Stable determination of polyhedral interfaces from boundary data for the Helmholtz equation

STABLE DETERMINATION OF POLYHEDRAL INTERFACES FROM BOUNDARY DATA FOR THE HELMHOLTZ EQUATION

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Abstract. We study an inverse boundary value problem for the Helmholtz equation using the Dirichlet-to-Neumann map. We consider piecewise constant wave speeds on an unknown tetrahedral partition and prove a Lipschitz stability estimate in terms of the Hausdorff distance between partitions.

Keywords. Inverse boundary value problem, Helmholtz equation, Lipschitz stability

MSC: 35R30, 35J08, 35J25

1. Introduction. We consider an inverse boundary value problem for the Helmholtz equation

$$\Delta u + \omega^2 q(x) u = 0 \quad \text{in } \Omega \subset \mathbb{R}^3,$$

where $q = c^{-2}$ and c is the wavespeed. The data are the Dirichlet-to-Neumann map and the objective is to recover the wavespeed. The uniqueness of this inverse problem was established by Sylvester and Uhlmann [20] for $q \in L^\infty(\Omega)$. Concerning stability, conditional logarithmic continuous dependence of the wavespeed on the Dirichlet-to-Neumann map has been proven in [2] in the case of wavespeeds in $H^s(\Omega)$ with $s > \frac{3}{2}$. We refer to Novikov [13] for a refinement of this stability estimate. The logarithmic rate of stability is optimal [12]. For the inverse conductivity problem the authors of [3] proposed restricting the class of unknown coefficients to a finite dimensional set to obtain Lipschitz stability estimates. The result was extended to complex-valued conductivities in [6]. In this finite dimensional setting, in [4, 5], a Lipschitz stability estimate for the recovery of piecewise constant wavespeeds for a given domain partition from boundary data for the Helmholtz equation, and an estimate for the stability constant in terms of the number of domains in the partition, were obtained.

Here, we study the problem of determining the finite partition from boundary data given a (possibly large) finite set of attainable values for the wavespeed. Due to the severe nonlinearity of the problem the derivation of Lipschitz stability estimates is more subtle. For this reason, we consider a partitoning of the domain with a (regular) unstructured tetrahedral mesh. In fact, an unstructured tetrahedral mesh admits a local refinement and, with piecewise constant wavespeeds, can accurately approximate realistic models in applications. In geophysics, we mention as an example the work of Rüger and Hale [16]. Here, knowledge of a set of attainable values for the wavespeed can be motivated by the general knowledge of relevant rock types. The deformation allows one to adjust the mesh and recover structures in the models. In geodynamics, these structures can be an imprint of the local geology and tectonics [18]. Moreover, one can parametrize major discontinuities at (polyhedral) surfaces by connecting boundaries of subdomains in the partition via a segmentation for example.

In this paper, we establish a Lipschitz stability estimate expressed in terms of the Hausdorff distance between partitions using tetrahedra from the Dirichlet-to-Neumann map. Lipschitz stability estimates provide a framework for optimization, specifically, iterative reconstruction of the wavespeed with a convergence radius determined by the stability constant [7, 8]. The recovery of polyhedral

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interfaces then becomes a shape optimization. The analysis in [7] makes explicit use of a Landweber iteration. Via successive approximations, and making use of estimates for the corresponding growth of the stability constant, the reconstruction can be cast into a multi-level scheme [8] effectively enlarging the radius of convergence. As an important application, we mention so-called time-harmonic full waveform inversion (FWI) developed in reflection seismology [14, 15, 19, 21] with the goal to image wavespeed variations in Earth's interior. The data, here, are essentially the single-layer potential operator. However, stability estimates for the Dirichlet-to-Neumann map directly carry over to stability estimates for this operator.

We give an outline of the paper. We first state the main result and the main assumptions (Section 2). Then we establish a rough stability estimate for the potentials using complex geometrical optics (CGO) solutions following the outline of an estimate in Beretta *et al.* [5] (Section 3). The CGO solutions were introduced by Sylvester and Uhlmann [20] in their proof of uniqueness of this inverse boundary value problem. The CGO solutions in our analysis differ slightly from theirs to obtain better constants in the stability estimates as proposed in [17]. We proceed with establishing the recovery of the number of tetrahedra in the mesh from the potential, and with expressing the Hausdorff distance between meshes in terms of the difference of piecewise constant potentials defined on these meshes. Naturally, the information on the Hausdorff distance between meshes can be transformed to information on the vertices of the tetrahedra forming the meshes (Section 4). The main part of the proof of our result pertains to obtaining a lower bound for the Gateaux derivative of the Dirichlet-to-Neumann map under mesh deformation (Section 5).

Notation. We use the Fourier transform convention,

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x) e^{ix\cdot\xi} dx.$$

If the function f is defined on a subset of $\mathbb{R}^3$, it is extended to $\mathbb{R}^3$ attaining the value zero. We denote by $\check{f}$ the inverse Fourier transform of f,

$$\check{f}(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} f(\xi) e^{-ix\cdot\xi} d\xi. \tag{1.1}$$

We introduce coordinates, $x = (x', x_3)$, in $\mathbb{R}^3$, where $x' \in \mathbb{R}^2$ and $x_3 \in \mathbb{R}$. We denote the open ball in $\mathbb{R}^3$ centered at x of radius r by $B_r(x)$, and the open ball in $\mathbb{R}^2$ centered at x' of radius r by $B'_r(x')$.

2. Assumptions and main result.

We let Ω be a bounded domain in $\mathbb{R}^3$ such that $\mathbb{R}^3 \setminus \Omega$ is connected,

$$\Omega \subset B_R(0) \text{ for some } R > 0, \tag{2.1}$$

and

$$\Omega \text{ has a Lipschitz boundary with constants } r_0 \text{ and } K_0, \tag{2.2}$$

that is, for any point $P \in \partial\Omega$, there exists a rigid transformation of coordinates under which $P = 0$ and

$$\Omega \cap \{(x', x_3) \in \mathbb{R}^3 : |x'| < r_0, |x_3| < K_0 r_0\} = \{(x', x_3) : |x'| < r_0, \ x_3 > \psi(x')\},$$

where ψ is a Lipschitz continuous (level set) function in B'_{r_0} such that

$$\psi(0) = 0 \text{ and } \|\nabla\psi\|_{L^\infty(B'_{r_0})} \le K_0.$$

We consider the boundary value problem for the Helmholtz equation,

$$\begin{cases} \Delta u + \omega^2 q u &= 0 \text{ in } \Omega, \\ u &= \phi \text{ on } \partial\Omega \end{cases} \tag{2.3}$$

for $\phi \in H^{1/2}(\partial\Omega)$, and introduce the Dirichlet-to-Neumann map

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega) \tag{2.4}$$

according to

$$\phi \to \Lambda_q(\phi) := \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}. \tag{2.5}$$

The normal derivative is defined in the weak sense as

$$\left\langle \frac{\partial u}{\partial \nu}, \psi|_{\partial\Omega} \right\rangle = \int_\Omega (\nabla u \cdot \nabla \psi - \omega^2 q u \psi) dx$$

for every $\psi \in H^1(\Omega)$. In the above, $q \in L^\infty(\partial\Omega)$ is identified with c^{-2} where c denotes the wavespeed. The solution of (2.3) exists in $H^1(\Omega)$ and is unique if ω is not in the Dirichlet spectrum of $q^{-1}\Delta$ on Ω.

We introduce ω_0, ω_1 such that $0 < \omega_0 < \omega_1$ and

$$\omega_1 \leq \sqrt{\frac{\lambda_1(B_R)}{2Q_0}}, \tag{2.6}$$

where $\lambda_1(B_R)$ is the first eigenvalue of $-\Delta$ on B_R. We recall that $\lambda_1(B_R) = \lambda_1(B_1)R^{-2}$. (If we detect the spectrum, we substitute the true first eigenfrequency for ω_1.) We then assume that

$$\omega_0 \leq \omega \leq \omega_1. \tag{2.7}$$

Unstructured tetrahedral mesh. We let $\{T_j\}_{j=1}^N$ be a regular partition of Ω into tetrahedra, namely a collection of closed tetrahedra such that

$$\overline{\Omega} = \cup_{j=1}^N T_j; \tag{2.8}$$

(2.9) for $j \neq k$ either $T_j \cap T_k = \emptyset$ or it consists of a common vertex, a common edge or a common facet;

(2.10) the radius of the insphere of each tetrahedron is larger than $r_1 > 0$.

We say that two different tetrahedra of such regular partition are adjacent if they share a common facet.

Remark 1. *Assumption* (2.10)*, together with* (2.1) *implies that the tetrahedra of the partition are not degenerate. In particular, there are two positive numbers d_1 and α_1 (depending on R and r_1 only) such that*

(2.11) *for each T_j the distance between vertices is greater than d_1 and internal angles of triangular facets are greater than α_1.*

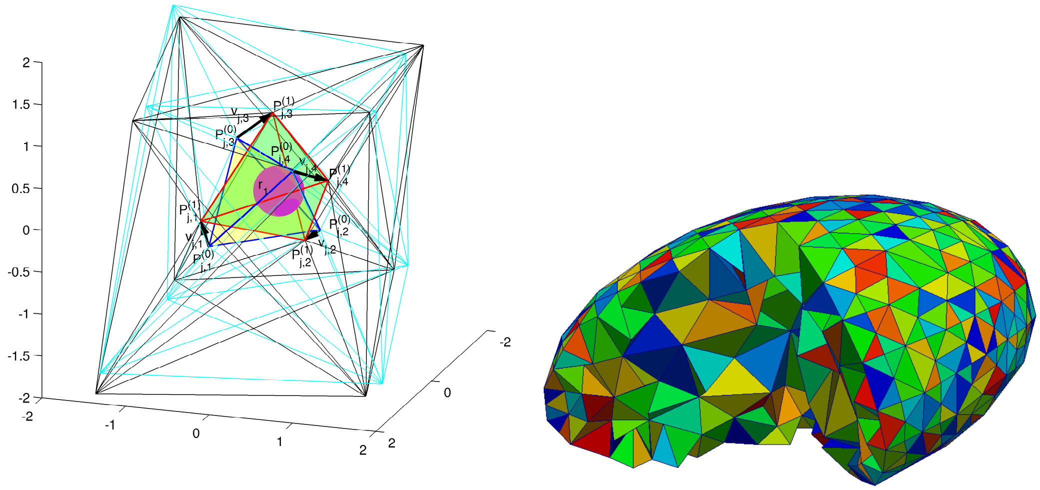

Figure 1: Left: Quantities associated with the assumptions, and deformation of the mesh (cf. (4.4)) Right: An example model, containing polyhedral interfaces, in the 'stable' class.

