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Abstract

By using the recently introduced framework of unilateral agents interactions, we provide tight graph-theoretical conditions ensuring asymptotic convergence of opinions in finite networks of cooperative agents towards equilibrium configurations where at most 2 distinct opinions persist. Such conditions extend previously known results on asymptotic agreement (or consensus).

Key words: Opinion dynamics, bimodal consensus, Monotone Systems, Consensus, Synchronization, Multi agent systems.

1 Introduction

Nowadays, there is an ongoing intense attention of the scientific community around the most appropriate methodological approaches to deal with the complex and intricate understanding of social dynamics. This requires a development of mathematical models that are sufficiently simple to be examined and capture, at the same time, the complex behavior of real social groups, where opinions and actions related to them may form clusters of different size. Unlike many natural and artificial multi agent systems, whose cooperative behavior is motivated by the attainment of some global coordination among the agents (i.e. robot coordination), actors in social networks usually disagree and may form irregular factions (clusters). A challenging problem is to develop models of opinion dynamics and associated theoretical tools sufficiently instructive to capture the main properties of real social networks and able to characterize the kind of emerging behaviour (consensus on a common opinion, clustering of opinions) that such models or form of interactions may allow.

A first model, in this respect, was introduced in [1] to characterize the process of a group of agents reaching opinion consensus on a common issue by pooling their subjective opinions (also referred to as the iterative opinion pooling). The (linear) interactions are described by a stochastic matrix and sufficient conditions for achieving opinion *consensus* are provided. The model proposed in [2] extends the idea of Degroot model by taking into account the actors prejudices, caused by some exogenous factors eventually leading to opinions disagreement. Specifically, in the F-J model some of the agents are stubborn in the sense that they never forget their prejudices, and thus remain persistently influenced by exogenous conditions under which those prejudices were formed ([3]). The F-J model and its variants have been largely analysed in several scenarios (i.e. [4–7,11,8] just to cite a few).
Another specific and more realistic feature introduced in models of opinion dynamics is the notion of bounded-confidence interaction, that is the possibility of a state-dependent interaction where two agents may influence each other if and only if the difference between their opinions is smaller than a given confidence bound. Two well known models, in this respect, are the nonlinear Hegselmann-Krause (H-K) model [9] and the Deffuant model ([10]). In the first, the dynamic evolution of an agent's opinion is affected by all its connected neighbors, while, in the latter, only the opinion of some randomly selected neighbors matters ("gossip" mechanism). Confidence-based models are technically challenging due to the strongly nonlinear nature of agents' interactions that make

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the network's topology state-dependent. Therefore standard graph theory fails to provide accurate and tight analysis of this kind of models. A variant of the model considers time-varying confidence bound with vanishing amplitude ([16,15]) showing that communities correspond to asymptotically connected components of the network, while another one numerically analyses the case of changing weights in a confidence range ([14]). In [26] a simple model of opinion polarization over structurally balanced graphs has been introduced. The standard diffusive-type interaction consensus scenario is extended by considering signed graph networks in which the edges may assume also negative weights. The model has been analysed and extended in several ways (see e.g. [13] and references therein). Due to the nature of opinion dynamic framework, many results solving multi agent consensus problems are applied to the opinion dynamic scenario. In this respect, many works relate topological property to the type of final equilibrium ([12], [19], [25], [20],[18] just to cite a few). Most of the above contributions assume the presence of *bilateral* agent interactions, in the sense that each node with a certain state value may feel the influence of the neighboring agents regardless of whether their current state is higher or lower than its own (precise definitions to be given later). In some applications, however, node interactions are likely to be unilateral or specifically designed to be such. Herein we consider an opinion dynamics model where an agent may be willing to update its own belief on the basis of the input from a certain neighbour only if its opinion happens to be "optimistic" [and/or "pessimistic"] (that is larger or smaller) when compared to his current state value. In this respect, the considered framework falls within the asymmetric confidence type, with the notable feature of unbounded confidence intervals (either or both in the positive and negative directions). As shown in [19], this feature allows a tight characterization of asymptotic dynamics on the basis of suitable graph theoretical concepts. The proposed model describes situations where confidence may be granted to opinions that are more polarized (in either direction) than the own (i.e. higher or lower; stronger or weaker; etc..). This may model unilateral agents' conservatism in diffusion of innovations in social networks [21], extremal conformity, social inertia, or preservation in opinion dynamics [22,23], drastic risk aversion and conformity in herding phenomena in economic and financial decision-making [24]. In our model interactions are nonlinear and, differently from symmetric bounded confidence models, opinions cannot always freely exchange in order (i.e. relative opinion changes the sign from positive to negative and viceversa) without affecting agents' interaction's strength. For instance, concerning political opinions, a Democratic voter interacting with a person that he believes to be Republican may accept influence from her/his opinions only if these reinforce own beliefs, and not otherwise. Therefore, the model could justify the survival of extremist opinion in social networks. It does not attempt to model their origin, however. Notice that our results are not trivial as they point out conditions guaranteeing that only at most 2 opinions survive in the long run. The contribution of this paper is to identify conditions under which opinions are guaranteed to converge asymptotically towards an equilibrium state with at most two clusters. This corresponds to a dichotomy where, depending upon the topology of interactions, their relative strength and the distribution of initial opinions, consensus may be reached, or, alternatively, a polarisation where only two opinions are allowed asymptotically. To the best of our knowledge no existing theory could predict asymptotic convergence to either bimodal or unimodal equilibria in the considered nonlinear time varying set-up. This may be of interest, for instance, in distributed decision making, where two alternative choices are possible and one would like each individual to opt for one or the other within some reasonable time.

2 Graph-theoretical preliminaries

In the following we make use of the framework of bicolored interaction graphs defined in [19]. In particular, a bicolored graph G is a triple $\{N, E_p, E_o\}$ Where N is a finite set of nodes, E_p and E_o are two (normally) distinct sets of directed edges, $E_p \subset N \times N$ and $E_o \subset N \times N$. The subscript refers to the denomination of such edges as *pessimistic* or (respectively) *optimistic*. Moreover we regard the edge (n_1, n_2) as being directed from node n_1 towards node n_2. Let a set N, $|N|$ denotes its cardinality.

Remark Notice that the graph theoretical tools needed for our subsequent analysis depart significantly from the notion of *signed* graph adopted to describe the Altafini model [26]. Indeed, we formulate our graph theoretical notions within the framework of bicolored graph, rather than signed graph, as edges are allowed to take both colors. Moreover, the notion of sign of a path, is never needed and would not correspond to the product of the sign of its edges. Rather, the color of a path (albeit never explicitly mentioned below), could be defined, for instance, only provided all its edges have the same color.

