$, we mean the distance of the point $x \in \Omega$ from the boundary Γ, defined by

$$d_\Gamma(x) = \inf\{|x - y| : y \in \Gamma\}, \; x \in \overline{\Omega}.$$

Moreover, in Theorem 3.6, we obtain the corresponding formula for the solution of the elliptic problem (6)-(7):

$$\lim_{\varepsilon \to 0^+} \varepsilon \log u^\varepsilon(x) = -\sqrt{p'} \, d_\Gamma(x), \; x \in \overline{\Omega}. \tag{10}$$

These pointwise asymptotic profiles extend known formulas in the linear case, first obtained by Varadhan by using analytic methods (see [56, 57]). See also [21], where

Evans and Ishii used arguments pertaining the theory of viscosity solutions, and [23, Section 10.1], for a treatment with probabilistic methods by Freidlin and Wentzell. These formulas are common in the context of *large deviations theory.* There, random differential equations with small noise intensities are considered. The profiles for small values of ε and t of the solutions of (6)-(7) and (3)-(5) are respectively related to the behavior of the exit time and to the probability to exit from Ω of a certain stochastic process (see [21] and [23]).

Formulas (9) and (10) are obtained by employing barrier arguments based on accurate estimates on radial solutions. In particular, all we need is to control solutions both in a ball or in the complement of a ball. In the elliptic case we are able to compute them in terms of *Bessel functions* whereas the parabolic case is more delicate, since we need to properly involve (8) and the existence of a global solution of (3).

We mention that proper versions of formula (9) are included in a series of works by Magnanini-Sakaguchi in linear cases (see [35, 36, 37, 38, 40, 42]), in certain nonlinear contexts (see [37, 39, 41]) and concerning initial-value problems ([33, 34]).

A second type of asymptotic formulas that we obtain, which deeply link solutions of (3)-(5) or (6)-(7) to the geometry of the domain, involve certain statistical quantities called *q-means.* Given $x \in \Omega$, $t > 0$ and $q \in (1, \infty]$, the q-mean $\mu_q(x,t)$ on $B_R(x) \subset \Omega$ of the solution u of (3)-(5) is the unique real number μ such that

$$\|u(\cdot,t) - \mu\|_{L^q(B_R(x))} \le \|u(\cdot,t) - \lambda\|_{L^q(B_R(x))} \quad \text{for any} \quad \lambda \in \mathbb{R}.$$

Observe that μ_q generalizes the mean value of u on $B_R(x)$, which is obtained when one chooses $q = 2$ in the above definition.

Consider a domain of class C^2. Suppose that there exists $y_x \in \Gamma$ such that $(\mathbb{R}^N \setminus \Omega) \cap \partial B_R(x) = \{y_x\}$, where $R = d_\Gamma(x)$. Assume that $\kappa_1(y_x), \dots, \kappa_{N-1}(y_x) < \frac{1}{R}$, where we have denoted with $\kappa_1, \dots, \kappa_{N-1}$ the principal curvatures with respect to the inward normal of Γ at y_x, and set

$$\Pi_\Gamma(y_x) = \prod_{j=1}^{N-1} \Big[1 - R\, \kappa_j(y_x)\Big].$$

In Theorem 4.7 we establish that, for any $q \in (1, \infty)$ and $p \in (1, \infty]$,

$$\lim_{t \to 0^+} \left(\frac{R^2}{t}\right)^{\frac{N+1}{4(q-1)}} \mu_q(x,t) = C_{N,p,q} \left\{\Pi_\Gamma(y_x)\right\}^{-\frac{1}{2(q-1)}}, \tag{11}$$

where $C_{N,q,p}$ is a positive constant only depending on the labelled parameters. In the extremal case $q = \infty$ and for any $p \in (1, \infty]$, we obtain that

$$\mu_\infty(x,t) \to \frac{1}{2} \quad \text{as } t \to 0^+.$$

Analogously, from an accurate improvement of barriers in the case of smooth domains we compute the asymptotic profile of $\mu_{q,\varepsilon}$, the q-mean of u^ε on $B_R(x)$. In fact, in

Theorem 4.11 we show that

$$\lim_{\varepsilon\to 0^+}\left(\frac{R}{\varepsilon}\right)^{\frac{N+1}{2(q-1)}}\mu_{q,\varepsilon}(x)=\widetilde{C}_{N,p,q}\left\{\Pi_\Gamma(y_x)\right\}^{-\frac{1}{2(q-1)}}. \tag{12}$$

Also in this case, we obtain that

$$\mu_{\infty,\varepsilon}(x)\to\frac{1}{2}\ \text{ as }\ \varepsilon\to 0^+.$$

We emphasize that (11) and (12) generalize, to each $p\in(1,\infty]$, and extend, to any $q\in(1,\infty)$, known formulas in the linear case, for $p=q=2$. In fact, we recall that in [37, Theorem 4.2], it has been given the following asymptotic formula for the so-called *heat content* on $B_R(x)$:

$$\lim_{t\to 0^+} t^{-\frac{N+1}{4}}\int_{B_R(x)}u(z,t)\,dz=\frac{C_N R^{\frac{N-1}{2}}}{\sqrt{\Pi_\Gamma(y_x)}}, \tag{13}$$

where u is the solution of the heat equation satisfying (4) and (5). A normalization makes apparent the connection of the heat content to $\mu_2(x,t)$. See also [35, Theorem 2.3] for a similar formula in the elliptic case. Similar formulas for the case $q=2$ in nonlinear settings can be found in [37] for the evolutionary p-Laplace equation, and in [41], for a class of non-degenerate fast diffusion equations.

It is worth noting that (11) and (12) are essentially based on two ingredients. The first is the geometrical lemma [37, Lemma 2.1] which determines the behavior of $\mathcal{H}_{N-1}(\Gamma_s\cap B_R(x))$ for vanishing $s>0$ in terms of the function Π_Γ. Here, $\Gamma_s=\{x\in\Omega : d_\Gamma(x)=s\}$, for $s>0$ and $\mathcal{H}_{N-1}$ is the $(N-1)$-Hausdorff measure. The second ingredient is the construction of sharp uniform estimates of Varadhan-type formulas (9) and (10), which are novelty even in the case $p=2$.

In the case Ω is of class $C^{0,\omega}$, that is Γ is locally a graph of a continuous function with modulus of continuity controlled by ω (see Section 3.4), we provide in Theorem 3.7 the following estimate:

$$4\,t\log u(x,t)+p'\,d_\Gamma(x)^2=O\left(t\,\log\psi_\omega(t)\right), \tag{14}$$

for $t\to 0^+$, uniformly on every compact subset of $\overline{\Omega}$. Here, $\psi_\omega(t)$ is a function that depends on ω and is positive and vanishes as $t\to 0^+$. In particular, if Ω is smooth enough, (14) gives a sharp estimate of the rate of convergence in (9). In fact, if for example Ω is a α-Hölder domain, for some $0<\alpha<1$, we obtain that the right-hand side of (14) is $O\left(t\log t\right)$, as $t\to 0^+$.

In the elliptic case, explicit barriers are available, we obtain more accurate uniform estimates. In fact, in Theorem 3.9, we prove that

$$\varepsilon\log u^\varepsilon(x)+\sqrt{p'}\,d_\Gamma(x)=\begin{cases}O\left(\varepsilon\right) & \text{if } \ p=\infty,\\ O\left(\varepsilon\log\varepsilon\right) & \text{if } \ p\in(N,\infty),\end{cases} \tag{15}$$

as $\varepsilon \to 0^+$, on every compact subset of $\overline{\Omega}$. In the case of a domain Ω of class $C^{0,\omega}$, it holds instead that

$$\varepsilon \log u^\varepsilon(x) + \sqrt{p'}\, d_\Gamma(x) = \begin{cases} O\left(\varepsilon \log|\log \psi_\omega(\varepsilon)|\right) & \text{if } \ p = N, \\ O\left(\varepsilon \log \psi_\omega(\varepsilon)\right) & \text{if } \ p \in (1, N), \end{cases} \tag{16}$$

for $\varepsilon \to 0^+$, uniformly on every compact subset of $\overline{\Omega}$.

We observe that the presence of the threshold for the exponent p in this last formula seems to be connected to the integrability of the global solution of (3) with respect to the variable $t \in (0, \infty)$. This suggests that even in the parabolic case we may expect this kind of behavior. We were not able to prove it so far.

Finally, notice that, by using comparison results, formulas (14) and (15)-(16) can be easily extended to the case of a prescribed non-constant data on the boundary. See Corollaries 3.8 and 3.10.

In Chapter 5, the obtained Varadhan-type formulas and formulas for q-means will find applications to geometric and symmetry results. The linearity of Δ was used in [35] to derive radial symmetry of compact *stationary isothermic surfaces*, that is those level surfaces of the temperature which are invariant in time. In Chapter 5, we will extend this type of result to the case $p \neq 2$. In the case $p = 2$, it was shown that the mean values $\mu_2(x, t)$ or $\mu_{2,\varepsilon}(x)$ do not depend on x if this lies on a stationary isothermic surface, and hence, for instance, (13) gives that

$$\Gamma \ni y \mapsto \Pi_\Gamma(y) \text{ is constant.}$$

The radial symmetry then ensues from Alexandrov's Soap Bubble Theorem for Weingarten surfaces (see [2]).

For $p \neq 2$, this approach is no longer possible. However, when Γ is compact, an approach based on the *method of moving planes* (see [55], as in [39] and [17]) is still feasible. We also treat a case in which Γ is unbounded, by using the *sliding method* (see [12]), as in [38], [39], [40] and [53] to obtain that Γ must be a hyperplane. We stress that a crucial step to apply the cited methods in our cases is the application of the (classical) strong comparison principle in a suitable subset (as done in [3] or in [9]), which is determined by an application of the strong maximum principle and Hopf lemma for viscosity solutions.

We conclude this introduction by a summary of this thesis. Chapter 1 recalls those technical tools of the theory of viscosity solutions which we will use in the remaining chapters.

In Chapter 2, we consider the cases of symmetric domains, in which Δ_p^G acts as a linear operator. This allows us to deal with explicit solutions of (3)-(5) and (6)-(7) and to compute their asymptotic profiles. The explicit formulas give sharp estimates that will be crucial to control the case of generic domains.

Formulas of Varadhan-type, that is (9) and (10), are collected in Chapter 3. There, also their uniform sharp versions (14) and (15)-(16) shall be given.

In Chapter 4, we provide our asymptotic formulas for q-means (11) and (12).

Finally, Chapter 5 contains a few applications of the formulas derived in Chapters 3 and 4. In particular, we provide generalization of symmetry results present in the literature.

Chapter 1

Preliminaries on the theory of viscosity solutions

Several results of this thesis are based on some important properties of viscosity solutions. In the next chapters we shall use comparison principles and we shall apply strong maximum principles and the Hopf-Oleinik Lemma. Since Δ_p^G has discontinuous coefficients, standard results for viscosity solutions cannot be directly applied, but they must be adapted. In this chapter we describe how, by pointing out the significant references.

A recent summary on aspects of viscosity solutions of our interest, which includes a quite complete list of references, is a dedicated chapter in [24] (where the theory is instrumental to the study of *surface evolution equations*). We adopt that approach. Useful references on definitions and relevant properties of viscosity solutions are also the classical surveys [16, 18], besides the beginner's guide [31]. For more recent and specific works on Δ_p^G, where viscosity solutions are adopted, we give the following (not complete) list of publications: [5, 6, 7, 8, 13, 14, 20, 29, 30, 44].

In Section 1.1, we shall begin with definitions and relevant properties of viscosity solutions of general singular differential equations. In this more general context, we shall also state those theorems (from [11, 19, 25, 54]) which we shall apply to our cases, in Section 1.2.

In Section 1.2, in the specific case of the game-theoretic p-laplacian, we shall collect those results which we will use in the next chapters of this thesis. In particular, in Subsection 1.2.1, for both equations (3) and (6), we will give comparison principles (see Corollaries 1.12 and 1.14) as well as corresponding strong maximum principles (see Corollaries 1.16 and 1.18).

Finally, in Subsection 1.2.2 we shall extend a sharp version of Hopf-Oleinik lemma (obtained by Mazya et al., in [4]), to viscosity solutions of (3)-(5) and (6)-(7) (see Corollaries 1.24 and 1.25).

1.1 Viscosity solutions of singular differential equations

To start with, we introduce some definitions on arbitrary functions, which we use in this chapter. Let f be a function on a metric space $\mathcal{X}$, with values in $\mathbb{R}$. We recall that a function f is called *lower semicontinuous* if

$$\liminf_{y \to x} f(y) \geq f(x)$$

and that a function f is called *upper semicontinuous* if $-f$ is lower semicontinuous.

For a function f, the *lower semicontinuous envelope* f_* is defined by

$$f_*(z) = \lim_{\delta \to 0^+} \inf\{f(z') : z' \in B_\delta(z) \subset \mathcal{X}\},$$

and the *upper semicontinuous envelope* f^* is defined by $f^* = -(-f)_*$. Note that the previous definition differs from that of lim inf, since the infimum is taken upon the whole ball $B_\delta(z)$. Of course, if f is continuous, f and its envelopes coincide.

For $N \geq 1$, let $\mathcal{S}^N$ be the linear space of $N \times N$ symmetric matrices. An operator $F : \Omega \times (0, \infty) \times \mathbb{R} \times \left(\mathbb{R}^N \setminus \{0\}\right) \times \mathcal{S}^N \to \mathbb{R}$, is called *(degenerate) elliptic* if it satisfies

$$F(x, t, u, \xi, X) \geq F(x, t, u, \xi, Y) \quad \text{if} \quad X \leq Y,$$

for any $(x, t, u, \xi) \in \Omega \times (0, \infty) \times \mathbb{R} \times \left(\mathbb{R}^N \setminus \{0\}\right)$ and $X, Y \in \mathcal{S}^N$. Here with $X \leq Y$ we mean that $\langle (X - Y)\eta, \eta \rangle \leq 0$, for any $\eta \in \mathbb{R}^N$.

We give the following definitions (see [24, Chapter 2]). A lower semicontinuous function, $u : \Omega \times (0, \infty) \to \mathbb{R}$, is a *viscosity subsolution* of

$$u_t + F\left(x, t, u, \nabla u, \nabla^2 u\right) = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

if for any $(x, t, \phi) \in \Omega \times (0, \infty) \times C^2\left(\Omega \times (0, \infty)\right)$, such that $u - \phi$ attains its maximum at (x, t), it holds that

$$\phi_t(x, t) + F_*\left(x, t, u(x, t), \nabla \phi(x, t), \nabla^2 \phi(x, t)\right) \leq 0. \tag{1.2}$$

Analogously, an upper semicontinuous function, $v : \Omega \times (0, \infty) \to \mathbb{R}$, is called a *viscosity supersolution* of (1.1), if for any $(x, t, \psi) \in \Omega \times (0, \infty) \times C^2\left(\Omega \times (0, \infty)\right)$ such that $v - \psi$ attains its minimum at (x, t), it holds that

$$\psi_t(x, t) + F^*\left(x, t, v(x, t), \nabla \psi(x, t), \nabla^2 \psi(x, t)\right) \geq 0. \tag{1.3}$$

Finally, we say that a function u is a *viscosity solution* of (1.1) if it is both a viscosity subsolution and a viscosity supersolution.

The next lemma affirms that the theory is consistent with the classical definition of solutions. The proof is straightforward.

Lemma 1.1 (Consistency)**.** *Assume that F is (degenerate) elliptic.*
Let $u \in C^2(\Omega \times (0,\infty))$ be such that

$$F_*\left(x,t,u,\nabla u,\nabla^2 u\right) \leq u_t \leq F^*\left(x,t,u,\nabla u,\nabla^2 u\right),$$

for every $(x,t) \in \Omega \times (0,\infty)$, then u is a viscosity solution of (1.1)*.*

For our purposes, we will need the following extension lemma, that claims that, for smooth functions, it suffices to check the definitions away from isolated critical points.

Lemma 1.2 (Extension)**.** *Assume that F is (degenerate) elliptic.*
Let $u \in C^2(\Omega \times (0,\infty))$ such that

(i) $(x_0,t) \in \Omega \times (0,\infty)$ is the unique point in $\Omega \times (0,\infty)$ such that

$$\nabla u(x_0,t) = 0.$$

(ii) In $(\Omega \setminus \{x_0\}) \times (0,\infty)$, it holds that

$$F_*\left(x,t,u,\nabla u,\nabla^2 u\right) \leq u_t \leq F^*\left(x,t,u,\nabla u,\nabla^2 u\right).$$

Then u is a solution of (1.1) *in $\Omega \times (0,\infty)$.*

Proof. Take a sequence of points $y_n \in \Omega \setminus \{x_0\}$ such that $y_n \to x_0$ as $n \to \infty$. From our assumption on u, we have that both of the following inequalities hold at (y_n,t):

$$u_t + F_*\left(x,t,u,\nabla u,\nabla^2 u\right) \leq 0,$$

and

$$u_t + F^*\left(x,t,u,\nabla u,\nabla^2 u\right) \geq 0.$$

Now, since F_* is lower semicontinuous and F^* is upper semicontinuous we have that

$$u_t(x_0,t) + F_*\left(x_0,t,u(x_0),\nabla u(x_0),\nabla^2 u(x_0)\right) \leq \liminf_{n\to\infty}\left\{u_t(y_n,t) + F_*\left(y_n,t,u(y_n),\nabla u(y_n),\nabla^2 u(y_n)\right)\right\} \leq 0$$

and

$$u_t(x_0,t) + F^*\left(x_0,t,u(x_0),\nabla u(x_0),\nabla^2 u(x_0)\right) \geq \limsup_{n\to\infty}\left\{u_t(y_n,t) + F^*\left(y_n,t,u(y_n),\nabla u(y_n),\nabla^2 u(y_n)\right)\right\} \geq 0.$$

The claim follows, thanks to Lemma 1.1. □

Remark 1.3. Obvious adjustments of definitions and Lemmas 1.1 and 1.2 are given in the case of singular elliptic differential equations (see for example [18]).

1.1.1 Comparison principles

For our purposes, we propose quite general comparison results, in a not necessarily bounded domain Ω. The relevant assumptions apply to the differential equations (3) and (6).

As a *modulus of continuity* we mean a continuous function $\omega : [0, \infty) \to [0, \infty)$, such that $\omega(0) = 0$. If $X \in \mathcal{S}^N$, by $|X|$ we intend the operator norm of X on $\mathbb{R}^N$.

Theorem 1.4 ([25, Theorem 2.1])**.** *Assume that $F : \mathbb{R}^N \setminus \{0\} \times \mathcal{S}^N \to \mathbb{R}$ is continuous and (degenerate) elliptic. In addition, let F satisfy the following properties:*

(a) it holds that
$$-\infty < F_* (0, 0) = F^* (0, 0) < +\infty;$$

(b) for every $R > 0$,
$$\sup\{|F (\xi, X) | : 0 < |\xi| \leq R, \ |X| \leq R\} < \infty.$$

Let u and v be, respectively, a subsolution and a supersolution of
$$u_t + F\left(\nabla u, \nabla^2 u\right) = 0$$
in $\Omega \times (0, \infty)$. Assume that

(i) $u(x, t) \leq K(|x| + 1)$, $v(x, t) \geq -K(|x| + 1)$, for some $K > 0$ independent of $(x, t) \in \Omega \times (0, \infty)$;

(ii) there is a modulus ω such that
$$u^*(x, t) - v_*(y, t) \leq \omega(|x - y|),$$
for all $(x, y, t) \in \partial (\Omega \times \Omega) \times (0, \infty) \cup (\Omega \times \Omega) \times \{0\}$;

(iii) $u^*(x, t) - v_*(y, t) \leq K(|x - y| + 1)$ on $\partial (\Omega \times \Omega) \times (0, \infty) \cup (\Omega \times \Omega) \times \{0\}$, for some $K > 0$ independent of (x, y, t).

Then, it holds that
$$u^* \leq v_* \quad on \;\; \overline{\Omega} \times (0, \infty).$$

Remark 1.5. Note that (ii) is equivalent to the condition $u^* \leq v_*$ on $\partial\Omega \times (0, \infty)$ and that (i) and (iii) are unnecessary, if Ω is bounded.

It is also evident that, if u^* and v_* are bounded, then (i) and (iii) are satisfied.

We now state a general comparison principle for elliptic equations, which is a corollary of [54, Theorem 2.2], that treats general equations.

Theorem 1.6. *Let $F : \mathbb{R}^N \setminus \{0\} \times \mathcal{S}^N \to \mathbb{R}$ be continuous, (degenerate) elliptic and such that:*

(a)
$$-\infty < F_*(0,0) = F^*(0,0) < +\infty;$$

(b) for every $R > 0$,
$$\sup\{|F(\xi,X)| : 0 < |\xi| \le R,\ |X| \le R\} < \infty;$$

(c) for every $R > \rho > 0$, *there exists a modulus* σ *such that*
$$|F(x,r,\xi,X) - F(x,r,\eta,X)| \le \sigma(|\xi-\eta|),$$
for all $x \in \Omega$, $r \in \mathbb{R}$, $\rho \le |\xi|, |\eta| \le R$, $|X| \le R$.

Let u *and* v *be, respectively, a subsolution and a supersolution of*
$$u + F\left(\nabla u, \nabla^2 u\right) = 0 \quad \textit{in} \quad \Omega.$$

Moreover, suppose that

(i) $u(x) \le K(|x|+1)$, $v(x) \ge -K(|x|+1)$, *for some* $K > 0$ *independent of* $x \in \Omega$*;*

(ii) there is a modulus ω *such that*
$$u^*(x) - v_*(y) \le \omega(|x-y|), \quad \textit{for all} \quad (x,y) \in \partial(\Omega \times \Omega);$$

(iii) $u^*(x) - v_*(y) \le K(|x-y|+1)$ *on* $\partial(\Omega\times\Omega)$, *for some* $K > 0$ *independent of* $(x,y) \in \partial(\Omega\times\Omega)$.

Then, it holds that
$$u^* \le v_* \quad \textit{on} \quad \Omega.$$

Remark 1.7. Observe that assumptions (a) and (b) of Theorem 1.6 are fulfilled, if both F_* and F^* are continuous in their variables. In particular, the supremum in (b) is less than or equal to
$$\sup\{|F_*(\xi,X)| + |F^*(\xi,X)| : |\xi|,\ |X| \le R\}.$$

1.1.2 Strong maximum principles

In Chapter 5 we shall use the strong maximum principle for (3) and (6). Here, we state results for a quite large class of differential operators.

We start with the next theorem, which can be seen as a corollary of [19, Corollary 2.3].

Theorem 1.8. *Assume that* $F : \mathbb{R}^N \setminus \{0\} \times \mathcal{S}^N \to \mathbb{R}$ *is lower semicontinuous and (degenerate) elliptic. Moreover, assume that* F *satisfies the following requirements:*

(a) there exists $\rho_0 > 0$ *such that , for any choice of* $0 < |s|, |\xi| < \rho_0$*, there exists* $\gamma_0 \geq 0$ *such that*

$$s + F(\xi, I - \gamma\xi \otimes \xi) > 0$$

holds for every $\gamma \geq \gamma_0 > 0$*;*

(b) for all $\eta > 0$ *there exist a function* $\varphi : (0,1) \to (0,\infty)$*,* $\varepsilon_\eta > 0$ *and* $\gamma_0 \geq 0$ *such that for all* $\lambda \in (0, \varepsilon_\eta]$ *and* $\gamma > \gamma_0$*,*

$$\lambda s + F_*(\lambda\xi, \lambda(I - \gamma\xi \otimes \xi)) \geq \varphi(\lambda)\left[s + F_*(\xi, (I - \gamma\xi \otimes \xi))\right]$$

holds for all $0 < |\xi| \leq \eta$*,* $|s| \leq \eta$*.*

(c) there exists $\delta_0 > 0$ *such that*

$$1 + F_*(0, \delta I) > 0$$

for all $0 < \delta < \delta_0$*.*

(d) for all $\eta > 0$ *there exist* $\varphi : (0,1) \to (0,\infty)$*,* $\varepsilon_\eta > 0$ *such that for each* $K > 0$ *and for each* $\lambda \in (0, \varepsilon_\eta]$*, then*

$$\lambda s + F_*(2K\lambda(y - x), 2K\lambda I) \geq \varphi(\lambda)\left[s + F_*(2K(y - x), 2KI)\right],$$

holds for all $-\eta \leq s \leq 0$*.*

Let u *be a viscosity subsolution of* (1.1) *in* $\Omega \times (0,\infty)$*. Suppose that* u *achieves a maximum at* $(x_0, t_0) \in \Omega \times (0,\infty)$*.*

Then u *is constant on the set of all points, which can be connected to* (x_0, t_0) *by a simple continuous curve in* $\Omega \times (0, t_0)$ *along which the* t*-coordinate is nondecreasing.*

Remark 1.9. Note that a *strong minimum principle* also holds for (1.1), once the obvious necessary changes in Theorem 1.8 are made.

