

**ON THE MACROSCOPIC LIMIT
OF QUANTUM SYSTEMS**

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UNIVERSITÀ
DEGLI STUDI
FIRENZE

**DOTTORATO DI RICERCA IN
FISICA E ASTRONOMIA**

CICLO XXXI

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*On the macroscopic limit of
quantum systems*

Settore Scientifico Disciplinare: FIS/02

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Anni 2015/2018

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List of Publications

Part of the research described in this thesis has been presented in the following journal papers. The content of Chap. 3, already on the arXiv, is the subject of the third reference, i.e., the *submitted* one. Chap. 4 and Chap. 5 are based on the preliminary analyses reported in the following Bachelor Theses, to which the author participated.

- **Chapter 2**

- “Quantum dynamics of a macroscopic magnet operating as an environment of a mechanical oscillator”, C. Foti, A. Cuccoli, and P. Verrucchi, *Phys. Rev. A* 94, 062127 (2016);
- “Effective description of the short-time dynamics in open quantum systems”, M. Rossi, C. Foti, A. Cuccoli, J. Trapani, P. Verrucchi, and M. Paris, *Phys. Rev. A* 96, 032116 (2017).

- **Chapter 3**

- “Whenever a quantum environment emerges as a classical system, it behaves like a measuring apparatus”, C. Foti, T. Heinosaari, S. Maniscalco, P. Verrucchi, arXiv:1810.10261; *submitted*.

- **Chapter 4**

- “Time & clocks”, Bachelor Thesis by Giulio Barni, a.y. 2016/2017; paper *in preparation*.

- **Chapter 5**

- “When the Maxwell’s dæmon stares at an event horizon”, Bachelor Thesis by Nicola Pranzini, a.y. 2015/2016.

Other paper not reported in this thesis to which the author contributed:

- “A dynamical model for Positive-Operator-Valued Measures”, A. De Pasquale, C. Foti, A. Cuccoli, V. Giovannetti, P. Verrucchi; *to be submitted*.

List of Symbols

\mathcal{Q}	Quantum Theory
\mathcal{H}	Hilbert space
\hat{H}	Hamiltonian
$O(\Lambda) \equiv \langle \Lambda \hat{O} \Lambda \rangle$	Symbols
\mathcal{C}	Classical Theory
\mathcal{C}	Phase space
h	Classical Hamiltonian
$\mathbb{X}(N)$	Global symmetry
\hat{U}	Symmetry transformations
\mathfrak{g}	(Lie) Algebra
\mathcal{G}	Dynamical group
\mathcal{F}	Maximum stability subgroup
\mathcal{G}/\mathcal{F}	Coset space
\mathcal{M}	Differentiable manifold
$ R\rangle$	Reference state
$ \Lambda\rangle, \eta\rangle$	Generalized coherent states
$\{\hat{H}_i, \hat{E}_\alpha, \hat{E}_{-\alpha}\}$	Cartan basis
$d\mu(\hat{\Lambda})$	Invariant measure on \mathcal{G}/\mathcal{F}
$d\mu(\Lambda)$	Invariant measure on \mathcal{M}
K	Set of classical operators
k	Quanticity parameter
\mathcal{Q}_k	Effective \mathcal{Q} theory
\mathcal{Q}_N	Many-body \mathcal{Q} theory
$ \alpha\rangle$	Field coherent states
\mathfrak{C}	Complex plane
$ \Omega\rangle$	Spin coherent states
S^2	2-sphere
\mathcal{E}	Dynamical map
M	Measure
\hat{M}	Measurement operator
r_S	Schwarzschild radius

Introduction

Since its first developments in the Twenties of the past century, Quantum Mechanics (QM) has established itself as a fundamental physical theory, although at the same time it has begun to raise a large amount of questions since its very early appearances. The issue of its interpretation is as old as the theory itself, and it is still subject of intense debate among scientific community members. Apart from the intriguing and neverending discussions about its ultimate significance in describing reality, QM is a tremendously successful theory: whenever it predicts something it seems to work and it has been verified experimentally to an extremely high degree of accuracy. Quantum theory has hereinbefore a history of about a hundred years, but it has become very active recently our ability to manipulate matter in situations where QM is important, not just at the atomic scale, but also at some large scale. The technological progress has made necessary a deeper comprehension of how we, macroscopic objects, can establish a dialogue with the quantum world of the microscopic ones, in order to interact and exert control upon small quantum devices. In fact, the analysis of the quantum evolution of a system interacting with its macroscopic environment has made clear that it is indeed in the peculiarities of one such evolution that the most debated issues in the theory of QM might find their solution. This is the case, in particular, of what is usually dubbed as the *emergence of classicality*, i.e., the mechanism that makes us to observe a classical reality, despite the fundamental laws of physics being quantum mechanical. Indeed, the passage from a *quantum and coherent* world to the *classical non-coherent* one which people experience in everyday life remains obscure, though great effort have been done in this sense.

Besides these conceptual difficulties, every system - apart from the whole Universe - is never completely isolated, since the very same possibility of texting its existence relies on the fact that it interacts with some *external world*, i.e., with some environment. Given an “all-along” isolated system $S = A + B$ its *exact* description is provided by the Open Quantum Systems formalism, which considers a purely quantum physical system interacting with its equally *quantum* environment. The compound system S is in a pure state described by a ket $|S\rangle$, while the notion of state for each subsystem can be recovered through the reduced density matrix approach, $\varrho_A = \text{Tr}_B [|S\rangle\langle S|]$ and $\varrho_B = \text{Tr}_A [|S\rangle\langle S|]$ respectively. It is indeed in such axiomatically exact representation that the very peculiar property of QM can be described, i.e., the entanglement can be defined. To tackle the problems related to the difficulties in managing the many variables by which the environment is usually represented,

people often exploit *effective* descriptions where the physical quantum system is described by a local Hamiltonian $\hat{H}_A(t; \{q_i\})$, acting only on its own Hilbert space and depending on external possibly time-dependent parameters $(t; \{q_i\})$, whose presence testifies the existence of a surrounding environment. In such description the quantum system is usually referred to as “closed”; its state is described by a ket $|\Phi_A(t; \{q_i\})\rangle$, and at the heart of this approach stands the approximation that the environment is *classical*, so that the operators acting on its Hilbert space are replaced by some field $f(t; \{q_i\})$. Actually some authors dub as “closed”, quantum systems which interact with a *classical* environment, but are in a mixed state. The most typical example is the so called “thermal state” describing a quantum system put into a thermal bath, acting indeed as its *classical* environment; in this case the closed system is described by a density matrix $\varrho(\beta)$ depending on the external parameter $\beta = 1/T$ with T the temperature of the bath. However, what takes place in this formalism are the geometrical effects of the parameters $(t; \{q_i\}), \beta$ indeed testifying the environmental existence. Quoting from the famous work by W. H. Zurek [1], “*The idea that “openness” of quantum systems may have anything to do with the transition from quantum to classical was resolutely ignored for a very long time, probably because in classical physics problems of fundamental importance were always settled in isolated systems, [...] and the understanding of how the environment distills the classical essence from quantum systems is more recent.*” In fact, from the above brief comparison between the exact whole quantum world and the effective quantum-classical one, it emerges that profound differences appear according to the description one decides to assume for the environment, being it *quantum* or *classical*, and, on the other hand, a proper connection between the two approaches is still missing.

In what follows, we focus indeed on the role taken on by the environment, always considering quantum environments which potentially exhibit an emergent classical behaviour in some limit. Having in mind that the fundamental interactions ruling the world are quantum and that classical mechanics is just an approximation of QM related to the human experience, words as “quantization” have no sense, as far as this thesis is conceived.

Both the environment and the open system can either be made of a unique system, or by several subsystems respectively. In the latter case, one can for example analyse both the effects of the environment on the properties of the open system, and the role played by the environment in determining the behaviour of each subsystem; whether the interaction between the open system and the environment be relevant or not depends on the setting and on the specific focus of the analysis one aims to develop. Notice, however, that by adopting the terms “open quantum system” and “environment”, an asymmetry between the twos is suggested. As a matter of fact, the former is usually referred to as “principal system”, whereas the environment is often left in the back, and considered as something which only complicates the analysis, to which get rid off as soon as it is possible. Nevertheless, beyond the already mentioned fundamental role the environment has in determining the different possible representations of the open and closed systems, it is today clear that all those situations where we want to understand the way quantum systems can be used and

controlled for conveying or extracting information, require a thorough analysis of how they interact with their, no matter how *big*, environment. Therefore, the essential role of the environment is completely restored, since its analysis matches the importance of the description of the principal system itself. In this context, there is no point in dubbing the open system the “principal” system, if not that of aligning ourselves with the common convention, as indeed we will do in this thesis as well.

The thesis is structured as follows:

- in **Chapter 1** we gather all the formal aspects and mathematical tools pertaining to the development of the approach we will use to define a proper classical limit of quantum theories and to combine quantum and classical formalism. In particular, we discuss the difference between *macroscopicity* and *classicality*, and design the minimal theoretical structure to move towards a classical limit. The fundamental role played by the *generalized coherent states* is introduced and their fundamental properties outlined. We then describe the *parametric representation with environmental coherent states*, and its use in the analysis of open quantum systems.
- **Chapter 2** is devoted to study hybrid schemes, which represent those particular situations where it is important to keep trace of the quanticity of the macroscopic, potentially classical, environment so that it is possible to develop a sensible description of the interaction with its quantum companion. We initially aim to understand 1) if, 2) to what extent, and possibly 3) how the evolution of the former testifies to the coupling with the latter. To this purpose we consider a magnetic environment made of a large number of spin-1/2 particles, coupled with a quantum mechanical oscillator, and we focus on the analysis of the so called *back-action*, i.e., the effects of the presence of the principal system on the way its environment evolves. We then address the dynamics of a bosonic system coupled to either a bosonic or a magnetic environment and derive a set of sufficient conditions that allow to describe the dynamics in terms of the effective interaction with a classical fluctuating field.
- **Chapter 3** aims to study the case of an environment acting effectively as a measuring apparatus. We consider a quantum principal system, with an environment made of N elementary quantum components. We demonstrate that whenever this behaves according to an effective classical theory such theory emerges from the quantum description of a measuring apparatus in the large- N limit, regardless of the actual interaction between the environment and the principal system.
- In **Chapter 4** we investigate if it is possible for an environment to act as a clock for its quantum companion. We deal therefore with the issue of properly defining time in quantum theory, seeking for a possible explanation of how it always emerges in the equations of motions as a classical parameter. Presuming that

what we sense as time is due to the internal interactions of the different subsystems of the Universe, we show that it is possible to obtain a von Neumann-like equation for any quantum system, once the classical limit of its environment is implemented.

- **Chapter 5** recapitulates the purposes of the thesis and sketches some further possible goals. In particular, having in mind to enlarge the environment as much as it is possible, applying the approach used through this thesis to (quantum?) gravity, we start from the Black Hole Information paradox, resulting indeed in the attempt of combining QM and General Relativity. We present an information theoretic model to deal with it, focusing on the role which has an external observer related with the quantum measurement process.
- Finally, in the **Appendices**, we present an algebraic algorithm to construct the generalized coherent states starting from the knowledge of the group associated to a quantum system, and discuss in detail their properties. Moreover, we provide the formal definition of the quantum measure and of the minimal interpretation of the QM measurement postulate.

Chapter 1

What A Wonderful (Quantum) World

Beyond the perception that we can have in our everyday life, our World is a quantum World. Real physical systems living in it are quantum systems, which continuously get in touch with each other. Open quantum systems (OQS) are non-isolated quantum systems whose description takes into account the interaction with quantum surroundings. Analysing such systems takes a significant role, since any physical realisation of a quantum system requests at least the presence of an environment which verifies its existence. However, although the modeling of any OQS inherently implies the one of its surroundings, knowing the quantum structure of the total Hamiltonian, including the details of the couplings between the principal system and its environment, does not usually suffice to develop a simple and meaningful model of the overall system. This is generally due to the environment being made of a very large number N of quantum components, a fact that indeed we will hereafter take as integrant to the definition itself of *environment*.

In this chapter we provide the fundamental tools employed in this thesis. We will deal with finding proper methods to move in a sort of *twilight zone* of a (quantum) environment, that, becoming macroscopic, can potentially exhibit a classical behaviour. In Sec. 1.1 macroscopic quantum systems are introduced, and the difference between *macroscopicity* and *classicality* is discussed. In Sec. 1.2 we present the general formalism, reviewing some fundamental works that have particularly inspired us, as far as the moving towards a classical limit of a quantum theory is concerned. A very special role is played by generalized coherent states, since they have some peculiar properties which allow us to formally define the classical limit, and Sec. 1.3 highlights indeed which are their major features used in this thesis. Sec. 1.4 is then devoted to present a theoretical method specifically designed for tackling OQS with macroscopic environments, and, lastly, the chapter ends with some comments, reported in Sec. 1.5, about the structure previously designed in Sec. 1.2.

1.1 Macroscopic quantum systems

A physical system is macroscopic when it is made of a large number of components. Since each component of any physical system is inherently quantum, we can also state this: a macroscopic system is a *quantum* system with many degrees of freedom. Whether the quantum properties of such a system manifest themselves or not, it depends on the specific case one is considering. As a first intuition, one can think that a *macroscopic* system displays a distinctly quantum behaviour when the Hilbert space effectively explored by the system during its evolution is finite. To make this sentence more explicit, let us consider the paradigmatic case of a macroscopic magnetic system, i.e., a spin-ring made of N particles of spin- $\frac{1}{2}$, each described by its Pauli matrices $\hat{\sigma}_i \equiv (\hat{\sigma}_i^x, \hat{\sigma}_i^y, \hat{\sigma}_i^z)$. Let $\hat{S} \equiv \frac{1}{2} \sum_i^N \hat{\sigma}_i$ be the total spin of the ring and $|\hat{S}|^2 = S(S+1)$, with S ranging from 0 to $\frac{N}{2}$ if N is even, or from $\frac{1}{2}$ to $\frac{N}{2}$ if N is odd. When $|\hat{S}|^2$ commutes with the system Hamiltonian, the total spin quantum number S is a conserved quantity, as in Fig. 1.1. The two extremal cases, of min-

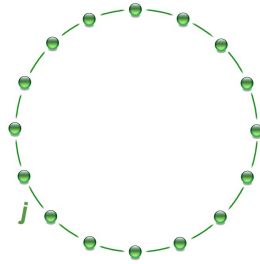


Figure 1.1: Graphical representation of a magnetic system made of distinguishable particles, distributed on a ring-shaped lattice (referred to as a “spin-ring” in the text). The total spin S is constant if, for instance, the spins are coupled amongst themselves via a homogeneous, isotropic (or Heisenberg) nearest-neighbour interaction, $j \sum_i \hat{\sigma}_i \cdot \hat{\sigma}_{i+1}$.

imum and maximum S , are not equivalent, as a profound difference can emerge in the behaviour of the system depending indeed on the value of S . While in the first case the spin-ring inherently behaves as a quantum system no matter how large N is, in the latter case it can behave as a classical system if N becomes large. In fact, the portion of the Hilbert space effectively explored by a magnetic system has dimension $2S+1$, and hence a classical-like dynamics is expected when $S \sim N \rightarrow \infty$. As disclosed above, it can be thus qualitatively understood that what actually matters, as far as the wiping of quantum features is concerned, it is the dimension of the portion of Hilbert space effectively explored by the state of the system during its evolution: if this dimension is large, a classical behaviour can emerge.

Nevertheless, the solution based on the clear statement that microscopic elements obey quantum rules while macroscopic objects follow the classical ones, it is by now unsatisfactory. We have learnt that macroscopic objects may well exhibit a distinctive quantum behaviour, and many are the examples theorized and discovered in the

last decades: in 1957 Bardeen, Cooper and Schrieffer assembled their BCS theory of superconductivity [2], in the 80s the topological order was understood [3], the condensation firstly predicted by Bose and Einstein in 1924-1925 [4, 5] was observed in the 90s when the first gaseous condensates were produced in the lab, and, in more recent years, magnetic properties of large molecules with spin $S = 1/2$ have been observed [6], as well as quantum effects in biology have started to be studied [7]. In any case, it is nowadays clear that the large- N condition is not sufficient for a system made of N quantum particles to behave classically.

As a matter of fact, there are assumptions which isolate the minimal structure any quantum theory should possess if it is to have a classical limit, as we will examine in depth in the next section. Although they have been variously expressed depending on the approach adopted by different authors, these assumptions imply precise physical constraints on the original quantum theory that describes a macroscopic quantum system if this has to behave classically.

1.2 Minimal structure towards a classical limit

Before introducing the general approach we are going to adopt, some useful notions should be recalled. A quantum description of a physical system, or a *quantum theory* \mathcal{Q} for short, is based on the introduction of (i) a Hilbert space \mathcal{H} ¹, (ii) a Lie product $[\cdot, \cdot]$, defining the commutation rules between the operators on \mathcal{H} , and (iii) a Hamiltonian \hat{H} . Trace class operators on \mathcal{H} representing physical observables usually make up a vector space; this space, together with the above Lie product, is the algebra \mathfrak{g} of the theory. The expectation values of Hermitian operators \hat{O} acting on \mathcal{H} , $O(\Lambda) \equiv \langle \Lambda | \hat{O} | \Lambda \rangle \in \mathbb{R}$, are the (only) physical outputs of the theory, i.e., the experimentally accessible properties of the system. On the other hand, a classical description of a physical system, or a *classical theory* \mathcal{C} ² for short, is defined by (i) a phase space \mathcal{C} , (ii) a Poisson bracket $\{\cdot, \cdot\}$, and (iii) a Hamiltonian $h(\zeta)$, with ζ representing the set of conjugate variables of the classical phase space \mathcal{C} . Real functions defined on \mathcal{C} are the (only) physical outputs of the theory, in the same sense as above.

As mentioned in the previous section, a physical system made by a large number N of quantum elements does not necessarily obey to the rules of classical physics, and the problem of whether or not a system made by quantum particles can be described by a classical theory has been extensively studied in the last decades of the last century, especially in the context of quantum-field theory (see for instance the thorough discussion on the relation between large- N limits and classical theories developed in Sec.VII of Ref. [8]). In every physical theory known to have a sensible large- N limit, the expectation values of any product $\hat{A}\hat{B}$ of physical observables on physical states satisfy the factorization relation, i.e., $\langle \hat{A}\hat{B} \rangle = \langle \hat{A} \rangle \langle \hat{B} \rangle + \mathcal{O}(1/N)$,

¹The Hilbert space of a quantum theory is *separable*, meaning that its dimension is finite or infinitely countable, and, hence, \mathcal{H} is spanned by a set of states whose cardinality is that of the integer numbers.

²More accurately, this is the definition of a *Hamiltonian classical theory*, but not all classical theories are Hamiltonian. Anyway, this kind of theories will be the only one considered in this thesis.

so that quantum fluctuations become irrelevant in the classical limit. Apart from the original framework to which many works were devoted, among them, we have been particularly inspired by the fundamental work by L. G. Yaffe, included in the review “Large- N limit as classical mechanics” [8]. This paper, in turn, generalizes previous results by other authors, amongst which we particularly mention that of E. H. Lieb “The classical limit of quantum spin systems” [9], specifically referring to spin systems indeed. In his work of 1982, Yaffe aims at finding a bridge from an arbitrary quantum theory \mathcal{Q} to a classical theory \mathcal{C} , starting from the question “Can one find a classical system whose dynamics is equivalent to the $N \rightarrow \infty$ limit of a given quantum theory?”, that is the same to ask if it be possible to find classical mechanics as the large- N limit of a given quantum theory. The answer, shown by many authors, is that a general scheme can be built if the quantum theory satisfies a small set of assumptions, which to be verified needs some essential ingredients.

First of all, the quantum theory must exhibit a *global symmetry* $\mathbb{X}(N)$, i.e., a symmetry whose transformations do not leave invariant subspaces in the Hilbert space \mathcal{H} of the corresponding system. If, for instance, \mathcal{Q} describes N spin-1/2 particles interacting via an isotropic Heisenberg-like magnetic exchange, as in the example of the spin-ring in Sec. 1.1, one such symmetry can be the one defined by operators that rotate the spin of each particle of the same angle. If one takes instead N particles whose interaction depends only on their distance, the symmetry might be defined by the same spatial translation of each particle. As for non-interacting, identical, but yet distinguishable particles, a possible global symmetry might be defined by the permutation operators. In any case the global symmetry is a key feature, since it is the ultimate responsible of the drastic reduction of the physical observables of the system analysed, and, thus, of the resulting treatment of the many N components of a macroscopic object as a whole.

What is then essential in order that a classical theory \mathcal{C} surface from \mathcal{Q} , is the possibility of defining the set of *Generalized Coherent States* (GCS), which are, in fact, the only states for which a proper classical limit can be formally defined, as they naturally provide the identification of a classical phase space \mathcal{C} . It is indeed through the set of GCS that the conditions, needed to find the classical limit \mathcal{C} from \mathcal{Q} , emerge. In extreme short, their construction goes as follows (see appendix A for explicit examples of the following procedure). Associated to any quantum system, depicted by a quantum theory \mathcal{Q} , there is a Hilbert space \mathcal{H} and a *dynamical group* \mathcal{G} , i.e., the group containing all the propagators that describe possible evolutions of the system (quite equivalently, \mathcal{G} is the group corresponding to the Lie-algebra \mathfrak{g} to which all the physical Hamiltonians of the system belong). The arbitrary choice of a reference state $|R\rangle \in \mathcal{H}$ defines the subgroup \mathcal{F} of the operators \hat{f} acting trivially on $|R\rangle$, i.e., such that $\hat{f}|R\rangle = e^{i\phi_f}|R\rangle$. \mathcal{F} is usually referred to as the *maximum stability subgroup*, and taking the coset of \mathcal{G} over \mathcal{F} , we get the coset space \mathcal{G}/\mathcal{F} . Given these three inputs, \mathcal{H} , \mathcal{G} and $|R\rangle$, the GCS can be constructed, according for instance to the procedure described in Ref. [10] or [11]. By acting on the reference state with operators $\hat{\Lambda}$ belonging to \mathcal{G}/\mathcal{F} , we get

$$|\Lambda\rangle \equiv \hat{\Lambda}|R\rangle, \quad (1.1)$$

that is *the general group definition of the GCS*. The operators $\hat{\Lambda} \in \mathcal{G}/\mathcal{F}$ are the elements of \mathcal{G} that do not belong to the maximum stability subgroup \mathcal{F} , and are called *displacement operators*. A useful representation of the algebra \mathfrak{g} is provided by the Cartan decomposition $\{\hat{H}_i, \hat{E}_\alpha, \hat{E}_{-\alpha}\}$, according to

$$\begin{aligned} [\hat{H}_i, \hat{H}_j] &= 0 & , & & [\hat{H}_i, \hat{E}_\alpha] &= \alpha_i \hat{E}_\alpha , \\ [\hat{E}_\alpha, \hat{E}_{-\alpha}] &= \alpha_i \hat{H}_i & , & & [\hat{E}_\alpha, \hat{E}_\beta] &= c_{\alpha\beta} \hat{E}_{\alpha+\beta} . \end{aligned} \quad (1.2)$$

In every irreducible representation, it is possible to choose the operators \hat{H}_i diagonal and Hermitian, i.e., $\hat{H}_i^\dagger = \hat{H}_i$, while the representatives of $(\hat{E}_\alpha, \hat{E}_{-\alpha})$ become shift-up and shift-down operators such that $\hat{E}_\alpha^\dagger = \hat{E}_{-\alpha}$. The diagonal operators have the reference state among their eigenvectors, whereas the shift ones appear in the displacement operators, $\hat{\Lambda}$, reading

$$\hat{\Lambda} = \exp \left(\sum_{\beta} \Lambda_{\beta} \hat{E}_{\beta} - \Lambda_{\beta}^* \hat{E}_{\beta}^{\dagger} \right) , \quad (1.3)$$

where the parameters Λ_{β} are complex numbers that can be recognized as the coordinates of a point Λ on a differentiable manifold \mathcal{M} . As a matter of fact, the definition (1.1) guarantees that the GCS be in one-to-one correspondence with the operators $\hat{\Lambda} \in \mathcal{G}/\mathcal{F}$. Moreover, as the quotient space \mathcal{G}/\mathcal{F} can be associated to a differentiable manifold \mathcal{M} , their construction actually establishes also a one-to-one correspondence between each coherent state $|\Lambda\rangle$ and a point Λ on \mathcal{M} , or, which is the same, between any such point and a displacement operator $\hat{\Lambda} \in \mathcal{G}/\mathcal{F}$. Therefore, the GCS preserve all the algebraic and topological properties of \mathcal{G}/\mathcal{F} and, hence, of \mathcal{M} . It is indeed the manifold \mathcal{M} that readily provides us with the classical phase space \mathcal{C} needed to define C , when the bridge between the theories \mathcal{Q} and C is in the running. A measure $d\mu(\hat{\Lambda})$ on \mathcal{G}/\mathcal{F} is also provided, and a resolution of the identity on \mathcal{H} is available in the form

$$\int_{\mathcal{G}/\mathcal{F}} d\mu(\hat{\Lambda}) |\Lambda\rangle \langle \Lambda| = \hat{\mathbb{I}}_{\mathcal{H}} , \quad (1.4)$$

meaning that GCS are an overcomplete set on \mathcal{H} , where the prefix “over” is due to their being non-orthogonal. Neither \mathcal{M} nor $d\mu(\hat{\Lambda})$ depend on the choice of the reference state, as choosing a different one is tantamount to changing the origin of the reference frame used to identify each point on \mathcal{M} by a set of coordinates. Moreover, $d\mu(\hat{\Lambda})$ is invariant under the action of any element \hat{g} of \mathcal{G} , i.e., $d\mu(\hat{\Lambda}) = d\mu(\hat{g}\hat{\Lambda})$, and, through the one-to-one correspondence between $|\Lambda\rangle$ and $\Lambda \in \mathcal{M}$, a measure $d\mu(\Lambda)$ on \mathcal{M} is consistently associated to the above introduced $d\mu(\hat{\Lambda})$, so that requiring GCS to be normalized, i.e., $\langle \Lambda|\Lambda\rangle = 1$, implies

$$\langle \Lambda|\Lambda\rangle = \langle \Lambda| \int_{\mathcal{G}/\mathcal{F}} d\mu(\hat{\Lambda}') |\Lambda'\rangle \langle \Lambda'|\Lambda\rangle = \int_{\mathcal{M}} d\mu(\Lambda') |\langle \Lambda'|\Lambda\rangle|^2 = 1 . \quad (1.5)$$

Once these ingredients are known, we are ready to identify which is the set of assumptions that we are looking for.

Let us consider a quantum theory \mathcal{Q}_k , characterized by some parameter k , assumed to take positive values including the limiting $k = 0$ one. The $k \rightarrow 0$ limit of \mathcal{Q}_k is actually the one which we are interested in, being k the real positive number, referred to as the *quanticity parameter*, such that all the commutators of the theory (or anticommutators, in the case of a fermionic one) vanish with it. Be \mathcal{G}_k and \mathfrak{g}_k a representation of the dynamical Lie group \mathcal{G} and of its associated Lie algebra \mathfrak{g} defining the quantum theory³. There exists a minimal set of conditions, emerging in terms of GCS built from the dynamical group \mathcal{G}_k ⁴, that \mathcal{Q}_k must fulfill to guarantee that its $k \rightarrow 0$ limit be a classical theory C .

Assumption 1 There must be no non trivial subspaces of \mathcal{H}_k left invariant under the action of all elements of \mathcal{G}_k , i.e., each representation \mathcal{G}_k of the dynamical group \mathcal{G} acts irreducibly on the corresponding Hilbert space.

The irreducibility of \mathcal{G}_k automatically provides us with the completeness relation

$$c_k \int_{\mathcal{G}/\mathcal{F}} dm(\hat{\Lambda}) |\Lambda\rangle \langle \Lambda| = \hat{\mathbb{I}}_{\mathcal{H}_k}, \quad (1.6)$$

where the constant c_k depends on the normalization of the group measure and must be computed explicitly. Notice that, looking at Eq. (1.4), the following relation holds

$$d\mu(\hat{\Lambda}) \equiv c_k dm(\hat{\Lambda}), \quad (1.7)$$

with the footprint of the quanticity of the theory contained in c_k through k .

For any operator \hat{A}_k acting on \mathcal{H}_k , let us now define its *symbol* $A_k(\Lambda)$, by

$$A_k(\Lambda) = \langle \Lambda | \hat{A}_k | \Lambda \rangle, \quad (1.8)$$

that, in words, means that a symbol of an operator is the set of GCS expectation values.

Assumption 2 Zero $\hat{\emptyset}_k$ is the only operator whose symbol identically vanishes, i.e., s.t.

$$\langle \Lambda | \hat{\emptyset}_k | \Lambda \rangle = 0.$$

This assumption implies that two different operators cannot have the same symbol, otherwise their difference would violate it. Therefore, any operator may be uniquely recovered by its symbol, which in turn implies that it is sufficient to study the behaviour of the symbols of the various operators to characterize the theory completely.

³Notice that the abstract group \mathcal{G} and its algebra \mathfrak{g} do not depend on k , which instead enters \mathcal{G}_k and its algebra \mathfrak{g}_k , because of the k dependence of the Hilbert space \mathcal{H}_k .

⁴Sometimes \mathcal{G}_k is dubbed *coherence group*, as for example in Ref. [8]. Nevertheless, whenever the Hamiltonian \hat{H}_k can be written as a linear combination of generators of a Lie algebra, which is the case in all quantum theories will be interested in, there is a one-to-one correspondence between the elements of the coherence group and those of the dynamical group, used above according to Ref. [10] or [11], as a starting point for the construction of the GCS.

Clearly, an arbitrary operator does not need to have a sensible limit when k goes to 0. We thus focus our attention on a restricted class K of operators, which can be identified requiring that their $k \rightarrow 0$ limits be well defined, yielding

$$\lim_{k \rightarrow 0} \frac{\langle \Lambda | \hat{A}_k | \Lambda' \rangle}{\langle \Lambda | \Lambda' \rangle} < \infty \quad \forall \hat{\Lambda}, \hat{\Lambda}' \in \mathcal{G}_k / \mathcal{F}_k. \quad (1.9)$$

Such operators are named *classical operators*, and, in relation, two coherent states $|\Lambda\rangle$ and $|\Lambda'\rangle$ are said to be *classically equivalent* if

$$\lim_{k \rightarrow 0} A_k(\Lambda) = \lim_{k \rightarrow 0} A_k(\Lambda'), \quad \forall \hat{A}_k \in K. \quad (1.10)$$

Hereafter, we will use the symbol \sim to indicate the *classical equivalence*, writing $\Lambda \sim \Lambda'$ for short.

Assumption 3 Classically inequivalent states become orthogonal when $k \rightarrow 0$.

Specifically, we require that

$$\lim_{k \rightarrow 0} \langle \Lambda | \Lambda' \rangle = e^{-\lim_{k \rightarrow 0} \frac{\Delta(\Lambda, \Lambda')}{k}}, \quad (1.11)$$

where $\lim_{k \rightarrow 0} \Delta(\Lambda, \Lambda')$ exists $\forall \hat{\Lambda}, \hat{\Lambda}' \in \mathcal{G}_k / \mathcal{F}_k$ and

$$\begin{aligned} i) \quad & \Re \Delta(\Lambda, \Lambda') > 0 \quad \text{if } \Lambda \not\sim \Lambda', \\ ii) \quad & \Re \Delta(\Lambda, \Lambda') = 0 \quad \text{if } \Lambda \sim \Lambda'. \end{aligned} \quad (1.12)$$

This shows that if $\Lambda \not\sim \Lambda'$, their overlap $\langle \Lambda | \Lambda' \rangle$ decreases exponentially in the small- k limit. From the above property it follows⁵

$$\lim_{k \rightarrow 0} \frac{1}{k} |\langle \Lambda | \Lambda' \rangle|^2 = \delta(\Lambda - \Lambda'), \quad (1.13)$$

that is a most relevant properties of GCS, i.e., that they become orthogonal in the classical limit. Thus, for any classical operator $\hat{A}_k \in K$, $\langle \Lambda | \hat{A}_k | \Lambda' \rangle$ becomes highly peaked about $\Lambda \sim \Lambda'$ as $k \rightarrow 0$, otherwise \hat{A}_k will not be a classical operator. Assumption 3 allows us to prove also that the factorization

$$\lim_{k \rightarrow 0} [(AB)_k(\Lambda) - A_k(\Lambda)B_k(\Lambda)] = 0 \quad (1.14)$$

holds for any pair of operators $\hat{A}, \hat{B} \in K$.

Lastly, to control the $k \rightarrow 0$ limit completely, we have to put some restraint on the dynamics, and the fourth assumption places indeed a condition on the quantum Hamiltonian.

Assumption 4 $k\hat{H}_k$ is a classical operator,

⁵We use the Dirac- δ representation $\delta(x - y) = \lim_{\epsilon \rightarrow 0} (1/\epsilon) \exp\{(x - y)^2/\epsilon\}$.

hence ensuring that the coupling constants in the Hamiltonian be scaled in such a way that a sensible dynamics be guaranteed as $k \rightarrow 0$. This restraint could seem artificial, but it is not. In fact, by requiring $k\hat{H}_k \in K$ one avoids that the Hamiltonian diverge or vanish, that is perfectly reasonable if one considers that, experimentally, infinite energies do not exist.

The four assumptions listed above suffice to show that the complete quantum theory reduces to classical mechanics when $k \rightarrow 0$. In particular, it is possible to demonstrate that a classical phase space \mathcal{C} exists. Being the Lie algebra \mathfrak{g} a linear space, its *dual space* \mathfrak{g}^* is well defined, and coadjoint orbits⁶ on \mathfrak{g}^* naturally provide the symplectic structure that we need to consistently define the Poisson brackets on the manifold \mathcal{M} , furnished inherently by GCS. The classical dynamics is then determined by the classical Hamiltonian h , gained through

$$h = \lim_{k \rightarrow 0} k\hat{H}_k. \quad (1.15)$$

We underline that proving any form of necessity for the above assumptions appears to be a very difficult task. Nevertheless, by this formalism, once bridged the $k \rightarrow 0$ limit with the large- N limit from which we started, one can show that for a vast class of theories, including essentially all known theories with sensible large- N limit, the $N \rightarrow \infty$ limit is a classical limit [8, 9, 12].

Let us now go back to the quantum theory \mathcal{Q}_N describing the macroscopic system. Considering its dynamical group \mathcal{G}_N we can construct the GCS $|\Lambda_N\rangle$, which, by definition, provide us with a differentiable manifold \mathcal{M}_N , and, exploiting them, find the classical limit \mathcal{C} of \mathcal{Q}_N . Theoretically the general scheme works, but in practice dealing with the large- N limit of any given theory is really hard, because of the difficulties one encounters in managing a vast amount of variables. Here is where the existence of the global symmetry $\mathbb{X}(N)$ emerges as a necessary ingredient, since it guarantees that the dimensionality of the representation \mathfrak{g}_k be significantly smaller than that of \mathfrak{g}_N . As a matter of fact, the two theories \mathcal{Q}_N and \mathcal{Q}_k are related because their respective Lie algebras, \mathfrak{g}_N and \mathfrak{g}_k , are representations with different dimensionality of the same abstract algebra \mathfrak{g} . The quantum many-body theory \mathcal{Q}_N is the microscopic quantum theory that would exactly describe the macroscopic physical system, were we able to determine the details of its internal interactions. Any such \mathcal{Q}_N defines a \mathcal{Q}_k , by this meaning that the latter can be explicitly defined from the former, such that $k = 1/N^\alpha$, with α a positive real number depending on specific features of \mathcal{Q}_N . Indeed, the global symmetry guarantees the existence of the simpler theory \mathcal{Q}_k (with k the real parameter defined by N) whose $k \rightarrow 0$ limit, hereafter indicated by $\mathcal{Q}_{k \rightarrow 0}$, is physically equivalent to the large- N limit, $\mathcal{Q}_{N \rightarrow \infty}$, of \mathcal{Q}_N , by this meaning that each expectation value that stays finite in the latter limit can be obtained as some expectation value provided by $\mathcal{Q}_{k \rightarrow 0}$. Resulting from the above is the following: to each $|\Lambda\rangle$, coherent state for \mathcal{Q}_k , is associated a set $\{|\Lambda_N\rangle\}_\sim$ of coherent states for \mathcal{Q}_N such that

$$\lim_{N \rightarrow \infty} \langle \Lambda_N^i | \hat{A}_N | \Lambda_N^i \rangle = \lim_{k \rightarrow 0} \langle \Lambda | \hat{A}_k | \Lambda \rangle \equiv A_{cl}(\Lambda), \quad (1.16)$$

⁶A *coadjoint orbit* is the set of points generated by the action of elements of the group \mathcal{G} on a given element of the dual space \mathfrak{g}^* .

for all $|\Lambda_N^i\rangle \in \{|\Lambda_N\rangle\}_\sim$ and any Hermitian operator $\hat{A}_k \in K$. Operators \hat{A}_k and \hat{A}_N are formally related, though it is not possible to express such relation in general, and A_{cl} is a real function on \mathcal{C} , with $A_{cl}(\Lambda)$ its value on the point that univocally corresponds to $|\Lambda\rangle$, according to the one-to-one correspondence characteristic of the GCS. Elements of the same set are related by $|\Lambda_N^i\rangle = \hat{U}^{ij} |\Lambda_N^j\rangle$, with \hat{U}^{ij} any unitary $\mathbb{X}(N)$ -symmetry transformation, and are dubbed *classically equivalent*, while operators \hat{A}_N such that their symbol keeps finite in the $N \rightarrow \infty$ limit are called *classical operators*, as in the definition (1.9) for \mathcal{Q}_k . On the other hand, if the four assumptions for \mathcal{Q}_k hold, $\mathcal{Q}_{k \rightarrow 0}$ is also a well defined classical theory C , with phase space \mathcal{C} and classical Hamiltonian $h(\zeta)$, that therefore provides an effective classical description of the original many-particles quantum system in its macroscopic limit, through the chain $\mathcal{Q}_{N \rightarrow \infty} = \mathcal{Q}_{k \rightarrow 0} = C$. Therefore, we have the following scheme, graphically represented in Fig. 1.2: there is an exact quantum theory \mathcal{Q}_N whose large- N limit defines a classical theory C , that is the same classical limit that we get from an effective theory \mathcal{Q}_k when k goes to 0. Again, it is thanks to the $\mathbb{X}(N)$ symmetry that we can reverse the logic, exploiting the $k \rightarrow 0$ limit in \mathcal{Q}_k to study the $N \rightarrow \infty$ limit of \mathcal{Q}_N , otherwise completely not-accessible. Summarizing, we can state that the large- N limit of \mathcal{Q} implies a classical behaviour of the macroscopic system it describes IF $N \rightarrow \infty$ implies $k \rightarrow 0$. However, for the sake of clarity, we will not hereafter use the vanishing of the quantity parameters, but rather refer to the $N \rightarrow \infty$, or large- N , limit.

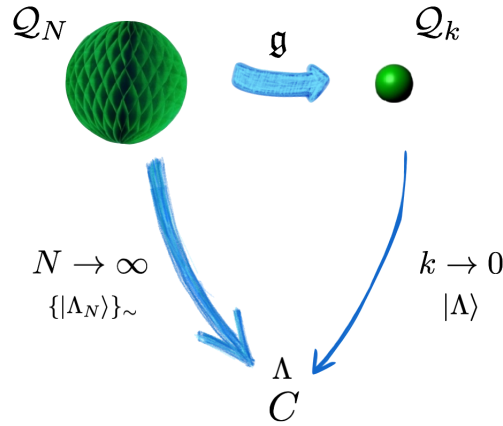


Figure 1.2: Graphical representation of the relation between microscopic and effective theories, \mathcal{Q}_N and \mathcal{Q}_k , from which the same classical theory C stems.

1.3 Generalized Coherent States in the large- N limit

In consideration of the cited results, it may appear now clear that GCS constitute a precious tool for the kind of situations that we are going to analyse. Amongst the above properties of GCS, that are better discussed in App. A, the one that will play a key role in this thesis regards quantities of the form $\langle \Lambda | \xi \rangle$, whose square modulus

represents the probability that a system Ξ in some generic pure state $|\xi\rangle$ be observed in the coherent state $|\Lambda\rangle$, when Ξ becomes macroscopic. These overlaps never vanish for finite N , due to the overcompleteness of GCS: as a consequence, if one considers two orthonormal states, say $|\xi'\rangle$ and $|\xi''\rangle$, there will always be a finite probability for a system in a GCS $|\Lambda\rangle$ to be observed either in $|\xi'\rangle$ or in $|\xi''\rangle$. This formally implies that, defined S_ξ the set of points on the manifold \mathcal{M} where $|\langle\Lambda|\xi\rangle| > 0$, it generally is $S_{\xi'} \cap S_{\xi''} \neq \emptyset$. On the other hand, the quantity

$$\lim_{N \rightarrow \infty} |\langle\Lambda|\xi\rangle|^2 \quad (1.17)$$

features some very relevant properties depending on the specific $|\xi\rangle$ considered. It is for instance known that it converges to the Dirac-delta distribution $\delta(\Lambda - \Lambda')$ if $|\xi\rangle$ is another GCS $|\Lambda'\rangle$, as we learnt from Assumption 3 (see Eq. (1.13)). Moreover, we have demonstrated the following. Given two elements, say $|\xi'\rangle$ and $|\xi''\rangle$, of any orthonormal basis $\{|\xi\rangle\}_{\mathcal{H}}$ for the Hilbert space \mathcal{H} of the system for which GCS have been constructed, it can be shown that

$$\lim_{N \rightarrow \infty} S_{\xi'} \cap S_{\xi''} = \emptyset. \quad (1.18)$$

Indeed, being $|\xi'\rangle$ and $|\xi''\rangle$ orthonormal, using the resolution of the identity (1.4) and Eq. (1.7), it is

$$\delta_{\xi'\xi''} = \langle\xi'|\xi''\rangle = c_k \int_{\mathcal{M}} dm(\Lambda) \langle\xi'|\Lambda\rangle \langle\Lambda|\xi''\rangle, \quad (1.19)$$

implying

$$\lim_{k \rightarrow 0} c_k \int_{\mathcal{M}} dm(\Lambda) \langle\xi'|\Lambda\rangle \langle\Lambda|\xi''\rangle = \delta_{\xi'\xi''} \quad \forall |\xi'\rangle, |\xi''\rangle \in \{|\xi\rangle\}_{\mathcal{H}}. \quad (1.20)$$

The constant c_k keeps the footprint of the quanticity of the theory, that is supposed to vanish as $k \rightarrow 0$, or, equivalently, $N \rightarrow \infty$. Even if it depends on the normalization of the group measure $d\mu(\hat{\Lambda})$ and should be computed on a case-by-case basis, the normalization of GCS (1.5) is guaranteed by construction, and from Eq.(1.13) it follows $|\langle\Lambda|\Lambda'\rangle|^2 \rightarrow k\delta(\Lambda - \Lambda')$ as k vanishes, so that

$$\lim_{k \rightarrow 0} c_k k \int_{\mathcal{M}} dm(\Lambda') \delta(\Lambda - \Lambda') = 1, \quad (1.21)$$

which implies $c_k = \frac{1}{k}$, as readily verified in those cases where an explicit form of GCS is available. Therefore, we observe that the only possibility to keep Eq. (1.19) meaningful in the large- N limit is that the product of the overlaps contained in the integral vanish $\forall \Lambda \in S_\xi \subset \mathcal{M}$, so that the result (1.20) hold true. In fact, due to the homogeneous divergence of c_k , and the requirement that the above result hold for whatever pair $|\xi'\rangle, |\xi''\rangle$ of whatever orthonormal basis $\{|\xi\rangle\}_{\mathcal{H}}$, the only possibility for it to hold is indeed that the two overlaps entering the integral are never simultaneously finite on \mathcal{M} or, more precisely, on a set of finite measure. In other terms, Eq. (1.20) implies Eq. (1.18), and vice-versa - which is trivial. Since result (1.18) will be essential in this thesis, we here discuss its actual meaning with two explicit examples.