For a set of edges $E \subset N \times N$ we denote its *mirrored* image $\overleftarrow{E}$ as the following set:

$$\overleftarrow{E} := \{(n_1, n_2) \in N \times N : (n_2, n_1) \in E\}.$$

We say that a graph $G = \{N, E\}$ is *undirected* if $\overleftarrow{E} = E$. The *composition* of two (possibly distinct) set of edges E_1, E_2 is denoted by $E_1 \cdot E_2$ and corresponds to the following set of edges:

$$E_1 \cdot E_2 := \{(n_1, n_3) \in N \times N : \exists n_2 \in N : (n_1, n_2) \in E_1 \text{ and } (n_2, n_3) \in E_2\}.$$

We remark that composition is an associative, but non-commutative, binary operation. Also, it is seen that: $\overleftarrow{E_1 \cdot E_2} = \overleftarrow{E_2} \cdot \overleftarrow{E_1}$. Composition of a set of edges with itself is also denoted by $E^2 := E \cdot E$. In general, without ambiguity, due to associativity of the operation, we may denote $E^i := E \cdot E \cdot \ldots \cdot E$ i times for any positive integer i. The *transitive closure* of a set of edges is denoted by E^* and amounts to:

$$E^* = \bigcup_{i=0}^{+\infty} E^i$$

where, regardless of E, E^0 denotes the diagonal $\{(n, n) : n \in N\}$ and E^1 denotes E. Notice that (n_1, n_2) belongs to E^* iff there exists a finite (possibly empty) path from n_1 to n_2 in the directed graph $G := \{N, E\}$. A graph $G = \{N, E\}$ is strongly connected iff $E^* = N \times N$. Moreover, $\overleftarrow{(E^*)} = (\overleftarrow{E})^*$ so that the notation $\overleftarrow{E}^*$ can be used without ambiguity. The following notion of connectness for bicolored graphs was seen to play a fundamental role in determining the asymptotic convergence towards consensus in the case of networks of agents with unilateral interaction, [19].

Definition 1 *We say that* $G := \{N, E_p, E_o\}$ *fulfils* bicolored quasi-strong connectedness *if and only if:* $\overleftarrow{E_p}^* \cdot E_o^* = N \times N$.

In general, however, the graph $\tilde{G} := \{N, \overleftarrow{E_p}^* \cdot E_o^*\}$, may have $q > 1$ strongly connected components. When such components are all-to-all we say that the graph fulfills q-modal bicolored quasi-strong connectedness as stated below.

Definition 2 *We say that a bicolored graph* $G = \{N, E_o, E_p\}$ *fulfills q-modal bicolored quasi-strong connectedness if* $\tilde{G} = \{N, \overleftarrow{E_p}^* \cdot E_o^*\}$ *admits q strongly connected components which are all to all.*

Notice that, as remarked in [19], a standard (monocolored) directed graph $G = \{N, E\}$ can be seen as a particular case of a bicolored graph in which $E_p = E_o = E$. In particular, as shown in [19], quasi-strong connectivity of $G := \{N, E\}$ amounts to strong connectivity of $G = \{N, \overleftarrow{E}^* \cdot E^*\}$. On the other hand, we see that $\overleftarrow{E} \cdot E = \overleftarrow{\overleftarrow{E} \cdot E}$, and therefore the graph $G = \{N, \overleftarrow{E} \cdot E\}$ is undirected. Similar considerations apply to: $G = \{N, \overleftarrow{E}^* \cdot E^*\}$. This means that, whenever $G = \{N, \overleftarrow{E}^* \cdot E^*\}$ has more than one all to all strongly connected component, it is also disconnected. This situation is not particularly interesting in the study of opinion dynamics, as it entails the presence of multiple groups of agents which do not interact with each other and can, without loss of generality, be recast as three separate multi-agent systems each one fulfilling quasi-strong connectedness. This is why conditions for q-modal asymptotic clustering have never been explicitly stated in the context of bilateral agents interactions.

The main result in [19] clarifies in what respect satisfaction of Definition 1 is a tight necessary and sufficient condition guaranteeing asymptotic consensus of any solutions of the associated network. With a bit of ingenuity, one may wonder if, in addition, fulfillment of Definition 2 for $q \geq 2$ might be enough to ensure convergence of solutions towards equilibrium or a corresponding bound on the number of clusters in the final equilibrium configurations. It turns out that this is not the case. Roughly speaking this is because individual all-to-all components may allow existence of "local leaders" (or group of leaders), normally referred to as *stubborn* agents in the opinion dynamics literature, who might preserve their opinion regardless of others' beliefs and, at the same time, propagate a direct or inderct influence over each others' local followers so as to induce persistent oscillatory behaviours or additional asymptotic opinion values, see counterexample in Section 4.

To this end we modify Definition 2 by strengthening the connectivity requirements within all-to-all components as follows:

Definition 3 *We say that a bicolored graph* $G = \{N, E_o, E_p\}$ *fulfills* 2*-modal bicolored strong connectedness if* $\tilde{G} = \{N, \overleftarrow{E_p}^* \cdot E_o^*\}$ *admits* 2 *strongly connected components which are all to all and, in addition, when denoting with* N_i $i = 1, 2$ *the set of nodes associated to each of the all to all components, the graphs* $G_p^1 := \{N_1, E_p \cap (N_1 \times N_1)\}$ *and* $G_o^2 := \{N_2, E_o \cap (N_2 \times N_2)\}$ *are strongly connected.*

Our first result is an alternative formulation of q-modal bicolored quasi-strong connectedness.

Lemma 4 *A bicolored graph* $G = \{N, E_p, E_o\}$ *fulfills q-modal bicolored quasi-strong connectedness if and only if there exists a partition of* $N = \bigcup_i N_i$, *with* $N_i \cap N_j = \emptyset$ *for all* $i \neq j$, *such that each associated subgraph*

$$G^i := \{N_i, E_p \cap (N_i \times N_i), E_o \cap N_i \times N_i\}$$

fulfills bicolored quasi-strong connectedness and, in addition, $E_p \cap (N_i \times N_j) = \emptyset$ *for all* $j > i$ *and* $E_o \cap (N_i \times N_j) = \emptyset$ *for all* $j < i$.