We now give the elliptic version of the strong maximum principle, which will be applied to solutions of equation (6). We report a corollary of [11, Corollary 1], for a general operator $G = G(u, \nabla u, \nabla^2 u)$. Then, in next section, we shall apply to the case of the game-theoretic p-laplacian.

Theorem 1.10. *Let* Ω *be a connected set,* $G : \mathbb{R} \times \mathbb{R}^N \setminus \{0\} \times \mathcal{S}^N \to \mathbb{R}$*. Assume that* G *is lower semicontinuous and satisfies*

(a) for any $s, r \in \mathbb{R}$*,* $\xi \in \mathbb{R}^N \setminus \{0\}$ *and* $X, Y \in \mathcal{S}^N$*, it holds that*

$$G(r, \xi, X) \leq G(s, \xi, Y) \quad \text{if} \quad s \geq r \quad \text{and} \quad Y \leq X;$$

(b) for every $\eta > 0$*, there exists a function* $\varphi : (0,1) \to (0,1)$ *such that*

$$G(\lambda r, \lambda\xi, \lambda X) \geq \varphi(\lambda) G(r, \xi, X)$$

holds for all $r \in [-1, 0]$*,* $0 < |\xi| \leq \eta$*,* $|X| \leq \eta$*;*

(c) there exists $\rho_0 > 0$ *such that for any choice of* $0 < |\xi| \leq \rho_0$*,*

$$G\,(0, \xi, I - \gamma\xi \otimes \xi) > 0 \quad \textit{for} \quad \gamma > \gamma_0$$

holds for some $\gamma_0 \geq 0$*.*

Suppose that u *be a viscosity subsolution of*

$$G\left(u, \nabla u, \nabla^2 u\right) = 0$$

in Ω *that achieves a nonnegative maximum in* Ω*. Then* u *is constant in* Ω*.*

1.2 The case of the game-theoretic p-laplacian

Now, we consider the case of the game-theoretic p-laplacian, that is the differential equations (3) and (6). To begin with, we observe that Δ_p^G can be formally seen as a singular, quasi-linear operator $F : \left(\mathbb{R}^N \setminus \{0\}\right) \times \mathcal{S}^N \to \mathbb{R}$, where

$$F(\xi, X) = -\operatorname{tr}[A(\xi)X], \tag{1.4}$$

with

$$A(\xi) = \frac{1}{p}\,I + \left(1 - \frac{2}{p}\right)\frac{\xi \otimes \xi}{|\xi|^2} \tag{1.5}$$

for $\xi \in \mathbb{R}^N \setminus \{0\}$. Here, I denotes the $N \times N$ identity matrix.

Observe that, if $\xi \neq 0$, $F = F_* = F^*$ while, if $\xi = 0$, we can explicitly calculate the semicontinuous envelopes of F (see [5] or [20]). It holds that

$$p\,F_*(0, X) = -\operatorname{tr}(X) - \min(p-2, 0)\,\lambda(X) - \max(p-2, 0)\,\Lambda(X), \tag{1.6}$$

$$p\,F^*(0, X) = -\operatorname{tr}(X) - \max(p-2, 0)\,\lambda(X) - \min(p-2, 0)\,\Lambda(X), \tag{1.7}$$

where $\lambda(X)$ and $\Lambda(X)$ are the maximum and minimum eigenvalue of X. Since F has a bounded discontinuity at $\xi = 0$, we have that

$$-\infty < F_*\,(\xi, X) \leq F^*\,(\xi, X) < \infty, \tag{1.8}$$

for any $(\xi, X) \in \mathbb{R}^N \times \mathcal{S}^N$.

Note that F is *uniformly elliptic*, in the case $p \in (1, \infty)$, since

$$\min(1/p', 1/p)\,I \leq A(\xi) \leq \max(1/p', 1/p)\,I,$$

and merely (degenerate) elliptic in the case $p = 1, \infty$. Moreover, for $\xi \neq 0$, F is a linear operator in the variable X.

In the case of (3), (1.2) and (1.3) are replaced by the following. We say that an upper semicontinuous function in $\Omega \times (0, \infty)$, $u : \Omega \times (0, \infty) \to \mathbb{R}$, is a *viscosity subsolution* of (3) if, for every $(x,t,\phi) \in \Omega \times (0,\infty) \times C^2(\Omega \times (0,\infty))$ such that $u - \phi$ attains its maximum at (x,t), then

$$\begin{cases} \phi_t(x,t) - \Delta_p^G \phi(x,t) \leq 0 & \text{if } \nabla\phi(x,t) \neq 0 \\ \phi_t(x,t) + F_*\left(0, \nabla^2\phi(x,t)\right) \leq 0 & \text{if } \nabla\phi(x,t) = 0, \end{cases} \tag{1.9}$$

where F_* is given by (1.6).

We say that a lower semicontinuous function in $\Omega \times (0,\infty)$, $v : \Omega \times (0,\infty) \to \mathbb{R}$, is a *viscosity supersolution* of (3) if, for every $(x,t,\psi) \in \Omega \times (0,\infty) \times C^2(\Omega \times (0,\infty))$, such that $v - \psi$ attains its minimum at (x,t), then

$$\begin{cases} \psi_t(x,t) - \Delta_p^G \psi(x,t) \geq 0 & \text{if } \nabla\psi(x,t) \neq 0 \\ \psi_t(x,t) + F^*\left(0, \nabla^2\psi(x,t)\right) \geq 0 & \text{if } \nabla\psi(x,t) = 0, \end{cases} \tag{1.10}$$

where F^* is given by (1.7).

A function u that is both a viscosity subsolution and viscosity supersolution is called a *viscosity solution* of (3).

Here is the case of the *resolvent equation* (6). We say that an upper semicontinuous function in Ω, $u : \Omega \to \mathbb{R}$, is a *viscosity subsolution* of (6) if, for every $(x,\phi) \in \Omega \times C^2(\Omega)$ such that $u - \phi$ attains its maximum at x, then

$$\begin{cases} u(x) - \varepsilon^2 \Delta_p^G \phi(x) \leq 0 & \text{if } \nabla\phi(x) \neq 0; \\ u(x) + \varepsilon^2 F_*\left(0, \nabla^2\phi(x)\right) \leq 0 & \text{if } \nabla\phi(x) = 0, \end{cases}$$

where F_* is given by (1.6).

We say that a lower semicontinuous function $v : \Omega \to \mathbb{R}$, is a *viscosity supersolution* of (6) if, for every $(x,\psi) \in \Omega \times C^2(\Omega)$, such that $v - \psi$ attains its minimum at x, then

$$\begin{cases} v(x) - \varepsilon^2 \Delta_p^G \psi(x) \geq 0 & \text{if } \nabla\psi(x) \neq 0, \\ v(x) + \varepsilon^2 F^*\left(0, \nabla^2\psi(x)\right) \geq 0 & \text{if } \nabla\psi(x) = 0, \end{cases}$$

where F^* is given by (1.7). A function u is a viscosity solution of (6) if it is both a viscosity subsolution and a viscosity supersolution.

Remark 1.11. Observe that, since Δ_p^G is degenerate elliptic, then both Lemmas 1.1 and 1.2 are valid.

1.2.1 Comparison and strong maximum principles

In this section, we provide the ad hoc comparison results that will be applied in the rest of the thesis.

Corollary 1.12 (Comparison principle for (3))**.** *Let Ω be a domain in $\mathbb{R}^N$, with non-empty boundary Γ. Let u and v be two bounded viscosity solutions of* (3) *in $\Omega \times (0,\infty)$. Assume that u and v are continuous on $\Gamma \times (0,\infty)$ and on $\Omega \times \{0\}$.*

Then, if $u \leq v$ *on* $\Gamma \times (0, \infty) \cup \Omega \times \{0\}$*, it holds that*

$$u \leq v \quad on \quad \overline{\Omega} \times (0, \infty).$$

Proof. It is enough to observe that we can apply Theorem 1.4 to (3). Indeed, (i), (ii) and (iii) of Theorem 1.4 are satisfied, thanks to Remark 1.5 and the assumption for $u - v$ on the parabolic boundary. On the other hand, (a) and (b) of Theorem 1.4, follow from (1.8) and Remark 1.7, since, in the case of F_* and F^* given by (1.6) and (1.7), both F_* and F^* are continuous in their variables. □

A standard obvious corollary to comparison principle is the uniqueness of solutions of initial-boundary problems. We state the one of our interest.

Corollary 1.13 (Uniqueness of solutions of (3)-(5))**.** *Let* Ω *be as in Corollary 1.12. Then, the bounded viscosity solution of* (3)-(5) *is unique.*

The comparison principle for equation (6) follows from Theorem 1.6, as follows.

Corollary 1.14 (Comparison principle for (6))**.** *Assume that* Ω *is a domain, with non-empty boundary* Γ*. Let* u *and* v *be two bounded viscosity solutions of* (6) *in* Ω*. Assume that* u *and* v *are continuous up to the boundary* Γ*.*

Then, if $u \leq v$ *on* Γ*, it holds that*

$$u \leq v \quad on \quad \overline{\Omega}.$$

Proof. We need only to ensure that we can apply Theorem 1.6 to the case in which F is given by (1.4). In virtue of Remark 1.7, since F_* and F^* given by (1.6) and (1.7) are continuous, it suffices to verify (c) of Theorem 1.6. The condition (c) of Theorem 1.6 is fulfilled since, away from $\xi = 0$, F is differentiable with respect to ξ and then, in particular, F is Lipschitz continuous (with respect to ξ) in the compact set $\{(\xi, X) : \rho \leq |\xi| \leq R, |X| \leq R\}$. □

Corollary 1.15 (Uniqueness of solutions of (6)-(7))**.** *Let* Ω *be as in Corollary 1.14. Then, the bounded viscosity solution of* (6)-(7) *is unique.*

We give, as corollary of Theorem 1.8, the following result. We observe that in [15] the *strong minimum principle* for (3) is proved, by means of a weak Harnack inequality.

Corollary 1.16 (Strong maximum principle for (3))**.** *Let* Ω *be connected. Let* u *be a viscosity subsolution of* (3) *in* $\Omega \times (0, \infty)$*. If* u *attains its maximum at a point* $(x_0, t_0) \in \Omega \times (0, \infty)$*, then* u *must be constant in* $\overline{\Omega} \times [0, t_0]$*.*

Proof. We need to check that F in (1.4) verifies assumptions (a)-(d) of Theorem 1.8. Conditions (b) and (d) are fulfilled by choosing $\varphi(\lambda) = \lambda$, since $-\Delta_p^G$ is one-homogeneous.

Given $s, \xi \neq 0$, condition (a) can be read, in the case of game-theoretic p-laplacian, as

$$ps - \operatorname{tr}\left\{\left(I + (p-2)\frac{\xi \otimes \xi}{|\xi|^2}\right)(I - \gamma \xi \otimes \xi)\right\} = ps - (N + p - 2) + \gamma(p-1)|\xi|^2 > 0,$$

which is true for any $\gamma > \gamma_0$, where $\gamma_0 = \left[\frac{N+p-2-ps}{(p-1)|\xi|^2}\right]_+$ and for $t \in \mathbb{R}$, $[t]_+ = \max\{0, t\}$.

Analogously, condition (c) it is satisfied, since here we have that

$$1 + F_* (0, \delta I) = 1 - \delta N/p' > 0,$$

for any $\delta < p'/N$. □

Remark 1.17. From Remark 1.9 we have that also a strong minimum principle holds for (3).

We shall also need an analogous result for equation (6), as application of Theorem 1.10. The proof is straightforward and follows the same line as that of Corollary 1.16.

Corollary 1.18 (Strong maximum principle for (6))**.** *Let Ω be connected and u be a nonnegative (viscosity) solution of* (6) *in Ω. If u attains a maximum at an interior point, then u must be constant in Ω.*

Proof. We apply Theorem 1.10 with $G\left(u, \nabla u, \nabla^2 u\right) = u + \varepsilon^2 F\left(\nabla u, \nabla^2 u\right)$, where F is given by (1.4). Indeed, conditions (a) and (c) of Theorem 1.10 follow from the uniform ellipticity of $-\Delta_p^G u$ and the condition (b) is due to the one-homogeneity of $-\Delta_p^G u$. □

Remark 1.19. Observe that we can apply Theorem 1.10 also to $G\left(u, \nabla u, \nabla^2 u\right) = F\left(\nabla u, \nabla^2 u\right)$ where F is given by (1.4). This gives the strong maximum principle for subsolutions of $-\Delta_p^G u = 0$.

1.2.2 Hopf-Oleinik lemma

We conclude this chapter with a version of Hopf-Oleinik boundary point lemma. This lemma will be used in Chapter 5. In the recent work [4, Theorem 4.4] it has been given a sharp version of it, involving the so-called *pseudo-balls*, defined as follows. Given $a, b, R > 0$ and $\omega : [0, R] \to [0, \infty)$, the pseudo-ball, $\mathscr{G}^\omega_{a,b}$, with apex 0, direction $e_N = (0, \ldots, 1) \in \mathbb{S}^{N-1}$ and shape function ω, is defined by

$$\mathscr{G}^\omega_{a,b} = \{x \in B_R : a\omega(|x|)|x| \le x_N \le b\}. \tag{1.11}$$

In [4, Theorem 4.4] it is considered the case of classical solutions of certain second order linear differential equations in a domain satisfying a pseudo-ball interior condition. Here, we give its adaptation only to the case of our interest.

We say that $\tilde\omega : [0, R] \to [0, \infty)$ is a *Dini continuous* function, if it is continuous, $\tilde\omega(t) > 0$, for $t \in (0, R]$, and satisfies

$$\int_0^R \frac{\tilde\omega(t)}{t}\, dt < \infty.$$

Lemma 1.20 (Sharp version of Hopf-Oleinik Lemma for $-\Delta_p^G$)**.** *Let $a, b, R > 0$ and $\Omega = \mathscr{G}^\omega_{a,b}$, as in* (1.11).

Assume that ω *is continuous on* $[0,R]$, $\omega(t)>0$, *for* $t\in(0,R]$, *it satisfies*

$$\sup_{0<t\leq R}\left(\frac{\omega(t/2)}{\omega(t)}\right)<\infty, \tag{1.12}$$

and there exists $\eta>0$, *such that*

$$\frac{\omega(s)}{s}\leq\eta\frac{\omega(t)}{t}\quad \text{for}\ \ 0<t\leq s\leq R.$$

Suppose that there exists a Dini continuous function $\tilde{\omega}:[0,R)\to[0,\infty)$, *such that*

$$\limsup_{\Omega\ni x\to 0}\left[\frac{\omega(|x|)}{|x|}\frac{x_N}{\tilde{\omega}(|x|)}\right]<\infty.$$

Let $p\in(1,\infty)$ *and* u *be a viscosity subsolution of*

$$-\Delta_p^G u=0, \tag{1.13}$$

such that $u(0)>u$ *in* Ω.

If it is assumed that u *is differentiable at* 0, *then* $\nabla u(0)\neq 0$.

The proof follows that of [4, Theorem 4.4]. We report its main steps.

Proof. Let $K>0$ be the finite number that realizes the supremum in (1.12). As in [4, Theorem 4.4], given $\gamma>1+\max\{0,\log_2 K\}$, $C_0,C_1>0$, we define the function $v:\mathscr{G}^{\omega}_{a,b}\to\mathbb{R}$, by

$$v(x)=-x_N-C_0\int_0^{x_N}\int_0^{\sigma}\frac{\tilde{\omega}(t)}{t}\,dt\,d\sigma+C_1\int_0^{|x|}\int_0^{\sigma}\frac{\omega(t)}{t}\left(\frac{t}{\xi}\right)^{\gamma-1}dt\,d\sigma. \tag{1.14}$$

We first list the relevant proprieties of v, which follow by the assumptions on both ω and $\tilde{\omega}$. The function v is of class $C^0\left(\overline{\Omega}\right)\cap C^2(\Omega)$, it satisfies $v(0)=0$ and $\nabla v(0)=-e_N$.

By combining (1.14), assumptions on ω and elementary integral inequalities, as shown in [4, Theorem 4.4], it holds that

$$v(x)\geq\left(\frac{C_1}{2\eta\gamma}-a-aC_0\int_0^b\frac{\tilde{\omega}(t)}{t}\,dt\right)|x|\omega(|x|)\quad\text{on}\quad \partial\mathscr{G}^{\omega}_{a,b}\setminus\{x_N=b\}.$$

For a fixed $C_1>2a\eta\gamma$, it is sufficient to choose $b_*>0$ such that

$$\int_0^{b_*}\frac{\tilde{\omega}(t)}{t}\,dt\leq\frac{C_1-2a\eta\gamma}{2a\eta\gamma C_0}$$

to obtain that $v\geq 0$ on $\partial\mathscr{G}^{\omega}_{a,b_*}\setminus\{x_n=b_*\}$. Moreover, since u and v are continuous functions and the fact that $u(0)>u$ in $\mathscr{G}^{\omega}_{a,b}$, on $\mathcal{K}=\{x\in\partial\mathscr{G}^{\omega}_{a,b}:x_N=b_*\}$, there exists $\lambda>0$ such that

$$\lambda<\frac{\min_{\mathcal{K}}|u-u(x_0)|}{\max_{\mathcal{K}}|v|}.$$

In particular, this last condition implies that $\lambda v \geq u - u(x_0)$ on $\mathcal{K}$. Hence, there exist $b_* > 0$ and $\lambda > 0$ such that

$$\lambda v \geq u - u(0) \quad \text{in} \quad \partial \mathscr{G}^{\omega}_{a,b_*}.$$

Now, in order to apply comparison, we only need to show, for some C_0, that v is a supersolution of (1.13).

By differentiating twice, we have that

$$\nabla^2 v(x) = -C_0 \frac{\tilde{\omega}(x_N)}{x_N} e_N \otimes e_N + C_1 \int_0^{|x|} \omega(t) t^{\gamma-2}\, dt \left[\frac{I}{|x|^{\gamma}} - \gamma \frac{x \otimes x}{|x|^{\gamma+2}} \right] + C_1 \frac{\omega(|x|)}{|x|} \frac{x \otimes x}{|x|^2}.$$

Since $\nabla v(0) = -e_N$, by possibly taking R sufficiently small, we can assume that $\nabla v(x) \neq 0$, for any $x \in \Omega$. After setting $\xi(x) = \frac{\nabla v(x)}{|\nabla v(x)|}$, we can calculate the following:

$$\begin{aligned} p \operatorname{tr} \left\{ A(\xi) \nabla^2 v \right\}(x) = -C_0 \frac{\tilde{\omega}(x_N)}{x_N} \left[1 + (p-2) \xi_N(x)^2 \right] + \\ \frac{C_1}{|x|^{\gamma}} \int_0^{|x|} \omega(t) t^{\gamma-2}\, dt \left\{ N + (p-2) - \gamma - \gamma(p-2) \frac{\langle x, \xi(x) \rangle^2}{|x|^2} \right\} + \\ C_1 \frac{\omega(|x|)}{|x|} \left[1 + (p-2) \frac{\langle x, \xi(x) \rangle^2}{|x|^2} \right], \end{aligned}$$

where $A(\xi)$, defined in (1.5), is the matrix of coefficients of Δ_p^G.

Then, by using also that $|\xi(x)| = 1$, we infer that

$$\begin{aligned} p \Delta_p^G v(x) \leq \\ \frac{\tilde{\omega}(x_N)}{x_N} \Bigg[-C_0 c_{1,p} + \frac{C_1}{|x|^{\gamma}} \int_0^{|x|} \omega(t) t^{\gamma-2}\, dt \frac{x_N}{\tilde{\omega}(x_N)} \left[N + p - 2 - \gamma c_{1,p} \right]_+ + \\ C_1 \frac{\omega(|x|)}{|x|} \frac{x_N}{\tilde{\omega}(x_N)} c_{2,p} \Bigg], \end{aligned}$$

where $c_{1,p} = \min\{1, p-1\}$, $c_{2,p} = \max\{1, p-1\}$ and $[t]_+ = \max\{0, t\}$.

The previous inequality gives that

$$\Delta_p^G v(x) \leq (1/p) \frac{\tilde{\omega}(x_N)}{x_N} \left\{ -C_0 c_{1,p} + C_1 \Lambda \left(M [N + p - 2 - \gamma c_{1,p}]_+ + c_{2,p} \right) \right\},$$

where

$$\Lambda = \sup_{x \in \Omega} \left(\frac{\omega(|x|)}{|x|} \frac{x_N}{\tilde{\omega}(x_N)} \right),$$

and

$$M = \sup_{x \in \Omega} \left(\frac{|x|^{1-\gamma}}{\omega(|x|)} \int_0^{|x|} \omega(t) t^{\gamma-2}\, dt \right).$$

Observe that Λ and M are finite as shown in the proof of [4, Theorem 4.4], from the assumptions on ω and $\tilde{\omega}$.

Thus, by choosing C_0 such that

$$C_0 > C_1 \frac{M\Lambda}{c_{1,p}} \left[N + p - 2 - \gamma c_{1,p}\right]_+ + \frac{c_{2,p}}{c_{1,p}},$$

then $\Delta_p^G v(x) \leq 0$. Hence, by the arbitrariness of x, we have that $-\Delta_p^G v \geq 0$ in $\mathscr{G}^\omega_{a,b_*}$.

Now, from

$$\lambda v \leq u - u(0) \quad \text{on} \quad \overline{\mathscr{G}^\omega_{a,b_*}}$$

and $\nabla v(0) = -e_N$, we can readily obtain the conclusion by a standard argument. $\square$

Remark 1.21. In [4], it has been observed that the conditions in Lemma 1.20 on ω are sharp (see [4] for details).

Remark 1.22. The class of domains for which Lemma 1.20 is valid is quite large and it is given implicitly by the assumptions that must be satisfied by ω.

In particular, if Ω is a domain of class $C^{1,\alpha}$, for some $\alpha \in (0,1)$, by choosing $\tilde{\omega}(t) = \omega(t) = t^\alpha$, the conclusion of Lemma 1.20 holds true.

In [4, Theorem 4.7], it has been observed that, if Ω is a domain of class $C^{1,\omega}$, where ω is Dini continuous and quasi-increasing, then the conclusion of Lemma 1.20 holds true. We say that ω is quasi-increasing if there exists $\tilde{\eta} > 0$ such that $\omega(s) \geq \tilde{\eta}\, \omega(t)$, for any $0 < s \leq t$.

In order to apply Lemma 1.20 to our case, we need to show that the solution of (3)-(5) is a (viscosity) subsolution of the equation (1.13) on every slice $\Omega \times \{t\}$, for $t > 0$.

Lemma 1.23. *Let u be the viscosity solution of* (3) *satisfying* (4)-(5)*.*

Then, for every $t > 0$, $u(\cdot, t) : \Omega \to \mathbb{R}$ is a viscosity subsolution of (1.13) *in Ω.*

Proof. Given $\tau > 0$, set

$$v(x,t) = u(x, t+\tau) \quad \text{for} \quad (x,t) \in \Omega \times (0,\infty).$$

Up to translate the test functions in (1.9) and (1.10), we verify that v is a viscosity solution of (3). After that, the boundary condition (5) is obviously satisfied by v, and we have that $v(x,0) = u(x,\tau) > 0$, by applying the strong minimum principle (Remark 1.17). Corollary 1.12 gives that $v \geq u$ on $\overline{\Omega} \times (0,\infty)$, which yields the following inequality

$$u(x, t+\tau) \geq u(x,t), \ x \in \overline{\Omega}, \ t, \tau > 0. \tag{1.15}$$

Now, for a fixed $t > 0$, let $(x, \varphi) \in \Omega \times C^2(\Omega)$ such that $u(\cdot, t) - \varphi$ attains its maximum at x. We show that

$$F_* \left(\nabla\varphi(x), \nabla^2\varphi(x)\right) \leq 0, \tag{1.16}$$

where F is that in (1.4).

