# A Model of Market Making with Heterogeneous Speculators

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**Abstract**

I introduce an optimizing monopolistic market maker in an otherwise standard setting *à la* Brock and Hommes (1998) (BH98). The market maker sets the price of a zero yielding asset taking advantage of her knowledge of speculators' demand, manages her inventory of the asset and eventually earns profits from trading. The resulting dynamic behavior is qualitatively identical to the one described in BH98, showing that the results of the latter are independent from the institutional framework of the market. At the same time I show that the market maker has conflicting effects. She acts as a stabilizer when she allows for market imbalances, while she acts as a destabilizer when she manages aggressively her inventories and when she trades, especially if she acts as fundamentalist or if she is a strong extrapolator. Indeed the more stable institutional framework is one in which the market makers is inventory neutral and doesn't trade but, even in this case, the typical complex behavior of BH98 occurs.

**Keywords**: Asset pricing model, heterogeneous beliefs, market architecture, market making, foreign exchange market.

**JEL codes**: G12, D84, D42, C62, F31

# 1 Introduction

Models of financial markets with heterogeneous bounded rational speculators generally come in two flavors. The seminal model of Brock and Hommes (1998) makes the standard assumption of market clearing. Other models assume instead the existence of a market maker, which accumulates inventories and adjusts the price in the same direction of net demand following a simple linear rule (Day and Huang, 1990; Lux, 1995).

The market maker hypothesis is widely considered a better description of the price adjustment mechanism of actual markets. In particular, it is the standard assumption for models of the FX market (Westerhoff, 2009). On the other hand, the standard linear adjustment rule, which is generally used in these models, might lead to large inventory unbalances. This implication is inconvenient, since the empirical evidence shows that FX dealers manage actively their inventories in order to end the trading day on a balanced position (Manaster and Mann, 1996; Bjønnes and Rime, 2005). To overcome this limitation, Westerhoff (2003) incorporates inventory management in the price adjustment rule, showing that a more aggressive control of inventories makes the market more volatile and less stable.

Equivalent results are obtained by Carraro and Ricchiuti (2015) and Zhu *et al.* (2009), who consider market makers that take speculative positions by adjusting their inventories towards a given exogenous target. These models don't derive the pricing rule from an optimizing behavior of the market maker, instead they rely on ad hoc pricing mechanisms. Following a similar path, Hommes *et al.* (2005) extend the BH98 framework to a market maker scenario, using a linear adjustment rule, and prove that the dynamic behavior of the system is pretty similar to BH98. Anufriev and Panchenko (2009) compare through simulations different market protocols (Walrasian auctioneer, market maker, batch auction and order book), allowing for the endogenous evolution of the proportions of speculators. They show that, no matter which type of market clearing is used, two different regimes with completely different dynamical properties occur depending on the value of the intensity of choice, and that the trading protocol strongly affects the critical value of the intensity of choice.

This paper presents a model which incorporates a more sophisticated representation of the behavior of the market maker than the current literature. In particular, I suppose that there are two types of speculators (fundamentalists and chartists) who submit their trades to a monopolistic market maker who is an optimizing bounded rational agent. I further suppose that the market maker knows the optimal demand of speculators, in accordance with the evidence that market makers profit from their knowledge of the market (see

Sec. 2). In the first version, profits come only from market making, while in the second version the market maker might trade on her own account with bounded rational expectations regarding the future price.

My model confirms the main result of Hommes *et al.* (2005) while, on the other hand, it shows that the market maker has conflicting effects on the stability of the market. From this perspective, my results converge with those of Zhu *et al.* (2009) and Carraro and Ricchiuti (2015), who also underline that the market maker destabilizes the market when she manages aggressively her inventory. On the other hand, these works provide analytical results which are restricted to the case of fixed proportions of speculators, while my results are derived using the heuristic switching mechanism of BH98 which allows these proportions to evolve endogenously. Moreover, since I derive the market clearing case of BH98 as a special case, I can prove analytically that the trading protocol affects the critical values of the intensity of choice, thus strengthening the results of Anufriev and Panchenko (2009).

The remaining of this paper is organized as follows. In sec. 2 I relate the hypotheses of this paper to the literature on the microstructure of FX markets. In sec. 3 I present the model with a "pure" market maker, and in sec. 4 the model with an "activist" market maker. Sec. 5 concludes.

# 2 Related Literature

In this paper I make three critical assumptions: 1) that the market maker has perfect knowledge of the demand schedule of her customers; 2) that the market maker is able to satisfy the net demand coming from customers by adjusting in advance her inventory at the previous market price, i.e. before she announces the new optimal price to her customers, or equivalently that she can offload fully its inventory at the previous price on a different market [1]; 3) that customers have no informational advantage over the market maker regarding the fundamental price of the asset.

These hypotheses contradict those of a number of well established models of market making which rely either on the idea that some customers are better informed than the market maker (Glosten and Milgrom, 1985; Kyle, 1985) or on the idea that the market maker sets the price to adjust her inventory level (Ho and Stoll, 1981; Huang and Stoll, 1997), or again on both mechanisms (Madhavan and Smidt, 1991). In this section I present some pieces of evidence which support the framework proposed in this paper.