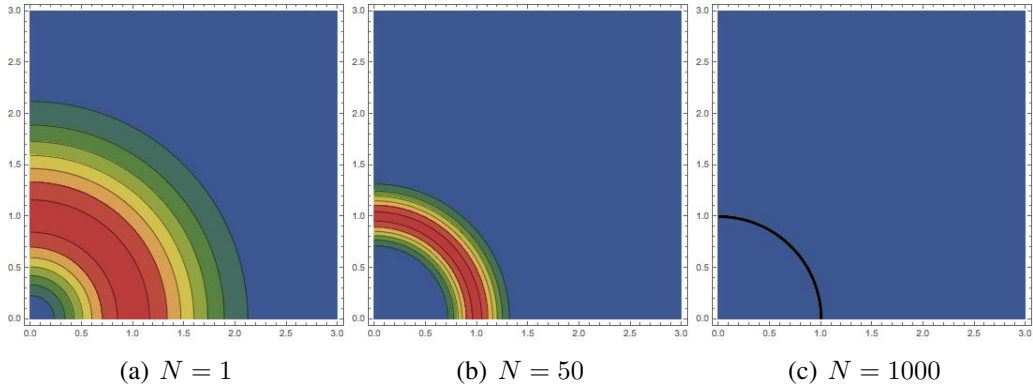


Figure 1.3: $|\langle\alpha|n\rangle|^2$ with $n = 1$ for $N = 1, 10, 1000$ (left to right): Contourplot on part of \mathcal{M} , which is now the complex plane \mathfrak{C} . Colours increasing from blue to red.

1.3.1 Field Coherent States

Consider a system Ξ whose Lie algebra is \mathfrak{h}_4 , i.e., the vector space spanned by $\{\hat{a}, \hat{a}^\dagger, \hat{n} \equiv \hat{a}^\dagger \hat{a}, \hat{\mathbb{I}}\}$, with Lie-brackets $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbb{I}}$, $[\hat{a}, \hat{n}] = \hat{a}$, $[\hat{a}^\dagger, \hat{n}] = -\hat{a}^\dagger$; its GCS are the well known field coherent states $|\alpha\rangle$, with $|0\rangle : \hat{n}|0\rangle = 0$ the reference state, and \mathcal{M} the complex plane \mathfrak{C} . The eigenstates of the diagonal operators $\{\hat{\mathbb{I}}, \hat{n}\}$ are the Fock states $|n\rangle$, and $\hat{D}(\alpha) \equiv \exp\{\alpha\hat{a} - \alpha^*\hat{a}^\dagger\}$ is the displacement operator such that $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$. The invariant measure defined by the GCS can be written as

$$d\mu(\alpha) = \frac{1}{\pi\hbar} d^2\alpha, \quad (1.22)$$

with α complex parameters on the plane \mathfrak{C} . The quanticity parameter k is defined in such a way that the theory becomes classical as $k \rightarrow 0$. The easiest way to determine its relation with N is to examine the Lie-brackets of the theory and find the condition that makes them vanish in the large- N limit. In the case of bosonic theories, this procedure is made less direct by the use of operators whose commutators are either numbers or proportional to \hbar , which is a finite constant. People sometimes speak of the $\hbar \rightarrow 0$ limit to refer to the bosonic classical limit, understanding \hbar as the quanticity parameter of the theory, which is quite a no-sense being its value fixed. However, one can restore dimensionful ladder operators, $\hat{a}^{(\dagger)} \rightarrow \sqrt{\frac{2\hbar}{M\omega}} \hat{a}^{(\dagger)}$, and observe that all the commutators vanish in the large- M limit. Further taking $M \propto N$ it is easily found that $k \sim 1/N$. Since \hbar always appears in dividing M , or N , one gets the same results letting \hbar vanish, so that, loosely speaking, the $\hbar \rightarrow 0$ limit is a classical limit.

We are interested in evaluating quantities $\langle\alpha|n'\rangle\langle n''|\alpha\rangle$, with $\langle\alpha|n\rangle$ given in App. A. First of all, we notice that the support of $\lim_{N \rightarrow \infty} |\langle\alpha|n\rangle|^2$ does not shrink into the neighbourhood of a point on \mathcal{M} , as is the case for $\lim_{N \rightarrow \infty} |\langle\alpha|\alpha'\rangle|^2 = \delta(\alpha - \alpha')$, but rather into that of the circle $|\alpha|^2 = n$, as shown in Fig. 1.3. In other terms, more field coherent states overlap with the same Fock state, but the reverse does not hold true in the large- N limit. In Fig. 1.4 we show $|\langle\alpha|n\rangle|^2$ as a function of $|\alpha|^2$, for

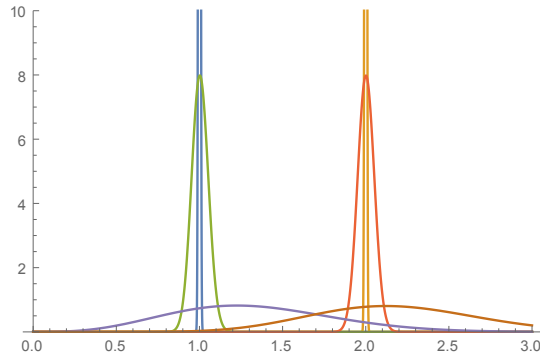


Figure 1.4: $|\langle\alpha|n\rangle|^2$ as a function of $\sqrt{\alpha\alpha^*}$, for $n = 1$ (left) and $n = 4$ (right), for $N = 1, 10, 1000$ (bottom to top).

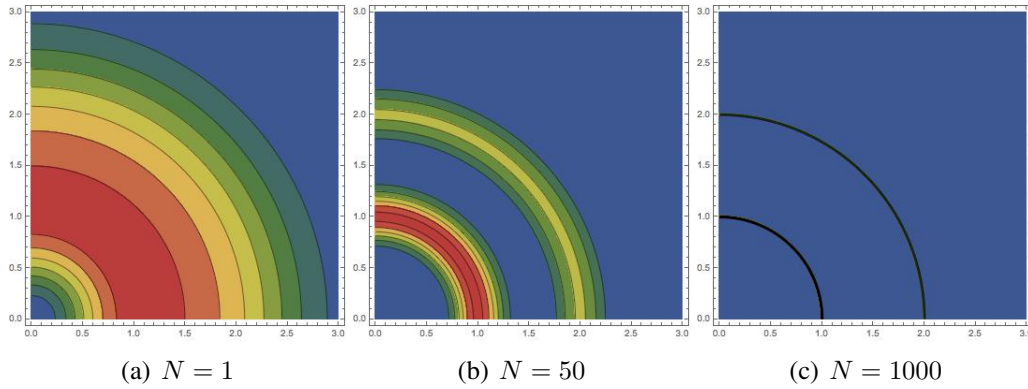


Figure 1.5: Sum $|\langle\alpha|n'\rangle|^2 + |\langle\alpha|n''\rangle|^2$ with $n' = 1$ and $n'' = 4$ for $N = 1, 10, 1000$ (left to right): Contourplot on part of \mathcal{C} . Colours as in Fig. 1.3.

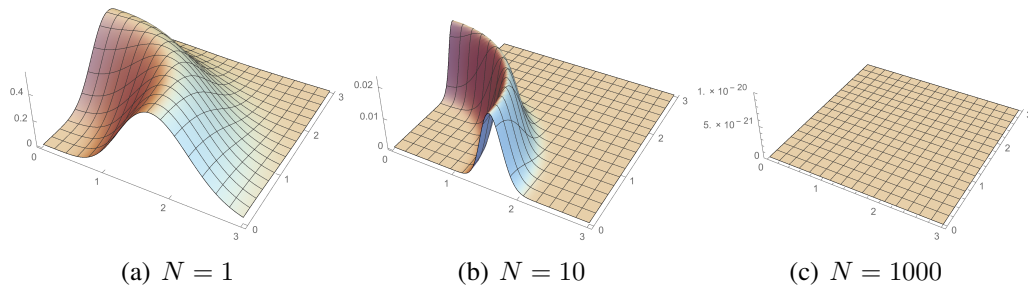


Figure 1.6: Product $|\langle\alpha|n'\rangle\langle\alpha|n''\rangle|^2$ with $n' = 1$ and $n'' = 4$ for $N = 1, 10, 1000$ (left to right), on part of the complex plane \mathcal{C} .

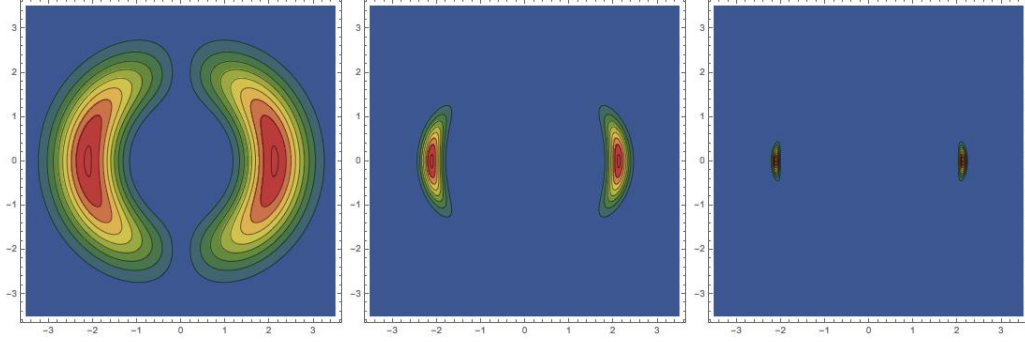


Figure 1.7: Sum $|\langle \alpha|+\rangle|^2 + |\langle \alpha|-\rangle|^2$ with $|\pm\rangle = (|1\rangle \pm |2\rangle)/\sqrt{2}$, for $N = 1, 10, 1000$ (left to right): Contourplot on part of the complex plane \mathcal{C} . Colours increasing from blue to red.

$n = 1, 4$ and different values of the quantity parameter, which is proportional to $1/N$ in this case. If the theory keeps its quantum character, $S_{n'} \cap S_{n''} \neq \emptyset$, while it is seen that $S_{n'} \cap S_{n''} \rightarrow \emptyset$ when $N \rightarrow \infty$, as we can also observe in Fig. 1.5, where we show the sum $|\langle \alpha|n'\rangle|^2 + |\langle \alpha|n''\rangle|^2$. This means that the product between the overlaps vanishes, unless $n' = n''$, as shown in Fig 1.6. Lastly, we observe that this picture holds not only for Fock states, but, as expressed by Eq. (1.19), for any set of orthonormal states. In Fig. 1.7, for instance, we contour-plot the sum $|\langle \alpha|+\rangle|^2 + |\langle \alpha|-\rangle|^2$ with $|\pm\rangle \equiv (|1\rangle \pm |2\rangle)/\sqrt{2}$: in this case S_+ and S_- are disjoint already for $N = 1$, and keep shrinking as N increases.

1.3.2 Spin Coherent States

A very similar scenario appears when studying a system Ξ whose Lie algebra is $\mathfrak{su}(2)$, i.e., the vector space spanned by $\{\hat{S}^+, \hat{S}^-, \hat{S}^z\}$, with Lie-brackets $[\hat{S}^+, \hat{S}^-] = 2\hat{S}^z$, $[\hat{S}^z, \hat{S}^\pm] = \pm\hat{S}^\pm$, and $|\hat{S}|^2 = S(S+1)$; its GCS are the spin (or atomic) coherent states $|\Omega\rangle$, with the reference state $|0\rangle : \hat{S}^z|0\rangle = -S|0\rangle$, and \mathcal{M} the unit sphere S^2 . The eigenstates of the diagonal operator \hat{S}^z are the states $|m\rangle : \hat{S}^z|m\rangle = (-S+m)|m\rangle$, and the displacement operators are $\hat{D}(\Omega(\zeta)) = \exp\{\zeta\hat{S}^- - \zeta^*\hat{S}^+\}$, with $\zeta = \frac{\vartheta}{2}e^{i\varphi}$, and $\vartheta \in [0, \pi)$, $\varphi \in [0, 2\pi)$ the spherical coordinates. The invariant measure defined by the GCS can be written as

$$d\mu(\Omega) = \frac{2S+1}{4\pi} d\Omega, \quad (1.23)$$

where $d\Omega = \sin\vartheta d\vartheta d\varphi$ is the solid-angle volume element at (ϑ, φ) on the sphere S^2 . The quantity parameter for a spin system is readily recognized as $1/S$ writing $\hat{S} = S\hat{s}$, so that for instance $[\hat{s}^+, \hat{s}^-] \propto \frac{1}{S}\hat{s}^z$, implying that, as S goes to infinity, the “normalized” spin operators become classical variables and the magnetic system is described by a classical vector of unitary modulus. Further taking $S \propto N$ it is easily found that $k \sim 1/N$.

The product of overlaps in Eq. (1.19) is now $\langle \Omega|m'\rangle\langle m''|\Omega\rangle$, with the analytical expression for $\langle \Omega|m\rangle$ available (see App. A). Notice that, as seen in the bosonic case,

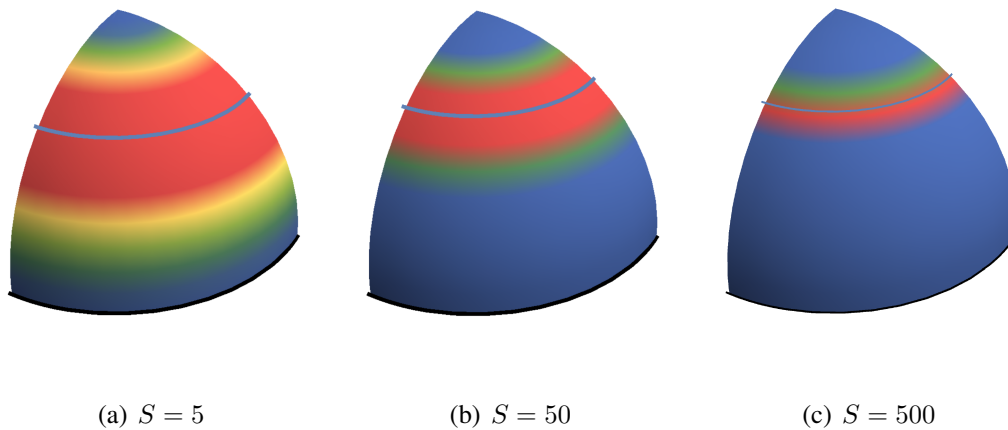


Figure 1.8: $|\langle \Omega | m \rangle|^2$ with $m/S = 0.8$ for $S = 5, 50, 500$ (left to right): Densityplot on part of \mathcal{M} , which is now the unit sphere S^2 . Colours increasing from blue to red.

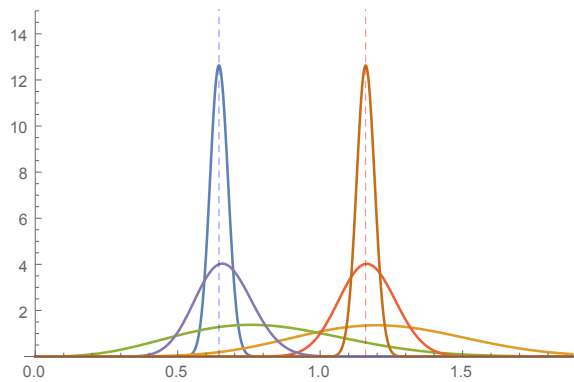


Figure 1.9: $|\langle \Omega | m \rangle|^2$ as a function of θ , for $m/S = 0.8$ (left) and 0.4 (right), for $S = 5, 50, 500$ (bottom to top).

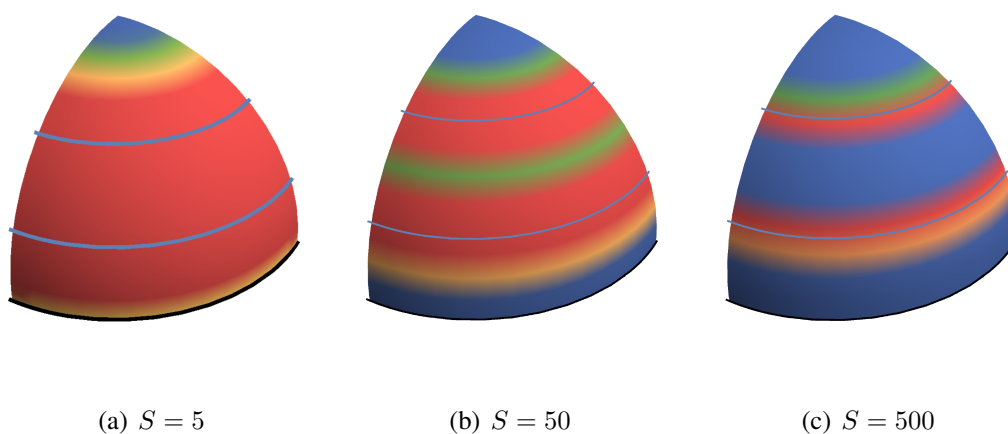


Figure 1.10: Sum $|\langle \Omega | m' \rangle|^2 + |\langle \Omega | m'' \rangle|^2$ with $m'/S = 0.8$ and $m''/S = 0.4$, for $S = 5, 50, 500$ (left to right): Densityplot on part of the unit sphere S^2 . Colours as in Fig. 1.8.

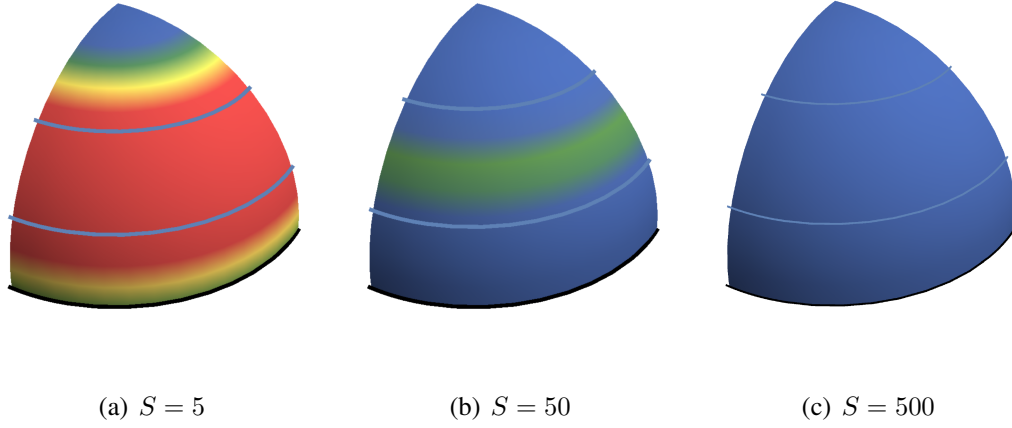


Figure 1.11: Product $|\langle \Omega | m' \rangle \langle \Omega | m'' \rangle|$ with $m'/S = 0.8$ and $m''/S = 0.4$, for $S = 5, 50, 500$ (left to right): Densityplot on part of the unit sphere S^2 . Colours as in Fig. 1.8.

the support of $\lim_{N \rightarrow \infty} |\langle \Omega | m \rangle|^2$ does not shrink into the neighbourhood of a point on the sphere, as is the case for $\lim_{N \rightarrow \infty} |\langle \Omega | \Omega' \rangle|^2 = \delta(\Omega - \Omega')$, but rather into that of the parallel $\cos \theta = m/S$ (Fig. 1.8). In Fig. 1.9 we show the sum $|\langle \Omega | m' \rangle|^2 + |\langle \Omega | m'' \rangle|^2$ for $m'/S = 0.8$ and $m''/S = 0.4$ as densityplot on the unit sphere, for different values of the quantity parameter, which is now proportional to $1/S$, and hence to $1/N$. Again, we see that if the theory keeps its quantum character, $S_{m'} \cap S_{m''} \neq \emptyset$, whereas $S_{m'} \cap S_{m''} \rightarrow \emptyset$ as $N \rightarrow \infty$, as evident in Fig. 1.10 where we show the sum $|\langle \Omega | m' \rangle|^2 + |\langle \Omega | m'' \rangle|^2$, and the product in Eq. (1.19) vanishes, unless $m' = m''$, as shown in Fig. 1.11.

1.4 Open Quantum Systems: combining MACRO with micro

The work done in the second half of the last century on the $N \rightarrow \infty$ limit of quantum theories is quite comprehensive, but it neglects the case when the large- N system is the big partner of a principal quantum system, that only indirectly experiences such limit. In fact, the large- N theories extensively developed and used in quantum-field theory, including the works by Yaffe and Lieb to which we particularly referred in Sec. 1.2, are not trivially applicable when the large- N system is not isolated, and, unless one decides that the quantum system is not “principal” at all and hence can be neglected, several foundational issues arise in this setting, due to the difficult coexistence of quantum and classical formalisms, possibly made worse by the presence of thermal baths or stochastic agents. On the other hand, this is an exemplary situation in quantum technologies and the research field of OQS, for which reason we must develop some proper tools to deal with those cases where “MACRO” and “micro” need to be combined, making classical and quantum coexist.

Consider a quantum environment Ξ made by N quantum components, interacting with a principal system Γ . Whereas the latter stays intrinsically quantum, the former will be characterized by the assumptions of Sec. 1.2, so as to guarantee it, and it

alone, feature a classical behaviour in the large- N limit, actually through the emergence of the effective theory Q_k and the $k \rightarrow 0$ limit of the quanticity parameter. We want to analyse what happens in the twilight zone of the quantum environment that is becoming classical. Therefore, a method is required such that a large- N limit for Ξ can be defined, without affecting Γ . We identify one such method in the **Parametric Representation with Environmental Coherent States** (PRECS), a theoretical tool that has been introduced [13, 14] to specifically address those bipartite quantum systems where one part, on its own, shows an emerging classical behaviour in becoming macroscopic. As the name suggests, the PRECS makes use of GCS for the system intended to become macroscopic, and we call them Environmental Coherent States (ECS), since they are relative to a system, Ξ , which is the environment of a principal system, Γ .

Before introducing the method, let us observe that the parametric representation pertains to the principal system, but, being exact, the information provided is sufficient to reconstruct the state of the whole system, $\Gamma + \Xi$. Exact parametric representations of OQS can always be constructed, providing an option to the reduced density matrix approach. This means that it is possible to adopt a parametric representation for Γ and yet keep a meaningful definition of the possible entanglement between Γ and Ξ ⁷. In fact, parametric representations provide an exact formalism for studying a composite, bipartite, quantum system in such a way that Γ is not represented by a density operator, but rather by a collection of pure states, each labelled by a parameter that univocally specifies a possible configuration for Ξ . The possibility of defining a parametric representation relies on two assumptions:

- The composite, global system $\Psi = \Gamma + \Xi$ is isolated, and hence described by a pure state.
- It is possible to define a resolution of the identity operator on \mathcal{H}_Ξ in terms of projectors onto normalized states.

Parametric representations can be *discrete* or *continuous* depending on whether the above identity resolution on \mathcal{H}_Ξ is given as a discrete sum or a continuous integral. Of course, for the pursuit of interpolating the fully quantum treatment of an OQS with that of closed ones, the appropriate candidates appear to be the continuous parametric representations, and, particularly, the PRECS, which indeed strongly relies on the special role played by GCS, as far as the classical limit of any quantum theory is concerned.

⁷To this respect, we observe that entanglement finds no position in those approaches where an OQS Γ with an environment Ξ is described as a “closed” one, i.e., as a quantum system interacting with a classical environment (there cannot be entanglement between a quantum and a classical system). Nevertheless, such representations have some relevant advantages, since representing Ξ by a small number of parameters proves extremely useful; this choice indeed characterizes essential formalisms, such as the quantum statistical mechanics in the canonical ensemble, where the density matrix ϱ_Γ parametrically depends on temperature, or the Born-Oppenheimer formalism for describing molecules, where electrons are described in terms of pure states parametrically dependent on the nuclear positions.

1.4.1 The Parametric Representation with Environmental Coherent States

Let us consider a bipartite, isolated system $\Psi = \Gamma + \Xi$ with Hilbert space $\mathcal{H}_\Psi = \mathcal{H}_\Gamma \otimes \mathcal{H}_\Xi$ in the pure state

$$|\psi\rangle = \sum_{\gamma\xi} c_{\gamma\xi} |\gamma\rangle \otimes |\xi\rangle \quad \text{with} \quad \sum_{\gamma\xi} |c_{\gamma\xi}|^2 = 1, \quad (1.24)$$

where $\{|\gamma\rangle\}_\Gamma$ and $\{|\xi\rangle\}_\Xi$ are local orthonormal bases for \mathcal{H}_Γ and \mathcal{H}_Ξ respectively. The dependence of the coefficients $c_{\gamma\xi}$ on both γ and ξ evidences that $|\psi\rangle$ is entangled. Once ECS are available, according to the procedure briefly introduced in Sec. 1.2 and discussed in App. A, the PRECS of whatever pure state $|\psi\rangle$ of Ψ is obtained by inserting an identity resolution in the form (1.4) into any decomposition of $|\psi\rangle$ with respect to the partition $\Psi = \Gamma + \Xi$. Explicitly, we can write the state (1.24) as

$$|\psi\rangle = \sum_{\gamma\xi} c_{\gamma\xi} |\gamma\rangle \left(\int_{\mathcal{G}/\mathcal{F}} d\mu(\hat{\Lambda}) |\Lambda\rangle \langle \Lambda| \right) |\xi\rangle = \int_{\mathcal{M}} d\mu(\Lambda) \sum_{\gamma} f_{\gamma}(\Lambda) |\gamma\rangle |\Lambda\rangle, \quad (1.25)$$

with the symbol \otimes understood - as hereafter done whenever convenient- and

$$f_{\gamma}(\Lambda) = \sum_{\xi} c_{\gamma\xi} \langle \Lambda | \xi \rangle. \quad (1.26)$$

In order to get a normalized state for Γ , we define

$$\chi(\Lambda) = \sqrt{\sum_{\gamma} |f_{\gamma}(\Lambda)|^2}, \quad (1.27)$$

which is a real function on \mathcal{M} , and write

$$|\psi\rangle = \int_{\mathcal{M}} d\mu(\Lambda) \chi(\Lambda) |\phi(\Lambda)\rangle |\Lambda\rangle, \quad (1.28)$$

such that $\langle \phi(\Lambda) | \phi(\Lambda) \rangle = 1 \quad \forall \Lambda$. Notice that, due to the normalization of $|\Psi\rangle$, it is

$$\int_{\mathcal{M}} d\mu(\Lambda) \chi(\Lambda)^2 = 1. \quad (1.29)$$

From Eqs. (1.26), (1.27) and (1.28),

- We define the PRECS of Γ , relative to $\Psi = \Gamma + \Xi$ in the state $|\psi\rangle$, the set $\{|\phi(\Lambda)\rangle, \chi(\Lambda)^2\}_{\Lambda \in \mathcal{M}}^{|\psi\rangle}$.

The parametric dependence of the pure, normalized Γ -states $|\phi(\Lambda)\rangle$ on the environmental parameter Λ is the fingerprint that Γ has an environment with which it may be possibly entangled. In fact, it is easily seen that if $|\psi\rangle$ were separable no such

dependence would survive. Moreover, we notice that $\chi(\Lambda)$ does not depend on γ and, from the property (1.29), a nice physical picture emerges: a parametric representation of an OQS derived from the pure state of a composite system is a collection of pure parametrized (and normalized) states $\{|\phi(\Lambda)\rangle \in \mathcal{H}_\Gamma\}$ whose occurrence is ruled by the probability distribution $\chi(\Lambda)^2$ on \mathcal{M} , that can be interpreted, thanks to the one-to-one correspondence between points on \mathcal{M} and coherent states $|\Lambda\rangle$, as the probability for Ξ to be in the state $|\Lambda\rangle$ when Ψ is in $|\psi\rangle$. This interpretation is further justified by noticing that the square of the real function $\chi(\Lambda)$ is the Husimi function of the environmental reduced density matrix $\varrho_\Xi \equiv \text{Tr}_\Gamma[|\psi\rangle\langle\psi|]$, i.e.,

$$\chi(\Lambda)^2 = \langle\Lambda|\varrho_\Xi|\Lambda\rangle, \quad (1.30)$$

that, being a well-behaved probability distribution on \mathcal{M} , uniquely identifies ϱ_Ξ ⁸. Let us now compare the PRECS and the reduced density matrix formalism: it is

$$\varrho_\Gamma \equiv \text{Tr}_\Xi[|\psi\rangle\langle\psi|] = \int_{\mathcal{M}} d\mu(\Lambda)\chi(\Lambda)^2 |\phi(\Lambda)\rangle\langle\phi(\Lambda)|, \quad (1.31)$$

and hence, for whatever local observable \hat{O}_Γ , one finds

$$\begin{aligned} \text{Tr}_\Gamma[\varrho_\Gamma\hat{O}_\Gamma] &= \\ &= \int_{\mathcal{M}} d\mu(\Lambda) \sum_{\gamma''} \langle\gamma''| \left[\sum_{\gamma} f_{\gamma}(\Lambda) |\gamma\rangle \sum_{\gamma'} f_{\gamma'}(\Lambda)^* \langle\gamma'| \hat{O}_\Gamma \right] |\gamma''\rangle \\ &= \int_{\mathcal{M}} d\mu(\Lambda) \sum_{\gamma''} \sum_{\gamma} f_{\gamma}(\Lambda) \langle\gamma''|\gamma\rangle \sum_{\gamma'} f_{\gamma'}(\Lambda)^* \langle\gamma'| \hat{O}_\Gamma |\gamma''\rangle \\ &= \int_{\mathcal{M}} d\mu(\Lambda) \left(\sum_{\gamma'} f_{\gamma'}(\Lambda)^* \langle\gamma'| \right) \hat{O}_\Gamma \left(\sum_{\gamma} f_{\gamma}(\Lambda) |\gamma\rangle \right) \\ &= \int_{\mathcal{M}} d\mu(\Lambda)\chi(\Lambda)^2 \langle\phi(\Lambda)| \hat{O}_\Gamma |\phi(\Lambda)\rangle, \end{aligned} \quad (1.32)$$

consistently with the above interpretation of the set $\{|\phi(\Lambda)\rangle, \chi(\Lambda)^2\}_{\Lambda \in \mathcal{M}}^{|\psi\rangle}$ as a statistical ensemble.

1.5 About the Global Symmetry

We end the chapter with some further considerations about the global symmetry. Amongst the analysed assumptions, being invariant under a *global symmetry* is key for the drastic reduction of physical observables characterizing a system made of a large number of quantum components w.r.t. those featured by one single macroscopic

⁸In fact, a *Husimi function* is in principle defined on a classical phase space, while \mathcal{M} is a differential manifold with a symplectic structure that should not be considered a phase space yet, i.e., before the large- N limit is taken; however, it is quite conventional to extend the term to the expectation value of density matrices ϱ on GCS.

object that behaves classically. Think again to the paradigmatic case of the spin-ring of Sec. 1.1: we started from assuming the total spin S to be constant, that is the same to say that we started from a symmetry property of the physical system considered. In fact, the central role of symmetry was indeed a primary lesson of the 20th century's first half physics: the essential point is that in QM theorems about the conservation of all kinds of quantities are related to the symmetries of the system, and the conserved quantities are typically the observables that we want, and we are able, to analyse. Therefore, as far as our aim is to investigate which kind of macroscopic quantum systems properties can emerge in our classical world, it is not surprising at all that the symmetry properties of the physical systems play an important role.

We stated that a global symmetry $\mathbb{X}(N)$ is a symmetry which does not leave invariant subspaces in the Hilbert space \mathcal{H} of the corresponding system. This latter requirement means that there must exist a group of unitary operators, each acting nontrivially on all of the N constituents, that leave the physical observables of the theory invariant. Putting in the other way around, we can define a collective physical quantity \hat{A}_N representing a global property of the system such that

$$[\hat{A}_N, \hat{U}] = 0 \quad \forall \hat{U} \in \mathbb{X}(N), \quad (1.33)$$

where the unitary operators \hat{U} acting on \mathcal{H} are the transformations of the symmetry group, i.e., the transformations under which the system is invariant defining the symmetry itself. The conservation laws produced by the above mechanism are sometimes dubbed *superselection rules*, and represent selection rules for composite systems [15]. A superselection rule can be generally thought to stem from two slightly different origins:

- A sort of “golden rule”, defining *a priori* the system itself: for instance, the electron e^- is defined as the elementary particle with mass m_{e^-} , charge $-e$ and spin $1/2$;
- A collective operator that commutes with all the operators describing the system (namely, with all its sensible observables), and that defines a conservation law holding for each system component.

These “second kind” superselection rules are the ones to which we will refer in this thesis, as they come from collective operators defining global properties for the macroscopic quantum system and satisfying Eq. (1.33). Via the symmetry transformations $\hat{U} \in \mathbb{X}(N)$ we can identify equivalence classes $\{|\Xi_N^i\rangle\}_\sim$ of states $|\Xi_N^j\rangle \in \mathcal{H}$ with $1 \leq i, j \leq \dim \mathcal{H}$ such that

$$\hat{U} |\Xi_N^j\rangle = |\Xi_N^i\rangle \quad \forall \hat{U} \in \mathbb{X}(N) \quad \text{and} \quad \langle \Xi_N^j | \hat{A}_N | \Xi_N^j \rangle = \langle \Xi_N^i | \hat{A}_N | \Xi_N^i \rangle . \quad (1.34)$$

States belonging to the same equivalence classes are dubbed *symmetry equivalent states*, i.e., translating in words Eq. (1.34), states connected by symmetry transformations and giving the same expectation value of the global property \hat{A}_N . Once defined the symmetry equivalence classes, we want to focus on what happens if we implement the large- N condition, and we discover that there are *some* symmetry

equivalent states that become *classically equivalent states* when N goes to infinity. Such states, $|\Lambda_N^i\rangle$, are those symmetry equivalent states (1.34) for which the expectation value of the global property \hat{A}_N has a well defined large- N limit, $A_{cl}(\Lambda)$, reading

$$\lim_{N \rightarrow \infty} \langle \Lambda_N^i | \hat{A}_N | \Lambda_N^i \rangle = A_{cl}(\Lambda) < \infty. \quad (1.35)$$

Looking this expression, we immediately recognize Eqs. (1.9) and (1.16): indeed, the *special* states satisfying Eqs. (1.34) and (1.35) are GCS, as we could expect thinking about the special role they play in dealing with large- N limits.

Chapter 2

Large Environment as *part of an hybrid quantum scheme*

Several physical realizations of OQS are being considered in the literature, usually modeled in terms of bipartite isolated systems $\Psi = \Gamma + \Xi$, with Hilbert space $\mathcal{H}_\Psi = \mathcal{H}_\Gamma \otimes \mathcal{H}_\Xi$. The two subsystems act respectively as the principal system Γ and its environment Ξ , usually implying by this choice that $\dim \mathcal{H}_\Gamma \ll \dim \mathcal{H}_\Xi$. In fact, although there is no general necessity for this condition to hold, a very large $\dim \mathcal{H}_\Xi$ most often stands as an actual definition of what is really meant by *environment* - see Chap. 1. Since Ξ usually needs to be described by a very large number N of quantum components, the modeling of an effective description of Ξ , and of its influence of Γ , usually stems from intuitive and phenomenological arguments, or even from an arbitrary choice, rather than from a formal derivation. During almost all of last century, the problem of how a principal quantum system behaves when interacting with a macroscopic environment has been considered assuming the latter to be a classical system. If this is the case, a quantum analysis of how the two subsystems evolve due to their reciprocal interaction is hindered, and this is quite a severe limitation being indeed macroscopic environments the tools by which we ultimately extract information about, or exercise control upon, any microscopic quantum system [16, 17, 18, 19, 20]. For example, the effects of the presence of Γ on the way Ξ evolves, often referred to as *back-action* in the literature, have no place in the description, and entanglement between the twos is completely neglected (there cannot be entanglement with a classical system).

Recently, however, hybrid schemes in which micro- and macroscopic systems coexist in a quantum device have been considered in different frameworks, from the analysis of foundational issues via optomechanical setups, to quantum thermodynamics or nanoelectronics [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. These situations are peculiar since two different levels of quanticity are present within the same system, and often semi-classical techniques, as for instance perturbative methods, involve the system as a whole, flattening unavoidably the quanticity difference, that is nevertheless fundamental, and must hence be preserved, in hybrid schemes. In fact, it is not completely clear why one should renounce a quantum description of

a macroscopic system: after all, this is nothing but a system made of many quantum particles that, for one reason or another, can be described regardless of its internal structure as if it was a single object with its own effective Hilbert space, as we pointed out in Chap. 1. Again, the exemplary case of such a situation is when Ξ is the spinning made of a large number N of spin- $\frac{1}{2}$ particles, featuring a global symmetry that guarantees the total spin S to be a constant of motion. No matter how large N is, the corresponding magnetic environment behaves, in general, like a quantum system, as it is clearly seen if its total spin equals, say, $S = 1/2$ or $S = 1$. On the other hand, when $S \sim N \rightarrow \infty$ a classical-like dynamics is expected [9], while large- S approximations are ideal tools for studying macroscopic, and yet quantum, magnetic systems. In general, models that are hybrid in the sense explained above must be studied with the toolkit of OQS enriched by specific accessories for dealing with the macroscopicity of some of their elements, so that they become the most typical example of a possible situation for which the PRECS introduced in Sec. 1.4.1 has been specifically designed [13, 34].

This Chapter is devoted to the analysis of environments which compose hybrid systems, and which hence need to keep trace of their quantum character to allow a sensible description of the interaction with their quantum companion¹. In Sec. 2.1 we consider a magnetic environment Ξ , made of a large number N of spin- $\frac{1}{2}$ particles, with the total spin S that is a constant of motion. As we mentioned above, until S is finite, such magnet is the prototype of a system that exhibits a distinct quantum behaviour despite being macroscopic ($N \gg 1$). The microscopic companion of the magnet is assumed to be a quantum mechanical oscillator Γ , and the Hamiltonian that describes the interaction between Γ and Ξ allows for energy-exchange between the system and its environment. Addressing the time evolution of the composite system $\Gamma + \Xi$, we obtain for the propagator a factorized expression that allows us to focus on the back-action of the principal system on its environment. In Sec. 2.2 we study a bosonic system Γ coupled to either a bosonic or a magnetic environment Ξ , investigating if be possible describing its dynamics by an effective Hamiltonian acting on Γ only, given that its environment is macroscopic. In particular, we critically inspect the conditions for the validity of this hypothesis as a tool to understand whether it stems from Ξ being macroscopic, or the temperature being high, or from enforcing some other specific condition. Taking a general viewpoint, we show that any quantum environment can be described by classical fields whenever global symmetries lead to the definition of environmental operators that remain well defined when the dynamical variables of the environment increase. Finally, in Sec. 2.3 we gather the

¹The results here presented have been published in two works, reported in Secs. 2.1 and 2.2 respectively:

- “Quantum dynamics of a macroscopic magnet operating as an environment of a mechanical oscillator”, C. Foti, A. Cuccoli, and P. Verrucchi, Phys. Rev. A 94, 062127 (2016);
- “Effective description of the short-time dynamics in open quantum systems”, M. Rossi, C. Foti, A. Cuccoli, J. Trapani, P. Verrucchi, and M. Paris, Phys. Rev. A 96, 032116 (2017).

results obtained in this chapter.

2.1 A macroscopic magnet dancing with a quantum partner

Let us start considering a physical system Ξ to be the spin-ring introduced in the first chapter and mentioned also above. Be $\hat{\mathbf{S}} \equiv \frac{1}{2} \sum_i^N \hat{\boldsymbol{\sigma}}_i$ its total spin and $|\hat{\mathbf{S}}|^2 \equiv (\hat{S}^x)^2 + (\hat{S}^y)^2 + (\hat{S}^z)^2 = S(S+1)$ its modulus, with S ranging from 0 to $N/2$ if N is even (from $1/2$ to $N/2$ if N is odd). As we will always understand \hbar to be finite, we can hereafter set $\hbar = 1$. When $|\hat{\mathbf{S}}|^2$ commutes with the propagator, the value S stays constant and Ξ can be seen as one single physical system described by the spin operators closed under the $\mathfrak{su}(2)$ commutation relations $[\hat{S}^\alpha, \hat{S}^\beta] = i\varepsilon_{\alpha\beta\gamma} \hat{S}^\gamma$, with $\alpha(\beta, \gamma) = x, y, z$. Notice that taking $|\hat{\mathbf{S}}|^2$ conserved implies assuming that a global symmetry exists in the Hamiltonian acting on Ξ , according to what we learnt in Chap. 1. Once it is guaranteed that the total spin has a constant value S , we can consider that a spin system displays a classical-like behaviour when $S \gg 1$. In fact, the quantum-to-classical crossover of a magnetic system can be naively understood by the following argument: defining the normalized spin operator $\hat{\mathbf{s}} \equiv \hat{\mathbf{S}}/S$, it is $[\hat{s}^\alpha, \hat{s}^\beta] = i\varepsilon_{\alpha\beta\gamma} \hat{s}^\gamma/S$, so that $\hat{\mathbf{s}}$ becomes a classical vector in the $S \rightarrow \infty$ limit. We already used this simple reasoning, below Eq. (1.23) in Sec. 1.3.2, to point out the quanticity parameter of a magnetic system, i.e., $1/S$, and in Sec. 2.1.2 we will show how to introduce a large- S approximation essentially based on the above argument.

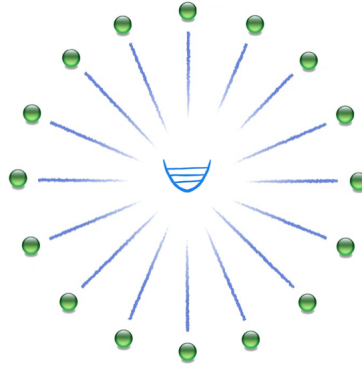


Figure 2.1: Graphical representation of a spin-ring coupled with a quantum mechanical oscillator and whose components are independent of each other.

The spin-ring is now identified as the magnetic environment Ξ of a quantum mechanical oscillator Γ - see Fig. 2.1. We choose the Hamiltonian of the overall system of the form

$$\hat{H} = \omega \hat{n} + \frac{f}{2} \sum_i^N \hat{\sigma}_i^z + \frac{1}{2} \sum_i^N g_i (\hat{a} \hat{\sigma}_i^+ + \hat{a}^\dagger \hat{\sigma}_i^-), \quad (2.1)$$

where ω is the (angular) frequency of the quantum oscillator, f is an external field defining the z axis, and $\hat{\sigma}_i^\pm \equiv \hat{\sigma}_i^x \pm i \hat{\sigma}_i^y$; the different g_i are the couplings between each

spin of the ring and the oscillator, and, being $\hbar = 1$, for the bosonic operators describing the principal system it holds that $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbb{1}}$. The Hamiltonian that describes the interaction between Γ and Ξ goes beyond the pure-dephasing model [35, 36], allowing for energy-exchange between the system and its environment and, possibly, dissipation - it will indeed describe the “exchange case” in Sec. 2.2. In order for the model (2.1) to describe a system whose environment can be made macroscopic, we need to guarantee the existence of a global symmetry such that the total spin is conserved. This can be accomplished by implementing different conditions, amongst which we choose $g_i = g \forall i$, leading to the Tavis-Cummings (TC) model [37, 38, 39]

$$\hat{H} = g(\hat{a}\hat{S}^+ + \hat{a}^\dagger\hat{S}^-) + (\omega\hat{n} + f\hat{S}_z) = \hat{Y} + \hat{X}, \quad (2.2)$$

where we have defined the free, $\hat{X} \equiv \omega\hat{n} + f\hat{S}_z$, and interacting, $\hat{Y} \equiv g(\hat{a}\hat{S}^+ + \hat{a}^\dagger\hat{S}^-)$, terms. This is an exactly solvable model [37], and analytic expressions for its eigenvectors and eigenvalues exist; however, these expressions are useless if one aims at writing the propagator in a form that lends to the recognition of different components in the overall dynamics, which is indeed our goal. In fact, the TC model is usually studied by taking the bosonic mode as the environment, for a principal system which is, in a way or another, described by the spin operator \hat{S} [40, 41]. If one tries to overturn the picture, analysing the TC dynamics regarding the magnetic system as the environment, formal problems due to the spin-operators algebra for large S immediately emerge. This is the reason why such choice most often trails behind itself that of a completely classical treatment of the environment, resulting in the replacement of the Hamiltonian’s spin-operators with a classical field $\mathbf{f}(t)$, that displays “ad hoc” time dependences [19, 42, 43, 44, 45, 46]. In this respect, we notice that describing a quantum system via a time-dependent Hamiltonian implies assuming that an environment exists, which is not, however, sensitive to the presence of the principal system itself. In fact, the time dependence of the field $\mathbf{f}(t)$ is arbitrarily chosen and does not change with the principal system’s evolution, a condition that defines the so called *no back-action* approximation. On the other hand, if one aims at studying specifically the back-action that the environment experiences because of its interaction with the principal system, it is necessary to consider the TC model with the spin system described as a genuinely quantum, magnetic environment.

