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# THE GEOMETRY OF RANK DECOMPOSITIONS OF MATRIX MULTIPLICATION I: $2 \times 2$ MATRICES 

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#### Abstract

This is the first in a series of papers on rank decompositions of the matrix multiplication tensor. In this paper we: establish general facts about rank decompositions of tensors, describe potential ways to search for new matrix multiplication decompositions, give a geometric proof of the theorem of [3] establishing the symmetry group of Strassen's algorithm, and present two particularly nice subfamilies in the Strassen family of decompositions.


## 1. Introduction

This is the first in a planned series of papers on the geometry of rank decompositions of the matrix multiplication tensor $M_{\langle\mathbf{n}\rangle} \in \mathbb{C}^{\mathbf{n}^{2}} \otimes \mathbb{C}^{\mathbf{n}^{2}} \otimes \mathbb{C}^{\mathbf{n}^{2}}$. Our goals for the series are to determine possible symmetry groups for potentially optimal (or near optimal) decompositions of the matrix multiplication tensor and eventually to derive new decompositions based on symmetry assumptions. In this paper we study Strassen's rank 7 decomposition of $M_{\langle 2\rangle}$, which we denote $\mathcal{S t r}$. In the next paper [1] new decompositions of $M_{\langle 3\rangle}$ are presented and their symmetry groups are described. Although this project began before the papers [3, 4] appeared, we have benefited greatly from them in our study.

We begin in $\S_{2}$ by reviewing Strassen's algorithm as a tensor decomposition. Then in $\$ 3$ we explain basic facts about rank decompositions of tensors with symmetry, in particular, that the decompositions come in families, and each member of the family has the same abstract symmetry group. While these abstract groups are all the same, for practical purposes (e.g., looking for new decompositions), some realizations are more useful than others. We review the symmetries of the matrix multiplication tensor in 84 . After these generalities, in $\$ 5$ we revisit the Strassen family and display a particularly convenient subfamily. We examine the Strassen family from a projective perspective in 96 , which renders much of its symmetry transparent. Generalities on the projective perspective enable a very short proof of the upper bound in Burichenko's determination of the symmetries of Strassen's decomposition [3]. The projective perspective and emphasis on symmetry also enable two geometric proofs that Strassen's expression actually is a decomposition of $M_{\langle 2\rangle}$, which we explain in $\$ 7$.

Notation and conventions. $A, B, C, U, V, W$ are vector spaces, $G L(A)$ denotes the group of invertible linear maps $A \rightarrow A$, and $P G L(A)=G L(A) / \mathbb{C}^{*}$ the group of projective transformations of projective space $\mathbb{P} A$. If $a \in A,[a]$ denotes the corresponding point in projective space. $\mathfrak{S}_{d}$ denotes the permutation group on $d$ elements. Irreducible representations of $\mathfrak{S}_{d}$ are indexed by partitions. We let $[\pi]$ denote the irreducible $\mathfrak{S}_{d}$ module associated to the partition $\pi$.

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## 2. Strassen's Algorithm

In 1968, V. Strassen set out to prove the standard algorithm for multiplying $\mathbf{n} \times \mathbf{n}$ matrices was optimal in the sense that no algorithm using fewer multiplications exists. Since he anticipated this would be difficult to prove, he tried to show it just for two by two matrices. His spectacular failure opened up a whole new area of research: Strassen's algorithm for multiplying $2 \times 2$ matrices $a, b$ using seven scalar multiplications [8] is as follows: Set

$$
\begin{aligned}
I & =\left(a_{1}^{1}+a_{2}^{2}\right)\left(b_{1}^{1}+b_{2}^{2}\right), \\
I I & =\left(a_{1}^{2}+a_{2}^{2}\right) b_{1}^{1}, \\
I I I & =a_{1}^{1}\left(b_{2}^{1}-b_{2}^{2}\right) \\
I V & =a_{2}^{2}\left(-b_{1}^{1}+b_{1}^{2}\right) \\
V & =\left(a_{1}^{1}+a_{2}^{1}\right) b_{2}^{2} \\
V I & =\left(-a_{1}^{1}+a_{1}^{2}\right)\left(b_{1}^{1}+b_{2}^{1}\right), \\
V I I & =\left(a_{2}^{1}-a_{2}^{2}\right)\left(b_{1}^{2}+b_{2}^{2}\right) .
\end{aligned}
$$

Set

$$
\begin{aligned}
& c_{1}^{1}=I+I V-V+V I I, \\
& c_{1}^{2}=I I+I V, \\
& c_{2}^{1}=I I I+V, \\
& c_{2}^{2}=I+I I I-I I+V I .
\end{aligned}
$$