[1] This assumption is consistent with the existence of a wholesale market where the monopolist market maker trades with other smaller, less informed, dealers.

These derive mostly from the market microstructure literature on the foreign exchange market.

Indeed the possibility of private information in the FX market has been questioned since the fundamental value of a currency is determined by macroeconomic information, which is publicly available . Instead, insider models like Kyle (1985) or Glosten and Milgrom (1985) draw their inspiration from the stock market, where a privileged access to information is more likely to occur.

On the other hand, net order flows are strong predictors of exchange rate movements (King *et al.*, 2013), and this is considered by most economists as a clear sign of asymmetric information. What could be the source of private information on the FX market if fundamentals are common knowledge? One possible answer is that private information that is most valuable does not concern fundamentals. One source of non fundamental information, according to King *et al.* (2013), stems from demand and supply themselves. In particular, if they have only finite elasticity, i.e. if the liquidity of markets is limited, market makers can leverage on their role to make profits. Following this line, Cai *et al.* (2001) show with high frequency data that customer order flows have an influence on rates distinct from macroeconomic announcements. Thus the anticipation of public announcements is not the only potential source of private information on the foreign exchange market.

This line of thought leads us to the next problem, namely: who is informed on the FX market? Using a detailed breakdown of customer typologies, Osler and Vandrovych (2009) show that only leveraged investors bring information to the market. All other types of customers appear to be uninformed, while banks themselves, which host the main foreign exchange dealers, appear to be better informed than their customers. Specifically, the price impact of bank trades remains strong for up to one week, while the price impact of leveraged-investors loses significance after six hours.

These result confirm the common view among FX dealers that big banks are better informed because they trade with the biggest customers (Cheung and Chinn, 2001). The intuition is that banks, by servicing their customers, collect dispersed information from the market which they put to a good use for their own trades. This view is supported also by the empirical evidence that spreads are narrower for financial customers and for larger trades (Osler *et al.*, 2011). This stylized fact is inconsistent with adverse selection models, according to which market makers should charge larger spreads to the most informed traders and on larger trades, which are more likely to be originated from informed counterparts. To motivate the actual pricing choices of market makers we must refer to factors like market power and strategic dealing. According to this view, FX dealers submit competitive quotes to attract large order flows because they seek to understand promptly the direction of the market.

The opportunity of profits for dealers arise because FX is a two-tier market and they may use the information gathered with customers in the first tier to profit from interdealer trades in the second tier. For instance, the results of Osler *et al.* (2011) show that dealers are more likely to trade aggressively on the interdealer market after trades with informed counterparts.

We might wonder what would be the optimal pricing and trading strategy for a dealer under these circumstances. Empirical evidence shows that FX dealers unload inventories quickly and do no adjust their quotes on the interdealer market following an inventory unbalance (Manaster and Mann, 1996; Bjønnes and Rime, 2005). Inventory and adverse selection models agree instead on the prediction that buyer-initiated trades will make the market maker raise prices, while seller-initiated trades will have the opposite effect. But the empirical evidence mentioned above suggests instead that FX dealers leave their quotes unchanged and profit from the future movement of price by trading as quickly as possible on the wholesale market in the same direction of their incoming trades. An optimal trading strategy of this sort is derived in the widely considered model of Evans and Lyons (2002) and is linked to the so called "hot potato trading" on the interdealer market (Lyons, 1997), i.e. the passing through of undesired inventory positions among FX dealers.

We need to remark at this point that the fact that market makers trade on the interdealer market before they adjust their price quotes does not exclude that market imbalances affect prices. In order book markets like the interdealer FX market, an excess of trading in one direction impacts price since liquidity is not unlimited, and price variations on the wholesale market are quickly transmitted to prices charged to customers. In the model of Evans and Lyons (2002) 'profit takers' clear the market and allow FX dealers to profit from end-of-day trades after a price adjustment occurs on the interdealer market which is transmitted to the retail market. Indeed, it is the trasmission of price variations from the wholesale to the retail market which makes interdealer trade profitable even for those dealers who are less informed and thus unlikely to profit when they trade with other dealers.

To sum up, according to the evidence collected from FX market data, we can state the following: 1) market making is a valuable source of information about demand which is used for taking speculative positions; 2) market makers do not set prices to manage their inventories because they can offload them on the interdealer market at favourable prices; 3) customers have no particular informational advantage over market makers. Besides supporting the assumptions of the models presented here, these pieces of evidence suggest that there cannot be a universal model of market making. Instead, the institutional specificities of each market must be taken carefully into consideration by the modeler.