Indeed, we point out that assumptions (2.10) and (2.1) are equivalent to the following

Assumption 1. *There exists a positive constant C_1 such that*

$$|B_r(P) \cap T_j| \geq C_1 r^3, \tag{2.12}$$

for every $j = 1, \dots, N$, every $P \in T_j$, and $r \leq r_1$.

We show an illustration of a typical model and the assumptions pertaining to the mesh in Figure 1.

We introduce a finite set of numbers,

$$\mathcal{Q} = \{\tilde{q}_1, \dots, \tilde{q}_L\}$$

representing the possible values which the wavespeed can attain in the domain Ω,

$$Q_0 = \max\{|\tilde{q}_j| \,:\, j = 1, \dots, L\}, \tag{2.13}$$

and

$$c_0 = \min\{|\tilde{q}_j - \tilde{q}_k| \,:\, j, k = 1, \dots, L,\, j \neq k\}. \tag{2.14}$$

Assumption 2. *The potentials are piecewise constant and of the form*

$$q(x) = \sum_{j=1}^{N} q_j \chi_{T_j}(x) \tag{2.15}$$

such that $\{T_j\}_{j=1}^N$ is a regular partition of Ω with

$$N \leq N_0 \tag{2.16}$$

for some N_0,

$$q_j \in \mathcal{Q} \text{ for every } j = 1, \dots, N, \tag{2.17}$$

and

$$q_j \neq q_k \text{ if } T_j \text{ is adjacent to } T_k. \tag{2.18}$$

We denote by $\|\cdot\|_\star$ the norm in $\mathcal{L}\left(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\right)$ defined by

$$\|T\|_\star = \sup\{\langle T\phi, \psi\rangle \,:\, \phi, \psi \in H^{1/2}(\partial\Omega),\ \|\psi\|_{H^{1/2}(\partial\Omega)} = \|\phi\|_{H^{1/2}(\partial\Omega)} = 1\}.$$

We refer to the values of R, r_0, K_0, r_1, Q_0, c_0, ω_0, ω_1 and N_0 as to the *a priori data*. In the sequel we will introduce a number of constants that we will always denote by C and, unless otherwise stated, will depend on a priori data only. The values of these constants might differ from one line to the other.

We state the main result

Theorem 2.1. *Given a domain* Ω *satisfying* (2.1) *and* (2.2), *a set of values* $\mathcal{Q}$, *and* $\omega \in [\omega_0, \omega_1]$, *there exist two positive constants* ε_0 *and* C_0 *depending on the a priori data and on* N_0 *only such that, for every pair of potentials*

$$q^{(0)} = \sum_{j=1}^{N} q_j^{(0)} \chi_{T_j^{(0)}} \text{ and } q^{(1)} = \sum_{k=1}^{M} q_k^{(1)} \chi_{T_k^{(1)}} \tag{2.19}$$

satisfying Assumptions 1 and 2, if

$$\|\Lambda_{q^{(0)}} - \Lambda_{q^{(1)}}\|_\star \leq \varepsilon_0, \tag{2.20}$$

then

$$N = M \tag{2.21}$$

and the order of the tetrahedra can be rearranged so that for every $j = 1, \dots, N$ *we have*

$$q_j^{(0)} = q_j^{(1)}, \tag{2.22}$$

and

$$d_{\mathcal{H}}(T_j^{(0)}, T_j^{(1)}) \leq C_0 \|\Lambda_{q^{(0)}} - \Lambda_{q^{(1)}}\|_\star, \tag{2.23}$$

where $d_{\mathcal{H}}$ *denotes the Hausdorff distance.*

3. A rough stability estimate. We begin with developing a rough stability estimate for the recovery of the potential or wavespeed.

Theorem 3.1. *Given* Ω, $q^{(0)}$, $q^{(1)}$ *and* ω *as in Theorem 2.1, there exist two positive constants* $\varepsilon_1 < 1$ *and* C_2 *depending on* R, r_0, K_0, Q_0, ω_0, ω_1 *such that, for* $\|\Lambda_{q^{(0)}} - \Lambda_{q^{(1)}}\|_\star < \varepsilon_1$,

$$\|q^{(0)} - q^{(1)}\|_{L^2(\Omega)} \leq C_2 \sqrt{N_0}\, \left|\log\left(\|\Lambda_{q^{(0)}} - \Lambda_{q^{(1)}}\|_\star\right)\right|^{-1/7}. \tag{3.1}$$

Proof. We proceed as in [5]. Alessandrini's identity states that

$$\omega^2 \int_\Omega (q^{(0)} - q^{(1)}) u_0 u_1 dx = \langle (\Lambda_0 - \Lambda_1)(u_0|_{\partial\Omega}), u_1|_{\partial\Omega} \rangle \tag{3.2}$$

for every pair of functions u_0 and u_1 such that

$$\Delta u_k + \omega^2 q^{(k)} u_k = 0 \text{ in } \Omega \text{ for } k = 0, 1,$$

where we use the shorthand notation, $\Lambda_k = \Lambda_{q^{(k)}}$.

We fix $\xi \in \mathbb{R}^3$ and let η_1 and η_2 be unit vectors in $\mathbb{R}^3$ such that $\{\xi, \eta_2, \eta_2\}$ is an orthogonal set of vectors. We let $\mu > 0$ be a parameter to be chosen later, and set, for $k = 0, 1$,

$$\zeta_k = \begin{cases} (-1)^{k+1} \frac{\mu}{\sqrt{2}} \left(\sqrt{1 - \frac{|\xi|^2}{2\mu^2}}\ \eta_1 + \frac{(-1)^k}{\sqrt{2}\mu}\, \xi + i\, \eta_2 \right) & \text{if} \quad \frac{|\xi|}{\mu\sqrt{2}} < 1, \\ (-1)^{k+1} \frac{\mu}{\sqrt{2}} \left(\frac{(-1)^k}{\sqrt{2}\mu}\, \xi + i \sqrt{\frac{|\xi|^2}{2\mu^2} - 1}\ \eta_1 + \eta_2 \right) & \text{if} \quad \frac{|\xi|}{\mu\sqrt{2}} \geq 1. \end{cases} \tag{3.3}$$

As can be easily checked,

$$\zeta_0 + \zeta_1 = \xi,$$

$$\zeta_k \cdot \zeta_k = 0 \text{ for } k = 0, 1$$

and

$$|\zeta_k| = \max\left\{ \mu, \frac{|\xi|}{\sqrt{2}} \right\}. \tag{3.4}$$

We use here complex geometrical optics (CGO) solutions of the Helmholtz equation and, in particular, the estimates in [17, Theorem 3.8] which are due to [9]. For $|\zeta_k| \geq \max\{{\omega_1}^2 Q_0, 1\} =: c_1$, there is a solution u_k of

$$\Delta u_k + \omega^2 q^{(k)} u_k = 0 \text{ in } \Omega$$

of the form

$$u_k(x) = e^{ix\cdot\zeta_k}(1 + \varphi_k(x)), \tag{3.5}$$

with

$$\begin{aligned} \|\varphi_k\|_{L^2(\Omega)} &\leq \frac{C{\omega_1}^2 Q_0}{|\zeta_k|} \leq \frac{C{\omega_1}^2 Q_0}{\mu}, \\ \|\nabla\varphi_k\|_{L^2(\Omega)} &\leq C{\omega_1}^2 Q_0, \end{aligned} \tag{3.6}$$

where $C = C(R)$.

Inserting (3.5) into (3.2), we get

$$\begin{aligned} \omega^2 \left| (\widehat{q}^{(0)} - \widehat{q}^{(1)})(\xi) \right| &\leq |\langle (\Lambda_0 - \Lambda_1)(u_0|_{\partial\Omega}), u_1|_{\partial\Omega} \rangle| \\ &\quad + \omega^2 \left| \int_\Omega (q^{(0)}(x) - q^{(1)}(x)) e^{i\xi\cdot x} (\varphi_0(x) + \varphi_1(x) + \varphi_0(x)\varphi_1(x)) dx \right| \\ &\leq \|\Lambda_0 - \Lambda_1\|_\star \|u_0\|_{H^1(\Omega)} \|u_1\|_{H^1(\Omega)} + 2\omega^2 Q_0 \left| \int_\Omega (\varphi_0 + \varphi_1 + \varphi_0\varphi_1) dx \right|. \end{aligned}$$

Hence,

$$
\begin{aligned}
&\left|(\widehat{q}^{(0)}-\widehat{q}^{(1)})(\xi)\right|^2\\
&\leq \frac{2}{{\omega_0}^4}\|\Lambda_0-\Lambda_1\|_\star^2\|u_0\|_{H^1(\Omega)}^2\|u_1\|_{H^1(\Omega)}^2+8Q_0^2\left|\int_\Omega(\varphi_0+\varphi_1+\varphi_0\varphi_1)dx\right|^2\\
&\leq \frac{2}{{\omega_0}^4}\|\Lambda_0-\Lambda_1\|_\star^2\|u_0\|_{H^1(\Omega)}^2\|u_1\|_{H^1(\Omega)}^2+8Q_0^2|\Omega|\left(\|\varphi_0\|_{L^2(\Omega)}+\|\varphi_1\|_{L^2(\Omega)}\right)\\
&\quad+8B_0^2\|\varphi_0\|_{L^2(\Omega)}\|\varphi_1\|_{L^2(\Omega)}.
\end{aligned}
$$