Define the function $\phi \in C^2(\Omega \times (0,\infty))$ by $\phi(y,s) = \varphi(y)$, for $(y,s) \in \Omega \times (0,\infty)$. From (1.15) and the assumption on φ, it follows that $u(x,t) - \phi(x,t) \geq u(y,s) - \phi(y,s)$, for any $y \in \Omega$ and $0 < s \leq t$. By proceeding as in [28, Theorem 1], this inequality is sufficient to infer that

$$\phi_t(x,t) + F_* \left(\nabla\phi(x,t), \nabla^2\phi(x,t)\right) \leq 0,$$

since F is a quasi-linear (degenerate) elliptic operator. Therefore, (1.16) follows and this concludes the proof. □

We say that Ω satisfies an interior ω-pseudo-ball condition in $x_0 \in \Gamma$, if there exist $a, b, R > 0$ and a modulus ω, such that, up to translate and rotate Ω, $\mathscr{G}^\omega_{a,b} \subset \Omega$ and $\partial\mathscr{G}^\omega_{a,b} \cap \Gamma = \{x_0\}$.

Corollary 1.24 (Hopf-Oleinik lemma for (3))**.** *Let Ω be a domain satisfying the interior ω-pseudo-ball condition, with ω that satisfies assumptions of Lemma 1.20.*

Let $p \in (1,\infty)$ and u be the viscosity solution of (3)-(5)*. Assume that there exist $x_0 \in \Gamma$ and $\bar{t} > 0$, such that*

$$u(x_0, \bar{t}) > u \quad \text{in} \quad \mathscr{G}^\omega_{a,b} \times \{\bar{t}\}.$$

If it is assumed that u is differentiable at x_0, then $\nabla u(x_0, \bar{t}) \neq 0$.

Proof. We apply first Lemma 1.23 and then Lemma 1.20 to $y \mapsto u(y, \bar{t})$. □

Corollary 1.25 (Hopf-Oleinik lemma for (6))**.** *Let Ω be a domain satisfying the interior ω-pseudo-ball condition, with ω that satisfies assumptions of Lemma 1.20.*

Let $p \in (1,\infty)$ and u^ε be the viscosity solution of (6)-(7)*. Assume that there exists $x_0 \in \Gamma$, such that*

$$u^\varepsilon(x_0) > u \quad \text{in} \quad \mathscr{G}^\omega_{a,b}.$$

If it is assumed that u^ε is differentiable at x_0, then $\nabla u^\varepsilon(x_0) \neq 0$.

Proof. Observe that, by setting $w \equiv 0$ on $\overline{\Omega}$, then w is a solution of (6) and $w < u^\varepsilon$ on Γ. Hence, by applying Corollary 1.14 we have that $u^\varepsilon \geq 0$ on $\overline{\Omega}$. This implies that u^ε is a viscosity subsolution of (1.13). We then conclude by applying Lemma 1.20. □

Chapter 2

Asymptotics for explicit solutions

In this chapter, we consider equations (3) and (6) and we deduce asymptotic formulas for global solutions and for solutions of certain boundary-value problems in symmetric domains, such as the half-space, the ball and the exterior of a ball.

The obtained solutions will later be used as barriers to extend the relevant asymptotic formulas to more general domains.

In the cases examined in this chapter, the (viscosity) solutions can be explicitly computed by taking advantage of the fact that they are smooth and that the relevant equations become linear. The corresponding boundary-value problems become considerably simpler, since they concern functions that depend on only one space variable.

Most of the explicit representations are based on Bessel functions, whose relevant properties are recalled in Section 2.1. We present the corresponding theorems in Sections 2.2, 2.3 and 2.4. Section 2.5 is then devoted to the asymptotic analysis of the obtained solutions as the relevant parameters tend to zero.

2.1 Formulas for Bessel functions

For an overview of this subject, we refer to [1, Chapter 9]. Here, we briefly collect the properties of our interest.

Given $\alpha \in \mathbb{C}$, the *Bessel's equation of order* α, is the following ordinary differential equation:

$$\sigma^2 y'' + \sigma y' + (\sigma^2 - \alpha^2) y = 0 \quad \text{for} \quad \sigma > 0. \tag{2.1}$$

Every solution of (2.1) can be written as

$$A\, J_\alpha(\sigma) + B\, Y_\alpha(\sigma),$$

where J_α is called a *Bessel function of first kind*, Y_α, is called a *Bessel function of second kind*, and A, B are constants. We know that J_α is finite at $\sigma = 0$, while Y_α is singular at $\sigma = 0$.

The following result can be found in [30].

Lemma 2.1 ([30, Theorem 2.2])**.** *Let $p \in (1, \infty)$. The eigenfunctions of the problem*

$$\begin{aligned} &v''(\sigma) + \frac{N-1}{p-1}\frac{v'(\sigma)}{\sigma} + \frac{p}{p-1}\lambda_n v(\sigma) = 0 \quad \textit{in} \quad (0,1), \\ &v'(0) = v(1) = 0, \end{aligned}$$

are given by

$$v_n(\sigma) = \sigma^{-\frac{N-p}{2(p-1)}} J_{\frac{N-p}{2(p-1)}}(\kappa_n \sigma), \tag{2.2}$$

where, κ_n is the n-th zero of $J_{\frac{N-p}{2(p-1)}}$ and $\lambda_n = \frac{\kappa_n^2}{p'}$, $n = 1, 2, \cdots$.

Moreover, after normalization, the set $\{v_n : n \in \mathbb{N}\}$ form a complete orthonormal system in the weighted space $L^2\left((0,1); \sigma^{\frac{N-1}{p-1}} d\sigma\right)$.

Remark 2.2. Observe that Lemma 2.1 can be extended to the case $p = \infty$. In this case, for $n = 1, 2, \cdots$, the eigenfunctions are given by

$$v_n(\sigma) = \sqrt{2}\cos(\lambda_n \sigma), \quad \lambda_n = \frac{(2n-1)\pi}{2}.$$

We will also use the *modified Bessel functions*, that can be defined by the formulas:

$$\begin{aligned} &I_\alpha(\sigma) = i^{-\alpha} J_\alpha(i\sigma), \quad \text{for} \quad \sigma \in \mathbb{R}, \\ &K_\alpha(\sigma) = \frac{\pi}{2\sin(\alpha\pi)}\left(I_\alpha(\sigma) - I_{-\alpha}(\sigma)\right), \quad \text{for} \quad \sigma \in \mathbb{R}, \end{aligned}$$

where here by J_α we mean the analytic extension to the complex plane. (Notice that the above definitions hold when α is not an integer; at integer points I_α and K_α are obtained as limits in the parameter α.)

We say that I_α is a *modified Bessel function of first kind* and K_α is a *modified Bessel function of second kind*. They are two linearly independent solutions of the *modified Bessel's equation* of order α:

$$\sigma^2 y'' + \sigma y' - (\sigma^2 + \alpha^2) y = 0. \tag{2.3}$$

We will use the following integral representations of I_α and K_α (see [1, formulas 9.6.18, 9.6.23]),

$$I_\alpha(\sigma) = \frac{(\sigma/2)^\alpha}{\sqrt{\pi}\,\Gamma\left(\frac{\alpha+1}{2}\right)} \int_0^\pi e^{\sigma\cos\theta} (\sin\theta)^{2\alpha}\, d\theta, \quad \text{for} \quad \mathrm{Re}(\alpha) > -\frac{1}{2}, \tag{2.4}$$

and

$$K_\alpha(\sigma) = \frac{\sqrt{\pi}(\sigma/2)^\alpha}{\Gamma\left(\frac{\alpha+1}{2}\right)} \int_0^\infty e^{-\sigma\cosh\theta} (\sinh\theta)^{2\alpha}\, d\theta, \qquad \text{for} \quad \mathrm{Re}(\alpha) > -\frac{1}{2}, \ |\arg(\sigma)| < \frac{\pi}{2}. \tag{2.5}$$

The next lemma will be used to obtain solutions related to the game-theoretic p-laplacian in radially symmetric domains and it can be seen as corollary of [1, Formula 9.1.52].

Lemma 2.3. *Let $p \in (1, \infty)$ and $\lambda > 0$. The functions*

$$\sigma^{\frac{p-N}{2(p-1)}}\, I_{\frac{N-p}{2(p-1)}}\left(\sqrt{\frac{p}{p-1}}\lambda\sigma\right) \quad \textit{and} \quad \sigma^{\frac{p-N}{2(p-1)}}\, K_{\frac{N-p}{2(p-1)}}\left(\sqrt{\frac{p}{p-1}}\lambda\sigma\right), \tag{2.6}$$

are linearly independent solutions of

$$y''(\sigma) + \frac{N-1}{p-1}\frac{y'(\sigma)}{\sigma} - \frac{p}{p-1}\lambda^2 y(\sigma) = 0 \quad \textit{for} \quad \sigma > 0. \tag{2.7}$$

Proof. The proof follows at once, by direct inspection. □

We conclude this section by deducing asymptotic formulas for the integrals involved in (2.4) and (2.5). The next lemma is essentially contained in [13].

Lemma 2.4 (Asymptotics for the modified Bessel functions)**.** *For $\alpha > -1$ and $\sigma > 0$, let*

$$g(\sigma) = \int_0^{\pi} e^{-\sigma(1-\cos\theta)}(\sin\theta)^{\alpha}\, d\theta,$$
$$f(\sigma) = \int_0^{\infty} e^{-\sigma(\cosh\theta-1)}(\sinh\theta)^{\alpha}\, d\theta.$$

Then, we have that

$$g(\sigma) = 2^{\frac{\alpha-1}{2}}\Gamma\left(\frac{\alpha+1}{2}\right)\sigma^{-\frac{\alpha+1}{2}}\left\{1 + O(1/\sigma)\right\} \tag{2.8}$$

$$f(\sigma) = 2^{\frac{\alpha-1}{2}}\Gamma\left(\frac{\alpha+1}{2}\right)\sigma^{-\frac{\alpha+1}{2}}\left\{1 + O(1/\sigma)\right\} \tag{2.9}$$

as $\sigma \to \infty$ and

$$f(\sigma) = \begin{cases} \sigma^{-\alpha}\,\Gamma(\alpha)\left\{1 + o(1)\right\} & \textit{if} \ \ \alpha > 0, \\ \log(1/\sigma) + O(1) & \textit{if} \ \ \alpha = 0, \\ \frac{\sqrt{\pi}}{2\,\sin(\alpha\pi/2)}\,\frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)} + o(1) & \textit{if} \ \ -1 < \alpha < 0, \end{cases} \tag{2.10}$$

as $\sigma \to 0^+$.

Proof. To start with, we establish formula (2.8). By the change of variable $\tau = \sigma\,(1 - \cos\theta)$ we get:

$$g(\sigma) = 2^{\frac{\alpha-1}{2}}\sigma^{-\frac{\alpha+1}{2}}\int_0^{\infty} e^{-\tau}\left(\tau - \frac{\tau^2}{2\sigma}\right)^{\frac{\alpha-1}{2}} d\tau.$$

This formula gives (2.8), once we observe that the integral $\int_0^{\infty} e^{-\tau}\tau^{\frac{\alpha+1}{2}}$ converges.

To deduce (2.9), we use the change of variable $\tau = \sigma(\cosh\theta - 1)$, obtaining

$$f(\sigma) = \frac{1}{\sigma}\int_0^\infty e^{-\tau}\left(\frac{2\tau}{\sigma} + \frac{\tau^2}{\sigma^2}\right)^{\frac{\alpha-1}{2}} d\tau.$$

When $\sigma \to \infty$, our claim follows, as above, by writing

$$f(\sigma) = 2^{\frac{\alpha-1}{2}}\sigma^{-\frac{\alpha+1}{2}}\int_0^\infty e^{-\tau}\left(\tau + \frac{\tau^2}{2\sigma}\right)^{\frac{\alpha-1}{2}} d\tau.$$

When $\sigma \to 0^+$ and $\alpha > 0$, our claim follows by writing

$$f(\sigma) = \sigma^{-\alpha}\int_0^\infty e^{-\tau}\left(\tau^2 + 2\sigma\,\tau\right)^{\frac{\alpha-1}{2}} d\tau,$$

since the integral $\int_0^\infty e^{-\tau}\tau^{\alpha-1}\,d\tau$ converges, from $\alpha > 0$.

For $-1 < \alpha \le 0$, we use [1, Formula 9.6.23], to infer that

$$f(\sigma) = \frac{1}{\sqrt{\pi}}\,\Gamma\left(\frac{\alpha+1}{2}\right)\left(\frac{\sigma}{2}\right)^{-\frac{\alpha}{2}} e^{\sigma} K_{\alpha/2}(\sigma).$$

Then, [1, Formulas 9.6.9 and 9.6.13] give our claims for $\alpha = 0$ and $-1 < \alpha < 0$, respectively. □

2.2 The global solutions

In this section, we present the global solutions of the parabolic equation (3) and of the resolvent equation (6). The former can be found in [7, Proposition 2.1] . We recall that $p' = p/(p-1)$ and we mean $p' = 1$ for $p = \infty$.

Proposition 2.5. *Let $p \in (1, \infty]$ and Φ be the function defined as*

$$\Phi(x,t) = \begin{cases} t^{-\frac{N+p-2}{2(p-1)}}\, e^{-p'\frac{|x|^2}{4t}} & \text{if } p \in (1,\infty), \\ t^{-\frac{1}{2}} e^{-\frac{|x|^2}{4t}} & \text{if } p = \infty, \end{cases} \tag{2.11}$$

for $x \in \mathbb{R}^N$ and $t > 0$.

Then, Φ is a viscosity solution of (3) *in $\mathbb{R}^N \times (0,\infty)$ and is bounded on $\left(\mathbb{R}^N \setminus B_\delta\right) \times (0,\infty)$, for $\delta > 0$.*

Proof. Observe that $\Phi \in C^\infty\left(\mathbb{R}^N \times (0,\infty)\right)$ and that $\Phi(x,t) = \phi(|x|,t)$, where $\phi = \phi(r,t)$ is clearly defined by (2.11). Equation (3) for a radially symmetric function u reads as

$$u_t = \frac{p-1}{p}u_{rr} + \frac{N-1}{p}\frac{u_r}{r}.$$

By simple computations, we verify that this equation is satisfied (pointwise) by $\phi(r,t)$ for $r \neq 0$ and $t > 0$.

Thus, Φ is a classical solution in $\left(\mathbb{R}^N \setminus \{0\}\right) \times (0,\infty)$, where its spatial gradient does not vanish. Then, we apply Lemma 1.2 to conclude.

The stated boundness of Φ easily follows by its definition. □

Proposition 2.6. *Let $p \in (1,\infty]$ and Φ_ε be the function defined, for $x \in \mathbb{R}^N$, by*

$$\Phi_\varepsilon(x) = \begin{cases} \displaystyle\int_0^\infty e^{-\sqrt{p'}\frac{|x|}{\varepsilon}\cosh\theta} (\sinh\theta)^{\frac{N-p}{p-1}}\, d\theta & \text{if } p \in (1,\infty), \\ e^{-\frac{|x|}{\varepsilon}} & \text{if } p = \infty. \end{cases}$$

Then Φ_ε is a viscosity solution of (6) *in $\mathbb{R}^N \setminus \{0\}$. Moreover, in the case $p \in (N,\infty]$, Φ_ε is bounded in $\mathbb{R}^N$ while, in the case $p \in (1,N]$, it is bounded on the complement of any ball centered at* 0*.*

Proof. The case $p = \infty$ is immediate. In the case $p \in (1,\infty)$, from (2.5), we have that $\Phi_\varepsilon = \phi(|x|)$, where $\phi(r)$ is a solution of (2.7), for $r > 0$. A direct check shows that (2.7) is simply the equation (6) for radial functions. Hence, Φ_ε is a solution of (6), outside the origin. The stated boundedness follows from the properties of the function $K_{\frac{N-p}{2(p-1)}}$. □

2.3 Elliptic solutions in symmetric domains

Lemma 2.7 (Elliptic solution in the ball, [14, Lemma 2.1])**.** *Set $p \in (1,\infty]$.*

Then, for $x \in \overline{B}_R$, the following function,

$$u^\varepsilon(x) = \begin{cases} \dfrac{\int_0^\pi e^{\sqrt{p'}\frac{|x|}{\varepsilon}\cos\theta} (\sin\theta)^{\frac{N-p}{p-1}}\, d\theta}{\int_0^\pi e^{\sqrt{p'}\frac{R}{\varepsilon}\cos\theta} (\sin\theta)^{\frac{N-p}{p-1}}\, d\theta} & \text{if } p \in (1,\infty), \\ \dfrac{\cosh(|x|/\varepsilon)}{\cosh(R/\varepsilon)} & \text{if } p = \infty, \end{cases} \tag{2.12}$$

is the (viscosity) solution of (6)-(7)*.*

Proof. We only consider the case $p \in (1,\infty)$, while the extremal case $p = \infty$ is similar and simpler. We have that $u^\varepsilon(x) = \phi(|x|)$, where $\phi(r) = r^{\frac{p-N}{2(p-1)}} I_{\frac{N-p}{2(p-1)}}(r)$. For Lemma 2.3, $\phi(r)$ is a solution of (2.7), which is the equation (6) for radial functions. Hence, u^ε is a classical solution of (6), away from the origin. Since u^ε is of class C^2 in B_R, then Lemma 1.2 informs us that it is also a viscosity solution in the whole ball. Finally, it is clear that $u^\varepsilon = 1$ on the boundary, then it is the solution of (6)-(7), by uniqueness (Corollary 1.15). □

Lemma 2.8 (Elliptic solution in the exterior of the ball, [13, Lemma 2.2])**.** *Set $p \in (1,\infty]$.*

Then, the following function, defined for $x \in \mathbb{R}^N \setminus B_R$, by

$$u^\varepsilon(x) = \begin{cases} \dfrac{\int_0^\infty e^{-\sqrt{p'}\frac{|x|}{\varepsilon}\cosh\theta}(\sinh\theta)^{\frac{N-p}{p-1}}\,d\theta}{\int_0^\infty e^{-\sqrt{p'}\frac{R}{\varepsilon}\cosh\theta}(\sinh\theta)^{\frac{N-p}{p-1}}\,d\theta} & \text{if } p \in (1,\infty), \\ e^{-\frac{|x|-R}{\varepsilon}} & \text{if } p = \infty, \end{cases} \tag{2.13}$$

is the bounded (viscosity) solution of (6)-(7).

Proof. We just need to observe that u^ε is just Φ_ε of Proposition 2.6, once it is normalized to the value 1 on the boundary of B_R. Hence, u^ε is a bounded solution of (6)-(7). By uniqueness (Corollary 1.15), we conclude. □

2.4 Parabolic solutions in symmetric domains

Before going on, we point out that, from now on, with Γ we indicate the boundary of the relevant set, which it will be evident by the context.

First, we focus our attention on the case of the half-space of $\mathbb{R}^N$. We will use the *complementary error function* defined by

$$\mathrm{Erfc}(\sigma) = \frac{2}{\sqrt{\pi}} \int_\sigma^\infty e^{-\tau^2} d\tau, \quad \sigma \in \mathbb{R}.$$

Proposition 2.9 (Parabolic solution in the half-space, [14, Proposition 2.3])**.** *Let $p \in (1,\infty]$ and H be the half-space in which $x_1 > 0$. The function Ψ, defined by*

$$\Psi(x,t) = \sqrt{\frac{p'}{4\pi}} \int_{\frac{x_1}{\sqrt{t}}}^\infty e^{-\frac{1}{4}p'\sigma^2}\,d\sigma = \mathrm{Erfc}\left(\frac{\sqrt{p'}x_1}{2\sqrt{t}}\right) \quad \text{for} \quad (x,t) \in \overline{H} \times (0,\infty).$$

is the bounded solution of (3)-(5).

Proof. In virtue of Corollary 1.13, the bounded solution of (3)-(5) is unique. The function Ψ is smooth with no critical points. To conclude, it is enough to note that, after an inspection, it satisfies pointwise (3) in $H \times (0,\infty)$ and both (4) and (5). □

Now, we establish a series representation of parabolic solutions in the ball.

Lemma 2.10 (Parabolic solution in the ball)**.** *Let $p \in (1,\infty]$.*

Then, the solution of (3)-(5) *has the following representation:*

(i) if $p \in (1,\infty)$, we have that, for $x \in B_R$ and $t > 0$,

$$u(x,t) = 2\left(\frac{R}{|x|}\right)^{\frac{N-p}{2(p-1)}} \sum_{n=1}^\infty \frac{J_{\frac{N-p}{2(p-1)}}\left(\frac{\kappa_n}{R}|x|\right)}{\kappa_n\, J_{\frac{N+p-2}{2(p-1)}}(\kappa_n)} \left(1 - e^{-\frac{\kappa_n^2}{p' R^2}t}\right). \tag{2.14}$$

where κ_n is the n-th positive zeros of $J_{\frac{N-p}{2(p-1)}}$.

(ii) If $p = \infty$, we get, for $x \in B_R$ and $t > 0$:

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos\left((2n-1)\frac{\pi|x|}{2R}\right)}{2n-1}\left(1 - e^{-\frac{(2n-1)^2\pi^2}{4R^2}\,t}\right). \tag{2.15}$$

Proof. Preliminarily, we observe that, since (3) is invariant if we add a constant to u, $1-u$ is also a solution of (3). Moreover, $1-u$ vanishes on $\Gamma \times (0,\infty)$ and is equal to 1 on $B_R \times \{0\}$. We want to find the series representation of $1-u$.

We use the same scheme, in all cases $p \in (1,\infty]$, clarifying that, in the extremal case $p = \infty$, we use Remark 2.2 instead of Lemma 2.1. Now, set $\alpha = \frac{N-p}{2(p-1)}$. By applying Lemma 2.1, we infer that $\{\sigma^{-\alpha} J_\alpha\left(\frac{\kappa_n}{R}\sigma\right) : n = 1, 2, \cdots\}$ is a complete system in $L^2\left((0,R); \sigma^{2\alpha+1}\, d\sigma\right)$. This fact implies that

$$\sum_{n=1}^{\infty} c_n \left\{\sigma^{-\alpha} J_\alpha\left(\frac{\kappa_n}{R}\sigma\right)\right\} = 1 \quad \text{for} \quad \sigma \in (0,R), \tag{2.16}$$

for

$$c_n = \frac{\int_0^R \sigma^{\alpha+1} J_\alpha\left(\frac{\kappa_n}{R}\sigma\right)\, d\sigma}{\int_0^R \sigma J_\alpha\left(\frac{\kappa_n}{R}\sigma\right)^2\, d\sigma} = \frac{\int_0^R \sigma^{\alpha+1} J_\alpha\left(\frac{\kappa_n}{R}\sigma\right)\, d\sigma}{R^2 J_{\alpha+1}(\kappa_n)^2/2},$$

where, in the last equality, we have used [1, formulas 11.4.5 and 9.5.4]. Now, since, by formula [1, formula 11.3.20] we know that

$$\int_0^1 \sigma^{\alpha+1} J_\alpha\left(\kappa_n\sigma\right)\, d\sigma = \frac{J_{\alpha+1}(\kappa_n)}{\kappa_n},$$

we obtain

$$c_n = \frac{2R^\alpha}{\kappa_n J_{\alpha+1}(\kappa_n)}.$$

We see that, for any $n = 1, 2, \cdots$, by reason of (2.2), the function

$$u_n(x,t) = |x|^{-\alpha} J_\alpha\left(\frac{\kappa_n}{R}|x|\right) e^{-\frac{\kappa_n^2}{p'\,R^2}t},$$

is a solution of (3) in $B_R \times (0,\infty)$ and it vanishes on $\Gamma \times (0,\infty)$.

Thus, by uniqueness (Corollary 1.13, we have that

$$1 - u(x,t) = \sum_{n=1}^{\infty} \frac{2R^\alpha}{\kappa_n J_{\alpha+1}\left(\kappa_n\right)} |x|^{-\alpha} J_\alpha\left(\frac{\kappa_n}{R}|x|\right) e^{-\frac{\kappa_n^2}{p'\,R^2}t},$$

which, by using (2.16), gives (2.14).

Finally, we observe that, thanks to Remark 2.2, in the case $p = \infty$, (2.14) is just (2.15). $\square$

Last, we report the following connection between elliptic and parabolic radial (viscosity) solutions, as stated in [14, Lemma 2.5].