# 3 "Pure" market maker

I assume that the market maker is a profit maximizing monopolist who trades a zero yielding asset with a large number of different types of speculators whose weights on the market evolve endogenously. The market maker trades also on a separate competitive wholesale market with other dealers. I assume that the market maker knows the optimal demand of each type of speculator and that she employs this information when she solves her objective. Then I suppose that the wholesale market is liquid enough to allow the market maker to adjust in advance her inventory at the current market price, in order to match the projected orders of speculators. After this adjustment she announces the optimal price, taking into account a quadratic cost of inventory maintenance. Once the new price is revealed to speculators, the latter trade according to their optimal demand in such a way that, at the end of the period, the net variation of the inventory position of the market maker is zero. Finally, the wholesale market price is updated to the optimal price of the market maker.[2]

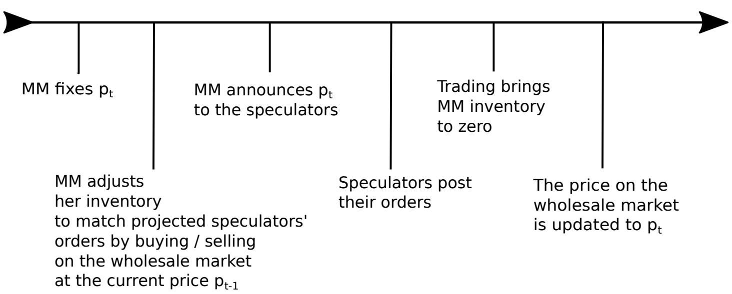



Figure 1: Timeline of events occuring within a single period of the model.

The timeline of events in each period of the model is pictured in Fig. 1. We remark that the assumption on market liquidity is analogous to the one made by Evans and Lyons (2002), who suppose that FX dealers trade on the wholesale market *before* the price adjustment on the retail market occurs. It is also in line with evidence from the FX market and in particular with the

[2] One might alternatively suppose that the optimal price is unknown to the participants of the wholesale market until the end of the trading period. In this setting, which is completely equivalent to the one described in the main text, the inventory is adjusted *after* trading with speculators.

practice of "hot potato" trading which allows FX dealers to profit from retail trading (see Sec. 2).

Recalling that we are considering the case of a zero yielding risky asset, the wealth of the market maker evolves according to the following equation:

$$W_{d,t} = R\,W_{d,t-1} + (p_t - p_{t-1})\,z_t - \omega z_t^2 \tag{1}$$

where $R$ is the gross return of a safe asset, $p_t$ is the price charged to speculators at $t$, $p_{t-1}$ is the price charged in the previous period which is equal to the current wholesale price, $z_t$ is market net demand and $\omega$ measures the impact of the quadratic inventory cost term on $W_t$. We remark that, according to our hypotheses, the inventory cost is linked to $z_t$ since at the end of each period the inventory is brought down to zero. In practice, it is the cost of holding an amount of risky asset equal to $z_t > 0$ until it is resold to the speculators[3].

Considering the heterogeneity of the speculators, market demand is defined as follows:

$$z_t = \sum_{h=0}^{H-1} n_{h,t} z_{h,t} \tag{2}$$

where $z_{h,t}$ is the demand of speculators of type $h$ and, following BH98, $n_{h,t} = \frac{e^{\beta\,U_{h,t-1}}}{Z_t}$ is the fraction of speculators of type $h$. In this formula $Z_t$ is a normalization factor and

$$U_{h,t-1} = (p_{t-1} - Rp_{t-2})\,z_{h,t-2} - C_h \tag{3}$$

is the fitness of the trading strategy $h$, where $C_h$ is a type specific fixed cost.

The wealth of speculators of type $h$ evolves in a standard way:

$$W_{h,t} = R\,W_{h,t-1} + (p_t - Rp_{t-1})\,z_{h,t-1} \tag{4}$$

where $R$ has the same meaning as above and the rightmost term on the right hand side of the equation above represents the excess profit obtained buying the amount $z_{h,t-1}$ at the price $p_{t-1}$ and reselling the same amount at the price $p_t$.

Both the speculators and the market maker act to maximize their current wealth, and this maximization is equivalent to a static profit maximization. Thus I follow the literature on asset pricing with heterogeneous speculators

[3] In the case in which $z_t < 0$, we might think that the risky asset must be exchanged against some other asset and that holding this other asset is costly too.

mentioned in Sec. 1 and exclude, for simplicity, a more complex forward looking setting like the one of Madhavan and Smidt (1991). In particular, speculators are myopic mean-variance maximizers, since their future wealth is uncertain. Their objective can be written as follows:

$$\max_{z_{h,t}} \left\{ E_{ht}\left[W_{t+1}\right] - \frac{1}{2D} V_{ht}\left[W_{t+1}\right] \right\} \tag{5}$$

where $D^{-1}$ is a risk aversion parameter linked to the variance of future wealth $V_{ht}\left[W_{t+1}\right]$. Indeed the wealth of speculators at $t+1$ is determined by their net demand at $t$ and they ignore the future market price when taking their decision. Then we have:

$$E_{ht}\left[W_{h,t+1}\right] = R\,W_{h,t} + \left(E_{ht}\left[p_{t+1}\right] - Rp_t\right) z_{h,t} \tag{6}$$

$$V_{ht}\left[W_{t+1}\right] = V_{ht}\left[p_{t+1}\right] z_{ht}^2 \tag{7}$$

It is convenient to rewrite the speculator problem in terms of deviations from a fundamental value $x_t = p_t - p_t^*$. It is possible to do so by assuming the standard pricing relationship $p_t^* = E_t[p_{t+1}^*]/R$. Thus, taking into account Eqs. (6) and (7) and further supposing for simplicity that $V_{ht}\left[p_{t+1}\right] = \sigma^2$ is equal and constant across all types, we may rewrite the objective as follows:

$$\max_{z_{h,t}} \left\{ R\,W_{h,t} + \left(E_{ht}\left[x_{t+1}\right] - Rx_t\right) z_{h,t} - \frac{1}{2D}\sigma^2 z_{ht}^2 \right\} \tag{8}$$