With (3.5) and (3.6) we find that there exists a constant c_2 depending only on R such that, for $\mu > c_2$,

$$
\|u_k\|_{H^1(\Omega)}\leq Ce^{2R(\mu+|\xi|)}, \tag{3.7}
$$

$k=0,1$, where $C=C(R,\omega_1,Q_0)$. Hence,

$$
\left|(\widehat{q}^{(0)}-\widehat{q}^{(1)})(\xi)\right|^2\leq C\left(e^{8R(\mu+|\xi|)}\|\Lambda_0-\Lambda_1\|_\star^2+\frac{1}{\mu^2}\right), \tag{3.8}
$$

where $C=C(R,\omega_0,\omega_1,Q_0)$. But then, for $\mu\geq\max(c_1,c_2)$,

$$
\begin{aligned}
\|q^{(0)}-q^{(1)}\|_{L^2(\Omega)}^2&=\int_{|\xi|\leq\rho}\left|(\widehat{q}^{(0)}-\widehat{q}^{(1)})(\xi)\right|^2d\xi+\int_{|\xi|>\rho}\left|(\widehat{q}^{(0)}-\widehat{q}^{(1)})(\xi)\right|^2d\xi\\
&\leq C\rho^3\left(e^{8R(\mu+\rho)}\|\Lambda_0-\Lambda_1\|_\star^2+\frac{1}{\mu^2}\right)\\
&\quad+\int_{|\xi|>\rho}\left|(\widehat{q}^{(0)}-\widehat{q}^{(1)})(\xi)\right|^2d\xi.
\end{aligned} \tag{3.9}
$$

To estimate the integral in (3.9) we show that for every $s\in(0,1/2)$

$$
\|q^{(0)}-q^{(1)}\|_{H^s(\Omega)}^2\leq C\sqrt{N_0}, \tag{3.10}
$$

where $C=C(R,r_0,Q_0)$. Indeed, by [11] we have

$$
\begin{aligned}
&\|q^{(0)}-q^{(1)}\|_{H^s(\Omega)}^2\leq 2\left(\|q^{(0)}\|_{H^s(\Omega)}^2+\|q^{(1)}\|_{H^s(\Omega)}^2\right)\\
&\leq 2\left(\sum_{j=1}^N|q_j^{(0)}|^2|T_j^{(0)}|^{1-2s}|\partial T_j^{(0)}|^{2s}+\sum_{k=1}^M|q_k^{(1)}|^2|T_k^{(1)}|^{1-2s}|\partial T_k^{(1)}|^{2s}\right)\\
&\leq CN_0,
\end{aligned}
$$

where $C=C(R,r_0,Q_0)$.

Using (3.10),

$$
\begin{aligned}
\int_{|\xi|>\rho}\left|\left(\widehat{q}^{(0)}-\widehat{q}^{(1)}\right)(\xi)\right|^2d\xi&\leq\frac{1}{\rho^{2s}}\int_{|\xi|>\rho}(1+|\xi|^s)^2\left|\left(\widehat{q}^{(0)}-\widehat{q}^{(1)}\right)(\xi)\right|^2d\xi\\
&\leq\frac{1}{\rho^{2s}}\|q^{(0)}-q^{(1)}\|_{H^s(\Omega)}^2\leq\frac{CN_0}{\rho^{2s}}.
\end{aligned} \tag{3.11}
$$

Finally, by inserting (3.11) into (3.9), we get that

$$
\|q^{(0)}-q^{(1)}\|_{L^2(\Omega)}^2\leq CN_0\left\{\rho^3\left(e^{8R(\mu+\rho)}\|\Lambda_0-\Lambda_1\|_\star^2+\frac{1}{\mu^2}\right)+\frac{1}{\rho^{2s}}\right\},
$$

where $C = C(R, r_0, \omega_0, \omega_1, Q_0)$. We then choose

$$\rho = \mu^{\frac{2}{3+2s}},$$

and observe that there is a constant c_3 depending only on R such that, for $\mu \geq c_3$,

$$\rho^3 e^{8R(\mu+\rho)} \leq e^{18R\mu}$$

so that

$$\|q^{(0)} - q^{(1)}\|^2_{L^2(\Omega)} \leq CN_0 \left(e^{18R\mu} \|\Lambda_0 - \Lambda_1\|^2_\star + \frac{1}{\mu^{\frac{4s}{3+2s}}} \right),$$

where $C = C(R, r_0, \omega_0, \omega_1, Q_0)$.

We now take

$$\mu = \frac{1}{18R} |\log \|\Lambda_0 - \Lambda_1\|_\star|$$

and assume that

$$\|\Lambda_0 - \Lambda_1\|_\star \leq e^{-18Rc_3} =: \varepsilon_1$$

so that $\mu \geq \max\{c_1, c_2, c_3\}$. Then

$$\|q^{(0)} - q^{(1)}\|^2_{L^2(\Omega)} \leq CN_0 \left(\|\Lambda_0 - \Lambda_1\|^2_\star + \left| \log \|\Lambda_0 - \Lambda_1\|_\star \right|^{-\alpha} \right),$$

where $\alpha = \frac{2s}{3+2s}$. The claim follows upon choosing $s = \frac{1}{4}$. □

Next, we establish an estimate for the Haussdorff distance between two domain partitions in terms of the difference of potentials defined on these partitions.

Proposition 3.2. *Given Ω, $q^{(0)}$ and $q^{(1)}$ as in Theorem 2.1, there exists a positive constant σ_1 depending on R, r_1, Q_0 and c_0 such that, if*

$$\|q^{(0)} - q^{(1)}\|_{L^2(\Omega)} \leq \sigma_1 \tag{3.12}$$

then

$$N = M \tag{3.13}$$

and the order of the tetrahedra can be rearranged so that for every $j = 1, \dots, N$

$$q_j^{(0)} = q_j^{(1)} \tag{3.14}$$

and

$$d_{\mathcal{H}}(T_j^{(0)}, T_j^{(1)}) \leq \frac{\|q^{(0)} - q^{(1)}\|^{2/3}_{L^2(\Omega)}}{(c_0^2 C_1)^{1/3}}, \tag{3.15}$$

where c_0 is given by (2.14) and C_1 by (2.12).

Proof. We write

$$\sigma = \|q^{(0)} - q^{(1)}\|_{L^2(\Omega)}. \tag{3.16}$$

For every $l \in \{1, \ldots, L\}$ we let

$$\mathcal{B}_l^{(0)} = \left\{ j \in \{1, \ldots, N\} \,:\, q_j^{(0)} = \tilde{q}_l \right\} \tag{3.17}$$

and

$$\mathcal{B}_l^{(1)} = \left\{ k \in \{1, \ldots, M\} \,:\, q_k^{(1)} = \tilde{q}_l \right\}. \tag{3.18}$$

We note that

$$\|q^{(0)} - q^{(1)}\|^2_{L^2(\Omega)} = \sum_{l=1}^{L} \left(\sum_{j \in \mathcal{B}_l^{(0)}} \sum_{k \notin \mathcal{B}_l^{(1)}} \left| q_j^{(0)} - q_k^{(1)} \right|^2 \left| T_j^{(0)} \cap T_k^{(1)} \right| \right). \tag{3.19}$$

If $j \in \mathcal{B}_l^{(0)}$ and $k \notin \mathcal{B}_l^{(1)}$ then, by (2.14),

$$\left| q_j^{(0)} - q_k^{(1)} \right| \geq c_0;$$

hence, by (3.19) and (3.16), we have

$$\sigma^2 \geq c_0^2 \sum_{l=1}^{L} \sum_{j \in \mathcal{B}_l^{(0)}} \sum_{k \notin \mathcal{B}_l^{(1)}} \left| T_j^{(0)} \cap T_k^{(1)} \right| \tag{3.20}$$

so that

$$\left| T_j^{(0)} \cap T_k^{(1)} \right| \leq \frac{\sigma^2}{c_0^2} \text{ for every } j, k \text{ such that } q_j^{(0)} \neq q_k^{(1)}. \tag{3.21}$$

By assumption (2.11), estimate (3.21) implies that $T_j^{(0)} \cap T_k^{(1)}$ is close to $\partial T_j^{(0)}$. To make this precise, we introduce

$$T_{j,\delta}^{(0)} = \left\{ x \in T_j^{(0)} \,:\, d(x, \partial T_j^{(0)}) > \delta \right\}$$

and prove that

$$T_k^{(1)} \cap T_{j,\delta_\sigma}^{(0)} = \emptyset \tag{3.22}$$

with

$$\delta_\sigma = \left(\frac{\sigma^2}{c_0^2 C_1} \right)^{1/3}. \tag{3.23}$$

Indeed, assume that $j \in \mathcal{B}_l^{(0)}$ for some $l \in \{1, \ldots, N\}$, $k \notin \mathcal{B}_l^{(1)}$ and that there is a point $P \in T_j^{(0)} \cap T_k^{(1)}$ such that

$$d(P, \partial T_j^{(0)}) \geq \delta, \tag{3.24}$$

that is, $B_\delta(P) \subset T_j^{(0)}$. Using assumption (2.11) and (2.12) in Remark 1, it then follows that

$$\left| T_j^{(0)} \cap T_k^{(1)} \right| \geq \left| B_\delta(P) \cap T_k^{(1)} \right| \geq C_1 \delta^3 \tag{3.25}$$

if $\delta < r_1$. By (3.21)

$$C_1 \delta^3 \leq \frac{\sigma^2}{c_0^2}. \tag{3.26}$$

Thus (3.22) holds provided that

$$\delta_\sigma = \left(\frac{\sigma^2}{c_0^2 C_1}\right)^{1/3} \leq r_1,$$

that is,

$$\sigma \leq \sigma_1 = \sqrt{r_1^3 c_0^2 C_1}. \tag{3.27}$$

Now we consider $T^{(0)}_{j,\delta_\sigma}$ for $\sigma \leq \sigma_1$ and $j \in \mathcal{B}^{(0)}_l$ for some l. Since $\{T^{(1)}_k\}_k$ is a partition of Ω, we can write

$$\begin{aligned} T^{(0)}_{j,\delta_\sigma} &= T^{(0)}_{j,\delta_\sigma} \cap \left(\bigcup_{k=1}^{M} T^{(1)}_k\right) \\ &= \bigcup_{k=1}^{M} \left(T^{(0)}_{j,\delta_\sigma} \cap T^{(1)}_k\right). \end{aligned}$$