2.1.1 The propagator

The evolution induced by the TC Hamiltonian is severely involuted, since not only the free \hat{X} and interacting \hat{Y} terms of Eq. (2.2) do not commute, but also the spin-commutation relations further prevent us from obtaining usable expressions via the BCH formula. In fact, it is quite clear that, as far as the coupling g in Eq. (2.2) is finite, any attempt to disentangle the propagator $\exp(-i\hat{H}t)$, by taking out factors separately acting on Γ and Ξ , will face the problem of dealing with infinitely nested commutators.

We take on the problem of studying the evolution

$$|\Psi(t)\rangle = e^{-i\hat{H}t} |\Psi(0)\rangle = e^{\lambda(\hat{Y}+\hat{X})} |\Psi(0)\rangle, \quad (2.3)$$

with $\lambda \equiv -it$, by means of the left-oriented version of the Zassenhaus formula, so as to make the free term \hat{X} act directly on the initial state $|\Psi(0)\rangle$, as will be done in Sec. 2.1.4. The left oriented Zassenhaus formula can be written [47] as

$$e^{\lambda(\hat{Y}+\hat{X})} = \dots e^{\lambda^n \tilde{C}_n} \dots e^{\lambda^3 \tilde{C}_3} e^{\lambda^2 \tilde{C}_2} e^{\lambda \hat{Y}} e^{\lambda \hat{X}}, \quad (2.4)$$

where $\tilde{C}_n = (-1)^{n+1} C_n$ with $n \geq 2$, and the Zassenhaus operators C_n are given in terms of commutators involving \hat{X} and \hat{Y}

$$ad_{\hat{X}}^0 \hat{Y} = \hat{Y}, \quad ad_{\hat{X}} \hat{Y} = [\hat{X}, \hat{Y}], \quad ad_{\hat{X}}^k \hat{Y} = \underbrace{[\hat{X}, [\hat{X} \dots [\hat{X}, \hat{Y}] \dots]]}_{k\text{-times}}, \quad (2.5)$$

and the same for $\hat{X} \leftrightarrow \hat{Y}$. In particular it is

$$C_{n+1} = \frac{1}{n+1} \sum_{i_0, i_1, \dots, i_n} \frac{(-1)^{i_0+i_1+\dots+i_n}}{i_0! i_1! \dots i_n!} ad_{C_n}^{i_n} \dots ad_{C_2}^{i_2} ad_{\hat{Y}}^{i_1} ad_{\hat{X}}^{i_0} \hat{Y}, \quad (2.6)$$

where each $(n+1)$ -tuple of non negative integers (i_0, i_1, \dots, i_n) must satisfy

$$\begin{aligned} i_0 + i_1 + 2i_2 + \dots + ni_n &= n \\ \text{and} & \\ i_0 + i_1 + 2i_2 + \dots + ji_j &\geq j + 1, \quad \text{for } j = 0, \dots, n-1. \end{aligned} \quad (2.7)$$

We underline that, as demonstrated in Ref. [47], the commutators defining the separate terms of the sum in Eq. (2.6) are all linearly independent; this means that, once the commutator defined by a certain $(n+1)$ -tuple has been determined, it is guaranteed that no other $(n+1)$ -tuple will give the same operator. Moreover, we notice that the time dependence of each exponential in Eq. (2.4) follows the ordering of the Zassenhaus terms in powers of t , so that t^m exclusively multiplies \tilde{C}_m , for all m . As for the order in g , it is easily seen that each commutator in Eq. (2.6) is proportional to g^l , where l is the number of operators \hat{Y} entering its definition. These features allow us to monitor the validity of the approximation scheme hereafter adopted, as extensively discussed at the end of the next section.

2.1.2 Large- S approximation

We have underlined that one of the features that characterizes a system as *environment* is that of being macroscopic, and when dealing with an environment described by spin operators, one can consistently implement macroscopicity by choosing a large value of S . On the other hand, if we take a large S and still want to maintain the original picture of a quantum system Γ interacting with its equally quantum environment Ξ , we must require that the interaction Hamiltonian stay finite for $S \gg 1$, implying that the coupling g in Eq. (2.2) scales as $1/S^2$. Therefore, we take gS

²It might seem that the same reasoning should hold for the external field f , but that is actually a different issue: the role of f is that of defining an energy scale for the magnetic system only, and the free Hamiltonian stays physical also in the $S \rightarrow \infty$ limit.

constant setting $gS = 1$ in what follows, and posit

$$g^m \prod_{i=1}^{n < m} \hat{S}^{\alpha_i} \sim 0, \quad (2.8)$$

with the symbol “ \sim ” that will be hereafter used to explicitly remind that condition (2.8) is assumed. We notice that this large- S approximation is utterly different from those required for making spin-boson transformations tractable by truncating square roots of operators, as done when using the Holstein-Primakoff or Villain transformations [48]. In those cases, in fact, the spin sphere S^2 , i.e., the isomorphic manifold of the $\mathfrak{su}(2)$ algebra, is projected onto a plane or a cylinder, respectively, parametrized by the usual conjugate coordinates. This implies that the algebra of the analysed quantum system is substantially altered. On the contrary, Eq. (2.8) keeps the spin-character of the magnetic operators without modifying their associated geometry, so that terms like *axial*, *planar*, *pole*, *equator*, ... simultaneously maintain their meaning.

Let us now get back to Eq. (2.4). In order to obtain the operators \tilde{C}_n , we define

$$\delta \equiv (\omega - f) \quad \text{and} \quad \hat{Y} \equiv g(\hat{a}\hat{S}^+ - \hat{a}^\dagger\hat{S}^-), \quad (2.9)$$

use

$$[\hat{X}, \hat{Y}] = \delta\hat{Y} \quad , \quad [\hat{X}, \hat{Y}] = \delta\hat{Y} \quad , \quad [\hat{Y}, \hat{Y}] = -2g^2(2\hat{a}^\dagger\hat{a}\hat{S}_z + \hat{S}^+\hat{S}^-), \quad (2.10)$$

and find that, due to condition (2.8), only two types of commutators survive

$$\underbrace{[\hat{X}, [\hat{X} \dots [\hat{X}, \hat{Y}] \dots]]}_{n\text{-times}} \equiv ad_{\hat{X}}^n \hat{Y} = g\delta^n \left(\hat{a}\hat{S}^+ + (-1)^n \hat{a}^\dagger\hat{S}^- \right) \quad (2.11)$$

and

$$\hat{Y}, \underbrace{[\hat{X}, [\hat{X} \dots [\hat{X}, \hat{Y}] \dots]]}_{(n-1)\text{-times}, n \text{ even}} \equiv ad_{\hat{Y}} ad_{\hat{X}}^{n-1} \hat{Y} = -2g^2\delta^{n-1}\hat{S}^+\hat{S}^-. \quad (2.12)$$

Consequently, referring to conditions (2.7), only the following $(n+1)$ -tuples remain in the sum entering Eq. (2.6), i.e.,

$$\begin{aligned} i_0 = n & \quad \text{with } i_k = 0 \quad \forall k \neq 0, \\ i_0 = n-1, i_1 = 1 & \quad \text{with } i_k = 0 \quad \forall k \neq 0, 1. \end{aligned} \quad (2.13)$$

Therefore, defining $\hat{Y}^+ \equiv g\hat{a}\hat{S}^+$ and $\hat{Y}^- \equiv g\hat{a}^\dagger\hat{S}^-$, the Zassenhaus operators are found to be

$$\begin{aligned} \tilde{C}_{2m+1} &= C_{2m+1} \sim \frac{1}{(2m+1)!} \delta^{2m} (\hat{Y}^+ + \hat{Y}^-) + \frac{2m}{(2m+1)!} \delta^{2m-1} (-2g^2\hat{S}^+\hat{S}^-) \\ \tilde{C}_{2m} &= -C_{2m} \sim \frac{1}{(2m)!} \delta^{2m-1} (\hat{Y}^+ - \hat{Y}^-). \end{aligned} \quad (2.14)$$

We underline that \tilde{C}_2 and \tilde{C}_3 only contain commutators of the form (2.11)-(2.12), meaning that expressions (2.14) are exact for $m = 1$. Finally, based on condition (2.8), we will hereafter use

$$\begin{aligned} [\hat{Y}^+, \hat{Y}^-] &= [g\hat{a}\hat{S}^+, g\hat{a}^\dagger\hat{S}^-] \\ &\sim g^2\hat{S}^+\hat{S}^- \sim g^2\hat{S}^-\hat{S}^+ \end{aligned} \quad (2.15)$$

$$\sim g^2 \left[S(S+1) - \hat{S}_z^2 \right], \quad (2.16)$$

and

$$[[\hat{Y}^+, \hat{Y}^-], \hat{Y}^\pm] \sim [[\hat{Y}^+, \hat{Y}^-], h\hat{S}^z] \sim 0, \quad (2.17)$$

yielding, as far as the evaluation of the propagator (2.3) is concerned,

$$e^{\hat{Y}^+ + \hat{Y}^-} \sim e^{\hat{Y}^+} e^{\hat{Y}^-} e^{-\frac{1}{2}g^2\hat{S}^+\hat{S}^-}. \quad (2.18)$$

We underline that $[\hat{Y}^+, \hat{Y}^-]$ does not vanish, despite condition (2.8) being enforced, because of the non-commutativity of \hat{a} and \hat{a}^\dagger , an evidence that we will comment on further at the end of Sec. 2.1.3.

We now get back to Eq. (2.3) and, by the repeated use of Eq. (2.18), proceed as follows.

i) Isolate $e^{\lambda\hat{X}}$:

$$\begin{aligned} \exp \left[\lambda(\hat{Y} + \hat{X}) \right] &\sim \dots \exp \left[\frac{\lambda^n(\delta)^n}{n!\delta} (\hat{Y}^+ - (-1)^n \hat{Y}^-) \right] \dots \exp \left[\lambda(\hat{Y}^+ + \hat{Y}^-) \right] \\ &\times \exp \left(-2g^2 K_{1\delta}(\lambda) \hat{S}^+ \hat{S}^- \right) \exp \left(\lambda \hat{X} \right), \end{aligned} \quad (2.19)$$

with

$$K_{1\delta}(\lambda) = \frac{1}{\delta^2} \sum_{m \geq 1} \frac{2m}{(2m+1)!} \lambda^{2m+1} \delta^{2m+1} = \frac{1}{\delta^2} (-it\delta \cos t\delta + i \sin t\delta). \quad (2.20)$$

ii) Factorize the exponentials containing both \hat{Y}^+ and \hat{Y}^- :

$$\begin{aligned} \exp \left[\lambda(\hat{Y} + \hat{X}) \right] &\sim \dots \exp \left(\frac{\lambda^n \delta^n}{n! \delta} \hat{Y}^+ \right) \exp \left[-\frac{(-\lambda)^n \delta^n}{n! \delta} \hat{Y}^- \right] \dots \\ &\times \exp \left(\lambda \hat{Y}^+ \right) \exp \left[-(-\lambda) \hat{Y}^- \right] \\ &\times \exp \left(-2g^2 K_{2\delta}(\lambda) \hat{S}^+ \hat{S}^- \right) \exp \left(\lambda \hat{X} \right), \end{aligned} \quad (2.21)$$

where

$$K_{2\delta}(\lambda) = K_{1\delta}(\lambda) - \frac{1}{4\delta^2} \sum_{n \geq 1} (-1)^n \left[\frac{\lambda^n \delta^n}{n!} \right]^2, \quad (2.22)$$

accounts for the commutators introduced via Eq. (2.18), while factoring all the exponentials of \hat{Y}^+ and \hat{Y}^- .

iii) Group together the operators \hat{Y}^+ (\hat{Y}^-), performing the necessary $\hat{Y}^+ \leftrightarrow \hat{Y}^-$ permutations in the infinite product of exponentials entering Eq. (2.21):

$$\begin{aligned} \exp \left[\lambda(\hat{Y} + \hat{X}) \right] &\sim \exp \left[\frac{1}{\delta} \sum_{n \geq 1} \frac{\lambda^n \delta^n}{n!} \hat{Y}^+ \right] \exp \left[-\frac{1}{\delta} \sum_{n \geq 1} \frac{(-\lambda)^n \delta^n}{n!} \hat{Y}^- \right] \\ &\times \exp \left[-2g^2 K_{2\delta}(\lambda) \hat{S}^+ \hat{S}^- \right] \exp \left[\gamma_\delta(\lambda) g^2 \hat{S}^+ \hat{S}^- \right] \exp \left(\lambda \hat{X} \right), \end{aligned} \quad (2.23)$$

where $\gamma_\delta(\lambda)$ is the coefficient resulting from the commutators $[\hat{Y}^+, \hat{Y}^-]$, introduced while moving all the \hat{Y}^- to the right.

In order to determine $\gamma_\delta(\lambda)$, we consider

$$\begin{aligned} e^{\mu \hat{Y}^-} e^{\pi \hat{Y}^+} &\sim e^{\mu \hat{Y}^- + \pi \hat{Y}^+ + \frac{1}{2} \mu \pi [\hat{Y}^-, \hat{Y}^+]} = e^{\pi \hat{Y}^+} e^{\mu \hat{Y}^-} e^{-\mu \pi [\hat{Y}^+, \hat{Y}^-]} = \\ &= e^{\pi \hat{Y}^+} e^{\mu \hat{Y}^-} e^{-\mu \pi g^2 \hat{S}^+ \hat{S}^-}, \end{aligned} \quad (2.24)$$

and define

$$\pi_n = \frac{\lambda^n \delta^n}{n! \delta}, \quad \mu_n = -\frac{(-\lambda)^n \delta^n}{n! \delta}, \quad (2.25)$$

so that the expression from which we will get $\gamma_\delta(\lambda)$ (see Eq. (2.23)) reads

$$\dots e^{\mu_{\ell+1} \hat{Y}^-} e^{\pi_\ell \hat{Y}^+} \dots e^{\mu_3 \hat{Y}^-} e^{\pi_2 \hat{Y}^+} e^{\mu_2 \hat{Y}^-} e^{\pi_1 \hat{Y}^+} e^{\mu_1 \hat{Y}^-}. \quad (2.26)$$

We then need to exchange every $\pi_n \hat{Y}^+$ with all the $\mu_\ell \hat{Y}^-$ of the following orders, i.e., such that $n > \ell$; after the first permutation, we get

$$\dots e^{\mu_{\ell+1} \hat{Y}^-} e^{\pi_\ell \hat{Y}^+} \dots e^{\pi_3 \hat{Y}^+} e^{\mu_3 \hat{Y}^-} \underbrace{e^{\pi_2 \hat{Y}^+} e^{\pi_1 \hat{Y}^+}}_{e^{(\pi_1 + \pi_2) \hat{Y}^+}} e^{\mu_2 \hat{Y}^-} e^{-\mu_2 \pi_1 g^2 \hat{S}^+ \hat{S}^-} e^{\mu_1 \hat{Y}^-}, \quad (2.27)$$

and one can easily check that successive permutations give the terms

$$\begin{aligned} &e^{-\mu_2 \pi_1 g^2 \hat{S}^+ \hat{S}^-} \\ &e^{-\mu_3 (\pi_1 + \pi_2) g^2 \hat{S}^+ \hat{S}^-} \\ &\vdots \\ &e^{-\mu_{\ell+1} (\pi_1 + \pi_2 + \pi_3 + \dots + \pi_\ell) g^2 \hat{S}^+ \hat{S}^-}, \end{aligned}$$

so that

$$\begin{aligned} \exp \left[\gamma_\delta(\lambda) g^2 \hat{S}^+ \hat{S}^- \right] &\sim \exp \left[- \left(\sum_{\ell \geq 2} \mu_\ell \sum_{1 \leq j < \ell} \pi_j \right) g^2 \hat{S}^+ \hat{S}^- \right] = \\ &= \exp \left[\frac{1}{\delta^2} \left(\sum_{\ell \geq 2} \frac{(-\lambda)^\ell \delta^\ell}{\ell!} \sum_{1 \leq j < \ell} \frac{\lambda^j \delta^j}{j!} \right) g^2 \hat{S}^+ \hat{S}^- \right]. \end{aligned} \quad (2.28)$$

Therefore, we get

$$\begin{aligned}
K_{3\delta}(\lambda) &= -2K_{2\delta}(\lambda) + \gamma_\delta(\lambda) = -2K_{2\delta}(\lambda) + \frac{1}{\delta^2} \sum_{\ell \geq 2} \frac{(-\lambda)^\ell \delta^\ell}{\ell!} \sum_{1 \leq j < \ell} \frac{\lambda^j \delta^j}{j!} = \\
&= -2K_{1\delta}(\lambda) + \frac{1}{2\delta^2} \sum_{n \geq 1} \frac{(-\lambda\delta)^n}{n!} \frac{(\lambda\delta)^n}{n!} + \frac{1}{\delta^2} \sum_{\ell \geq 2} \frac{(-\lambda\delta)^\ell}{\ell!} \sum_{1 \leq j < \ell} \frac{(\lambda\delta)^j}{j!},
\end{aligned} \tag{2.29}$$

yielding

$$\begin{aligned}
\exp \left[\lambda(\hat{Y} + \hat{X}) \right] &\sim \exp \left[\frac{1}{\delta} \sum_{n \geq 1} \frac{\lambda^n \delta^n}{n!} \hat{Y}^+ \right] \exp \left[-\frac{1}{\delta} \sum_{n \geq 1} \frac{(-\lambda)^n \delta^n}{n!} \hat{Y}^- \right] \\
&\quad \times \exp \left(K_{3\delta}(\lambda) g^2 \hat{S}^+ \hat{S}^- \right) \exp \left(\lambda \hat{X} \right).
\end{aligned} \tag{2.30}$$

We are now in the position of summing up the series in Eq. (2.30), which are equal to $(e^{\pm \lambda \delta} - 1)$, and finally get the global propagator in the form

$$\exp(-i\hat{H}t) \sim \exp \left\{ g \left[f_\delta(t) \hat{a} \hat{S}^+ - f_\delta^*(t) \hat{a}^\dagger \hat{S}^- \right] \right\} \exp \left[g^2 G_\delta(t) \hat{S}^+ \hat{S}^- \right] \exp \left(-it\hat{X} \right), \tag{2.31}$$

where the real time $t = i\lambda$ is back, $f_\delta(t) \equiv (e^{-it\delta} - 1)/\delta$, and the function $G_\delta(t) \equiv K_3(-it) - |f_\delta(t)|^2/2$ is pure imaginary. In fact, since $\lambda = -it$, we notice that $-\lambda = \lambda^*$ and, setting $x = \lambda\delta$, $x^* = \lambda^*\delta$, the last two terms of Eq. (2.29) can be written as

$$\begin{aligned}
&\frac{1}{\delta^2} \left(\frac{1}{2} \sum_{n \geq 1} \frac{x^{*n} x^n}{n! n!} + \sum_{\ell \geq 2} \frac{x^{*\ell}}{\ell!} \sum_{1 \leq j < \ell} \frac{x^j}{j!} \right) = \\
&= \frac{1}{2\delta^2} \sum_{n \geq 1} \frac{x^n}{n!} \sum_{\ell \geq 1} \frac{x^{*\ell}}{\ell!} - \frac{1}{2\delta^2} \sum_{n \geq 1} \frac{x^n}{n!} \sum_{\ell \neq n} \frac{x^{*\ell}}{\ell!} + \frac{1}{\delta^2} \sum_{\ell \geq 2} \frac{x^{*\ell}}{\ell!} \sum_{1 \leq j < \ell} \frac{x^j}{j!} = \\
&= \frac{1}{2} |f_\delta(\lambda)|^2 + M_\delta(\lambda),
\end{aligned} \tag{2.32}$$

where $f_\delta(\lambda) \equiv (e^{\lambda\delta} - 1)/\delta$, i.e., the function defined in Eq. (2.31), and

$$M_\delta(\lambda) = -\frac{1}{2\delta^2} \sum_{n \geq 1} \frac{(\lambda\delta)^n}{n!} \sum_{\ell \neq n} \frac{(\lambda^*\delta)^\ell}{\ell!} + \frac{1}{\delta^2} \sum_{\ell \geq 2} \frac{(\lambda^*\delta)^\ell}{\ell!} \sum_{1 \leq j < \ell} \frac{(\lambda\delta)^j}{j!}. \tag{2.33}$$

The propagator (2.30) is then

$$\begin{aligned}
\exp \left[\lambda(\hat{Y} + \hat{X}) \right] &\sim \exp \left[f_\delta(\lambda) \hat{Y}^+ \right] \exp \left[-f_\delta^*(\lambda) \hat{Y}^- \right] \\
&\quad \times \exp \left(K_{3\delta}(\lambda) g^2 \hat{S}^+ \hat{S}^- \right) \exp \left(\lambda \hat{X} \right) \\
&\sim \exp \left[f_\delta(\lambda) \hat{Y}^+ - f_\delta^*(\lambda) \hat{Y}^- \right] \exp \left(-\frac{1}{2} |f_\delta(\lambda)|^2 g^2 \hat{S}^+ \hat{S}^- \right) \\
&\quad \times \exp \left(K_{3\delta}(\lambda) g^2 \hat{S}^+ \hat{S}^- \right) \exp \left(\lambda \hat{X} \right),
\end{aligned} \tag{2.34}$$

where in the last step we used Eq. (2.15) and

$$e^{f_\delta(\lambda)\hat{Y}^+} e^{-f_\delta^*(\lambda)\hat{Y}^-} \sim e^{f_\delta(\lambda)\hat{Y}^+ - f_\delta^*(\lambda)\hat{Y}^- - \frac{1}{2}|f_\delta(\lambda)|^2[\hat{Y}^+, \hat{Y}^-]}. \quad (2.35)$$

Looking at Eq. (2.31), we therefore have

$$G_\delta(\lambda) = K_{3\delta}(\lambda) - \frac{1}{2}|f_\delta(\lambda)|^2 = -2K_{1\delta}(\lambda) + M_\delta(\lambda). \quad (2.36)$$

In order to show that this is a pure imaginary function we notice that $K_{1\delta}(\lambda) \in \mathfrak{S}$, and concentrate upon $M_\delta(\lambda)$. Restoring $x = \lambda\delta$ and $x^* = \lambda^*\delta$ for the sake of a lighter notation, it is

$$\begin{aligned} M(x) &= -\frac{1}{2\delta^2} \sum_{n \geq 1} \frac{x^n}{n!} \sum_{\ell \neq n} \frac{x^{*\ell}}{\ell!} + \frac{1}{\delta^2} \sum_{\ell \geq 2} \frac{x^{*\ell}}{\ell!} \sum_{1 \leq j < \ell} \frac{x^j}{j!} \\ &= -\frac{1}{2\delta^2} \sum_{n \geq 1} \frac{x^n}{n!} \sum_{\ell > n} \frac{x^{*\ell}}{\ell!} - \frac{1}{2\delta^2} \sum_{n \geq 2} \frac{x^n}{n!} \sum_{1 \leq \ell < n} \frac{x^{*\ell}}{\ell!} \\ &\quad + \frac{1}{2\delta^2} \sum_{\ell \geq 2} \frac{x^{*\ell}}{\ell!} \sum_{1 \leq j < \ell} \frac{x^j}{j!} + \frac{1}{2\delta^2} \sum_{\ell \geq 2} \frac{x^{*\ell}}{\ell!} \sum_{1 \leq j < \ell} \frac{x^j}{j!}, \end{aligned} \quad (2.37)$$

and one can easily verify that the first and the third terms sum up to zero. We get

$$M(x) = -\frac{1}{2\delta^2} \sum_{n \geq 2} \frac{x^n}{n!} \sum_{1 \leq \ell < n} \frac{x^{*\ell}}{\ell!} + \frac{1}{2\delta^2} \sum_{\ell \geq 2} \frac{x^{*\ell}}{\ell!} \sum_{1 \leq j < \ell} \frac{x^j}{j!}, \quad (2.38)$$

and, noticing that the above expression is the sum of $-\frac{1}{2\delta^2} \sum_{n \geq 2} \frac{x^n}{n!} \sum_{1 \leq \ell < n} \frac{x^{*\ell}}{\ell!}$ with its complex conjugate, $M(x) \in \mathfrak{S}$. This in turn implies that, looking at the exponents $n + \ell$ as powers of $t\delta$, only the odd terms survive in the sum of the two series above. Finally, Eq. (2.36) becomes

$$G_\delta(t) = i \left\{ -\frac{1}{\delta^2} \left[-2t\delta \cos t\delta + 2 \sin t\delta + \text{Im} \left(\sum_{n \geq 2} \frac{(-it\delta)^n}{n!} \sum_{1 \leq \ell < n} \frac{(it\delta)^\ell}{\ell!} \right) \right] \right\}, \quad (2.39)$$

which evidently is a pure imaginary quantity. This is the *back-action term* defined in Eq. (2.31), upon which we will concentrate in the next section.

Before moving forward in our analysis, the conditions under which the final form of the propagator (2.31) holds should be determined. Since according to condition (2.8) products of n spin operators have been neglected if multiplied by g^m with $m > n$, it must be $g \ll 1$, consistently with the large- S assumption with gS finite. As for the time dependence, we remind that condition (2.8) does not affect \tilde{C}_2 and \tilde{C}_3 , and Eq. (2.4) with Zassenhaus coefficients given by Eqs. (2.14) is exact up to the third order in T . Moreover, we notice that, through steps (i)-(iii), terms linear in whatever spin-operator \hat{S}^* appear as $g^n t^n \hat{S}^*$ and are kept only for $n = 1$, which is a valid choice if $gt \ll 1$. Since we have set $gS = 1$, this is tantamount to say $t \ll S$. On the whole, the condition $t \ll S$, with S large, defines the proper time-scale in which our results hold true.

2.1.3 The back-action

The final form of the propagator given by expression (2.31) most relevant feature is the appearance of the term $g^2 G_\delta(t) \hat{S}^+ \hat{S}^-$ which has no equivalent in the original Hamiltonian and, despite regarding the magnetic system only, is effectively generated, as made evident by its being proportional to the square of the coupling, by its interaction with the mechanical oscillator, thus standing as the type of back-action we were actually aiming at describing. In fact, if one reviews the way the above term is obtained, it becomes clear that condition (2.8) can be enforced without wiping the back-action off the global dynamics, if and only if $[\hat{a}, \hat{a}^\dagger]$ does not vanish, as we pointed out below Eq. (2.18). In other terms, it is the quantum character of the oscillator that keeps the back-action alive in the large- S limit, i.e., when the magnet becomes macroscopic.

In order to better understand the effects of the $\hat{S}^+ \hat{S}^-$ term, we notice that Eq. (2.17) ensures that $[g^2 G_\delta(t) \hat{S}^+ \hat{S}^- - it \hat{X}]$ commutes with itself at different times, and set

$$g^2 G_\delta(t) = -i \int_0^t A_\delta(\tau) d\tau, \quad (2.40)$$

with $A_\delta(t)$ real, being $G_\delta(t) \in \mathfrak{S}$. This allows us to define the effective time-dependent free Hamiltonian

$$\hat{X}_\delta^{\text{eff}}(t) \equiv A_\delta(t) \hat{S}^+ \hat{S}^- + \hat{X}, \quad (2.41)$$

such that

$$\exp(-i\hat{H}t) \sim \exp \left\{ g \left[f_\delta(t) \hat{a} \hat{S}^+ - f_\delta^*(t) \hat{a}^\dagger \hat{S}^- \right] \right\} \exp \left[-i \int_0^t \hat{X}_\delta^{\text{eff}}(\tau) d\tau \right]. \quad (2.42)$$

Notice that an effective Hamiltonian for Ξ that contains a term proportional to $\hat{S}^+ \hat{S}^-$ is also found in Ref. [39], where, however, different approximations are considered that do not encompass any time dependence for such an effective term. As for the interaction term, we notice that despite being $f_\delta(t) = -i \int_0^t d\tau e^{-i\delta\tau}$ one is unable to find an effective time-dependent interaction Hamiltonian, $\hat{Y}_\delta^{\text{eff}}(t)$ analogous to $\hat{X}_\delta^{\text{eff}}(t)$, as the argument of the first exponential in Eq. (2.42) does not commute with itself at different times, unless $\delta = 0$. If this is the case, however, $f_0(t) = -it$ and such exponential transforms into the propagator generated by $g(\hat{a} \hat{S}^+ + \hat{a}^\dagger \hat{S}^-)$. Moreover, from the general form of $G_\delta(t)$ (2.39), one easily finds $G_0(t) = 0$, implying that a genuine interaction picture for Ψ emerges. In other terms, when the free evolutions of Γ and Ξ are resonant ($\delta = 0 \Leftrightarrow \omega = f$) there is no back-action whatsoever, and information is not transferred from one system to the other.

2.1.4 Effective evolution of the environment

The operator $\hat{X}_\delta^{\text{eff}}(t)$ can be interpreted as the sum of the original free Hamiltonian for the bosonic mode, $\hat{H}_\Gamma = \omega \hat{a}^\dagger \hat{a}$, plus an effective time-dependent environmental one

$$\hat{H}_\Xi^{\text{eff}}(t) \equiv f \hat{S}^z + A_\delta(t) \hat{S}^+ \hat{S}^- \sim f \hat{S}^z - A_\delta(t) (\hat{S}^z)^2 - \epsilon_\delta(t), \quad (2.43)$$

where we have used Eqs. (2.15)-(2.16) and set $\epsilon_\delta(t) = A_\delta(t)S(S+1)$. In this way, we see that the presence of Γ makes the environment feel an effective magnetic anisotropy $-A_\delta(t)(\hat{S}^z)^2$ that favours or hinders the alignment of its spin along the quantization axis, depending on the sign of $A_\delta(t)$. The time dependence of $A_\delta(t)$ represents the continuous updating of the back-action, which is ruled by the energy exchange between Γ and Ξ . In particular, it is $A_\delta(t) \propto t^2 + \mathcal{O}(t^4)$ [from Eq. (2.40) and the analytical expression of $G_\delta(t)$ (2.39)], meaning that there exists an initial time-interval during which the environment is not affected by the presence of Γ in any way other than that due to their explicit interaction. After some time, however, the energy exchange implied by that very same interaction becomes so costly as to cause a reaction that switches on the back-action, in the form of a magnetic anisotropy. We analyse this phenomenology in details with the help of some figures, where lines fade if the conditions that guarantee the validity of our results ($t \ll S$) are not rigorously met.

In Fig. 2.2 we show the time evolution of the effective anisotropy $A_\delta(t)$ for $S = 10$ and some negative values of δ . We see that $A_\delta(t)$ initially works against the magnetic field, favoring the spread of the environmental magnetic moment on the xy -plane. As time goes by, however, $A_\delta(t)$ changes its sign (for $t \simeq 1/|\delta|$), thus preventing the dynamics from freezing by reverting its character into an easy-axis one. As for the dependence on the detuning, we observe that $A_\delta(t)$ stays negative for longer time and displays a deeper minimum for smaller values of $|\delta|$; we understand this evidence by noticing that small values of the detuning entail energy scales for the two subsystems comparable to each other, which implies that the environment closely follows the beat of its quantum partner for a longer time interval.

In Fig. 2.3 we set $\delta = -0.5$ and consider different values of S : we find that $|A_\delta(t)|$ decreases as S increases, to represent the growing inefficacy of Γ in altering the dynamics of its environment as this becomes macroscopic, thus approaching a classical system. In fact, as briefly discussed at the beginning of the chapter, a classical-like dynamics, with no back-action at all, must characterize the magnetic environment when $S \rightarrow \infty$, which conforms to the vanishing of the anisotropy observed for large S in the plot.

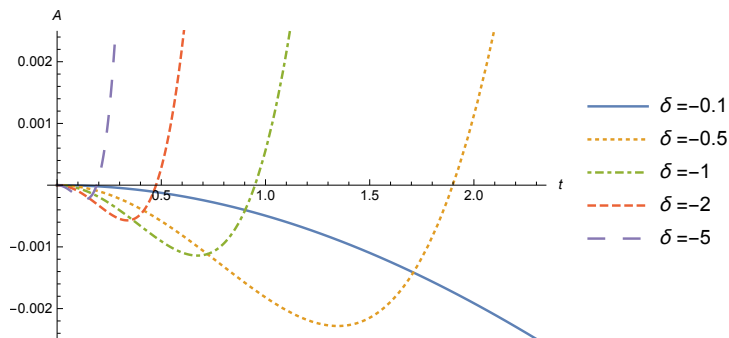


Figure 2.2: Effective anisotropy $A_\delta(t)$ as a function of T , for $S = 10$ and different values of negative δ , as indicated.

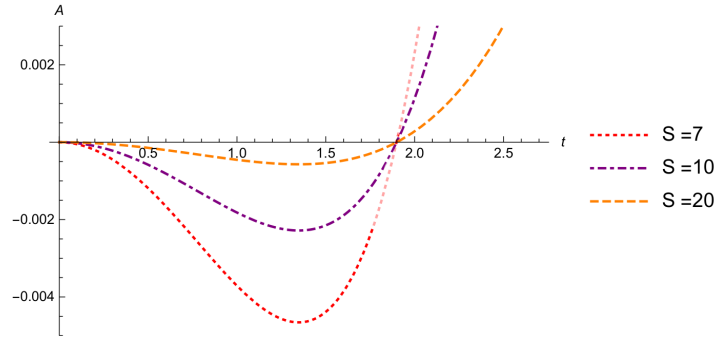


Figure 2.3: Effective anisotropy $A_\delta(t)$ as a function of T , for $\delta = -0.5$ and different values of S , as indicated. The curve for $S = 7$ fades when the validity of the results is not fully under control.

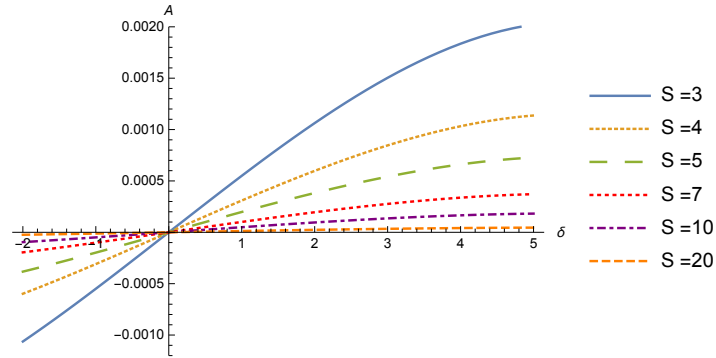


Figure 2.4: Effective anisotropy $A_\delta(t)$ as a function of δ , for $t = 0.1$ and different values of S , as indicated.

In the above comments, and Figs. 2.2 and 2.3, we have considered the case of negative detuning, $f < \omega$. The opposite case, $f > \omega$, trivially follows from $A_\delta(t) = -A_{-\delta}(t)$, as seen from the expression of $G_\delta(t)$ in the Appendix, as well as from Fig. 2.4, where we see that the effective anisotropy at a given time is an odd function of δ , for all values of S .

Observe now how the factorized form of the propagator (2.31) allows us to identify, amongst the overall effects of the interaction between Γ and Ξ , those that do not generate entanglement between the two. This is better seen and understood considering the evolved state for the entire system $\Psi = \Gamma + \Xi$, assuming that its initial state $|\Psi(0)\rangle$ is separable, i.e., $|\Psi(0)\rangle = |\Gamma\rangle \otimes |\Xi\rangle$ (we will hereafter understand the symbol \otimes whenever possible). From Eq. (2.42) we get

$$\begin{aligned} |\Psi(t)\rangle = e^{-i\hat{H}t} |\Gamma\rangle |\Xi\rangle &\sim e^{g(f_\delta(t)\hat{a}\hat{S}^+ - f_\delta^*(t)\hat{a}^\dagger\hat{S}^-)} e^{-i\omega\hat{n}t} |\Gamma\rangle e^{-i\int_0^t \hat{X}_\Xi^{\text{eff}}(\tau) d\tau} |\Xi\rangle \\ &\sim e^{g(f_\delta(t)\hat{a}\hat{S}^+ - f_\delta^*(t)\hat{a}^\dagger\hat{S}^-)} |\Gamma(t)\rangle |\tilde{\Xi}(t)\rangle, \end{aligned} \quad (2.44)$$

where $|\Gamma(t)\rangle = e^{-i\omega\hat{n}t} |\Gamma\rangle$ and $|\tilde{\Xi}(t)\rangle = \exp[-i\int_0^t \hat{X}_\Xi^{\text{eff}}(\tau) d\tau] |\Xi\rangle$ describe the free evolution of the bosonic system and the effective free evolution of the magnetic one, respectively. We have used the notation $|\tilde{\Xi}(t)\rangle$ to underline that while $|\Gamma(t)\rangle$ does

not depend on the interaction between Γ and Ξ , the evolution of $|\tilde{\Xi}(t)\rangle$ is induced not only by the free environmental Hamiltonian $f\hat{S}_z$, but also by the back-action $g^2G_\delta(t)\hat{S}^+\hat{S}^-$ that follows from its coupling with Γ .

In the above expression (2.44) we can recognize a sort of interaction picture with two distinct rotating frames, one for the principal system and one for the environment, that do not move independently. In particular, it is the latter that changes its pace according to the continuous update of the non-commuting components of the environmental magnetic moment implied by an interaction of the TC form. It is worth noticing, in this respect, that the spin commutation relations, that in our case constitute the obstacle to the adoption of an exact interaction picture and the reason why an approximation scheme must be adopted, effectively manifest themselves in the non trivial time dependence of the back-action, to represent their essential role in the quantum dynamics generated by the Hamiltonian (2.2).

Reminding that G is pure imaginary, in Fig. 2.5 we plot $g^2|G_\delta(t)|$ as a function of time for $\delta = -0.5$ and $S = 3, 10$. Its behavior qualitatively shows that the back-action has its maximum effect, at least as far as the time-interval where our approximation holds, for $t \simeq 1/|\delta|$ and vanishes for $t \gtrsim 1/|\delta|$, no matter the value of S .

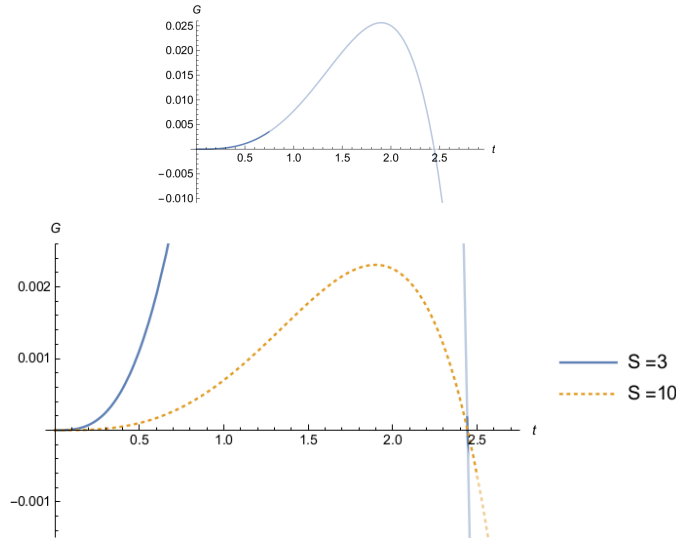


Figure 2.5: The back-action $g^2|G_\delta(t)|$ for $\delta = -0.5$ and different values of S ; the inset shows the $S = 3$ case in its proper plot-range. The curve for $S = 3$ fades when the validity of the results is not fully under control.

Focus finally the attention on the environmental reduced density matrix. Writing the projector $\varrho(t) = |\Psi(t)\rangle \langle \Psi(t)|$ and tracing out the Γ -degrees of freedom, we get

$$\varrho_\Xi(t) \sim \sum_{\gamma} \hat{O}_{\Xi}^{\gamma, \Gamma}(t) |\tilde{\Xi}(t)\rangle \langle \tilde{\Xi}(t)| \hat{O}_{\Xi}^{\gamma, \Gamma \dagger}(t), \quad (2.45)$$

where we have defined the operators

$$\hat{O}_{\Xi}^{\gamma}(t) \equiv \hat{O}_{\Xi}^{\gamma}(t; |\Gamma(t)\rangle) \equiv \langle \gamma | e^{g(f_\delta(t)\hat{a}\hat{S}^+ - f_\delta^*(t)\hat{a}^\dagger\hat{S}^-)} | \Gamma(t) \rangle \quad (2.46)$$

and $\{|\gamma\rangle\}_{\mathcal{H}_\Gamma}$ is an orthonormal basis on \mathcal{H}_Γ . The set of operators $\{\hat{O}_\Xi^\gamma(t)\}$ acting on the Hilbert space of the environment can be interpreted as one possible set of Kraus operators [49], since the completeness relation

$$\sum_\gamma \hat{O}_\Xi^{\gamma\dagger}(t) \hat{O}_\Xi^\gamma(t) = \hat{\mathbb{I}}_\Xi \quad (2.47)$$

holds for all t , as one can easily verify via

$$\begin{aligned} & \sum_\gamma \langle \Gamma(t) | e^{g(f_\delta^*(t)\hat{a}^\dagger\hat{S}^- - f_\delta(t)\hat{a}\hat{S}^+)} |\gamma\rangle \langle \gamma| e^{g(f_\delta(t)\hat{a}\hat{S}^+ - f_\delta^*(t)\hat{a}^\dagger\hat{S}^-)} | \Gamma(t) \rangle = \\ & = \langle \Gamma(t) | e^{g(f_\delta^*(t)\hat{a}^\dagger\hat{S}^- - f_\delta(t)\hat{a}\hat{S}^+)} e^{g(f_\delta(t)\hat{a}\hat{S}^+ - f_\delta^*(t)\hat{a}^\dagger\hat{S}^-)} | \Gamma(t) \rangle \sim \hat{\mathbb{I}}_\Xi . \end{aligned} \quad (2.48)$$

Notice that, as the symbol “ \sim ” reminds us, the results above holds in the limit of our approximation (2.8). The fact that the emerging Kraus operators do not depend on $G_\delta(t)$ is fully consistent with the fact that the back-action does not generate entanglement, as commented above, and rather dynamically renormalizes the environmental free Hamiltonian $\hat{H}_\Xi^{\text{eff}}(t)$. We do also notice that, in order for the back-action to have a non-trivial effect on the environment, the initial state $|\Xi(0)\rangle$ must be different from whatever eigenstate of \hat{S}^z , to avoid the anisotropy term in $\hat{X}_\delta^{\text{eff}}(t)$ to affect $|\Xi(t)\rangle$ only by a phase factor.

We report in Sec. 2.3 the concluding remarks about the analysis of the TC model (2.2) so far developed.

2.2 Effective description of quantum environments as classical fields

In this second part of the chapter, we scrutinize the general idea that the dynamics of a quantum system with a macroscopic environment may be effectively described by a non-autonomous, i.e., time-dependent, Hamiltonian acting on the principal system only. More specifically, we analyse the situation where a principal quantum system Γ interacts with an equally quantum environment Ξ , put into contact with a further external system \mathbb{T} (see Figs. 2.6(a) and 2.6(b)). If Ξ is macroscopic and \mathbb{T} is a thermal bath at high temperature, it may appear intuitive, and naively understood, that Γ effectively evolves as if it were under the influence of a classical fluctuating field. This statement, however, has the nature of an ansatz as far as it is not formally inferred, and conditions ensuring its validity are not given.