Finally, we drop the irrelevant term $R\,W_{h,t}$, set the expectation of speculators of type $h$ on $x_t$ to be $E_h\left[x_{t+1}\right] = b_h + g_h x_{t-1}$, and let the constant variance $\sigma^2$ be absorbed by the parameter $D^{-1}$. Thus we obtain the definitive formulation of the speculator objective:

$$\max_{z_{h,t}} \left\{ \left(b_h + g_h x_{t-1} - Rx_t\right) z_{h,t} - \frac{z_{h,t}^2}{2D} \right\} \tag{9}$$

The market maker instead is not subject to uncertainty. Moreover she is aware of the optimal demand of speculators $z_{h,t}^*$ when she optimizes. This assumption is consistent with the monopoly position which market makers have, especially on markets where the largest part of transactions are over the counter, like the FX market (see sec. 2). Following these arguments and dropping the irrelevant term $R\,W_{d,t-1}$ the objective of the market maker in terms of price deviation $x_t$ can be written as follows:

$$\max_{x_t} \left\{ (x_t - x_{t-1}) \sum_{h=0}^{H-1} n_{h,t} z^*_{h,t} - \omega \left( \sum_{h=0}^{H-1} n_{h,t} z^*_{h,t} \right)^2 \right\} \tag{10}$$

The optimal demand of speculators of type $h$, obtained from the FOC of eq. (9), is the standard one among this type of models:

$$z^*_{h,t} = D\left(b_h + g_h x_{t-1} - R x_t\right) \tag{11}$$

Substituting eq. (11) in (10) and deriving we obtain the FOC for the market maker which can be solved for $x_t$:

$$x_t = \frac{1}{DR\omega + 1} \left[ \frac{x_{t-1}}{2} + \left( D\omega + \frac{1}{2R} \right) \sum_{h=0}^{H-1} \left[ \left(b_h + g_h x_{t-1}\right) n_{h,t} \right] \right] \tag{12}$$

Taking into account that the optimal market demand is

$$z^*_t = D \sum_{h=0}^{H-1} \left[ \left(b_h + g_h x_{t-1}\right) n_{h,t} \right] - D\, R x_t \tag{13}$$

we obtain from eq. (12) the following form:

$$x_t = \frac{1}{DR\omega + 1} \left[ \left( D\omega + \frac{1}{2R} \right) \left( \frac{z^*_t}{D} + R x_t \right) + \frac{x_{t-1}}{2} \right] \tag{14}$$

from which we get the linear price adjusting rule that is used in many works (see Sec. 1):

$$x_t - x_{t-1} = \left( 2\omega + \frac{1}{DR} \right) z^*_t \tag{15}$$

In the two type case of fundamentalists ($g_0 = 0$,$b_0 = 0$, $C_0 = C > 0$) and chartists ($g_1 = g > 0$,$b_1 = 0$,$C_1 = 0$), eq. (12) becomes

$$x_t = \frac{1}{DR\omega + 1} \left[ \frac{1}{2} + g \left( D\omega + \frac{1}{2R} \right) n_{1,t} \right] x_{t-1} \tag{16}$$

In this case, after replacing $z_{ht}$ in (3) with (11), $n_{1,t}$ reads as follows:

$$n_{1,t} = \frac{1}{e^{-\beta[Dg(Rx_{t-2} - x_{t-1})x_{t-3} - C]} + 1} \tag{17}$$

It is convenient to introduce $m_t \equiv n_{0,t} - n_{1,t}$. Then eq. (16) becomes

$$x_t = \frac{1}{DR\omega + 1}\left[\frac{1}{2} - g\left(D\omega + \frac{1}{2R}\right)\left(\frac{m_t - 1}{2}\right)\right]x_{t-1} \tag{18}$$

where

$$m_t = \tanh\left(\frac{\beta}{2}\left[Dg_c\left(Rx_{t-2} - x_{t-1}\right)x_{t-3} - C\right]\right) \tag{19}$$

The market clearing case of BH98 is obtained for $\omega \to \infty$:

$$x_t = -\frac{g}{2R}\left(m_t - 1\right)x_{t-1} \tag{20}$$

From eqs.(18) and (20) we see that $x = 0$ is always a steady state of the system. This is usually called the fundamental steady state solution because the actual price equals the fundamental price. One of the main result of BH98 is to show that, for the system described by eq. (20), there are additional non fundamental steady states under appropriate conditions, and that the fundamental steady state becomes unstable under the same conditions. This is a key result because it states that the actual price might deviate systematically from the fundamental price under appropriate conditions, i.e. that the market may exhibit, under these conditions, a "bubble and burst" behavior.