Using (3.22),

$$T^{(0)}_{j,\delta_\sigma} \cap T^{(1)}_k = \emptyset \text{ for } k \notin \mathcal{B}^{(1)}_l,$$

and we then obtain

$$T^{(0)}_{j,\delta_\sigma} = \bigcup_{k \in \mathcal{B}^{(1)}_l} \left(T^{(0)}_{j,\delta_\sigma} \cap T^{(1)}_k\right). \tag{3.28}$$

If k_1 and $k_2 \in \mathcal{B}^{(1)}_l$, then $T^{(1)}_{k_1}$ and $T^{(1)}_{k_2}$ cannot be adjacent by assumption (2.18). This means that there is a unique $k \in \mathcal{B}^{(1)}_l$ such that

$$T^{(0)}_{j,\delta_\sigma} \cap T^{(1)}_k \neq \emptyset \tag{3.29}$$

and, with (3.28),

$$T^{(0)}_{j,\delta_\sigma} = T^{(0)}_{j,\delta_\sigma} \cap T^{(1)}_k \subset T^{(1)}_k.$$

Thus we proved that for every $j \in \{1, \ldots, N\}$ there is a unique index $\overline{k}(j) \in \{1, \ldots, M\}$ such that

$$q^{(0)}_j = q^{(1)}_{\overline{k}(j)} \tag{3.30}$$

and

$$T^{(0)}_{j,\delta_\sigma} \subset T^{(1)}_{\overline{k}(j)}. \tag{3.31}$$

In particular, this implies that $M \geq N$.

By interchanging the roles of $q^{(0)}$ and $q^{(1)}$ it follows that $M = N$, $\overline{k}$ is a permutation on $\{1, \ldots, N\}$ and

$$T^{(0)}_{j,\delta_\sigma} \subset T^{(1)}_{\overline{k}(j)} \text{ and } T^{(1)}_{\overline{k}(j),\delta_\sigma} \subset T^{(0)}_j$$

that, by (3.23), gives (3.15). □

Combining Theorem 3.1 and Proposition 3.2, we obtain the following logarithmic stability estimate

Corollary 3.3. *Under the assumptions of Theorem 3.1, there is a constant $\varepsilon_2 < 1$ depending only on the a priori data such that, if*

$$\|\Lambda_{q^{(0)}} - \Lambda_{q^{(1)}}\|_\star \leq \varepsilon_2$$

then

$$N = M$$

and the order of tetrahedra can be rearranged so that

$$q^{(0)}_j = q^{(1)}_j$$

and

$$d_{\mathcal{H}}(T^{(0)}_j, T^{(1)}_j) \leq \left(\frac{C_2^2 N_0}{c_0^2 C_1}\right)^{1/3} \left|\log\left(\|\Lambda_{q^{(0)}} - \Lambda_{q^{(1)}}\|_\star\right)\right|^{-2/21}. \tag{3.32}$$

4. Geometric estimates, construction of an intermediate partition and augmenting the domain. Here, we map the information on the Haussdorff distance of tetrahedra in information on the distance between vertices of these tetrahedra. It is straightforward to see that if $T^{(k)}$, $k = 0, 1$, are tetrahedra generated by vertices $P^{(k)}_i$, $i = 1, 2, 3, 4$, that then

$$d_{\mathcal{H}}(T^{(0)}, T^{(1)}) \leq \min_{\wp} \max_{1\leq i\leq 4} \left|P^{(0)}_i - P^{(1)}_{\wp(i)}\right|, \tag{4.1}$$

where $\wp$ denotes a permutation on the set $\{1, 2, 3, 4\}$. Moreover, if $T^{(k)} \subset B_R(0)$ and satisfies assumption (2.10) for $k = 0, 1$, then there exists a positive constant A_1, depending on R and r_1 only, such that

$$\min_{\wp} \max_{1\leq i\leq 4} \left|P^{(0)}_i - P^{(1)}_{\wp(i)}\right| \leq A_1 d_{\mathcal{H}}(T^{(0)}, T^{(1)}). \tag{4.2}$$

Using Corollary 3.3 we then obtain

Proposition 4.1. *Under the assumptions of Theorem 3.1, there is a positive constant $\varepsilon_3 < 1$ such that if*

$$\|\Lambda_{q^{(0)}} - \Lambda_{q^{(1)}}\|_\star \leq \varepsilon_3$$

then for every vertex $P^{(0)}_{j,i}$ of $T^{(0)}_j$ (with $i = 1, 2, 3, 4$) there is a unique vertex $P^{(1)}_{j,i}$ of $T^{(1)}_j$ such that

$$d(P^{(0)}_{j,i}, P^{(1)}_{j,i}) \leq \frac{d_1}{4} \tag{4.3}$$

for d_1 as in 2.11.

Proof. It is sufficient to consider $\varepsilon_3 < 1$, such that

$$A_1 \left(\frac{C_2^2 N_0}{c_0^2 C_1} \right)^{1/3} |\log(\varepsilon_3)|^{-2/21} < \frac{d_1}{4},$$

and the statement follows. □

We introduce a deformation of the tetrahedra forming the partition of Ω. To this end, for each $j \in \{1, \dots, N\}$, we define tetrahedra $T_j^{(t)}$ by its vertices,

$$P_{j,i}^{(t)} = P_{j,i}^{(0)} + t v_{j,i} \text{ for } t \in [0,1], \tag{4.4}$$

where

$$v_{j,i} = P_{j,i}^{(1)} - P_{j,i}^{(0)}. \tag{4.5}$$

The resulting partition $\{T_j^{(t)}\}_j$ is a regular partition of Ω satisfying condition (2.10). We point out that, by (4.1) and (4.2), there is a positive constant $A_2 > 1$ such that

$$A_2^{-1} \left(\sum_{i=1}^{4} |v_{j,i}|^2 \right)^{1/2} \leq d_{\mathcal{H}} \left(T_j^{(0)}, T_j^{(1)} \right) \leq A_2 \left(\sum_{i=1}^{4} |v_{j,i}|^2 \right)^{1/2}. \tag{4.6}$$

We define

$$q^{(t)} = \sum_{j=1}^{N} q_j \chi_{T_j^{(t)}},$$

where we denoted by $q_j = q_j^{(0)} = q_j^{(1)}$. A suggestion of Alessandrini ([1]) allows us to avoid the assumption thet q is known on $\partial\Omega$. To this aim we extend our domain and introduce a regular domain $\tilde{\Omega}$ containing Ω; we extend each potential $q^{(t)}$, for $t \in [0,1]$, to $\tilde{\Omega}$ with the same constant value, $\tilde{q}_0$. The particular choice of value $\tilde{q}_0$ for this extension does not matter, as long as we are able to ensure well-posedness of the corresponding Dirichlet problem. For this reason we choose a special value. We take $\tilde{R} = \frac{2}{\sqrt{3}} R$, so that

$$\lambda_1(B_{\tilde{R}}) = \frac{3}{4} \lambda_1(B_R), \tag{4.7}$$

and choose

$$\tilde{\Omega} = B_{\tilde{R}}(0). \tag{4.8}$$

We then define

$$\tilde{q}^{(t)} = \tilde{q}_0 + (q^{(t)} - \tilde{q}_0)\chi_\Omega \text{ for } t \in [0,1], \tag{4.9}$$

with $\tilde{q}_0 = Q_0$ (cf. (2.13)). For $\omega \leq \omega_1$ and $t \in [0,1]$, we have

$$\left| \omega^2 \tilde{q}^{(t)} \right| \leq {\omega_1}^2 Q_0 \leq \frac{1}{2} \lambda_1(B_R) = \frac{2}{3} \lambda_1(\tilde{\Omega}),$$

cf. (4.7) and (2.6), whence the Dirichlet problem

$$\begin{cases} \Delta u + \omega^2 \tilde{q}^{(t)} u &= 0 \text{ in } \tilde{\Omega}, \\ u &= \phi \text{ on } \partial\tilde{\Omega}, \end{cases} \tag{4.10}$$

has a unique solution $u \in H^1(\tilde\Omega)$ for every $\phi \in H^{1/2}(\partial\tilde\Omega)$. Thus the one-parameter family of Dirichlet-to-Neumann maps,

$$\tilde\Lambda_t = \Lambda_{\tilde q^{(t)}}, \text{ for } t \in [0,1] \tag{4.11}$$

is well defined in $\mathcal{L}(H^{1/2}(\partial\tilde\Omega), H^{-1/2}(\partial\tilde\Omega))$. We denote the norm in this space by $\|T\|_{\tilde\star}$.