Then $c=a b$.
To better see symmetry, view matrix multiplication as a trilinear map $(X, Y, Z) \mapsto \operatorname{trace}(X Y Z)$ and in tensor form. To view it more invariantly, let $U, V, W=\mathbb{C}^{2}$, let $A=U^{*} \otimes V, B=V^{*} \otimes W$, $C=W^{*} \otimes U$ and consider $M_{\langle 2\rangle} \in\left(V \otimes U^{*}\right) \otimes\left(W \otimes V^{*}\right) \otimes\left(U \otimes W^{*}\right)$, where $M_{\langle 2\rangle}=\operatorname{Id}_{U} \otimes \operatorname{Id}_{V} \otimes \operatorname{Id}_{W}$ with the factors re-ordered (see, e.g., 7, §2.5.2]). Write

$$
\begin{equation*}
u_{1}=\binom{1}{0}, u_{2}=\binom{0}{1}, u^{1}=(1,0) u^{2}=(0,1) \tag{1}
\end{equation*}
$$

and set $v_{j}=w_{j}=u_{j}$ and $v^{j}=w^{j}=u^{j}$. Then Strassen's algorithm becomes the following tensor decomposition

$$
\begin{align*}
M_{\langle 2\rangle}= & \left(v_{1} u^{1}+v_{2} u^{2}\right) \otimes\left(w_{1} v^{1}+w_{2} v^{2}\right) \otimes\left(u_{1} w^{1}+u_{2} w^{2}\right)  \tag{2}\\
& {\left[v_{1} u^{1} \otimes w_{2}\left(v^{1}-v^{2}\right) \otimes\left(u_{1}+u_{2}\right) w^{2}\right.}  \tag{3}\\
& +\left(v_{1}+v_{2}\right) u^{2} \otimes w_{1} v^{1} \otimes u_{2}\left(w^{1}-w^{2}\right) \\
& \left.+v_{2}\left(u^{1}-u^{2}\right) \otimes\left(w_{1}+w_{2}\right) v^{2} \otimes u_{1} w^{1}\right] \\
& +\left[v_{2} u^{2} \otimes w_{1}\left(v^{2}-v^{1}\right) \otimes\left(u_{1}+u_{2}\right) w^{1}\right.  \tag{4}\\
& +\left(v_{1}+v_{2}\right) u^{1} \otimes w_{2} v^{2} \otimes u_{1}\left(w^{2}-w^{1}\right) \\
& \left.+v_{1}\left(u^{2}-u^{1}\right) \otimes\left(w_{1}+w_{2}\right) v^{1} \otimes u_{2} w^{2}\right]
\end{align*}
$$

Note that this is the sum of seven rank one tensors, while the standard algorithm in tensor format has eight rank one summands.

Introduce the notation

$$
\left\langle v_{i} u^{j} \otimes w_{k} u^{l} \otimes u_{p} w^{q}\right\rangle_{\mathbb{Z}_{3}}:=v_{i} u^{j} \otimes w_{k} u^{l} \otimes u_{p} w^{q}+v_{k} u^{l} \otimes w_{p} u^{q} \otimes u_{i} w^{j}+v_{p} u^{q} \otimes w_{i} u^{j} \otimes u_{k} w^{l}
$$

Then the decomposition becomes

$$
\begin{align*}
M_{\langle 2\rangle}= & \left(v_{1} u^{1}+v_{2} u^{2}\right) \otimes\left(w_{1} v^{1}+w_{2} v^{2}\right) \otimes\left(u_{1} w^{1}+u_{2} w^{2}\right)  \tag{5}\\
& +\left\langle v_{1} u^{1} \otimes w_{2}\left(v^{1}-v^{2}\right) \otimes\left(u_{1}+u_{2}\right) w^{2}\right\rangle_{\mathbb{Z}_{3}}  \tag{6}\\
& -\left\langle v_{2} u^{2} \otimes w_{1}\left(v^{1}-v^{2}\right) \otimes\left(u_{1}+u_{2}\right) w^{1}\right\rangle_{\mathbb{Z}_{3}} \tag{7}
\end{align*}
$$

From this presentation we immediately see there is a cyclic $\mathbb{Z}_{3}$ symmetry by cyclically permuting the factors $A, B, C$. The $\mathbb{Z}_{3}$ acting on the rank one elements in the decomposition has three orbits If we exchange $u_{1} \leftrightarrow u_{2}, u^{1} \leftrightarrow u^{2}, v^{1} \leftrightarrow v^{2}$, etc., the decomposition is also preserved by this $\mathbb{Z}_{2}$, with orbits (5) and the exchange of the triples, call this an internal $\mathbb{Z}_{2}$. These symmetries are only part of the picture.

## 3. Symmetries and families

Let $T \in\left(\mathbb{C}^{N}\right)^{\otimes k}$. We say $T$ has rank one if $T=a_{1} \otimes \cdots \otimes a_{k}$ for some $a_{j} \in \mathbb{C}^{N}$. Define the symmetry group of $T, G_{T} \subset\left(G L_{N}^{\times k}\right) \ltimes \mathfrak{S}_{k}$ to be the subgroup preserving $T$, where $\mathfrak{S}_{k}$ acts by permuting the factors.

For a rank decomposition $T=\sum_{j=1}^{r} t_{j}$ with each $t_{j}$ of tensor rank one, define the set $\mathcal{S}:=$ $\left\{t_{1}, \ldots, t_{r}\right\}$, which we also call the decomposition, and the symmetry group of the decomposition $\Gamma_{\mathcal{S}}:=\left\{g \in G_{T} \mid g \cdot \mathcal{S}=\mathcal{S}\right\}$. Let $\Gamma_{\mathcal{S}}^{\prime}=\Gamma_{\mathcal{S}} \cap(G L(A) \times G L(B) \times G L(C))$. Let Str denote Strassen's decomposition of $M_{\langle 2\rangle}$.