Several OQS have been indeed investigated to appraise whether an effective description is viable; general arguments valid also for bipartite systems have been discussed [50, 51, 52], and the effects of the interaction with a classical field have been studied in detail [53, 54, 43, 55, 56, 57, 58]. Parametric representations have also been used to show that classical variables can emerge in quantum Hamiltonians as environmental degrees of freedom [59, 60, 61, 62, 63].

We here study two specific models that go beyond pure dephasing and whose analysis will also serve as an explicit guidance for the most abstract approach that we

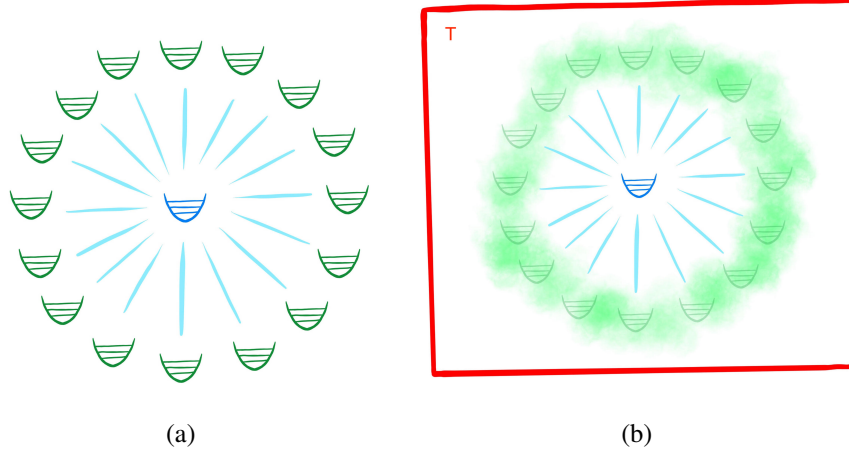


Figure 2.6: Graphical representation of the model. In panel (a) the system is made of a principal system Γ and an environment Ξ ; in panel (b) the system $\Gamma + \Xi$ is put into contact with a further system T .

will adopt in Sec. 2.2.3. In particular, we consider the case where Γ is a bosonic mode coupled with an equally bosonic environment, hereafter called B, which is made of N distinguishable modes that do not interact amongst themselves. The Hamiltonian reads

$$\hat{H} = \nu \hat{a}^\dagger \hat{a} + \sum_k^N (\lambda_{1k} \hat{a}^\dagger + \lambda_{2k} \hat{a}) \hat{b}_k + \sum_k^N (\lambda_{1k}^* \hat{a} + \lambda_{2k}^* \hat{a}^\dagger) \hat{b}_k^\dagger + \sum_k^N \omega_k \hat{b}_k^\dagger \hat{b}_k \quad (2.49)$$

where $[\hat{a}, \hat{a}^\dagger] = 1$ and $[\hat{b}_k, \hat{b}_{k'}^\dagger] = \delta_{kk'}$, with $\nu, \omega_k \in \mathbb{R}$ and $\lambda_{1k}, \lambda_{2k} \in \mathbb{C}$, $\forall k$. Also, we have set $\hbar = 1$, as done throughout this thesis as far as it is not explicitly pointed out. This model has a sibling that describes the case of a spin environment, hereafter called S, made by N distinguishable spin- $\frac{1}{2}$ particles that do not interact amongst themselves. Its dynamics is described by the Hamiltonian

$$\hat{H}^S = \nu \hat{a}^\dagger \hat{a} + \sum_i^N (g_{1i} \hat{a}^\dagger + g_{2i} \hat{a}) \hat{\sigma}_i^- + \sum_i^N (g_{1i}^* \hat{a} + g_{2i}^* \hat{a}^\dagger) \hat{\sigma}_i^+ + \sum_i^N f_i \hat{\sigma}_i^z \quad (2.50)$$

where $[\hat{\sigma}_i^+, \hat{\sigma}_i^-] = 2\delta_{ii} \hat{\sigma}_i^z$, $[\hat{\sigma}_i^z, \hat{\sigma}_i^\pm] = \pm \delta_{ii} \hat{\sigma}_i^\pm$, $f_i \in \mathbb{R}$ and $g_{1i}, g_{2i} \in \mathbb{C}$, $\forall i$. Despite the differences between the two cases, essentially due to the specific algebras of bosonic and spin operators, we study the evolution of the reduced density matrix for the principal system, and we show that the short-time dynamics, defined by Eq. (2.49) and Eq. (2.50) respectively, can be properly described by an effective Hamiltonian $\hat{H}_\Gamma^{\text{eff}}(\zeta)$ acting on Γ only, for the two cases both. The functions ζ embody the remnants of B or S in the form of classical, possibly fluctuating fields, depending on external parameters such as time and temperature.

2.2.1 Bosonic environment

We consider the Hamiltonian (2.49), for either 1) $\lambda_{2k} = 0$, with $\lambda_k \equiv \lambda_{1k}$ finite (linear exchange), or 2) $\lambda_{1k} = 0$, with $\lambda_k \equiv \lambda_{2k}$ finite (parametric hopping), $\forall k$, i.e.,

$$\hat{H}_1 = \nu \hat{a}^\dagger \hat{a} + \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k + \sum_k \left(\lambda_k^* \hat{a} \hat{b}_k^\dagger + \lambda_k \hat{a}^\dagger \hat{b}_k \right), \quad (2.51)$$

$$\hat{H}_2 = \nu \hat{a}^\dagger \hat{a} + \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k + \sum_k \left(\lambda_k^* \hat{a}^\dagger \hat{b}_k^\dagger + \lambda_k \hat{a} \hat{b}_k \right). \quad (2.52)$$

We hereafter use the index $j = 1, 2$ to refer to the exchange and hopping case, respectively. The Heisenberg equations of motion (EOM) for the mode operators are

$$\text{Exchange:} \quad \dot{\hat{a}} = i[\hat{H}_1, \hat{a}] = -i\nu \hat{a} - i \sum_k \lambda_k \hat{b}_k, \quad (2.53a)$$

$$\dot{\hat{b}}_k = i[\hat{H}_1, \hat{b}_k] = -i\omega_k \hat{b}_k - i\lambda_k^* \hat{a}, \quad (2.53b)$$

$$\text{Hopping:} \quad \dot{\hat{a}} = i[\hat{H}_2, \hat{a}] = -i\nu \hat{a} - i \sum_k \lambda_k^* \hat{b}_k^\dagger, \quad (2.54a)$$

$$\dot{\hat{b}}_k^\dagger = i[\hat{H}_2, \hat{b}_k^\dagger] = i\omega_k \hat{b}_k^\dagger + i\lambda_k \hat{a}. \quad (2.54b)$$

If the spectrum of the environment is narrow enough to write $\omega_k \simeq \omega \forall k$, the above EOM can be written as

$$\text{Exchange:} \quad \dot{\hat{a}} = -i\nu \hat{a} - i\Lambda \hat{b}, \quad \dot{\hat{b}} = -i\omega \hat{b} - i\Lambda \hat{a}, \quad (2.55)$$

$$\text{Hopping:} \quad \dot{\hat{a}} = -i\nu \hat{a} - i\Lambda \hat{b}^\dagger, \quad \dot{\hat{b}}^\dagger = i\omega \hat{b}^\dagger + i\Lambda \hat{a}, \quad (2.56)$$

where the bosonic operator \hat{b} is defined as

$$\hat{b} \equiv \frac{1}{\Lambda} \sum_k \lambda_k \hat{b}_k, \quad \text{with} \quad \Lambda^2 \equiv \sum_k |\lambda_k|^2. \quad (2.57)$$

The above Eqs. (2.55-2.56) are the same EOM that one would obtain starting from the two-mode bosonic Hamiltonians

$$\text{Exchange:} \quad \nu \hat{a}^\dagger \hat{a} + \omega \hat{b}^\dagger \hat{b} + \Lambda (\hat{a} \hat{b}^\dagger + \hat{a}^\dagger \hat{b}), \quad (2.58)$$

$$\text{Hopping:} \quad \nu \hat{a}^\dagger \hat{a} + \omega \hat{b}^\dagger \hat{b} + \Lambda (\hat{a}^\dagger \hat{b}^\dagger + \hat{a} \hat{b}), \quad (2.59)$$

describing two oscillators, with different frequencies ν and ω , exchanging quanta through a linear interaction. Notice, though, that such direct relation only exists if $\omega_k \sim \omega, \forall k$, a condition to which we will refer as defining a *narrow energy spectrum*. Both systems of Eqs. (2.55) and (2.56) can be solved by Laplace transform, using the rule $\tilde{a}(s) = s\tilde{a}(s) - a(0)$ to obtain algebraic equations from differential ones. Few

calculations lead us, after back-transforming and recalling that the index $j = 1, 2$ refers to the exchange and hopping respectively, to the solutions

$$\begin{aligned}\hat{a}(t) &= e^{-i\hat{H}_j t} \hat{a} e^{i\hat{H}_j t} = \left[\mu_j(t) \hat{a} + \pi_j(t) \hat{\mathcal{B}}_j \right] e^{-i\omega_j t}, \\ \hat{\mathcal{B}}_j(t) &= e^{-i\hat{H}_j t} \hat{\mathcal{B}}_j e^{i\hat{H}_j t} = \left[(-)^j \pi_j^*(t) \hat{a} + \mu_j^*(t) \hat{\mathcal{B}}_j \right] e^{-i\omega_j t},\end{aligned}\quad (2.60)$$

where $\hat{\mathcal{B}}_1 = \hat{b}$, $\hat{\mathcal{B}}_2 = \hat{b}^\dagger$,

$$\mu_j(t) = \cos(\Delta_j t) - i \frac{\delta_j}{\Delta_j} \sin(\Delta_j t), \quad (2.61a)$$

$$\pi_j(t) = -i \frac{\Lambda}{\Delta_j} \sin(\Delta_j t), \quad (2.61b)$$

with

$$\delta_j = \frac{1}{2} (\nu + (-)^j \omega), \quad (2.62a)$$

$$\omega_j = \frac{1}{2} (\nu - (-)^j \omega), \quad (2.62b)$$

$$\Delta_j^2 = |\delta_j^2 - (-)^j \Lambda^2|, \quad (2.62c)$$

and we have used $\mu_j^*(t) = \mu_j(-t)$. The overall phase factors in the rightmost terms of Eqs. (2.60) suggest that a natural interaction picture exists, corresponding to frames rotating at frequency ω_j . We will use these frames in the following, so as to omit those phase factors. Further notice that $|\mu_j(t)|^2 - (-)^j |\pi_j(t)|^2 = 1$, ensuring that $[\hat{a}(t), \hat{a}^\dagger(t)] = [\hat{b}(t), \hat{b}^\dagger(t)] = 1, \forall t$ and also that $|\mu_j(t)|^2 + (-)^j \pi_j^2(t) = 1$, meaning that the evolutions correspond to rotations in the rotating frames.

Our goal is now to obtain an effective Hamiltonian $\hat{H}_\Gamma^{\text{eff}}(\zeta)$, acting on Γ only, without renouncing to the quantum character of its companion B . This means that we can consider nothing but the time dependence of the reduced density matrix for the principal system Γ

$$\rho_\Gamma(t) = \text{Tr}_B \left[e^{-i\hat{H}_j t} \rho_\Gamma \otimes \rho_B e^{i\hat{H}_j t} \right] \equiv \mathcal{E}_j[\rho_\Gamma](t), \quad (2.63)$$

with the notation $\rho_X \equiv \rho_X(0)$ used hereafter. In particular, as already implied by Eq. (2.63), we want to derive the explicit form of the dynamical map $\mathcal{E}_j[\rho_\Gamma]$ upon assuming that at $t = 0$ the system $\Gamma + B$ is in a factorized state $\rho_\Gamma \otimes \rho_B$. Moreover, we specifically take B initially prepared in the state at thermal equilibrium

$$\rho_B = \frac{1}{1 + n_T} \left(\frac{n_T}{1 + n_T} \right)^{b^\dagger b}, \quad (2.64)$$

where $n_T = (e^{\omega/T} - 1)^{-1}$ is the thermal number of photons, and we have set the Boltzmann constant equal to 1. This implicitly means that B further interacts with a third system T , that is specifically a thermal bath due to the choice of the state (2.64).

We can then positively move towards the derivation of the field ζ entering $\hat{H}_\Gamma^{\text{eff}}$, and of its possible dependence on some external parameter. To this aim we first write the initial state of $\Gamma + B$ using the Glauber formula,

$$\rho_\Gamma \otimes \rho_B = \iint \frac{d^2\gamma' d^2\gamma''}{\pi^2} \chi[\rho_\Gamma](\gamma') \chi[\rho_B](\gamma'') \hat{D}_a^\dagger(\gamma') \otimes \hat{D}_b^\dagger(\gamma''), \quad (2.65)$$

where $\chi[\rho](\gamma) = \text{Tr}[\rho \hat{D}(\gamma)]$ is the characteristic function of the state ρ , and $\hat{D}_x(\gamma) = \exp\{\gamma \hat{x}^\dagger - \gamma^* \hat{x}\}$, with $[\hat{x}, \hat{x}^\dagger] = 1$, is the bosonic displacement operator. In order to get the argument of the partial trace in Eq. (2.63), we use Eqs. (2.60) to write the evolution of the displacement operators entering Eq. (2.65),

$$e^{-i\hat{H}_j t} \hat{D}_a^\dagger(\gamma') \otimes \hat{D}_b^\dagger(\gamma'') e^{i\hat{H}_j t} = \hat{D}_a^\dagger[\mu_j^*(t)\gamma' + \pi_j^*(t)\gamma''] \otimes \hat{D}_b^\dagger[\pi_j^*(t)\gamma' + \mu_j(t)\gamma'']. \quad (2.66)$$

We then perform the partial trace using $\text{Tr}[\hat{D}(\gamma)] = \pi \delta^{(2)}(\gamma)$, so as to get

$$\begin{aligned} \mathcal{E}_j[\rho_\Gamma](t) &= \int \frac{d^2\gamma'}{\pi} \chi[\rho_\Gamma](\gamma') \chi[\rho_B] \left(-\frac{\gamma' \pi_j^*(t)}{\mu_j(t)} \right) \hat{D}^\dagger \left(\frac{\gamma'}{\mu_j(t)} \right) \\ &= \int \frac{d^2\gamma}{\pi} |\mu_j(t)|^2 \chi[\rho_\Gamma](\gamma \mu_j(t)) \chi[\rho_B](-\gamma \pi_j^*(t)) \hat{D}^\dagger(\gamma), \end{aligned} \quad (2.67)$$

where, in the last step, we made the substitution $\gamma' \rightarrow \gamma \mu_j(t)$. Upon expanding the coefficients (2.61) for $\Delta_j t \ll 1$,

$$\mu_j(t) \simeq 1 - i\delta_j t + O(t^2), \quad (2.68a)$$

$$\pi_j(t) \simeq -i\Lambda t + O(t^2), \quad (2.68b)$$

$$|\mu_j(t)|^2 \simeq 1 + O(t^2), \quad (2.68c)$$

and using the explicit form of the characteristic function of the thermal state, $\chi[\rho_B](\gamma) = \exp\{-|\gamma|^2(n_T + \frac{1}{2})\}$, we finally write

$$\rho_\Gamma(t) = \mathcal{E}_j[\rho_\Gamma](t) = \int \frac{d^2\gamma}{\pi} \chi[\rho_\Gamma](\gamma) e^{-|\gamma|^2 \sigma^2(t)} \hat{D}^\dagger(\gamma), \quad (2.69)$$

with $\sigma^2(t) = \Lambda^2 t^2 (n_T + \frac{1}{2})$. We now wonder whether the above map is realized by some known unitary evolution involving the interaction with a classical environment only. Indeed, by first noticing that for any state ϱ it is

$$\chi[\varrho](\gamma) e^{-|\gamma|^2 \sigma^2} = \chi[\varrho_{\text{GN}}](\gamma), \quad (2.70)$$

with

$$\varrho_{\text{GN}} \equiv \int \frac{d^2\alpha}{\pi \sigma^2} e^{-\frac{|\alpha|^2}{\sigma^2}} \hat{D}(\alpha) \varrho \hat{D}^\dagger(\alpha), \quad (2.71)$$

we recognize in Eq. (2.71) the Kraus decomposition corresponding to a Gaussian noise (GN) channel, namely a random displacement with Gaussian modulated amplitude³. The same map [64, 46] describes the evolution of a bosonic system in the presence of a classical fluctuating field, i.e., governed by a non-autonomous Hamiltonian of the form

$$\hat{H}_{\text{stoc}}(t) = \nu \hat{a}^\dagger \hat{a} + \hat{a} \zeta^*(t) e^{i\omega_\zeta t} + \hat{a}^\dagger \zeta(t) e^{-i\omega_\zeta t}, \quad (2.72)$$

where $\zeta(t)$ is a random classical field described by a Gaussian stochastic process $\zeta(t) = \zeta_x(t) + i\zeta_y(t)$ with zero mean $[\zeta_x(t)]_\zeta = [\zeta_y(t)]_\zeta = 0$ and diagonal structure of the autocorrelation function

$$[\zeta_x(t_1)\zeta_x(t_2)]_\zeta = [\zeta_y(t_1)\zeta_y(t_2)]_\zeta = K(t_1, t_2), \quad (2.73a)$$

$$[\zeta_x(t_1)\zeta_y(t_2)]_\zeta = [\zeta_y(t_1)\zeta_x(t_2)]_\zeta = 0. \quad (2.73b)$$

The function $\sigma(t)$ in Eq. (2.70) is in this case

$$\sigma(t) = \int_0^t \int_0^t dt_1 dt_2 \cos[\delta_\zeta(t_1 - t_2)] K(t_1, t_2), \quad (2.74)$$

where $\delta_\zeta = \omega_\zeta - \nu$ is the detuning between the natural frequency ν of Γ and the central frequency ω_ζ of the classical field $\zeta(t)$. The map (2.70) may be obtained, for instance, upon considering the classical environment fluctuating according to a Gaussian Ornstein-Uhlenbeck stochastic process [65] characterized by the autocorrelation function

$$K_\tau^{\text{OU}}(t_1 - t_2) = \frac{G}{2\tau} e^{-\frac{1}{\tau}|t_1 - t_2|}, \quad (2.75)$$

where τ is the correlation time, and G is the amplitude of the process. In the short-time limit, one easily finds that

$$\sigma(t) = \frac{G}{2\tau} t^2. \quad (2.76)$$

In conclusion, we have shown that, as far as $t \ll |\Delta_j|^{-1}$, the effective Hamiltonian $\hat{H}_\Gamma^{\text{eff}}(\zeta(t))$ equals $\hat{H}_{\text{stoc}}(t)$, meaning that

$$\hat{H}_\Gamma^{\text{eff}}(\zeta(t)) = \nu \hat{a}^\dagger \hat{a} + \hat{a} \zeta^*(t) e^{i\omega_\zeta t} + \hat{a}^\dagger \zeta(t) e^{-i\omega_\zeta t}, \quad (2.77)$$

with the field $\zeta(t)$ as from Eqs. (2.73)-(2.76), and $G = 2\tau\Lambda^2(n_T + \frac{1}{2})$. Notice that the dynamical map for Γ in the short-time limit, Eq. (2.69), is the same in the exchange and hopping cases. However, due to the j dependence of Δ_j , the condition defining the above short-time limit is different in the two cases. In fact, the difference is

³To prove the equivalence in Eq. (2.70), start from $\text{Tr}[\rho_{\text{GN}} \hat{D}(\gamma)]$, insert the definition of ρ_{GN} , use the composition rule $\hat{D}^\dagger(\alpha)\hat{D}(\gamma)\hat{D}(\alpha) = \hat{D}(\gamma)e^{\alpha^*\gamma - \gamma^*\alpha}$ and perform the resulting Fourier transform.

removed when the number of environmental modes becomes large, and the effective coupling $\Lambda = \sqrt{\sum_k \lambda_k^2}$ increases accordingly, so that

$$t \ll \frac{1}{\sqrt{(\nu \mp \omega)^2 \pm \Lambda^2}} \xrightarrow{\text{large-}N} t \ll \frac{1}{\Lambda}, \quad (2.78)$$

which establishes a relation between the short-time constraint and some large- N condition that will be further discussed later on. Overall, we have that the interaction (either exchange or hopping) of an oscillator with a bosonic environment induces a dynamics that is amenable to a description in terms of the interaction with a fluctuating classical field if the following conditions can be, at least approximately, met:

- (i) narrow environmental energy spectrum ($\omega_k \simeq \omega \forall k$)
- (ii) short interacting times
- (iii) environment at thermal equilibrium.

It is worth noticing that, if conditions (i)-(iii) hold, the above description in terms of classical fields is valid at all temperatures.

2.2.2 Magnetic environment

We now consider the situation described by the Hamiltonian (2.50), i.e., that of a bosonic mode Γ interacting linearly with a magnetic system \mathbf{S} , made of N spin- $\frac{1}{2}$ particles, each described by its respective Pauli matrices $(\hat{\sigma}_i^x, \hat{\sigma}_i^y, \hat{\sigma}_i^z) \equiv \hat{\boldsymbol{\sigma}}_i$. As in Sec. 2.2.1, we consider both the exchange and the hopping case. Setting 1) $g_{2i} = 0$, with $g_i \equiv g_{1i}$ finite, and 2) $g_{1i} = 0$, with $g_i \equiv g_{2i}$ finite, $\forall i$, from Eq. (2.50) we get

$$\hat{H}_1^S = \nu \hat{a}^\dagger \hat{a} + \sum_i f_i \hat{\sigma}_i^z + \sum_i (g_i^* \hat{a} \hat{\sigma}_i^+ + g_i \hat{a}^\dagger \hat{\sigma}_i^-), \quad (2.79)$$

$$\hat{H}_2^S = \nu \hat{a}^\dagger \hat{a} + \sum_i f_i \hat{\sigma}_i^z + \sum_i (g_i^* \hat{a}^\dagger \hat{\sigma}_i^+ + g_i \hat{a} \hat{\sigma}_i^-), \quad (2.80)$$

where the superscript S refers to the magnetic nature of the environment. Setting $f_i = f, \forall i$, and further choosing $f > 0$, the EOM in the Heisenberg picture are

$$\text{Exchange:} \quad \dot{\hat{a}} = i[\hat{H}_1^S, \hat{a}] = -i\nu \hat{a} - i \sum_{i=1}^N g_i \hat{\sigma}_i^-, \quad (2.81a)$$

$$\dot{\hat{\sigma}}_i^- = i[\hat{H}_1^S, \hat{\sigma}_i^-] = -if \hat{\sigma}_i^- + i\hat{a} g_i^* 2\hat{\sigma}_i^z, \quad (2.81b)$$

$$\text{Hopping:} \quad \dot{\hat{a}} = i[\hat{H}_2^S, \hat{a}] = -i\nu \hat{a} - i \sum_{i=1}^N g_i^* \hat{\sigma}_i^+ \quad (2.82a)$$

$$\dot{\hat{\sigma}}_i^+ = i[\hat{H}_2^S, \hat{\sigma}_i^+] = if \hat{\sigma}_i^+ - i\hat{a} g_i 2\hat{\sigma}_i^z, \quad (2.82b)$$

where we have related the index of the Hamiltonians $\hat{H}_{1,2}^S$ with the exchange and hopping cases, respectively. Despite that Eqs. (2.81)-(2.82) have the same form as

Eqs. (2.53)-(2.54) of the bosonic case, they cannot be solved exactly, due to the different algebra of the spin operators. However, restricting ourselves to physical situations where the operator $\hat{S}^z \equiv \sum_{i=1}^N \hat{\sigma}_i^z$ can be replaced by some reasonable expectation value $\langle \hat{S}^z \rangle \equiv \frac{N}{2} \langle \hat{\sigma}^z \rangle \equiv -\frac{N}{2} m$ (with $m > 0$, due to f being positive), we can rewrite the above EOM in the form

$$\text{Exchange:} \quad \dot{\hat{a}} = -i\nu\hat{a} - i\Lambda^S \hat{S}^-, \quad (2.83a)$$

$$\dot{\hat{S}}^- = -if\hat{S}^- - i\Lambda^S \hat{a} \quad (2.83b)$$

$$\text{Hopping:} \quad \dot{\hat{a}} = -i\nu\hat{a} - i\Lambda^S \hat{S}^+, \quad (2.84a)$$

$$\dot{\hat{S}}^+ = if\hat{S}^+ + i\Lambda^S \hat{a}, \quad (2.84b)$$

with $g = \sqrt{\sum_{i=1}^N |g_i|^2}$, $\Lambda^S = g\sqrt{2m}$, and

$$\hat{S}^+ = \frac{1}{\Lambda^S} \sum_{i=1} g_i \hat{\sigma}_i^+, \quad \hat{S}^- = (\hat{S}^+)^\dagger. \quad (2.85)$$

In fact, these equations can be derived from the Hamiltonians

$$\text{Exchange:} \quad \nu\hat{a}^\dagger\hat{a} + f\hat{S}^z + \Lambda^S (\hat{a}\hat{S}^+ + \hat{a}^\dagger\hat{S}^-), \quad (2.86)$$

$$\text{Hopping:} \quad \nu\hat{a}^\dagger\hat{a} + f\hat{S}^z + \Lambda^S (\hat{a}^\dagger\hat{S}^+ + \hat{a}\hat{S}^-), \quad (2.87)$$

upon further assuming that the commutation relations

$$[\hat{S}^+, \hat{S}^-] = -1, \quad [\hat{S}^z, \hat{S}^+] = \hat{S}^+, \quad [\hat{S}^z, \hat{S}^-] = -\hat{S}^-,$$

hold, meaning that the spin algebra is simplified into a bosonic one. Notice that replacing the total spin operator $\sum_i \hat{\sigma}_i^z$ with an expectation value $\langle \hat{S}^z \rangle = \frac{N}{2} \langle \hat{\sigma}^z \rangle$ we imply that the field f selects the same expectation value $\langle \hat{\sigma}^z \rangle$ for every spin-1/2, in the spirit of the usual random phase approximation. Once linearized, the EOM (2.83)-(2.84) can be solved as in the bosonic case, to get solutions formally analogous to Eqs. (2.60) for the operators \hat{a} and \hat{S}_j , with the replacement $\hat{B}_j(t) \rightarrow \hat{S}_j(t)$ with $\hat{S}_1 = \hat{S}^-$, $\hat{S}_2 = \hat{S}^+$, and $\omega \rightarrow f$ in the magnetic expressions corresponding to Eqs. (2.62). Whatever follows Eq. (2.60) in Sec. 2.2.1 can be easily retraced until the choice of the initial environmental state ρ_S appears into the equation defining the evolution of the reduced density matrix of the principal system, i.e.,

$$\rho_\Gamma(t) = \text{Tr}_S \left[e^{-i\hat{H}_j^S t} \rho_\Gamma \otimes \rho_S e^{i\hat{H}_j^S t} \right] \equiv \mathcal{E}_j^S(\rho_\Gamma). \quad (2.88)$$

Assuming that S is initially prepared in a state at thermal equilibrium, we take

$$\rho_S = \frac{1}{1 + n_T^S} \left(\frac{n_T^S}{1 + n_T^S} \right)^{\hat{S}^+ \hat{S}^-}, \quad (2.89)$$

with $n_T^S \equiv \frac{N}{2}(1-m)$. Despite the formal analogy with Eq. (2.64), it is important to notice that the temperature dependence of n_T^S , and hence that of the dynamical map, is generally different from what we get in the bosonic case, where the thermal number of photons is $n_T = (\exp\{\frac{\omega}{T}\} - 1)^{-1}$. We can, for example, suppose that the magnetic environment thermalizes with the thermal bath so that $\langle \hat{S}^z \rangle = -\text{sign}(f)\hat{S}B_S(x) = -\frac{N}{2}\text{sign}(f)m$, where $S = N/2$ and $B_S(x) = m$ is the Brillouin function

$$B_S(x) = \frac{2S+1}{2S} \coth\left(\frac{2S+1}{2S}x\right) - \frac{1}{2S} \coth\left(\frac{x}{2S}\right), \quad (2.90)$$

with $x = S|f|/T$. With this choice, it is $n_T^S \equiv S(1 - B_S(x))$ and the dependence on T of the bosonic model is only recovered when $T \rightarrow 0$, being $B_S(x) \rightarrow 1 - e^{-x}$ the low temperature limit of Eq. (2.90). Notice that, in order for the above representation to stay meaningful in the large- S limit, temperature must scale as $T \sim S$ so as to guarantee a finite x . Performing the large- S limit, the Brillouin function turns into the Langevin one $L(x) = \coth(x) - \frac{1}{x}$, indeed the classical limit of Eq. (2.90). We observe that the approximations introduced for the spin system are consistent with our aim of finding an effective classical description for the environment. As a matter of fact, once the total spin is guaranteed a constant value S , a classical-like behaviour is expected for a spin-system when $S \gg 1$, and the bosonic expansion given by the Holstein-Primakoff transformation can be safely truncated at its lowest order $\hat{S}^+ \sim \hat{b}^\dagger$ (if $f > 0$, \hat{b}^\dagger being a generic bosonic creation operator) [66]. We can now write the initial state $\rho_\Gamma \otimes \rho_S$ using the Glauber formula as in Eq. (2.65), with the spin displacement operator defined as $\hat{D}_{\hat{S}}(\gamma) = \exp\{\gamma\hat{S}^+ - \gamma^*\hat{S}^-\}$ due to the choice $f > 0$, and hence $\langle \hat{\sigma}^z \rangle < 0$ (had we taken $f < 0$ it would be $\hat{D}_{\hat{S}}(\gamma) = \exp\{\gamma\hat{S}^- - \gamma^*\hat{S}^+\}$). Using the solutions of the EOM (2.83)-(2.84), one can write the evolution of the displacement operators and proceed as done in the previous section up to Eq. (2.69), obtaining that the dynamical map in the magnetic case does also correspond to a Gaussian noise channel. With the additional requirement of a random phase approximation, an effective Hamiltonian of the form of Eq. (2.77) can thus be written again, allowing us to conclude that the set of conditions sufficient to find an effective classical description is the same as in the bosonic model, the only difference being in the temperature dependence of the standard deviation σ^2 , due to the different definition of n_T^S in the magnetic case.

2.2.3 Deriving the classical fields

Upon inspecting the dynamics of both systems in order to retrace the derivation of the short-time dynamics, we notice that no explicit condition on the value of N is involved. This is somehow surprising, given that B and S are named *environment* insofar as the number N of their quantum components is large, virtually infinite in the case of a macroscopic environment. Therefore, in order to understand whether a relation exists between a large value of N and the assumptions of short-time and narrow energy spectrum $\omega_k \simeq \omega$ used in Secs. 2.2.1-2.2.2, we take on the model (2.49) from a more abstract viewpoint. More specifically, we generalize the procedures for

deriving classical theories as large- N limit of quantum ones [8, 9], to the case of composite quantum systems, and find that replacing quantum operators by classical fields for $N \rightarrow \infty$ requires that environmental operators stay well defined in such limit. This, in turn, implies that the environment must feature some global symmetry, as we know it has to be if a classical limit of \mathcal{Q} exists. Our aim is to understand whether the emergence of an effective Hamiltonian $\hat{H}_F^{\text{eff}}(\zeta(t))$, as in Eq. 2.77, is a general feature of OQS with macroscopic environments. We also aim at further clarifying the meaning of the conditions (i)-(iii) given at the end of Sec. 2.2.1, and the reasons why they seem to be utterly necessary in order to obtain an effective Hamiltonian description. Thinking also to what we learnt from the analysis developed in the first part 2.1 of this chapter, the main idea is to show that the emergence of $\hat{H}_F^{\text{eff}}(\zeta)$ is related to the crossover from a quantum to a classical environment, possibly observed when the number of components becomes very large. In fact, were the environment described by a classical theory, its effects on the system would naturally be represented by the classical field ζ .

According to the procedure described in Sec. 1.2 for deriving the classical theory C that formally represents $\mathcal{Q}_{N \rightarrow \infty}$, the first step is that of identifying \mathfrak{g}_N , exploiting the knowledge of the Hilbert space $\mathcal{H} = \otimes_i^N \mathcal{H}_i$, the Lie product, and the Hamiltonian \hat{H}_N (we will equip quantities with the index N to indicate their being relative to the \mathcal{Q}_N theory). As the Hamiltonian \hat{H}_N represents a physical observable, an effective strategy to identify \mathfrak{g}_N is that of writing \hat{H}_N as a linear combination of operators, and see if they belong to some minimal set that generates a representation \mathfrak{g}_N of some abstract Lie algebra \mathfrak{g} . The second step of the procedure is that of finding an irreducible representation \mathfrak{g}_k of \mathfrak{g}_N , which stands as the Lie algebra for \mathcal{Q}_k (notice that this most often implies that an explicit expression for \hat{H}_k does also become available). Here is where the existence of the global symmetry emerges as a necessary ingredient. In fact, the way \mathfrak{g}_k can be most often identified, is writing the original Hamiltonian as a linear combination of some global (i.e., acting non trivially upon each subsystem of the Hilbert space of the theory) operators that are invariant under the symmetry-operations, and generate a representation of the same abstract algebra \mathfrak{g} which is also represented by \mathfrak{g}_N .

In order to be used in the framework of OQS dynamics, this procedure and the results described in Sec. 1.2 need to be generalized, as we deal with the quantum theory of a bipartite system where just one of the two components, namely the environment, is intended to become macroscopic. However, due to the linear structure of the interactions entering Eq. (2.49), the procedure mentioned above can still be applied as follows. Keeping in mind that we have to deal with physically meaningful Lie algebras, we first notice that the coupling terms in Eq. (2.49) can be written as $\hat{a}(\hat{a}^\dagger)$ tensor-times some sum over k of operators acting on \mathcal{H}_B iff either $\lambda_{1k} = \lambda_{2k}$ or $\lambda_{1(2)k} = 0$, for all k . Taking one or the other of the above conditions true is quite equivalent, as far as the following construction is concerned. For the sake of clarity, and at variance with what done in Secs. 2.2.1-2.2.2, we specifically choose $\lambda_{2k} = 0$ and set $\lambda_k \equiv \lambda_{1k}$ finite, for all k , meaning that we explicitly consider the exchange case only. Further taking $\omega_k = \omega \forall k$, as done in Secs. 2.2.1-2.2.2, we can define the

global operators

$$\hat{E} \equiv \frac{1}{N} \sum_k^N \hat{b}_k^\dagger \hat{b}_k \quad \text{and} \quad \hat{B} \equiv \frac{1}{\sqrt{N\Lambda^2}} \sum_k^N \lambda_k \hat{b}_k, \quad (2.91)$$

with $\Lambda^2 \equiv \sum_k^N |\lambda_k|^2$ as in Eq. (2.57), and write the Hamiltonian (2.49) as

$$\hat{H} = \nu \hat{a}^\dagger \hat{a} + N \left[\frac{\Lambda}{\sqrt{N}} (\hat{a}^\dagger \hat{B} + \hat{a} \hat{B}^\dagger) + \omega \hat{E} \right]. \quad (2.92)$$

The way N enters Eqs. (2.91)-(2.92) is designed to recognize $\frac{1}{N}$ as the parameter to quantify quantumness of the environment B, and let all the operators, no matter whether acting on Γ , B, or $\Gamma+B$, independent of the number of environmental modes. The operators (2.91), together with the identity, are easily seen to generate an algebra \mathfrak{h}_4 on \mathcal{H}_B , being

$$[\hat{B}, \hat{B}^\dagger] = \frac{1}{N}, \quad [\hat{B}, \hat{E}] = \frac{1}{N} \hat{B}, \quad [\hat{B}^\dagger, \hat{E}] = -\frac{1}{N} \hat{B}^\dagger. \quad (2.93)$$

However, this cannot be regarded as the Lie algebra \mathfrak{g}_N of some environmental theory, due to the presence of non commuting operators acting on Γ in Eq. (2.92), unless the $N \rightarrow \infty$ limit is taken, as shown below. We now introduce the set of antihermitian operators

$$\{\hat{L}(\epsilon, \beta) \equiv iN(\epsilon \hat{E} + \beta^* \hat{B} + \beta \hat{B}^\dagger)\}, \quad (2.94)$$

where $\beta \in \mathbb{C}$, with $|\beta| \propto \frac{1}{\sqrt{N}}$, while the coefficients $\epsilon \in \mathbb{R}$ do not depend on N . In the large- N limit, where terms which are bilinear in β and β^* can be neglected due to their dependence on N , it is $[\hat{L}_1, \hat{L}_2] = \hat{L}_3$, with $\hat{L}_i \equiv \hat{L}_i(\epsilon_i, \beta_i)$, $\beta_3 = i(\epsilon_1 \beta_2 - \epsilon_2 \beta_1)$, and $\epsilon_3 = 0$, meaning that the set (2.94) is a Lie Algebra. This is indeed the algebra \mathfrak{g}_N whose recognition represents the first step towards the large- N limit of the quantum theory that describes B. It is easily checked that a possible representation \mathfrak{g}_k , of the same abstract algebra represented by \mathfrak{g}_N , is given by the 2×2 matrices

$$\left\{ \ell(\epsilon, \beta) \equiv i \begin{pmatrix} 0 & \beta^* \\ 0 & \epsilon \end{pmatrix} \right\}, \quad (2.95)$$

being $[\ell_1, \ell_2] = \ell_3$, with $\ell_i \equiv \ell_i(\epsilon_i, \beta_i)$, and β_3, ϵ_3 as above. We underline that the choice of a representation \mathfrak{g}_k that contains only either β or β^* is the simplest way to make the presence of non-commuting operators on \mathcal{H}_Γ in the Hamiltonian (2.92) harmless, as far as the following construction is concerned. The matrices $\ell(\epsilon, \beta)$ allow us to write

$$\left[\hat{L}, \begin{pmatrix} 1 \\ \hat{B} \end{pmatrix} \right] \equiv \begin{pmatrix} [\hat{L}, 1] \\ [\hat{L}, \hat{B}] \end{pmatrix} = \begin{pmatrix} 0 \\ -i(\epsilon \hat{B} + \beta^*) \end{pmatrix} \quad (2.96)$$

as

$$\left[\hat{L}, \begin{pmatrix} 1 \\ \hat{B} \end{pmatrix} \right] = \ell^\dagger \begin{pmatrix} 1 \\ \hat{B} \end{pmatrix}, \quad (2.97)$$

with $\ell^\dagger \equiv (\ell^*)^t$, and, quite equivalently,

$$\left[\hat{L}, \begin{pmatrix} 1 & \hat{B}^\dagger \end{pmatrix} \right] = \begin{pmatrix} 1 & \hat{B}^\dagger \end{pmatrix} \ell. \quad (2.98)$$

Let us now consider the unitary operators $\hat{U}(\epsilon, \beta) \equiv \exp\{\hat{L}(\epsilon, \beta)\}$. Given that, for any pair of operators \hat{O} and \hat{P} , it holds

$$e^{-\hat{P}} \hat{O} e^{\hat{P}} = \sum_n \frac{(-1)^n}{n!} \underbrace{[\hat{P}, [\hat{P}, [\dots[\hat{P}, \hat{O}]\dots]]]}_{n\text{-times}}, \quad (2.99)$$

from Eqs. (2.97) and (2.98) it follows

$$\hat{U}^{-1} \begin{pmatrix} 1 \\ \hat{B} \end{pmatrix} \hat{U} = u(\phi, \zeta) \begin{pmatrix} 1 \\ \hat{B} \end{pmatrix} \quad \text{and} \quad \hat{U}^{-1} \begin{pmatrix} 1 & \hat{B}^\dagger \end{pmatrix} \hat{U} = \begin{pmatrix} 1 & \hat{B}^\dagger \end{pmatrix} u^\dagger(\phi, \zeta), \quad (2.100)$$

with

$$u(\phi, \zeta) \equiv \begin{pmatrix} 1 & 0 \\ \zeta & \phi \end{pmatrix}, \quad (2.101)$$

where

$$\phi = e^{i\epsilon} \quad \text{and} \quad \zeta = \frac{\beta}{\epsilon} (e^{i\epsilon} - 1) \quad (2.102)$$

are obtained by explicitly summing the series in Eq. (2.99).

The fact that the set (2.94) is a Lie algebra in the large- N limit reflects upon the unitary operators $\hat{U}(\phi, \zeta)$, in that they form a group in the same limit. This is just the Lie group corresponding to \mathfrak{g}_k , sometimes dubbed *dynamical* [10] or *coherence* [11] group, that defines, together with the arbitrary choice of a reference state $|0\rangle \in \mathcal{H}_B$, the GCS $|u(\phi, \zeta)\rangle \equiv \hat{U}(\phi, \zeta)|0\rangle$ for the theory \mathcal{Q}_k (see App. A). In fact, GCS for \mathcal{Q}_k , hereafter indicated by $|u\rangle \in \mathcal{H}_k$, enter the procedure in the third step due to their being in one-to-one correspondence with points u on a manifold \mathcal{M}_k , whose cotangent bundle is a classical phase space \mathcal{C} . As we know from Sec. 1.2, each GCS $|u\rangle$ of the theory \mathcal{Q}_k defines a point $u \in \mathcal{M}_k$ and a set of conjugate variables $\zeta \in \mathcal{C}$, and $\mathcal{Q}_{k \rightarrow 0}$ is a classical theory \mathcal{C} , with phase space the above cotangent bundle \mathcal{C} , and Hamiltonian $h(\zeta) = \langle u | \hat{H}_k | u \rangle / N$. The last step of the procedure is that of deriving, possibly without knowing the explicit form of the GCS, the expectation values $\langle u | \hat{H}_k | u \rangle$, and finally obtain the effective classical Hamiltonian describing the original quantum system in the $N \rightarrow \infty$ limit. Notice that the role of the parameters N and k , which has been here understood, becomes evident when explicitly employing the procedure, as in the case here considered where it is $k = 1/N$.

Getting back to our model, the operators \hat{B} and \hat{E} transform, as N goes to infinity, into $B(u) \equiv \langle u | \hat{B} | u \rangle / N$ and $E(u) \equiv \langle u | \hat{E} | u \rangle / N$, respectively, since they necessarily are classical operators as far as a sensible classical limit of the theory \mathcal{Q}_N is concerned, implying hence Assumption 4 of Sec. 1.2 hold true. Therefore, in order to find the large- N limit of the Hamiltonian (2.92), we now only need to evaluate $B(u)$ and $E(u)$, even without knowing the explicit form of the GCS, to obtain

$H_N^{\text{eff}}(\zeta)$ from

$$\hat{H} \xrightarrow{N \rightarrow \infty} \nu \hat{a}^\dagger \hat{a} + N \left[\frac{\Lambda}{\sqrt{N}} (\hat{a}^\dagger B(u) + \hat{a} B^*(u)) + \omega E(u) \right] \equiv \hat{H}_N^{\text{eff}}(\zeta), \quad (2.103)$$

where the relation between $|u\rangle$ and ζ is made explicit below. To proceed accordingly, we choose the reference state for the GCS to be $|0\rangle = \Pi_k |0\rangle_k$, with $|0\rangle_k$ such that $\hat{b}_k |0\rangle_k = 0$. This implies, given the separable structure of the operators $\hat{U}(\phi, \zeta)$, that the states $|u\rangle$ are tensor products of single-mode pure states. As a consequence, it is $\langle u | \hat{B} \hat{B}^\dagger | u \rangle = \langle u | \sum_{k'k} \hat{b}_{k'} \hat{b}_k^\dagger | u \rangle = \langle u | \sum_k \hat{b}_k \hat{b}_k^\dagger | u \rangle = NE(u)$, which allows us to determine $B(u)$ and $E(u)$ via

$$\langle u | \begin{pmatrix} 1 \\ \hat{B} \end{pmatrix} \otimes (1 \quad \hat{B}^\dagger) | u \rangle = N \begin{pmatrix} 1 & B^*(u) \\ B(u) & E(u) \end{pmatrix}, \quad (2.104)$$

and finally obtain, by Eqs. (2.100) and again neglecting terms bilinear in β and β^* ,

$$\begin{aligned} \langle 0 | u(\phi, \zeta) \begin{pmatrix} 1 \\ \hat{B} \end{pmatrix} \otimes (1 \quad \hat{B}^\dagger) u^\dagger(\phi, \zeta) | 0 \rangle &= \\ &= \langle 0 | \begin{pmatrix} 1 & \zeta^* + \phi^* \hat{B}^\dagger \\ \zeta + \phi \hat{B} & \zeta \zeta^* + \zeta \phi^* \hat{B}^\dagger + \zeta^* \phi \hat{B} + \phi \phi^* \hat{B} \hat{B}^\dagger \end{pmatrix} | 0 \rangle \\ &= \begin{pmatrix} 1 & \zeta^* \\ \zeta & 1 \end{pmatrix}, \end{aligned} \quad (2.105)$$

i.e., $E(u) = 1/N$ and $B(u) = \zeta/N$.