To proceed, we take $\phi, \psi \in H^{1/2}(\partial\tilde\Omega)$ and let $\tilde u_0$ and $\tilde u_1$ be the solutions to

$$\begin{cases} \Delta\tilde u_0 + \omega^2\tilde q^{(0)}\tilde u_0 &= 0 \text{ in } \tilde\Omega, \\ \tilde u_0 &= \phi \text{ on } \partial\tilde\Omega, \end{cases} \quad\text{and}\quad \begin{cases} \Delta\tilde u_1 + \omega^2\tilde q^{(1)}\tilde u_1 &= 0 \text{ in } \tilde\Omega, \\ \tilde u_1 &= \psi \text{ on } \partial\tilde\Omega. \end{cases}$$

We then use Alessandrini's identity and write

$$\begin{aligned}&\langle(\tilde\Lambda_1 - \tilde\Lambda_0)(\phi), \psi\rangle = \int_{\tilde\Omega}(\tilde q^{(1)} - \tilde q^{(0)})\tilde u_0\tilde u_1 dx = \int_\Omega (q^{(1)} - q^{(0)})\tilde u_0 \tilde u_1 dx \\ &= \langle(\Lambda_1 - \Lambda_0)(\tilde u_0|_{\partial\Omega}), \tilde u_1|_{\partial\Omega}\rangle \le \|\Lambda_1 - \Lambda_0\|_\star \|\tilde u_0\|_{H^{1/2}(\partial\Omega)}\|\tilde u_1\|_{H^{1/2}(\partial\Omega)}.\end{aligned}$$

Moreover, by trace and regularity estimates, we have

$$\|\tilde u_k\|_{H^{1/2}(\partial\Omega)} \le C\|\tilde u_k\|_{H^1(\tilde\Omega)} \le C\|\tilde u_k\|_{H^{1/2}(\partial\tilde\Omega)} \text{ for } k = 0, 1,$$

where C depends on the a priori data. We have then shown that

$$\|\tilde\Lambda_1 - \tilde\Lambda_0\|_{\tilde\star} \le C_3\|\Lambda_1 - \Lambda_0\|_\star. \tag{4.12}$$

5. Proof of Lipschitz stability. In this section, we give the proof of Lipschitz stability starting from the logarithmic estimate obtained in Corollary 3.3. We split the proof into three steps:

First step. We show that for any pair of functions ϕ and ψ in $H^{1/2}(\partial\tilde\Omega)$, the function

$$\mathcal{F}(t, \phi, \psi) = \langle\tilde\Lambda_t(\phi), \psi\rangle$$

is differentiable.

Second step. We show that there is a positive constant L_1 and a number $\alpha \in (0,1)$ depending on the a-priori data such that for any ϕ and ψ in $H^{1/2}(\partial\tilde\Omega)$,

$$\left|\frac{d}{dt}\mathcal{F}(t,\phi,\psi) - \frac{d}{dt}\mathcal{F}(t,\phi,\psi)_{|t=0}\right| \le L_1 d_T^{1+\alpha}\|\phi\|_{H^{1/2}(\partial\tilde\Omega)}\|\psi\|_{H^{1/2}(\partial\tilde\Omega)}. \tag{5.1}$$

Third step. Finally, we prove that there is a positive constant m_1 such that, for special choices of non-zero functions ϕ_0 and ψ_0, we have

$$\left|\frac{d}{dt}\mathcal{F}(t,\phi_0,\psi_0)_{|t=0}\right| \ge m_1 d_T \|\phi_0\|_{H^{1/2}(\partial\tilde\Omega)}\|\psi_0\|_{H^{1/2}(\partial\tilde\Omega)}. \tag{5.2}$$

Here, $d_T = \sum_{j=1}^N d_{\mathcal{H}}\left(T_j^{(0)}, T_j^{(1)}\right)$.

Once these three steps have been proven we conclude that

$$\begin{aligned}&\left|\langle(\tilde\Lambda_1 - \tilde\Lambda_0)(\phi_0), \psi_0\rangle\right| = |\mathcal{F}(1,\phi_0,\psi_0) - \mathcal{F}(0,\phi_0,\psi_0)| = \left|\int_0^1 \frac{d}{dt}\mathcal{F}(t,\phi_0,\psi_0)\right| \\ &\ge \left|\frac{d}{dt}\mathcal{F}(t,\phi_0,\psi_0)_{|t=0}\right| - \int_0^1\left|\frac{d}{dt}\mathcal{F}(t,\phi_0,\psi_0) - \frac{d}{dt}\mathcal{F}(t,\phi_0,\psi_0)_{|t=0}\right| \\ &\ge \|\phi_0\|_{H^{1/2}}\|\psi_0\|_{H^{1/2}} d_T\left(m_1 - L_1 d_T^\alpha\right),\end{aligned}$$

that is,

$$\|\tilde{\Lambda}_1 - \tilde{\Lambda}_0\|_\star \geq d_T \left(m_1 - L_1 d_T^\alpha\right). \tag{5.3}$$

By Corollary 3.3, there exists a positive constant $\varepsilon_0 \leq \varepsilon_3$ such that, if

$$\|\Lambda_1 - \Lambda_0\|_\star \leq \varepsilon_0$$

then

$$(m_1 - L_1 d_T^\alpha) \geq \frac{m_1}{2}$$

and, hence, by (4.12)

$$d_T \leq \frac{m_1}{2}\|\tilde{\Lambda}_1 - \tilde{\Lambda}_0\|_{\tilde{\star}} \leq \frac{m_1 C_3}{2}\|\Lambda_1 - \Lambda_0\|_\star,$$

which implies (2.23).

5.1. First step: Differentiability of $\mathcal{F}(t, \phi, \psi)$. Let $\phi, \psi \in H^{1/2}(\partial\tilde{\Omega})$ and let $t_0 \in [0, 1]$. For $h \neq 0$ such that $t_0 + h \in [0, 1]$ we introduce the finite difference

$$R(h) = \frac{1}{h}\left(\mathcal{F}(t_0 + h, \phi, \psi) - \mathcal{F}(t_0, \phi, \psi)\right). \tag{5.4}$$

For $t \in [0, 1]$ fixed, we let $u(x; t)$ and $v(x; t)$ be the (unique) solutions in $H^1(\tilde{\Omega})$ to the boundary value problems,

$$\begin{cases} \Delta u(x;t) + \omega^2 \tilde{q}^{(t)}(x) u(x;t) &= 0 \text{ for } x \in \tilde{\Omega}, \\ u(x;t) &= \phi(x) \text{ for } x \in \partial\tilde{\Omega} \end{cases}$$

and

$$\begin{cases} \Delta v(x;t) + \omega^2 \tilde{q}^{(t)}(x) v(x;t) &= 0 \text{ for } x \in \tilde{\Omega}, \\ v(x;t) &= \psi(x) \text{ for } x \in \partial\tilde{\Omega}. \end{cases}$$

Applying Alessandrini's identity and the definition of $\tilde{q}^{(t)}$, we find that

$$\begin{aligned} R(h) &= \frac{\omega^2}{h}\int_\Omega \left(q^{(t_0+h)}(x) - q^{(t_0)}(x)\right) u(x; t_0 + h) v(x; t_0) dx \\ &= \frac{\omega^2}{h}\sum_{j=1}^N q_j \left\{\int_{T_j^{(t_0+h)}} u(x; t_0 + h) v(x; t_0) dx - \int_{T_j^{(t_0)}} u(x; t_0 + h) v(x; t_0) dx\right\}. \end{aligned}$$

For any index $j \in \{1, \ldots, N\}$ we define $\Phi_{j,t_0} : \mathbb{R}^3 \to \mathbb{R}^3$ as the affine map with the property that

$$\Phi_{j,t_0}(P_{j,i}^{(0)} + t_0 v_{j,i}) = v_{j,i} \text{ for } i = 1, 2, 3, 4, \tag{5.5}$$

where $P_{j,i}^{(0)}$ is defined in (4.4) and $v_{j,i}$ in (4.5). We let

$$F_{j,\tau}^{t_0}(x) = x + \tau \Phi_{j,t_0}(x) \tag{5.6}$$

so that $F_{j,\tau}^{t_0}(T_j^{(t_0)}) = T_j^{(t_0+\tau)}$. We note that with assumption (2.11)

$$|\Phi_{j,t_0}| + |\text{div}\, \Phi_{j,t_0}| \leq C(R, r_1). \tag{5.7}$$

By using $F^{t_0}_{j,h}$ as a change of variable, we get

$$R(h) = \frac{\omega^2}{h} \sum_{j=1}^{N} q_j \int_{T_j^{(t_0)}} \mu_j(x,t_0)dx, \tag{5.8}$$

where

$$\mu_j(x,t_0) = u(F^{t_0}_{j,h}(x); t_0+h)v(F^{t_0}_{j,h}(x); t_0)|\det DF^{t_0}_{j,h}(x)| - u(x;t_0+h)v(x;t_0). \tag{5.9}$$

We proceed with the analysis on each tetrahedron $T_j^{(t_0)}$ in the same way and for simplicity of notation drop the index j.

By standard regularity estimates for solutions of elliptic equations, we know that $u(\cdot,t)$ and $v(\cdot,t)$ belong to $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$ and that

$$\|u(\cdot;t)\|_{C^{1,\alpha}(\Omega)} \le C\|\phi\|_{H^{1/2}(\partial\tilde{\Omega})}, \tag{5.10}$$

$$\|v(\cdot,t)\|_{C^{1,\alpha}(\Omega)} \le C\|\psi\|_{H^{1/2}(\partial\tilde{\Omega})}, \tag{5.11}$$

where C depends on the a priori data. Thus,

$$u(F^{t_0}_h(x); t_0+h) - u(x;t_0+h) = h\nabla u(x;t_0+h)\cdot\Phi_{t_0}(x) + \eta_1(h). \tag{5.12}$$

For some ξ between x and $F^{t_0}_h(x) = x + h\Phi_{t_0}(x)$,

$$\begin{aligned}
|\eta_1(h)| &= |h\nabla u(\xi;t_0+h)\cdot\Phi_{t_0}(x) - h\nabla u(x;t_0+h)\cdot\Phi_{t_0}(x)| \\
&\le |h|\|u(\cdot,t_0+h)\|_{C^{1,\alpha}(\Omega)}|\xi - x|^{\alpha}\,|\Phi_{t_0}(x)| \\
&\le C\|\phi\|_{H^{1/2}(\partial\tilde{\Omega})}\,(|h|)^{1+\alpha}\,|\Phi_{t_0}(x)| \\
&\le C\|\phi\|_{H^{1/2}(\partial\tilde{\Omega})}|h|^{1+\alpha},
\end{aligned} \tag{5.13}$$

where we used (5.7) in the last estimate. A similar estimate holds for $v(F^{t_0}_h(x); t_0+h) - v(x;t_0+h)$. Moreover, by direct calculation,

$$\left|\det DF^{t_0}_h(x)\right| = 1 + h\ \mathrm{div}\,(\Phi_{t_0}) + o(h). \tag{5.14}$$