If $g \in G_{T}$, then $g \cdot \mathcal{S}:=\left\{g t_{1}, \ldots, g t_{r}\right\}$ is also a rank decomposition of $T$. Moreover:
Proposition 3.1. For $g \in G_{T}, \Gamma_{g \cdot \mathcal{S}}=g \Gamma_{\mathcal{S}} g^{-1}$.
Proof. Let $h \in \Gamma_{\mathcal{S}}$, then $g h g^{-1}\left(g t_{j}\right)=g\left(h t_{j}\right) \in g \cdot \mathcal{S}$ so $\Gamma_{g \cdot \mathcal{S}} \subseteq g \Gamma_{\mathcal{S}_{t}} g^{-1}$, but the construction is symmetric in $\Gamma_{g \cdot \mathcal{S}}$ and $\Gamma_{\mathcal{S}}$.

Similarly for a polynomial $P \in S^{d} \mathbb{C}^{N}$ and a Waring decomposition $P=\ell_{1}^{d}+\cdots+\ell_{r}^{d}$ for some $\ell_{j} \in \mathbb{C}^{N}$, and $g \in G_{P} \subset G L_{N}$, the same result holds where $\mathcal{S}=\left\{\ell_{1}, \ldots, \ell_{r}\right\}$.

In summary, algorithms come in $\operatorname{dim}\left(G_{T}\right)$-dimensional families, and each member of the family has the same abstract symmetry group.

We recall the following theorem of de Groote:
Theorem 3.2. [5] The set of rank seven decompositions of $M_{\langle 2\rangle}$ is the orbit $G_{M_{\langle 2\rangle}}$. $\mathcal{S t r}$.

## 4. Symmetries of $M_{\langle\mathbf{n}\rangle}$

We review the symmetry group of the matrix multiplication tensor

$$
G_{M_{\langle\mathbf{n}\rangle}}:=\left\{g \in G L_{n^{2}}^{\times 3} \times \mathfrak{S}_{3} \mid g \cdot M_{\langle\mathbf{n}\rangle}=M_{\langle\mathbf{n}\rangle}\right\}
$$

One may also consider matrix multiplication as a polynomial that happens to be multi-linear, $M_{\langle\mathbf{n}\rangle} \in S^{3}(A \oplus B \oplus C)$, and consider

$$
\tilde{G}_{M_{\langle\mathbf{n}\rangle}}:=\left\{g \in G L(A \oplus B \oplus C) \mid g \cdot M_{\langle\mathbf{n}\rangle}=M_{\langle\mathbf{n}\rangle}\right\}
$$

Note that $(G L(A) \times G L(B) \times G L(C)) \times \mathfrak{S}_{3} \subset G L(A \oplus B \oplus C)$, so $G_{M_{\langle\mathbf{n}\rangle}} \subseteq \tilde{G}_{M_{\langle\mathbf{n}\rangle}}$.

It is clear that $P G L_{\mathbf{n}} \times P G L_{\mathbf{n}} \times P G L_{\mathbf{n}} \times \mathbb{Z}_{3} \subset G_{M_{\langle\mathbf{n}\rangle}}$, the $\mathbb{Z}_{3}$ because trace $(X Y Z)=$ $\operatorname{trace}(Y Z X)$, and the $P G L_{\mathbf{n}}$ 's appear instead of $G L_{\mathbf{n}}$ because if we rescale by $\lambda \operatorname{Id}_{U}$, then $U^{*}$ scales by $\frac{1}{\lambda}$ and there is no effect on the decomposition. Moreover since trace $(X Y Z)=$ $\operatorname{trace}\left(Y^{T} X^{T} Z^{T}\right)$, we have $P G L_{n}^{\times 3} \ltimes D_{3} \subseteq G_{M_{\langle\mathbf{n}\rangle}}$, where the dihedral group $D_{3}$ is isomorphic to $\mathfrak{S}_{3}$, but we denote it by $D_{3}$ to avoid confusion with a second copy of $\mathfrak{S}_{3}$ that will appear. We emphasize that this $\mathbb{Z}_{2}$ is not contained in either the $\mathfrak{S}_{3}$ permuting the factors or the $P G L(A) \times P G L(B) \times P G L(C)$ acting on them. In $\tilde{G}_{M_{\langle\mathbf{n}\rangle}}$ we can also rescale the three factors by non-zero complex numbers $\lambda, \mu, \nu$ such that $\lambda \mu \nu=1$, so we have $\left(\mathbb{C}^{*}\right)^{\times 2} \times P G L_{n}^{\times 3} \ltimes D_{3} \subseteq \tilde{G}_{M_{\langle\mathbf{n}\rangle}}$,

We will be primarily interested in $G_{M_{\langle\mathbf{n}\rangle}}$. The first equality in the following proposition appeared in [5, Thms. 3.3,3.4] and [4, Prop. 4.7] with ad-hoc proofs. The second assertion appeared in [6]. We reproduce the proof from [6], as it is a special case of the result there.
Proposition 4.1. $G_{M_{\langle\mathbf{n}\rangle}}=P G L_{n}^{\times 3} \ltimes D_{3}$ and $\tilde{G}_{M_{\langle\mathbf{n}\rangle}}=\left(\mathbb{C}^{*}\right)^{\times 2} \times P G L_{n}^{\times 3} \ltimes D_{3}$.
Proof. It will be sufficient to show the second equality because the $\left(\mathbb{C}^{*}\right)^{\times 2}$ acts trivially on $A \otimes B \otimes C$. For polynomials, we use the method of [2, Prop. 2.2] adapted to reducible representations. A straight-forward Lie algebra calculation shows the connected component of the identity of $\tilde{G}_{M_{\langle\mathbf{n}\rangle}}$ is $\tilde{G}_{M_{\langle\mathbf{n}\rangle}}^{0}=\left(\mathbb{C}^{*}\right)^{\times 2} \times P G L_{n}^{\times 3}$. As was observed in [2] the full stabilizer group must be contained in its normalizer $N\left(\tilde{G}_{M_{\langle\mathbf{n}\rangle}}^{0}\right)$. But the normalizer is the automorphism group of the marked Dynkin diagram for $A \oplus B \oplus C$, which in our case is