Lemma 2.11 (Laplace transform and symmetric solutions)**.** *Let $p \in (1, \infty]$. Let $u(x,t)$ be given by Lemma 2.10. Then, it holds*

$$u^\varepsilon(x) = \varepsilon^{-2} \int_0^\infty u(x,\tau) e^{-\tau/\varepsilon^2}\, d\tau \quad \textit{for} \quad x \in \overline{B_R}, \tag{2.17}$$

where u^ε is the solution of (6)-(7)*, in B_R.*

Proof. We prove that both sides of (2.17) have the same eigenfunction expansion. Here, we treat the case $p \in (1, \infty)$. In the extremal case $p = \infty$ we have only to utilize the expressions of eigenfunctions in Remark 2.2.

Let κ_n be given by Lemma 2.1, then, for any $\varepsilon > 0$, it holds that

$$\varepsilon^{-2} \int_0^\infty \left(1 - e^{\frac{\kappa_n^2 \tau}{p' R^2}}\right) e^{-\tau/\varepsilon^2}\, d\tau = \frac{\varepsilon^2 \kappa_n^2/(p'R^2)}{\varepsilon^2 \kappa_n^2/(p'R^2) + 1} = \frac{\varepsilon^2 \kappa_n^2}{\varepsilon^2 \kappa_n^2 + p'R^2}.$$

Hence, from (2.14), we obtain that the right-hand side of (2.17) equals

$$2\,(R/|x|)^{\frac{N-p}{2(p-1)}}\, R^{-2} \sum_{n=1}^\infty \frac{\kappa_n}{J_{\frac{N+p-2}{2(p-1)}}(\kappa_n)} \left\{ \frac{p'}{\varepsilon^2} + \frac{\kappa_n^2}{R^2} \right\}^{-1} J_{\frac{N-p}{2(p-1)}}\left(\frac{\kappa_n}{R}|x|\right), \tag{2.18}$$

for any $x \in B_R$ and $\varepsilon > 0$.

Now, we observe that the sum of the last series is the function u^ε. Indeed, by comparing (2.12) and (2.4), we have that u^ε is given by

$$u^\varepsilon(x) = (R/|x|)^{\frac{N-p}{2(p-1)}}\, I_{\frac{N-p}{2(p-1)}}\left(\frac{\sqrt{p'}}{\varepsilon}|x|\right) / I_{\frac{N-p}{2(p-1)}}\left(\frac{\sqrt{p'}}{\varepsilon}R\right)$$

and then the relevant coefficients can be calculated by applying [1, Formula 11.3.29], that is

$$\int_0^R \sigma I_{\frac{N-p}{2(p-1)}}\left(\frac{\sqrt{p'}\sigma}{\varepsilon}\right) J_{\frac{N-p}{2(p-1)}}\left(\frac{\kappa_n}{R}\sigma\right) d\sigma = \kappa_n I_{\frac{N-p}{2(p-1)}}\left(\frac{\sqrt{p'}R}{\varepsilon}\right) J_{\frac{N+p-2}{2(p-1)}}(\kappa_n) \left\{ \frac{p'}{\varepsilon^2} + \frac{\kappa_n^2}{R^2} \right\}^{-1},$$

which gives (2.18). □

2.5 Asymptotics

In this section, we collect asymptotic formulas for the functions presented in Sections 2.3 and 2.4.

2.5.1 The elliptic case

Theorem 2.12 (Asymptotics in the ball, [13, Lemma 2.1])**.** *Let $p \in (1, \infty]$. Assume that u^ε be the (viscosity) solution of* (6)-(7) *in B_R.*

Then, it holds that

$$\varepsilon \log u^\varepsilon + \sqrt{p'} d_\Gamma = \begin{cases} O(\varepsilon \, \log \varepsilon) & \text{if } \ 1 < p < \infty, \\ O(\varepsilon) & \text{if } \ p = \infty, \end{cases} \tag{2.19}$$

uniformly on $\overline{B_R}$ as $\varepsilon \to 0^+$.

Proof. First, we observe that $d_\Gamma(x) = R - |x|$. The case $p = \infty$ follows at once, since (2.12) gives:

$$\varepsilon \log \{u^\varepsilon(x)\} + d_\Gamma(x) = \varepsilon \log \left[\frac{1 + e^{-2\frac{|x|}{\varepsilon}}}{1 + e^{-2\frac{R}{\varepsilon}}} \right].$$

If $1 < p < \infty$, by (2.12) we have that

$$\varepsilon \log \{u^\varepsilon(x)\} + \sqrt{p'} \, d_\Gamma(x) = \varepsilon \, \log \left[\frac{\int_0^\pi e^{-\sqrt{p'}(1-\cos\theta) \, \frac{|x|}{\varepsilon}} (\sin\theta)^{\frac{N-p}{p-1}} \, d\theta}{\int_0^\pi e^{-\sqrt{p'}(1-\cos\theta) \, \frac{R}{\varepsilon}} (\sin\theta)^{\frac{N-p}{p-1}} \, d\theta} \right]$$

and the right-hand side is decreasing in $|x|$, so that

$$0 \le \varepsilon \, \log \{u^\varepsilon(x)\} + \sqrt{p'} \, d_\Gamma(x) \le \varepsilon \, \log \left[\frac{\int_0^\pi (\sin\theta)^{\frac{N-p}{p-1}} \, d\theta}{\int_0^\pi e^{-\sqrt{p'} \, (1-\cos\theta) \, \frac{R}{\varepsilon}} (\sin\theta)^{\frac{N-p}{p-1}} \, d\theta} \right].$$

This formula gives (2.19), since we have that

$$\int_0^\pi e^{-\sqrt{p'} \, (1-\cos\theta) \frac{R}{\varepsilon}} (\sin\theta)^{\frac{N-p}{p-1}} \, d\theta = 2^{\frac{N-2p+1}{2(p-1)}} \Gamma \left(\frac{N-1}{2p-2} \right) \left(\frac{R\sqrt{p'}}{\varepsilon} \right)^{-\frac{N-1}{2p-2}} \quad [1 + O(\varepsilon)]$$

as $\varepsilon \to 0^+$, by using (2.8), with $\sigma = \sqrt{p'} R / \varepsilon$. ☐

The next theorem is contained in [13, Lemma 2.2].

Theorem 2.13 (Asymptotics in the exterior of the ball)**.** *Let $p \in (1, \infty]$. Assume that u^ε be the (viscosity) solution of* (6)-(7) *in $\mathbb{R}^N \setminus \overline{B_R}$.*

Then, it holds that

$$\varepsilon \log u^\varepsilon + \sqrt{p'} \, d_\Gamma = O(\varepsilon) \quad as \quad \varepsilon \to 0^+, \tag{2.20}$$

uniformly on every compact subset of $\mathbb{R}^N \setminus B_R$.

Proof. Notice that $d_\Gamma(x) = |x| - R$. If $p = \infty$, (2.20) holds exactly as

$$\varepsilon \log \{u^\varepsilon(x)\} + d_\Gamma(x) \equiv 0.$$

If $1 < p < \infty$, we write that

$$\varepsilon \log \{u^\varepsilon(x)\} + \sqrt{p'}\, d_\Gamma(x) = \varepsilon \log \left[\frac{\int_0^\infty e^{-\sqrt{p'}\frac{|x|}{\varepsilon}(\cosh\theta - 1)} (\sinh\theta)^{\frac{N-p}{p-1}}\, d\theta}{\int_0^\infty e^{-\sqrt{p'}\frac{R}{\varepsilon}(\cosh\theta - 1)} (\sinh\theta)^{\frac{N-p}{p-1}}\, d\theta} \right]$$

and hence, by monotonicity, we have that

$$\varepsilon \ \log \left\{ \frac{\int_0^\infty e^{-\sqrt{p'}\frac{R'}{\varepsilon}(\cosh\theta - 1)} (\sinh\theta)^{\frac{N-p}{p-1}}\, d\theta}{\int_0^\infty e^{-\sqrt{p'}\frac{R}{\varepsilon}(\cosh\theta - 1)} (\sinh\theta)^{\frac{N-p}{p-1}}\, d\theta} \right\} \le \varepsilon \log \{u^\varepsilon(x)\} + \sqrt{p'}\, d_\Gamma(x) \le 0,$$

for every x such that $R \le |x| \le R'$, with $R' > R$. Our claim then follows by an inspection on the left-hand side, after applying (2.9), with $\sigma = \sqrt{p'}R'/\varepsilon$ and $\sigma = \sqrt{p'}R/\varepsilon$. □

2.5.2 The parabolic case

We state the following theorem, contained essentially in [14, Proposition 2.3].

Theorem 2.14 (Asymptotics in the half-space)**.** *Let $p \in (1, \infty]$ and let H be the half-space in which $x_1 > 0$. Let Ψ be given by*

$$\Psi(x, t) = \sqrt{\frac{p'}{4\pi}} \int_{\frac{x_1}{\sqrt{t}}}^\infty e^{-\frac{1}{4}p'\sigma^2}\, d\sigma,$$

for $x \in H$ and $t > 0$.

Then, it holds that

$$4t \log \{\Psi(x, t)\} = -p'\, d_\Gamma(x)^2 + O(t \log t),$$

uniformly for x in every strip $\{x \in \mathbb{R}^N : 0 \le x_1 \le \delta\}$ with $\delta > 0$.

Proof. Observe that $d_\Gamma(x) = x_1$. By employing a change of variables in the expression of Ψ, we get that

$$\Psi(x, t) = e^{-\frac{p' x_1^2}{4t}} \sqrt{\frac{p'}{4\pi}} \int_0^\infty e^{-\frac{1}{2}p'\frac{x_1}{\sqrt{t}}\sigma - \frac{1}{4}p'\sigma^2}\, d\sigma,$$

and hence

$$4t \log \Psi(x, t) + p' x_1^2 = 4t \ \log \left(\sqrt{\frac{p'}{4\pi}} \int_0^\infty e^{-\frac{1}{2}p'\frac{x_1}{\sqrt{t}}\sigma - \frac{1}{4}p'\theta^2}\, d\sigma \right).$$

Thus, for $0 \le x_1 \le \delta$, we have that

$$4t\,\log\left(\sqrt{\frac{p'}{4\pi}}\int_0^\infty e^{-\frac{p'\delta}{2\sqrt{t}}\sigma-\frac{p'\sigma^2}{4}}\,d\sigma\right) \le 4t\log\Psi(x,t)+p'x_1^2 \le 0. \tag{2.21}$$

Integrating by parts on the two functions

$$e^{-\frac{p'\delta}{2\sqrt{t}}\sigma} \quad\text{and}\quad e^{-\frac{p'\sigma^2}{4}}$$

and a change of variable give that

$$\int_0^\infty e^{-\frac{p'\delta}{2\sqrt{t}}\sigma-\frac{p'\sigma^2}{4}}\,d\sigma = \frac{2}{p'\delta}\sqrt{t}\,[1+O(t)]$$

as $t \to 0^+$ (see also [1, Formula 7.1.23]). Then (2.21) gives the desired uniform convergence.

$\square$

In the case of the ball, the series representation of $u(x,t)$, established in Lemma 2.10, is not convenient to obtain an asymptotic formula for $t \to 0$. We then proceed differently. To start with, we state the next lemma.

Lemma 2.15. *Let $p \in (1,\infty]$. Assume that u is the solution of (3)-(5).*
If $p \in (1,\infty)$, then, for every $x \in \overline{B_R}$, $t > 0$ and $\lambda > 0$, it holds that

$$4t\log u(x,t) \le \frac{4}{\lambda^2} - \frac{4}{\lambda}\sqrt{p'}d_\Gamma(x) + 4t\log\left[\frac{\int_0^\pi (\sin\theta)^{\frac{N-p}{p-1}}\,d\theta}{\int_0^\pi e^{-\frac{\sqrt{p'}}{\lambda t}(1-\cos\theta)d_\Gamma(x)}\,(\sin\theta)^{\frac{N-p}{p-1}}\,d\theta}\right]. \tag{2.22}$$

If $p = \infty$, for every $x \in \overline{B_R}$, $t > 0$ and $\lambda > 0$, we have that

$$4t\log u(x,t) \le \frac{4}{\lambda^2} - \frac{4}{\lambda}d_\Gamma(x) + 4t\log\left[\frac{2}{1+e^{-2\frac{d_\Gamma(x)}{\lambda t}}}\right]. \tag{2.23}$$

Proof. Preliminarly, observe that if $x \in \Gamma$, (2.22) and (2.23) are obviously satisfied.

For a fixed $x \in B_R$, we argue as follows. By Lemma 2.11, for every $\varepsilon > 0$ the function u^ε defined in (2.17) is the solution of (6)-(7) in B_R. Moreover, from (1.15) in the proof of Lemma 1.23, we have that $t \mapsto u(x,t)$ is increasing.

Thus, it holds that

$$\varepsilon^2 u(x,t)\,e^{-t/\varepsilon^2} \le \int_t^\infty u(x,\tau)\,e^{-\tau/\varepsilon^2}\,d\tau \le \int_0^\infty u(x,\tau)e^{-\tau/\varepsilon^2}\,d\tau = \varepsilon^2 u^\varepsilon(x),$$

and hence

$$u(x,t) \le u^\varepsilon(x)\,e^{t/\varepsilon^2};$$

the last inequality holds for any $t, \varepsilon > 0$. Next, we choose $\varepsilon = \lambda\, t$ and obtain that

$$u(x,t) \leq u^{\lambda t}(x)\, e^{1/(\lambda^2 t)} \quad \text{for any} \quad t > 0.$$

Therefore,

$$4t\, \log u(x,t) \leq 4t\, \log u^{\lambda t}(x) + \frac{4}{\lambda^2}.$$

Now, let $B^x = B_{d_\Gamma(x)}(x)$, be the ball centered at x, with radius $d_\Gamma(x)$. Let $u^\varepsilon_{B^x}$ be the solution of (6)-(7) in B^x. By the comparison principle (Corollary 1.14), we have that, for any $t, \lambda > 0$ and $x \in B_R$,

$$u^{\lambda t} \leq u^{\lambda t}_{B^x} \quad \text{on} \quad \overline{B^x}$$

and, in particular,

$$u^{\lambda t}(x) \leq u^{\lambda t}_{B^x}(x).$$

By using this fact and the previous inequality, we obtain that, for any $x \in B_R$, $t > 0$ and $\lambda > 0$,

$$4t\, \log u(x,t) \leq 4t\, \log u^{\lambda t}_{B^x}(x) + \frac{4}{\lambda^2},$$

which, together with (2.12), gives (2.22) and (2.23). Indeed, we observe that, by translating, $u^{\lambda t}_{B^x}(x)$ is simply the solution of (6)-(7) in $B_{d_\Gamma(x)}$, with $\varepsilon = \lambda t$, evaluated at the origin. □

We are ready to obtain the uniform asymptotics for $u(x,t)$ in the ball. The pointwise formula is given in [14, Theorem 2.6].

Theorem 2.16 (Asymptotics in the ball)**.** *Set $p \in (1, \infty]$ and let $u(x,t)$ be the viscosity solution of* (3)-(5)*.*

Then, it holds that

$$4t \log u(x,t) + p'\, d_\Gamma(x)^2 = O\left(t \log t\right), \tag{2.24}$$

as $t \to 0^+$, uniformly on $\overline{B_R}$.

Proof. Given $x \in B_R$ there exists $y \in \Gamma$ such that $|x - y| = d_\Gamma(x)$ (y is unique unless $x = 0$). Let H be the half-space containing B_R and such that $\partial H \cap \Gamma = \{y\}$; notice that $d_\Gamma(x) = d_{\partial H}(x)$.

Let Ψ^y be the solution of (3)-(5) in $H \times (0, \infty)$; since B_R is contained in H, Ψ^y obviously satisfies (3) and (4) for B_R and, also, $\Psi^y \leq 1$ on $\Gamma \times (0, \infty)$. By the comparison principle (Corollary 1.12), we get that $u \geq \Psi^y$ and hence

$$4t \log u(x,t) \geq 4t \log \Psi^y(x,t) \quad \text{for} \quad (x,t) \in \overline{B_R} \times (0, \infty).$$

Thus, Theorem 2.14 with $\delta = 2R$, implies that

$$4t \log u(x,t) \geq -p'\, d_{\partial H}(x)^2 + O(t \log t), \tag{2.25}$$

uniformly on $\overline{B_R}$, as $t \to 0^+$, which is (2.24) by one side, since $d_{\partial H}(x) = d_\Gamma(x)$.

On the other hand, by applying Lemma 2.15 with $\lambda = \lambda^* > 0$ such that

$$-\frac{4\sqrt{p'}d_\Gamma(x)}{\lambda^*} + \frac{4}{(\lambda^*)^2} = -p'\, d_\Gamma(x)^2, \quad \text{that is} \quad \lambda^* = \frac{2}{\sqrt{p'}\, d_\Gamma(x)},$$

we obtain, if $p \in (1, \infty)$,

$$4\,t \log u(x,t) \le -p'\, d_\Gamma(x)^2 + 4\,t \log \left[\frac{\int_0^\pi (\sin\theta)^{\frac{N-p}{p-1}}\, d\theta}{\int_0^\pi e^{-\frac{p'}{2t}(1-\cos\theta) d_\Gamma(x)^2} (\sin\theta)^{\frac{N-p}{p-1}}\, d\theta} \right],$$

and the obvious corresponding inequality, in the case $p = \infty$.

Hence, by applying (2.8) with $\sigma = p' d_\Gamma(x)^2/(2\,t)$, we conclude that

$$4\,t \log u(x,t) \le -p'\, d_\Gamma(x)^2 + O\left(t \log t\right), \tag{2.26}$$

uniformly on $\overline{B_R}$, as $t \to 0^+$. Putting together (2.25) and (2.26), we conclude the proof. $\square$

Chapter 3

Varadhan-type formulas

In his paper [57], S.R.S. Varadhan considered the following problems for the heat and resolvent equations:

$$\begin{aligned} u_t - \tfrac{1}{2}\operatorname{tr}\left[A\,\nabla^2 u\right] = 0 &\quad \text{in } \ \Omega \times (0,\infty), \\ u = 1 &\quad \text{on } \ \Gamma \times (0,\infty), \\ u = 0 &\quad \text{on } \ \Omega \times \{0\}, \end{aligned}$$

and

$$\begin{aligned} \varepsilon^{-2} u^\varepsilon - \tfrac{1}{2}\operatorname{tr}\left[A\,\nabla^2 u^\varepsilon\right] = 0 &\quad \text{in } \ \Omega, \\ u^\varepsilon = 1 &\quad \text{on } \Gamma, \end{aligned}$$

where $A = A(x)$, for $x \in \Omega$, is a symmetric and positive definite $N \times N$ matrix.

When A is uniformly elliptic and uniformly Hölder continuous, Varadhan proved the following asymptotic formulas:

$$-\lim_{t\to 0^+} 2t \log u(x,t) = d_\Gamma^A(x)^2$$

and

$$-\lim_{\varepsilon\to 0^+} \varepsilon \log u^\varepsilon(x) = \sqrt{2}\, d_\Gamma^A(x),$$

where d_Γ^A is the distance from the boundary of Ω, induced by a Riemannian metric related to the matrix A. In particular, when A is the identity matrix, $d_\Gamma^A = d_\Gamma$ coincides with the usual euclidean distance, defined by

$$d_\Gamma(x) = \inf\{|x-y| : y \in \Gamma\}.$$

In this chapter, we establish asymptotic Varadhan-type formulas for the solutions of problems (3)-(5) and (6)-(7), in quite general domains. See Theorems 3.5 and 3.6.

In particular, we prove the pointwise formulas (9) and (10), that is

$$-\lim_{t\to 0^+} 4t \log u(x,t) = p'\, d_\Gamma(x)^2$$

and

$$-\lim_{\varepsilon\to 0^+} \varepsilon \log u^\varepsilon(x) = \sqrt{p'}\, d_\Gamma(x).$$

These formulas hold for a (not necessarily bounded) domain Ω which merely satisfies the topological assumption $\Gamma = \partial\left(\mathbb{R}^N \setminus \overline{\Omega}\right)$.

We also compute the uniform rate of convergence in (9) and (10). In fact, we show that

$$4t \log u(x,t) + p'\, d_\Gamma(x)^2 = O(t \log \psi(t)) \quad \text{as} \quad t \to 0^+,$$

and, for $\varepsilon \to 0$, that

$$\varepsilon \log u^\varepsilon(x) + \sqrt{p'}\, d_\Gamma(x) = \begin{cases} O(\varepsilon) & \text{if} \;\; p = \infty, \\ O(\varepsilon \log \varepsilon) & \text{if} \;\; p \in (N, \infty), \end{cases}$$

and

$$\varepsilon \log u^\varepsilon(x) + \sqrt{p'}\, d_\Gamma(x) = \begin{cases} O(\varepsilon \log |\log \psi(\varepsilon)|) & \text{if} \;\; p = N, \\ O(\varepsilon \log \psi(\varepsilon)) & \text{if} \;\; p \in (1, N), \end{cases}$$

uniformly on every compact subset of $\overline{\Omega}$. The function ψ depends on appropriate regularity assumptions on Γ. See Theorems 3.7 and 3.9.

The results presented in this chapter are based on the construction of suitable barriers, which essentially employ the radial solutions, deduced in Chapter 2. In Sections 3.1 and 3.2, we present such barriers. We stress the fact that no regularity assumption on Ω is needed.

Sections 3.3 and 3.4 are then dedicated to deduce the already mentioned pointwise and uniform asymptotic formulas.

3.1 Barriers in the parabolic case

The next two lemmas give global barriers from below and above for the solution of (3)-(5).

Lemma 3.1 (A pointwise barrier from below)**.** *Set $1 < p \le \infty$. Let $\Omega \subset \mathbb{R}^N$ be a domain and let $z \in \mathbb{R}^N \setminus \overline{\Omega}$. Assume that $u(x,t)$ is the bounded (viscosity) solution of* (3)-(5).

Then, for every $x \in \overline{\Omega}$ and $t > 0$, it holds that

$$4t \log\{u(x,t)\} + p'\, |x - z|^2 \ge 4t\, \log E^-\left(d_\Gamma(z), t\right). \tag{3.1}$$

Here, for $\sigma > 0$ and $t > 0$, we mean that

$$E^-\left(\sigma, t\right) = A_{N,p} t^{-\frac{N+p-2}{2(p-1)}} \sigma^{\frac{N+p-2}{p-1}}, \tag{3.2}$$

where

$$A_{N,p} = \left\{\frac{pe}{2(N+p-2)}\right\}^{\frac{N+p-2}{2(p-1)}}.$$

Proof. The function U^z, defined, for $x \in \overline{\Omega}$ and $t > 0$, by

$$U^z(x,t) = \left\{ A_{N,p} d_\Gamma(z)^{\frac{N+p-2}{p-1}} \right\} \Phi(x-z,t) \quad \text{for} \quad (x,t) \in \overline{\Omega} \times (0,\infty),$$

is a solution of (3) in $\Omega \times (0,\infty)$, since Φ is a global solution (Proposition 2.5). Moreover, $U^z(x,0^+) = 0$, for any $x \in \Omega$ and, in virtue of the fact that

$$\max\{\Phi(x-z,t) : x \in \Gamma,\, t > 0\} = \left(A_{N,p} d_\Gamma(z)^{\frac{N+p-2}{p-1}} \right)^{-1},$$

we also have that $U^z \leq 1$ on $\Gamma \times (0,\infty)$.

Hence, we just apply the comparison principle (Corollary 1.12), to conclude that $U^z \leq u$, on $\overline{\Omega} \times (0,\infty)$, which implies (3.1), by recalling the definition of Φ. $\square$

Lemma 3.2 (An uniform barrier from above)**.** *Set $1 < p \leq \infty$. Let $\Omega \subset \mathbb{R}^N$ be a domain. Let u be the bounded (viscosity) solution of* (3)-(5)*.*

Then, for every $x \in \overline{\Omega}$ and $t > 0$, it holds that

$$4t \log\{u(x,t)\} + p'\, d_\Gamma(x)^2 \leq 4t \log E^+ \left(d_\Gamma(x), t\right), \tag{3.3}$$

where $E^+(\sigma,t)$ is given by

$$E^+(\sigma,t) = \begin{cases} \dfrac{\int_0^\pi (\sin\theta)^{\frac{N-p}{p-1}}\, d\theta}{\int_0^\pi e^{-\frac{p'}{2t}(1-\cos\theta)\sigma^2} (\sin\theta)^{\frac{N-p}{p-1}}\, d\theta} & \text{if } p \in (1,\infty), \\[2ex] \dfrac{2}{1+e^{-\frac{p'}{t}\sigma^2}} & \text{if } p = \infty, \end{cases}$$

for $\sigma \geq 0$ and $t > 0$.