Using (5.9), (5.12), (5.13) and (5.14), we get

$$\mu(x,t_0) = h\ \mathrm{div}\,(u(x;t_0+h)v(x;t_0)\Phi_{t_0}(x)) + \eta(h) \tag{5.15}$$

with

$$|\eta(h)| \le C|h|^{1+\alpha}, \tag{5.16}$$

where C depends on the a priori data and on $\|\phi\|_{H^{1/2}(\partial\tilde{\Omega})}$ and $\|\psi\|_{H^{1/2}(\partial\tilde{\Omega})}$. By inserting estimates (5.15) and (5.16) into (5.8) we obtain

$$R(h) = \omega^2 \sum_{j=1}^{N} q_j \int_{T_j^{(t_0)}} \mathrm{div}\,(u(x;t_0+h)v(x;t_0)\Phi_{j,t_0}(x))\,dx + O(h^{\alpha}). \tag{5.17}$$

Applying usual energy estimates, we find that

$$\|u(\cdot,t_0+h) - u(\cdot,t_0)\|_{H^1(\Omega)} \le C\omega^2\|q^{(t_0+h)} - q^{(t_0)}\|_{L^2(\Omega)}\|\phi\|_{H^{1/2}(\partial\tilde{\Omega})} \tag{5.18}$$

and, hence,

$$\lim_{h\to 0} R(h) = \omega^2 \sum_{j=1}^{N} q_j \int_{T_j^{(t_0)}} \operatorname{div}\left(u(x;t_0)v(x;t_0)\Phi_{j,t_0}(x)\right) dx.$$

This implies that $\mathcal{F}(t,\phi,\psi)$ is differentiable and that

$$\frac{d}{dt}\langle \tilde{\Lambda}_t(\phi),\psi\rangle_{t=t_0} = \omega^2 \sum_{j=1}^{N} q_j \int_{T_j^{(t_0)}} \operatorname{div}\left(u(x;t_0)v(x;t_0)\Phi_{j,t_0}(x)\right) dx. \tag{5.19}$$

Using the divergence theorem, we obtain

$$\frac{d}{dt}\langle \Lambda_t(\phi),\psi\rangle_{t=t_0} = \omega^2 \sum_{j=1}^{N} q_j \int_{\partial T_j^{(t_0)}} u(x;t_0)v(x;t_0)\left(\Phi_{j,t_0}(x)\cdot \nu_j\right) d\sigma_x, \tag{5.20}$$

where ν_j is the exterior normal to $\partial T_j^{(t_0)}$ and $d\sigma_x$ is the surface measure.

5.2. Second step: Behavior of $\frac{d}{dt}\mathcal{F}(t,\phi,\psi)$ with respect to t. In this subsection, we estimate, for any fixed $t\in[0,1]$, the quantity

$$\tilde{J} = \frac{d}{dt}\mathcal{F}(t,\phi,\psi) - \frac{d}{dt}\mathcal{F}(t,\phi,\psi)|_{t=0}.$$

By (5.19), we can write

$$\tilde{J} = \omega^2 \sum_{j=1}^{N} q_j J_j \tag{5.21}$$

where

$$J_j = \int_{T_j^{(t)}} \operatorname{div}\left(u(x;t)v(x;t)\Phi_{j,t}(x)\right) dx - \int_{T_j^{(0)}} \operatorname{div}\left(u(x;0)v(x;0)\Phi_{j,0}(x)\right) dx.$$

We write

$$V_j = \sum_{i=1}^{4} |v_{j,i}|\,. \tag{5.22}$$

Since, here, we focus on each tetrahedron separately, we drop the index j from J_j, $T_j^{(t)}$, $T_j^{(0)}$, $\Phi_{j,t}$, $\Phi_{j,0}$, and V_j, again, for simplicity of notation. We use the change of variable $F_t(x) = F_{j,t}$ as defined in (5.6), and get

$$J = \int_{T^{(0)}} \Big(\operatorname{div}_y\left(u(y;t)v(y;t)\Phi_t(y)\right)_{y=F_t(x)} |\det DF_t(x)| - \operatorname{div}_x\left(u(x;0)v(x;0)\Phi_0(x)\right)\Big) dx$$

We introduce the quantity

$$G(y,t) = \operatorname{div}_y\left(u(y;t)v(y;t)\Phi_t(y)\right),$$

and estimate J,

$$\begin{aligned} J &= \left|\int_{T^{(0)}} \left(G(F_t(x),t)\,|\det DF_t(x)| - G(x,0)\right) dx\right| \\ &\le \int_{T^{(0)}} |G(F_t(x),t) - G(x,0)|\,|\det DF_t(x)|\,dx + \int_{T^{(0)}} |G(x,0)|\,|\det DF_t(x) - 1|\,dx \\ &= J^{(1)} + J^{(2)}, \end{aligned}$$

in which

$$J^{(1)} \le C \left\{ \int_{T^{(0)}} \left| \nabla_y \left(u(y;t)v(y;t) \right)_{|_{y=F_t(x)}} - \nabla \left(u(x;0)v(x;0) \right) \right| |\Phi_0(x)| \, dx \right.$$
$$\left. + \int_{T^{(0)}} \left| u(F_t(x);t)v(F_t(x);t) \left(\operatorname{div} \Phi_t(y) \right)_{|_{y=F_t(x)}} - u(x;0)v(x;0) \left(\operatorname{div} \Phi_0(x) \right) \right| dx \right\},$$

using that $\Phi_t(F_t(x)) = \Phi_0(x)$. A straightforward calculation gives

$$\left(\operatorname{div} \Phi_t(y) \right) |_{y=F_t(x)} = \operatorname{div} \Phi_0(x) - t \operatorname{tr} \left(D\Phi_t(F_t(x)) D\Phi_0(x) \right).$$

Hence, writing

$$w(y;t) = u(y;t)v(y;t)$$

we obtain the estimate

$$J^{(1)} \le C \left\{ \int_{T^{(0)}} |\nabla w(F_t(x);t) - \nabla w(x;0)| \, |\Phi_0(x)| \, dx \right.$$
$$+ \int_{T^{(0)}} |w(F_t(x);t) - w(x;0)| \, |\operatorname{div} \Phi_0(x)| \, dx$$
$$\left. + t \int_{T^{(0)}} |w(F_t(x);t)| \, |\operatorname{tr} \left(D\Phi_t(F_t(x)) D\Phi_0(x) \right)| \, dx \right\}.$$

Using (5.5) and (5.22), we find that

$$|\Phi_t(x)| + |D\Phi_t(x)| \le CV \tag{5.23}$$

and, hence,

$$J^{(1)} \le CV \left\{ \int_{T^{(0)}} |\nabla w(F_t(x);t) - \nabla w(x;0)| + |w(F_t(x);t) - w(x;0)| \, dx \right\}$$
$$+ CV^2 \|\phi\|_{H^{1/2}(\partial\tilde{\Omega})} \|\psi\|_{H^{1/2}(\partial\tilde{\Omega})}.$$

We analyze the term containing ∇w. By combining (5.10), (5.11) and (5.18) and using the fact that $|F_t(x) - x| = t \, |\Phi_t(x)| \le CV$, we obtain

$$\int_{T^{(0)}} |\nabla w(F_t(x);t) - \nabla w(x;0)| \, dx$$
$$\le \int_{T^{(0)}} \left(|\nabla w(F_t(x);t) - \nabla w(x;t)| + |\nabla w(x;t) - \nabla w(x;0)| \right) dx$$
$$\le C \|\phi\|_{H^{1/2}(\partial\tilde{\Omega})} \|\psi\|_{H^{1/2}(\partial\tilde{\Omega})} \left(V^\alpha + \omega^2 \|q^{(t)} - q^{(0)}\|_{L^2(\Omega)} \right).$$

Then, by (2.13), (2.6) and (4.6),

$$\omega^2 \|q^{(t)} - q^{(0)}\|_{L^2(\Omega)} \le C \sum_{j=1}^N V_j$$

and, so,

$$\int_{T^{(0)}} |\nabla w(F_t(x);t) - \nabla w(x;0)| \, dx \le C \|\phi\|_{H^{1/2}(\partial\tilde{\Omega})} \|\psi\|_{H^{1/2}(\partial\tilde{\Omega})} \left(V^\alpha + \sum_{j=1}^N V_j \right).$$

An analogous estimate holds for $\int_{T^{(0)}} |w(F_t(x);t) - \nabla w(x;0)|\,dx$. Finally, by recalling (5.22), we obtain

$$J^{(1)} \le C\|\phi\|_{H^{1/2}}\|\psi\|_{H^{1/2}} \left(\sum_{j=1}^N V_j\right)^{1+\alpha}. \tag{5.24}$$

The integral, $J^{(2)}$, can be estimated in a similar way by observing that, by (5.10), (5.11) and (5.23),

$$|\mathrm{div}\,(u(x,0)v(x,0)\Phi_0(x))| \le C\|\phi\|_{H^{1/2}(\partial\tilde{\Omega})}\|\psi\|_{H^{1/2}(\partial\tilde{\Omega})}V \tag{5.25}$$

and, by (5.6) and (5.23),

$$|\det DF_t(x) - 1| \le CV. \tag{5.26}$$

By combining (5.21), (5.24), (5.25) and (5.26) and adding up the contributions from all the tetrahedra, we get

$$\tilde{J} \le L_1\|\phi\|_{H^{1/2}(\partial\tilde{\Omega})}\|\psi\|_{H^{1/2}(\partial\tilde{\Omega})} \left(\sum_{j=1}^N V_j\right)^{1+\alpha}$$

and, by (5.21), (2.13), (2.6) and (4.6), we finally arrive at estimate (5.1) and conclude the proof of second step.