There are three triples of marked diagrams. Call each column consisting of 3 marked diagrams a group. The automorphism group of the picture is $D_{3}=\mathbb{Z}_{2} \ltimes \mathbb{Z}_{3}$, where the $\mathbb{Z}_{2}$ may be seen as flipping each diagram, exchanging the first and third diagram in each group, and exchanging the first and second group. The $\mathbb{Z}_{3}$ comes from cyclically permuting each group and the diagrams within each group.

Regarding the symmetries discussed in $\S 2$, the $\mathbb{Z}_{3}$ is in the $\mathfrak{S}_{3}$ in $P G L_{2}^{\times 3} \times \mathfrak{S}_{3}$ and the internal $\mathbb{Z}_{2}$ is in $\Gamma_{\mathcal{S} t r}^{\prime} \subset P G L_{2}^{\times 3}$.

Thus if $\mathcal{S}$ is (the set of points of) a rank decomposition of $M_{\langle\mathbf{n}\rangle}$, then $\Gamma_{\mathcal{S}} \subset[(G L(U) \times G L(V) \times$ $\left.G L(W)) \ltimes \mathbb{Z}_{3}\right] \ltimes \mathbb{Z}_{2}$.

We call a $\mathbb{Z}_{3} \subset \Gamma_{\mathcal{S}}$ a standard cyclic symmetry if it corresponds to (Id, Id, Id) $\cdot \mathbb{Z}_{3} \subset(G L(U) \times$ $G L(V) \times G L(W)) \ltimes \mathbb{Z}_{3}$.

We call a $\mathbb{Z}_{2} \subset \Gamma_{\mathcal{S}}$ a convenient transpose symmetry if it corresponds to the symmetry of $M_{\langle\mathbf{n}\rangle}$ given by $a \otimes b \otimes c \mapsto a^{T} \otimes c^{T} \otimes b^{T}$. The convenient transpose symmetry lies in $(G L(A) \times G L(B) \times$ $G L(C)) \times \mathfrak{S}_{2} \subset(G L(A) \times G L(B) \times G L(C)) \times \mathfrak{S}_{3}$, where the component of the transpose in $\mathfrak{S}_{2}$ switches the last two factors and the component in $G L(A) \times G L(B) \times G L(C)$ sends each matrix to its transpose.

Remark 4.2. Since $M_{\langle\mathbf{n}\rangle} \in\left(U^{*} \otimes U\right)^{\otimes 3}$ one could consider the larger symmetry group considering $M_{\langle\mathbf{n}\rangle} \in U^{\otimes 3} \otimes U^{* \otimes 3}$ as is done in [3].

## 5. The Strassen family

Since $P G L_{2}^{\times 3} \subset G_{M_{\langle 2\rangle}}$, we can replace $u_{1}, u_{2}$ by any basis of $U$ in Strassen's decomposition, and similarly for $v_{1}, v_{2}$ and $w_{1}, w_{2}$. In particular, we need not have $u_{1}=v_{1}$ etc... When we do that, the symmetries become conjugated by our change of basis matrices. If we only use elements of the diagonal $P G L_{2}$ in $P G L_{2}^{\times 3}$, the $\mathbb{Z}_{3}$-symmetry remains standard. More subtly, the $\mathbb{Z}_{3}$-symmetry remains the standard cyclic permutation of factors if we apply elements of $\mathbb{Z}_{3}$ in any of the $P G L_{2}$ 's, i.e., setting $\omega=e^{\frac{2 \pi i}{3}}$, we can apply any of

$$
\rho(\omega)=\left(\begin{array}{ll}
0 & -1 \\
1 & -1
\end{array}\right) \text { and } \rho\left(\omega^{2}\right)=\left(\begin{array}{ll}
-1 & 1 \\
-1 & 0
\end{array}\right)
$$

to $U, V$ or $W$.
For example, if we apply the change of basis matrices

$$
g_{U}=\left(\begin{array}{cc}
1 & -1 \\
0 & 1
\end{array}\right) \in G L(U), g_{V}=\left(\begin{array}{ll}
-1 & 0 \\
-1 & 1
\end{array}\right) \in G L(V), g_{W}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right) \in G L(W),
$$

and take the image vectors as our new basis vectors, then setting $u_{3}=-\left(u_{1}+u_{2}\right)$ and $u^{3}=u^{1}-u^{2}$ and similarly for the $v$ 's and $w$ 's, the decomposition becomes:

$$
\begin{align*}
M_{\langle 2\rangle}= & -\left(v_{1} u^{2}+v_{2} u^{3}\right) \otimes\left(w_{1} v^{2}+w_{2} v^{3}\right) \otimes\left(u_{1} w^{2}+u_{2} w^{3}\right)  \tag{8}\\
& +v_{1} u^{1} \otimes w_{1} v^{1} \otimes u_{1} w^{1}  \tag{9}\\
& +v_{3} u^{2} \otimes w_{3} v^{2} \otimes u_{3} w^{2}  \tag{10}\\
& +v_{2} u^{3} \otimes w_{2} v^{3} \otimes u_{2} w^{3}  \tag{11}\\
& -\left\langle v_{1} u^{2} \otimes w_{2} v^{1} \otimes u_{3} w^{3}\right\rangle_{\mathbb{Z}_{3}} . \tag{12}
\end{align*}
$$

Remark 5.1. The matrices in (12) are all nilpotent, and none of the other matrices appearing in this decomposition are.