In particular, it holds that

$$t \log E^+ \left(d_\Gamma(x), t\right) = \begin{cases} O(t \log t) & \text{if } p \in (1,\infty), \\ O(t) & \text{if } p = \infty, \end{cases}$$

as $t \to 0^+$, uniformly on every subset of $\overline{\Omega}$ in which d_Γ is bounded.

Proof. Let u_B be the solution of (3)-(5) in the unit ball B. We prove that $u(x,t) \leq u_B(0, t/d_\Gamma(x)^2)$, for $(x,t) \in \overline{\Omega} \times (0,\infty)$, where we mean that $u_B(0,t/d_\Gamma(x)^2) = 1$ when $x \in \Gamma$.

Indeed, for $x \in \Gamma$, the inequality is satisfied as an equality. Let $x \in \Omega$ and let $v^x = v^x(y,t)$ be the solution of (3)-(5) in $B^x \times (0,\infty)$, where B^x is the ball centered at x with radius $d_\Gamma(x)$. Corollaries 1.16 and 1.12 give that

$$u(y,t) \leq v^x(y,t) \quad \text{for every} \quad (y,t) \in \overline{B^x} \times (0,\infty),$$

and hence, in particular, $u(x,t) \le v^x(x,t)$ for every $t > 0$. Since x is arbitrary in Ω, we infer that

$$u(x,t) \le v^x(x,t) \quad \text{for} \quad (x,t) \in \Omega \times (0,\infty). \tag{3.4}$$

Now, for fixed $x \in \Omega$, consider the function defined by

$$w(y,t) = v^x(x + d_\Gamma(x)\, y, d_\Gamma(x)^2 t) \quad \text{for} \quad (y,t) \in \overline{B} \times (0,\infty).$$

By the translation and scaling invariance of (3), we have that w satisfies the problem (3)-(5) in B, and hence equals u_B on $\overline{B} \times (0,\infty)$, by uniqueness.

Therefore, evaluating u_B at $(0, t/d_\Gamma(x)^2)$ gives that

$$u_B(0, t/d_\Gamma(x)^2) = w(0, t/d_\Gamma(x)^2) = v^x(x,t) \ge u(x,t),$$

by (3.4).

We conclude and obtain (3.3) by using (2.22) with $R = 1$, $x' = 0$, $t' = \frac{t}{d_\Gamma(x)^2}$. Indeed, in the case $p \in (1,\infty)$, we have that

$$4\left\{\frac{t}{d_\Gamma(x)^2}\right\} \log v^x(x,t) = 4\left\{\frac{t}{d_\Gamma(x)^2}\right\} \log u_B\left(0, \frac{t}{d_\Gamma(x)^2}\right) \le$$
$$\frac{4}{\lambda^2} - \frac{4}{\lambda}\sqrt{p'} + 4\left\{\frac{t}{d_\Gamma(x)^2}\right\} \log\left[\frac{\int_0^\pi (\sin\theta)^{\frac{N-p}{p-1}}\, d\theta}{\int_0^\pi e^{-\frac{\sqrt{p'}}{\lambda t}(1-\cos\theta)} (\sin\theta)^{\frac{N-p}{p-1}}\, d\theta}\right].$$

Hence, by choosing $\lambda = \frac{2}{\sqrt{p'}}$, we conclude that

$$4t \log u_B\left(0, \frac{t}{d_\Gamma(x)^2}\right) \le -p' d_\Gamma(x)^2 + 4t \log\left[\frac{\int_0^\pi (\sin\theta)^{\frac{N-p}{p-1}}\, d\theta}{\int_0^\pi e^{-\frac{p'}{2t}(1-\cos\theta) d_\Gamma(x)^2} (\sin\theta)^{\frac{N-p}{p-1}}\, d\theta}\right],$$

which implies (3.3). In the case $p = \infty$ we just need to replace (2.22) with (2.23) to obtain the conclusion. □

3.2 Barriers in the elliptic case

The next two lemmas give explicit barriers for the solution u^ε of (6)-(7) in a general domain Ω. Compared to the parabolic case, here we obtain a sharper barrier from below, since we have a more convenient formula for the solution in the exterior of a ball.

Lemma 3.3 (An elliptic barrier from below, [13, Lemma 2.4]). *Let $\Omega \subset \mathbb{R}^N$ be a domain and $p \in (1,\infty]$. Let u^ε be the bounded (viscosity) solution of (6)-(7). Pick $z \in \mathbb{R}^N \setminus \overline{\Omega}$.*

Then, we have that

$$\varepsilon \log\{u^\varepsilon(x)\} + \sqrt{p'}\,\{|x-z| - d_\Gamma(z)\} \ge \varepsilon \log e^\varepsilon_{p,z}(x) \quad \textit{for any} \quad x \in \overline{\Omega},$$

where

$$e^{\varepsilon}_{p,z}(x) = \begin{cases} \dfrac{\int_0^\infty e^{-\sqrt{p'}\frac{\cosh\theta-1}{\varepsilon}|x-z|}\,(\sinh\theta)^{\frac{N-p}{p-1}}\,d\theta}{\int_0^\infty e^{-\sqrt{p'}\frac{\cosh\theta-1}{\varepsilon}d_\Gamma(z)}\,(\sinh\theta)^{\frac{N-p}{p-1}}\,d\theta} & \text{if } 1<p<\infty, \\ 1 & \text{if } p=\infty. \end{cases} \tag{3.5}$$

Proof. We consider the ball $B = B_R(z)$ with radius $R = d_\Gamma(z)$ and let v^ε be the bounded solution of (6)-(7) relative to $\mathbb{R}^N \setminus \overline{B} \supset \Omega$. From the fact that $z \in \mathbb{R}^N \setminus \overline{\Omega}$, we have that $\Gamma \subset \mathbb{R}^N \setminus B$, which implies that

$$v^\varepsilon \le 1 \ \text{ on } \ \Gamma,$$

by the explicit expression of v^ε given in (2.13). Thus, by the comparison principle, we infer that $v^\varepsilon \le u^\varepsilon$ on $\overline{\Omega}$. The desired claim then follows by easy manipulations on (2.13). □

Lemma 3.4 (An elliptic barrier from above, [13, Lemma 2.3])**.** *Let $\Omega \subset \mathbb{R}^N$ be a domain and $p \in (1,\infty]$. Let u^ε be the bounded (viscosity) solution of* (6)-(7)*.*

Then, we have that

$$\varepsilon \log\{u^\varepsilon(x)\} + \sqrt{p'}\,d_\Gamma(x) \le \varepsilon \log E^\varepsilon_p\left(d_\Gamma(x)\right),$$

for every $x \in \overline{\Omega}$, where, for $\sigma \ge 0$,

$$E^\varepsilon_p(\sigma) = \begin{cases} \dfrac{\int_0^\pi (\sin\theta)^{\frac{N-p}{p-1}}\,d\theta}{\int_0^\pi e^{-\sqrt{p'}\frac{1-\cos\theta}{\varepsilon}\sigma}(\sin\theta)^{\frac{N-p}{p-1}}\,d\theta} & \text{if } 1<p<\infty, \\ \dfrac{2}{1+e^{-\frac{2\sigma}{\varepsilon}}} & \text{if } p=\infty. \end{cases}$$

In particular, it holds that

$$\varepsilon \log E^\varepsilon_p\left(d_\Gamma\right) = \begin{cases} O(\varepsilon\,\log\varepsilon) & \text{if } 1<p<\infty, \\ O(\varepsilon) & \text{if } p=\infty, \end{cases}$$

as $\varepsilon \to 0^+$, on every subset of $\overline{\Omega}$ in which d_Γ is bounded.

Proof. For a fixed $x \in \Omega$, we consider the ball $B^x = B_R(x)$ with $R = d_\Gamma(x)$ and denote by $u^\varepsilon_{B^x}$ the solution of (6)-(7) with $\Omega = B^x$. The comparison principle gives that

$$u^\varepsilon \le u^\varepsilon_{B^x} \ \text{ on } \ \overline{B^x}$$

and, in particular,

$$u^\varepsilon(x) \le u^\varepsilon_{B^x}(x). \tag{3.6}$$

Observe that the uniqueness of the solution of (6)-(7) and the scaling properties of Δ_p^G imply that

$$u^\varepsilon_{B^x}(x) = u^{\varepsilon/R}_B(0),$$

where u^ε_B is the solution of (6)-(7) with $\Omega = B$, the unit ball. The explicit expressions in (2.12) and (3.6) then yield the pointwise estimate, since $R = d_\Gamma(x)$.

The last uniform formula then follows from (2.19) in Theorem 2.12. □

3.3 Pointwise Varadhan-type formulas

Theorem 3.5. *Set $p \in (1, \infty]$. Let Ω be a domain in $\mathbb{R}^N$, with boundary Γ such that $\Gamma = \partial(\mathbb{R}^N \setminus \overline{\Omega})$, and let u be the bounded (viscosity) solution of* (3)-(5).

Then, we have that

$$\lim_{t \to 0^+} 4t \log\{u(x,t)\} = -p'\, d_\Gamma(x)^2 \quad \textit{for every } x \in \overline{\Omega}. \tag{3.7}$$

Proof. It is clear that (3.7) holds for $x \in \Gamma$. Let $x \in \Omega$. We only need to apply Lemmas 3.1 and 3.2. Indeed, for $z \in \mathbb{R}^N \setminus \overline{\Omega}$, for every and $t > 0$, Lemmas 3.1 and 3.2 lead to

$$p'\left(d_\Gamma(x)^2 - |x-z|^2\right) + 4t\, \log E^-\left(d_\Gamma(z), t\right) \le \\ 4t \log\{u(x,t)\} + p'\, d_\Gamma(x)^2 \le 4t \log E^+\left(d_\Gamma(x), t\right). \tag{3.8}$$

The last chain of inequalities implies at once that

$$p'\left(d_\Gamma(x)^2 - |x-z|^2\right) \le \liminf_{t \to 0^+} \left[4t \log u(x,t) + p'\, d_\Gamma(x)^2\right] \le \\ \limsup_{t \to 0^+} \left[4t \log u(x,t) + p'\, d_\Gamma(x)^2\right] \le 0,$$

where we have used Lemma 3.2 and the fact that $t \log E^-\left(d_\Gamma(z), t\right) \to 0$, as $t \to 0^+$.

Since $\Gamma = \partial\left(\mathbb{R}^N \setminus \overline{\Omega}\right)$, we can find always a sequence of z_n that converges to a point $y \in \Gamma$ such that $d_\Gamma(x) = |x - y|$; by taking $z = z_n$ in the last formula and letting $n \to \infty$, we obtain the desired claim. □

Theorem 3.6 ([13, Theorem 2.5])**.** *Let $p \in (1, \infty]$ and Ω be a domain in $\mathbb{R}^N$ satisfying $\Gamma = \partial\left(\mathbb{R}^N \setminus \overline{\Omega}\right)$; assume that u^ε is the bounded (viscosity) solution of* (6)-(7).

Then, it holds that

$$\lim_{\varepsilon \to 0^+} \varepsilon \log\{u^\varepsilon(x)\} = -\sqrt{p'}\, d_\Gamma(x) \quad \textit{for any} \quad x \in \overline{\Omega}. \tag{3.9}$$

Proof. Given $z \in \mathbb{R}^N \setminus \overline{\Omega}$ and $\varepsilon > 0$, combining Lemmas 3.4 and 3.3 gives at $x \in \overline{\Omega}$ that

$$\sqrt{p'}\,\{-|x-z| + d_\Gamma(x) + d_\Gamma(z)\} + \varepsilon \log e^\varepsilon_{p,z}(x) \le \\ \varepsilon \log\{u^\varepsilon(x)\} + \sqrt{p'}\, d_\Gamma(x) \le \varepsilon \log E^\varepsilon_p\left(d_\Gamma(x)\right). \tag{3.10}$$

Letting $\varepsilon \to 0^+$ then gives that

$$\sqrt{p'}\,\{-|x-z| + d_\Gamma(x) + d_\Gamma(z)\} \le \\ \liminf_{\varepsilon \to 0^+} \left[\varepsilon \log\{u^\varepsilon(x)\} + \sqrt{p'}\, d_\Gamma(x)\right] \le \\ \limsup_{\varepsilon \to 0^+} \left[\varepsilon \log\{u^\varepsilon(x)\} + \sqrt{p'}\, d_\Gamma(x)\right] \le 0,$$

where we have used Lemma 3.4 and the fact that $\varepsilon \log e^\varepsilon_{p,z}(x)$ vanishes, as $\varepsilon \to 0^+$. This follows by applying (2.9) to (3.5).

We conclude the proof as in the previous theorem, by letting z tend to $y \in \Gamma$ such that $|x - y| = d_\Gamma(x)$. □

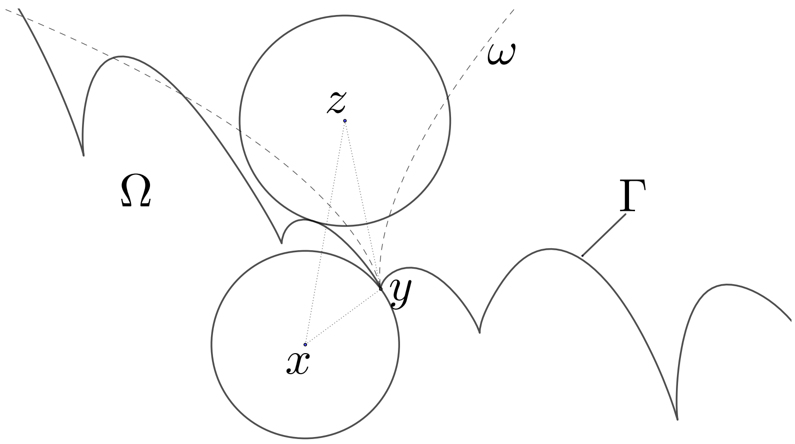

Figure 3.1: The geometric description of the argument in the proof of Theorems 3.7 and 3.9.

3.4 Quantitative uniform formulas

For an open set of class C^0, we mean that its boundary is locally the graph of a continuous function. For the sequel, it is convenient to specify the modulus of continuity, by the following definition (used in [14, 13]). Let $\omega : (0, \infty) \to (0, \infty)$ be a strictly increasing continuous function such that $\omega(\tau) \to 0$ as $\tau \to 0^+$. We say that an open set Ω is of class $C^{0,\omega}$, if there exists a number $r > 0$ such that, for every point $x_0 \in \Gamma$, there is a coordinate system $(y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, and a function $\zeta : \mathbb{R}^{N-1} \to \mathbb{R}$ such that

(i) $B_r(x_0) \cap \Omega = \{(y', y_N) \in B_r(x_0) : y_N < \zeta(y')\}$;

(ii) $B_r(x_0) \cap \Gamma = \{(y', y_N) \in B_r(x_0) : y_N = \zeta(y')\}$

(iii) $|\zeta(y') - \zeta(z')| \le \omega(|y' - z'|)$ for all $(y', \zeta(y')), (z', \zeta(z')) \in B_r(x_0) \cap \Gamma$.

In the sequel, it will be useful the function defined by

$$\psi_\omega(\sigma) = \min_{0 \le s \le r} \sqrt{s^2 + [\omega(s) - \sigma]^2}, \quad \text{for} \quad \sigma \ge 0.$$

This is the distance of the point $z = (0', \sigma) \in \mathbb{R}^{N-1} \times \mathbb{R}$ from the graph of the function ω. Notice that

$$\psi(\sigma) = \sigma \quad \text{if } \zeta \in C^k \quad \text{with} \quad k \ge 2 \tag{3.11}$$

and, otherwise, $\psi(\sigma) \sim C\, \omega^{-1}(\sigma)$, for some positive constant C, where ω^{-1} is the inverse function of ω. For instance, if Ω is of class C^α, with $0 < \alpha < 1$ — that means that Γ is locally a graph of an α-Hölder continuous function — then $\psi(\sigma) \ge a\, \sigma^{1/\alpha}$ as $\sigma \to 0^+$.

Figure 3.1 describes the geometric setting considered throughout Section 3.3.

Theorem 3.7. *Let $p \in (1, \infty]$ and suppose that Ω is a domain of class $C^{0,\omega}$. Let u be the bounded (viscosity) solution of* (3)-(5)*.*

Then, it holds that

$$4t\, \log u(x,t) + p'\, d_\Gamma(x)^2 = O(t \log \psi_\omega(t)) \quad as \quad t \to 0^+, \tag{3.12}$$

uniformly on every compact subset of $\overline{\Omega}$. In particular, if $t \log \psi_\omega(t) \to 0$ as $t \to 0^+$, then the solution u of (3)-(5) *satisfies* (3.7) *uniformly on every compact subset of $\overline{\Omega}$.*

Proof. We need to choose z, in (3.8), uniformly with respect to $x \in \overline{\Omega}$. For every $x \in \overline{\Omega}$, we fix a coordinate system $(y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, with its origin at a point in Γ at minimal distance $d_\Gamma(x)$ from x. In this coordinate system, we choose $z(t) = (0', t)$ that, if t is small enough is by construction a point in $\mathbb{R}^N \setminus \overline{\Omega}$, since $t > \zeta(0')$. Also, by our assumptions on Ω, $d_\Gamma(z(t))$ is bounded from below by the distance of $z(t)$ from the graph of the function $y' \mapsto \omega(|y'|)$ defined for $y' \in \{y \in B_r(0) : y_N = 0\}$, that is

$$d_\Gamma(z(t)) \geq \min_{0 \leq s \leq r} \sqrt{s^2 + [\omega(s) - t]^2}.$$

It is clear that this construction does not depend on the particular point $x \in \overline{\Omega}$ chosen, but only on the regularity assumptions on Ω.

Then, (3.8) reads as

$$p' \left(d_\Gamma(x)^2 - |x - z(t)|^2 \right) + 4t\, \log E^-(d_\Gamma(z(t)), t) \leq$$
$$4t \log \{u(x,t)\} + p'\, d_\Gamma(x)^2 \leq 4t \log E^+ \left(d_\Gamma(x), t \right),$$

for every $x \in \overline{\Omega}$ and $t > 0$.

Observe that $d_\Gamma(z(t)) \geq \psi_\omega(t)$ and $|x - z(t)| \leq d_\Gamma(x) + |y - z(t)|$. Hence, if x is such that $d_\Gamma(x) \leq \delta$, we have that

$$- p' \left(|y - z(t)|^2 + 2\delta |y - z(t)| \right) + 4t\, \log E^-(\psi_\omega(t), t) \leq$$
$$4t \log \{u(x,t)\} + p'\, d_\Gamma(x)^2 \leq 4t \log E^+ \left(\delta, t \right).$$

From Lemma 3.2, the last term is $O(t \log t)$, as $t \to 0^+$. Whereas, from the choice of $z(t)$, the first term can be read as $-p'(t^2 + 2\delta t) + 4t \log E^-(\psi_\omega(t), t)$ and hence its leading term is due to $4t \log E^-(\psi_\omega(t), t)$, which is a $O\left(t \log \psi_\omega(t)\right)$, thanks to (3.2).

□

The same assertion of Theorem 3.7 holds true even if we replace 1 in (5) by a bounded time-dependent non-constant boundary data, provided that this is bounded away from zero.

Corollary 3.8. *Let w be the bounded solution of* (3), (4) *satisfying*

$$w = h \quad on \quad \Gamma \times (0, \infty),$$

where the function $h : \Gamma \times (0, \infty) \to \mathbb{R}$ *is such that*

$$\underline{h} \le h \le \overline{h} \quad on \quad \Gamma \times (0, \infty),$$

for some positive numbers $\underline{h}, \overline{h}$.

Then, we have that

$$4t \log w(x,t) = -p' d_\Gamma(x)^2 + O(t \log \psi_\omega(t)) \quad as \quad t \to 0^+,$$

uniformly on every compact subset of $\overline{\Omega}$.

Proof. Since $\underline{h}\, u \le w \le \overline{h}\, u$ on $\Gamma \times (0, \infty)$, we can apply Corollary 1.12 to get:

$$\underline{h}\, u(x,t) \le w(x,t) \le \overline{h}\, u(x,t) \quad \text{on} \quad \overline{\Omega} \times (0, \infty).$$

This implies that, for every $x \in \overline{\Omega}$ and $t > 0$,

$$4t \log \underline{h} + 4t \log u(x,t) \le 4t \log w(x,t) \le 4t \log \overline{h} + 4t \log u(x,t).$$

The conclusion then easily follows from Theorem 3.7. □

Theorem 3.9 ([13, Theorem 2.6]). *Let* $p \in (1, \infty]$ *and* Ω *be a domain of class* C^0. *Suppose that* u^ε *is the bounded (viscosity) solution of* (6)-(7).

Then, as $\varepsilon \to 0^+$, *we have that*

$$\varepsilon \log \{u^\varepsilon(x)\} + \sqrt{p'}\, d_\Gamma(x) = \begin{cases} O(\varepsilon) & \text{if } p = \infty, \\ O(\varepsilon \log \varepsilon) & \text{if } p > N. \end{cases} \tag{3.13}$$

Moreover, if Ω *is of class* $C^{0,\omega}$, *it holds that*

$$\varepsilon \log \{u^\varepsilon(x)\} + \sqrt{p'}\, d_\Gamma(x) = \begin{cases} O(\varepsilon \log |\log \psi_\omega(\varepsilon)|) & \text{if } N = p, \\ O(\varepsilon \log \psi_\omega(\varepsilon)) & \text{if } 1 < p < N. \end{cases} \tag{3.14}$$

The formulas (3.13) *and* (3.14) *hold uniformly on the compact subsets of* $\overline{\Omega}$.

In particular, if $\varepsilon \log \psi_\omega(\varepsilon) \to 0$ *as* $\varepsilon \to 0^+$, *then the convergence in* (3.9) *is uniform on every compact subset of* $\overline{\Omega}$.

Proof. For any fixed compact subset K of $\overline{\Omega}$ we let d be the positive number, defined as

$$d = \max_{x' \in K} \{d_\Gamma(x'), |x'|\}.$$

To obtain the uniform convergence in (3.9) we will choose $z = z_\varepsilon$ independently on $x \in K$, as follows.

If Ω is of class $C^{0,\omega}$, fix $x \in K$, take $y \in \Gamma$ minimizing the distance to x, and consider a coordinate system in $\mathbb{R}^{N-1} \times \mathbb{R}$ such that $y = (0', 0)$. If we take $z_\varepsilon = (0', \varepsilon)$, then $z_\varepsilon \in \mathbb{R}^N \setminus \overline{\Omega}$ when ε is sufficiently small. With this choice, (3.10) reads as

$$\sqrt{p'}\, \{-|x - z_\varepsilon| + d_\Gamma(x) + d_\Gamma(z_\varepsilon)\} + \varepsilon \log e^\varepsilon_{p, z_\varepsilon}(x) \le \\ \varepsilon \log u^\varepsilon(x) + \sqrt{p'}\, d_\Gamma(x) \le \varepsilon \log E^\varepsilon_p (d_\Gamma(x)).$$

Hence, we get:

$$-\sqrt{p'}\,\varepsilon+\varepsilon\log e^{\varepsilon}_{p,z_\varepsilon}(x)\le\varepsilon\log u^{\varepsilon}(x)+\sqrt{p'}\,d_\Gamma(x)\le\varepsilon\log E^{\varepsilon}_{p}\left(d_\Gamma(x)\right),$$

since $d_\Gamma(z_\varepsilon)\ge 0$ and $|x-z_\varepsilon|\le d_\Gamma(x)+\varepsilon$.

Thus, if $p=\infty$, Lemmas 3.4 and (3.3) give that

$$-\varepsilon\le\varepsilon\log\left\{u^{\varepsilon}(x)\right\}+d_\Gamma(x)\le\varepsilon\log\left\{\frac{2}{1+e^{-\frac{d}{\varepsilon}}}\right\},$$

being $d_\Gamma(x)\le d$, and (3.13) follows at once.