Regarding the system described by eq. (18), the following proposition holds:

**Proposition 1.** *(Existence and stability of steady states). Let* $\overline{m}_f = \tanh\left(-\frac{\beta C}{2}\right)$, $\overline{m}_{nf} = 1 - \frac{2R}{g}$ *and* $x^*$ *be the positive solution (if it exists) of* $\tanh\left[\frac{\beta}{2}\left(Dg\left(R-1\right)x^2 - C\right)\right] = \overline{m}_{nf}$. *Then:*

1. *For* $0 < g < R$ *the fundamental equilibrium* $E_0 = (0, \overline{m}_f)$ *is the unique, globally stable steady state*

2. *For* $g \geq 2R$ *the fundamental equilibrium is locally unstable and two other steady states exists:* $E_1 = (x^*, \overline{m}_{nf})$ *and* $E_2 = (-x^*, \overline{m}_{nf})$

3. *For* $R < g < 2R$ *there exists* $\beta^* = \frac{1}{C}\log\left(\frac{1}{g/R-1}\right) \in (0, \infty)$ *such that*

   (a) *if* $\beta < \beta^*$ *the fundamental equilibrium* $E_0 = (0, \overline{m}_f)$ *is the unique, globally stable steady state*

   (b) *if* $\beta > \beta^*$ *the fundamental equilibrium is locally unstable and two other steady states exists:* $E_1 = (x^*, \overline{m}_{nf})$ *and* $E_2 = (-x^*, \overline{m}_{nf})$

   (c) *if* $\beta = \beta^*$ *the fundamental and non fundamental equilibria coincide*

*Proof.* See Appendix A. □

We remark that with the proposition above we recover exactly, for the system described by eq. (18), the result of BH98, Lemma 2, which is obtained under the assumption of market clearing, i.e. for the system described by eq. (20). This includes the fact that at $\beta = \beta^*$ a pitchfork bifurcation occurs (Fig. 3). Thus we see that the institutional framework of the market has no effect on the existence of non fundamental equilibria and on the stability of the fundamental equilibrium. In particular, the existence and stability of the steady states in BH98 does not depend on the assumption of market clearing. Moreover, the value of $\beta^*$ is independent from the market setting.

Another important result of BH98 (Lemma 3) is to show that there exists a critical value $\beta^{**}$ above which the two non fundamental steady states become themselves unstable. When $\beta > \beta^{**}$, the system (20) might follow non trivial dynamical paths and even exhibit complex or chaotic trajectories. The equivalent of this lemma for the system decribed by eq. (18) is stated as follows:

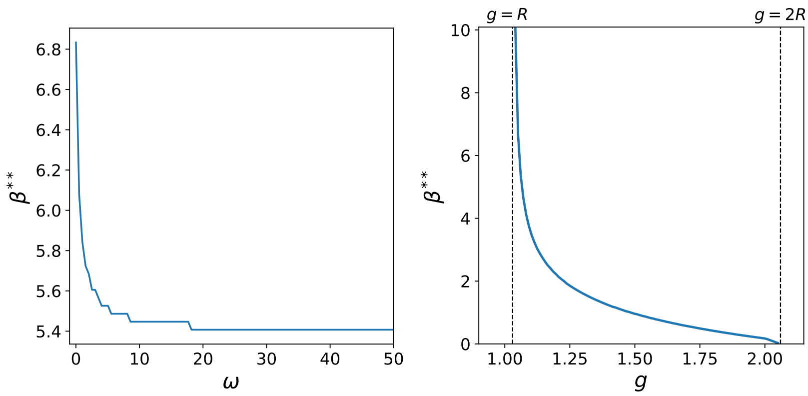
**Proposition 2.** *(Secondary Bifurcation). Let $E_1$ and $E_2$ be the non fundamental steady states and $\beta^*$ be the pitchfork bifurcation value. Assume $R < g < 2R$ and $C > 0$ and further suppose that $R \in \left(1, \frac{4}{3}\right]$. Then there exists $\beta^{**}$ such that $E_1$ and $E_2$ are stable for $\beta^* < \beta < \beta^{**}$ and unstable for $\beta > \beta^{**}$. For $\beta = \beta^{**}$ $E_1$ and $E_2$ exhibit a Hopf bifurcation.*

*Proof.* See Appendix A. □

From the bifurcation plot of Eq. (18) (Fig. 3) we see that at $\beta^{**}$ the (positive) non fundamental steady state becomes unstable and that for higher values of $\beta$ there is an oscillating behavior between the fundamental and non fundamental steady state. The resulting dynamics is qualitatively identical to the one of the system in BH98 described by Eq. (20).

As stated above, the critical value $\beta^*$ of the system described by Eq. (18) remains the same of Eq. (20). Instead, we see from Fig. 2 that $\beta^{**}$ is decreasing in $\omega$. In other terms, the likelihood that the market will settle at the non fundamental price is lower in the market clearing case ($\omega \to \infty$). Symmetrically, the more the market maker is inventory neutral ($\omega \to 0$), the more the market is likely to settle at the non fundamental steady state.