This lemma clarifies that q-modal quasi-strong connectedness corresponds to quasi-strong connectedness holding on the subgraphs induced by a suitable partition of the sets of agents (which we may call the quasi-strongly connected components). Such partition, however, has to also comply with the requirement that pessimistic and optimistic edges in between quasi-strongly connected components are only allowed to exist in the direction of ascending or descending index respectively.

Proof Consider the graph $G = \{N, \overleftarrow{E_p}^* \cdot E_o^*\}$. Assume that this has q strongly connected components which are all to all. Let N_i, for $i = 1, \dots, q$ denote the set of nodes belonging to each component. It is well known that N_is can be ordered such that all arcs only happen between N_i and N_j if $j > i$. In particular then, no edges occur in $N_i \times N_j$ for $j < i$, that is $N_i \times N_j \cap (\overleftarrow{E_p}^* \cdot E_o^*) = \emptyset$. Notice that:

$$E_o \subset E_o^* \subset \overleftarrow{E_p}^0 \cdot E_o^* \subset \overleftarrow{E_p}^* \cdot E_o^*.$$

Hence, for all $j < i$:

$$N_i \times N_j \cap E_o \subset N_i \times N_j \cap (\overleftarrow{E_p}^* \cdot E_o^*) = \emptyset.$$

Similarly:

$$\overleftarrow{E_p} \subset \overleftarrow{E_p}^* \subset \overleftarrow{E_p}^* \cdot E_o^*.$$

Again, for all $j < i$, we see that:

$$N_i \times N_j \cap \overleftarrow{E_p} \subset N_i \times N_j \cap (\overleftarrow{E_p}^* \cdot E_o^*) = \emptyset.$$

The latter inclusion is equivalent to:

$$N_j \times N_i \cap E_p = \emptyset, \quad \forall\, j < i.$$

Moreover, by assumption, $N_i \times N_i \subset (\overleftarrow{E_p}^* \cdot E_o^*)$, for all $i = 1 \dots q$. This completes the proof of the claim in one direction. Conversely, given the partition of N as $\bigcup_i N_i$, by quasi-strong bicolored connectivity of the G_i subgraphs we have:

$$(\overleftarrow{E_p \cap N_i \times N_i})^* \cdot (E_o \cap N_i \times N_i)^* = N_i \times N_i.$$

Moreover:

$$(\overleftarrow{E_p})^* \cdot E_o^* = \left(\overleftarrow{E_p \cap \bigcup_{j \geq i} N_i \times N_j} \right)^* \cdot (E_o \cap \bigcup_{j \geq i} N_i \times N_j)^*$$

$$\subset \left(\bigcup_i \left(\overleftarrow{E_p \cap N_i \times N_i} \right)^* \cdot (E_o \cap N_i \times N_i)^* \right) \cup \bigcup_{j>i} (N_i \times N_j)$$

$$= \bigcup_i (N_i \times N_i) \cup \bigcup_{j>i} (N_i \times N_j).$$

Conversely:

$$(\overleftarrow{E_p})^* \cdot E_o^* \supset \left(\bigcup_i \overleftarrow{E_p \cap N_i \times N_i}^* \cdot (E_o \cap N_i \times N_i) \right)$$

$$= \bigcup_i N_i \times N_i.$$

This shows that $\{N, E_p, E_o\}$ fulfills n-modal quasi-strong bicolored connectedness.

A consequence of the previous result is also an alternative interpretation of *strong* q-modal bicolored connectedness, as clarified by the following result.

Corollary 5 *A bicolored graph* $G = \{N, E_p, E_o\}$ *fulfills* 2*-modal bicolored strong connectedness if and only if there exists a partition of* $N = N_1 \cup N_2$, *with* $N_1 \cap N_2 = \emptyset$, *such that each associated subgraph*

$$G^i := \{N_i, E_p \cap N_i \times N_i, E_o \cap N_i \times N_i\}$$

fulfills bicolored quasi-strong connectedness, $E_p \cap (N_1 \times N_2) = \emptyset$, $E_o \cap (N_2 \times N_1) = \emptyset$ *and, in addition, the graphs* $G_p^1 := \{N_1, E_p \cap (N_1 \times N_1)\}$ *and* $G_o^2 := \{N_2, E_o \cap (N_2 \times N_2)\}$ *are strongly connected.*

3 Set-up and problem formulation

Throughout this paper we consider nonlinear, possibly time-varying networks of interacting agents. Each agent is characterized by an *opinion*, that is a variable ranging in $\mathbb{R}$ which quantifies his orientation on a particular topic or issue. Its absolute value, normally, does not carry any specific meaning, but its relationships to the opinions of his neighbors does and is seen a measure of relative agreement on the specific subject. More closely, we consider networks of individuals whose interactions pull individuals towards mutual agreement and this may happen with time-varying or time-invariant intensity, according to different set-ups. For a finite number n of individuals we collect their opinions in the state vector $x = [x_1, x_2, \dots, x_n]'$ and consider its time evolution according to the differential equation given below:

$$\dot{x}(t) = f(t, x(t)). \tag{1}$$

The function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz in x, uniformly in t, and piecewise continuous in t, for each frozen value of x. Moreover, it fulfills the following condition:

$$f(t, \alpha \mathbf{1}) = 0 \qquad \forall\, t, \alpha \in \mathbb{R}.$$

This is simply to make sure that consensus states (where each individual carries the same opinion) are indeed equilibria of (1). Besides consensus states, more general configuration of opinions are possible. In particular, it is useful to classify equilibrium states on the basis of the different number of opinions that these exhibit. Specifically, we define the multiplicity of a vector $x = [x_1, x_2, \dots, x_n]$ as follows:

$$m(x) = \text{card}(\{x_1, x_2, \dots, x_n\}).$$

where repeated elements are only counted once. Of course $m(x) \le n$ and $m(x) = 1$ iff x is a consensus state. Additional assumptions are typically considered for f in (1) to guarantee that interactions are pulling individuals towards a mutual agreeent. One such assumption, which is sometimes only implicitly assumed, but will play a major role in the subsequent analysis, is the notion of *cooperativity* ([27]).

Definition 6 *A network as in (1) is cooperative if for all i, $f_i(t,x)$ is monotonically non-decreasing with respect to x_j for all $j \neq i$ and all t.*

Our problem of interest is to provide sufficient conditions for networks as in (1) to exhibit solutions that, regardless of initial conditions, converge towards equilibrium. Moreover, we wish to characterize for all equilibrium states x_e an upper-bound to the number $m(x_e)$, as this is then an estimate of the number of distinct opinions that the system can afford asymptotically. Our criteria will be based on graph-theoretical descriptions of social interactions according to the framework estabilished in [19]. In particular, to every network we associate a bicolored graph according to the following procedure. Every agent is a node of the graph. In particular we deal with a finite set of nodes $N = \{1, 2, \dots, n\}$. Arcs between nodes represent agents interactions. Accordingly, an optimistic arc from node i to node j signifies that agent j is susceptible to the influence of node i whenever the latter has an opinion which is higher in value than the current opinion of node j. Similarly, a pessimistic edge from node i to node j signals susceptibility of agent j to the influence of agent i, whenever the latter has a lower opinion than node's j. In the case of cooperative networks the following definitions (originally proposed in [19]) are appropriate.