Notice that for the first term $v_{1} u^{2}+v_{2} u^{3}=v_{2} u^{1}+v_{3} u^{1}=v_{3} u^{2}+v_{2} u^{1}$. Here there is a standard $\mathbb{Z}_{3} \subset \mathfrak{S}_{3}$. There are four fixed points for this standard $\mathbb{Z}_{3}$ : (8), (9) (10), (11). (In any element of the Strassen family there will be some $\mathbb{Z}_{3}$ with four fixed points, but the $\mathbb{Z}_{3}$ need not be standard.) There is also a standard $\mathbb{Z}_{3} \subset P G L_{2}^{\times 3}$ embedded diagonally, that sends $u_{1} \rightarrow u_{2} \rightarrow u_{3}$, and acting by the inverse matrix on the dual basis, and similarly for the $v$ 's and $w$ 's. Under this action (8) is fixed and we have the cyclic permutation (9) $\rightarrow$ (11) $\rightarrow$ (10).

If we take the standard vectors of (1) in each factor we get
$M_{\langle 2\rangle}=\left(\begin{array}{ll}0 & -1 \\ 1 & -1\end{array}\right)^{\otimes 3}+\left(\begin{array}{ll}1 & 0 \\ 0 & 0\end{array}\right)^{\otimes 3}+\left(\begin{array}{ll}0 & 1 \\ 0 & 1\end{array}\right)^{\otimes 3}+\left(\begin{array}{cc}0 & 0 \\ -1 & 1\end{array}\right)^{\otimes 3}+\left\langle\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right) \otimes\left(\begin{array}{ll}0 & 0 \\ 1 & 0\end{array}\right) \otimes\left(\begin{array}{ll}1 & -1 \\ 1 & -1\end{array}\right)\right\rangle_{\mathbb{Z}_{3}}$
If we want to see the $\mathbb{Z}_{3} \subset P G L_{2}^{\times 3}$ more transparently, it is better to diagonalize the $\mathbb{Z}_{3}$ action so the first matrix becomes

$$
a=\left(\begin{array}{cc}
\omega & 0 \\
0 & \omega^{2}
\end{array}\right) .
$$

where $\omega:=\exp \left(\frac{-2 \pi i}{3}\right)$. Then for $\iota:=i / \sqrt{3}, \sigma:=\exp \left(\frac{2 \pi i}{12}\right) / \sqrt{3}$ we get

$$
M_{\langle 2\rangle}=a^{\otimes 3}+b^{\otimes 3}+(\varrho(b))^{\otimes 3}+\left(\varrho^{2}(b)\right)^{\otimes 3}+\left\langle c \otimes \varrho(c) \otimes \varrho^{2}(c)\right\rangle_{\mathbb{Z}_{3}}
$$

where

$$
b:=\left(\begin{array}{cc}
\sigma & \bar{\iota} \\
\iota & \bar{\sigma}
\end{array}\right), \quad c:=\left(\begin{array}{cc}
\iota & \iota \\
\bar{\iota} & \bar{\iota}
\end{array}\right), \quad \varrho: \mathbb{C}^{2 \times 2} \rightarrow \mathbb{C}^{2 \times 2}, \varrho(X)=a X a^{-1} .
$$

Note that $a+b+c=0$.

## 6. Projective perspective

Although the above description of the Strassen family of decompositions for $M_{\langle 2\rangle}$ is satisfying, it becomes even more transparent with a projective perspective.
6.1. $M_{\langle 2\rangle}$ viewed projectively. Recall that $P G L_{2}$ acts simply transitively on the set of triples of distinct points of $\mathbb{P}^{1}$. So to fix a decomposition in the family, we select a triple of points in each space. We focus on $\mathbb{P} U$. Call the points $\left[u_{1}\right],\left[u_{2}\right],\left[u_{3}\right]$. Then these determine three points in $\mathbb{P} U^{*},\left[u^{1 \perp}\right],\left[u^{2 \perp}\right],\left[u^{3 \perp}\right]$. We choose representatives $u_{1}, u_{2}, u_{3}$ satisfying $u_{1}+u_{2}+u_{3}=0$. We could have taken any linear relation, it just would introduce coefficients in the decomposition. We take the most symmetric relation to keep all three points on an equal footing. Similarly, we fix the scales on the $u^{j \perp}$ by requiring $u^{j \perp}\left(u_{j-1}\right)=1$ and $u^{j \perp}\left(u_{j+1}\right)=-1$, where indices are considered $\bmod \mathbb{Z}_{3}$, so $u_{3+1}=u_{1}$ and $u_{1-1}=u_{3}$.