BH98 prove (Lemma 4) that, for $\beta \to \infty$, if $g > R^2$ the system described by eq. (20) is globally unstable, while if $R < g < R^2$ the same system displays *homoclinic orbits*, i.e. closed trajectories which join the fundamental steady state (which in this case is a saddle point) to itself. This result is important because it suggests that, for high values of $\beta$, the system can




(a) $g = 1.05$ (b) $\omega = 1$

Figure 2: Value of $\beta^{**}$ with $D = C = 1$, $R = 1.03$.

be characterized by complex and eventually chaotic trajectories around the fundamental equilibrium. The existence of homoclinic orbits depends on the fact that the unstable manifold around the fundamental steady state is bounded, where the unstable manifold may be loosely defined as the set of points that are sent away from a fixed point as time moves forward. On the other hand, if the unstable manifold is unbounded and the system has no homoclinic orbits, this means that any orbit starting at the fundamental steady state will never come back in a neighborhood of the steady state itself. Since in this case the typical paths of $x_t$ are diverging to infinity, this is also an important situation to consider from an economic point of view because it suggests that, under these conditions, the market is going to crash.

On this matter, in the case of Eq. (18) it's possible to prove the following proposition:

**Proposition 3.** *Assume $C > 0$ and $\beta \to \infty$. For $g > R$ the fundamental steady state $E_0 = (0, -1)$ is locally unstable, with eigenvalues 0 and $\frac{R+(2DR\omega+1)g}{2R(DR\omega+1)}$. Let's fix $\bar{g}_0 = \frac{R\left(2DR^2\omega+2R-1\right)}{2DR\omega+1}$. There are two possibilities for the unstable manifold $W^u(E_0)$:*

1. *if $g > \bar{g}_0$ then $W^u(E_0)$ equals the unstable eigenvector and is unbounded*
2. *if $R < g < \bar{g}_0$ then $W^u(E_0)$ is bounded and all orbits converge to $E_0$*

*Proof.* See Appendix A. □

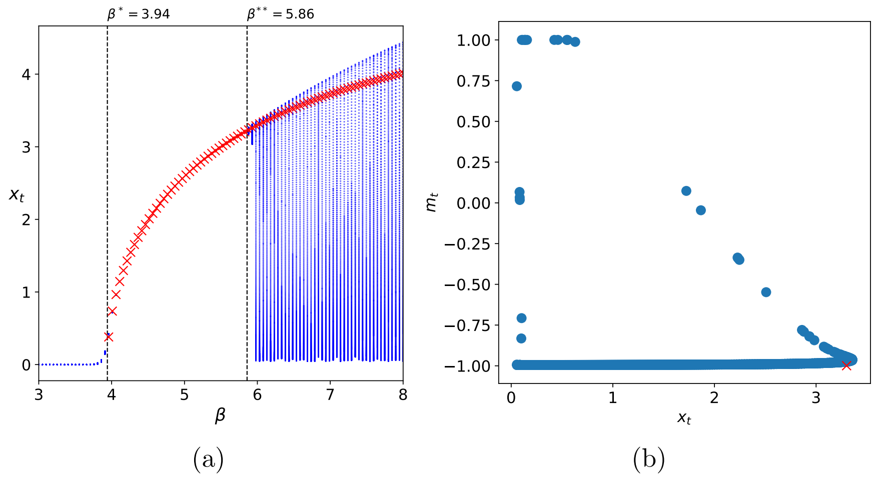



Figure 3: (a) Bifurcation diagram with $3 \leq \beta \leq 8$ and (b) periodic orbit for $\beta = 6$. The value of the other parameters are fixed as in Fig. 2, panel (a). The red crosses stand for the value of the non fundamental solution $x^*$ obtained from eq. (31)

Apart from the value of the threshold $\overline{g}_0$, the proposition above replicates Lemma 4 of BH98, showing that there are no deep mathematical differences, under this respect, between Eqs. (18) and (20).

From Fig. 4 we see that the threshold $\overline{g}_0$ is decreasing in $\omega$, which means that the global stability of the market depends on the behavior of the market maker. We recover the result of BH98 considering the limit $\omega \to \infty$, where $\bar{g}_0 = R^2$. Instead for $\omega \to 0$ we obtain the upper bound $\bar{g}_0 = R(2R-1)$. Thus the activity of the market maker, which absorbs the imbalance of supply and demand, makes the market less likely to suffer from severe instability which might call for an external intervention.

As final comment, I would like to stress the remarkable difference between the three results of this section. According to Proposition 1, the market maker has no impact whatsoever on the existence and stability of the fundamental steady state, as well as on the existence of the non fundamental steady states. Instead, according to Fig. 2, the market maker increases the critical value $\beta^{**}$ above which the non fundamental steady state become unstable. On the other hand, from Fig. 2 we see also that any value of $\beta^{**}$ that can be obtained by choosing a value of $\omega$ can be also obtained by choosing a value of $g$. Thus, in this case, what we can obtain with a market maker which reacts more strongly to inventory imbalances we can obtain as well with a chartist speculator with a stronger extrapolative behavior. According to Proposition 3, finally, the existence of the market maker improves the global stability of the system for any given behavioral attitude of the chartist speculators. Therefore we might consider the last one as the stronger of our results.