Definition 7 (*Pessimistic edge*) *We say that $(j,i) \in E_p \subset N^2$ is a pessimistic edge connecting j to i for a cooperative network* (1)*, if for all compacts $\mathcal{K} \subseteq \mathbb{R}$, there exist $\varepsilon_{\mathcal{K}} > 0$ and sufficiently large $T_{\mathcal{K}} > 0$ so that for any $t \ge 0$, for all pairs $x_i > x_j \in \mathcal{K}^2$ it holds:*

$$\int_t^{t+T_{\mathcal{K}}} f_i(\tau, x_i \mathbf{1} + e_j(x_j - x_i))\, d\tau \le -\varepsilon_{\mathcal{K}}(x_i - x_j). \tag{2}$$

Definition 8 (*Optimistic edge*) *We say that $(j,i) \in E_o \subset N^2$ is an optimistic edge connecting j to i for a cooperative network* (1)*, if for all compacts $\mathcal{K} \subseteq \mathbb{R}$, there exist $\varepsilon_{\mathcal{K}} > 0$ and sufficiently large $T_{\mathcal{K}} > 0$ so that for any $t \ge 0$, for all pairs $x_i < x_j \in \mathcal{K}^2$ it holds:*

$$\int_t^{t+T_{\mathcal{K}}} f_i(\tau, x_i \mathbf{1} + e_j(x_j - x_i))\, d\tau \ge \varepsilon_{\mathcal{K}}(x_j - x_i). \tag{3}$$

It is worth pointing out that inequalities (2) and (3) are a uniform lower bound of how much a displaced j-th agent (with respect to an agreement configuration where all agents are equal to x_i) can influence, either from below (in the pessimistic case) or above (in the optimistic one) the current state of agent i over a sufficiently long time interval ($T_{\mathcal{K}}$). Only when these are fulfilled the edge is introduced in the interaction graph. Therefore, while $f_i(t,x)$ is time varying, the graph is time-invariant and captures connectivity over the whole interval $[0, +\infty[$. Notice, also, that integration is performed on frozen state variables and not along solutions of (1). Intuitively, conditions (2) and (3) represent a way to quantify existence of attraction of agent i towards agent j when the latter is either below or. respectively, above the former. Moreover, such attraction grows, over a uniform and sufficiently long time-window, at least linearly in the current distance of opinions.

In traditional formulations of consensus protocols absence of an edge between two agents does not have any implications besides lack of fulfillment of the conditions for existence of that very edge (for instance if two agents are interacting too sporadically for the condition to be fulfilled or with vanishing intensity) or, of course, are not interacting at all. In the present set-up, however, we require a dichotomy of behaviour concerning agents interactions, namely, either (pessimistic or optimistic) interactions between two agents exist, and in this case they fulfill the edge existence conditions, or they don't. Hence, absence of an edge between nodes signifies absence of influence between the corresponding agents.

Assumption 9 (*Dichotomy of interactions*) *If edge $(j,i) \notin E_p$ then the following holds:*

$$\frac{\partial f_i}{\partial x_j}(t,x) = 0 \qquad \forall\, t, \forall\, x : x_j \le x_i.$$

If edge $(j,i) \notin E_o$ then the following holds:

$$\frac{\partial f_i}{\partial x_j}(t,x) = 0 \qquad \forall\, x : x_j \ge x_i.$$

Our main result for this Section is stated below.

Theorem 10 *Let network (1) admit an associated* *bicolored interaction graph that fulfills 2-modal bicolored strong connectedness. Then, for all initial conditions $x(0) \in \mathbb{R}^n$, the corresponding solution $x(t)$ admits a limit x_e as $t \to +\infty$ and the latter equilibrium fulfills:*

$$m(x_e) \leq 2. \tag{4}$$

Before proceeding to the proof of Theorem 10, we introduce some notations and formulate preliminary Lemmas. For a vector $x = [x_1, x_2, \ldots, x_n]^T$, we denote by x_M and x_m the following scalars:

$$x_M = \max_{i \in \{1,\ldots,n\}} x_i \qquad x_m = \min_{i \in \{1,\ldots,n\}} x_i, \tag{5}$$

moreover we assume that, according to the partition $N = N_1 \cup N_2$, the state vector x is similarly arranged as $x = [x^{1^T}, x^{2^T}]^T$ and the vector field as $f = [f^{1^T}, f^{2^T}]^T$. We also introduce, $x^j_M = \max_{i \in N_j} x_i$, $x^j_m = \min_{i \in N_j} x_i$, $j = 1, 2$ and, with a slight abuse of notation, $\mathbf{1}^1$ and $\mathbf{1}^2$ as the vectors of ones of dimension equal to $|N_1|$ and $|N_2|$ respectively. In particular, $\mathbf{1} = [\mathbf{1}^{1^T}, \mathbf{1}^{2^T}]^T$. Let $j \in N$ be arbitrary and $\mathcal{K} \subset \mathbb{R}$ a compact interval. For any $x \in \mathcal{K}^n$ denote by $\bar{x}_j$ and $\underline{x}_j$ the following:

$$\bar{x}_j = x^1_M \, \mathbf{1} + \Big(x_j - x^1_M\Big) e_j,$$

$$\underline{x}_j = x^2_m \, \mathbf{1} + \Big(x_j - x^2_m\Big) e_j.$$

Let, with a slight abuse of notation (as dependence upon j and t is not emphasized), $\tilde{x}(\cdot)$ and $\underset{\sim}{x}(\cdot)$ denote the solutions of equation (1) from initial state $\bar{x}_j$ at time t (viz. $\tilde{x}(\cdot) = \phi(\cdot, t, \bar{x}_j)$) and from initial state $\underline{x}_j$, respectively (viz. $\underset{\sim}{x}(\cdot) = \phi(\cdot, t, \underline{x}_j)$).