Next, if $1<p<\infty$, we recall that $\varepsilon\log E^{\varepsilon}_{p}\left(d_\Gamma(x)\right)=O(\varepsilon\log\varepsilon)$ on K as $\varepsilon\to 0^+$, by Lemma 3.4. On the other hand, by observing that $d_\Gamma(z_\varepsilon)\ge\psi(\varepsilon)$, by our assumption on Ω, and that also $|x-z_\varepsilon|\le 2d$ for $\varepsilon\le d$, (3.5) gives on K that

$$e^{\varepsilon}_{p,z_\varepsilon}\ge\frac{\int_0^\infty e^{-\frac{2d\sqrt{p'}}{\varepsilon}(\cosh\theta-1)}\,(\sinh\theta)^{\frac{N-p}{p-1}}\,d\theta}{\int_0^\infty e^{-\frac{\sqrt{p'}\psi(\varepsilon)}{\varepsilon}(\cosh\theta-1)}\,(\sinh\theta)^{\frac{N-p}{p-1}}\,d\theta}.$$

Now, after setting $\alpha=\frac{N-p}{2(p-1)}$, to this formula we apply (2.9) with $\sigma=2d\sqrt{p'}/\varepsilon$ at the numerator and (2.10) $\sigma=\sqrt{p'}\psi(\varepsilon)/\varepsilon$ at the denominator. Thus, since the sign of α is that of $N-p$, on K we have as $\varepsilon\to 0$ that

$$\varepsilon\log\left(e^{\varepsilon}_{p,z_\varepsilon}\right)\ge\alpha\,\varepsilon\log\psi(\varepsilon)-\frac{\alpha-1}{2}\,\varepsilon\log\varepsilon+O(\varepsilon)=\alpha\varepsilon\log\psi(\varepsilon)+O(\varepsilon\log\varepsilon),$$

if $p<N$,

$$\varepsilon\log\left(e^{\varepsilon}_{p,z_\varepsilon}\right)\ge-\varepsilon\,\log|\log\psi(\varepsilon)|+O(\varepsilon\log\varepsilon),$$

if $p=N$, and

$$\varepsilon\log\left(e^{\varepsilon}_{p,z_\varepsilon}\right)\ge\frac{\alpha+1}{2}\,\varepsilon\log\varepsilon+O(\varepsilon),$$

if $p>N$. □

Corollary 3.10. *Let $v^{\varepsilon}:\Omega\to\mathbb{R}$ be the bounded solution of (6) satisfying*

$$v^{\varepsilon}=h_\varepsilon\quad\textit{on}\quad\Gamma\times(0,\infty),$$

where, for any $\varepsilon>0$, the function $h_\varepsilon:\Gamma\to\mathbb{R}$ is such that

$$\underline{h}\le h_\varepsilon\le\overline{h}\quad\textit{on}\quad\Gamma,$$

for some positive numbers $\underline{h},\overline{h}$.

Then, we have that

$$\varepsilon\log v^{\varepsilon}(x)+\sqrt{p'}\,d_\Gamma(x)=\begin{cases}O(\varepsilon)\quad\textit{if}\quad p=\infty,\\ O\left(\varepsilon\log\varepsilon\right)\quad\textit{if}\quad p\in(N,\infty),\\ O\left(\varepsilon\log|\log\psi_\omega(\varepsilon)|\right)\quad\textit{if}\quad p=N,\\ O\left(\varepsilon\log\psi_\omega(\varepsilon)\right)\quad\textit{if}\; p\in(1,N).\end{cases}$$

uniformly on every compact subset of $\overline{\Omega}$.

Proof. The proof runs similarly to that of Corollary 3.8, having in mind Theorem 3.9 instead of Theorem 3.7. $\square$

Chapter 4

Asymptotics for q-means

In this chapter, we mainly consider domains of class C^2. For these domains, we are able to provide further asymptotic formulas involving solutions of (3)-(5) and (6)-(7). In particular, we consider statistical nonlinear quantities, called *q-means*, defined, in the case of our interest, as follows. Given $q \in [1, \infty]$, B a ball in $\mathbb{R}^N$ and a function $u : B \to \mathbb{R}$, the q-mean of u on B is the unique real value μ, such that

$$\|u - \mu\|_{L^q(B)} \leq \|u - \lambda\|_{L^q(B)}, \quad \text{for} \quad \lambda \in \mathbb{R}. \tag{4.1}$$

These quantities generalize the standard mean value, which corresponds to the case $q = 2$. These means (there named p-means) have also been studied by Ishiwata, Magnanini and Wadade, in [26], in connection with asymptotic mean value properties for p-harmonic functions.

Formulas that we give here are proper generalizations of those due to Magnanini and Sakaguchi (see [35] and [37]) concerning the linear cases. In [37], the solution of the heat equation subject to conditions (4) and (5) is considered and the following formula for the mean value of u on a ball touching the boundary Γ at only one point $y_x \in \Gamma$ is proved:

$$\lim_{t \to 0^+} \left(\frac{R^2}{t} \right)^{\frac{N+1}{4}} \fint_{B_R(x)} u(z, t)\, dz = \frac{C_N}{\sqrt{\Pi_\Gamma(y_x)}}.$$

Here, C_N is a positive constant, $x \in \Omega$, $R = d_\Gamma(x)$ and $\{y_x\} = \overline{B_R(x)} \cap \Gamma$. Also,

$$\Pi_\Gamma(y) = \prod_{j=1}^{N-1} \left[1 - R\kappa_j(y)\right], \quad \text{for} \quad y \in \Gamma, \tag{4.2}$$

where $\kappa_1(y), \dots, \kappa_{N-1}(y)$ denote the principal curvatures of Γ at y with respect to the interior normal direction to Γ. In [35], a corresponding elliptic case is considered.

In Section 4.3, we consider the q-mean $\mu_q(x, t)$ on $B_R(x)$ of the solution of (3)-(5) and obtain the formula:

$$\lim_{t \to 0^+} \left(\frac{R^2}{t} \right)^{\frac{N+1}{4(q-1)}} \mu_q(x, t) = C_{N,p,q}\, \left\{\Pi_\Gamma(y_x)\right\}^{-\frac{1}{2(q-1)}},$$

This formula holds for any $p \in (1, \infty]$ and $q \in (1, \infty)$. The positive constant $C_{N,p,q}$ will be specified in Theorem 4.7.

In the elliptic case, we consider the q-mean of the solution of (6)-(7) and, for the same values of p and q, we compute:

$$\lim_{\varepsilon \to 0^+} \left(\frac{R}{\varepsilon}\right)^{\frac{N+1}{2(q-1)}} \mu_{q,\varepsilon}(x) = \widetilde{C}_{N,p,q} \, \{\Pi_\Gamma(y_x)\}^{-\frac{1}{2(q-1)}},$$

The value of $\widetilde{C}_{N,p,q}$ can be found in Theorem 4.11.

In Theorems 4.7 and 4.11 also the extremal case in which $q = \infty$ will be treated obtaining that

$$\lim_{t \to 0^+} \mu_\infty(x,t) = \lim_{\varepsilon \to 0^+} \mu_{\infty,\varepsilon}(x) = \frac{1}{2}.$$

The above limits are obtained by using improved versions of the barriers we have constructed in Chapter 3. These versions, that are valid for C^2-regular domains, are presented in Sections 4.1 and 4.2. In Section 4.3, we first prove the asymptotic formulas for the improved barriers (see Lemmas 4.5 and 4.9) and hence thanks to appropriate properties of monotonicity of the q-mean, we extend the formulas to the relevant solutions.

It is worth noting that the results of this chapter are based on Lemma 4.4, a geometrical lemma proved in [37]. In Section 4.3, we recall it, from [37], with its complete proof.

4.1 Improving of barriers in the parabolic case

The following lemma is a consequence of Theorem 3.7.

Lemma 4.1 ([14, Corollary 2.12])**.** *Set $p \in (1, \infty]$. Let Ω be a domain of class $C^{0,\omega}$. Let $v : \overline{\Omega} \times (0, \infty) \to \mathbb{R}$ be defined by*

$$\operatorname{Erfc}\left(\frac{\sqrt{p'}\, v(x,t)}{2\sqrt{t}}\right) = u(x,t) \quad \text{for} \quad (x,t) \in \overline{\Omega} \times (0, \infty),$$

where $u(x,t)$ is the (bounded) viscosity solution of (3)-(5)*.*

Then,

$$v(x,t) = d_\Gamma(x) + O(t \log \psi_\omega(t)) \quad \text{as} \quad t \to 0^+,$$

uniformly on every compact subset of $\overline{\Omega}$.

Proof. From the definition of $v(x,t)$, operating as in the proof of Theorem 2.14 yields that

$$4t \, \log u(x,t) + p' \, v(x,t)^2 = 4t \, \log\left(\sqrt{\frac{p'}{4\pi}} \int_0^\infty e^{-\frac{1}{2} p' \frac{v(x,t)}{\sqrt{t}} \sigma - \frac{1}{4} p' \sigma^2} d\sigma\right) \leq 0.$$

By this inequality, since the first summand at the left-hand side converges uniformly on every compact $K \subset \overline{\Omega}$ as $t \to 0^+$, we can infer that there exist $\bar{t} > 0$ and $\delta > 0$ such that $0 \le v(x,t) \le \delta$ for any $x \in K$ and $0 < t < \bar{t}$.

Thus, for $x \in K$ we have that

$$-\left[4t\,\log u(x,t) + p'\,d_\Gamma(x)^2\right] + 4t\,\log\left(\sqrt{\frac{p'}{\pi}}\int_0^\infty e^{-\frac{1}{2}p'\frac{\delta}{\sqrt{t}}\sigma - \frac{1}{4}p'\sigma^2}d\sigma\right) \le$$
$$p'\,\left[v(x,t)^2 - d_\Gamma(x)^2\right] \le -\left[4t\,\log u(x,t) + p'\,d_\Gamma(x)^2\right],$$

which implies the desired uniform estimate, by means of (3.12). $\square$

We can now refine the barriers given in Section 3.1. We define a function of t by

$$\eta_{u,K}(t) = \frac{1}{\sqrt{t}}\max\{|v(x,t) - d_\Gamma(x)| : x \in K\} \quad \text{for} \quad t > 0. \tag{4.3}$$

Corollary 4.2. *Set $p \in (1,\infty]$. Let Ω be a C^2 domain. For any compact set $K \subseteq \overline{\Omega}$, we have that*

$$\operatorname{Erfc}\left(\sqrt{\frac{p'}{4t}}d_\Gamma(x) + \eta_{u,K}(t)\right) \le u(x,t) \le \operatorname{Erfc}\left(\sqrt{\frac{p'}{4t}}d_\Gamma(x) - \eta_{u,K}(t)\right) \tag{4.4}$$

for $(x,t) \in K \times (0,\infty)$. It holds that $\eta_{u,K}(t) = O\left(\sqrt{t}\log t\right)$, as $t \to 0^+$.

Proof. We use Lemma 4.1 and (4.3) to obtain (4.4). The asymptotic profile of $\eta_{u,K}$ follows from (3.11). $\square$

4.2 Improving of barriers in the elliptic case

To start with, we recall that a domain Ω of class C^2 satisfies both the uniform exterior and interior ball conditions, i.e. there exist $r_i, r_e > 0$ such that every $y \in \Gamma$ has the property that there exist $z_i \in \Omega$ and $z_e \in \mathbb{R}^N \setminus \overline{\Omega}$ for which

$$B_{r_i}(z_i) \subset \Omega \subset \mathbb{R}^N \setminus \overline{B}_{r_e}(z_e) \ \text{ and } \ \overline{B}_{r_i}(z_i) \cap \overline{B}_{r_e}(z_e) = \{y\}. \tag{4.5}$$

We will also use two families of probability measures on the intervals $[0,\infty)$ and $[0,\pi]$ with densities defined, respectively, by

$$d\nu^\tau(\theta) = \frac{e^{-\tau\,(\cosh\theta - 1)}(\sinh\theta)^\alpha}{\int_0^\infty e^{-\tau(\cosh\theta-1)}(\sinh\theta)^\alpha\,d\theta}\,d\theta,$$
$$d\mu^\tau(\theta) = \frac{e^{-\tau\,(1-\cos\theta)}(\sin\theta)^\alpha}{\int_0^\pi e^{-\tau(1-\cos\theta)}(\sin\theta)^\alpha\,d\theta}\,d\theta.$$

Lemma 4.3 ([13, Lemma 3.1]). *Set $p \in (1,\infty]$. Let $\Omega \subset \mathbb{R}^N$ be a C^2 domain. Assume that u^ε is the bounded (viscosity) solution of* (6)-(7)*.*

If $p \in (1,\infty)$, we set for $\tau_\varepsilon = \sqrt{p'}r_e/\varepsilon$:

$$U^\varepsilon(\sigma) = \int_0^\infty e^{-\sigma\cosh\theta} d\nu^{\tau_\varepsilon}(\theta), \quad \sigma \geq 0,$$

and

$$V^\varepsilon(\sigma) = \begin{cases} \displaystyle\int_0^\pi e^{-\sigma\cos\theta} d\mu^{\tau_\varepsilon}(\theta) & \text{if } 0 \leq \sigma < \tau_\varepsilon, \\ \displaystyle\left\{\int_0^\pi e^{-\sigma\cos\theta} d\mu^0(\theta)\right\}^{-1} & \text{if } \sigma \geq \tau_\varepsilon. \end{cases}$$

If $p = \infty$, we set $U^\varepsilon(\sigma) = e^{-\sigma}$ and

$$V^\varepsilon(\sigma) = \begin{cases} \dfrac{\cosh(\tau_\varepsilon - \sigma)}{\cosh\tau_\varepsilon} & \text{if } 0 \leq \sigma < \tau_\varepsilon, \\ 1/\cosh\sigma & \text{if } \sigma \geq \tau_\varepsilon. \end{cases}$$

Then, we have that

$$U^\varepsilon\left(\frac{d_\Gamma(x)}{\varepsilon/\sqrt{p'}}\right) \leq u^\varepsilon(x) \leq V^\varepsilon\left(\frac{d_\Gamma(x)}{\varepsilon/\sqrt{p'}}\right), \tag{4.6}$$

for any $x \in \overline{\Omega}$.

Proof. Let $p \in (1,\infty)$. For any $x \in \Omega$ we can consider $y \in \Gamma$ such that $|x-y| = d_\Gamma(x)$. From the assumptions on Ω there exists $z_e \in \mathbb{R}^N \setminus \overline{\Omega}$ such that (4.5) holds for y. As seen in the proof of Lemma 3.3, by using the comparison principle and the explicit expression (2.13), we obtain

$$u^\varepsilon(x) \geq \frac{\int_0^\infty e^{-\sqrt{p'}|x-z_e|/\varepsilon\,\cosh\theta}(\sinh\theta)^\alpha\, d\theta}{\int_0^\infty e^{-\sqrt{p'}r_e/\varepsilon\,\cosh\theta}(\sinh\theta)^\alpha\, d\theta}.$$

Thus, the fact that $|x-z_e| = d_\Gamma(x) + r_e$ gives the first inequality in (4.6), by recalling the definition of U^ε.

To obtain the second inequality in (4.6) we proceed differently whether $x \in \Omega_{r_i}$ or not. Indeed, if $x \in \Omega_{r_i}$, there exists $z_i \in \Omega$ such that (4.5) holds for some $y \in \Gamma$ and $x \in B_{r_i}(z_i)$; moreover, since $\partial B_{d_\Gamma(x)}(x) \cap \partial B_{r_i}(z_i) = \{y\}$, we observe that x lies in the segment joining y to z_i, and hence $|x-z_i| = r_i - d_\Gamma(x)$. Again, by using the comparison principle and the expression in (2.12), we get that

$$u^\varepsilon(x) \leq \frac{\int_0^\pi e^{\sqrt{p'}\,\cos\theta\,\frac{|x-z_i|}{\varepsilon}}(\sin\theta)^\alpha\, d\theta}{\int_0^\pi e^{\sqrt{p'}\,\cos\theta\,\frac{r_i}{\varepsilon}}(\sin\theta)^\alpha\, d\theta},$$

that, by using the definition of V^ε and the fact that $|x-z_i| = r_i - d_\Gamma(x)$, leads to the second inequality in (4.6).

If $x \in \Omega \setminus \overline{\Omega}_{r_i}$, we just note that the expression of V^ε was already obtained in Lemma 3.4.

The case $p = \infty$ can be treated with similar arguments. □

4.3 Asymptotics for q-means

Before going on, we recall from [26] some preliminary facts about q-means and a geometrical lemma from [37].

For any continuous function u there exists an unique μ satisfying (4.1) (see [26, Theorem 2.1]). For $1 \leq q < \infty$, μ can be characterized by the equation

$$\int_B |u(z) - \mu|^{q-2} \left[u(z) - \mu\right] \, dz = 0.$$

This equation is equivalent to

$$\int_B [u(z) - \mu]_+^{q-1} \, dz = \int_B [\mu - u(z)]_+^{q-1} \, dz, \tag{4.7}$$

where $[s]_+ = \max\{0, s\}$, for $s \in \mathbb{R}$.

The q-mean is monotonically increasing with respect to u, in the sense that

$$\mu^u \leq \mu^v \quad \text{if} \quad u \leq v \quad \text{in} \quad B. \tag{4.8}$$

where, μ^u and μ^v are respectively the q-mean of u and of v.

The next lemma is a version of [37, Lemma 2.1] slightly adapted to our notations. For the reader's convenience, we also report its proof. We recall that by Π_Γ we mean the function in (4.2).

Lemma 4.4. *Let $x \in \Omega$ and assume that, for $R > 0$, there exists $y_x \in \Gamma$ such that $\overline{B_R(x)} \cap (\mathbb{R}^N \setminus \Omega) = \{y_x\}$ and that $\kappa_j(y_x) < 1/R$ for $j = 1, \dots, N-1$. Set $\Gamma_s = \{y \in \Omega : d_\Gamma(y) = s\}$, for $s > 0$.*

Then, it holds that

$$\lim_{s \to 0^+} s^{-\frac{N-1}{2}} \mathcal{H}_{N-1}(\Gamma_s \cap B_R(x)) = \frac{\omega_{N-1} \, (2R)^{\frac{N-1}{2}}}{(N-1)\sqrt{\Pi_\Gamma(y_x)}},$$

where $\mathcal{H}_{N-1}$ denotes $(N-1)$-dimensional Hausdorff measure and ω_{N-1} is the surface area of a unit sphere in $\mathbb{R}^{N-1}$.

Proof from [37]. Without loss of generality, we can suppose that we are working with a coordinate system $\{z_1, \dots, z_N\}$, such that $y_x = 0$, the tangent plane to Γ at y_x coincides with the plane $\{z_N = 0\}$ and $x = (0, \dots, 0, R)$. We can also suppose that $z_1, \dots, z_{N-1}$ are chosen such that

$$d_\Gamma(z) = z_N - \frac{1}{2} \sum_{j=1}^{N-1} \kappa_j(y_x) z_j^2 + o(|z|^2) \tag{4.9}$$

$$\frac{\partial d_\Gamma}{\partial z_N}(z) = 1 + o(|z|). \tag{4.10}$$

Note that, with these choices, $B_R(x)$ is represented by the inequality $|z'|^2 + (z_N - R)^2 < R^2$, where $z' = (z_1, \dots, z_{N-1})$. Hence, near the origin, $\partial B_R(x)$ is represented by

$$z_N = \frac{1}{2}|z'|^2 + O(|z'|^3). \tag{4.11}$$

Combining (4.9) with (4.11), gives

$$d_\Gamma(z) = \frac{1}{2}\sum_{j=1}^{N-1}\left(\frac{1}{R} - \kappa_j(y_x)\right) z_j^2 + o(|z'|^2) \quad \text{for} \quad z \in B_R \cap \partial B_R(x). \tag{4.12}$$

Since $\overline{B}_R(x) \cap \left(\mathbb{R}^N \setminus \Omega\right) = \{0\}$, for every $\varepsilon > 0$, there exists $s_\varepsilon > 0$ such that

$$\Gamma_s \cap B_R(x) \subseteq B_\varepsilon \quad \text{if} \quad 0 < s < s_\varepsilon. \tag{4.13}$$

Hence, from (4.10), if $\varepsilon > 0$ is sufficiently small and $0 < s < s_\varepsilon$, $\Gamma_s \cap B_R(x)$ is represented by the graph of a smooth function $z_N = \psi(z')$. Differentiating $d_\Gamma(z', \psi(z')) = s$ with respect to z_j yields

$$d_{z_N} \nabla_{z'}\psi + \nabla_{z'} d = 0,$$

which together with $|\nabla d_\Gamma| = 1$ implies that

$$\sqrt{1 + |\nabla_{z'}\psi|^2} = 1/d_{z_N}. \tag{4.14}$$

Projecting $\Gamma_s \cap B_R(x)$ orthogonally on $\{z_N = 0\}$ yields a domain $A_s \subseteq \mathbb{R}^{N-1}$. Let $\eta > 0$ be sufficiently small. From (4.12) and (4.13), there exists $\varepsilon_0 > 0$ such that, for every $0 < s < s_{\varepsilon_0}$, we have

$$E_s^+ \subseteq A_s \subseteq E_s^- \tag{4.15}$$

where

$$E_s^\pm = \left\{ z' \in \mathbb{R}^{N-1} : \frac{1}{2}\sum_{j=1}^{N-1}\left(\frac{1}{R} - \kappa_j(y_x) \pm \eta\right) z_j^2 < s \right\}.$$

Moreover, combining (4.10) and (4.14) yields

$$1 \le \sqrt{1 + |\nabla_{z'}\psi|^2} \le 1 + \eta, \tag{4.16}$$

for very $0 < s < s_{\varepsilon_0}$. Hence, it follows from (4.15) and (4.16) that

$$\int_{E_s^+} 1\, dz' \le \mathcal{H}_{N-1}\left(\Gamma_s \cap B_R(x)\right) \le \int_{E_s^-} (1 + \eta)\, dz', \tag{4.17}$$

for every $0 < s < s_{\varepsilon_0}$, since

$$\mathcal{H}_{N-1}\left(\Gamma_s \cap B_R(x)\right) = \int_{A_s} \sqrt{1 + |\nabla_{z'}\psi|^2}\, dz'.$$

Hence, from (4.17) we see that

$$\frac{\omega_{N-1}\, 2^{\frac{N-1}{2}}}{(N-1)} \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(y_x) + \eta \right] \right\}^{-1/2} \leq$$
$$s^{-\frac{N-1}{2}} \mathcal{H}_{N-1}\left(\Gamma_s \cap B_R(x)\right) \leq$$
$$\frac{\omega_{N-1}\, 2^{\frac{N-1}{2}}}{(N-1)} \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(y_x) - \eta \right] \right\}^{-1/2}$$

for every $0 < s < s_{\varepsilon_0}$. Since $\eta > 0$ is arbitrarily small, we conclude the proof. □

4.3.1 Short-time asymptotics for q-means

Lemma 4.5 (Asymptotics for the q-mean of a barrier, [14, Lemma 3.4])**.** *Set $1 < q < \infty$, let $x \in \Omega$, and assume that, for $R > 0$, there exists a point $y_x \in \Gamma$ such that $\overline{B_R(x)} \cap (\mathbb{R}^N \setminus \Omega) = \{y_x\}$ and $\kappa_j(y_x) < 1/R$ for $j = 1, \ldots, N-1$.*

Let $\xi, \eta : (0,\infty) \to (0,\infty)$ be two functions of time such that $\xi(t)$ is positive in $(0,\infty)$, and

$$\lim_{t\to 0^+} \xi(t) = \lim_{t\to 0^+} \eta(t) = 0.$$

For a non-negative, decreasing and continuous function f on $\mathbb{R}$ such that

$$\int_0^\infty f(\sigma)^{q-1} \sigma^{\frac{N-1}{2}} d\sigma < \infty,$$

set

$$w(z,t) = f\left(\frac{d_\Gamma(z)}{\xi(t)} + \eta(t)\right) \quad \text{for} \quad (z,t) \in \overline{\Omega} \times (0,\infty).$$

If $\mu_q^w(x,t)$ is the q-mean of $w(\cdot,t)$ on $B_R(x)$, then the following formula holds:

$$\lim_{t\to 0^+} \left(\frac{R}{\xi(t)}\right)^{\frac{N+1}{2(q-1)}} \mu_q^w(x,t) = \left\{ \frac{2^{-\frac{N+1}{2}} N! \int_0^\infty f(\sigma)^{q-1} \sigma^{\frac{N-1}{2}} d\sigma}{\Gamma\left(\frac{N+1}{2}\right)^2 \sqrt{\Pi_\Gamma(y_x)}} \right\}^{\frac{1}{q-1}}. \tag{4.18}$$

Proof. We know from (4.7) that $\mu(t) = \mu_q^w(x,t)$ is the unique root of the following equation

$$\int_{B_R(x)} [w(z,t) - \mu(t)]_+^{q-1} dz = \int_{B_R(x)} [\mu(t) - w(z,t)]_+^{q-1} dz. \tag{4.19}$$

Firstly, we compute the short-time behavior of the left-hand side of (4.19). Let $\Gamma_s = \{z \in B_R(x) : d_\Gamma(z) = s\}$. By the co-area formula, we get that

$$\int_{B_R(x)} [w(z,t) - \mu(t)]_+^{q-1} dz = \int_0^{2R} \left[f\left(\frac{s}{\xi(t)} + \eta(t)\right) - \mu(t) \right]_+^{q-1} \mathcal{H}_{N-1}\left(\Gamma_s\right) ds.$$

By the change of variable $s = \xi(t)\,[\sigma - \eta(t)]$, we obtain that

$$\int_{B_R(x)} [w(z,t) - \mu(t)]_+^{q-1} dz = \xi(t) \int_{\eta(t)}^{\beta(t)} [f(\sigma) - \mu(t)]_+^{q-1} \mathcal{H}_{N-1}\left(\Gamma_{\xi(t)[\sigma-\eta(t)]}\right) d\sigma,$$

where we set $\beta(t) = \frac{2R}{\xi(t)} + \eta(t)$.