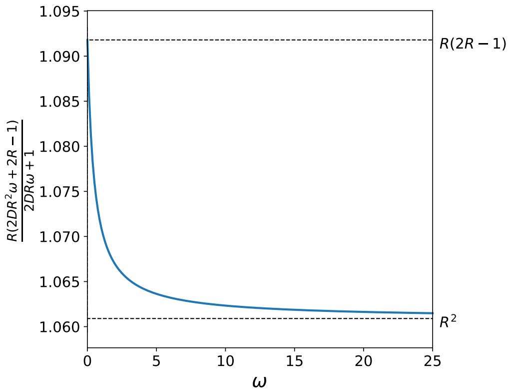



Figure 4: $\overline{g}_0$ for different values of $\omega$. The values of the other parameters are: $D = 1$, $R = 1.03$.

# 4 "Activist" market maker

We extend the model of the previous section supposing that the market maker acts as a mean-variance optimizing speculator too. Her objective becomes

$$\max_{x_t,y_t}\left\{(x_t - x_{t-1})\sum_{h=0}^{H-1} n_{h,t}z^*_{h,t} - \omega\left(\sum_{h=0}^{H-1} n_{h,t}z^*_{h,t}\right)^2 + (g_d - R)\,x_t y_t - \frac{y_t^2}{2D}\right\} \tag{21}$$

The first two terms represent the variation of wealth $W_{d,t}$ obtained with market making, which is equal to Eq. ( 10 ). The two additional terms represent the risk-adjusted expected net profit derived from trading and can be obtained through the same arguments which we have used for speculators in Sec. 3. In particular $y_t$ represents the speculative demand of the market maker at $t$ and $g_d$ is her extrapolation parameter for the future price deviation, i.e. $E_d[x_{t+1}] = g_d x_t$. In fact we assume that the market maker has bounded rational expectations regarding the future price. Finally, we assume that the risk aversion parameter and the variance of the future price are the same as those of the speculators so that both factors are still absorbed by the parameter $D$.

Unlike the model of the previous section, the market maker in this case may end the trading period with a net variation of her inventory of the risky asset, which is equal to $y_t$. Moreover, the two components of the objective function cannot be decoupled because the price set by the market maker impacts on her expectation of future prices. Excluding this impact would have made the market maker's rationality appear really too limited: since she sets the current price based on market demand, she can use this information to forecast the future price.

Of course we might conceive the market maker as more far-sighted in considering in her expectation the fact that future prices depend not only on her current price setting decisions but also on the future ones. However an explicit inclusion of this factor would have made the model much more complicated. The actual formulation $E_d[x_{t+1}] = g_d x_t$ might be considered instead a reduced form for the belief that the excess demand of speculators is autocorrelated enough to make future price deviations autocorrelated too. Indeed the market maker is likely to be aware that a positive net demand $z_t$ is pushing herself to raise the price and that this higher price will feed back positively into the future demand of chartist speculators. Thus we may conclude that the mechanism of expectation formation we have conjectured for the market maker, although it is of a heuristic kind, is consistent with the actual working of the model. Furthermore, this formulation allows to examine the case of a disagreement of expectations between the speculators and the market maker which might lead to instability.

Substituting the optimal demand (11) in the objective (21) and deriving for $x_t$ and $y_t$ we obtain the FOCs. In particular from the FOC with respect to $y_t$ we obtain the optimal trading strategy for the market maker:

$$y_t = D\left(g_d - R\right)x_t \tag{22}$$

Thus the market maker will trade in the same direction of $x_t$ only if the extrapolative component of her expectation is strong enough. Substituting (22) in the other FOC and solving for $x_t$ we obtain the following price equation:

$$x_t = \frac{Rx_{t-1} + \left(2DR\omega + 1\right)\sum_{h=0}^{H-1}\left(b_h + g_h x_{t-1}\right)n_{h,t}}{2R\left(DR\omega + 1\right) - \left(R - g_d\right)^2} \tag{23}$$

We see that the price equation is defined only if $g_d \neq R \pm \sqrt{2R(DR\omega + 1)}$, a fact which hints at the importance of the expectations of the market maker for the stability of the market. The results that follow are derived supposing that $g_d$ belongs to the interval $\left(R - \sqrt{R(2DR\omega + 1)}, \quad R + \sqrt{R(2DR\omega + 1)}\right)$, so that these two singularities are ruled out. Within this interval we see that price deviations $x_t$ are increasing in past price deviations $x_{t-1}$ as it happens in eqs. (12), (18) and (20).

Proceeding as in the previous model, from eqs. (23) and (13) we get the following form:

$$x_t = \frac{R}{R-(R-g_d)^2}x_{t-1} + \frac{2DR\omega+1}{D\left[R-(R-g_d)^2\right]}z_t^* \tag{24}$$

We see that the linear rule of the previous model is recovered for $g_d = R$, i.e. when the market maker doesn't trade (see eq. (22)). If $g_d = R \pm \sqrt{R}$ the coefficients of $x_{t-1}$ and $z_t$ diverge but this singularity is removable since the reduced form dynamics is still defined. If $g_d \notin \left[R-\sqrt{R}, \quad R+\sqrt{R}\right]$ the same coefficients become negative, while for $g_d \in \left(R-\sqrt{R}, \quad R+\sqrt{R}\right)$ they are positive. In the latter case, the behavior of the market maker may be still described by a linear pricing rule, with resulting price deviations at $t$, for the same past price deviations $x_{t-1}$ and market excess demand $z_t$, which will be larger than in the previous model. A fundamentalist market maker ($g_d = 0$), instead, will reply to a positive price deviation and excess demand with a price decrease, following her own beliefs, and thus she will be more likely to cause instability over the market (see below).