Lemma 11 *The functions $x^1_M(t)$ and $x^2_m(t)$ are (respectively) monotonically non-increasing and non-decreasing.*

Proof Equivalently we show that the set:

$$\mathcal{M}_c := \{x : \max_{i \in N_1} x_i \leq c\},$$

is forward invariant for all $c \in \mathbb{R}$. Let x in $\mathcal{M}_c$ be arbitrary. Since $\mathcal{M}_c$ is convex, its tangent cone at x is simply given by $TC_x\mathcal{M}_c = \{z : z_i \leq 0, \forall\, i \in N_1 : x_i = c\}$ (see Proposition 5.5, [28]). Moreover, for any t, any $x \in \mathcal{M}_c$ and all $i \in N_1$ such that $x_i = c$, it holds:

$$f_i(t, x^1, x^2) = f_i(t, x^1, \min\{x^2, c\mathbf{1}^2\}) \leq$$

$$\leq f_i(t, x^1, c\mathbf{1}^2) \leq f_i(t, c\mathbf{1}) = 0,$$

where the first equality follows from Assumption 9 and the latter inequalities by monotonicity. Hence $f(t, x) \in TC_x\mathcal{M}_c$. As this holds for all $x \in \mathcal{M}_c$ it proves forward invariance of $\mathcal{M}_c$ (by Nagumo's Theorem - [17]) and monotonicity of $x^1_M(t)$. A symmetric argument can be used to prove monotonicity of $x^2_m(t)$ by proving forward invariance of $\mathcal{N}_c = \{x : \min_{i \in N_2} x_i \geq 0\}$. ■

Lemma 12 *Assume there exist a finite positive integer $\bar{k}$, $\mu > 0$ (uniform in t and x) such that for each given $j \in N_1$ the corresponding $\tilde{x}$ and $\underset{\sim}{x}$ solutions fulfill for all $i \in N_1$:*

$$\tilde{x}_i(t + \bar{k}T) \leq x^1_M(t) - \mu|x^1_M(t) - \tilde{x}_j(t)| \tag{6}$$

and, respectively, $\forall i \in N_2$, having fixed any $j \in N_2$:

$$\underset{\sim}{x}_i(t + \bar{k}T) \geq x^2_m(t) + \mu|x^2_m(t) - \underset{\sim}{x}_j(t)|, \tag{7}$$

then, similar inequalities hold for the solution $x(\cdot) = \phi(\cdot, t, x)$, viz.:

$$x_i(t + \bar{k}T) \leq x^1_M(t) - \mu|x^1_M(t) - x_j(t)| \tag{8}$$

and:

$$x_i(t + \bar{k}T) \geq x^2_m(t) + \mu|x^2_m(t) - x_j(t)|. \tag{9}$$

Proof Fix $j \in N_1$ and let $\tilde{x}$ be the solution corresponding to initial condition $\bar{x}_j(t)$. Clearly, $\bar{x}_j(t) \geq x(t)$, and therefore, by monotonicity of system (1) we have: $\tilde{x}(\tau) \geq x(\tau)$ for all $\tau \geq t$. In particular, then,

$$x_i(t+\bar{k}T) \leq \tilde{x}_i(t+\bar{k}T) \leq x^1_M(t) - \mu|x^1_M(t) - \tilde{x}_j(t)|$$

$$= x^1_M(t) - \mu|x^1_M(t) - x_j(t)|.$$

A symmetric argument applies to $\underset{\sim}{x}(\cdot)$.

The Lemma asserts that, for monotone networks, there exists a "worst-case" scenarios for the initial distribution of agents with respect to convergence speed towards equilibrium. These are described by vectors $\bar{x}_j$ (or $\underline{x}_j$, respectively): initial conditions in which all agents have maximum (respectively minimum) and equal value, except for a single node j, which may take any lower (or, respectively higher) value.

Proof of Theorem 10 Let the compact interval $\mathcal{K}$ be given and $x_0 \in \mathcal{K}^n$ arbitrary. Fix any $j \in N_1$. We denote by $x(t) \doteq \phi(t, 0, x_0)$. For any assigned $t \geq 0$ we let:

$$\tilde{x}(\cdot) = \phi(\cdot, t, \bar{x}_j(t)),$$

according to the notations of Lemma 12. By assumption, there exist a path in G^1_p between any couple of nodes i and j in N_1 (or, respectively, in G^2_o among nodes in N_2). Let $d(q) : N_1 \to \mathbb{N}$ denote the distance from j to node q along such path. We need to verify that, for any node $i \in N_1$ the following inequality holds:

$$\tilde{x}_i(t + 2d(i)T) \leq x_M^1(t) - \mu|x_M^1(t) - \tilde{x}_j(t)|. \quad (10)$$

We use an induction argument.

STEP 1

The statement is trivial for $i = j$, viz. $d(i) = 0$. In this case, $\tilde{x}_j^1(t) \leq x_M^1(t)$, it results for any $\mu_0 \in (0, 1)$:

$$\tilde{x}_j^1(t) - x_M^1(t) \leq -\mu_0|x_M^1(t) - \tilde{x}_j^1(t)|.$$

STEP 2

Now we make the inductive step and prove that, if the statement (6) holds for nodes q at distance $d(q)$ from j, then it holds for nodes k at distance $d(k) = d(q) + 1$ with $(q, k) \in E_p \cap (N_1 \times N_1)$. From Assumption 9, and remarking that $\tilde{x}_k(\tau) \leq x_M^1(\tau) \leq x_M^1(t)$ for all $\tau \geq t$, the following equalities holds for $\tau \in [t + 2d(q)T, t + 2d(k)T]$:

$$\tilde{x}_k(\tau) - \tilde{x}_k(t + 2d(q)T) = \int_{t+2d(q)T}^{\tau} f_k(\theta, \tilde{x}(\theta))\, d\theta$$
$$= \int_{t+2d(q)T}^{\tau} f_k(\theta, \tilde{x}^1(\theta), \min\{\tilde{x}^2(\theta), x_M^1(t)\mathbf{1}^2\})\, d\theta.$$