Hence,

$$\xi(t)^{-\frac{N-1}{2}} \int_{B_R(x)} [w(z,t) - \mu(t)]_+^{q-1} dz = \int_{\eta(t)}^{\beta(t)} \frac{\mathcal{H}_{N-1}\left(\Gamma_{\xi(t)[\sigma-\eta(t)]}\right)}{\{\xi(t)\,[\sigma - \eta(t)]\}^{\frac{N-1}{2}}} [\sigma - \eta(t)]^{\frac{N-1}{2}} \{f(\sigma) - \mu(t)\}^{q-1} \, d\sigma.$$

Now, as $t \to 0^+$ we have that $\eta(t), \xi(t), \mu(t) \to 0$, $\beta(t) \to \infty$ and that $\xi(t)\,[\sigma - \eta(t)] \to 0$ for almost every $\sigma \geq 0$. Thus, we can infer that

$$\lim_{t \to 0^+} \xi(t) \int_{B_R(x)} [w(z,t) - \mu(t)]_+^{q-1} dz = \frac{\omega_{N-1}(2R)^{\frac{N-1}{2}}}{(N-1)\sqrt{\Pi_\Gamma(y_x)}} \int_0^\infty f(\sigma)^{q-1} \sigma^{\frac{N-1}{2}} \, d\sigma, \quad (4.20)$$

by Lemma 4.4 and an application of the dominated convergence theorem, as an inspection of the integrand function reveals.

Secondly, we treat the short-time behavior of the right-hand side of (4.19). By again performing the co-area formula and after some manipulations, we have that

$$\int_{B_R(x)} [\mu(t) - w(z,t)]_+^{q-1} \, dz = \mu(t)^{q-1} \int_0^{2R} \left[1 - f\left(\frac{s}{\xi(t)} + \eta(t)\right) \Big/ \mu(t)\right]_+^{q-1} \mathcal{H}_{N-1}(\Gamma_s) \, ds \quad (4.21)$$

which, on one hand, leads to

$$\int_{B_R(x)} [\mu(t) - w(z,t)]_+^{q-1} \, dz \leq \mu(t)^{q-1} |B_R(x)|.$$

Notice in particular that, by using both (4.25) and (4.20), the last inequality informs us that

$$\mu(t) \geq c\,\xi(t)^{\frac{N+1}{2(q-1)}},$$

for some positive constant c. Hence, after setting $\beta(s,t) = \frac{s}{\xi(t)} + \eta(t)$, the assumptions on f give the following chain of inequalities:

$$\int_{\beta(s,t)/2}^{\infty} f(\sigma)^{q-1} \sigma^{\frac{N-1}{2}} \, d\sigma \geq \int_{\beta(s,t)/2}^{\beta(s,t)} f(\sigma)^{q-1} \sigma^{\frac{N-1}{2}} \, d\sigma \geq \frac{2\left(1 - 2^{-\frac{N+1}{2}}\right)}{N+1} \frac{f(\beta(s,t))^{q-1}}{\xi(t)^{\frac{N+1}{2}}} [s + \eta(t)\,\xi(t)]^{\frac{N+1}{2}} \geq \frac{2\left(1 - 2^{-\frac{N+1}{2}}\right)}{c(N+1)} \left[f\left(\frac{s}{\xi(t)} + \eta(t)\right) \Big/ \mu(t)\right]^{q-1} [s + \eta(t)\,\xi(t)]^{\frac{N+1}{2}}.$$

Since, for almost every $s \geq 0$, the first term of the chain vanishes as $t \to 0^+$, we have that

$$\lim_{t\to 0^+} \frac{f\left(\frac{s}{\xi(t)} + \eta(t)\right)}{\mu(t)} = 0,$$

for almost every $s \geq 0$. Thus, (4.21) gives at once that

$$\lim_{t\to 0^+} \mu(t)^{1-q} \int_{B_R(x)} [\mu(t) - w(z,t)]_+^{q-1}\, dz = |B_R(x)|. \tag{4.22}$$

Finally, (4.25), (4.20) and (4.22) tell us that

$$\mu(t)^{q-1} = \xi(t)^{\frac{N+1}{2}} \frac{\omega_{N-1}\,(2R)^{\frac{N-1}{2}}}{(N-1)\sqrt{\Pi_\Gamma(y_x)}} \frac{\int_0^\infty f(\sigma)^{q-1}\,\sigma^{\frac{N-1}{2}} d\sigma + o(1)}{|B_R(x)| + o(1)},$$

that gives (4.18), after straightforward calculations involving Euler's gamma function. □

Remark 4.6. If $q = \infty$, we know that

$$\mu_\infty^w(x,t) = \frac{1}{2}\left\{\min_{B_R(x)} w(\cdot,t) + \max_{B_R(x)} w(\cdot,t)\right\} = \frac{1}{2}\left[f\left(\frac{\overline{d}}{\xi(t)} + \eta(t)\right) + f(\eta(t))\right],$$

where $\overline{d}$ is positive, being the maximum of d_Γ on $\overline{B_R(x)}$. Hence, it is easy to compute:

$$\lim_{t\to 0^+} \mu_\infty^w(x,t) = \frac{1}{2}\, f(0).$$

Thus, formula (4.18) does not extend continuously to the case $q = \infty$.

Theorem 4.7 (Short-time asymptotics for q-means, [14, Theorem 3.5])**.** *Let $x \in \Omega$, and assume that, for $R > 0$, there exists a point $y_x \in \Gamma$ such that $\overline{B_R(x)} \cap (\mathbb{R}^N \setminus \Omega) = \{y_x\}$ and $\kappa_j(y_x) < 1/R$ for $j = 1, \dots, N-1$.*

Set $1 < p \leq \infty$ and suppose that u is the bounded (viscosity) solution of (3)-(5) *and, for $1 < q \leq \infty$, $\mu_q(x,t)$ is the q-mean of $u(\cdot,t)$ on $B_R(x)$.*

Then, if $1 < q < \infty$, the following formulas hold:

$$\lim_{t\to 0^+} \left(\frac{R^2}{t}\right)^{\frac{N+1}{4(q-1)}} \mu_q(x,t) =$$

$$\left\{\frac{N!\int_0^\infty \operatorname{Erfc}(\sigma)^{q-1}\sigma^{\frac{N-1}{2}} d\sigma}{\Gamma\left(\frac{N+1}{2}\right)^2}\right\}^{\frac{1}{q-1}} \left\{p'^{\frac{N+1}{2}}\, \Pi_\Gamma(y_x)\right\}^{-\frac{1}{2(q-1)}}, \tag{4.23}$$

and

$$\lim_{t\to 0^+} \mu_\infty(x,t) = \frac{1}{2}.$$

Proof. By using (4.4) and (4.8), the limit in (4.23) will result from Lemma 4.5, where we choose:

$$w(x,t) = \operatorname{Erfc}\left(\sqrt{\frac{p'}{4t}}\, d_\Gamma(y) \pm \eta(t)\right),$$

that is we choose $\xi(t) = \sqrt{4t/p'}$ and $\eta(t)$ is given by (4.3), with $K = \overline{B}_R(x)$. Thus, (4.23) will follow at once from (4.18), where $f(\sigma) = \operatorname{Erfc}(\sigma)$.

By the same argument, we also get the case $q = \infty$, since $f(0) = 1$. □

Remark 4.8. Notice that

$$\left\{\int_0^\infty \operatorname{Erfc}(\sigma)^{q-1} \sigma^{\frac{N-1}{2}} d\sigma\right\}^{\frac{1}{q-1}}$$

can be seen as the $(q-1)$-norm of Erfc in $(0,\infty)$ with respect to the weighed measure $\sigma^{\frac{N-1}{2}} d\sigma$.

4.3.2 Asymptotics for q-means in the elliptic case

The next lemma gives the asymptotic formula for $\varepsilon \to 0^+$ for the q-mean on $B_R(x)$ of a quite general class of functions, which includes both the barriers U^ε and V^ε of Lemma 4.3.

Lemma 4.9 ([13, Lemma 3.3])**.** *Set $1 < q < \infty$. Let $x \in \Omega$ and assume that, for $R > 0$, there exists $y_x \in \Gamma$ such that $\overline{B_R(x)} \cap (\mathbb{R}^N \setminus \Omega) = \{y_x\}$ and that $\kappa_j(y_x) < 1/R$ for $j = 1, \dots, N-1$.*

Let $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{f_n\}_{n\in\mathbb{N}}$ be sequences such that

(i) $\xi_n > 0$ and $\xi_n \to 0$ as $n \to \infty$;

(ii) $f_n : [0,\infty) \to [0,\infty)$ are decreasing functions;

(iii) f_n converges to a function f almost everywhere as $n \to \infty$;

(iv) it holds that

$$\lim_{n\to\infty} \int_0^\infty f_n(\sigma)^{q-1}\, \sigma^{\frac{N-1}{2}}\, d\sigma = \int_0^\infty f(\sigma)^{q-1}\, \sigma^{\frac{N-1}{2}}\, d\sigma,$$

and the last integral converges.

For some $1 < q < \infty$, let $\mu_{q,n}(x)$ be the q-mean of $f_n(d_\Gamma/\xi_n)$ on $B_R(x)$.

Then we have:

$$\lim_{n\to\infty} \left(\frac{R}{\xi_n}\right)^{\frac{N+1}{2(q-1)}} \mu_{q,n}(x) = \left\{\frac{2^{-\frac{N+1}{2}} N! \int_0^\infty f(\sigma)^{q-1} \sigma^{\frac{N-1}{2}}\, d\sigma}{\Gamma\left(\frac{N+1}{2}\right)^2 \sqrt{\Pi_\Gamma(y_x)}}\right\}^{\frac{1}{q-1}}. \tag{4.24}$$

Proof. From (4.7), we know that $\mu_n = \mu_{q,n}(x)$ is the only root of the equation

$$\int_{B_R(x)} [f_n(d_\Gamma/\xi_n) - \mu_n]_+^{q-1}\, dz = \int_{B_R(x)} [\mu_n - f_n(d_\Gamma/\xi_n)]_+^{q-1}\, dz, \tag{4.25}$$

where we mean $[t]_+ = \max(0,t)$.

Thus, if we set

$$\Gamma_\sigma = \{y \in B_R : d_\Gamma(y) = \sigma\},$$

by the co-area formula we get that

$$\int_{B_R(x)} [f_n(d_\Gamma/\xi_n) - \mu_n]_+^{q-1}\, dz = \int_0^{2R} [f_n(\sigma/\xi_n) - \mu_n]_+^{q-1}\, \mathcal{H}_{N-1}(\Gamma_\sigma)\, d\sigma,$$

that, after the change of variable $\sigma = \xi_n \tau$ and easy manipulations, leads to the formula:

$$\int_{B_R(x)} [f_n(d_\Gamma/\xi_n) - \mu_n]_+^{q-1}\, dz = \xi_n^{\frac{N+1}{2}} \int_0^{2R/\xi_n} [f_n(\tau) - \mu_n]_+^{q-1}\, \tau^{\frac{N-1}{2}} \left[\frac{\mathcal{H}_{N-1}(\Gamma_{\xi_n \tau})}{(\xi_n \tau)^{\frac{N-1}{2}}}\right] d\tau.$$

Therefore, since $\mu_n \to 0$ as $n \to \infty$, an inspection of the integrand at the right-hand side, assumptions (i)-(iv), and Lemma 4.4 make it clear that we can apply the generalized dominated convergence theorem (see [32]) to infer that

$$\lim_{n\to\infty} \xi_n^{-\frac{N+1}{2}} \int_{B_R(x)} [f_n(d_\Gamma/\xi_n) - \mu_n]_+^{q-1}\, dz = \frac{(2R)^{\frac{N-1}{2}} \omega_{N-1}}{(N-1)\sqrt{\Pi_\Gamma(y_x)}} \int_0^\infty f(\sigma)^{q-1} \sigma^{\frac{N-1}{2}}\, d\sigma. \tag{4.26}$$

Next, by employing again the co-area formula, the right-hand side of (4.25) can be re-arranged as

$$\int_{B_R(x)} [\mu_n - f_n(d_\Gamma/\xi_n)]_+^{q-1}\, dz = \mu_n^{q-1} \int_0^{2R} \left[1 - \frac{f_n(\sigma/\xi_n)}{\mu_n}\right]_+^{q-1} \mathcal{H}_{N-1}(\Gamma_\sigma)\, d\sigma,$$

that leads to the formula

$$\lim_{\varepsilon\to 0^+} \mu_n^{1-q} \int_{B_R(x)} [\mu_n - f_n(d_\Gamma/\xi_n)]_+^{q-1}\, dz = |B_R|, \tag{4.27}$$

by dominated convergence theorem, if we can prove that

$$\frac{f_n(\sigma/\xi_n)}{\mu_n} \to 0 \quad \text{as} \quad n \to \infty, \tag{4.28}$$

for almost every $\sigma \geq 0$. Then, after straightforward computations, (4.24) will follow by putting together (4.25), (4.26) and (4.27).

We now complete the proof by proving that (4.28) holds. From (4.25), (4.26), and the fact that
$$\int_{B_R(x)} [\mu_n - f_n(d_\Gamma/\xi_n)]_+^{q-1}\, dz \le \mu_n^{q-1}|B_R|,$$
we have that there is a positive constant c such that
$$\mu_n^{1-q} \le c\,\xi_n^{-\frac{N+1}{2}}.$$
Also, for every $\tau > 0$ we have that
$$\int_{\tau/2\xi_n}^{\infty} f_n(\sigma)^{q-1}\,\sigma^{\frac{N-1}{2}}\,d\sigma \ge \int_{\tau/2\xi_n}^{\tau/\xi_n} f_n(\sigma)^{q-1}\,\sigma^{\frac{N-1}{2}}\,d\sigma \ge$$
$$\frac{2(1-2^{-\frac{N+1}{2}})}{N+1}\, f_n(\tau/\xi_n)^{q-1}\left(\frac{\tau}{\xi_n}\right)^{\frac{N+1}{2}} \ge$$
$$\frac{2(1-2^{-\frac{N+1}{2}})}{c\,(N+1)}\,\tau^{\frac{N+1}{2}}\left\{\frac{f_n(\tau/\xi_n)}{\mu_n}\right\}^{q-1}.$$
Thus, (4.28) follows, since the first term of this chain of inequalities converges to zero as $n \to \infty$, under our assumptions on f_n and ξ_n, in virtue of the generalized dominated convergence theorem. □

Remark 4.10. The case $q = \infty$ is simpler. From [26] and then the monotonicity of f_n we obtain that:
$$\mu_{\infty,n}(x) = \frac{1}{2}\left\{\min_{B_R(x)} f_n\left(d_\Gamma/\xi_n\right) + \max_{B_R(x)} f_n\left(d_\Gamma/\xi_n\right)\right\} = \frac{1}{2}\left\{f_n\left(2R/\xi_n\right) + f_n(0)\right\}.$$
Thus, if we replace the assumptions (iii) and (iv) by $f_n(0) \to f(0)$ as $n \to \infty$, we conclude that $\mu_{\infty,n}(x) \to f(0)/2$, since $f_n\left(2R/\xi_n\right) \to 0$ as $n \to \infty$.

Theorem 4.11 ([13, Theorem 3.5])**.** *Set $1 < p \le \infty$. Let $x \in \Omega$ be such that $B_R(x) \subset \Omega$ and $\overline{B_R(x)} \cap (\mathbb{R}^N \setminus \Omega) = \{y_x\}$; suppose that $k_j(y_x) < \frac{1}{R}$, for every $j = 1, \dots, N-1$.*

Let u^ε be the bounded (viscosity) solution of (6)-(7) and, for $1 < q \le \infty$, let $\mu_{q,\varepsilon}(x)$ be the q-mean of u^ε on $B_R(x)$.

Then, if $1 < q < \infty$, we have that
$$\lim_{\varepsilon\to 0^+}\left(\frac{\varepsilon}{R}\right)^{-\frac{N+1}{2(q-1)}} \mu_{q,\varepsilon}(x) = \left\{\frac{2^{-\frac{N+1}{2}}N!}{(q-1)^{\frac{N+1}{2}}\Gamma\left(\frac{N+1}{2}\right)}\right\}^{\frac{1}{q-1}} \left\{p'^{\frac{N+1}{2}}\Pi_\Gamma(y_x)\right\}^{-\frac{1}{2(q-1)}}. \quad (4.29)$$

If $q = \infty$, we simply have that
$$\lim_{\varepsilon\to 0^+} \mu_{\infty,\varepsilon}(x) = \frac{1}{2}.$$

Proof. We have that $\mu^{U^\varepsilon}_{q,\varepsilon}(x) \le \mu_{q,\varepsilon}(x) \le \mu^{V^\varepsilon}_{q,\varepsilon}(x)$ by the monotonicity of the q-means, where with $\mu^{U_\varepsilon}_{q,\varepsilon}$ and $\mu^{V^\varepsilon}_{q,\varepsilon}$ we denote the q-mean of $U^\varepsilon(d/\varepsilon)$ and $V^\varepsilon(d/\varepsilon)$ on $B_R(x)$. Hence, in order to prove (4.29), we only need to apply Lemma 4.9 to $f_n = U^{\varepsilon_n}$ and $f_n = V^{\varepsilon'_n}$, where the vanishing sequences ε_n and ε'_n are chosen so that the lim inf and lim sup of $(\varepsilon/R)^{-\frac{N+1}{2(q-1)}}\mu_{q,\varepsilon}(x)$ as $\varepsilon \to 0$ are attained along them, respectively.

By an inspection, it is not difficult to check that $f_n = U^{\varepsilon_n}$ and $f_n = V^{\varepsilon'_n}$, with $\xi_\varepsilon = \varepsilon/\sqrt{p'}$ and $f(\sigma) = e^{-\sigma}$, satisfy the relevant assumptions of Lemma 4.9, by applying, in particular, Lemma 2.4 for (iii) and the dominated convergence theorem for (iv). $\square$

Chapter 5

Geometric and symmetry results

The goal of this chapter is to collect some geometric and symmetry results for solutions (3)-(5) or (6)-(7), in the spirit of those given by Magnanini and Sakaguchi in [17, 35, 37, 38, 41, 42]. We obtain characterizations of balls, spheres and hyperplanes as applications of Varadhan-type formulas of Chapter 3 and of formulas for q-means of Chapter 4.

We introduce the problems that we consider. We say that an $(N-1)$-dimensional surface Σ is a *time-invariant level surface* for the solution u of (3)-(5), if there exists a function $a_\Sigma : (0,\infty) \to (0,\infty)$ such that

$$u(x,t) = a_\Sigma(t) \quad \text{for any} \quad (x,t) \in \Sigma \times (0,\infty). \tag{5.1}$$

In the case of the heat equation, a time-invariant level surface is commonly called *stationary isothermic surface*. We list two results concerning stationary isothermic surfaces, which are relevant for the analysis carried out in this chapter. In [35, Theorem 1.1], for a bounded domain Ω that satisfies the exterior sphere condition, if (5.1) holds for $\Sigma = \partial D$, where D satisfies the interior cone condition and $\overline{D} \subset \Omega$, then Ω must be a ball. See also [17] and [42]. In [38], the case of domains with non-compact boundary is considered. In fact in [38, Theorem 3.4], the authors have shown that if Γ is the graph of function defined on the whole $\mathbb{R}^{N-1}$, satisfying certain sufficient assumptions, and (5.1) holds, then Γ must be a hyperplane. See also [39, 40, 53].

In Section 5.2, by employing the method of moving planes (see [55]) as in [17] and [42], we give a proper version of [35, Theorem 1.1] in the case $p \in (1,\infty)$. See Corollary 5.5. This corollary is actually a consequence of a quite more general theorem in which one obtains the spherical symmetry under the weaker condition that there exist $\overline{t} > 0$ and $R > 0$ such that

$$x \mapsto u(x,\overline{t}) \quad \text{is constant on} \quad \Gamma_R,$$

where $\Gamma_R = \{x \in \Omega : d_\Gamma(x) = R\}$. See Theorem 5.3.

In the case (6)-(7) we consider surfaces Σ that are level sets of u^ε, for any $\varepsilon > 0$, i.e. Σ satisfies the requirement:

$$u^\varepsilon \quad \text{is constant on} \quad \Sigma, \quad \text{for any } \varepsilon > 0. \tag{5.2}$$

We point out that, in the linear case, from (8) it follows that Σ is a stationary isothermic surface if and only if (5.2) holds true.

In Section 5.2, we present results which are analogous to those obtained for $p \neq 2$ in the parabolic case, by just adapting the proofs in the elliptic context.

Section 5.3 contains our result for the case of non-compact boundaries. There, we generalize the result in [38, Theorem 3.4] to a generic $p \in (1, \infty)$ and to the elliptic case.

Section 5.4 contains another type of symmetry result for the solution of (3)-(5). It concerns the following condition for the q-mean $\mu_q(x,t)$ of a ball $B_R(x)$ such that $R = d_\Gamma(x)$. Let Ω be a domain with bounded and connected boundary in which there exists a parallel surface Γ_R, such that

$$x \mapsto \mu_q(x,t) \text{ is constant on } \Gamma_R, \tag{5.3}$$

for any fixed $t > 0$. In the spirit of [37, Theorem 1.2], in Theorem 5.12 we show that if (5.3) holds, then Γ must be a sphere. Theorem 5.13 takes care of the elliptic counterpart of (5.3).

5.1 Parallel surfaces

An interesting geometric property that invariant surfaces enjoy is that they are parallel to Γ, as the following results show for both the parabolic and elliptic case.

Theorem 5.1 ([14, Theorem 3.6])**.** *Let Ω be a domain in $\mathbb{R}^N$ satisfying $\Gamma = \partial\left(\mathbb{R}^N \setminus \overline{\Omega}\right)$ and suppose that, for $1 < p \leq \infty$, u is the solution of* (3)-(5)*.*

If $\Sigma \subset \Omega$ is a time-invariant level surface for u, then there exists $R > 0$ such that

$$d_\Gamma(x) = R \quad \textit{for every} \quad x \in \Sigma. \tag{5.4}$$

Proof. Let $R = \operatorname{dist}(\Sigma, \Gamma)$ and let x_0 be a point in Σ such that $d_\Gamma(x_0) = R$. If $y \in \Gamma$, we have that $u(x_0,t) = u(y,t)$ and hence $4t \log u(x_0,t) = 4t \log u(y,t)$ for every $t > 0$. By Theorem 3.5, we infer that $d_\Gamma(x_0) = d_\Gamma(y)$ and hence we obtain our claim. □

Theorem 5.2. *Let Ω be a domain in $\mathbb{R}^N$ satisfying $\Gamma = \partial\left(\mathbb{R}^N \setminus \overline{\Omega}\right)$ and suppose that, for $1 < p \leq \infty$, u^ε is the solution of* (6)-(7)*.*

If $\Sigma \subset \Omega$ is an level surface of u^ε, for every $\varepsilon > 0$, then there exists $R > 0$ such that (5.4) *holds true.*

Proof. To conclude, it is sufficient to properly modify the proof of Theorem 5.1, having in mind Theorem 3.6. □

5.2 Spherical symmetry for invariant level surfaces

We recall that bounded viscosity solutions of (3) and (6) are of class $C^{1,\beta}_{loc}$, for some $0 < \beta < 1$. See [6, Theorem 2.1] and [5, Theorem 2.1].