The above result implies that the original Hamiltonian (2.92) formally transforms, according to Eq.(2.103), as

$$\hat{H} \xrightarrow{N \rightarrow \infty} \hat{H}_\Gamma^{\text{eff}}(\zeta) = (\nu \hat{a}^\dagger \hat{a} + \omega) + \zeta^* \hat{a} + \zeta \hat{a}^\dagger, \quad (2.106)$$

where we have rescaled $\zeta \rightarrow \zeta \Lambda / \sqrt{N}$ and $(\zeta, \zeta^*) \in \mathbb{R}^2$ is any point of the classical phase space \mathcal{M}_B with canonical variables $q \equiv (\zeta + \zeta^*)/2$ and $p \equiv (\zeta - \zeta^*)/(2i)$. Notice that $|\zeta| \propto \Lambda / \sqrt{N}$, which is independent of N by definition. Once Eq. (2.106) is obtained, we can maintain with confidence that the Hamiltonian (2.49), originally acting on $\Gamma + B$, formally transforms, as $N \rightarrow \infty$, into one that exclusively acts on Γ . However, the presence of the classical field ζ is the remnant of the underlying quantum interaction between Γ and the huge number of elementary constituents of B , namely the bosonic modes $\{\hat{b}_k\}_{k=1}^N$. To this respect, notice that the Hilbert space $\mathcal{H}_B = \otimes_k \mathcal{H}_{b_k}$ is replaced by a two-dimensional classical phase space, $\mathcal{M}_B \equiv \mathcal{C}$, with \mathcal{C} the complex plane, implying an impressive reduction of dynamical variables. This reduction is the most striking consequence of the global symmetry that the quantum theory for B must exhibit in order to flow into a well defined classical theory when B is macroscopic. In our case, although we did not explicitly use it, the symmetry is that under permutation of the bosonic modes \hat{b}_k , and that is why we have set $\omega_k = \omega \forall k$. In fact, one can easily check that this is an essential condition for the very

same definition of global operators obeying commutation rules of the form (2.93), which on their turn are necessary to proceed to the definition of the Lie Algebra, and all the rest.

At this point, we notice that $\omega_k = \omega \forall k$ is just the “narrow environmental energy-spectrum condition” (i), discussed at the end of Sec. 2.2.1. In fact, it immediately strikes that the effective Hamiltonian in Eq. (2.106) has the same structure of that in Eq. (2.77), given that the latter refers to an interaction picture that hides the environmental frequency ω . On the other hand, it is somehow puzzling that time does not enter the above construction, which leave us clueless, so far, concerning the relation $\zeta \rightarrow \zeta(t)e^{-i\omega\zeta t}$. Looking for the possible origin of a time dependence in the classical field ζ , we reckon that the results of this section imply the following. Suppose there exists another macroscopic system T that is not coupled with Γ , but interacts with B in such a way that the global symmetry between Γ and B is preserved. The presence of T manifests itself in terms of some parameter τ (think about time and/or temperature, for instance) upon which ζ depends, according to the rule $\zeta = \zeta(\tau)$ provided by the classical theory describing $B + T$. This dependence can be safely imported into the effective description of Γ via $\zeta \rightarrow \zeta(\tau)$ in $\hat{H}_\Gamma^{\text{eff}}(\zeta)$, Eq. (2.106), as far as the direct interaction between Γ and T can be neglected, at least on the time scales one is interested in, defining the evolution of Γ induced by B.

Finally, we notice that the detuning $\nu - \omega$ does not play any role in this section, which brings us back to Eq. (2.78) and the possible relation between the large- N condition here enforced and the short-time approximation previously adopted.

2.3 Conclusions

The case where the (large) environment of an OQS is part of an hybrid quantum scheme has been considered here.

In the first part of the chapter, we specifically referred to a magnetic environment Ξ , made of a large number N of spin-1/2 particles, coupled with a quantum mechanical oscillator Γ , and studied its dynamics by a large- S approximation that represents the macroscopicity of Ξ , since $N \geq 2S$ holds, without totally wiping its quantum character, since S is finite. Moreover, such an approximation allowed us to obtain a factorized expression for the propagator of the composite system and find that, due to the coupling between Γ and Ξ , a specific term appears, effectively representing the back-action of the principal system on its environment. In fact, as mentioned at the beginning of the chapter, if Ξ is the measuring instrument used for getting information on, or exert our control upon, the quantum system Γ , a strategy that makes the dynamics of Ξ the most sensitive to its interaction with Γ might reveal details, or allow a steering precision, otherwise inaccessible. In this respect, the lessons learnt in Sec 2.1 are as follows.

(1) Detuning: $\delta = \omega - f$ must be finite if one wants to observe footprints of Γ into an effectively-free evolution of Ξ , i.e., without further interacting with Γ itself. Off-resonance is a necessary condition for the back-action to switch on.

(2) Timing: Depending on the value of δ and S , there is a finite time interval, that can be well within the range of validity of our results as shown in Figs. 2.2-2.5, where the back-action is stronger, meaning that effects of Γ on the dynamics of Ξ could be more pronounced.

(3) Magnetic properties: Although our results are obtained in the large- S approximation, it is important that S stays finite, to avoid the disentangled dynamics of Ξ being just a silent Larmor precession. For the same reason, it is important that Ξ be prepared in an initial state which is not an eigenstate of \hat{S}^z ; significantly, we have seen [67] that spin coherent states [9, 68, 66] might be a particularly significant choice.

We conclude the report on Sec 2.1 by mentioning that the method here used for implementing the large- S approximation, i.e., making explicit the dependence of the spin algebra on the quantity parameter $1/S$, and then requiring the interaction Hamiltonian to stay finite as such parameter drops, is general and might turn useful in studying other quantum systems with several interacting components, amongst which is a macroscopic one, furthermore preserving the geometry of the spin sphere.

In the second part of the chapter, we have addressed the dynamics of a bosonic system Γ coupled to either a bosonic B or a magnetic environment S. In particular, we have discussed the conditions under which the dynamics of the system Γ may be described in terms of the effective interaction with a classical fluctuating field.

Our results show that for both kinds of environments an effective, time-dependent, Hamiltonian description may be obtained for short interaction time and environments with a narrow energy spectrum at thermal equilibrium. The corresponding dynamics is described by a Gaussian noise channel independently of the kind of environment, their magnetic or bosonic nature entering only the form of the noise variance. Moreover, exploiting a general treatment based on the large- N limit of the environment, we have clarified the origin and the meaning of the narrow-environmental-spectrum and short-time conditions. In fact, we find that $\omega_k \simeq \omega \forall k$ is needed for a global symmetry to emerge and characterize the environment, which is a necessary ingredient for the environment to be described by a small number of macroscopic variables. On the other hand, the large energy scale implied by whatever coupling with a macroscopic environment limits any effective description to short times only. Overall, the results obtained in Sec. 2.2 indicate that quantum environments may be described by classical fields whenever global symmetries allows one to define environmental operators that remain well defined when increasing the size, i.e., the number of dynamical variables, of the environment. This is a quite general criterion that may serve as a guideline for further analysis, e.g. for fermionic principal systems and/or environments compound of both bosonic and spin parts.

Chapter 3

Large environment as *measuring apparatus*

There are two closely-related questions about the quantum mechanical nature of our universe that keep being intriguing: why we do not experience states superpositions, and why we all observe the same world around us. Whether these questions be open or not depends on the viewpoint one takes on the answers that have been proposed in decades of thought processing. It is a fact, however, that the subject of both questions is “we”, which underlines the importance of being big as we are, and that the verbs are *experience* and *observe*, which brings into the play the quantum measurement process. Indeed, the reason why we cannot even see states superposition when observing quantum systems, is somehow assumed to be due to the continuous measurement process acted upon by the environment they interact with. However, despite often considered as a satisfactory answer, this argument is not a formal result, with attempts to make it as such only recently proposed [69, 70]. Indeed, the most recent analysis of the quantum measurement process [71], its hamiltonian description [72, 34], as well as its characterization in the framework of the OQS dynamics [73] has revealed the qualitative nature of the above argument, thus making it ever more urgent a rigorous approach to the original question, which is in fact what we aim at providing in this chapter.

We have already highlighted that macroscopicity and classicality go together, since there cannot be the latter without the former, but they are not the same thing, since macroscopic quantum systems do actually exist, implying that the large- N condition is not sufficient per-sé for a system made of N quantum particles to behave classically - see the discussion of Sec. 1.1. In what follows the assumptions, isolating the minimal structure that any quantum theory should possess if it has to have a classical limit, will formally characterize the quantum environment, so that its large- N limit embody its macroscopic and classical limit in unison. On the other hand, since we want to investigate around the observation of quantum systems, the principal system needs to stay intrinsically quantum, a fact that immediately sets us again on one of those exemplary situations to which this thesis is indeed devoted and for which the PRECS presented in Sec. 1.4.1 has been designed. Exploiting the results

of Secs. 1.2 and 1.3 for the large- N limit of quantum theories in the framework of the OQS dynamics, we here show that details of the interaction between a quantum principal system Γ and its environment Ξ are irrelevant in determining the state of Ξ at any time τ in the large- N limit, as long as such limit implies a classical behaviour for Ξ itself¹. If this is the case, in fact, such state can always be recognized as the one of an apparatus that measures some observable of the principal system, and this observable is the same for each macroscopic subsystem into which one can think Ξ to be divided. The relation between our findings and the two questions that open this section is evident.

The chapter is structured as follows. In the first section we describe the measurement process focusing on its dynamical description. In Sec. 3.2 we define the dynamical maps characterizing the different evolutions that we aim at comparing, goal that we get in Sec. 3.3 exploiting the large- N limit for the environment. In Sec. 3.4 we report some comments about the results obtained and the assumptions made, and finally we sum up the meaning of what we will find in this chapter in Sec. 3.5.

3.1 Measure-like dynamics

A measurement is an operation performed on a quantum system Γ to produce a set of results, once assumed the existence of a quantum apparatus Ξ that, interacting with Γ , allows us to obtain the results themselves (see App. B for the formal definition). To build a consistent theory of the quantum measurement process, one needs therefore to fit the OQS framework in order to define the scheme suitable for defining a proper dynamical process. Such scheme was originally introduced by von Neumann [74] and later characterized by Ozawa [72] under the name of *conventional measuring process* of non-degenerate sharp observables². In this picture the measuring apparatus plays the role of the quantum environment Ξ of the observed system Γ and the interaction between the two is considered quantum as well. Before describing it, notice that although the Ozawa's scheme allow us to understand many of the process features, the discontinuous change of the Γ -state vector, due to the measurement of a physical quantity and usually named *collapse of the state*, clashes with the otherwise unitary dynamics of the overall isolated system $\Gamma + \Xi$ and it is not included in such description. In fact, this part of the measurement process is still debated, together with the so called *output production*, i.e., the mechanism through which only one result is ultimately selected among the many ones simultaneously available. One possibility, exploiting the PRECS to describe the measurement process [34], is that of understanding them as due to a mechanism of global symmetry breaking ruled by

¹The results here presented have been mainly reported in

- “Whenever a quantum environment emerges as a classical system, it behaves like a measuring apparatus”, C. Foti, T. Heinosaari, S. Maniscalco, P. Verrucchi, arXiv:1810.10261.

²An observable is called *sharp* when associated to a projective measure.

the macroscopic character of the measuring apparatus. In any case, the collapse and the output production are different processes with respect to the dynamical evolution described by the conventional model, and, albeit representing its natural continuation, they are not included in the minimal interpretation described in App. B. We will not further consider them in this thesis.

The key players of the measurement process are exactly those we have dealt with so far, i.e., a principal system Γ and its environment Ξ . It is understood that Ξ represents the measurement apparatus and Γ the system to be measured, both taken as genuinely quantum. In what follows we will use the symbol “ \rightarrow ” to indicate the transformation caused by the measurement process.

Consider now a state of the total system $\Psi = \Gamma + \Xi$ that before the measurement takes place ($t \leq 0$) be separable, i.e., such that $\forall t \leq 0$ be

$$|\Psi(t \leq 0)\rangle = |\Gamma\rangle |\Xi\rangle, \quad (3.1)$$

so that the result of the measurement can be ascribed to the system Γ in the pure state $|\Gamma\rangle$. According to von Neumann, we then postulate the existence of a *preferred basis* $\{|p\rangle\}_{\mathcal{H}_\Gamma}$ for Γ , such that its elements are not disturbed by the measurement, yielding

$$|p\rangle |\Xi\rangle \rightarrow |p\rangle |\Xi^p\rangle \quad \forall |p\rangle \in \{|p\rangle\}_{\mathcal{H}_\Gamma}. \quad (3.2)$$

We assume that what holds just for one state of $\{|p\rangle\}_{\mathcal{H}_\Gamma}$ holds also when a linear combination of $|p\rangle$ is taken, so that for every initial separable state of $\Gamma + \Xi$ we can always write $|\Gamma\rangle = \sum_p a_p |p\rangle$, and get

$$|\Gamma\rangle |\Xi\rangle = \sum_p a_p |p\rangle |\Xi\rangle \rightarrow \sum_p a_p |p\rangle |\Xi^p\rangle. \quad (3.3)$$

Finally, assuming the orthogonality of the states $\{|\Xi^p\rangle\}$, a one-to-one correspondence between elements of the preferred basis and states of Ξ is established. Notice that the final state appearing in the above relation (3.3) is a fictitious state, in no way accessible, and, since the last step of extracting the relevant information from the apparatus Ξ is not considered, such state is often named *unselected measurement*, while the corresponding stage of the measurement process is the *premeasurement process*. This scheme not only needs to assume the existence of a preferred basis, but also its behaviour according to relation (3.3), which seems quite a top-down assumption and that is why it is considered a problem, namely the *problem of the preferred basis* [16]. In order to translate the transformation “ \rightarrow ” into a dynamical description, as originally done by Ozawa [72], let us consider a projective measurement (PVM) with measurement operators $\{|\pi\rangle\langle\pi|\}$ acting on \mathcal{H}_Γ . According to the Ozawa’s model, the first step corresponds to a unitary evolution of Ψ , that is determined by the propagator $\exp[-it\hat{H}^M]$, with

$$\hat{H}^M = g\hat{O}_\Gamma \otimes \hat{O}_\Xi + \hat{\mathbb{I}}_\Gamma \otimes \hat{H}_\Xi, \quad (3.4)$$

where g is the coupling constant, $\hat{O}_\Gamma = \sum_\pi \omega^\pi |\pi\rangle\langle\pi|$ is the hermitian operator on \mathcal{H}_Γ associated to the measured observable, while \hat{O}_Ξ is the operator on \mathcal{H}_Ξ conjugate to the one which is usually referred to as the pointer observable - see App. B; moreover,

\hat{H}_Ξ acts on \mathcal{H}_Ξ only, being $\hat{\mathbb{1}}_\Gamma$ the identity on \mathcal{H}_Γ . The Hamiltonian (3.4) defines the *standard model* [71] for describing premeasurements as dynamical processes. Writing the initial state of Γ in the basis $\{|\pi\rangle\}_{\mathcal{H}_\Gamma}$ of the \hat{O}_Γ eigenvectors, $|\Gamma\rangle = \sum_\pi a_\pi |\pi\rangle$, from Eq. (3.4) we get

$$|\Psi(t \leq 0)\rangle = |\Gamma\rangle |\Xi\rangle \rightarrow |\Psi(t)\rangle = e^{-it\hat{H}^M} |\Gamma\rangle |\Xi\rangle = \sum_\pi a_\pi |\pi\rangle |\Xi^\pi(t)\rangle, \quad (3.5)$$

at any time during the premeasurement $0 \leq t < T$ with T representing the time of the output production, where we have defined the environmental states

$$|\Xi^\pi(t)\rangle \equiv e^{-i\hat{H}_\Xi^\pi t} |\Xi\rangle, \quad (3.6)$$

and

$$\hat{H}_\Xi^\pi \equiv g\omega^\pi \hat{O}_\Xi + \hat{H}_\Xi. \quad (3.7)$$

In what follows, we will name the evolution (3.5) *measure-like dynamics* and the Hamiltonian \hat{H}^M , defined in Eq. (3.4), *measure-like Hamiltonian*. Despite the formal analogy of Eqs. (3.3) and (3.5), in order that the latter evolution correspond to the transformation idealized by von Neumann, the states $|\Xi^\pi(t)\rangle$ need to be orthogonal. This is not true in general, but the time dependence of the states $|\Xi^\pi(t)\rangle$ allows us to require that such condition hold in a specific time interval from which we will then take the instant t . From Eq. (3.5) we have that the reduced density matrix of Γ at a generic time t is

$$\begin{aligned} \varrho_\Gamma(t) &= \text{Tr}_\Xi \left[\sum_{\pi\pi'} a_\pi a_{\pi'}^* |\pi\rangle \langle \pi'| \otimes |\Xi^\pi(t)\rangle \langle \Xi^{\pi'}(t)| \right] = \\ &= \sum_\pi |a_\pi|^2 |\pi\rangle \langle \pi| + \sum_{\pi\pi'} a_\pi a_{\pi'}^* |\pi\rangle \langle \pi'| \langle \Xi^\pi(t) | \Xi^{\pi'}(t) \rangle, \end{aligned} \quad (3.8)$$

and, thus, requiring the orthogonality of the states $|\Xi^\pi(t)\rangle$ implies that the off-diagonal terms of $\varrho_\Gamma(t)$ in the basis $\{|\pi\rangle\}_{\mathcal{H}_\Gamma}$ be null. On the other hand, the cancellation of such terms embodies the well known phenomenon of the decoherence with respect to the basis $\{|\pi\rangle\}_{\mathcal{H}_\Gamma}$, corresponding to the information loss about the phase relationship between the components of the initial state $|\Gamma\rangle$ written as a linear combination of the elements of the basis $\{|\pi\rangle\}_{\mathcal{H}_\Gamma}$ [75, 1]. This is the reason why decoherence plays a fundamental role in the measurement process, since indeed it is such phenomenon that allows us to identify the time interval prior of the output production as the one during which the observed system does not show coherence with respect to the basis $\{|\pi\rangle\}_{\mathcal{H}_\Gamma}$, that in other words means the interval during which decoherence has already occurred.

Getting back to the final state in Eq. (3.5), we notice that if more operators \hat{O}_{Γ_i} appear in \hat{H}^M , they must satisfy

$$[\hat{O}_{\Gamma_i}, \hat{O}_{\Gamma_j}] = 0 \quad \forall i, j. \quad (3.9)$$

This condition implies that a basis $\{|\pi\rangle\}_{\mathcal{H}_\Gamma}$ of common \hat{O}_Γ -eigenvectors does actually exist, so that to guarantee the process does not imply the simultaneous measurement of non-commuting observables, and a one-to-one correspondence between π and $|\Xi^\pi(t)\rangle$ is still univoquely established. In fact, the possibility of extracting information about Γ reporting on Ξ , relies on such correlation, i.e., on the dynamical entanglement generation induced by the interaction $g\hat{O}_\Gamma \otimes \hat{O}_\Xi$, if and only if $|\Xi\rangle$ is not an eigenvector of \hat{O}_Ξ . On the other hand, the measuring apparatus is expected to be in a stationary state before the above interaction is switched on. Therefore, it is usually taken $\hat{H}_\Xi |\Xi\rangle = E_\Xi |\Xi\rangle$ and $[\hat{O}_\Xi, \hat{H}_\Xi] \neq 0$.

From the above description the von Neumann scheme is recognized as a unitary evolution of the overall isolated system Ψ , and the preferred basis $\{|p\rangle\}_{\mathcal{H}_\Gamma}$ naturally emerges as that of the eigenvectors of the Γ -operators entering the interaction between Γ and the measuring apparatus. The problem of the preferred basis is shifted thus in the choice of such operators. However, this is not a further logic problem for the theory, since this latter choice is inherently dictated by the choice of the physical observable one wants to measure, which is indeed a legitimate decision of the observers.

We end this section mentioning that giving a hamiltonian description of more general quantum measurement processes, i.e., identifying the appropriate propagator for the dynamics of such processes up to the output production, it is a very relevant problem that has recently attracted the interest of several authors, including people from my research group and myself.

3.2 Dynamical maps

Let us now consider the unitary evolution of an isolated bipartite system $\Psi = \Gamma + \Xi$, with Hilbert space $\mathcal{H}_\Gamma \otimes \mathcal{H}_\Xi$; being Ψ isolated, it is

$$|\Psi(t)\rangle = e^{-it\hat{H}} |\Psi\rangle , \quad (3.10)$$

where $\hbar = 1$ and \hat{H} is any Hamiltonian, describing whatever interaction between Γ and Ξ , as long as physically meaningful. The state $|\Psi\rangle$ is assumed separable

$$|\Psi\rangle = |\Gamma\rangle |\Xi\rangle , \quad (3.11)$$

meaning that we begin studying the evolution at a time $t = 0$ when both Γ and Ξ are in pure states. This is not a neutral assumption, and we will get back to it in Sec. 3.4. The evolution described by Eq.(3.10) will be hereafter dubbed *true*, by this referring to its being determined by the actual Hamiltonian \hat{H} .

Consider now that, at any fixed time τ , it exists a Schmidt decomposition of the state (3.10),

$$|\Psi(\tau)\rangle = \sum_{\gamma} c_{\gamma} |\gamma\rangle |\xi_{\gamma}\rangle , \quad (3.12)$$

with $\gamma = 1, \dots, \gamma_{\max} \leq \dim \mathcal{H}_\Gamma$, $c_\gamma \in \mathbb{R}^+$ and $\sum_\gamma c_\gamma^2 = 1$ ³. The states $\{|\gamma\rangle\}_{\mathcal{H}_\Gamma}$, and $\{|\xi_j\rangle\}_{\mathcal{H}_\Xi}$ with $j = 1, \dots, \dim \mathcal{H}_\Xi$, make up what we will hereafter call the τ -Schmidt bases, to remind that the Schmidt decomposition is state-specific and therefore depends on the time τ appearing in the left hand side of Eq.(3.12) (in whose right hand side we have instead understood the τ -dependence of c_γ , $|\gamma\rangle$, and $|\xi_\gamma\rangle$, for the sake of a lighter notation). Consistently with the idea that Ξ is a macroscopic system, we take $\gamma_{\max} < \dim \mathcal{H}_\Xi$: therefore, the states $\{|\xi_\gamma\rangle\}_{\mathcal{H}_\Xi}$ entering Eq. (3.12) are a subset of the pertaining τ -Schmidt basis. Given that $|\Gamma\rangle$ is fully generic, the unitary evolution (3.10) defines, via $\varrho_\Xi = \text{Tr}_\Gamma \varrho_\Psi$, the CPTP linear map (from Γ - to Ξ -states)

$$\mathcal{E}: |\Gamma\rangle\langle\Gamma| \rightarrow \varrho_\Xi = \sum_\gamma c_\gamma^2 |\xi_\gamma\rangle\langle\xi_\gamma|. \quad (3.13)$$

In principle, being the output ϱ_Ξ a convex sum of orthogonal projectors, Eq. (3.13) might describe a projective measurement acted upon by Ξ on the principal system Γ , by what is often referred to as *measure and prepare* (m&p) map in the literature. However, for this being the case, the *probability reproducibility condition* must also hold (see the last section in App. B), meaning that, written

$$|\Gamma\rangle = \sum_\gamma a_\gamma |\gamma\rangle, \quad (3.14)$$

it should also be $c_\gamma^2 = |a_\gamma|^2, \forall \gamma$, which cannot be generally true, if only for the τ -dependence of the Schmidt coefficients $\{c_\gamma\}$ which is not featured by the set $\{a_\gamma\}$. In fact, a dynamical model exists for which $|c_\gamma|^2 = |a_\gamma|^2, \forall \gamma$ and $\forall \tau$: the Ozawa's model for projective measurements described in the previous section. Such model is defined by the Hamiltonian (3.4), that is not generic, but it is identified by condition (3.9), i.e., that of containing operators acting on Γ which must commute. This is

³A very useful tool for studying composite systems is provided by the famous *Schmidt decomposition theorem*, an important result that pertains the tensor product structure and that can be proven using standard linear algebra [76, 49].

Theorem (Schmidt decomposition). *If $|\Psi\rangle$ is a vector of a composite system $\mathcal{H}_\Psi = \mathcal{H}_\Gamma \otimes \mathcal{H}_\Xi$, there always exist an orthonormal basis $\{|i_\Gamma\rangle\}_{\mathcal{H}_\Gamma}$ for \mathcal{H}_Γ , an orthonormal basis $\{|i_\Xi\rangle\}_{\mathcal{H}_\Xi}$ for \mathcal{H}_Ξ , and non negative real numbers $\{r_i\}$ satisfying $\sum_{i=1}^N r_i^2 = 1$, such that*

$$|\Psi\rangle = \sum_{i=1}^N r_i |i_\Gamma\rangle |i_\Xi\rangle,$$

where $N \equiv \min\{\dim \mathcal{H}_\Gamma, \dim \mathcal{H}_\Xi\}$.

The bases $\{|i_\Gamma\rangle\}_{\mathcal{H}_\Gamma}$ and $\{|i_\Xi\rangle\}_{\mathcal{H}_\Xi}$ are called *Schmidt bases*, while the coefficients $\{r_i\}$ are called *Schmidt coefficients*. Notice that the Schmidt decomposition is state dependent and ensures that, given a state $|\Psi\rangle$, there exists a canonical local basis in each subsystem Hilbert space thanks to which the double sum, usually appearing in a generic state of a bipartite system $|\Psi\rangle = \sum_{\gamma,\xi} c_{\gamma\xi} |\gamma\rangle |\xi\rangle$, is replaced by a single sum that, moreover only involves, at most, a number of elements equal to the dimension of the smallest Hilbert space in the tensor product. The number of non-zero Schmidt coefficients is called *Schmidt rank* and in some sense quantify the amount of entanglement between Γ and Ξ ; a state is separable iff its rank is equal to 1.

the reason why we specifically labeled it \hat{H}^M , with the apex M hinting at the corresponding measurement process, and distinguishing it from the generic Hamiltonian defining the true evolution (3.10).

Once established that Eq. (3.13) does not define a m&p map, we can nonetheless use the elements provided by the Schmidt decomposition as ingredients to construct a measure-like Hamiltonian \hat{H}^M , whose corresponding m&p map $\mathcal{E}^M : |\Gamma\rangle\langle\Gamma| \rightarrow \varrho_{\Xi}^M$ be the “nearest” possible to the actual \mathcal{E} defined in Eq. (3.13).

To this aim, we first use the τ -Schmidt bases, $\{|\gamma\rangle\}_{\mathcal{H}_\Gamma}$ and $\{|\xi_j\rangle\}_{\mathcal{H}_\Xi}$, to define the hermitian operators

$$\hat{O}_\Gamma = \sum_{\gamma} \varepsilon_{\gamma} |\gamma\rangle\langle\gamma| \quad , \quad \hat{O}_\Xi = \sum_j E_j |\xi_j\rangle\langle\xi_j| \quad , \quad (3.15)$$

with $\varepsilon_{\gamma}, E_j$ arbitrary real numbers, and the interaction Hamiltonian

$$\hat{H}^M = g\hat{O}_\Gamma \otimes \hat{O}_\Xi \quad , \quad (3.16)$$

according to the form (3.4) prescribed by the Ozawa’s model. Further using the Schmidt coefficients, we construct the separable state

$$|\Psi^M\rangle = |\Gamma\rangle |\Xi^M\rangle \quad , \quad (3.17)$$

where $|\Gamma\rangle$ is the same as in Eq. (3.11), while $|\Xi^M\rangle = \sum_{\gamma} c_{\gamma} |\xi_{\gamma}\rangle$, with c_{γ} and $|\xi_{\gamma}\rangle$ as in Eq. (3.12). Finally, we define

$$|\Psi_{\tau}^M\rangle \equiv e^{-i\tau\hat{H}^M} |\Psi^M\rangle \quad , \quad (3.18)$$

that reads, using $\hat{O}_\Gamma |\gamma\rangle = \varepsilon_{\gamma} |\gamma\rangle$, $\hat{O}_\Xi |\xi_{\gamma}\rangle = E_{\gamma} |\xi_{\gamma}\rangle$, and $|\Gamma\rangle = \sum_{\gamma} a_{\gamma} |\gamma\rangle$,

$$|\Psi_{\tau}^M\rangle = e^{-i\tau\hat{H}^M} \sum_{\gamma} a_{\gamma} |\gamma\rangle \sum_{\gamma'} c_{\gamma'} |\xi_{\gamma'}\rangle = \sum_{\gamma, \gamma'} a_{\gamma} |\gamma\rangle c_{\gamma'} e^{-i\varphi_{\gamma\gamma'}} |\xi_{\gamma'}\rangle \quad , \quad (3.19)$$

with $\varphi_{\gamma\gamma'} \equiv \tau g \varepsilon_{\gamma} E_{\gamma'} \in \mathbb{R}$.

Given that $|\Gamma\rangle$ is fully generic, equation (3.18) defines, via $\varrho_{\Xi} = \text{Tr}_{\Gamma} \varrho_{\Psi}$, the CPTP map from Γ - to Ξ -states

$$\mathcal{E}^M: |\Gamma\rangle\langle\Gamma| \rightarrow \varrho_{\Xi}^M = \sum_{\gamma\gamma'\gamma''} |a_{\gamma}|^2 c_{\gamma'} c_{\gamma''} e^{i(\varphi_{\gamma\gamma''} - \varphi_{\gamma\gamma'})} |\xi_{\gamma'}\rangle\langle\xi_{\gamma''}| \quad . \quad (3.20)$$

Comparing Eqs. (3.20) and (3.13) we see that \mathcal{E}^M has the right coefficients $\{|a_{\gamma}|^2\}$ but the wrong form, i.e., it is not a sum of orthogonal projectors, while \mathcal{E} has the correct form but with the wrong coefficients, $\{c_{\gamma}^2\}$. In fact, were these two maps equal in some limit, it would mean that, for each time τ , it exists an observable for Γ such that the state into which Ξ has evolved due to its true interaction with Γ is the same, in such limit, as if Ξ were some measuring apparatus proper to that observable, which is quite a statement. On the other hand, being \mathcal{E} and \mathcal{E}^M linear, they are the same map iff the output states ϱ_{Ξ} and ϱ_{Ξ}^M are equal for whatever input $|\Gamma\rangle$. We can therefore concentrate upon the structure of such output states, which we do in the next section by the PRECS.

3.3 A macroscopic environment that behaves classically

One important aspect of the GCS construction, that we have seen in Sec. 1.2, is that it ensures the Husimi function $\langle \Lambda | \varrho | \Lambda \rangle$ for whatever state ϱ is a well-behaved probability distribution on \mathcal{M} that uniquely identifies ϱ itself. As a consequence, studying $\langle \Lambda | \varrho | \Lambda \rangle$ on \mathcal{M} is fully equivalent to perform a state-tomography of ϱ on the Hilbert space. In other terms, once GCS are available one can analyse any state ϱ of the system by studying its Husimi function on \mathcal{M} , which is what we will do in the following.

The PRECS of whatever pure state is given by Eq. (1.28), that we rewrite here for convenience

$$|\psi\rangle = \int_{\mathcal{M}} d\mu(\Lambda) \chi(\Lambda) |\phi(\Lambda)\rangle |\Lambda\rangle, \quad (3.21)$$

where $|\phi(\Lambda)\rangle$ is a normalized state for Γ that parametrically depends on Λ and $\chi(\Lambda)$ is the real function on \mathcal{M} , whose square $\chi(\Lambda)^2 = \langle \Lambda | \varrho_{\Xi} | \Lambda \rangle$ is the environmental Husimi function relative to $\varrho_{\Xi} = \text{Tr}_{\Gamma} [|\psi\rangle\langle\psi|]$.

In particular, for the states (3.12) and (3.19), it is

$$\chi(\Lambda)^2 = \sum_{\gamma} c_{\gamma}^2 |\langle \Lambda | \xi_{\gamma} \rangle|^2, \quad (3.22)$$

and

$$\chi^{\mathcal{M}}(\Lambda)^2 = \sum_{\gamma\gamma'\gamma''} |a_{\gamma}|^2 c_{\gamma'} c_{\gamma''} e^{i(\varphi_{\gamma\gamma''} - \varphi_{\gamma\gamma'})} \langle \Lambda | \xi_{\gamma'} \rangle \langle \xi_{\gamma''} | \Lambda \rangle, \quad (3.23)$$

respectively. Comparing $\chi(\Lambda)^2$ and $\chi^{\mathcal{M}}(\Lambda)^2$ is equivalent to compare ϱ_{Ξ} and $\varrho_{\Xi}^{\mathcal{M}}$, and hence the maps (3.13) and (3.20), defining the true and the measure-like evolutions respectively. However, despite the very specific construction leading to $|\Psi_{\tau}^{\mathcal{M}}\rangle$, the only thing to say about $\chi(\Lambda)^2$ and $\chi^{\mathcal{M}}(\Lambda)^2$ at this level is that they are different. On the other hand, we still have to exploit the fact that the environment is doomed to be big and behave classically, which is why ECS turn out to be so relevant to the final result (remember that when GCS are relative to a system Ξ which is the environment of a principal system Γ , we call them environmental coherent states).

We have demonstrated in Sec. 1.3 that

$$\langle \xi' | \xi'' \rangle = \delta_{\xi' \xi''} \Leftrightarrow \lim_{N \rightarrow \infty} S_{\xi'} \cap S_{\xi''} = \emptyset, \quad (3.24)$$

meaning that orthonormal states are put together by distinguishable sets of GCS. In other terms, the large- N limit makes it emerge a one-to-one correspondence between elements of any orthonormal basis $\{|\xi\rangle\}_{\mathcal{H}}$ and disjoint sets of GCS, in such a way that the distinguishability of the former is reflected into the disjunction of the latter. We already revealed the relevance of this property, whose meaning has been discussed through the examples in Secs. 1.3.1 and 1.3.2, and here it is where we will explicitly exploit Eq. (3.24).

In fact, getting back to Eq. (3.23), the states $|\xi'_{\gamma}\rangle$ and $|\xi''_{\gamma}\rangle$ are orthonormal by definition, being elements of the τ -Schmidt basis $\{|\xi_j\rangle\}_{\mathcal{H}_{\Xi}}$ introduced in Sec. 3.2.

Therefore, Eq. (3.24) holds, implying

$$\lim_{N \rightarrow \infty} \langle \Lambda | \xi_{\gamma'} \rangle \langle \xi_{\gamma''} | \Lambda \rangle = \lim_{N \rightarrow \infty} |\langle \Lambda | \xi_{\gamma'} \rangle|^2 \delta_{\gamma' \gamma''}, \quad (3.25)$$

and hence

$$\lim_{N \rightarrow \infty} \chi^M(\Lambda)^2 = \sum_{\gamma \gamma'} |a_\gamma|^2 c_{\gamma'}^2 \lim_{N \rightarrow \infty} |\langle \Lambda | \xi_{\gamma'} \rangle|^2. \quad (3.26)$$

Using $\sum_\gamma |a_\gamma|^2 = 1$, and replacing γ' with γ for the sake of a lighter notation, we finally obtain

$$\lim_{N \rightarrow \infty} \chi^M(\Lambda)^2 = \lim_{N \rightarrow \infty} \chi(\Lambda)^2, \quad (3.27)$$

which is the result to which we were aiming, namely that the the dynamical maps of the true evolution (3.13) and of the measure-like one (3.20) are equal when Ξ is a quantum macroscopic system whose behaviour can be effectively described classically. Echoing what we learnt in Sec. 1.2, we will express the equality (3.27) by saying that \mathcal{E} and \mathcal{E}^M , are *classically equivalent*.

3.4 Discussion and Development

Aim of this section is to comment upon some specific aspects of our results, with possible reference to the way other authors have recently tackled the same subject. We then set the bases for a work in progress regarding the objectivity of outcomes, that is a fundamental aspect of Quantum Darwinism, i.e., why many observers do actually get the same results when looking at the world.

Let us first consider the assumption that the initial state (3.11) of the total system $\Psi = \Gamma + \Xi$ be separable. If this is not the case, as it may happen, one must look for the different partition $\Psi = A + B$, such that $|\Psi\rangle = |A\rangle \otimes |B\rangle$. If this partition is still such that the subsystem B is macroscopic and behaves classically, the change is harmless and the whole construction can be repeated with A the quantum system being observed and B its observing environment. On the other hand, if the new partition is such that neither A nor B meet the conditions for being a classical environment, then the problem reduces to the usual one of studying the dynamics of two interacting quantum systems, for which any approach based on effective descriptions is incongruous, as details of the true Hamiltonian will always be relevant. Notice that this analysis is fully consistent with the results presented in Ref. [70], which are embodied into inequalities whose meaning wears off as $\dim \mathcal{H}_B$ decreases. The case when Ψ is not initially in a pure state is similarly tackled by enlarging $\Psi \rightarrow \tilde{\Psi}$ as much as necessary for $\tilde{\Psi}$ to be in a pure state: a proper choice of a new partition of $\tilde{\Psi}$ will then follow.

We then want to clarify in what sense the Hamiltonian (3.16) is said to induce a “measure-like” dynamics or, which is quite equivalent, the channel (3.20) to define a “measure and prepare” map: the quotes indicate that the actual output production, which happens at a certain time according to some process whose nature we do not discuss, is not considered and it only enters the description via the requirement that the probability, for each output to occur, be that predicted by the Born’s rule. To this

respect, one might also ask what is the property of Γ which is observed by Ξ : this is the one associated with the operator \hat{O}_Γ by the Ozawa's model, and it therefore depends on the true evolution via the Schmidt decomposition of the evolved state. Put it the other way, details of the interaction do not modify the measure-like nature of the dynamics in the large- N limit, but they do affect what actual measurement is performed by the environment.

We finally close this Section discussing the connection between our results and Quantum Darwinism [69, 70]. As we know from Chap. 1, a sufficient condition for a quantum theory to have a large- N limit which is a classical theory is that of featuring a global symmetry. In fact, a few simple examples show that quantum theories with different global symmetries can flow into the same classical theory in the large- N limit: in other words, different quantum theories can be classically equivalent. Moreover, from this chapter, we have learnt that any quantum theory with a proper classical limit can be effectively described by a theory defining a measure-like dynamics, when such limit is considered. If one further argue that amongst classically equivalent quantum theories it always exists a *free theory*, describing N non-interacting environmental subsystems, we believe it is possible to show that each macroscopic fragment of Ξ can be effectively described as if it were the same measurement apparatus. From this observation we get a clue, and we therefore start by the following side implication: given any theory Q_k^M with a well defined classical limit C , describing a measure-like evolution as the label M specifies, we show that we can always find a Q_N^{free} theory, describing a measure-like dynamics for N non-interacting environmental subsystems, whose classical limit corresponds to C . Restoring the abstract scheme of Sec. 1.2 and App. A, we want a theory whose dynamical group be defined as

$$\mathcal{G}_N = \{ \hat{U} = \bigotimes_{\ell=1}^N e^{-i \sum_i b_i^\ell \hat{X}_i^\ell} \}, \quad (3.28)$$

where ℓ runs over each subsystem ξ_ℓ of the environment, that indeed we now assume to be made of N subsystems, namely $\Xi = +_{\ell=1}^N \xi_\ell$. Choosing a separable reference state $|R\rangle = |R_1\rangle \otimes \cdots \otimes |R_N\rangle$, the GCS obtained by applying elements of the group (3.28) will be separable as well. In fact, such a dynamical group can correspond to a measure-like Hamiltonian

$$\begin{aligned} \hat{H}_N^{\text{Mfree}} &\equiv \hat{O}_\Gamma \otimes \hat{O}_\Xi^{\text{free}} = \\ &= \hat{O}_\Gamma \otimes \sum_{\ell=1}^N \hat{\mathbb{I}}_{\xi_1} \otimes \cdots \otimes g_\ell \hat{O}_{\xi_\ell} \cdots \otimes \hat{\mathbb{I}}_{\xi_N} + \hat{\mathbb{I}}_\Gamma \otimes \sum_{\ell=1}^N \hat{\mathbb{I}}_{\xi_1} \otimes \cdots \otimes \hat{H}_{\xi_\ell} \cdots \otimes \hat{\mathbb{I}}_{\xi_N}, \end{aligned} \quad (3.29)$$

which in turn defines a family of environmental theory, one $\forall \gamma$, with Hamiltonians

$$\hat{H}^{\text{free } \gamma} \equiv \sum_{\ell=1}^N \hat{\mathbb{I}}_{\xi_1} \otimes \cdots \otimes \hat{H}_{\xi_\ell}^\gamma(\{\hat{X}_i^\ell\}) \cdots \otimes \hat{\mathbb{I}}_{\xi_N}, \quad (3.30)$$

where in every $\hat{H}_{\xi_\ell}^\gamma$ we have included the interaction term between Γ and each ξ_ℓ , and the local term acting on ξ_ℓ only; moreover, notice that each $\hat{H}_{\xi_\ell}^\gamma$ depends on the

set of operators $\{\hat{X}_i^\ell\}$ in such a way that the global symmetry be still preserved. An Hamiltonian of the form (3.29) pictures a model where Γ interacts with all the ξ_ℓ subsystems, which instead do not interact amongst themselves. This is the reason why the environmental theory defined by $\hat{H}^{\text{free}\gamma}$ is labelled *free* and is indeed associated to Q_N^{free} . Notice that the choice of the reference state $|R\rangle = |R_1\rangle \otimes \cdots \otimes |R_N\rangle$ is consistent with the form (3.30), since the initial state $|\Xi\rangle$ must be separable, being not interaction between the different subsystems of Ξ . In addition to assume $\Xi = +_{\ell=1}^N \xi_\ell$, we hereafter take the subsystems ξ_ℓ identical and interacting with Γ via the same set of operators $\{\hat{X}_i^\ell\}$, even if with arbitrary coupling constants. We do not think that this is a necessary condition, but we implement it for the sake of a lighter notation. Therefore, to every subsystem we can associate the same Lie algebra spanned by $\{\hat{X}_i^\ell\}$ with commutation relations

$$[\hat{X}_i^\ell, \hat{X}_j^\ell] = \sum_k c_{ij}^k \hat{X}_k^\ell \quad \forall \ell. \quad (3.31)$$

If we suppose the existence of g generators in each $\{\hat{X}_i^\ell\}$, the dynamical group \mathcal{G}_N in Eq. (3.28) will be associated to the algebra

$$\mathfrak{g}_N = \bigoplus_{\ell=1}^N \{\hat{X}_1^\ell, \dots, \hat{X}_g^\ell\}. \quad (3.32)$$

Let us now consider the effective theory Q_k^M , whose classical limit C for $k \rightarrow 0$ is the same of that defined by $Q_{N \rightarrow \infty}^{\text{free}}$. Its Lie algebra \mathfrak{g}_k is nothing, but the algebra spanned by

$$\mathfrak{g}_k = \{\hat{X}_1, \dots, \hat{X}_g\}, \quad (3.33)$$

with dynamical group \mathcal{G}_k and GCS $|\Lambda\rangle$ built from a reference state $|R_\Lambda\rangle$. Therefore, the algebra of the free theory (3.32) can be written as

$$\mathfrak{g}_N \equiv \bigoplus_{\ell=1}^N \mathfrak{g}_k^\ell, \quad (3.34)$$

where each ℓ labels a subsystem ξ_ℓ of Ξ . Its Hamiltonian will consistently be of the form (3.30), and defining for each $i = 1, \dots, g$ the Ξ -operator

$$\hat{\mathcal{X}}_i = \sum_{\ell=1}^N \hat{X}_i^\ell, \quad (3.35)$$

the set $\{\hat{\mathcal{X}}_1, \dots, \hat{\mathcal{X}}_g\}$ is still the algebra \mathfrak{g}_N , since the Hamiltonian of the theory Q_N^{free} can be rewritten in terms of its elements. We thus have defined a free theory with an algebra \mathfrak{g}_N representation of the abstract algebra \mathfrak{g} whose \mathfrak{g}_k is a representation too. Taking as reference state

$$|R\rangle = \bigotimes_{\ell=1}^N |R_\Lambda\rangle_\ell, \quad (3.36)$$

the same holds for the dynamical group \mathcal{G}_N and \mathcal{G}_k , so that the theories Q_N^{free} and Q_k^M are classical equivalent, according to the scheme outlined in Sec. 1.2. In particular,

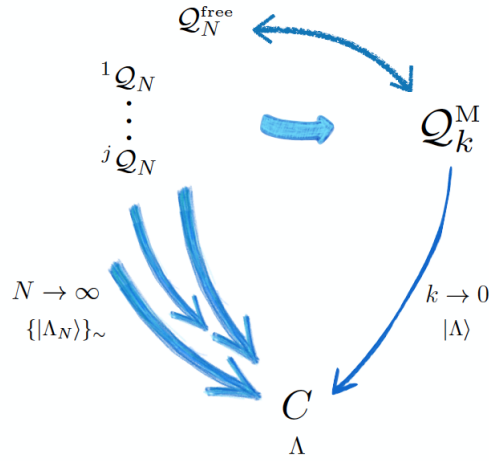


Figure 3.1: Graphical representation of the relation between classically equivalent theories: given a set of classically equivalent theories ${}^1Q_N, \dots, {}^jQ_N$, there always exists a theory Q_N^{free} , which is connected to the effective theory Q_k^M by a double arrow to symbolise that we can explicitly build Q_N^{free} from Q_k^M , where the latter represents the measure-like dynamics effectively describing any generic dynamics of an OQS in the classical limit.

the differentiable manifold \mathcal{M}_N will be the Cartesian product of N differentiable manifold \mathcal{M}_k .