Next, we exploit non-decreasingness of f_k with respect to all x_is, with $i \neq k$, in order to derive:

$$\begin{aligned}
&\tilde{x}_k(\tau) - \tilde{x}_k(t + 2d(q)T) \leq \\
&\int_{t+2d(q)T}^{\tau} f_k(\theta, x_M^1(t)\mathbf{1}^1 + (\tilde{x}_q(\theta) - x_M^1(t))e_q \\
&+ (\tilde{x}_k(\theta) - x_M^1(t))e_k, \min\{\tilde{x}^2(\theta), x_M^1(t)\mathbf{1}^2\})\, d\theta \\
&\leq \int_{t+2d(q)T}^{\tau} f_k(\theta, x_M^1(t)\mathbf{1}^1 + (\tilde{x}_q(\theta) - x_M^1(t))e_q \\
&+ (\tilde{x}_k(\theta) - x_M^1(t))e_k, x_M^1(t)\mathbf{1}^2))\, d\theta \\
= &\int_{t+2d(q)T}^{\tau} f_k(\theta, x_M^1(t)\mathbf{1} + (\tilde{x}_q(\theta) - x_M^1(t))e_q \\
&+ (\tilde{x}_k(\theta) - x_M^1(t))e_k)\, d\theta \\
&\leq \int_{t+2d(q)T}^{\tau} f_k(\theta, x_M^1(t)\mathbf{1} + (\tilde{x}_q(\theta) - x_M^1(t))e_q)\, d\theta \\
&- L\int_{t+2d(q)T}^{\tau} [\tilde{x}_k(\theta) - x_M^1(t)]\, d\theta.
\end{aligned}$$

Let $\hat{x}_q = \max_{\theta \in [t+2d(q)T, t+2d(k)T]} \tilde{x}_q(\theta)$, it results:

$$\begin{aligned}
&\tilde{x}_k(\tau) - \tilde{x}_k(t + 2d(q)T) \leq \\
&\int_{t+2d(q)T}^{\tau} f_k(\theta, x_M^1(t)\mathbf{1} + (\hat{x}_q - x_M^1(t))e_q)\, d\theta \\
&- L\int_{t+2d(q)T}^{\tau} [\tilde{x}_k(\theta) - x_M^1(t)]\, d\theta.
\end{aligned}$$

In particular then, for all $\tau \in [t + (2d(q) + 1)T, t + 2d(k)T]$ we see that:

$$\begin{aligned}
&\tilde{x}_k(\tau) - x_M^1(t) \leq \tilde{x}_k(\tau) - \tilde{x}_k(t + 2d(q)T) \\
&\leq -\varepsilon_{\mathcal{K}}(x_M^1(t) - \hat{x}_q) \\
&- L\int_{t+2d(q)T}^{\tau} [\tilde{x}_k(\theta) - x_M^1(t)]\, d\theta.
\end{aligned}$$

From now on the derivations may be conducted along the same lines of Lemma 16 in [19], yielding:

$$\tilde{x}_i(t + \bar{k}_1T) - x_M^1(t) \leq -\mu_1|x_M^1(t) - \tilde{x}_j^1(t)|.$$

where $\bar{k}_1$ and $\mu_1 \in (0, 1)$ are positive real (uniform for x_0 in $\mathcal{K}^n$) function of $|N_1|$. This concludes the proof of inequality (6) that, by Lemma 12, yields the inequality in (8). Given the arbitrariness of nodes $(i, j) \in N_1 \times N_1$ and the assumption of strong connectivity of G_p^1 we may conclude:

$$x_M^1(t + \bar{k}_1T) - x_M^1(t) \leq -\mu_1|x_M^1(t) - x_m^1(t)|. \quad (11)$$

Similar considerations for the agents in N_2 and the graph of optimistic interactions G_o^2 yield:

$$x_m^2(t + \bar{k}_2T) - x_m^2(t) \geq \mu_2|x_m^2(t) - x_M^2(t)|. \quad (12)$$

Finally, inequalities (11) and (12), imply that:

$$\lim_{t\to+\infty} x_M^1(t) = \lim_{t\to+\infty} x_m^1(t)$$

and

$$\lim_{t\to+\infty} x_M^2(t) = \lim_{t\to+\infty} x_m^2(t).$$

Hence, all agents in N_1 asymptotically converge towards $x_M^1(\infty)$ and all agents in N_2 towards $x_m^2(\infty)$. As a consequence, only two possibilities arise, namely: $m(x_e) = 1$ (consensus: $x_M^1(\infty) = x_m^1(\infty) = x_M^2(\infty) = x_m^2(\infty)$) or $m(x_e) = 2$ (2-cluster consensus: $x_M^1(\infty) = x_m^1(\infty) \neq x_M^2(\infty) = x_m^2(\infty)$). This proves the claim (4) and completes the proof of Theorem 10.

Notice the resemblance of the result with the conditions for asymptotic agreement formulated in [19]. Notice that in such paper (in the case of cooperative networks), only *quasi-strong* (1-modal) bicolored connectedness is required for asymptotic consensus to hold.

4 Examples and counterexamples

In the following we will consider examples (and counterexamples) designed in order to illustrate, in the simplest possible instances, the graph theoretical assumptions of our main result.

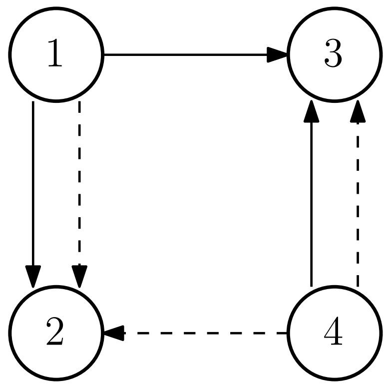

Fig. 1. A network fulfilling 2-modal bicolored quasi-strong connectedness

4.1 Stubborn agents and strong connectedness

We start by considering the following 4 agents time-invariant network:

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= (x_1 - x_2) + \min\{x_4 - x_2, 0\} \\ \dot{x}_3 &= (x_4 - x_3) + \max\{x_1 - x_3, 0\} \\ \dot{x}_4 &= 0. \end{aligned} \tag{13}$$

The associated bicolored graph is shown in Fig. 1. Notice that we may partition $N = \{1, 2, 3, 4\}$ as $N_1 \cup N_2$, with $N_1 = \{1, 2\}$ and $N_2 = \{3, 4\}$. These fulfill the conditions of Lemma 4; hence, the network's topology fulfills quasi-strong 2-modal bicolored connectedness. It does not, however, fulfill strong 2-modal bicolored connectedness, as the subgraphs associated with N_1 and N_2 are trees, and are not strongly connected (neither the optimistic nor the pessimistic one). In particular, agents 1 and 4, can be seen to be *stubborn* agents, namely agents who do not change their opinion in time. While existence of 2 stubborn agents might seem compatible with achieving at most two clusters at equilibrium, it is easily seen that this is not the case. In fact, stubborn agents may influence other agents and compete with each other in order to spread their influence in such a way that additional asymptotic opinion values are created. For instance, state $x_e = [2, 1, 1, 0]'$ is an equilibrium for which $m(x_e) = 3 > 2$. Moreover, this example can easily be modified and made time-varying in order to create oscillating solutions that never settle to a particular equilibrium value. This, in a sense, justifies strengthening Definition 2 to strong bicolored 2-modal connectedness as in Definition 3.