5.3. Third step: Lower bound of $\frac{d}{dt}\mathcal{F}(t,\phi,\psi)|_{t=0}$. With (5.20) the Gateaux derivative is given by

$$\frac{d}{dt}\mathcal{F}(t,\phi,\psi)|_{t=0} = \omega^2 \sum_{j=1}^N q_j \int_{\partial T_j^{(0)}} u(x)w(x)\left(\Phi_{j,0}(x)\cdot\nu_j\right)d\sigma_x,$$

where u and w solve problems

$$\begin{cases} \Delta u + \omega^2 q^{(0)} u &= 0 \text{ in } \tilde{\Omega}, \\ u &= \phi \text{ on } \partial\tilde{\Omega}. \end{cases}$$

and

$$\begin{cases} \Delta w + \omega^2 q^{(0)} w &= 0 \text{ in } \tilde{\Omega}, \\ w &= \psi \text{ on } \partial\tilde{\Omega}. \end{cases},$$

respectively. We introduce

$$\tilde{v}_{j,i} = \frac{v_{j,i}}{\sum_{j=1}^N V_j} \text{ for } j \in \{1,\dots,N\}, \quad i = 1,2,3,4, \tag{5.27}$$

where V_j is defined as in (5.22), and note that

$$\sum_{j=1}^N \sum_{i=1}^4 |\tilde{v}_{j,i}| = 1 \tag{5.28}$$

We also let

$$\tilde{\Phi}_j(x) = \frac{\Phi_{j,0}(x)}{\sum_{l=1}^N V_l}, \tag{5.29}$$

and consider the bilinear operator

$$\mathcal{G}(\phi,\psi)=\sum_{j=1}^{N} q_j \int_{\partial T_j^{(0)}} u(x)w(x)(\tilde{\Phi}_j(x)\cdot \nu_j) \tag{5.30}$$

in $\mathcal{L}(H^{1/2}(\partial\tilde{\Omega}), H^{-1/2}(\partial\tilde{\Omega}))$. Now, for every ϕ and ψ in $H^{1/2}(\partial\tilde{\Omega})$, we have

$$|\mathcal{G}(\phi,\psi)| \le m_0\|\phi\|_{H^{1/2}}\|\psi\|_{H^{1/2}}, \tag{5.31}$$

where

$$m_0 = \|\mathcal{G}\|_{\tilde{\star}}. \tag{5.32}$$

We choose boundary values corresponding to CGO solutions: Let ξ be any vector in $\mathbb{R}^3$ and let μ be a positive parameter to be chosen later, and let ζ_0 and ζ_1 as in (3.3). We form

$$\tilde{u}_0 = e^{ix\cdot\zeta_0}(1+\varphi_0(x)) \tag{5.33}$$

and

$$\tilde{w}_0 = e^{ix\cdot\zeta_1}(1+\varphi_1(x)), \tag{5.34}$$

which are both solutions of the equation $\Delta u + \omega^2 q^{(0)} u = 0$ in $\tilde{\Omega}$, such that

$$\begin{aligned} \|\varphi_k\|_{L^2(\tilde{\Omega})} &\le \frac{C_0{\omega_1}^2 Q_0}{\mu}, \\ \|\nabla\varphi_k\|_{L^2(\tilde{\Omega})} &\le C_0{\omega_1}^2 Q_0, \end{aligned} \tag{5.35}$$

and $\zeta_0+\zeta_1=\xi$. Substituting these functions into (5.31), by (2.13) and (5.23), we get

$$\begin{aligned} \left|\sum_{j=1}^{N} q_j \int_{\partial T_j^{(0)}} e^{ix\cdot\xi}(\tilde{\Phi}_j(x)\nu_j)\right| &\le m_0\|\phi\|_{H^{1/2}(\partial\tilde{\Omega})}\|\psi\|_{H^{1/2}(\partial\tilde{\Omega})} \\ &\quad + C\sum_{j=1}^{N}\int_{\partial T_j^{(0)}}\left(|\varphi_0|+|\varphi_1|+|\varphi_0||\varphi_1|\right). \end{aligned} \tag{5.36}$$

We now estimate last term in (5.36). We recall the interpolation estimate for $0<\tau<1$

$$\|\varphi_k\|_{H^\tau(\tilde{\Omega})} \le C\|\varphi_k\|_{L^2(\tilde{\Omega})}^{1-\tau}\left(\|\varphi_k\|_{L^2(\tilde{\Omega})}+\|\nabla\varphi_k\|_{L^2(\tilde{\Omega})}\right)^\tau \text{ for } k=0,1, \tag{5.37}$$

and the trace estimate, for $1/2<\tau<1$,

$$\|\varphi_k\|_{L^2(\partial T_j^{(0)})} \le \|\varphi_k\|_{H^{\tau-1/2}(\partial T_j^{(0)})} \le C_\tau\|\varphi_k\|_{H^\tau(\tilde{\Omega})}, \text{ for } k=0,1. \tag{5.38}$$

The estimates (5.37) and (5.38) combined with (5.35) give, for $k=0,1$,

$$\|\varphi_k\|_{L^2(\partial T_j^{(0)})} \le C_\tau\mu^{\tau-1}, \tag{5.39}$$

and, hence,

$$\int_{\partial T_j^{(0)}}\left(|\varphi_0|+|\varphi_1|+|\varphi_0||\varphi_1|\right) \le C_\tau\mu^{2(\tau-1)} \tag{5.40}$$

for any fixed $\tau \in (1/2, 1)$. By using (5.36), (3.7) and (5.40) we have the estimate

$$\left| \sum_{j=1}^{N} q_j \int_{\partial T_j^{(0)}} e^{ix\cdot\xi} (\tilde{\Phi}_j(x)\nu_j) \right| \leq C \left(m_0 e^{C(|\xi|+\mu)} + \mu^{-2(1-\tau)} \right). \tag{5.41}$$

We write the integral on the left-hand side of (5.41) in a slightly different form. We denote by $\{F_k\}_{k=1}^{M_1}$ the collection of facets of tetrahedra. We note that the set $\bigcup_{k=1}^{M_1} F_k$ contains special a priori information which is implied by the a priori information on the mesh of tetrahedra.

Each facet F_k not contained on $\partial\Omega$ belongs to two tetrahedra and the outer normal directions with respect to these two tetrahedra are opposite one to another. We denote by ν_k one of these two directions and denote by q_k^- the coefficient defined in the tetrahedron where ν_k is pointing towards and q_k^+ the one defined in the other tetrahedron. By assumption (2.18) and by (2.14) we have that

$$|q_k^+ - q_k^-| \geq c_0. \tag{5.42}$$

For any $k \in \{1, \ldots, M_1\}$ we let

$$f_k(x) = \begin{cases} 0 & \text{if } F_k \text{ is contained in } \partial\Omega \\ \left(q_k^+ - q_k^-\right) (\tilde{\Phi}_k(x) \cdot \nu_k) & \text{otherwise.} \end{cases} \tag{5.43}$$

We know that the f_k are affine functions on each facet, F_k, and that

$$\sum_{k=1}^{M_1} \|f_k\|_{H^{1/2}(F_k)} \leq E, \tag{5.44}$$

where E depends on a priori information. We denote by H the measure,

$$H = \sum_{k=1}^{M_1} h_k := \sum_{k=1}^{M_1} f_k d\sigma_k,$$

where $d\sigma_k$ is the surface element on F_k for $k \in \{1, \ldots, M_1\}$. More precisely, each h_k is defined as follows:

$$C_0^0\left(\mathbb{R}^3\right) \ni \phi \to \langle h_k, \phi\rangle = \int f_k \phi \, d\sigma_k \in \mathbb{R}.$$

Estimate (5.41) implies that

$$|\widehat{H}(\xi)| \leq C\gamma\left(|\xi|, \mu, m_0\right), \tag{5.45}$$

where

$$\gamma\left(t, \mu, m_0\right) = m_0 e^{C(t+\mu)} + \mu^{-2(1-\tau)} \text{ for every } t > 0, \mu > 0. \tag{5.46}$$

We estimate, for $s > 1$,

$$\left(\int_{\mathbb{R}^3} \left(1 + |\xi|^2\right)^{-s/2} |\widehat{H}(\xi)|^2 d\xi \right)^{1/2} \leq \sum_{k=1}^{M_1} \left(\int_{\mathbb{R}^3} \left(1 + |\xi|^2\right)^{-s/2} |\widehat{h}_k(\xi)|^2 d\xi \right)^{1/2}. \tag{5.47}$$

For each k we write

$$\int_{\mathbb{R}^3} \left(1+|\xi|^2\right)^{-s/2} |\widehat{h}_k(\xi)|^2 d\xi = \int_{|\xi|\leq 1} \left(1+|\xi|^2\right)^{-s/2} |\widehat{h}_k(\xi)|^2 d\xi$$
$$+\sum_{j=1}^{\infty} \int_{2^j\leq|\xi|\leq 2^{j+1}} \left(1+|\xi|^2\right)^{-s/2} |\widehat{h}_k(\xi)|^2 d\xi$$
$$\leq \int_{|\xi|\leq 1} |\widehat{h}_k(\xi)|^2 d\xi + \sum_{j=1}^{\infty} 2^{-js} \int_{|\xi|\leq 2^{j+1}} |\widehat{h}_k(\xi)|^2 d\xi. \tag{5.48}$$

Using [10, Theorem 7.1.26, p.173], estimate (5.48) gives

$$\int_{\mathbb{R}^3} \left(1+|\xi|^2\right)^{-s/2} |\widehat{h}_k(\xi)|^2 d\xi \leq C \left(1+2\sum_{j=1}^{\infty} 2^{-(s-1)j}\right) \int_{F_k} |f_k|^2 d\sigma_k,$$

and, by (5.44) and (5.47),

$$\left(\int_{\mathbb{R}^3} \left(1+|\xi|^2\right)^{-s/2} |\widehat{H}\left(\xi\right)|^2 d\xi\right)^{1/2} \leq CE. \tag{5.49}$$

We consider a single facet, for instance, the facet F_1. To simplify the notation, we assume that $F_1 \subset \mathbb{R}^2\times\{0\}$ and that 0 is a point of F_1 such that $B'_{2d}\left(0\right) \subset F_1$ where d depends on the a priori information only. We let $\eta \in C_0^\infty\left(\mathbb{R}^2\right)$ such that $0\leq\eta\leq 1$ and $\eta = 1$ on $B'_d\left(0\right)$.