In comparison with what we had before, letting the old indices be hatted, we have $\hat{u}_{1}=u_{1}$, $\hat{u}_{2}=u_{2}, \hat{u}_{3}=-u_{3}$ and $\hat{u}^{1}=u^{2 \perp}, \hat{u}^{2}=-u^{1 \perp}$, and $\hat{u}^{3}=-u^{3 \perp}$. The effect is to make the symmetries of the decomposition more transparent. Our identifications of the ordered triples $\left\{u_{1}, u_{2}, u_{3}\right\}$ and $\left\{v_{1}, v_{2}, v_{3}\right\}$ exactly determine a linear isomorphism $a_{0}: U \rightarrow V$, and similarly for the other pairs of vector spaces. Note that $a_{0}=v_{j} \otimes u^{j+1 \perp}+v_{j+1} \otimes u^{j+2 \perp}$ for any $j=1,2,3$.

Then

$$
\begin{align*}
M_{\langle 2\rangle} & =a_{0} \otimes b_{0} \otimes c_{0}  \tag{13}\\
& +\left\langle\left(v_{2} \otimes u^{1 \perp}\right) \otimes\left(w_{1} \otimes v^{3 \perp}\right) \otimes\left(u_{3} \otimes w^{2 \perp}\right)\right\rangle_{\mathbb{Z}_{3}} \\
& +\left\langle\left(v_{3} \otimes u^{1 \perp}\right) \otimes\left(w_{1} \otimes v^{2 \perp}\right) \otimes\left(u_{2} \otimes w^{3 \perp}\right)\right\rangle_{\mathbb{Z}_{3}} .
\end{align*}
$$

With this presentation, the $\mathfrak{S}_{3} \subset P G L_{2} \subset P G L_{2}^{\times 3}$ acting by permuting the indices transparently preserves the decomposition, with two orbits, the fixed point $a_{0} \otimes b_{0} \otimes c_{0}$ and the orbit of $\left(v_{2} \otimes u^{1 \perp}\right) \otimes\left(w_{1} \otimes v^{3 \perp}\right) \otimes\left(u_{3} \otimes w^{2 \perp}\right)$.
Remark 6.1. Note that here there are no nilpotent matrices appearing.
Remark 6.2. The geometric picture of the decomposition of $M_{\langle 2\rangle}$ can be rephrased as follows. Consider the space of linear isomorphisms $U \rightarrow V$ (mod scalar multiplication) as the projective space $\mathbb{P}^{3}$ of $2 \times 2$ matrices, in which we fix coordinates, coming from the choice of basis for $U, V$. The choice of basis also determines an identification between $U$ and $V$. Then $a_{0}$ represents in $\mathbb{P}^{3}$ a point of rank 2 , which can be taken as the identity in the choice of coordinates. The other 6 points $Q_{i}=u_{i} \otimes u^{j \perp}$ appearing in the first factor of the decomposition can be determined as follows. The points $P_{i}=u_{i} \otimes u^{i \perp}$ (in the identification) represent the choice of 3 points in the conic obtained by cutting with a plane (e.g. the plane of traceless matrices) the quadric $q=\operatorname{Seg}\left(\mathbb{P}^{1} \times \mathbb{P}^{1}\right)$ of matrices of rank 1 . Through each $P_{i}$ one finds lines of the two rulings of $q$, call then $L_{i}, M_{i}$. Then the six points $Q_{i}$ are given by:

$$
\begin{aligned}
& Q_{1}=L_{1} \cap M_{2}, \quad Q_{2}=L_{2} \cap M_{3}, \quad Q_{3}=L_{3} \cap M_{1} \\
& Q_{4}=M_{1} \cap L_{2}, \quad Q_{5}=M_{2} \cap L_{3}, \quad Q_{6}=M_{3} \cap L_{1} .
\end{aligned}
$$

An analogue of the construction determines the seven points in the other two factors of the tensor product, so that the 7 final summands can be determined combinatorially and the $\mathbb{Z}_{2}, \mathbb{Z}_{3}$ symmetries can be easily recognized.

The geometric construction can be generalized to higher dimensional spaces, so it could insight for extensions to larger matrix multiplication tensors. The difficult part is to determine how one should combine the points constructed in each factor of the tensor product, in order to produce a decomposition of $M_{\langle\mathbf{n}\rangle}$.

When we view (8) projectively, we get

$$
\begin{align*}
M_{\langle 2\rangle}= & \left(v_{1} u^{1 \perp}+v_{2} u^{3 \perp}\right) \otimes\left(w_{1} v^{1 \perp}+w_{2} v^{3 \perp}\right) \otimes\left(u_{1} w^{1 \perp}+u_{2} w^{3 \perp}\right)  \tag{14}\\
& +v_{1} u^{2 \perp} \otimes w_{1} v^{2 \perp} \otimes u_{1} w^{2 \perp}  \tag{15}\\
& +v_{3} u^{1 \perp} \otimes w_{3} v^{1 \perp} \otimes u_{3} w^{1 \perp}  \tag{16}\\
& +v_{2} u^{3 \perp} \otimes w_{2} v^{3 \perp} \otimes u_{2} w^{3 \perp}  \tag{17}\\
& \left\langle v_{1} u^{1 \perp} \otimes w_{2} v^{2 \perp} \otimes u_{3} w^{3 \perp}\right\rangle_{\mathbb{Z}_{3}} . \tag{18}
\end{align*}
$$