We think that the theory Q_N^{free} that we built from the theory Q_k^M , could be the microscopic theory that allows us to analyse the evolution of the different subsystems ξ_ℓ . In fact, we recognize in it the idea that each ξ_ℓ component evolve with Γ according to a measure-like dynamics. For construction, Q_N^{free} represents indeed a theory where each subsystem ξ_ℓ of Ξ is described by an effective theory Q_k^M , and, moreover, from the results here found, it can be considered classically equivalent to an original theory Q_N , that is a generic microscopic theory characterized by whatever dynamics where the environmental subsystems generally interact. We thus think that through this connections - illustrated in Fig. 3.1 - one can show that, in the large- N limit, several observers perform the same measurement on Γ , independently of their actual interaction with Γ itself, that can be possibly different from a measure-like one. Such statement is easily understood as tantamount to the objectivity of outcomes, included in the Quantum Darwinism.

The work on this last point is still in progress, based on the preliminary analysis above reported, on the quantum de Finetti theorem (see for instance Ref. [77]), and some results of Refs. [70, 78].

3.5 Conclusions

We have here analysed the case in which the environment doomed to be large, turns out to be a measuring apparatus.

Using the dynamical maps for OQS, we considered a generic interaction between a principal system and its environment, proposing a possible interpretation of the fact that we do not observe states superpositions due to the continual measurement acted upon by whatever macroscopic environment. This idea is crucial for making sense of our everyday experience w.r.t. the quantum description of nature. However, the formal analysis of this idea has been unsatisfactory for decades, due to several reasons, amongst which we underline the following. First of all such analysis requires a clean procedure for taking the large- N limit of the quantum theory that describes the environment in a way such that it formally transforms into a proper classical one. Moreover it must be possible to implement such limit only upon the environment, without affecting the microscopic nature of the principal quantum system. Finally, the analysis must not imply assumptions on the state of the observed system before the interaction starts, or on the form of the interaction itself.

The above three issues have been addressed here combining approaches from quantum field theory, formally describing the conditions for a classical theory to emerge as the large- N limit of a quantum one, with tools of OQS theory, such as the dynamical map description of the environmental evolution. In particular, exploiting the PRECS, we take the large- N limit so that a comparison between different evolutions of the environment, totally independent on the initial state of the principal system, becomes possible in terms of environmental dynamical maps.

Our approach allows us to tackle the so-called quantum to classical crossover [16] by a rigorous mathematical formulation that provides a physically intuitive picture of the underlying dynamical process. In fact, we have to keep in mind that not *every* theory has a proper defined classical limit. Indeed, we did not want to show that a measure-like dynamics *always* emerges whenever the environment of an OQS becomes macroscopic, rather than a measure-like dynamics *can* emerge for *any* generic dynamics *if* some conditions occur, which are indeed the ones assuring that a classical limit does actually exist when the large- N limit is taken. In other words, *if* some dynamics emerges in the classical world, it *necessarily* is a measure-like one.

We already mentioned the phenomenon known as Quantum Darwinism, introduced in [69] and recently considered in [70] from an information theoretic viewpoint. Our work provides a way of understanding Quantum Darwinism as a dynamical process, and its generality as deriving from the versatility of the Hamiltonian model for the quantum measurement process, and the loss of resolution inherent in the classical description.

Lastly, we stress again the relevance of the classical limit, that in our proposal inherently emerges as a fundamental ingredient via the PRECS, and that, in any case, we should expect to be essential given the scenario with which we dealt throughout these sections.

Chapter 4

Large environment as *clock*

The idea of time is so deeply settled in the way we sense the nature, that it seems obvious for us describing every experience we have of the world, either about our everyday life or concerning the models of the Universe, as events that *follow one after the other* and happen at *certain times*. Time, with space¹, is certainly established as a foundational concept in the human perception of reality. Nevertheless, its profound essence has still to be understood. Physically, it is a relative quantity, defined by the method used to measure it, and the problem of its measurement has intrigued the scientists for centuries, from both a philosophical and a practical points of view. This raises the issue of what time really is in a physical model, whether it is an actual phenomenon, a property, an observable or a parameter to which we can not attribute a further interpretation.

In Classical Mechanics time appears in the laws of dynamics down from above, as an extrinsic parameter external to the physical system which evolves. The same happens in conventional Quantum Mechanics, where time still enters the formalism as a classical parameter, a priori given with a certain value. This comes together with an even deeper problem: time is not a quantum observable, and yet quantum observables depend on it. In fact, the question of an energy-time uncertainty principle has been one of the earliest issues in QM. As well known, whereas time is indeed a classical parameter, the energy of a quantum system is represented by a hermitian operator. The Hamiltonian \hat{H} generates the evolution of the states through unitary operators that take the well known form $\hat{U} = \exp(-\frac{i}{\hbar}\hat{H}t)$. Indeed, many authors tried at first to look for some operator \hat{T} conjugate to the Hamiltonian with eigenvalues exactly those t , appearing in the propagators, that we call *instants* and interpret as *time*. However, there is apparently no operator \hat{T} canonically conjugate to \hat{H} , in the sense of a commutation relation of the form $[\hat{H}, \hat{T}] = -i\hbar$, from which a corresponding uncertainty relation can be derived [80].

A different and fascinating approach stems from the idea that time and dynamics must be some elemental properties of the systems, and should come out not because

¹There is a profound *asymmetry* between the two “building blocks” of our Universe [79], and whereas people have been able to satisfactorily understand space, updating its representation during the evolution of different physical theories, the story of time is completely different.

related to some external coordinate, rather because dependent on more fundamental elements that work as some *internal clock time*. This is the “timeless approach to time” and it was firstly proposed in the 1960s, at the early stage of the research on quantum gravity, by the physicist B. DeWitt, who states “*Other times are just special cases of other Universes*”. Only twenty years later, in 1983, D. Page and W. Wootters formalized the timeless approach in [81], introducing the so called Page and Wootters (PaW) mechanism. Nothing in their construction relies on defining a time operator, and at its core there is the idea that both time and dynamics can emerge as entanglement properties. Thus, as beautifully said in [82], quantum theory supplies the tools to solve the problem of time via one of its most profound properties, i.e., permitting entanglement between subsystems of the Universe. In 2013, at the Istituto Nazionale di Ricerca Metrologica in Turin, E. Moreva et al. implemented the first experimental test of Page and Wootters ideas, confirming the validity of the PaW mechanism [83].

Having this issue in mind, we here take the clue from the timeless approach as interpreted by Page and Wootters, and we show that it is possible to reproduce the dynamic evolution of a quantum system as dependent upon some internal clock readings. In particular, exploiting once again the peculiar properties of GCS, we focus on the reason why time always emerge as a classical parameter, and on the resulting necessity of a well-defined classical limit that allow us to understand time as a coordinate on a classical phase space.

In this Chapter we tackle the problem of time, whose features, that must be preserved seeking its proper quantum origin, are summed up in Sec. 4.1. In the second section we define the internal clock and show how the concept of *time* can emerge from the measurement process. In Sec. 4.3 we present our proposal to formalize the PaW mechanism, and show how a von Neumann-like equation is naturally obtained through the PRECS, once the classical limit is implemented. In Sec. 4.4 we discuss the assumptions made, reporting the final comments in Sec. 4.5.

4.1 Towards a Quantum Time

The PaW mechanism consists in promoting all the physical variables present in a model to operators, among which there will be one operating on the Hilbert space of a quantum system that act as an *internal clock*. The evolution of a (sub)system emerges thus without evolution, since the temporal behaviour we observe comes from the entanglement between the rest of the system and the clock. Infact, despite the whole Universe being isolated and in a stationary state of the total Hamiltonian \hat{H} that describes its energy, i.e., despite the Universe not evolving, an internal observer will describe the evolution of the (sub)system through the probabilities of getting certain results from measures conditioned on a reading of the clock that gives as result the value t . In other words, the dynamical evolution requires a partition of the Universe to emerge, so that we can interpret time *in relation* to the observation of a system, i.e., through a measurement process on it. The key point introduced

by the timeless approach is indeed that time is *relational*. As mentioned, what we detect as time is the change due to the internal quantum correlations of different subsystems of the Universe with respect to each other, and what does actually make sense it is not the flow of time, rather the values t at which different measurements are performed. It is indeed when we detect such relative changes that time emerges to our senses: whenever we measure, we build a temporal order, whereas time does not go by between different measurements. Therefore, the scheme to describe this scenario is again that of the OQS; we will consider a principal system Γ , whose evolution has to be observed, and its environment, that in this chapter will act as a clock and will be thus named C . The evolution of any state is described according to a label which changes, while the state is changing in turn. As a consequence, the Γ state will be parametrized by a clock label receptive to the environmental dynamics. This apparently nested picture becomes instead clear if one takes into account that the time evolution of Γ loses importance if its environment is stuck. In fact, such situation would be rather bizarre; suppose to observe - like in a movie - the main character dancing, walking down the street, acting, whereas all the people around her are frozen in a fixed world: we would immediately recognize an anomalous situation, and probably we would not by chance think to the lack of time flow.

Before moving towards a quantum time, the features of what *time* is should be gathered, in order to understand which are the properties we must require for a sensible description. In the equations of motion time is a label “ t ”. It is thus quite natural starting from parametric representations to describe it, since first of all time is a *i) parameter*. Moreover, time is *ii) continuous*, and many are the continuous parametric representations developed in different contexts to analyse physical systems. The most famous parametric representations are probably the momentum and position representations of QM; another one is, for instance, the parametric representation with field coherent states often exploited by Walter T. Strunz and collaborators to study several OQS dynamics, as in Refs. [84, 85, 86, 87, 88]. Recently, a new continuous parametric representation has been introduced by V. Giovannetti, S. Lloyd and L. Maccone [89] precisely in the attempt of giving a consistent description of quantum time, based, as our will be, on the PaW mechanism. Nevertheless, we must also consider that time is lastly *iii) classical*. As mentioned, time indeed appears in the laws of dynamics, classical and quantum both, like a label featuring the properties *i) – iii)*. The attentive reader will have already got to the point: as a matter of fact, all the features required for a sensible description of time are inherently provided by the PRECS, which appears hence once again to be a good tool for our aim.

Our proposal to formalize the PaW mechanism is presented in Sec. 4.3; in the next section we focus instead on the role played by the internal clock, which is indeed at the heart of the timeless approach.

4.2 The Internal Clock

Time in QM cannot be considered an observable, since it does not exist a measure, i.e., a set of positive operators $\{\hat{M}(T)\}$, which, acting on the same system, gives as

results the values t understood by us as values of *time*. The operators representing system properties which can be observed are Hermitian and commute with the system Hamiltonian \hat{H} - see App. B. On the other hand, when a projective measure M corresponding to the operator \hat{M}_Γ is performed on the system Γ and a specific result $\bar{\gamma}$ is obtained, the system state reduces itself to the state $|\bar{\gamma}\rangle$, which is an eigenstate of \hat{M}_Γ and \hat{H} simultaneously, and thus stationary. If we then perform another measure M on this state, the result will always be $\bar{\gamma}$, being $|\bar{\gamma}\rangle$ stationary. So, a spontaneous question arises: why do we observe a dynamics in the observable properties of the quantum systems? The solution offered by the timeless approach is that the observed time-dependence is not related to an external coordinate, but rather to the state of a system belonging to a compound system $\Psi = \Gamma + C$. This is indeed the system to which we will refer to as *internal clock*. By *internal clock* we mean a system whose state can vary, and with respect to which we describe the change of the other subsystems states.

Consider now an isolated bipartite system $\Psi = \Gamma + C$, in a separable state $|\Psi\rangle = |\Gamma\rangle |C\rangle$. Suppose we want to measure the observables E and T on the systems Γ and C . Their possible results are $\{\varepsilon_i\}$ and $\{\xi_j\}$ respectively, and we exploit the PVM

$$M(E) : \{\varepsilon_i\} \rightarrow \{|i\rangle\langle i|_\Gamma \otimes \hat{\mathbb{I}}_C\} \quad (4.1)$$

$$M(T) : \{\xi_j\} \rightarrow \{\hat{\mathbb{I}}_\Gamma \otimes |j\rangle\langle j|_C\}. \quad (4.2)$$

By the minimal interpretation, the probability that the measurement of E and T on the total state $|\Psi\rangle$ give as result the couple $(\varepsilon_\ell, \xi_k)$ is

$$\begin{aligned} p_{|\Psi\rangle}^{E,T}(\varepsilon_\ell, \xi_k) &= \langle \Gamma | \langle C | (|\ell\rangle\langle \ell|_\Gamma \otimes \hat{\mathbb{I}}_C) (\hat{\mathbb{I}}_\Gamma \otimes |k\rangle\langle k|_C) | \Gamma \rangle | C \rangle = \\ &= \langle \Gamma | \langle C | |\ell\rangle\langle \ell|_\Gamma \otimes |k\rangle\langle k|_C | \Gamma \rangle | C \rangle. \end{aligned} \quad (4.3)$$

Defining the state

$$|\Psi_\Gamma(\xi_k)\rangle \equiv \frac{c \langle k | C \rangle | \Gamma \rangle}{|c \langle k | C \rangle|}, \quad (4.4)$$

Eq. (4.3) becomes

$$p_{|\Psi_\Gamma(\xi_k)\rangle}^E(\varepsilon_\ell) = \langle \Psi_\Gamma(\xi_k) | \ell \rangle \langle \ell | \Psi_\Gamma(\xi_k) \rangle = |\langle \ell | \Psi_\Gamma(\xi_k) \rangle|^2, \quad (4.5)$$

that can be read as the probability that the measurement of E related to the system Γ on the state $|\Psi_\Gamma(\xi_k)\rangle$ give as result the value ε_ℓ at the “time” ξ_k .

Notice that the state reduction does not play any role in what stated above. In fact, if the initial state of the system Ψ is not separable, then the probability that the measurement of E and T on the entangled state $|\Phi\rangle$ give as result the couple $(\varepsilon_\ell, \xi_k)$ is

$$p_{|\Phi\rangle}^{E,T}(\varepsilon_\ell, \xi_k) = \langle \Phi | (|\ell\rangle\langle \ell|_\Gamma \otimes \hat{\mathbb{I}}_C) (\hat{\mathbb{I}}_\Gamma \otimes |k\rangle\langle k|_C) | \Phi \rangle, \quad (4.6)$$

and the state

$$|\Phi_\Gamma(\xi_k)\rangle \equiv \frac{c \langle k | \Phi \rangle}{|c \langle k | \Phi \rangle|}, \quad (4.7)$$

can be interpreted as the state of Γ at the “time” ξ_k . We can therefore still recognize the probability that the measurement of E on the state $|\Phi_\Gamma(\xi_k)\rangle$ give as result the value ε_ℓ at the “time” ξ_k , in the expression

$$p_{|\Phi_\Gamma(\xi_k)\rangle}^E(\varepsilon_\ell) = \langle \Phi_\Gamma(\xi_k) | \ell \rangle \langle \ell | \Phi_\Gamma(\xi_k) \rangle = |\langle \ell | \Phi_\Gamma(\xi_k) \rangle|^2. \quad (4.8)$$

From this discussion, it seems that the parameter *time* can emerge from the measurement process as a parameter related indeed to the internal clock, whose state allows us to observe the change of all the rest. As mentioned, Page and Wootters were the first to develop this idea, and we refer the reader to Sec. II of Ref. [81] for an explicit example of the above construction.

4.3 Evolution without evolution by the PRECS

Once recognized that our perception of time stems during the measurement process, through which we realise and quantify the change observed in a quantum system, we want to investigate whether the evolution of the subsystem Γ can be understood indeed as the change related to another subsystem, chosen as the internal clock. In particular, we want to study if this kind of evolution can always reproduce the evolution of Γ as obtained by the equations of motion.

Let us consider an isolated bipartite system $\Psi = \Gamma + C$ with Hilbert space $\mathcal{H}_\Psi = \mathcal{H}_\Gamma \otimes \mathcal{H}_C$, where Γ is the system whose dynamics is under analysis, and C the clock system. Suppose Ψ be in the state $|\Psi\rangle \in \mathcal{H}_\Psi$ ², that we rewrite using the Schmidt decomposition as

$$|\Psi\rangle = \sum_{\gamma} c_{\gamma} |\gamma\rangle |\xi_{\gamma}\rangle \quad (4.9)$$

with $\gamma = 1, \dots, \gamma_{\max} \leq \dim \mathcal{H}_\Gamma$, $c_{\gamma} \in \mathbb{R}^+$, $\sum_{\gamma} c_{\gamma}^2 = 1$. The sets $\{|\gamma\rangle\}_{\mathcal{H}_\Gamma}$ and $\{|\xi_j\rangle\}_{\mathcal{H}_C}$ with $j = 1, \dots, \dim \mathcal{H}_C$ are the Schmidt bases, as we know from Sec. 3.2. Suppose the chosen partition be the one where the two subsystems Γ and C do not dynamically interact with each other, so that we can consider the operator

$$\hat{H} = \hat{H}_\Gamma \otimes \hat{\mathbb{I}}_C + \hat{\mathbb{I}}_\Gamma \otimes \hat{H}_C, \quad (4.10)$$

where \hat{H}_Γ and \hat{H}_C act locally on \mathcal{H}_Γ and \mathcal{H}_C respectively. In particular, we take \hat{H}_Γ as the free Hamiltonian of Γ , and we define \hat{H}_C as the hermitian operator

$$\hat{H}_C \equiv \sum_j E_j |\xi_j\rangle \langle \xi_j| \quad \text{with } E_j \in \mathbb{R}, \quad (4.11)$$

determined by the Schmidt decomposition (4.9), in a way similar to what we did in Sec. 3.2 of the previous chapter. Additionally, we assume \hat{H}_C bounded from below, so that the operator (4.10) be bounded from below as well. Noticing that \hat{H} is also hermitian, we can thus understand it as the total Hamiltonian of the composite system

²We preserve in this chapter the double-ket notation introduced in [89] to remind us that $|\Psi\rangle$ is defined on $\mathcal{H}_\Psi = \mathcal{H}_\Gamma \otimes \mathcal{H}_C$.

Ψ . We then suppose that the state (4.9) be an eigenvector of the total Hamiltonian, i.e.,

$$\hat{H} |\Psi\rangle = 0, \quad (4.12)$$

and from the hermiticity of \hat{H} we get

$$\langle \eta | \hat{H} |\Psi\rangle \langle \Psi | \eta \rangle = 0 \quad \Leftrightarrow \quad \langle \eta | \Psi \rangle \langle \Psi | \hat{H} | \eta \rangle = 0, \quad (4.13)$$

where $|\eta\rangle$ is an ECS of the set $\{|\eta\rangle\}_{\mathcal{H}_c}$ built for the clock, according to the procedure described in App. A, through an algebra whose Cartan decomposition includes \hat{H}_c as a diagonal operator. The operation $\langle \cdot | \cdot \rangle$ is formally defined by

$$\langle \cdot | \cdot \rangle : \mathcal{H}_\Gamma \otimes \mathcal{H}_c \rightarrow \mathcal{H}_\Gamma. \quad (4.14)$$

Using Eqs. (4.9) and (4.10), we get

$$\begin{aligned} 0 &= \langle \eta | \hat{H} |\Psi\rangle \langle \Psi | \eta \rangle = \\ &= \hat{H}_\Gamma \sum_{\gamma, \gamma'} c_\gamma c_{\gamma'} |\gamma\rangle \langle \gamma' | \langle \eta | \xi_\gamma \rangle \langle \xi'_{\gamma'} | \eta \rangle + \sum_{\gamma, \gamma'} c_\gamma c_{\gamma'} |\gamma\rangle \langle \gamma' | \langle \eta | \hat{H}_c | \xi_\gamma \rangle \langle \xi'_{\gamma'} | \eta \rangle = \\ &= \hat{H}_\Gamma \tilde{\varrho}_\Gamma(\eta) + \underbrace{\sum_{\gamma, \gamma'} c_\gamma c_{\gamma'} |\gamma\rangle \langle \gamma' | \langle \eta | \hat{H}_c | \xi_\gamma \rangle \langle \xi'_{\gamma'} | \eta \rangle}_a, \end{aligned} \quad (4.15)$$

and similarly

$$0 = \langle \eta | \Psi \rangle \langle \Psi | \hat{H} | \eta \rangle = \tilde{\varrho}_\Gamma(\eta) \hat{H}_\Gamma + \underbrace{\sum_{\gamma, \gamma'} c_\gamma c_{\gamma'} |\gamma\rangle \langle \gamma' | \langle \eta | \xi_\gamma \rangle \langle \xi'_{\gamma'} | \hat{H}_c | \eta \rangle}_b, \quad (4.16)$$

where we defined

$$\tilde{\varrho}_\Gamma(\eta) = \sum_{\gamma, \gamma'} c_\gamma c_{\gamma'} |\gamma\rangle \langle \gamma' | \langle \eta | \xi_\gamma \rangle \langle \xi'_{\gamma'} | \eta \rangle. \quad (4.17)$$

Notice that $\tilde{\varrho}_\Gamma(\eta)$ is not the reduced density matrix of Γ , since, for instance, its trace is not equal to 1. To explicitly write Eqs. (4.15) and (4.16) we need to work on the overlaps $\langle \eta | \hat{H}_c | \xi_\gamma \rangle$ and $\langle \xi'_{\gamma'} | \hat{H}_c | \eta \rangle$.

For this purpose, consider the Cartan decomposition $\{\hat{H}_i, \hat{E}_\alpha, \hat{E}_\alpha^\dagger\}$, introduced in Sec. 1.2, for the algebra describing the environmental theory, i.e., the clock; as mentioned, the operator \hat{H}_c appearing in \hat{H} is such that $\hat{H}_c \in \{\hat{H}_i\}$, being a diagonal operator itself, and, in particular we remind

$$[\hat{H}_c, \hat{E}_\alpha] = \alpha_c \hat{E}_\alpha \quad \Leftrightarrow \quad [\hat{H}_c, \hat{E}_\alpha^\dagger] = -\alpha_c \hat{E}_\alpha^\dagger, \quad (4.18)$$

according to the commutation rules (1.2) reported in the first chapter. Then, assuming that there is only one couple of shift-up/shift-down operators $(\hat{E}_\alpha, \hat{E}_\alpha^\dagger)$, a generic

ECS $|\eta\rangle$ can be obtained acting with the displacement operator on the chosen reference state $|R\rangle \equiv |0\rangle$, and, exploiting the BCH formula - see App. A and Ref. [10] -, we get

$$|\eta\rangle = e^{\eta\hat{E}_\alpha^\dagger - \eta^*\hat{E}_\alpha} |0\rangle = e^{\tau\hat{E}_\alpha^\dagger} e^{\sum_i \gamma_i \hat{H}_i} e^{-\tau^*\hat{E}_\alpha} |0\rangle = e^{\sum_i \gamma_i \varepsilon_i} e^{\tau\hat{E}_\alpha^\dagger} |0\rangle, \quad (4.19)$$

where τ is a function of $\eta \in \mathbb{C}$, $\hat{E}_\alpha |0\rangle = 0$ and $\hat{H}_i |0\rangle = \varepsilon_i^i |0\rangle$. Consequently, it is

$$\langle \eta | \hat{H}_c | \xi_\gamma \rangle = \langle 0 | e^{\sum_i \gamma_i \varepsilon_i} e^{\tau^*\hat{E}_\alpha} \hat{H}_c | \xi_\gamma \rangle = \langle 0 | e^{\sum_i \gamma_i \varepsilon_i} \sum_k \frac{(\tau^*\hat{E}_\alpha)^k}{k!} \hat{H}_c | \xi_\gamma \rangle, \quad (4.20)$$

and similarly

$$\langle \xi'_\gamma | \hat{H}_c | \eta \rangle = \langle \xi'_\gamma | \hat{H}_c e^{\sum_i \gamma_i \varepsilon_i} e^{\tau\hat{E}_\alpha^\dagger} |0\rangle = \langle \xi'_\gamma | \hat{H}_c \sum_k \frac{(\tau\hat{E}_\alpha^\dagger)^k}{k!} e^{\sum_i \gamma_i \varepsilon_i} |0\rangle, \quad (4.21)$$

where in the last steps of the above equations we used the Taylor expansion of the exponential. Via Eq. (4.18), we make \hat{E}_α^k and \hat{H}_c commute, yielding

$$\hat{E}_\alpha^k \hat{H}_c = \hat{H}_c \hat{E}_\alpha^k - \alpha_c k \hat{E}_\alpha^k \Leftrightarrow \hat{H}_c \hat{E}_\alpha^{\dagger k} = \hat{E}_\alpha^{\dagger k} \hat{H}_c - \alpha_c k \hat{E}_\alpha^{\dagger k}, \quad (4.22)$$

so that

$$\sum_k \frac{(\tau^*\hat{E}_\alpha)^k}{k!} \hat{H}_c = \hat{H}_c e^{\tau^*\hat{E}_\alpha} - \alpha_c \tau^* \hat{E}_\alpha e^{\tau^*\hat{E}_\alpha}, \quad (4.23)$$

and analogously

$$\hat{H}_c \sum_k \frac{(\tau\hat{E}_\alpha^\dagger)^k}{k!} = e^{\tau\hat{E}_\alpha^\dagger} \hat{H}_c - \alpha_c \tau \hat{E}_\alpha^\dagger e^{\tau\hat{E}_\alpha^\dagger}. \quad (4.24)$$

The assumption of a single couple of shift-up/shift-down operators ($\hat{E}_\alpha, \hat{E}_\alpha^\dagger$) implies that the parameter η appearing in the displacement operator is a complex number, that can thus be written as $\eta = \bar{\eta} e^{i\phi}$ with $\bar{\eta}$ representing the modulus and ϕ its phase. Through the relations between η and τ - see Eqs. (A.37) and (A.38) of App. A -, it follows $\tau = \bar{\tau}(\bar{\eta}) e^{i\phi}$, i.e., the modulus of τ appearing in the BCH formula is a function of the modulus $\bar{\eta}$ only, whereas the phase ϕ is the same. Therefore, being

$$\frac{d}{d\phi} e^{\tau^*\hat{E}_\alpha} = -i\tau^* \hat{E}_\alpha e^{\tau^*\hat{E}_\alpha}, \quad (4.25)$$

the overlaps (4.20) and (4.21) can be rewritten as derivative of ϕ , i.e.,

$$\begin{aligned} \langle \eta | \hat{H}_c | \xi_\gamma \rangle &= \langle 0 | e^{\sum_i \gamma_i \varepsilon_i} \left(\hat{H}_c e^{\tau^*\hat{E}_\alpha} - i\alpha_c \frac{d}{d\phi} e^{\tau^*\hat{E}_\alpha} \right) | \xi_\gamma \rangle = \\ &= \left(E_0 - i\alpha_c \frac{d}{d\phi} \right) \langle \eta | \xi_\gamma \rangle, \end{aligned} \quad (4.26)$$

and similarly

$$\langle \xi_{\gamma'} | \hat{H}_c | \eta \rangle = \left(E_0 + i\alpha_c \frac{d}{d\phi} \right) \langle \xi_{\gamma'} | \eta \rangle, \quad (4.27)$$

where we used $\hat{H}_c |0\rangle = E_0 |0\rangle$, following from the fact that, being \hat{H}_c one of the diagonal operators of the Cartan decomposition, $|0\rangle$ is one of its eigenvector, namely $|0\rangle \in \{|\xi_\gamma\rangle\}_{\mathcal{H}_c}$ as we supposed that the eigenbasis of \hat{H}_c is the Schmidt basis, i.e., Eq. (4.11). Inserting the above relations in the terms a and b defined in Eqs. (4.15) and (4.16) respectively, we get

$$\begin{aligned} a - b &= \sum_{\gamma, \gamma'} c_\gamma c_{\gamma'} |\gamma\rangle \langle \gamma'| \left[\left(-i\alpha_c \frac{d}{d\phi} \langle \eta | \xi_\gamma \rangle \right) \langle \xi_{\gamma'} | \eta \rangle + \langle \eta | \xi_\gamma \rangle \left(-i\alpha_c \frac{d}{d\phi} \langle \eta | \xi_{\gamma'} \rangle \right) \right] = \\ &= -i\alpha_c \frac{d}{d\phi} \sum_{\gamma, \gamma'} c_\gamma c_{\gamma'} |\gamma\rangle \langle \gamma'| \langle \eta | \xi_\gamma \rangle \langle \xi_{\gamma'} | \eta \rangle = \\ &= -i\alpha_c \frac{d}{d\phi} \tilde{\varrho}_\Gamma(\eta). \end{aligned} \quad (4.28)$$

Thus, subtracting Eq. (4.16) from Eq. (4.15), it is

$$0 = \hat{H}_\Gamma \tilde{\varrho}_\Gamma(\eta) - \tilde{\varrho}_\Gamma(\eta) \hat{H}_\Gamma - i\alpha_c \frac{d}{d\phi} \tilde{\varrho}_\Gamma(\eta), \quad (4.29)$$

i.e.,

$$i\alpha_c \frac{d}{d\phi} \tilde{\varrho}_\Gamma(\eta) = [\hat{H}_\Gamma, \tilde{\varrho}_\Gamma(\eta)]. \quad (4.30)$$

This expression resembles the von Neumann equation - also known as the Liouville–von Neumann equation -, which describes how a density operator evolves in time. Nevertheless, we must notice that, first of all, $\tilde{\varrho}_\Gamma(\eta)$ is not well-defined as a reduced density matrix, being not normalized - see comments below Eq. (4.17). Moreover, we would like to recognize ϕ as *time*, implying that the parameter on which a proper Γ density matrix should depend is (only) ϕ itself. In fact, the von Neumann equation reads

$$i\hbar \frac{d}{dt} \varrho_S(t) = [\hat{H}_S, \varrho(t)] \quad (4.31)$$

where $\varrho_S(t)$ is a well-defined density matrix of a generic system S with Hamiltonian \hat{H}_S . However, we know by the PRECS that $\tilde{\varrho}_\Gamma(\eta)$ must be divided by $\chi(\eta)^2$ to be normalized. Therefore, we multiply and divide Eq. (4.29), yielding

$$0 = \chi(\eta)^2 \left\{ [\hat{H}_\Gamma, |\Phi(\eta)\rangle \langle \Phi(\eta)|] - i\alpha_c \frac{1}{\chi(\eta)^2} \frac{d}{d\phi} \tilde{\varrho}_\Gamma(\eta) \right\}, \quad (4.32)$$

with $|\Phi(\eta)\rangle \langle \Phi(\eta)| \equiv \tilde{\varrho}_\Gamma(\eta) / \chi(\eta)^2$. Notice that we keep $\chi(\eta)^2$ out the derivative, being, in general, $\frac{d}{d\phi} \chi(\eta)^2 \neq 0$.

In order to go forward in the calculations, we should be able to show instead that $\chi(\eta)^2$ exclusively depends on the modulus $\bar{\eta}$, at least if the assumptions hereinbefore

considered hold true. Although we think that this can be done in general, we here report two specific examples, postponing to future work the general demonstration.

Consider the two cases where $\{|\eta\rangle\}_{\mathcal{H}_C}$ represents the sets of field coherent states $\{|\alpha\rangle\}$ and of spin coherent states $\{|\Omega\rangle\}$, respectively introduced in Secs. 1.3.1-1.3.2 and described in App. A. In other words, we consider the two paradigmatic cases of a bosonic and a magnetic clock.

If C is a bosonic system, \hat{H}_C in Eq. (4.10) is the number operator $\hat{n} \equiv \hat{a}^\dagger \hat{a}$, and the Schmidt basis $\{|\xi_\gamma\rangle\}_{\mathcal{H}_C} = \{|n_\gamma\rangle\}_{\mathcal{H}_C}$ with $|n_\gamma\rangle$ the Fock states. The field CS are built acting with the displacement operator on the vacuum, i.e., $|R\rangle = |0\rangle$, the complex parameter α can be parametrized by polar coordinates, i.e.,

$$\alpha = \bar{\alpha} e^{-i \arctan\left(\frac{\Im(\alpha)}{\Re(\alpha)}\right)}, \quad (4.33)$$

and, through the PRECS,

$$\chi(\alpha)^2 = \sum_{\gamma} c_{\gamma}^2 \langle \alpha | n_{\gamma} \rangle \langle n_{\gamma} | \alpha \rangle = \sum_{\gamma} c_{\gamma}^2 e^{-\bar{\alpha}^2} \frac{\bar{\alpha}^2}{n_{\gamma}!}, \quad (4.34)$$

where in the last step we used the overlap $\langle n | \alpha \rangle$ reported in Sec. A.2 of App. A.

When C is instead a magnetic system, \hat{H}_C in Eq. (4.10) is \hat{S}^z , and the Schmidt basis $\{|\xi_\gamma\rangle\}_{\mathcal{H}_C} = \{|m_\gamma\rangle\}_{\mathcal{H}_C}$ with $|m_\gamma\rangle$ the \hat{S}^z -eigenvectors. Acting with the displacement operator on the reference state $|R\rangle = |-S\rangle$, one gets the spin CS, and the complex parameters ζ , labeling points on the two-dimensional sphere S^2 , can be rewritten by the usual polar angles $\vartheta \in [0, \pi]$, $\varphi \in [0, 2\pi)$, according to

$$\zeta = \frac{\vartheta}{2} e^{-i\varphi}. \quad (4.35)$$

Through the PRECS, we get

$$\chi(\Omega)^2 = \sum_{\gamma} c_{\gamma}^2 \langle \Omega | m_{\gamma} \rangle \langle m_{\gamma} | \Omega \rangle = \sum_{\gamma} c_{\gamma}^2 \binom{2S}{m_{\gamma}} \left(\cos \frac{\vartheta}{2} \right)^{4S-2m_{\gamma}} \left(\sin \frac{\vartheta}{2} \right)^{2m_{\gamma}}, \quad (4.36)$$

where in the last step we used the overlap $\langle m | \Omega \rangle$ reported in Sec. A.3 of App. A.

Looking at Eqs. (4.34) and (4.36), it is evident that $\chi(\alpha)^2$ and $\chi(\Omega)^2$ are both independent of the phases appearing in Eqs. (4.33) and (4.35) respectively, that is, they are both independent of the *time* ϕ . The derivative in Eq. (4.32) does not thus affect the normalization $\chi(\eta)^2$, so that we can rewrite Eq. (4.32) with $|\Phi(\eta)\rangle\langle\Phi(\eta)|$ in the second term too.

For what concerns the parametric dependence of the Γ density matrix, we would like to define the analogous of $\varrho_S(t)$, i.e., $\varrho_{\Gamma}(\phi)$. Thus, in order to remove the modulus dependence, we integrate Eq. (4.32) in $d\bar{\alpha}/\pi$ for the case of the bosonic clock, and in $\frac{2S+1}{4\pi} \sin \vartheta d\vartheta$ for the case of the magnetic one, yielding

$$0 = \int \frac{d\bar{\alpha}}{\hbar\pi} \chi(\alpha)^2 \{ [\hat{H}_{\Gamma}, |\Phi(\alpha)\rangle\langle\Phi(\alpha)|] - i\alpha_c \frac{d}{d\varphi} |\Phi(\alpha)\rangle\langle\Phi(\alpha)| \}, \quad (4.37)$$

and, analogously,

$$0 = \frac{2S+1}{4\pi} \int \sin \vartheta d\vartheta \chi(\Omega)^2 \{ [\hat{H}_\Gamma, |\Phi(\Omega)\rangle\langle\Phi(\Omega)|] - i\alpha_c \frac{d}{d\varphi} |\Phi(\Omega)\rangle\langle\Phi(\Omega)| \}. \quad (4.38)$$

At this point, another important feature of time must be considered: in fact, to be finally able to recognize ϕ as a *classical* continuous parameter, we still have indeed to take the classical limit. Through the properties of GCS [14], and considering what we learnt in Chap. 1, it is thus

$$\lim_{N \rightarrow \infty} \frac{\chi(\alpha)^2}{\hbar\pi} = \sum_{\gamma} c_{\gamma}^2 \delta(\bar{\alpha} - \bar{\alpha}^{\gamma}) \quad \text{and} \quad \lim_{N \rightarrow \infty} \frac{2S+1}{4\pi} \sin \vartheta \chi(\Omega)^2 = \sum_{\gamma} c_{\gamma}^2 \delta(\vartheta - \vartheta^{\gamma}), \quad (4.39)$$

so that Eqs. (4.37) and (4.38) in the large- N limit become

$$0 = \sum_{\gamma} c_{\gamma}^2 \{ [\hat{H}_\Gamma, \varrho_{\Gamma}^{\gamma}(\phi)] - i\alpha_c \frac{d}{d\phi} \varrho_{\Gamma}^{\gamma}(\phi) \}, \quad (4.40)$$

with $\phi = \arctan\left(\frac{\Im(\alpha)}{\Re(\alpha)}\right)$ and $\phi = \varphi$ for the bosonic and the magnetic clock, respectively; $\varrho_{\Gamma}^{\gamma}(\phi)$ is the projector $|\Phi(\Omega)\rangle\langle\Phi(\Omega)|$ integrated and, hence, with the variable conjugate to ϕ fixed by the delta function of Eq. (4.39).

Noticing that $\text{Tr}_{\Gamma}[\varrho_{\Gamma}^{\gamma}(\phi)] = 1$, we are finally allowed to state that every γ -component of the sum in Eq. (4.40), evolves in *time* according to a properly defined von Neumann equation. Concluding, once the classical limit is taken, it is indeed the phase ϕ which can be understood as the temporal parameter *time* inherently emerging through the PRECS as a *classical continuous parameter*, in both cases.

4.4 Discussion

This section is devoted to comment upon some specific aspects of our results and on the assumptions made.

Let us start by the state $|\Psi\rangle\rangle$ of the isolated system, i.e., in a sense, of our Universe. We used the Schmidt decomposition, that, as already highlighted, is state dependent. However, notice that this is a peculiar situation, since, being the state $|\Psi\rangle\rangle$ unique, the Schmidt decomposition (4.9) is actually fixed. Moreover, the coefficients $c_{\gamma} \neq 0$ must be more than one, as we want to show that the evolution of the system Γ is indeed induced by the entanglement with the internal clock C. This is well understood through the PRECS: indeed, the parametric representation is meaningful if and only if the two subsystems are entangled, otherwise there would not be any clock label on the Γ states. It should then be stressed that the assumption of an entangled state of two subsystems which do not further interact, it is not at all too exotic. Think for example to the Bell experiments [90, 91, 92], specifically designed to test the theory of QM. They usually start with the preparation of two atoms with spin 1/2 both, combined in a molecule singlet state - or, quite equivalently, two generic spin-1/2 particles in a singlet state. The two particles are then separated, without influencing

the angular momentum, and they do not interact any longer; a similar scenario is the one supposed here for the two subsystems Γ and C . Moreover, we can also think that if there is a third system whose state is factorized with respect to $\Psi = \Gamma + C$, i.e., if there exists a system Φ such that its state is $|\Psi\rangle\rangle \otimes |\Phi\rangle$, separable from $|\Psi\rangle\rangle$, we simply have no way to detect it: it would then represent a sort of *another Universe*, with which we cannot communicate in any way and that would be therefore invisible to our observations.

Secondly, notice that the form (4.10) of the total Hamiltonian with the request of a stationary state (4.12) implies that the energy of the two subsystems are inherently bound together: fixing the energy of Γ is tantamount to establish a temporal scale, i.e., the energy of C , and viceversa. In other words, a posteriori we can state that the temporal resolution, with which an event is observed, determines which is the state of the diagonal operator \hat{H}_C selected in Eq. (4.9). In fact, the diagonal operator, associated to the variable conjugate to the *time* ϕ , and the shift-operators are maximally non-commuting, so that the uncertainty principle hold true. For example, in the case of the magnetic clock, it is $\Delta\vartheta\Delta\varphi \geq 1/S$, with ϑ the variable associated to $\Delta S^z \equiv \langle \hat{S}^z \rangle$. From the constraint (4.12), it then follows $\Delta S^z + \Delta E_\Gamma = 0$, with $\Delta E_\Gamma \equiv \langle \hat{H}_\Gamma \rangle$. Putting together these two relations we get thus $\Delta E_\Gamma \Delta\varphi \geq 1/S$, that can be understood as a time-energy uncertainty relation.

About the assumption of considering an eigenstate of \hat{H} , we agree with the picture provided by Page and Wootters: since any isolated system interacts only with itself *by definition*, there does not exist an external observer according to whom a sort of *global external time* can be defined. In addition, any isolated system is in an eigenstate of energy, which, however, cannot be meaningfully named *stationary*, as a global external time translation is indeed completely unobservable. Apart from the fascinating philosophical discussions, for which we refer the reader to Refs.[81, 89, 82], notice that General Relativity demands the Hamiltonian of the whole system to be null, in order to guarantee the general covariance of the system. This implies that “nothing moves around”, i.e., that “there is no time”, and Eq. (4.10) can be understood as a sort of Wheeler-DeWitt equation [93].

We also assumed that there is only one couple of shift-up/shift-down operators ($\hat{E}_\alpha, \hat{E}_\alpha^\dagger$). This implies that the manifold \mathcal{M} provided by the GCS has $\dim \mathcal{M} = 2$, so that the parameter emerging as *time* is unavoidably one-dimensional, as it should be. Indeed, besides the usual way that we have to represent time, considering it as a one-dimensional parameter allows us to directly find a connection with thermodynamics and the consequent definition of the so called thermodynamic arrow of time, provided by the second law³. Nevertheless, although the introduction of a single

³The *arrow of time* is the concept of the *one-way direction* or *asymmetry* of time, beautifully described in the book *The Nature of the Physical World* by Arthur Eddington, who first developed it: “Let us draw an arrow arbitrarily. If as we follow the arrow we find more and more of the random element in the state of the world, then the arrow is pointing towards the future; if the random element decreases the arrow points towards the past. That is the only distinction known to physics. This follows at once if our fundamental contention is admitted that the introduction of randomness is the only thing which cannot be undone. I shall use the phrase ‘time’s arrow’ to express this one-way property of time which has no analogue in space.”

couple $(\hat{E}_\alpha, \hat{E}_\alpha^\dagger)$ simplifies the problem a lot, we do not think it is a necessary condition for the definition of *time* as a parameter inherently provided by the PRECS. For instance, given a manifold \mathcal{M} with arbitrary dimension, one could identify on it a curve, which by definition is associated to a one-dimensional parameter, no matter how large is $\dim \mathcal{M}$.