In Theorems 5.3 and 5.6, we apply the method of moving planes to a subset D of a bounded domain Ω. The idea is to show that D is mirror symmetric in every direction. We introduce some notations.

Given a direction $\theta \in \mathbb{S}^{N-1}$ and $\lambda \in \mathbb{R}$, let $\pi_\lambda = \{x \in \mathbb{R}^N : \langle x, \theta \rangle = \lambda\}$. For a fixed point $x \in \mathbb{R}^N$, we define x^*, the reflection of x in π_λ, by

$$x^* = x - 2\left(\langle x, \theta \rangle - \lambda\right)\theta.$$

Let $H_\lambda = \{x \in \mathbb{R}^N : \langle x, \theta \rangle > \lambda\}$ so that $\pi_\lambda = \partial H_\lambda$. Also, let $D_\lambda = D \cap H_\lambda$ and D^*_λ its reflection in π_λ. We set $\Lambda = \sup\{\langle x, \theta \rangle : x \in D\}$. Suppose that D is of class C^2. From [22, Theorem 5.7], for $\lambda < \Lambda$ sufficiently close to Λ, we have that $D^*_\lambda \subseteq D$. Here, we observe that, from [17, Lemma 2.8], if $D^*_\lambda \subset D$, then $\Omega^*_\lambda \subseteq \Omega$.

Let λ^* be the number defined by

$$\lambda^* = \inf\{\lambda < \Lambda : D^*_\mu \subseteq D : \quad \text{for any } \lambda < \mu < \Lambda\}.$$

Eventually, one of the following two cases occurs:

(1) the boundary of $D^*_{\lambda^*}$ becomes tangent to that of D, and the set of tangency contains points not belonging to π_{λ^*}.

(2) π_{λ^*} is orthogonal to the boundary of D and of $D^*_{\lambda^*}$, at some point of the intersection.

Theorem 5.3. *Set $p \in (1, \infty)$. Let Ω be a bounded domain of class C^2 and let u be the solution of* (3)-(5). *Suppose that D is a C^2 domain such that $\overline{D} \subset \Omega$, $\partial D = \Gamma_R$, for some $R > 0$, and there exists $\bar{t} > 0$ for which u is constant on $\Gamma_R \times \{\bar{t}\}$.*

Then, D and Ω must be concentric balls.

Proof. Since $\Omega^*_{\lambda^*} \subset \Omega$, we can define the function $u^* : \Omega^*_{\lambda^*} \times (0, \infty) \to \mathbb{R}$, by $u^*(x, t) = u(x^*, t)$, for $x \in \Omega^*_{\lambda^*}$ and $t > 0$. It is easy to see that

$$\begin{cases} u^*_t - \Delta^G_p u^* = 0 & \text{in } \ \Omega^*_{\lambda^*} \times (0, \infty), \\ u^* = 0 & \text{on } \ \Omega^*_{\lambda^*} \times \{0\}, \\ u^* \geq u & \text{on } \ \partial\Omega^*_{\lambda^*} \times (0, \infty). \end{cases}$$

Moreover, by the (weak) comparison principle (see Corollary 1.12), we have that the function $w = u - u^*$ is non-positive on the whole $\overline{\Omega^*_{\lambda^*}} \times (0, \infty)$.

Now, following the proof of [17, Theorem 1.1] (see also [3, Theorem 1] [9, Theorem 1.1]), we can apply the strong maximum principle to w, in a proper sub-cylinder of $\Omega \times (0, \infty)$. Consider $v(x) = u(x, \bar{t})$, for $x \in D$. From Lemma 1.23, we have that v is a non-constant viscosity subsolution of $-\Delta^G_p v = 0$ in D. Moreover, v equals a constant on Γ_R. Thus, by applying the strong maximum principle (see Remark 1.19) and Corollary 1.25, we obtain that

$$\nabla v \neq 0 \quad \text{on} \quad \Gamma_R,$$

where we use that u is differentiable in $\Omega \times (0, \infty)$.

Also, we have that there exist $\delta > 0$ and $t_1, t_2 > 0$, with $t_1 < \bar{t} < t_2$, such that

$$\nabla u \neq 0 \quad \text{in} \quad S_\delta \times (t_1, t_2), \tag{5.5}$$

where $S_\delta = \{y \in \Omega : d_{\Gamma_R}(y) < \delta\}$. Observe that the last set displayed in (5.5) is a neighborhood of Γ_R.

Hence, by using (5.5), we have that $|\nabla u|, |\nabla u^*| > 0$ in $(S_\delta \cap \Omega^*_{\lambda^*}) \times (t_1, t_2)$. Therefore, by a standard procedure, we can infer that w is a (non-positive) solution of the following uniformly parabolic equation with smooth coefficients

$$w_t - \operatorname{tr}\left\{A' \nabla^2 w\right\} - b \cdot \nabla w = 0 \quad \text{in} \quad (S_\delta \cap \Omega^*_{\lambda^*}) \times (t_1, t_2), \tag{5.6}$$

where A' and b are defined, respectively, by the following expressions:

$$A' = \int_0^1 \nabla_X F\left(\sigma \nabla u + (1-\sigma)\nabla u^*, \sigma \nabla^2 u + (1-\sigma)\nabla^2 u^*\right) d\sigma$$

and

$$b = \int_0^1 \nabla_\xi F\left(\sigma \nabla u + (1-\sigma)\nabla u^*, \sigma \nabla^2 u + (1-\sigma)\nabla^2 u^*\right) d\sigma.$$

Here, F is defined in (1.4). Observe that (as shown in [9, Theorem 1.1]) the matrix A' is uniformly elliptic and the coefficients b are bounded in $(S_\delta \cap \Omega^*_{\lambda^*}) \times (t_1, t_2)$.

Applying the classical strong maximum principle to (5.6) (see [52]) yields that either $w \equiv 0$ on $\overline{(S_\delta \cap \Omega^*_{\lambda^*})} \times [t_1, t_2]$ or $w < 0$ in $(S_\delta \cap \Omega^*_{\lambda^*}) \times (t_1, t_2)$.

Now, we conclude as in [17, Theorem 1.1]. Suppose that $D^*_{\lambda^*} \subset D \setminus \overline{D_{\lambda^*}}$. Then $S_\delta \cap \Omega^*_{\lambda^*}$ contains points that are in $\partial D^*_{\lambda^*} \setminus \partial D$. This implies that

$$w < 0 \quad \text{in} \quad (S_\delta \cap \Omega^*_{\lambda^*}) \times (t_1, t_2) \tag{5.7}$$

and that

$$\frac{\partial w}{\partial \theta} > 0 \quad \text{on} \quad (S_\delta \cap \pi_{\lambda^*}) \times (t_1, t_2). \tag{5.8}$$

If (1) occurs, then there exists $y \in \partial D^*_{\lambda^*} \cap \partial D$. This implies that $u(y, \bar{t}) = u^*(y, \bar{t})$ since ∂D is a level surface of $u(\cdot, \bar{t})$ and hence that $w(y, \bar{t}) = 0$ which contradicts (5.7).

If (2) occurs, then ∂D must be orthogonal to π_{λ^*} at some $z \in \partial D$, then $\frac{\partial u}{\partial \theta}(z, \bar{t}) = 0$. Also, we have that $\frac{\partial u^*}{\partial \theta}(z, \bar{t}) = 0$ and then that $\frac{\partial w}{\partial \theta}(z, \bar{t}) = 0$. This contradicts (5.7), since $\frac{\partial w}{\partial \theta}(z, \bar{t}) > 0$, by (5.8).

Hence, D must be symmetric with respect to every direction and hence D must be a ball. Since Ω and D are C^2, then we conclude that Ω must be a ball. □

Remark 5.4. We point out that in Theorem 5.3, we can replace the C^2 regularity of Γ_R with a weaker assumption. Indeed, we can apply the Hopf-Oleinik lemma to infer that $\nabla u \neq 0$ on Γ_R by assuming that Γ_R admits an interior ω-pseudo ball condition, where ω satisfies the assumptions of Lemma 1.20. For example, if we assume $\Gamma_R \in C^{1,\alpha}$, for some $\alpha \in (0, 1)$, then Theorem 5.3 still holds true.

Now, we are ready to obtain the following characterization of balls, as a consequence of Theorem 5.3.

Corollary 5.5. *Set $p \in (1, \infty)$. Let Ω be a bounded domain of class C^2 and u be the solution of* (3)-(5)*. Suppose that D is a C^2 domain such that $\overline{D} \subset \Omega$ and with boundary $\partial D = \Sigma$ satisfying* (5.1)*.*

Then, D and Ω must be concentric balls.

Proof. It is enough to apply Theorem 5.3. Indeed, we just note that, from Theorem 5.1, there exists $R > 0$, such that $\Sigma = \Gamma_R$. □

With some adjustments, we obtain the same conclusions of Theorem 5.3 and Corollary 5.5 in the case of the elliptic problem (6)-(7).

Theorem 5.6. *Set $p \in (1, \infty)$. Let Ω be a bounded domain of class C^2 and u^ε be the solution of* (6)-(7)*. Suppose that D is a C^2 domain such that $\overline{D} \subset \Omega$, $\partial D = \Gamma_R$, for some $R > 0$, and there exists $\overline{\varepsilon} > 0$ for which $u = u^{\overline{\varepsilon}}$ is constant on Γ_R.*

Then, D and Ω must be concentric balls.

Proof. It is well defined the function $u^* : \Omega^*_{\lambda^*} \to \mathbb{R}$, by $u^*(x) = u(x^*)$, for $x \in \Omega^*_{\lambda^*}$. Then, it is an easy check to prove that

$$\begin{cases} u^* - \overline{\varepsilon}^2 \Delta_p^G u^* = 0 & \text{in } \ \Omega^*_{\lambda^*}, \\ u^* \geq u & \text{on } \ \partial\Omega^*_{\lambda^*}. \end{cases}$$

Hence, by applying the comparison principle (see Corollary 1.14), we have that $w = u - u^* \leq 0$ on $\overline{\Omega^*_{\lambda^*}}$.

Now, proceeding as in the proof of Theorem 5.3, by applying Corollary 1.18 and 1.25, we have that $|\nabla u|, |\nabla u^*| > 0$ in S_δ, the set defined in the proof of Theorem 5.3. By using standard elliptic regularity theory, we have that w is a (non-positive) solution of a smooth uniformly elliptic equation

$$w - \overline{\varepsilon}^2 \operatorname{tr}\left\{A' \nabla^2 w\right\} - \overline{\varepsilon}^2 \langle b, \nabla w \rangle = 0 \ \text{ in } \ S_\delta, \tag{5.9}$$

where A' and b have the same structure of those of the proof of Theorem 5.3. Since $w \leq 0$, we can apply to w the classical strong maximum principle (see [52]), which implies that either $w \equiv 0$ on $\overline{S_\delta \cap \Omega^*_{\lambda^*}}$ or $w < 0$ in $S_\delta \cap \Omega^*_{\lambda^*}$. Hence, we conclude as in the proof of Theorem 5.3. □

Corollary 5.7. *Set $p \in (1, \infty)$. Let Ω be a bounded domain of class C^2 and u^ε be the solution of* (6)-(7)*. Suppose that D is a domain of class C^2, such that $\overline{D} \subset \Omega$. Suppose that ∂D is a level surface of u^ε, for any $\varepsilon > 0$.*

Then, D and Ω must be concentric balls.

Proof. We conclude from Theorems 5.2 and 5.6. □

5.3 The case of non-compact boundaries

Let $f : \mathbb{R}^{N-1} \to \mathbb{R}$ be a continuous function. In this Subsection, we consider domains of the following form

$$\Omega = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > f(x')\}. \tag{5.10}$$

Theorem 5.8. *Set $p \in (1, \infty)$. Let Ω be defined by* (5.10)*, with f of class C^2. Suppose that there exists a basis $\{\xi^1, \ldots, \xi^{N-1}\} \subset \mathbb{R}^{N-1}$ such that for every $j = 1, \ldots, N-1$, the function $f(x' + \xi^j) - f(x')$ has either a maximum or a minimum in $\mathbb{R}^{N-1}$. Let u be the bounded solution of* (3)-(5)*. Suppose that there exists $R > 0$ such that Γ_R is of class C^2 and $\bar{t} > 0$ for which one of the following occurs:*

(i) $u(\cdot, \bar{t})$ is constant on Γ_R.

(ii) for some $q \in [1, \infty)$, the function $x \mapsto \mu_q(x, \bar{t})$ is constant on Γ_R.

Then, f is affine and so Γ is a hyperplane.

Proof. From the assumption on f, for a fixed $j = 1, \ldots, N-1$, the function of x' defined by $f(x'+\xi^j)-f(x')$ has either a maximum or a minimum in $\mathbb{R}^{N-1}$. Say $f(x'+\xi^j)-f(x')$ has a maximum M in $\mathbb{R}^{N-1}$. Then there exists $z' \in \mathbb{R}^{N-1}$ such that

$$f(x' + \xi^j) - f(x') \le M = f(z' + \xi^j) - f(z') \text{ for any } x' \in \mathbb{R}^{N-1}.$$

We apply the sliding method, a variant of the method of moving planes, introduced in [12]. Here, we adapt the proof given with that method in [53, Theorem 1.1]. Also see [38, 40, 39]. Set

$$\Omega_{\xi^j,M} = \{(x', x_N) \in \mathbb{R}^N : (x' + \xi^j, x_N + M) \in \Omega\}.$$

We have that $\Omega \subseteq \Omega_{\xi^j,M}$ and that $z = (z', f(z')) \in \Gamma \cap (\Gamma)_{\xi^j,M}$. In particular, from the regularity of both Γ and Γ_R we can infer that, if ν_Γ represents the unit outward normal to Γ, then $y = z - \nu_\Gamma(z)R \in \Gamma_R \cap (\Gamma_R)_{\xi^j,M}$, which means that $y + (\xi^j, M) \in \Gamma_R$.

Since $\Omega \subseteq \Omega_{\xi^j,M}$, we can define the function $u^* : \Omega \times (0, \infty) \to \mathbb{R}$, by $u^*(x, t) = u(x + (\xi^j, M), t)$, for $x \in \Omega$ and $t > 0$. It easy to see that

$$\begin{cases} u_t^* - \Delta_p^G u^* = 0 & \text{in } \Omega \times (0, \infty), \\ u^* = 0 & \text{on } \Omega \times \{0\}, \\ u^* \le u & \text{on } \Gamma \times (0, \infty). \end{cases}$$

Thus, from the comparison principle (Corollary 1.12), $u^* \le u$ on $\overline{\Omega} \times (0, \infty)$.

Now, suppose that $\Omega \subset \Omega_{\xi^j,M}$. As we have done in the proof of Theorem 5.3, we show that we can use the strong comparison, in a proper sub-cylinder of $\Omega \times (0, \infty)$. By using Corollaries 1.18 and 1.25 and the continuity of ∇u, there exist $\delta > 0$ and $t_1 < \bar{t} < t_2$ such that

$$\nabla u \neq 0 \text{ in } S_\delta \times (t_1, t_2),$$

where here S_δ is the compact set $B_R(y) \cap (\Gamma_R + B_\delta)$.

Thus, $u^* - u$ is a solution in $S_\delta \times (t_1, t_2)$ of the uniformly parabolic equation with smooth coefficients (5.6) and hence we can apply to $u^* - u$ the strong maximum principle. In particular, since Ω does not coincide with $\Omega_{\xi^j, M}$, we infer that

$$u^* < u \quad \text{in} \quad S_\delta \times (t_1, t_2). \tag{5.11}$$

Denoting with $\mu_q^*(y,t)$ the q-mean of $u^*(\cdot, \bar{t})$ on $B_R(y)$, the last inequality implies that

$$\mu_q^*(y, \bar{t}) < \mu_q(y, \bar{t}), \tag{5.12}$$

since $|\{u^* < u\} \cap B_R(y)| > 0$. Indeed, from (4.8), we have that $\mu^* \le \mu$ and from the fact that the function $u \mapsto |u - \mu^*|^{q-2}(u - \mu^*)$ is strictly increasing, we have that

$$0 = \int_{B_R(y)} |u^*(\zeta, \bar{t}) - \mu^*|^{q-2}(u^*(\zeta, \bar{t}) - \mu^*)\, d\zeta < \int_{B_R(y)} |u(\zeta, \bar{t}) - \mu^*|^{q-2}(u(\zeta, \bar{t}) - \mu^*)\, d\zeta$$

which implies that $\mu^* \neq \mu$.

Now, we prove that if either (i) or (ii) holds, we find a contradiction to (5.11) or (5.12). If (i) holds then we have that $u(y, \bar{t}) = u(y + (\xi^j, M), \bar{t}) = u^*(y, \bar{t})$, which contradicts (5.11) at once. If (ii) holds, then

$$\mu_q(y, \bar{t}) = \mu_q(y + (\xi^j, M), \bar{t}) = \mu_q^*(y, \bar{t}), \tag{5.13}$$

where the first equality is due to (ii) and the latter one is based on the following argument. Employing a change of variable $\zeta = \zeta' + (\xi^j, M)$ yields that

$$\int_{B_R(y + (\xi^j, M))} |u(\zeta, \bar{t}) - \lambda|^{q-2} \left(u(\zeta, \bar{t}) - \lambda\right)\, d\zeta = \int_{B_R(y)} |u^*(\zeta', \bar{t}) - \lambda|^{q-2} \left(u^*(\zeta', \bar{t}) - \lambda\right)\, d\zeta',$$

for any $\lambda \in \mathbb{R}$, which implies the latter equation in (5.13), by the definition of q-means. Hence, we have found again a contradiction. Hence, we must have $\Omega_{\xi^j, M} = \Omega$.

Now, we conclude as in [53, Theorem 1.1]. Indeed, we have that for any $1 \le j \le N-1$, for any $x' \in \mathbb{R}^{N-1}$,

$$f(x' + \xi^j) - f(x') = a_j,$$

for some $a_j \in \mathbb{R}$. The continuity of f, the fact that $\{\xi^1, \dots, \xi^{N-1}\}$ is a basis of $\mathbb{R}^{N-1}$ and an iteration of the sliding method imply that

$$f(x' + y') - f(x') = f(z' + y') - f(z') \quad \text{for any} \quad x', y', z' \in \mathbb{R}^{N-1}.$$

Since f is continuous, solving the latter system of functional equations yields that f must be affine. $\square$

Corollary 5.9. *Set $p \in (1, \infty)$. Let Ω be defined by* (5.10)*, with f of class C^2. Suppose that there exists a basis $\{\xi^1, \dots, \xi^{N-1}\} \subset \mathbb{R}^{N-1}$ such that for every $j = 1, \dots, N-1$, the function $f(x' + \xi^j) - f(x')$ has either a maximum or a minimum in $\mathbb{R}^{N-1}$. Let u be the bounded solution of* (3)-(5)*. Suppose that there exists a surface Σ of class C^2 such that* (5.1) *holds true.*

Then f is affine and Γ is a hyperplane.

Proof. It is sufficient to observe that, from Theorem 5.1, there exists $R > 0$ such that $\Sigma = \Gamma_R$. Hence, we apply Theorem 5.8. □

Also in this case, we can give the corresponding theorem in the case (6)-(7).

Theorem 5.10. *Set $p \in (1, \infty)$. Let Ω be defined by* (5.10)*, with f of class C^2. Suppose that there exists a basis $\{\xi^1, \dots, \xi^{N-1}\} \subset \mathbb{R}^{N-1}$ such that for every $j = 1, \dots, N-1$, the function $f(x' + \xi^j) - f(x')$ has either a maximum or a minimum in $\mathbb{R}^{N-1}$. Let u^ε be the bounded solution of* (6)-(7)*. Suppose that there exists $R > 0$ such that Γ_R is of class C^2 and $\overline{\varepsilon} > 0$ for which one of the following occurs:*

(i) $u^{\overline{\varepsilon}}$ is constant on Γ_R.

(ii) for some $q \in [1, \infty)$, the function $x \mapsto \mu_{q,\overline{\varepsilon}}(x)$ is constant on Γ_R.

Then, f is affine and so Γ is a hyperplane.

Proof. With the same notations as in the proof of Theorem 5.8, we have that, for $j = 1, \dots, N-1$, we can define the function $u^* : \Omega \to \mathbb{R}$, by $u^*(x) = u^{\overline{\varepsilon}}\left(x + (\xi^j, M)\right)$. For the comparison principle and the maximum principle (Corollaries 1.12 and 1.16), we have that $u^* - u^{\overline{\varepsilon}} \le 0$, on $\overline{\Omega}$.

Moreover, there exists $\delta > 0$ such that $|\nabla u^*|, |\nabla u| > 0$ in S_δ. Thus, $u^* - u^{\overline{\varepsilon}}$ satisfies the uniformly elliptic equation with smooth coefficients (5.9) in S_δ.

Supposing that $\Omega \subset \Omega_{\xi^j, M}$ yields that $u^* - u^{\overline{\varepsilon}} < 0$, in S_δ, from the strong maximum principle. Thus, we conclude as in the proof of Theorem 5.8. □

Corollary 5.11. *Set $p \in (1, \infty)$. Let Ω be defined by* (5.10)*, with $f \in C^2$. Suppose that there exists a basis $\{\xi^1, \dots, \xi^{N-1}\} \subset \mathbb{R}^{N-1}$ such that for every $j = 1, \dots, N-1$, the function $f(x' + \xi^j) - f(x')$ has either a maximum or a minimum in $\mathbb{R}^{N-1}$. Let u^ε be the bounded solution of* (6)-(7)*. Suppose that there exists a C^2 surface Σ that is level surface of u^ε, for any $\varepsilon > 0$.*

Then, Γ must be a hyperplane.

Proof. We apply together Theorems 5.10 and 5.2 . □

5.4 Spherical symmetry for q-means-invariant surfaces

In this section we give applications of Theorems 4.7 and 4.11. We give characterizations of spheres, based on q-means, in the spirit of [37, Theorem 1.2]. These results are new, even for the case $p = 2$.

Here, we consider those parallel surfaces Γ_R sufficiently near Γ, such that for every $x \in \Gamma_R$ there exists an unique y_x such that $\overline{B_R(x)} \cap \Gamma = \{y_x\}$, for every y_x. Also, we suppose that $\kappa_1(y_x), \cdots \kappa_{N-1}(y_x) < \frac{1}{R}$.

Theorem 5.12 ([14, Theorem 3.7])**.** *Set $1 < p \leq \infty$ and let Ω be a domain of class C^2 with bounded and connected boundary Γ. Let u be the bounded (viscosity) solution of* (3)-(5)*.*

Suppose that Σ is a C^2-regular surface in Ω, that is a parallel surface to Γ at distance $R > 0$.

If, for some $1 < q < \infty$ and every $t > 0$, the function

$$\Sigma \ni x \mapsto \mu_q(x, t)$$

is constant, then Γ must be a sphere.

Proof. Since Σ is of class C^2 and is parallel to Γ, for every $y \in \Gamma$, there is a unique $x \in \Sigma$ at distance R from y. Thus, owing to Theorem 4.7, we can infer that

$$\Pi_\Gamma = \text{constant on } \Gamma.$$

Our claim then follows from a variant of Alexandrov's Soap Bubble Theorem (see [2]), [37, Theorem 1.2], or [35, Theorem 1.1]. □

The next is the elliptic counterpart to Theorem 5.12. Here, we intend that $\mu_{q,\varepsilon}$ is the q-mean of u^ε on $B_R(x)$.

Theorem 5.13. *Set $1 < p \leq \infty$ and let Ω be a domain of class C^2 with bounded and connected boundary Γ. Let u^ε be the bounded (viscosity) solution of* (6)-(7)*.*

Suppose that Σ is a C^2-regular surface in Ω, that is a parallel surface to Γ at distance $R > 0$.

If, for some $1 < q < \infty$ and every $\varepsilon > 0$, the function

$$\Sigma \ni x \mapsto \mu_{q,\varepsilon}(x)$$

is constant, then Γ must be a sphere.

Proof. The proof runs similarly to that of Theorem 5.12, once we replace Theorem 4.7 by Theorem 4.11. □

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