With this presentation, $\mathfrak{S}_{3} \subset \Gamma_{\mathcal{S}}^{\prime}$ is again transparent.
6.2. Symmetries of $\Gamma_{\mathcal{S t r}}$. Let $M_{\langle\mathbf{n}\rangle}=\sum_{j=1}^{r} t_{j}$ be a rank decomposition for $M_{\langle\mathbf{n}\rangle}$ and write $t_{j}=a_{j} \otimes b_{j} \otimes c_{j}$. Let $\mathbf{r}_{j}=\left(r_{A, j}, r_{B, j}, r_{C, j}\right):=\left(\operatorname{rank}\left(a_{j}\right), \operatorname{rank}\left(b_{j}\right), \operatorname{rank}\left(c_{j}\right)\right)$, and let $\tilde{\mathbf{r}}_{j}$ denote the unordered triple. The following proposition is clear:
Proposition 6.3. Let $\mathcal{S}$ be a rank decomposition of $M_{\langle\mathbf{n}\rangle}$. Partition $\mathcal{S}$ by un-ordered rank triples into disjoint subsets: $\mathcal{S}:=\left\{\mathcal{S}_{1,1,1}, \mathcal{S}_{1,1,2}, \ldots, \mathcal{S}_{n, n, n}\right\}$. Then $\Gamma_{\mathcal{S}}^{\prime}$ preserves each $\mathcal{S}_{s, t, u}$.

We can say more about rank one elements:
If $a \in U^{*} \otimes V$ and $\operatorname{rank}(a)=1$, then there are unique points $[\mu] \in \mathbb{P} U^{*}$ and $[v] \in \mathbb{P} V$ such that $[a]=[\mu \otimes v]$.

Now given a decomposition $\mathcal{S}$ of $M_{\langle\mathbf{n}\rangle}$, define $\mathcal{S}_{U^{*}} \subset \mathbb{P} U^{*}$ and $\mathcal{S}_{U} \subset \mathbb{P} U$ to correspond to the elements appearing in $\mathcal{S}_{1,1,1}$. Then $\Gamma_{\mathcal{S}}^{\prime}$ preserves $\mathcal{S}_{U}$ and $\mathcal{S}_{U^{*}}$.

In the case of Strassen's decomposition $\mathcal{S t r}_{U}$ is a configuration of three points in $\mathbb{P}^{1}$, so $a$ priori we must have $\Gamma_{\mathcal{S t r}}^{\prime} \cap P G L(U) \subset \mathfrak{S}_{3}$. If we insist on the standard $\mathbb{Z}_{3}$-symmetry (i.e., restrict to the subfamily of decompositions where there is a standard cyclic symmetry), there is just one $P G L_{2}$ and we have $\Gamma_{\mathcal{S t r}}^{\prime} \subseteq \mathfrak{S}_{3}$. Recall that this is no loss of generality as the full symmetry group is the same for all decompositions in the family. We conclude $\Gamma_{\mathcal{S t r}} \subseteq \mathfrak{S}_{3} \times D_{3}$. We have already seen $\mathfrak{S}_{3} \times \mathbb{Z}_{3} \subset \Gamma_{\mathcal{S} t r}$, Burichenko [3] shows that in addition there is a nonconvenient $\mathbb{Z}_{2}$ obtained by taking the convenient $\mathbb{Z}_{2}$ (which sends the decomposition to another decomposition in the family) and then conjugating by $\left(\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right) \subset P G L_{2} \subset P G L_{2}^{\times 3}$ which sends the decomposition back to $\mathcal{S} t r$. We recover (with a new proof of the upper bound) Burichenko's theorem:
Theorem 6.4. 3] The symmetry group of Strassen's decomposition of $M_{\langle 2\rangle}$ is $\mathfrak{S}_{3} \times D_{3} \subset$ $P G L_{2}^{\times 3} \times D_{3}=G_{M_{\langle 2\rangle}}$ 。

## 7. How to prove Strassen's decomposition is actually matrix multiplication

The group $\Gamma_{\mathcal{S t r}}$ acts on $\left(U^{*} \otimes U\right)^{\otimes 3}$ (in different ways, depending on the choice of decomposition in the family). Say we did not know $\mathcal{S} t r$ but did know its symmetry group. Then we could look for it inside the space of $\Gamma_{\mathcal{S} t r}$ invariant tensors. In future work we plan to take candidate symmetry groups for matrix multiplication decompositions and look for decompositions with elements from these subspaces. In this paper we simply illustrate the idea by going in the other direction: furnishing a proof that $\mathcal{S t r}$ is a decomposition of $M_{\langle 2\rangle}$, by using the invariants to reduce the computation to a simple verification. We accomplish this in $\$ 7.2$ below. We first give yet another proof that Strassen's decomposition is matrix multiplication using the fact that $M_{\langle 2\rangle}$ is characterized by its symmetries.
7.1. Proof that Strassen's algorithm works via characterization by symmetries. Here is a proof that illustrates another potentially useful property of $M_{\langle\mathbf{n}\rangle}$ : it is characterized by its symmetry group [6] Any $T \in\left(U^{*} \otimes V\right) \otimes\left(V^{*} \otimes W\right) \otimes\left(W^{*} \otimes U\right)$ that is invariant under $P G L(U) \times$ $P G L(V) \times P G L(W) \ltimes D_{3}$ is up to scale to $M_{\langle\mathbf{n}\rangle}$. Any $T \in\left(U^{*} \otimes V\right) \otimes\left(V^{*} \otimes W\right) \otimes\left(W^{*} \otimes U\right)$ that is invariant under a group isomorphic to $P G L(U) \times P G L(V) \times P G L(W) \ltimes D_{3}$ is $G L(A) \times G L(B) \times$ $G L(C) \times \mathfrak{S}_{3}$-equivalent up to scale to $M_{\langle\mathbf{n}\rangle}$.