Lastly, observe that, in the large- N limit, a von Neumann equation is derived for every single γ -component of the state $|\Psi\rangle\rangle$. Although we think that a deeper analysis of the meaning of this result can be worth and could possibly open the scenes to fascinating theories such as the Many Worlds one, it suffices here to notice that which is the actual γ that describes the evolution of the observed system Γ is ultimately determined by the state (4.9) of the Universe.

4.5 Conclusions

The large environment is here considered to act as an internal clock C for a physical system Γ , whose evolution we want to describe.

We used the PRECS to parametrize Γ , through the ECS defined for the quantum system which embodies the internal clock. Such choice allows the parameter that we call *time* to naturally emerge in the laws of dynamics when we perform the classical limit of the clock.

In particular, we considered the two paradigmatic cases of a bosonic and a magnetic clock, respectively described by the sets of field coherent states $\{|\alpha\rangle\}$ and spin coherent states $\{|\Omega\rangle\}$, already encountered in this thesis.

Our proposal to formalize the PaW mechanism understands the phase ϕ of the ECS as the temporal parameter. Indeed, being the Universe isolated, a global external time coordinate does not exist, and any observed dynamical evolution needs a partition of the Universe to stem. What we sense as the flow of time is the change that takes place in the different subsystems, related to the subsystems themselves and due to the internal quantum correlations. Since we detect time through changes, it is straightforward that measurement processes play a key-role: this is indeed the only way we have to observe any change, and in QM observation and measurement are tightly tied together. Different, successive measurements build a temporal order, so that time does not go by if any measurement is performed.

We end this chapter stressing once again the fundamental role of the ECS: via the PRECS a proper von Neumann equation can be defined to describe all the possible evolutions of Γ represented by the index γ and parametrically depending by a clock label, which inherently emerges, once the large- N limit is performed, as a *classical continuous parameter* that can be consequently properly understood as *time*.

Chapter 5

Large environment... at most!

Leitmotif of this thesis is the analysis of macroscopic quantum environments, which potentially exhibit a classical behaviour. Combining the tools of the OQS theory with approaches from quantum field theory, formally describing the conditions for a classical theory to emerge as the large- N limit of a quantum one, we deal with the fundamental issue of the quantum-to-classical crossover and we investigate the general idea that a classical environment is a macroscopic quantum environment.

As the World is inherently quantum, we need a scheme that makes the classical one emerge, yet such that simultaneously allows us to preserve the quantum origin of the analysed systems: indeed, we want to keep trace of the quantum interactions from which ultimately our classical effective descriptions of the phenomena we observe everyday stem. Having this issues in mind, we initially faced the problem of identifying the proper mathematical instruments to interpolate the quantum and the classical formalisms. We analysed the very peculiar properties of the generalized coherent states, finding out that they play a very essential role in this framework, and we discussed what is the minimal structure to define a proper classical limit of a quantum theory. We thus concluded and formally showed that macroscopicity is a necessary - but not sufficient - condition to make classicality emerge.

According to this general idea, we studied three different situations, each corresponding to a different role taken on by the environment. We firstly considered the environment as part of an hybrid quantum scheme, focusing on two different aspects: i) the study of the back-action, i.e., of the environmental dynamics induced by the interaction with the quantum principal system; ii) the analysis of the effective description of macroscopic quantum environments as classical fields. Secondly, the environment was assumed as a measuring apparatus, or, more precisely, the reason why a classical environment can *always* be assumed as tantamount to a measuring apparatus was investigated. Thirdly, we looked at the environment acting as a clock, in order to deal with the intriguing question of a proper quantum definition of time.

In all the situations and the models considered, we provided formal analyses and we presented possible solutions to some general questions, often naively taken for granted. Notice however that, beyond the fascinating fundamental aspects of the connection between the quantum and classical worlds, recent developments in quantum technologies increasingly require a deeper comprehension of how we, as “macro-

scopic objects”, can achieve a higher level of control upon small quantum devices, so that to better discern the quantum-to-classical crossover acquires also an increasingly practical aim. In fact, reversing the point of view, the technological progress has made also necessary a deeper understanding of those mechanisms that might cause quantum features to be spoiled by the presence of some environment: indeed from the theoretical ongoing investigation on the OQS dynamics, there emerge several ways of approximately accounting for the environment, that evidently ease the above understanding.

In the last decade, one set of fundamental applications that has recently emerged as a far-reaching application for Quantum Information is to Quantum Gravity. There is now a variety of different connections between the two theories that have been explored and people coming from different communities have started to talk [94, 95, 96]. By the background of OQS, the very first consideration we can make is the following: there is only one system really isolated, the Universe. Addressing the problem of time in Chap. 4, we already mentioned this idea, and we started indeed to deal with the largest possible environments. In the future, we would like to enlarge the environment at most and to apply the tools presented in this thesis to approach the research of quantum gravity. Translating in our language, this implies to seek for a quantum theory whose proper classical limit is a classical theory whose Poisson brackets be tantamount to the Einstein field equations of GR, or similarly, according to the scheme here followed, a quantum theory classically equivalent to GR. For example, inspired by the results of Chap. 4, we aim to investigate the role of the parametrization used to make time emerge as a classical continuous parameter defined on the manifold provided by the ECS of the clock. The parametrization we used is profoundly asymmetric: one variable, i.e., the modulus associated to the energy of the system, runs from zero to infinity¹, while the other one, i.e., the phase, is limited in the interval $[0, 2\pi)$. Although this all hereinbefore is only speculation, we stress that, beyond easing our understanding of the emergence of time, this parametrization could hide a deeper physical meaning. In our analysis it could be indeed recognized an implicit use of the so called action-angle coordinates, that could pave the way for the definition of a manifold whose classical parameters are associated to 1) time, that we have already recognized as the phase, i.e., potentially the angle, and 2) temperature, that we could maybe associate to the modulus, i.e., potentially to the action, and which is indeed defined as a positive quantity running from zero to infinity. In this picture, the *space* would be what actually determines the *partition* between what is in that point that we observe - the principal system Γ - and all the rest - the environment Ξ to which the clock C belongs.

Having these issues in mind as perspectives for future work, we conclude this thesis devoting the second part of this chapter to what is presented as one of the biggest open challenge in modern physics, i.e., the Black Hole Information Paradox. In Sec. 5.1 we introduce the paradox, briefly recalling some features of these special

¹Remember that for the magnetic clock the classical limit is defined by the large- S condition and \hat{S}^z -eigenvalues can be defined in $[0, 2S]$.

objects named Black Holes (BHs). In Sec. 5.2 we remind the tools of Classical and Quantum Information Theory. In particular we focus on the connection between thermodynamics and information provided by the Landauer principle, analysing the resolution of the Maxwell dæmon paradox. In Sec. 5.3 we introduce the concept of the Hawking radiation and then we calculate its von Neumann entropy, related to the concept of the evaporation and to the consequent BH Information Paradox respectively. In Sec. 5.4 we develop our information theoretic model for describing the evaporation of a BH, aiming to discuss the role of an observer in the paradox. Lastly, we report the concluding remarks in Sec. 5.5.

5.1 When the Maxwell dæmon stares at an event horizon

When we talk about a *paradox*, we generally mean a situation that gives seemingly contradictory results, though based on postulates considered well defined and which work in other contexts. Such situations are usually not paradoxical at all, and their resolutions consist quite often in finding the gap of the ill-defined or self-referring concepts from which they actually stem. Many times, in the history of physics, paradoxes arose in the attempt of unifying different theories and their crossing gave birth to deep scientific revolutions.

The *Black Hole Information Paradox* was formulated by Stephen Hawking in 1975 and, since its very first appearance, has been considered one of the most important open questions that fundamental physics has to face today. The issue sprouts from a clash between General Relativity (GR) and QM, whose reconciliation is indeed one of the main goal of modern physics. According to Hawking's calculations, a BH emits a thermal radiation that makes it evaporate; such process, named *evaporation*, seems to correspond to a non-unitary evolution of the states describing the emitted particles, and, thus, to a violation of the evolution postulate of quantum mechanics.

Before discussing the paradox, some history and properties of the classical BH will be briefly revised. The concept of a body so massive to create a gravitational field such intense that neither light can escape, was firstly proposed at the end of the 18th century by the English natural scientist Reverend John Michell. It took however more than a whole century before Karl Schwarzschild formally described the BHs as geometrical objects, in the framework of GR. Despite the initial resistance of the community, including Einstein himself, with respect to these too much exotic objects, from the second half of the last century the BHs have become very popular, largely thanks to astrophysical discoveries and improved simulations of collapsing matter which made the scientific community strongly believe in their existence. As a matter of fact, further investigations have then followed and people have started to focus on them, as they could play a key-role in the attempt of unifying QM and GR. It is indeed in the presence of a BH that the effects of quantum gravity are supposed to manifest themselves.

The BHs represent very special regions of the spacetime, from which *nothing is able to escape*. The boundary of this region is named *event horizon*, which, according

to Schwarzschild metric, is placed at distance $r = r_S$ from the actual singularity of the spacetime at $r = 0$, ultimately generating the BH. The quantity $r_S \equiv 2GM/c^2$ is the Schwarzschild radius, where G is the universal gravitational constant, M the mass of the matter considered and c the vacuum speed of light. The boundary is not a physical surface, and if a person fell through the event horizon - before being torn apart by tidal forces -, she would not notice any physical surface at that position; it is indeed only a mathematical surface, significant in determining the BH's properties². The area of a BH never decreases, and, moreover, there exists the so called *no-hair theorem* guaranteeing that we just need to know three features in order to recover the information about the nature of a collapsing body which originally formed a BH: its mass, its electric charge and its angular momentum. Thinking about this result, there is a resemblance with the statistical description of an object at thermal equilibrium, for which it is indeed sufficient to define a few macroscopic parameters. At the beginning of the 1970s, people have actually realised that the laws ruling the BHs behaviour are incredibly similar to the four fundamental laws of thermodynamics [97, 98]. In particular, this resemblance led to the view that a quantity of the BH, named *surface gravity* k , is a measure of its temperature $T_{BH} \sim k/2\pi$, and that the event horizon area, A , is a measure of the BH entropy $S_{BH} \sim A/4$. Nevertheless, it is not yet completely clear if the analogy between the two theories is just formal or if it hides a deeper physical meaning. A step forward in transforming such perception was done indeed when in 1975 Stephen Hawking discovered that a BH emits as a black body with temperature T_{BH} [99]; from that moment, the possible scenarios that have been uncovered are still being extensively studied. By Hawking's result, our understanding of BHs physics has dramatically changed with respect to the classical view. Firstly, we learn that there is this radiation, so a BH is not so "black" after

²The Schwarzschild metric is the most general spherically symmetric vacuum solution of the Einstein field equations, and it describes, indeed, the gravitational field outside a spherical mass, on the assumption that its electric charge, its angular momentum, and the universal cosmological constant are all equal to zero. Introducing the coordinates $(t, r, \vartheta, \varphi)$, the length element for the Schwarzschild metric takes the form

$$ds^2 = - \left(1 - \frac{r_S}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{r_S}{r}\right)} dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2),$$

with t and r having the meaning of *time* and *radius* only asymptotically, i.e., where the spacetime becomes flat. Nevertheless, fixed r and t , the surface of a sphere S^2 is $4\pi r^2$, so that the meaning of the radius r is recovered in a sense. Notice that this metric has two singularities, in $r = 0$ and in $r = r_S$. Anyway, their nature is profoundly different, since the latter depends on the choice of coordinates not able to recover the region $r \leq r_S$ and it is hence removable changing the coordinate system, while the former is an actual spacetime singularity, where the curvature of the metric diverges. When the solutions described are related to little dense astronomical objects, such as stars or planets, the two singularities do not disturb, since the Schwarzschild metric is valid only outside the field source and $r > r_S$ in these situations. When instead gravitational collapses take place, ending with the formation of superdense objects, whose radii are smaller than r_S , the Schwarzschild metric needs to be modified in order to describe the spacetime geometry for $r \leq r_S$. The results of these collapses, with indeed a *physical* singularity at $r = 0$, form the BHs. A Schwarzschild BH, or static BH, is a BH that has neither electric charge nor angular momentum. A Schwarzschild BH is described by the Schwarzschild metric, and cannot be distinguished from any other Schwarzschild BH except by its mass M .

all; secondly, it is not clear what is the meaning of S_{BH} , since classically a BH has only one state - the state described by its mass, its electrical charge and its angular momentum -, and, consequently, its entropy should be null. Moving to a quantum scenario, some things can be interpreted, but even deeper questions arise, and among them the BH information paradox takes arrogantly the principal role.

In the following sections, we will discuss the presence of an observer looking at the Hawking radiation, focusing on the resulting connection with the measurement process of QM. As we want to analyse the paradox from an informative theoretic point of view, some useful definitions should be reminded.

5.2 Information Theory

The foundations of what we name today *theory of information* were laid in 1948 by C. E. Shannon, who built a formal apparatus to deal with problems concerning the signals codification and transmission [100]. The *source* is the system used to generate information, and it can be thought as a mechanism able to emit an element, called the *symbol* x , once having chosen it among a set of symbols, the *alphabet* \mathbb{X} . A source is *discrete* if the alphabet is a numerable set, *continuous* otherwise. A *string* of length n is a sequence of n symbols emitted by the source. The symbol x of the i -th emission is chosen with probability $p_i(x)$, where the subscript i , running from 1 to n , indicates that the function could depend from the whole history of the source. If instead the choice of every symbol is independent from the previous emission, the source is called *memoryless*. In this case the probability associated to a generic string (x_1, \dots, x_n) is simply

$$p(x_1, \dots, x_n) = p(x_1) \cdots p(x_n) , \quad (5.1)$$

where all the $p(x)$ are equal, but they give different values being valued for different symbols. We will refer always to discrete memoryless sources in what follows.

To formalize how a definite quantity of information can be stored in the shortest string possible, the *information* and the *entropy* associated to a symbol, a string or a source must be introduced. Consider a source emitting symbols x of an alphabet \mathbb{X} with probability $p_i(x)$. The information associated to the choice of every symbol, or, similarly, the surprise that we feel reading the single emission of the source, is

$$i(x) = -\log_2 p_i(x) . \quad (5.2)$$

This expression vanishes in the case of a certain emission, and it takes arbitrarily high values in the case of a rare emission. The basis of the logarithm stands for the choice of a information measurement unit. In Eq. (5.2) we used 2, according to the analysis that will follow in the next sections, based indeed on binary choices. As we will always adopt such convention, we will omit the 2 hereafter. The information is additive for symbols emitted by a memoryless source, being

$$i(x_1, x_2) = -\log p(x_1, x_2) = -\log (p(x_1)p(x_2)) = \quad (5.3)$$

$$= -\log p(x_1) - \log p(x_2) = i(x_1) + i(x_2) . \quad (5.4)$$

Given a generic string $s = (x_1, x_2, \dots, x_n)$, we can define the *information* contained in it as

$$i(s) \equiv \sum_{x \in s} i(x). \quad (5.5)$$

Then, the *entropy* of a source Σ is the mean quantity \mathbb{E} of emitted information, or, analogously, the uncertainty that we have about its emission, i.e.,

$$H_\Sigma = \mathbb{E}[i] = \sum_{x \in \mathbb{X}} p(x) i(x) = - \sum_{x \in \mathbb{X}} p(x) \log(p(x)). \quad (5.6)$$

with $0 \leq H_\Sigma \leq \log M$, where M is the alphabet dimension, that is the number of symbols forming the alphabet. In other words, Eq. (5.6) defines a semipositive function which takes null value in the case of certain emission and its maximum value in the case of uniform emission ($p_1 = \dots = p_M = 1/M$). When Shannon introduced this quantity, initially named *uncertainty*, he did not immediately recognize the resemblance with the expression representing the thermodynamic entropy S_T of an isolated system at the thermodynamic equilibrium, i.e.,

$$S_T = -k_B \sum_i p_i \log p_i, \quad (5.7)$$

where p_i are the probabilities that the system is in the i -th configuration and k_B is the Boltzmann constant. For many years people wondered if such correspondence was only formal or if it was hiding a deeper physical meaning, exactly as in the more recent case of the analogy between the classical thermodynamics and the BH thermodynamics previously mentioned. The doubt was solved in 1961 by Landauer [101], via the development of his principle, which states that *any logically irreversible manipulation of information must be accompanied by a corresponding entropy increase of the information-processing apparatus, or of its environment*. In order to erase a bit, the corresponding entropy increase is, for instance,

$$S_T = k_B \log 2. \quad (5.8)$$

Year after year, the Landauer principle has taken on a more and more important role, both from a practical viewpoint, since it establishes an inherent heat production by real machines for the elaboration of information, and from a theoretic viewpoint, since it promotes the information theory, considered for decades a mathematical theory, to the status of physical theory.

5.2.1 The Maxwell dæmon stares at the box

It was only thanks to the Landauer principle that the link between thermodynamics and information theory was definitely established, and the Maxwell dæmon paradox, which has provoked debate and theoretical work extending to the present day, was finally solved. The analysis of this issue is so general, that we will refer to it for the BH information paradox, as we will report in Sec. 5.4. Let us now revise the paradox

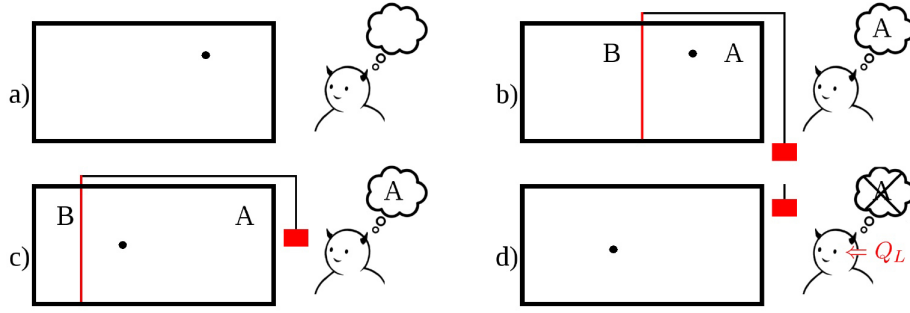


Figure 5.1: Graphical representation of the Szilard machine, with the Maxwell dæmon staring at the box.

formulation according to the Szilard machine model as presented in [102], which is quite different from the original one, proposed by Maxwell in 1872.

Suppose to have a box with a unique molecule of gas inside it - Fig. 5.1 a). Assume the existence of a dæmon able to divide through a moving wall the box in two sides, A and B, and to recognize where the molecule is, if in A or in B. Once realized the particle position, the dæmon can hang a weight to the wall - Fig. 5.1 b), and, putting the box in contact with a heat source, he can make the gas expand through an isothermal process. If the gas is perfect, this process produces work, according to

$$W = k_B T \log \frac{V_{initial}}{V_{final}} = k_B T \log \frac{V}{V/2} = k_B T \log 2, \quad (5.9)$$

so that it is possible to lift the weight - Fig. 5.1 c). When the expansion is finished, the gas comes back at its initial condition and a new cycle can be started, shaping therefore a process which can convert heat into work completely. This apparently violates the second principle of thermodynamics, stating that *it is impossible to devise a cyclically operating machine, the sole effect of which is to produce work absorbing heat from a single thermal reservoir*. Looking for the possible glitch from which the paradox stems, one realizes that the point here is not having included the dæmon in the picture. In fact, in order to really close the cycle after the gas expansion, going back at the initial condition of the system, the memory of the dæmon needs to be erased - Fig. 5.1 d) - as it was empty when the cycle started - Fig. 5.1 a). This operation is an irreversible one and it entails a heat dissipation of

$$Q_L = S_T T = k_B T \log 2, \quad (5.10)$$

as the Landauer principle (5.8) thought us. The work produced through the cycle is hence exactly the same that the system has to absorb in the form of heat to erase the dæmon memory, i.e.,

$$\Delta W = W - Q_L = 0. \quad (5.11)$$

Therefore, through the erasure, not only the cycle is an actual thermodynamic transformation between two identical states, but also it is not possible to create a thermal

machine capable of generating work via a single thermal reservoir, since the work produced by the process (5.11) is null. In other words, considering the real total system, which inherently includes the dæmon as an essential part, there is no paradox at all and, hence, the problem is solved.

5.2.2 Quantum Information Theory

Moving to a quantum world, some concepts of the classical picture introduced in the previous section need to be extended in order to define the Quantum Information Theory, that in the last decade entered the scene as one of the leading branches of modern physics, from a theoretical and a practical point of view both [49, 103]. A *quantum source* Σ is a system defined by the statistical ensemble $\{|\varphi_y\rangle, p(y)\}$ able to prepare a physical system, with Hilbert space \mathcal{H} , in the states $|\varphi_y\rangle \in \mathcal{H}$, each with probability $p(y)$. To deal with composite quantum systems, it is useful to introduce the density operator formalism, defining an operator which represents the expectation value of the prepared states $|\varphi_y\rangle$, i.e.,

$$\varrho_\Sigma = \mathbb{E}\{|\varphi_y\rangle \langle \varphi_y|\} = \sum_y p(y) |\varphi_y\rangle \langle \varphi_y| , \quad (5.12)$$

where \mathbb{E} is the expectation value associated to the statistical ensemble. Notice that the states $|\varphi_y\rangle$ are not necessarily orthogonal and, therefore, not necessarily distinguishable. Nevertheless, since ϱ_Σ is hermitian by construction, there always exist a diagonal form in which we can rewrite it, yielding

$$\varrho_\Sigma = \sum_{x \in \mathbb{X}} p(x) |x\rangle \langle x| , \quad (5.13)$$

where the states $\{|x\rangle\}_{x \in \mathbb{X}}$ form an orthonormal basis of \mathcal{H} . Thanks to this transformation, the states of the ensemble $\{|x\rangle, p(x)\}$ are distinguishable, so that we can restore the Shannon picture of the information, associating to each state a different symbol of a classical alphabet. In fact, given the density operator ϱ_Σ , its *von Neumann entropy* can be defined,

$$S \equiv -\text{Tr}(\varrho_\Sigma \log \varrho_\Sigma) , \quad (5.14)$$

and using the diagonal form (5.13), we get

$$S = -\text{Tr}(\varrho_\Sigma \log \varrho_\Sigma) = -\sum_{x \in \mathbb{X}} p(x) \log p(x) , \quad (5.15)$$

so that the von Neumann entropy associated to the ensemble of quantum states $\{|\varphi_y\rangle, p(y)\}$ is equal to the Shannon entropy associated to the ensemble of classical values $\{x, p(x)\}$. As the latter, also the first one is a semipositive defined function and it takes its values in the range

$$0 \leq S \leq \log N , \quad (5.16)$$

where the two extreme cases correspond respectively to a pure state $|\psi\rangle \in \mathcal{H}$ that has null S , and to the so called maximum mixed state, i.e., a state of the form (5.13) with all the $p(x) = 1/N$ being N the dimension of \mathcal{H} . Typical schemes in Information Theory regard composite systems made of independent subsystems. If this is the case, the density operator can be factorized in a tensor product of the density operators acting on the respective Hilbert subspaces, and, thus, the von Neumann entropy can be written as the sum of the entropies of the different subsystems. This means that, if we have a system $S = +A_i$, with $i = 1, \dots, N$ labelling the i -th subsystem, we get

$$S\left(+_{i=1}^N A_i\right) = \sum_{i=1}^N S(A_i) , \quad (5.17)$$

which is the *additivity* property of the von Neumann entropy, and it will be fundamental for our exposition of the BH information paradox.

5.3 Hawking radiation

The Hawking radiation is an outgoing flux of particles emitted by a BH as a consequence of particles creation connected to the formation of the BH itself. Such production is probably the most famous among the effects arising by the application of quantum field theory on curved spacetimes. In GR, the gravitational field is indeed associated to the curvature of the spacetime metric, so that it is not possible to describe the spacetime like flat in the presence of a BH. Performing this analysis, one learns that a time-dependent gravitational field creates particles and such production can not be ignored in the case of a BH, because of the rapid time-variation of the metric describing the spacetime near the event horizon. Just to give a naive idea of the phenomenon, we here present a simpler version in terms of ordinary QM, referring the interested reader to the several works presented in the Literature which properly describe the Hawking radiation [104, 105, 106, 107]. Consider a one dimension harmonic oscillator with frequency $\omega(t)$ and mass m . The system Hamiltonian is

$$\hat{H} = \hbar\omega(t) \left(\hat{n} + \frac{1}{2} \right) , \quad (5.18)$$

and if the system is described by the eigenstate $|\psi\rangle$ such that $\hat{n}|\psi\rangle = n|\psi\rangle$, the expression (5.18) can be seen whether as the energy of a single quantum harmonic oscillator in the n -th energy level, or as due to the presence of n particles in the unique excited state available, i.e., the state at energy $\hbar\omega(t)$. In the latter interpretation it is the number operator \hat{n} itself which tells us how many particles are detectable and, therefore, existent. Notice that this procedure of using a physical observable to evaluate the existence of a particle seems here to be quite logical: nevertheless, it is not a smooth point and we will go back to it in Sec. 5.4 about the discussion on the information paradox. Taking the $\omega(t)$ time-dependence as

$$\omega(t) = \omega\vartheta(-t) + \tilde{\omega}\vartheta(t) , \quad (5.19)$$

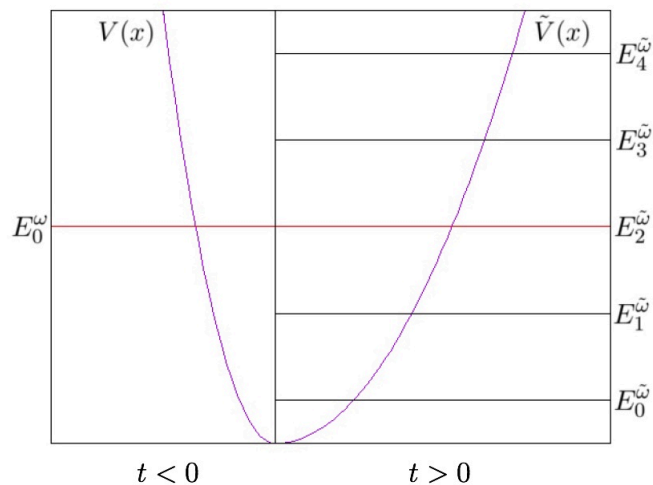


Figure 5.2: Energy spectrum before ($t < 0$) and after ($t > 0$) the variation of the potential: we chose $\tilde{\omega} = \frac{1}{5}\omega$ in the figure, so that to have an exact correspondence between the ground state for $t < 0$ and the second excited level for $t > 0$. In this case we will say that the time dependence of the potential creates 2 particles.

with $\vartheta(t)$ the Heaviside function, we get a potential that suddenly changes from $V(x) = \frac{1}{2}m\omega^2x^2$ to $\tilde{V}(x) = \frac{1}{2}m\tilde{\omega}^2x^2$ at $t = 0$. Since its physical observables can not vary in a discontinuous way, the physical system keeps the energy at the value it had at $t < 0$ for longer time than the time needed to the potential to vary (that is equal to zero in our case, since the potential variation is assumed instantaneous). This very quick variation represents the opposite case to what generally follows from the adiabatic theorem, which describes the evolution of states when the potential changes slowly, and it is so drastic that the system is not able to evolve according to it. Thus, if the system at $t < 0$ has an energy corresponding after the variation to one of the new stationary energy levels, it will set on that level at $t > 0$. Assume that the energy of the system is due to the presence of n particles; then, the measure of \hat{n} will give a result that changes *instantaneously* at $t = 0$, as well as the number of current particles. For example, if the frequency of the harmonic oscillator *instantaneously* decreases, as in Fig. 5.2, after the potential variation the number of particles increases with respect to the one at $t < 0$: we will say that these new extra particles have been therefore *created* by the time dependence of the potential.

Through a similar mechanism, couples of particles are continuously created on the BH event horizon, i.e., at distance $r = r_S$ from the singularity lying at $r = 0$ and ultimately generating the BH itself. Then, because of the continuous “stretching” of the spacetime that goes on in the horizon vicinity, the particles can be pulled apart from each other. In fact, due to the peculiar BH geometry, if a particle is created at distance $r = r_S$, the particle stays on the horizon, but, if the particle is created slightly inside/outside the horizon, it falls/escapes inexorably towards smaller/bigger r ; lastly, if the two particles of the couple are created by the same side of the horizon

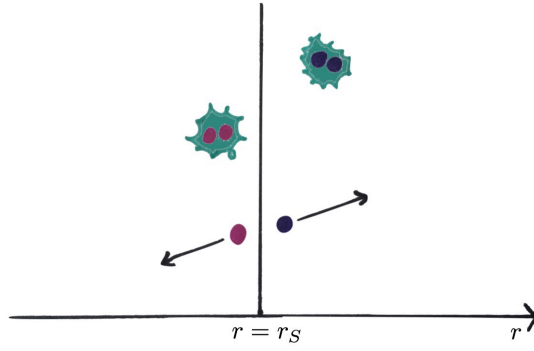


Figure 5.3: Graphical sketch of the particles production near the horizon. Couples created on the same side of the horizon quickly annihilate with each other; couples created with a particle per side inherently separate and the escaping one - on the right side of $r = r_S$ in the figure - forms the Hawking radiation.

they immediately annihilate with each other. With the locution *Hawking radiation* we name the set of particles which fly away from the event horizon, once separated from their partner that is instead created a little inside of $r = r_S$ - Fig.5.3.

5.3.1 Hawking radiation's entropy

Every couple of particles is created by the metric variation from the vacuum. This is the reason why its state is usually represented by some superposition of eigenstates of a binary physical observable that assumes opposite values in the particle and antiparticle states of the couple, so that it sums up to zero when the couple is considered as a whole. Using the computational basis $\{|0\rangle, |1\rangle\}$, the couple state $|\psi\rangle_{pair}$ can be for example written as

$$|\psi\rangle_{pair} = \frac{1}{\sqrt{2}} (|0\rangle_b |0\rangle_a + |1\rangle_b |1\rangle_a) , \quad (5.20)$$

where the subscripts a and b refer to the outside ($r > r_S$) and the inside ($0 < r < r_S$) BH regions respectively. Being the two regions causally disconnected, the Hilbert space of the couple is $\mathcal{H}_{pair} = \mathcal{H}_b \otimes \mathcal{H}_a$, and operators acting locally on \mathcal{H}_b and \mathcal{H}_a commute with each other. This state is usually introduced thinking about two electrons at different locations, with $|0\rangle$ and $|1\rangle$ representing the eigenvalues $\pm 1/2$ of the spin z -component operator; we follow such assumption, but underlining that, as we stated, we can actually consider any other binary observable and the created particles are not necessary electrons [108]. Notice that Eq. (5.20) represents one of the four Bell's states, i.e., one of the maximum entangled states available for a qubit, and it makes the entangled nature of the couple evident. As time goes by, new couples are produced until there will be N couples each in the state (5.20), composed by one particle detectable as Hawking radiation and the other one confined inside the BH region. During the evaporation the spacetime around the horizon keeps to stretch, so that every couple, being created very far away from the rest, is slightly influenced

by the others, and the state describing all the particles can be therefore written as the tensor product of N pair states, each identified by the apex i of the i -th couple, i.e.,

$$|\psi\rangle = \bigotimes_{i=1}^N \frac{1}{\sqrt{2}} (|0\rangle_{b^i} |0\rangle_{a^i} + |1\rangle_{b^i} |1\rangle_{a^i}) . \quad (5.21)$$

The total state of the BH contains then the matter whose collapse originally formed it. Since the event horizon is well separated from the singularity at $r = 0$, we can suppose with good approximation that the matter interact very weakly with the couples generated on it. If we then presume that the matter has never interacted with the region complementary to the BH, and that, thus, the whole state of the BH is isolated, we can consider it described by a pure state $|T\rangle$, with density operator

$$\varrho = |T\rangle \langle T| \sim |M\rangle \langle M| \otimes |\psi\rangle \langle \psi| , \quad (5.22)$$

where $|M\rangle$ is the state of the matter collapsed in the singularity and $|\psi\rangle$ the state (5.21) of the N couples. Being interested in the Hawking radiation, we need to get its reduced density operator ϱ_{rad} performing two partial traces, both on the Hilbert space of the matter, \mathcal{H}_M , and on the Hilbert space associated to all the particles inside the BH, i.e., $\mathcal{H}_B \equiv \bigotimes_i \mathcal{H}_{b^i}$. The first operation goes straightforward and, from Eq. (5.21), it is

$$\begin{aligned} \varrho_\psi &= \text{Tr}_M(\varrho) = \text{Tr}_M(|M\rangle \langle M|) |\psi\rangle \langle \psi| = \\ &= \frac{1}{2^N} \bigotimes_{i=1}^N (|00\rangle_i + |11\rangle_i) (\langle 00|_i + \langle 11|_i) , \end{aligned} \quad (5.23)$$

where we introduced the compact notation $|\beta\alpha\rangle_i \equiv |\beta\rangle_{b^i} |\alpha\rangle_{a^i}$. The state of the particles flying away from the event horizon and constituting the Hawking radiation, will then be given by

$$\varrho_{rad} = \text{Tr}_B(\varrho_\psi) , \quad (5.24)$$

so that the Hawking radiation entropy can be finally written applying definition (5.14), i.e.,

$$S_{rad} = -\text{Tr}(\varrho_{rad} \log \varrho_{rad}) . \quad (5.25)$$

In order to explicitly calculate it, let us focus on the contribution of the first couple generated. The state (5.22) reads hence $\varrho = |M\rangle \langle M| \otimes |\psi\rangle_{pair} \langle \psi|$, and, through the traces over \mathcal{H}_M and \mathcal{H}_B , we get the density operator of the first flying away particle, as

$$\varrho_{rad}^{(1)} = \frac{1}{2} (|0\rangle_a \langle 0| + |1\rangle_a \langle 1|) = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad (5.26)$$

where in the last step we used the basis $\{|0\rangle_a, |1\rangle_a\}$, so that from Eq. (5.25) it follows straightforward

$$S_{rad}^{(1)} = -\left(\frac{1}{2} \log \frac{1}{2} + \frac{1}{2} \log \frac{1}{2}\right) = \log 2 . \quad (5.27)$$

Since we supposed that each couple is independent from the others, the generalization of Eq. (5.27) to the case of an arbitrarily high number of created couples is directly provided by the additivity property (5.17). When N couples are formed it is thus

$$S_{rad} = N \log 2 , \quad (5.28)$$

simply summing every contribution tantamount to $\rho_{rad}^{(1)}$.

In this result, it is already contained the essence itself of the BH information paradox. A very wide number of particles is generated by the Hawking mechanism, and half of it is emitted in the region external to the BH in the form of an observable radiation. Until the BH exists, Eq. (5.28) can be understood as the entanglement entropy of the radiation particles with their inside partners. Nevertheless, Hawking's famous result proposes a dramatic modification in our expectation for the ultimate fate of a BH [109]: the evaporation suggests indeed that the BH, losing energy and correspondingly decreasing in mass, can disappear in a finite amount of time. There are hence two possibilities:

- 1) The BH evaporates away completely and the Hawking radiation ends up in a mixed state ρ_{rad} ;
- 2) The evolution stops when the BH mass reaches a certain value m_{rem} , that represents the mass of a small set of stable objects, the *remnants*, arbitrarily highly entangled with far away systems.

The first possibility entails a loss of unitarity and a consequent violation of the time evolution postulate of QM, since only a non-unitary dynamics allows an initial pure state to evolve to a final mixed state. On the contrary, the second one preserves QM postulates, but it opens the doors to a new unknown physics, that should be rather unusual. In fact, in order to solve the paradox and encode all the information about the initial state, the remnants should allow a very large number of possible states, i.e., $n \geq N$, so that they could have an entanglement entropy (5.28) with the Hawking radiation. Estimating m_{rem} from considerations about the scales at which the effects of quantum gravity are expected to become relevant, one concludes that a small object with mass of order of the Plank's mass - $\approx 2 \times 10^{-5}g$ - should have the same number of possible internal states of the original mass M collapsed to form the BH, that of course can be as large as one can imagine. This is not at all the expected behaviour for quantum systems and such unusual thermodynamical properties that these remnants should possess, make usually people consider this solution not feasible [110]. Therefore, we focus on the first case, excluding the possibility of remnants hereafter. However, we underline that Hawking's argument does not say which of the two possibilities 1) or 2) can occur, since the so called *niceness conditions*, on which he based his calculations, are violated near the endpoint of the BH evolution.

Before presenting our proposal, let us notice that the result (5.28), which includes indeed the core of the paradox, is derived at the leading order, since we have considered the pairs to be independent from each other and not entangled neither with

respect to themselves or to the original mass M . Many people have therefore tried to introduce corrections to these assumptions, consequently adjusting the state (5.22). Discussing this point goes beyond the scope of this thesis, but it is possible to prove that Hawking's problem is not solved by small corrections. The paradox will persist unless an order unity modification is made to the leading order, but this is forbidden by the whole scheme of GR itself. Moreover, although it is true that some delicate aspects of the classical theory, e.g., giving an accurate description of BH singularities, are expected to be solved by quantum gravity, the dispute about the fate of information during the Hawking evaporation of a BH seems to endure in that framework as well [108].

5.4 The Maxwell dæmon stares at the evaporation

In this section we build a model to describe the evaporation of a BH from an informatic point of view, inspired by the solution of the Maxwell dæmon paradox provided by the Szilard machine.

Imagine the Schwarzschild spacetime as a box whose left wall identifies $r = 0$, while the event horizon is represented by a moving wall, initially placed at distance $r = r_S^{(0)}$ from the singularity - Fig. 5.4 a). Suppose then the existence of a dæmon able to look inside the right part of the box. On the contrary, the left part of the box is not accesible, being causally disconnected by the right side of the box, which represents indeed the region of the spacetime external to the BH. The BH evaporation is due to the particles emission, in this model depicted as the particles addition in the right part of the box with the resulting pulling back of the moving wall; every time that a particle is emitted appearing in the right part, the dæmon updates its memory in order to account for the variation of the particles number. Figs. 5.4 b) and 5.4 c) correspond to the emission of the first and of the second particle respectively. Be the mass of the particles $m \simeq M/N$: at every emission the event horizon, i.e., the moving wall, will back up of the quantity

$$\delta = \frac{2Gm}{c^2}, \quad (5.29)$$

and after N emissions we will have

$$r_S^{(N)} = \frac{2GM}{c^2} - N\delta \simeq 0, \quad (5.30)$$

that identifies the moment at which the BH fades out completely. At this point we will say that the BH is evaporated away, since admitting that it can disappear completely implies that the partition of the spacetime, provided by the event horizon, vanishes as well. In our model this means that the moving wall reaches the left one, so that it is not possible to distinguish between the left and the right side of the box any longer - see Fig. 5.4 d) ³.

³We here considered only massive particles; however, even if the particles would be massless, we could apply the same argument invoking the mass-energy equivalence, at least at this level of the analysis.

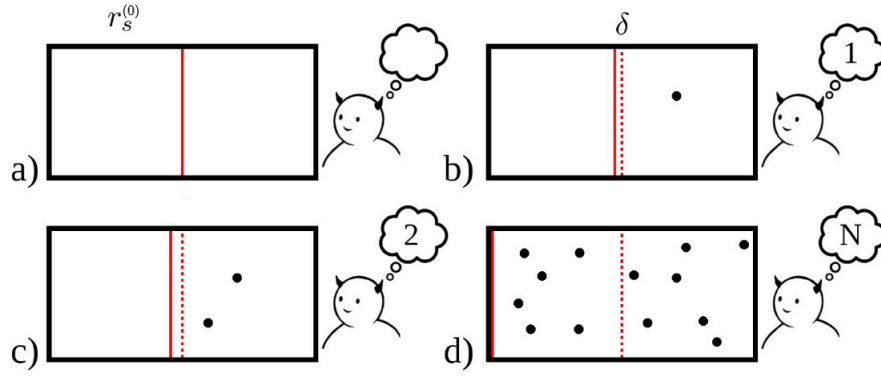


Figure 5.4: Graphical representation of the Maxwell dæmon staring at the BH evaporation.

As seen in Sec. 5.3.1, the Hawking radiation, i.e., the N particles in the right side of the box, end up in a maximally mixed state with entropy $S_{rad} = N \log 2$. Nevertheless, exactly as in the case of the Maxwell dæmon paradox, we must remember to include the entropy contribution given by the information that the dæmon acquires, while it is observing the radiation and, particle after particle, registering its existence. The only way that the dæmon has to realise the actual presence of a particle is via the measurement of a physical observable characterizing the particle itself, and, therefore, given the state (5.20), it necessary has to resort to the spin z -component operator - or to the arbitrary binary observable whose eigenstates are $\{|0\rangle, |1\rangle\}$ - to get such awareness. Assume now the BH be a classical source of information emitting a string chosen from an alphabet $\mathbb{X} = \{|0\rangle, |1\rangle\}$ with probability $p(|0\rangle) = p(|1\rangle) = 1/2$. From Eq. (5.2), the information acquired by the dæmon through the observation of the radiation is

$$i_{rad} = N \log 2 , \quad (5.31)$$

and its corresponding Shannon entropy is

$$H = N \left(p(|0\rangle) \log \frac{1}{p(|0\rangle)} + p(|1\rangle) \log \frac{1}{p(|1\rangle)} \right) = N \log 2 \quad (5.32)$$

as follows from Eq. (5.6), where the sum runs indeed on the whole alphabet. Although the radiation evolves from a state with zero entropy to a state with maximum entropy, from Eq. (5.32) the entropic balance results null for the whole process, both during the evaporation and at its end, as soon as we include the dæmon in the description of the total system. In fact, naming S the total entropy, we get

$$S = S_{rad} - H = 0 , \quad (5.33)$$

which implies that the total system, compound of both the radiation and the dæmon, remains in a state with $S = 0$, i.e., in a pure state, during the whole evolution, so that the paradox disappears.

5.5 Conclusions

In this last chapter we reported our concluding remarks and sketched some ideas for possible future perspectives. Having (quantum) gravity in mind, we then focused on the BH information paradox, resulting from the combination of QM and GR.

Exploiting the tools provided by the Information Theory, both classical and quantum, we presented our proposal inspired by the Maxwell dæmon paradox in the Szilard machine formulation. The evaporation model developed in Sec. 5.4 shows an entropic balance always equal to zero thanks to the dæmon presence, as expressed by Eq. (5.33). This result made us think about the importance of the observer in the Hawking radiation process. As a matter of fact, in our model the measurement of an observable is not only the mechanism through which we acquire information about a certain property of the system, but it is also the unique way that we have to empirically confirm the possible existence of the system itself.

The key-role played by the observer, that taking a set of measurements becomes aware of the existence of the radiation and, thus, of the actual evaporation of the BH, led us to ascribe the non-unitarity of the information paradox to the measurement process effectuated by the dæmon. Even though a non-unitary evolution represent a problem in the absence of an observer, these kinds of processes not only are envisaged by QM, but also establish the foundation of such theory, being intrinsic to the measurement postulate itself. It is not a mystery that one of the deepest difference between the classical and the quantum description of the world lays exactly in the measurement representation: if in the first case it is possible to conceive an ideal observer who acquires information about a physical system without modifying it, in the latter case such option is not even contemplable and every time that we want to look at a physical system to extract information from it, we have to consider the necessary interaction required by the theory itself between the system and some measuring apparatus. Therefore, the total isolated system for which we must expect a unitary evolution that starting from a pure state ends in a pure state, is the one that includes the observer as an essential subsystem of it. In other words, a correct analysis of the information paradox demands the introduction of an active observer and a formal description of the measurement process. We underline that, although the traditional approach to the problem does not consider the instrumental character of quantum theory, this aspect arised in recent years as fundamental to realize a complete analysis of the paradox according to several authors [111, 112, 110].