4.2 A simulative example

We consider the network composed of eight agents in Fig. 2. We see that agents 4 (resp. 2 and 3) is influenced by agents 8 (resp. 6 and 5) provided this is supplying an 'optimistic' information. This

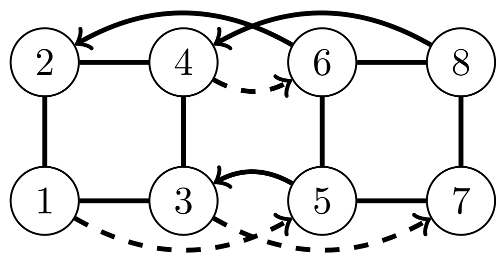

Fig. 2. Graph fulfilling bicolored strong connectedness: continuous and dashed arrows respectively highlight optimistic and pessimistic edges. Continuous line highlights bidirectional and bilateral edges.

translates into a continuous arrow from 8 to 4. Agent 2, instead, is bidirectionally influenced by agent 1, and viceversa. Influence of agent 1 towards agent 5 is of pessimistic nature only, and this is therefore modeled as dashed arrow (see the scheme in Fig 2). Notice that, if not allowing pessimistic and optimistic influences to be accounted for separately, the above network's equation would only afford a single bilateral influence from node 1 to 2, (which is clearly insufficient for achieving consensus). In other words the actual graph of influences between neighboring agents would be heavily underestimated.

We can cluster the agents in two subgroups: $N_2 = \{1, 2, 3, 4\}$ and $N_1 = \{5, 6, 7, 8\}$, with $N_1 \cap N_2 = \emptyset$. Moreover, $E_o \cap N_2 \times N_1 = \emptyset$, viz. no optimistic edges from nodes in N_2 towards nodes in N_1 and, symmetrically, $E_p \cap N_1 \times N_2 = \emptyset$, no pessimistic arcs from nodes in N_1 towards nodes in N_2. Hence, by virtue of Lemma 4, quasi-strong bicolored connectedness holds. Moreover, each subraph G_p^1, G_p^2, G_o^1 and G_o^2 is strongly connected, hence the topological conditions of our Theorem 10 are fulfilled, and one should expect, for a corresponding system of differential equations, that all solutions will asymptotically converge to equilibrium states with at most 2 clusters.

Different orderings of initial state variables may induce different equilibrium. For instance starting from the initial condition $x_i \leq x_j$, $i = 1, \ldots, 4$, $j = 5, \ldots, 8$ (i.e. $x(0) = [1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8]'$) the optimistic and pessimistic edges activation induces a topology that guarantees consensus (simulations in Fig. 3). Now we consider the node initial conditions $x(0) = [6 \quad 2 \quad 8 \quad 4 \quad 1 \quad 7 \quad 3 \quad 5]'$ such that in the scheme of Fig 2 there are not pessimistic arcs connecting nodes 1 and 3 to 5 and 7, respectively. Being $x_3(0) > x_5(0)$ also the optimistic edge from node 5 to 3 is inactive. Finally the opinion network dynamic evolution induces at some instants t_1, t_2 $x_4(t_1) > x_8(t_1)$ and $x_2(t_2) > x_6(t_2)$ such that the network splits into two connected components. This yields an equilibrium configuration with two clusters (simulations in Fig. 4).

4.3 The case $n \geq 3$

While graph theoretical notions of n-modal connectedness have been defined, it turns out that Theorem 10 only holds for strong 2-modal bicolored connectedness, and cannot be generalized to the case of $n > 2$. As a counterexample take the graph with $N = \{1, 2, 3\}$, $E_p = \{(3, 2)\}$ and $E_o = \{(1, 2)\}$. Then $\overleftarrow{E}_p = \{(2, 3)\}$ and $\overleftarrow{E}^*_p = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$. Similarly $E^*_o = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$. Composing the two edge sets yields: $\overleftarrow{E}^*_p \cdot E^*_o = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$. This is a graph with 3 strongly connected components, associated with $N_1 = \{1\}$, $N_2 = \{2\}$ and $N_3 = \{3\}$, moreover, there are no edges in $N_i \times N_j$ whenever $j < i$. Hence 3-modal strong bicolored connectedness is fulfilled (trivially, as each subgraph is a singleton). One can easily build an associated set of differential equations as follows:

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= a_{12}(t) \max\{0, x_1 - x_2\} + a_{23}(t) \min\{0, x_3 - x_2\} \\ \dot{x}_3 &= 0. \end{aligned} \tag{14}$$

where $a_{12}(\cdot)$ and $a_{23}(\cdot)$ are non-negative measurable locally essentially bounded functions, fulfilling for some $T > 0$ and all $t \geq 0$

$$\int_t^{t+T} a_{12}(\tau) d\tau \geq 1 \qquad \int_t^{t+T} a_{23}(\tau) d\tau \geq 1.$$

Indeed, for initial conditions $x_1(0) \leq x_2(0) \leq x_3(0)$ we have, $x_i(t) = x_i(0)$ for all $t \geq 0$ and all $i = 1, 2, 3$ and therefore $m(x_e) \leq 3$. On the other hand, the opposite condition in which $x_1(0) > x_2(0) > x_3(0)$ results in solutions of the following form:

$$\begin{aligned} x_1(t) &= x_1(0), \\ x_2(t) &= e^{\int_0^t a_{12}(\tau)+a_{23}(\tau) d\tau} x_2(0) + \\ &\int_0^t e^{\int_\tau^t a_{12}(s)+a_{23}(s)\, ds} [x_1(0)a_{12}(\tau) + x_3(0)a_{23}(\tau)]\, d\tau, \\ x_3(t) &= x_3(0). \end{aligned} \tag{15}$$

Such solutions oscillate forever and are asymptotically periodic provided a_{12} and/or a_{23} are (for instance) non-constant periodic functions. Therefore, asymptotic convergence towards equilibrium is in all such cases violated. This example illustrates the difficulty in generalizing (in a time-varying set-up) results predicting existence of equilibria with $n > 2$ opinion clusters. The same arguments used to prove our main result, can, however be used to prove the following Theorem.