Remark 7.1. $M_{\langle\mathbf{n}\rangle}$ is also characterized as a polynomial by its symmetry group $\tilde{G}_{M_{\langle\mathbf{n}\rangle}}$, and any $T \in\left(U^{*} \otimes V\right) \otimes\left(V^{*} \otimes W\right) \otimes\left(W^{*} \otimes U\right)$ that is invariant under $P G L(U) \times P G L(V) \times P G L(W)$ is up to scale to $M_{\langle\mathbf{n}\rangle}$. However, it is not characterized up to $G L(A) \times G L(B) \times G L(C)$-equivalence by $G_{M_{\langle\mathbf{n}\rangle}}^{\prime}$ in the strong sense of up to isomorphism because $(X, Y, Z) \mapsto \operatorname{trace}(Y X Z)$ has an isomorphic symmetry group but is not $G L(A) \times G L(B) \times G L(C)$-equivalent.

By the above discussion, we only need to check the right hand side of (13) is invariant under $P G L(U) \times P G L(V) \times P G L(W)$ and to check its scale. But by symmetry, it is sufficient to check it is invariant under $P G L(U)$. For this it is sufficient to check it is annihilated by $\mathfrak{s l}(U)$, and again by symmetry, it is sufficient to check it is annihilated by $u_{1} \otimes u^{1 \perp}$, which is a simple calculation.
7.2. Spaces of invariant tensors. As an $\mathfrak{S}_{3}$-module $A=U^{*} \otimes V=[21] \otimes[21]=[3] \oplus[21] \oplus\left[1^{3}\right]$. In what follows we use the decompositions:

$$
\begin{aligned}
S^{2}[21] & =[3] \oplus[21] \\
\Lambda^{2}[21] & =\left[1^{3}\right] \\
S^{3}[21] & =[3] \oplus[21] \oplus\left[1^{3}\right] .
\end{aligned}
$$

The space of standard cyclic $\mathbb{Z}_{3}$-invariant tensors in $A^{\otimes 3}=S^{3} A \oplus S_{21} A^{\oplus 2} \oplus \Lambda^{3} A$ is $S^{3} A \oplus \Lambda^{3} A$. Inside the space of $\mathbb{Z}_{3}$-invariant vectors we want to find instances of the trivial $\mathfrak{S}_{3}$-module [3] in $S^{3}\left([3] \oplus[2,1] \oplus\left[1^{3}\right]\right) \oplus \Lambda^{3}\left([3] \oplus[2,1] \oplus\left[1^{3}\right]\right)$. We have

$$
\begin{aligned}
S^{3}\left([3] \oplus[2,1] \oplus\left[1^{3}\right]\right)= & S^{3}[3] \oplus S^{2}[3] \otimes[2,1] \oplus S^{2}[3] \otimes\left[1^{3}\right] \oplus[3] \otimes S^{2}[2,1] \oplus[3] \otimes[21] \otimes\left[1^{3}\right] \\
& \oplus[3] \otimes S^{2}\left[1^{3}\right] \otimes S^{3}[21] \oplus S^{2}[21] \otimes[13] \oplus[21] \otimes S^{2}\left[1^{3}\right] \oplus S^{3}\left[1^{3}\right]
\end{aligned}
$$

and four factors contain (or are) a trivial representation: $S^{3}[3],[3] \otimes S^{2}[2,1],[3] \otimes S^{2}\left[1^{3}\right], S^{3}[21]$ Similarly

$$
\Lambda^{3}\left([3] \oplus[21] \oplus\left[1^{3}\right]\right)=\Lambda^{2}[21] \otimes[3] \oplus \Lambda^{2}[21] \otimes\left[1^{3}\right] \oplus[3] \otimes[21] \otimes\left[1^{3}\right]
$$

of which $\Lambda^{2}[21] \otimes\left[1^{3}\right]$ is the unique trivial submodule.
In summary:
Proposition 7.2. The space of $\mathfrak{S}_{3} \times \mathbb{Z}_{3}$ invariants in $\left(U^{*} \otimes U\right)^{\otimes 3}$ when $\operatorname{dim} U=2$ is five dimensional.

By a further direct calculation we obtain:
Proposition 7.3. The space of $\mathfrak{S}_{3} \times D_{3}$ invariants in $\left(U^{*} \otimes U\right)^{\otimes 3}$ when $\operatorname{dim} U=2$ is four dimensional.

So if we knew there were an $\mathfrak{S}_{3} \times D_{3}$ invariant decomposition of $M_{\langle 2\rangle}$, it would be a simple calculation to find it as a linear combination of four basis vectors of the $\mathfrak{S}_{3} \times D_{3}$-invariant tensors. In future work we plan to assume similar invariance for larger matrix multiplication tensors to shrink the search space to manageable size.

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