We conclude with a brief consideration about our description: whereas the formulation of the Maxwell dæmon paradox is independent from the classical or quantum setting in which it is presented, our proposal is exclusively quantum. We indeed built it on the importance of the quantum measurement process and on the active role played in it by the observer, as well as on the quantum formalism itself. Finding a classical corresponding for our description of the information paradox could be interesting, as it would allow to ponder about the connections between classical and BH thermodynamics.

Appendices

Appendix A

Generalized Coherent States

As the name itself suggests, Generalized Coherent States (GCS) are an extension of the field coherent states, firstly introduced by Glauber in 1963 to study the electromagnetic correlation functions of the harmonic oscillator [113, 114]. Yet, the idea of what is now called “coherent state” was conceived well before: back in 1926, immediately after the birth of QM, Schrödinger first proposed the concept in a pioneering paper that was indeed applicated only more than thirty-five years later. Lastly, after additional ten years, the complete construction of the GCS of Lie groups was achieved in two works, independently developed by Gilmore [68] and Perelomov [11], connecting the CS with the dynamical group for each physical problem one can consider, so that they no longer needed to be restricted to the harmonic oscillator. The construction based on Lie groups makes one important property arise: all GCS are inherently in one-to-one correspondence with the coset-space, that can be defined by the group itself. Since coset spaces are known to have geometrical properties, GCS must also be naturally endowed with geometry.

A.1 From algebra to geometry, and back

From the above considerations already, one can imagine why GCS play such a special role when classical limits are concerned. As a matter of fact, we need a bridge from the algebraic world of quantum mechanics to the geometrical one of classical physics, so that GCS, being provided *by construction* with both algebraic and geometrical tools, appear to be the right candidates. Quoting from J. R. Klauder, “*Coherent states are the natural language of quantum theory*”. In particular, as the reality is inherently quantum, we want to let the C theory emerge from the Q theory, and this means that we have to start from the algebra.

An algebra without a qualifier is just a vector space. In physics the relevant algebras are the Lie algebras, to which we will refer here and to which we referred through all this thesis. A Lie algebra is a vector space \mathfrak{g} over a field \mathbb{F} with a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, named Lie bracket, satisfying the following axioms:

- Bilinearity,
 $[aX + bY, Z] = a[X, Z] + b[Y, Z]$, $[Z, aX + bY] = a[Z, X] + b[Z, Y]$

for all scalars $a, b \in \mathbb{F}$ and all X, Y, Z elements of \mathfrak{g} ;

- Alternativity,
 $[X, X] = 0 \quad \forall X \in \mathfrak{g}$;
- Anticommutativity,
 $[X, Y] = -[Y, X] \quad \forall X, Y \in \mathfrak{g}$;
- Jacobi identity,
 $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \forall X, Y, Z \in \mathfrak{g}$.

Given a certain Lie algebra, physicists always work with its representations, that are a way of writing the Lie algebras as sets of matrices (or endomorphisms of a vector space) in such a way that the Lie bracket be given by the commutator. In other words, one looks for a vector space V together with a collection of operators on V satisfying some fixed set of commutation relations, as for instance in the case of the algebra $\mathfrak{su}(2)$ characterizing the angular momentum operators. Let $gl(V)$ denote the space of endomorphisms of V , i.e., the space of all linear maps of V to itself. Be $gl(V)$ a Lie algebra with Lie bracket defined by the commutator $[X, Y] = XY - YX$. Then a representation of \mathfrak{g} on V is a Lie algebra homomorphism $\varrho : \mathfrak{g} \rightarrow gl(V)$, that is ϱ assigns to each X in \mathfrak{g} an operator $\varrho(X)$ on V , with preservation of linearity and bracket. Explicitly, this means that ϱ must be a linear map and must satisfy

$$\varrho([X, Y]) = \varrho(X)\varrho(Y) - \varrho(Y)\varrho(X) \quad \forall X, Y \in \mathfrak{g}. \quad (\text{A.1})$$

The most basic example, although very important, of a Lie algebra representation is the *adjoint* representation, denoted by the symbol ad , of a Lie algebra \mathfrak{g} on itself, i.e., such that the vector space on which \mathfrak{g} operates is \mathfrak{g} itself:

$$ad : \mathfrak{g} \rightarrow gl(\mathfrak{g}), \quad X \rightarrow ad_X, \quad ad_X(Y) = [X, Y]. \quad (\text{A.2})$$

There is a very common use of \mathfrak{g} as if it were one \mathfrak{g} its representation, so that we say that a representative \hat{O} of \mathfrak{g} belongs to the algebra \mathfrak{g} itself. Associated to any Lie algebra there is then a Lie group, that is exactly what we need to go from algebra to geometry, and, thus, from Q to C . A set \mathcal{G} with an operation “ \cdot ”, named multiplication, combining two elements a and b of the set, is a group (\mathcal{G}, \cdot) , hereafter indicated as \mathcal{G} , if the following axioms are satisfied:

- Closure, $\forall a, b \in \mathcal{G}, \quad a \cdot b$ is also in \mathcal{G} ;
- Associativity, $\forall a, b, c \in \mathcal{G}, \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$;
- Identity element, $\forall a \in \mathcal{G} \exists! e \in \mathcal{G}$ s.t. $a \cdot e = e \cdot a = a$;
- Inverse element, $\forall a \in \mathcal{G} \exists! b \in \mathcal{G}$, usually denoted as a^{-1} , s.t. $a \cdot b = b \cdot a = e$.

A Lie group is a group in which the operations of multiplication and inversion are *smooth maps*, i.e., they have the derivatives of all orders well defined in their whole domain. Lie groups are said to be “groups which are also differentiable manifolds”.

A differentiable manifold \mathcal{M} is a topological manifold¹ equipped with an equivalence class of atlases whose transition maps are all differentiable. The way to move from the Lie group \mathcal{G} to the manifold \mathcal{M} is provided by the action. A group action $\varphi_{\mathcal{G}}$ of \mathcal{G} on \mathcal{M} is a function $\varphi : \mathcal{G} \times \mathcal{M} \rightarrow \mathcal{M}$, usually denoted $(g, x) \rightarrow \varphi_g(x)$ that satisfies the following axioms:

- Compatibility, $\forall g, k \in \mathcal{G}$ and $x \in \mathcal{M}$ it is $\varphi_g \cdot \varphi_k(x) = \varphi_{gk}(x)$
- Identity, $\forall x \in \mathcal{M}$ it is $\varphi_e(x) = x$, where e denotes the identity element of the group \mathcal{G} .

The group \mathcal{G} is said to act (left) on \mathcal{M} , and from the two axioms above it follows that for every $g \in \mathcal{G}$, the function which maps $x \in \mathcal{M}$ to $\varphi_g(x)$ is a bijective map from \mathcal{M} in \mathcal{M} , with its inverse given by the function which maps x into $\varphi_{g^{-1}}(x)$. In other words, the action of a group \mathcal{G} on \mathcal{M} is a group homomorphism from \mathcal{G} to the group of all the bijections from \mathcal{M} to \mathcal{M} itself. Moreover it is possible to prove that smooth, free and proper group actions always lead to smooth manifolds as orbit spaces. The Quotient Manifold Theorem states that

Theorem (Quotient Manifold Theorem). *If \mathcal{K} is a Lie group acting smoothly, freely and properly on a smooth manifold \mathcal{M} , then the topological quotient space \mathcal{M}/\mathcal{K} is a topological manifold of dimension equal to $\dim \mathcal{M} - \dim \mathcal{K}$ and has a unique smooth structure with the property that the projection $\pi : \mathcal{M} \rightarrow \mathcal{M}/\mathcal{K}$ is a covering map.*

This theorem guarantees that the quotient space of the Lie group, from which the generalized construction of GCS starts, is a topological manifold providing us with the orbit space that can be ultimately recognized as the classical phase space of a classical theory. Thus, thanks to this result, we can identify the one-to-one correspondence typical of GCS between states and points on the manifold, provided by the quotient space. It is indeed such property that award to GCS a very special role in the classical limit of quantum theories, since it contains the connection between algebra and geometry we are looking for.

As a matter of fact, through the Lie group-Lie algebra correspondence, we can study geometric objects, i.e., Lie groups, in terms of linear ones, i.e., Lie algebras, and vice-versa. The typical map that one defines from the Lie algebra \mathfrak{g} of a Lie group \mathcal{G} to the group, is the exponential map $\exp : \mathfrak{g} \rightarrow \mathcal{G}$. Given $A \in \mathfrak{g}$ and a scalar $a \in \mathbb{F}$, we have

$$\exp(aA) \sim \mathbb{I} + aA + \dots, \quad (\text{A.3})$$

where on the left side the exponentiation of A is an element of \mathcal{G} , while on the right side we are back to linear elements of \mathfrak{g} . The expression above can be understood as a description of the group starting by an open neighbourhood of the identity. In fact, the number of parameters that we need to describe the open neighbourhood are the same needed for describing the whole group itself, and, being all the elements of

¹A topological space is a set of points, each with its neighbourhood, with axioms relating points and neighbourhoods. A manifold is a topological space locally homeomorphic to a Euclidean space.

the group equivalent from a geometric point of view, we can start from the identity. To move instead from the group to the Lie algebra, we need the so called Baker-Campbell-Hausdorff (BCH) formula, which is the solution to the equation

$$Z = \log(e^X \cdot e^Y), \quad (\text{A.4})$$

where X, Y are elements of \mathfrak{g} and \cdot denotes the multiplication of \mathcal{G} . This formula expresses the logarithm of the product of two Lie group elements as an infinite sum of elements of \mathfrak{g} . Notice that, since the infinite series may or may not converge, Z is not defined as an actual element of \mathfrak{g} . Whenever the solution of this form is defined, the multiplication in the group can be expressed entirely in Lie algebraic terms. It is possible to express the BCH formula as a combinatorial formula, where the first terms are well known:

$$Z(X, Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) - \frac{1}{24}[Y, [X, [X, Y]]] + \dots, \quad (\text{A.5})$$

with all higher-order terms involving $[X, Y]$ and commutators nesting thereof, thus in the Lie algebra \mathfrak{g} . Let us notice that if the hamiltonian of a physical system belongs to

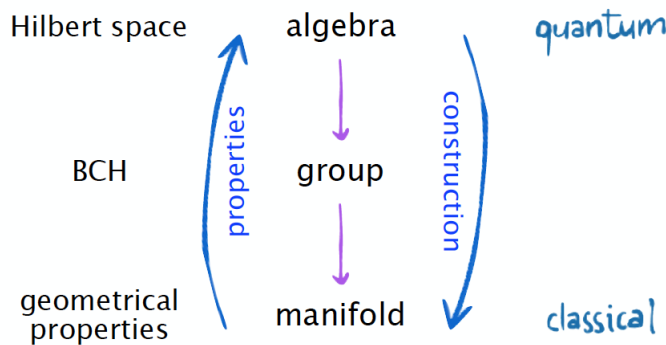


Figure A.1: Graphical representation of the relations between the quantum algebraic picture and the classical geometrical one.

a Lie algebra. i.e., if $\hat{H} \in \mathfrak{g}$, as indeed it is in all the cases of physical interest, then the corresponding propagator belongs to the associated Lie group, i.e., $\exp(-i\hat{H}t) \in \mathcal{G}$. In other words, the dynamical group, that is the group of all possible propagators describing the dynamics of a physical system, is a Lie group. Summarizing, the scheme for moving from the quantum algebraic world to the classical geometric one, must involve the connections depicted in Fig. A.1.

After the framework is set, we are ready to introduce the group construction of the so called Generalized Coherent States (GCS), i.e., the CS for any physical system. Time by time, we must find the specific CS that properly suit the system of our interest, making the generalization of the concept of CS to different systems a necessary prerequisite for our analysis. As mentioned at the beginning of the Appendix, Gilmore [68] and Perelomov [11] independently developed the general recipe for

their construction. Here we will mainly adopt Gilmore's procedure, following the review made by Zhang, Feng and Gilmore himself [10]. We will start explicitly considering two paradigmatic cases: the case of a bosonic system, for which the *field coherent states* are defined, and the case of a magnetic system, for which the *spin coherent states* are set.

A.2 Field Coherent States

According to Glauber [115], field coherent states can be defined in three equivalent ways: the best known is that the set $\{|\alpha\rangle\}$ of CS is the set of eigenstates of the annihilation operator \hat{a} describing a quantum mechanical oscillator, i.e., $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, where being \hat{a} not Hermitian α is a complex number. Another definition is that CS are the quantum states for which holds the equality in the minimum-uncertainty principle, i.e., s.t. $\Delta q \Delta p = \frac{\hbar}{2}$, where the coordinate and momentum operators (\hat{q}, \hat{p}) are $\hat{q} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)$ and $\hat{p} = \frac{1}{i\sqrt{2}}(\hat{a} - \hat{a}^\dagger)$. This second definition is by no means unique, as the minimum-uncertainty relation does not provide a unique solution for $(\Delta q, \Delta p)$. Nevertheless the definition paving the way to the generalization of CS to arbitrary dynamical systems, is that the states $|\alpha\rangle$ are the states obtained by applying the *displacement operator* $\hat{D}(\alpha)$ to the Fock vacuum $|0\rangle$:

$$|\alpha\rangle \equiv \hat{D}(\alpha)|0\rangle, \quad \hat{D}(\alpha) \equiv e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}. \quad (\text{A.6})$$

The idea underlying the extension of the concept of CS, yielding their construction for whatever quantum dynamical system, consists in resorting to a group-theoretic framework, where the expression (A.6) enters as the last defining step of a self-consistent procedure (or “algorithm” as dubbed by Gilmore himself). Such procedure only needs the specification of a dynamical system, in the sense that a **Hamiltonian** acting on some **Hilbert space** is given. Glauber's and Sudarshan's CS are just a particular case of Gilmore's and Perelomov's construction, based on the Lie algebra spanned by the operators $\{\hat{a}, \hat{a}^\dagger, \hat{n} \equiv \hat{a}^\dagger\hat{a}, \hat{\mathbb{I}}\}$, and denoted as \mathfrak{h}_4 . Hamiltonians describing bosonic systems are usually linear combinations of \hat{n}, \hat{a}^\dagger and \hat{a} : for instance, in the case of a single mode field interacting with some external time dependent source $\zeta(t)$ it is

$$\hat{H} = \omega\hat{n} + \zeta(t)\hat{a}^\dagger + \zeta^*(t)\hat{a}. \quad (\text{A.7})$$

The corresponding Lie Group is the Heisenberg-Weyl group H_4 , induced by the exponentiation of all possible linear combinations of the \mathfrak{h}_4 generators. This implies that H_4 is actually induced by the exponentiation of all possible system Hamiltonians, and, hence, that is the group of all possible propagators, generally dubbed *dynamical group*. In order to proceed towards the definition of GCS via the group-theoretic algorithm, we need to set a last ingredient into place, the **reference state** $|R\rangle$, that can be chosen as any state of the Hilbert space considered. In the case of the harmonic oscillator, a common, although not mandatory, choice is that of taking $|R\rangle$ as the ground state of the number operator \hat{n} , i.e., $|R\rangle \equiv |0\rangle$ s.t. $\hat{n}|0\rangle = 0$. We then get the field CS in three steps:

- a) Maximum stability subgroup. This is the subgroup of H_4 which leaves $|R\rangle$ unchanged up to a phase factor, i.e., the subgroup $U(1) \otimes U(1)$ whose algebra is spanned by $\{\hat{n}, \hat{\mathbb{I}}\}$, and that consists of all operators of the form

$$\hat{h} = e^{i(\delta\hat{n} + \eta\hat{\mathbb{I}})}. \quad (\text{A.8})$$

- b) Coset space. The coset $H_4/U(1) \otimes U(1)$ is the set of elements D providing a unique decomposition of any element $g \in H_4$ in the form $g = hD$, with $h \in U(1) \otimes U(1)$. Its generic element can be represented by

$$\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}), \quad (\text{A.9})$$

with $\alpha \in \mathbb{C}$.

- c) Coherent states. Acting on $|R\rangle$ with coset elements, we get the H_4 CS, labelled by the complex parameter α , yielding

$$|\alpha\rangle \equiv \hat{D}(\alpha)|0\rangle = e^{(\alpha\hat{a}^\dagger - \alpha^*\hat{a})}|0\rangle, \quad (\text{A.10})$$

where we recognize Eq. (A.6) and $\hat{D}(\alpha)$ as the displacement operator.

In the case of \mathfrak{h}_4 the BCH formula (A.5) simplifies a lot, and decoupling the displacement operator we get

$$|\alpha\rangle = e^{(\alpha\hat{a}^\dagger - \alpha^*\hat{a})}|0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger}|0\rangle, \quad (\text{A.11})$$

as the field CS are usually written. GCS inherently provide us with a geometric structure, that in this case is got by the homomorphism between the coset space $H_4/U(1) \otimes U(1)$ and the complex plane \mathfrak{C} . This implies that there is a one-to-one correspondence between the CS $|\alpha\rangle$ and points $\alpha \in \mathfrak{C}$. The field CS form an overcomplete set, since they provide a resolution of the identity being not orthogonal even if normalized. The overlap between two field CS is

$$\langle\alpha|\alpha'\rangle = \exp\left(\alpha^*\alpha' - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\alpha'|^2\right), \quad (\text{A.12})$$

from which $\langle\alpha|\alpha\rangle = 1$, and a common useful resolution can be written as

$$\int_{H_4/U(1) \otimes U(1)} \frac{d\alpha d\alpha^*}{\pi\hbar} |\alpha\rangle \langle\alpha| = \hat{\mathbb{I}}. \quad (\text{A.13})$$

Lastly, sometimes it can be useful expanding field CS in terms of the Fock states $|n\rangle$, yielding

$$|\alpha\rangle = \sum_{n=0}^{\infty} \langle n|\alpha\rangle |n\rangle, \quad (\text{A.14})$$

where the overlap between a generic Fock state and $|\alpha\rangle$ is given by

$$\langle n|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}}. \quad (\text{A.15})$$

A.3 Spin Coherent States

The *spin coherent states*, often called also *atomic coherent states*, are the CS defined for a magnetic system, i.e., when the physical system under analysis is described by single-spin operators. Be \mathcal{H} the **Hilbert space** of the system and \hat{H} its **Hamiltonian**, constituted by a linear combinations of spin angular momentum operators $\{\hat{S}^\pm = \hat{S}^x \pm i\hat{S}^y, \hat{S}^z\}$, which span the $\mathfrak{su}(2)$ algebra, obeying the commutation relations

$$[\hat{S}^+, \hat{S}^-] = 2\hat{S}^z \quad , \quad [\hat{S}^z, \hat{S}^\pm] = \pm\hat{S}^\pm . \quad (\text{A.16})$$

The dynamical group is thus $SU(2)$ and the Hilbert space has dimension $\dim \mathcal{H} = 2S + 1$, with S the spin modulus. A common, although not mandatory, choice is that of taking the **reference state** $|R\rangle$ as one of the two extremal states of the \hat{S}^z -eigenvectors, i.e., $|R\rangle \equiv |0\rangle$ and $|R\rangle \equiv |2S\rangle$ such that $\hat{S}^z |0\rangle = -S |0\rangle$ and $\hat{S}^z |0\rangle = S |2S\rangle$. We here choose the lowest one, setting $|R\rangle = |0\rangle$. Again, we then can get the spin CS in three steps:

- a) Maximum stability subgroup. The subgroup of $SU(2)$ which leaves $|R\rangle$ unchanged up to a phase factor, is the subgroup $U(1)$ whose elements are generated by \hat{S}^z , and that consists of all operators of the form

$$\hat{h} = e^{i\delta\hat{S}^z} , \quad (\text{A.17})$$

i.e., the rotations around the direction identified by z .

- b) Coset space. The coset $SU(2)/U(1)$ is the set of elements D such that $g \in SU(2)$ can be uniquely decomposed in the form $g = hD$, with $h \in U(1)$. Its generic element can be represented by

$$\hat{D}(\Omega(\zeta)) = \exp(\zeta\hat{S}^- - \zeta^*\hat{S}^+) , \quad (\text{A.18})$$

with ζ a complex parameter.

- c) Coherent states. Acting on $|R\rangle$ with coset elements, we get the $SU(2)$ CS, labelled by the complex parameter Ω , yielding

$$|\Omega\rangle \equiv \hat{D}(\Omega(\zeta)) | -S \rangle = e^{(\zeta\hat{S}^- - \zeta^*\hat{S}^+)} | -S \rangle , \quad (\text{A.19})$$

where $\hat{D}(\Omega(\zeta))$ is the displacement operator.

The geometric structure is provided by the homomorphism of the coset space $SU(2)/U(1)$ with the two dimensional sphere S^2 , so that there exists a one-to-one correspondence between the spin CS $|\Omega\rangle$ and points $\Omega \in S^2$. We can then parametrize S^2 in spherical coordinates, setting $\zeta = \frac{\vartheta}{2}e^{i\varphi}$, with $\vartheta \in [0, \pi]$, $\varphi \in [0, 2\pi)$. The resolution of the identity can then be written as

$$\frac{2S + 1}{4\pi} \int_{SU(2)/U(1)} d\Omega |\Omega\rangle \langle\Omega| = \hat{\mathbb{I}} , \quad (\text{A.20})$$

where $d\Omega = \sin\vartheta d\vartheta d\varphi$ is the solid-angle volume element at (ϑ, φ) on the sphere S^2 . The spin CS are not orthogonal, and the overlap reads

$$\langle \Omega' | \Omega \rangle = \left[\cos \frac{\vartheta}{2} \cos \frac{\vartheta'}{2} + \sin \frac{\vartheta}{2} \sin \frac{\vartheta'}{2} e^{i(\varphi - \varphi')} \right]^{2S}. \quad (\text{A.21})$$

Expanding in terms of the \hat{S}^z -eigenvectors

$$|m\rangle : \hat{S}^z |m\rangle = (-S + m) |m\rangle \text{ with } m = 0, \dots, 2S, \quad (\text{A.22})$$

often named *Dicke states*, we get

$$|\Omega\rangle = \sum_{m=0}^{2S} \langle m | \Omega \rangle |m\rangle, \quad (\text{A.23})$$

where the overlap between a generic \hat{S}^z -eigenvector and $|\Omega\rangle$ as function of (ϑ, φ) is given by

$$\langle m | \Omega \rangle = \binom{2S}{m}^{\frac{1}{2}} \left(\cos \frac{\vartheta}{2} \right)^{2S-m} \left(\sin \frac{\vartheta}{2} \right)^m e^{-i\varphi m}. \quad (\text{A.24})$$

A.4 The general recipe

Once the group-theoretic arguments above are generalized to an arbitrary dynamical group \mathcal{G} induced by the generators entering the system Hamiltonian (or, which is the same, its algebra \mathfrak{g}), one is naturally led to the notion of *generalized coherent states*. We emphasize that in Gilmore's construction \mathcal{G} not needs to be a Lie-group. Nevertheless, since in practice \mathcal{G} is most often a Lie group, we will adopt such restriction in the following. Considering an arbitrary quantum system, its **Hamiltonian** \hat{H} can be generally written in the abstract form

$$\hat{H} = \hat{H}(\{\hat{X}_i\}), \quad \hat{X}_i \in \mathfrak{g}, \quad (\text{A.25})$$

where, in practical applications, \mathfrak{g} is a closed Lie algebra so that \mathcal{G} is a Lie Group. The Lie algebra \mathfrak{g} is characterized by the commutation relations

$$[\hat{X}_i, \hat{X}_j] = \sum_k c_{ij}^k \hat{X}_k, \quad \{\hat{X}_i\} \in \mathfrak{g} \quad (\text{A.26})$$

where the coefficient c_{ij}^k are the structure constants. For a given Hamiltonian the physical states are described as vectors in a (separable) **Hilbert space** \mathcal{H} carrying a unitary irreducible representation of the dynamical group \mathcal{G} . Lastly, we need to choose a **reference state** $|R\rangle \in \mathcal{H}$, s.t. $\langle R | R \rangle = 1$. It is important to stress that such choice is arbitrary, since, although their structure strongly depends on it, GCS possess the same properties regardless of which state is chosen as $|R\rangle$ [11]. Provided with these three inputs (the dynamical group, the Hilbert space and the reference state), the group-theoretic procedure gives us GCS in three steps:

1. Maximum stability subgroup: one first finds the maximum stability subgroup $\mathcal{F} \subset \mathcal{G}$, that is the set of group elements \hat{f} which leave the reference state invariant *up to a phase factor*, i.e., such that

$$\hat{f} |R\rangle = e^{i\phi_f} |R\rangle . \quad (\text{A.27})$$

2. Quotient or coset space: one can then determine the coset space \mathcal{G}/\mathcal{F} so that every $\hat{g} \in \mathcal{G}$ can be written as a unique decomposition of two group elements, one belonging to \mathcal{F} and the other to \mathcal{G}/\mathcal{F} :

$$\hat{g} = \hat{\Lambda} \hat{f} \quad \text{with} \quad \hat{g} \in \mathcal{G}, \quad \hat{f} \in \mathcal{F}, \quad \hat{\Lambda} \in \mathcal{G}/\mathcal{F} . \quad (\text{A.28})$$

3. Generalized coherent states: GCS are finally obtained applying a generic group element \hat{g} to the reference state, yielding

$$\hat{g} |R\rangle = \hat{\Lambda} \hat{f} |R\rangle = |\Lambda\rangle e^{i\phi_f} , \quad (\text{A.29})$$

and

$$|\Lambda\rangle \equiv \hat{\Lambda} |R\rangle \quad (\text{A.30})$$

is the general group definition of coherent states.

The definition (A.30) establishes a one-to-one correspondence between CS $|\Lambda\rangle$ and elements $\hat{\Lambda}$ of the coset space \mathcal{G}/\mathcal{F} : this is the reason why GCS preserve *by construction* all the algebraic and topological properties of the coset \mathcal{G}/\mathcal{F} . To be precise, Eq. (A.30) defines the mapping $\pi : \mathcal{G} \rightarrow \tilde{\mathcal{M}}$ where $\tilde{\mathcal{M}}$ is the fiber bundle the base of which is $\mathcal{M} = \mathcal{G}/\mathcal{F}$ with fiber \mathcal{F} . It is worth to notice that this mapping is continuous, i.e., for any given ε , there exists a ε' such that

$$||\Lambda\rangle - |\Lambda'\rangle| < \varepsilon' \quad (\text{A.31})$$

$$\text{if} \quad |\Lambda - \Lambda'| < \varepsilon , \quad (\text{A.32})$$

where the distances are determined with respect to the intrinsic metrics. The metric in Eq. (A.31) is thus determined from the Hilbert space inner product, whereas the one in Eq. (A.32) is determined from the metric in the manifold \mathcal{M} , associated to \mathcal{G}/\mathcal{F} .

If the Lie algebra \mathfrak{g} associated to \mathcal{G} is semisimple, the generators can be written in the so called Cartan basis $\{\hat{H}_i, \hat{E}_\alpha, \hat{E}_{-\alpha}\}$, with commutation relations

$$\begin{aligned} [\hat{H}_i, \hat{H}_j] &= 0 & , & & [\hat{H}_i, \hat{E}_\alpha] &= \alpha_i \hat{E}_\alpha , \\ [\hat{E}_\alpha, \hat{E}_{-\alpha}] &= \alpha_i \hat{H}_i & , & & [\hat{E}_\alpha, \hat{E}_\beta] &= c_{\alpha\beta} \hat{E}_{\alpha+\beta} . \end{aligned} \quad (\text{A.33})$$

Therefore, if the Hamiltonian (A.25) is linear in the generators and \mathfrak{g} is a semisimple Lie algebra, we get

$$\hat{H} = \sum_i \epsilon_i \hat{H}_i + \sum_\alpha \left(\gamma_\alpha \hat{E}_\alpha + \gamma_\alpha^* \hat{E}_{-\alpha} \right) , \quad (\text{A.34})$$

where the part containing the generators \hat{H}_i is usually associated to the free term, while the generators $\hat{E}_\alpha, \hat{E}_{-\alpha}$ embody the perturbations. From the theory of representation, it is possible to choose \hat{H}_i diagonal and Hermitian, i.e., such that $\hat{H}_i^\dagger = \hat{H}_i$, while the representatives of $(\hat{E}_\alpha, \hat{E}_{-\alpha})$ become shift-up and shift-down operators such that $\hat{E}_\alpha^\dagger = \hat{E}_{-\alpha}$. Every group element $\hat{g} \in \mathcal{G}$ can be written as the exponential of an anti-Hermitian complex linear combination of \hat{H}_i and \hat{E}_α . The general expression of the representation $\hat{\Lambda}$ can then be put in the *generalized displacement operator* form

$$\hat{\Lambda} \equiv \hat{D}(\Lambda) = \exp \left\{ \sum'_\beta (\Lambda_\beta \hat{E}_\beta - \Lambda_\beta^* \hat{E}_{-\beta}) \right\}, \quad (\text{A.35})$$

with the sum \sum'_β running over those shift-up operators which do not annihilate the reference state, and the complex parameters Λ_β coordinates on the differentiable manifold \mathcal{M} . Coherent states can thus be explicitly expressed as

$$|\Lambda\rangle \equiv \hat{D}(\Lambda) |R\rangle = \exp \left\{ \sum'_\beta (\Lambda_\beta \hat{E}_\beta - \Lambda_\beta^* \hat{E}_{-\beta}) \right\} |R\rangle, \quad (\text{A.36})$$

where we recognize the generalized expression corresponding to Eqs. (A.10) and (A.19) in the case of $\mathfrak{g} = \mathfrak{h}_4$ and $\mathfrak{g} = \mathfrak{su}(2)$ respectively.

A.4.1 General properties

For the sake of simplicity, let us now consider the case of a single couple of shift-up/shift-down operators, i.e., $\beta = 1$ in Eq. (A.36), so that there exists a unique complex parameter Λ_1 , hereafter indicated as $\Lambda \in \mathbb{C}$. We can use other coordinates systems such as

$$\begin{cases} z = \Lambda \frac{\sin \sqrt{\Lambda^* \Lambda}}{\sqrt{\Lambda^* \Lambda}} & \text{if } \mathcal{M} \text{ is compact;} \\ z = \Lambda \frac{\sinh \sqrt{\Lambda^* \Lambda}}{\sqrt{\Lambda^* \Lambda}} & \text{if } \mathcal{M} \text{ is noncompact,} \end{cases} \quad (\text{A.37})$$

where the two solutions - a sine function and a sine hyperbolic function - manifest the compactness and noncompactness of the respective geometry of the coset spaces \mathcal{G}/\mathcal{F} for a compact and a noncompact Lie group \mathcal{G} respectively. If we instead introduce a complex projective representation of \mathcal{G}/\mathcal{F} , it is

$$\tau = z(1 \mp z^* z)^{-1/2}, \quad (\text{A.38})$$

where the $-$ and $+$ signs correspond to the compact and noncompact cases respectively. Notice that here all the parameters Λ, z, τ are complex numbers for the sake of simplicity. We refer the reader to Ref. [10] for the general case. It is then possible to show that the manifold \mathcal{M} is endowed with a natural metric structure, inherited by the dynamical group \mathcal{G} , so that it is possible to define a canonical volume form on

it, in the sense that such form is invariant under reparametrization. The corresponding volume element, namely the measure, can be written for whatever coordinates system as [116]

$$d\mu(\Lambda) = \text{const} \cdot \det m dz dz^* , \quad (\text{A.39})$$

where m is the metric tensor and the constant follows from the normalization of CS. Moreover, from the definition (A.36), CS have a natural symplectic structure, so that \mathcal{M} can be considered as a phase space over which Poisson brackets can be defined; indeed, under general assumptions, \mathcal{M} turns out to be the phase space of the classical system corresponding to the quantum system once a proper classical limit is taken. All GCS share two important properties, which follow from the irreducibility of the representation of \mathcal{G} on \mathcal{H} . First, the CS are complete, since any state can be obtained by superposing CS. They actually form an *overcomplete* set, being much more numerous than the elements of an orthonormal set would be, and thus they are not orthogonal and do in general overlap. Second, the CS provide a resolution of the identity, in the form

$$\int_{\mathcal{G}/\mathcal{F}} d\mu(\hat{\Lambda}) |\Lambda\rangle \langle \Lambda| = \hat{\mathbb{I}}_{\mathcal{H}} , \quad (\text{A.40})$$

where the group invariant measure $d\mu(\hat{\Lambda})$ is defined by opportunely choosing the constant prefactor in Eq. (A.39). Because of the CS overcompleteness, for any operator \hat{B} acting on \mathcal{H} , it is

$$\begin{aligned} \text{Tr} \hat{B} &= \sum_{\xi} \langle \xi | \hat{B} | \xi \rangle = \sum_{\xi} \int_{\mathcal{M}} d\mu(\Lambda) d\mu(\Lambda') \langle \xi | \Lambda' \rangle \langle \Lambda' | \hat{B} | \Lambda \rangle \langle \Lambda | \xi \rangle = \\ &= \int_{\mathcal{M}} d\mu(\Lambda) d\mu(\Lambda') \sum_{\xi} \langle \Lambda | \xi \rangle \langle \xi | \Lambda' \rangle \langle \Lambda' | \hat{B} | \Lambda \rangle = \\ &= \int_{\mathcal{M}} d\mu(\Lambda) \langle \Lambda | \int_{\mathcal{M}} d\mu(\Lambda') |\Lambda'\rangle \langle \Lambda' | \hat{B} | \Lambda \rangle = \int_{\mathcal{M}} d\mu(\Lambda) \langle \Lambda | \hat{B} | \Lambda \rangle , \end{aligned} \quad (\text{A.41})$$

where $\{|\xi\rangle\}_{\mathcal{H}}$ is an orthonormal basis for \mathcal{H} . This relation tells us that the CS basis is sufficiently overcomplete that any operator can be completely represented by its diagonal matrix elements alone.

Finally, we specify a general feature related to GCS dynamics. A consequence of their construction algorithm can be resumed by the motto “*Once a coherent state, always a coherent state*”. In other words, whatever be the system Hamiltonian, a coherent state remains a coherent state evolving under the dynamics defined by the dynamical group. It is indeed because of this property that \mathcal{G} is named *dynamical group*, and note that, being basically of kinematical origin, it is not supposed to be a symmetry group. In fact, if the system Hamiltonian is a linear combination of generators of the Lie algebra \mathfrak{g} and the initial state $|\Psi(t_0)\rangle$ is one of the corresponding GCS, i.e. $|\Psi(t_0)\rangle = |\Lambda_0\rangle$, then the time-dependent Schrödinger equation

$$i\hbar \frac{d}{dt} |\Psi\rangle = \hat{H} |\Psi\rangle , \quad (\text{A.42})$$

can be exactly solved by

$$|\Psi(t)\rangle = |\Lambda_t\rangle = \hat{\Lambda}_t |\Lambda_0\rangle e^{i\varphi(t)}, \quad (\text{A.43})$$

where

$$\varphi(t) = i \int_{t_0}^t \langle \Lambda_0 | \hat{\Lambda}_\tau^\dagger \left(i\hbar \frac{\partial}{\partial \tau} - \hat{H} \right) \hat{\Lambda}_\tau |\Lambda_0\rangle d\tau, \quad (\text{A.44})$$

and

$$\hat{\Lambda}_t = \exp \left\{ \sum_{\beta} \Lambda_{\beta}(t) \hat{E}_{\beta} - \Lambda_{\beta}^*(t) \hat{E}_{-\beta} \right\} \quad (\text{A.45})$$

with $\Lambda_{\beta}(t)$ determined by classical equations of motion, describing orbits on \mathcal{M} . These results establish that in this special case the quantum and the classical dynamics can be described by the same equation, once the suitable set of GCS is provided. However, when the Hamiltonian is a non-linear combination of generators of the algebra, it is in general difficult to obtain the exact solution of Eq. (A.42), and some approximate methods must be developed in order to proceed.

Appendix B

Measures

A specific experimental observation, repeated many times on the same physical system always assumed to be in the same state ϱ , produces a collection of results $\{x\}_\Omega$, where Ω labels the set to which the results belong. Since each result represents an observable value, the elements x will be open intervals in the set of the real numbers. We define the probability of getting such results through the definition of a measurement on Ω , as follows [71]. Let \mathfrak{B} be a family of Ω subsets satisfying the following conditions:

- $\emptyset \subseteq \mathfrak{B}$, with \emptyset the empty-set, and $\Omega \subseteq \mathfrak{B}$;
- If $x \in \mathfrak{B}$, then so its complement, $\Omega/x \in \mathfrak{B}$;
- If $x_1, \dots, x_n \in \mathfrak{B}$, then $\cup_{i=1}^n x_i \in \mathfrak{B}$.

The couple (Ω, \mathfrak{B}) forms a measurable space that we will name space of the results. Being $\mathcal{L}(\mathcal{H})$ the vector space of the trace class positive operators acting on the Hilbert space \mathcal{H} , we define Positive Operator Valued Measure (POVM) the application

$$M : \mathfrak{B} \rightarrow \mathcal{L}(\mathcal{H}), \quad (\text{B.1})$$

which connects to every element x an operator $\hat{M}(x) \in \mathcal{L}(\mathcal{H})$ such that

- $\hat{M}(x) \geq 0 \quad \forall x \in \mathfrak{B}$;
- $\hat{M}(\cup_i x_i) = \sum_i \hat{M}(x_i)$ for disjoint elements of \mathfrak{B} ;
- $\hat{M}(\Omega) = \hat{\mathbb{1}}$.

Moreover, if the following relation holds

$$\hat{M}(x \cap y) = \hat{M}(x)\hat{M}(y) \quad \forall x, y \in \mathfrak{B}, \quad (\text{B.2})$$

the measure is said Projector-Valued Measure (PVM), or projective measure. Indeed, from Eq.(B.2) we get

$$\hat{M}(x) = \hat{M}(x \cap x) = \hat{M}(x)^2 \quad \forall x \in \mathfrak{B}, \quad (\text{B.3})$$

and thus it is $\hat{M}(x) = |x\rangle\langle x|$, for all x .

B.1 Minimal Interpretation

Let us now define the application $p_\varrho^M : \mathfrak{B} \rightarrow [0, 1]$, such that

$$p_\varrho^M(x) \equiv \text{Tr} \left[\hat{M}(x) \varrho \right]. \quad (\text{B.4})$$

Through the properties of the measure M , the linearity of the trace and the condition $\text{Tr}[\varrho] = 1$ holding $\forall \varrho \in \mathcal{H}$, one can show that p_ϱ^M is a measure on (Ω, \mathfrak{B}) with values in $[0, 1]$ for each state ϱ . The minimal interpretation of QM states that

The number $p_\varrho^M(x)$ is the probability that the experimental observation associated to the measure M give a result in x when the system is in the state ϱ .

Such interpretation is called *minimal* to highlight that hereinbefore all the other interpretations of QM contain it. The minimal interpretation includes as a particular case the Born rule, that is shared by all the interpretations developed so far¹.

The notion of observable is introduced in QM to briefly represent all the possible results of a specific experimental observation on a system, independently of its state. Thanks to this reason and to the minimal interpretation, an observable can be defined as, and identified with, a measure M on the space of the results (Ω, \mathfrak{B}) . The observables whose measure is a PVM, are uniquely associated to Hermitian operators acting on the Hilbert space of the observed system, making the usual formalism of QM arise. In fact, let us consider a PVM whose space of results (Ω, \mathfrak{B}) be formed by discrete and countable values $\{x_1, \dots, x_n\}$. This defines the projectors $\hat{M}(x_i) = |\pi_i\rangle\langle\pi_i|$, through which we can build the operator

$$\hat{O}^M = \sum_i \omega(x_i) |\pi_i\rangle\langle\pi_i| = \sum_i \omega_i |\pi_i\rangle\langle\pi_i|, \quad (\text{B.6})$$

where $\omega(x)$ is an invertible function. Being the results $x_i \in \mathbb{R}$, and given that $\omega(x)$ is invertible, the values ω_i will be real in turn, so that Eq. (B.6) describe an hermitian operator. On the other hand, each hermitian operator \hat{O}^M defines a PVM through its spectral decomposition. Notice that each operator is uniquely associated to a measure M , but more hermitian operators can be associated to the same PVM. Moreover, different observables associated to the same PVM commute, being

$$\hat{M}(x_i)\hat{M}(x_j) = \hat{M}(x_i \cap x_j) = \hat{M}(x_j \cap x_i) = \hat{M}(x_j)\hat{M}(x_i). \quad (\text{B.7})$$

Therefore, the terms *observable* and *measure* are often used indistinctly when PVM are considered.

¹To find the Born rule, one can simply consider a PVM on a pure state $|\psi\rangle$, and, calculating the trace in Eq. (B.4) on a basis of \mathcal{H} containing $|\psi\rangle$, one gets

$$p_{|\psi\rangle}^M(x) = \text{Tr} \left[\hat{M}(x) |\psi\rangle\langle\psi| \right] = \langle\psi|\hat{M}(x)|\psi\rangle\langle\psi|\psi\rangle = \langle\psi|x\rangle\langle x|\psi\rangle = |\langle\psi|x\rangle|^2. \quad (\text{B.5})$$

B.2 Measurements

We now describe the measurement process performed on a system Γ through an apparatus Ξ , and be the compound system $\Psi = \Gamma + \Xi$ isolated. Suppose that Γ be initially in a pure state, so that the measurement be performed on a well defined state. Being Ψ isolated, this implies that also Ξ be in a pure state. Before the beginning of the measurement, i.e., $\forall t \leq 0$, the total state is therefore $|\Psi(t \leq 0)\rangle = |\Gamma\rangle |\Xi\rangle$, that is Γ and Ξ are initially not entangled. We then need to assume that the numbers that will be read on the apparatus at the end of the process, i.e., the actual result of the measurement, form a measurable space $(\Omega_\Xi, \mathfrak{B})$ connected with the space (Ω, \mathfrak{B}) associated to the observable M via the correlation function $f : \Omega_\Xi \rightarrow \Omega$. The function f is invertible and it is sometimes dubbed *calibration function*.

A POVM Π_Ξ on $(\Omega_\Xi, \mathfrak{B})$ will be finally the observable, named *pointer observable*, that will provide us with the probability $p_{|\Gamma\rangle\langle\Gamma|}^M(X)$. In fact, given the apparatus Ξ , its Hilbert space \mathcal{H}_Ξ , its initial pure state $|\Xi\rangle\langle\Xi|$, the pointer observable Π_Ξ , and the calibration function f , the minimal interpretation allows us to understand the measure

$$p_{\varrho_\Xi(t)}^{\Pi_\Xi}(f^{-1}(x)), \quad (\text{B.8})$$

as the probability that a measurement of Π_Ξ performed when Ξ is in the state $\varrho_\Xi(t)$ give us a result correlated with X . Notice that, even though the state $\varrho_\Xi(t)$ appearing in Eq. (B.8) is totally generic, we would like to understand it as the time evolution of the initial state $|\Xi\rangle\langle\Xi|$, as we have indeed illustrated in Sec. 3.1 of the third chapter. Suffice here to say that to describe the measurement process we need now to ask that the time evolution guarantee the existence of a time T such that

$$p_{|\Gamma\rangle\langle\Gamma|}^M(x) = p_{\varrho_\Xi(T)}^{\Pi_\Xi}(f^{-1}(x)), \quad (\text{B.9})$$

for all initial state $|\Gamma\rangle \in \mathcal{H}_\Gamma$ and $|\Xi\rangle \in \mathcal{H}_\Xi$. This condition, called *probability reproducibility condition* [71], is at the heart of the measurement process, being the actual connection through which the information about Γ is transferred to the apparatus Ξ , thanks to the calibration function f and to the map \mathcal{E}_t such that $\mathcal{E}_t[|\Xi\rangle\langle\Xi|] = \varrho_\Xi(t)$.

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