We choose a $g_1 \in H^1\left(\mathbb{R}^3\right)$ such that

$$g_1(x', 0) = \left(\eta f_1\right)(x'),\ x' \in \mathbb{R}^2,$$

$$\mathrm{supp} g_1 \cap F_k = \emptyset, \text{ for } k \neq 1 \tag{5.50}$$

and

$$\|g_1\|_{H^1(\mathbb{R}^3)} \leq CE, \tag{5.51}$$

where C depends on the a priori information only. Taking into account (5.50), we obtain

$$\int_{\mathbb{R}^3} \widehat{H}(\xi)\check{g}_1(\xi) d\xi = \int_{\mathbb{R}^3} d\xi \check{g}_1(\xi) \sum_{k=1}^{M_1} \int_{F_k} e^{ix\cdot\xi} f_k(x) d\sigma_k$$
$$= \sum_{k=1}^{M_1} \int_{F_k} f_k(x) d\sigma_k \int_{\mathbb{R}^3} e^{ix\cdot\xi} \check{g}_1(\xi) d\xi$$
$$= \sum_{k=1}^{M_1} \int_{F_k} f_k(x) g_1(x) d\sigma_k = \int_{F_1} \eta \left|f_1\right|^2 d\sigma_1,$$

that is,

$$\int_{F_1} \eta \left|f_1\right|^2 d\sigma_1 = \int_{\mathbb{R}^3} \widehat{H}(\xi)\check{g}_1(\xi) d\xi. \tag{5.52}$$

Moreover by (5.51) we have

$$\int_{\mathbb{R}^3} \left(1+|\xi|^2\right) \left|\check{g}_1(\xi)\right|^2 d\xi \leq CE^2. \tag{5.53}$$

We write

$$\int_{\mathbb{R}^3} \left|\widehat{H}(\xi)\check{g}_1(\xi)\right| d\xi = \int_{|\xi|\leq\rho} \left|\widehat{H}(\xi)\check{g}_1(\xi)\right| d\xi + \int_{|\xi|>\rho} \left|\widehat{H}(\xi)\check{g}_1(\xi)\right| d\xi \tag{5.54}$$

By (5.45) and (5.53) we have

$$\begin{aligned}
\int_{|\xi|\leq\rho} \left|\widehat{H}(\xi)\check{g}_1(\xi)\right| d\xi &\leq \gamma(\rho,\mu,m_0) \int_{|\xi|\leq\rho} |\check{g}_1(\xi)|\, d\xi \\
&\leq C\gamma(\rho,\mu,m_0) \left(\int_{|\xi|\leq\rho} (1+|\xi|^2)^{-1} d\xi\right)^{1/2} \left(\int_{|\xi|\leq\rho} (1+|\xi|^2)\,|\check{g}_1(\xi)|^2\, d\xi\right)^{1/2} \\
&\leq C\gamma(\rho,\mu,m_0)\sqrt{\rho}E
\end{aligned} \tag{5.55}$$

Using the Cauchy-Schwarz inequality and (5.49) we have

$$\begin{aligned}
\int_{|\xi|>\rho} \left|\widehat{H}(\xi)\check{g}_1(\xi)\right| d\xi &= \int_{|\xi|>\rho} \left(1+|\xi|^2\right)^{-s/4} \left|\widehat{H}(\xi)\right| \left(1+|\xi|^2\right)^{s/4} |\check{g}_1(\xi)|\, d\xi \\
&\leq \left(\int_{|\xi|>\rho} \left(1+|\xi|^2\right)^{-s/2} \left|\widehat{H}(\xi)\right|^2 d\xi\right)^{1/2} \left(\int_{|\xi|>\rho} \left(1+|\xi|^2\right)^{s/2} |\check{g}_1(\xi)|^2\, d\xi\right)^{1/2} \\
&\leq CE \left(\int_{|\xi|>\rho} \left(1+|\xi|^2\right)^{s/2} |\check{g}_1(\xi)|^2\, d\xi\right)^{1/2}.
\end{aligned} \tag{5.56}$$

Then, using (5.53), we find that for $1 < s < 2$,

$$\begin{aligned}
&\int_{|\xi|>\rho} \left(1+|\xi|^2\right)^{s/2} |\check{g}_1(\xi)|^2\, d\xi \\
&\quad= \int_{|\xi|>\rho} \left(1+|\xi|^2\right)^{-\frac{2-s}{2}} \left(1+|\xi|^2\right) |\check{g}_1(\xi)|^2\, d\xi \\
&\quad\leq \left(1+\rho^2\right)^{-\frac{2-s}{2}} \int_{\mathbb{R}^3} \left(1+|\xi|^2\right) |\check{g}_1(\xi)|^2\, d\xi \leq CE^2\rho^{-(2-s)}
\end{aligned} \tag{5.57}$$

and with (5.56) and (5.57),

$$\int_{|\xi|>\rho} \left|\widehat{H}(\xi)\check{g}_1(\xi)\right| d\xi \leq CE^2\rho^{-\frac{2-s}{2}}. \tag{5.58}$$

By (5.52), (5.54), (5.55) and (5.58) we have

$$\int_{F_1} \eta\,|f_1|^2\, d\sigma_1 \leq CE\sqrt{\rho}\left(m_0 e^{C\rho} e^{C\mu} + \mu^{-2(1-\tau)}\right) + CE^2\rho^{-\frac{2-s}{2}}. \tag{5.59}$$

We choose $\mu = \rho^{1/(1-\tau)}$ and get, for every $\rho \geq 1$,

$$\begin{aligned}
\int_{F_1} \eta\,|f_1|^2\, d\sigma_1 &\leq CE\left(m_0\sqrt{\rho}e^{C\left(\rho+\rho^{1/(1-\tau)}\right)} + \rho^{-3/2} + E\rho^{-\frac{2-s}{2}}\right) \\
&\leq C\,(E+m_0+1)^2 \left(\left(\frac{m_0}{E+m_0+1}\right) e^{C_\star \rho^{1/(1-\tau)}} + \rho^{-\frac{2-s}{2}}\right),
\end{aligned} \tag{5.60}$$

where $C_\star$ depends on the a priori data only.

We then choose

$$\rho = \left(\frac{1}{2C_\star} \left| \log \frac{m_0}{E + m_0 + 1} \right| \right)^{1-\tau}$$

so that

$$\int_{B'_d} |f_1|^2 d\sigma_1 \leq C \left(E + m_0 + 1 \right)^2 \left| \log \frac{m_0}{E + m_0 + 1} \right|^{-\frac{2-s}{2}}, \tag{5.61}$$

where C depends on s and the a priori information only. Because f_1 is an affine function on F_1 with a bounded gradient, and the size of B'_d is bounded from below with a constant depending only on a priori information, we have

$$|f_1(x)| \leq C \left(E + m_0 + 1 \right) \left| \log \frac{m_0}{E + m_0 + 1} \right|^{-\frac{2-s}{4}} \quad \text{for every } x \in F_1. \tag{5.62}$$

By repeating the same procedure on each facet, and recalling (5.42) and the fact that $\tilde{\Phi}_k(x) \cdot \nu_k = 0$ if $F_k \subset \partial\Omega$, we have

$$\left| \tilde{\Phi}_k(x) \cdot \nu_k \right| \leq C \varsigma_1(m_0) \text{ for } x \in F_k, \tag{5.63}$$

where

$$\varsigma_1(m_0) = \left(E + m_0 + 1 \right) \left| \log \frac{m_0}{E + m_0 + 1} \right|^{-\frac{2-s}{4}}.$$

We fix a tetrahedron $T_j^{(0)}$ and let P_1, P_2, P_3, P_4 be its vertices. We label the facets so that F_k, for $k = 1, 2, 3, 4$ is the facet of $T_j^{(0)}$ that does not contain P_k. We let $\nu^{(k)}$ be the unit outward normal to F_k. Each point on $x \in F_k$ can be written as

$$x = \sum_{i=1}^{4} s_i P_i,$$

where $0 \leq s_i \leq 1$, $s_k = 0$, and $\sum_{i=1}^{4} s_i = 1$. With this notation,

$$\tilde{\Phi}_k(x) \cdot \nu_k = \sum_{i=1}^{4} s_i \tilde{v}_i \cdot \nu^{(k)}$$

and using (5.63)

$$\left| \sum_{i=1}^{4} s_i \tilde{v}_i \cdot \nu^{(k)} \right| \leq C \varsigma_1(m_0).$$

This implies that

$$\left| \tilde{v}_i \cdot \nu^{(k)} \right| \leq C \varsigma_1(m_0) \text{ for every } i \neq k.$$

In particular, this means that for every vector $\tilde{v}_{j,i}$ we have

$$\left| \tilde{v}_{j,i} \cdot \nu^{(k)} \right| \leq C \varsigma_1(m_0)$$

for every direction $\nu_j^{(k)}$ orthogonal to the facet of $T_j^{(0)}$ that contains P_i. By the regularity of the partition, this implies that

$$|\tilde{v}_{j,i}| \leq C_3\varsigma_1(m_0), \tag{5.64}$$

where C_3 depends on the a priori information.

By adding together inequalities (5.64) and applying (5.28), we get

$$1 = \sum_{j=1}^{N}\sum_{i=1}^{4}|\tilde{v}_{j,i}| \leq 4C_3\varsigma_1(m_0)$$

that yields

$$m_0 \geq \varsigma_1^{-1}\left(\frac{1}{4C_3}\right). \tag{5.65}$$

From the definition of m_0 (see (5.32)), there exist a pair of boundary values ϕ_0 and ψ_0 such that

$$|\mathcal{G}(\phi_0,\psi_0)| \geq \frac{m_0}{2}\|\phi_0\|\|\psi_0\|$$

and, hence,

$$\left|\frac{d}{dt}\mathcal{F}(t,\phi_0,\psi_0)_{|t=0}\right| \geq \omega^2\sum_{j=1}^{N}V_j\frac{m_0}{2}\|\phi_0\|\|\psi_0\|$$

that, together with (4.6), gives (5.2) for

$$m_1 = \frac{1}{2}\omega_1 A_2^{-1}\varsigma_1^{-1}\left(\frac{1}{4C_3}\right).$$

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