Theorem 13 *Let network (1) admit an associated bicolored interaction graph that fulfills q-modal bicolored quasi-strong connectedness. Let moreover, the two graphs,* $G^1_p := \{N_1, E_p \cap (N_1 \times N_1)\}$ *and* $G^2_o := \{N_q, E_o \cap (N_q \times N_q)\}$ *are strongly connected. Then, for all initial conditions* $x(0) \in \mathbb{R}^n$*, the corresponding solution* $x(t)$ *admits an* ω*-limit* $\omega(x(0))$ *such that the following bound is fullfilled:*

$$\forall\, x \in \omega(x(0)), \qquad m(x) \leq 2 + \sum_{i=2}^{q-1} |N_i|. \tag{16}$$

5 Symmetric networks

Consider networks with additive dynamics of the following type:

$$\dot{x}_i = \sum_{j=1}^{n} f^p_{ij}(t, x_i, x_j) + f^o_{ij}(t, x_i, x_j) \tag{17}$$

where the o and p explicitly remark the pessimistic or optimistic nature of each interaction term. A very special case arises wen the following piecewise linear expression is valid:

$$\dot{x}_i = \sum_{j=1}^{n} a^p_{ij}(t) \min\{x_j - x_i, 0\} + a^o_{ij}(t) \max\{x_j - x_i, 0\}. \tag{18}$$

Notice that sums in (17) and (18) are taken for $j = 1 \ldots n$. This does not imply an all-to-all network of pessimistic and optimistic interactions as we allow for some of these additive terms to be identically 0 (and therefore the corresponding edge won't exist in the graph). We say that the network is symmetric if for all $t \geq 0$, all $x_i, x_j \in \mathbb{R}$ and $i, j \in N^2$ the following holds:

$$f^o_{ij}(t, x_i, x_j) = -f^p_{ji}(t, x_j, x_i).$$

Similarly, in the context of piecewise-linear networks, symmetric networks fulfill:

$$a^o_{ij}(t) = a^p_{ji}(t) \quad \forall\, t \geq 0, \forall\, i, j \in N^2.$$

Notice that, under such conditions, agent i attracts agent j iff agent j is attracting agent i, with exactly the same strength. This generalizes the well known condition of symmetric networks guaranteeing average consensus. In particular, in fact:

$$\sum_{i=1}^{n} \dot{x}_i = 0,$$

is fulfilled and therefore the average value of agents opinions doesn't change in time. Understanding how initial distributions of opinions affect the asymptotic number of clusters appears to be a very hard task, in general. Intuitively, mixing of opinions from both subgroups should promote consensus, whereas, separation into disjoint subgroups could more likely result into bimodal asymptotic equilibria. For general networks, one may conclude asymptotic coexistence of two opinions as soon as the following inequality is fulfilled:

$$\min_{i \in N_1} x_i(t) > \max_{i \in N_2} x_i(t).$$

A stronger result holds for symmetric networks.

Theorem 14 *Let $x(0)$ be the initial vector of agents' opinions. Assume that:*

$$\frac{\sum_{i \in N_1} x_i(0)}{|N_1|} > \frac{\sum_{i \in N_2} x_i(0)}{|N_2|}, \qquad (19)$$

where $|N_i|$ denotes the cardinality of set N_i, $i = 1, 2$. Then, under the same assumptions of Theorem 10, the following limit exists $\lim_{t \to +\infty} x(t) = x_e$ and $m(x_e) = 2$.

Proof By virtue of symmetry and 9 we see that:

$$\frac{d}{dt} \sum_{i \in N_1} x_i(t) = \sum_{i \in N_1} \dot{x}_i(t) =$$

$$= \sum_{i \in N_1} \sum_{j} f^o_{ij}(t, x_i(t), x_j(t)) + f^p_{ij}(t, x_i(t), x_j(t)) =$$

$$= \sum_{i \in N_1} \sum_{j \in N_2} f^o_{ij}(t, x_i(t), x_j(t)) + f^p_{ij}(t, x_i(t), x_j(t)) =$$

$$= \sum_{i \in N_1} \sum_{j \in N_2} f^o_{ij}(t, x_i(t), x_j(t)) \geq 0.$$

Hence,

$$\lim_{t \to +\infty} \frac{\sum_{i \in N_1} x_i(t)}{|N_1|} \geq \frac{\sum_{i \in N_1} x_i(0)}{|N_1|}.$$

A similar argument can be used to infer that:

$$\lim_{t \to +\infty} \frac{\sum_{i \in N_2} x_i(t)}{|N_2|} \leq \frac{\sum_{i \in N_2} x_i(0)}{|N_2|}.$$

By the inequality (19) then:

$$\lim_{t \to +\infty} \frac{\sum_{i \in N_1} x_i(t)}{|N_1|} > \lim_{t \to +\infty} \frac{\sum_{i \in N_2} x_i(t)}{|N_2|},$$

which proves the claim.

6 Conclusions

A model of nonlinear networks affording equilibria with a single opinion value (*consensus*) or multiple clustered opinions are introduced and analyzed. In particular, multiple opinions may result as a consequence of unilateral agents interactions (viz. asymmetric confidence where influence of one agent upon another is conditioned to a particular order relationship between their opinions). Suitable connectedness properties are proposed to guarantee, in the presence of possibly time-varying interactions, convergence towards equilibria with at most n distinct opinions. Simulations are proposed to show how initial conditions may affect the final outcome for the same pattern and strength of interactions.

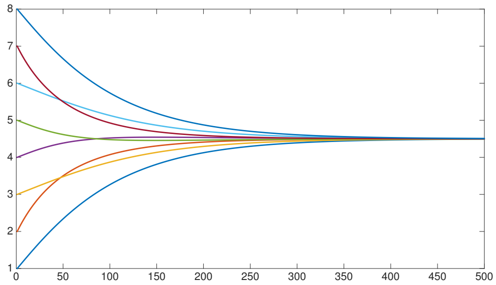

Fig. 3. Dynamic state evolution and convergence to consensus equilibria

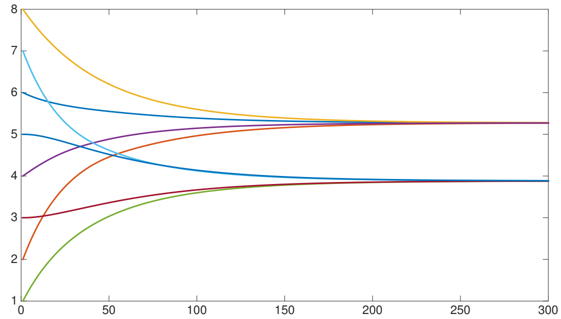

Fig. 4. Dynamic state evolution and convergence to cluster equilibria

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