The remaining part of this section will be dedicated to examine the two type case of fundamentalists ($g_0 = 0, b_0 = 0$, $C_0 = C > 0$) and chartists ($g_1 = g_c > 0, b_1 = 0, C_1 = 0$) which has been already examined in the previous section, with the purpose to extend the results of that section to the actual framework. In the two types case eq. (23) becomes

$$x_t = \frac{R + g_c\,(2DR\omega+1)\;\, n_{1,t}}{2R\,(DR\omega+1) - (R-g_d)^2}x_{t-1} \tag{25}$$

where $n_{1,t}$ is still given by eq. (17). After having replaced $n_{1,t} = \frac{1-m_t}{2}$ we obtain

$$x_t = \frac{R - g_c\,(2DR\omega+1)\,\dfrac{m_t-1}{2}}{2R\,(DR\omega+1) - (R-g_d)^2}x_{t-1} \tag{26}$$

where $m_t$ is still given by eq. (19). The BH98 equation (20) is obtained for $\omega \to \infty$ and $g_d \to R$, i.e. when the market maker doesn't accumulate inventories and doesn't trade.

The equation (26) has a fundamental steady state solution $x = 0$, like the model of the previous section, which becomes likewise unstable due to a pitchfork bifurcation which occurs when the following equality holds:

$$g_c = \left(1 + e^{-\beta C}\right)\left[R - \frac{(g_d - R)^2}{2DR\omega + 1}\right] \tag{27}$$

Accordingly, we can extend to the actual framework, with some adjustments, the results that there are additional non fundamental steady states under appropriate conditions, and that the fundamental steady state becomes unstable under the same conditions. In particular, the following proposition holds:

**Proposition 4.** *(Existence and stability of steady states). Let*

$$\overline{m}_{nf} = 1 - \frac{2R}{g_c} + \frac{2\left(R - g_d\right)^2}{g_c\left(2DR\omega + 1\right)}$$

*and* $x^*$ *be the positive solution (if it exists) of* $\tanh\left[\frac{\beta}{2}\left(Dg_c\left(R-1\right)x^2 - C\right)\right] = \overline{m}_{nf}$. *Further suppose that* $g_d \in \left(R - \sqrt{R(2DR\omega+1)},\quad R + \sqrt{R(2DR\omega+1)}\right)$. *Then:*

1. *For* $0 < g_c < R - \frac{(R-g_d)^2}{(2DR\omega+1)}$ *the fundamental equilibrium* $E_0 = (0, \overline{m}_f)$ *is the unique, globally stable steady state*
2. *For* $g_c > 2\left[R - \frac{(R-g_d)^2}{(2DR\omega+1)}\right]$ *the fundamental equilibrium is locally unstable and two other steady states exist:* $E_1 = (x^*, \overline{m}_{nf})$ *and* $E_2 = (-x^*, \overline{m}_{nf})$
3. *For* $0 < R - \frac{(R-g_d)^2}{(2DR\omega+1)} < g_c < 2\left[R - \frac{(R-g_d)^2}{(2DR\omega+1)}\right]$ *there exists* $\beta^* = \frac{1}{C}\log\left(\frac{1-\overline{m}_{nf}}{1+\overline{m}_{nf}}\right) \in (0,\infty)$ *such that*
   - *(a) if* $\beta < \beta^*$ *the fundamental equilibrium* $E_0 = (0, \overline{m}_f)$ *is the unique, globally stable steady state*
   - *(b) if* $\beta > \beta^*$ *the fundamental equilibrium is locally unstable and two other steady states exist:* $E_1 = (x^*, \overline{m}_{nf})$ *and* $E_2 = (-x^*, \overline{m}_{nf})$
   - *(c) if* $\beta = \beta^*$ *the fundamental and non fundamental equilibria coincide*

*Proof.* See Appendix A. □

The combinations of values of $g_c$ and $g_d$ which are consistent with the hypotheses of Proposition 4 are represented in Fig. (5). The light grey area represents the combinations for which the fundamental steady state is always stable, the dark grey area the combinations for which $\beta^* \in (0, \infty)$ exists. We see that the chartist expectation of the market maker lowers the thresholds of $g_c$ for which the fundamental steady state becomes unstable and those for which $\beta^*$ exists, when compared with the thresholds of the previous model, except when $g_d = R$.

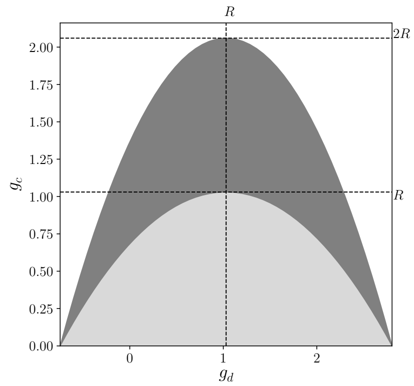
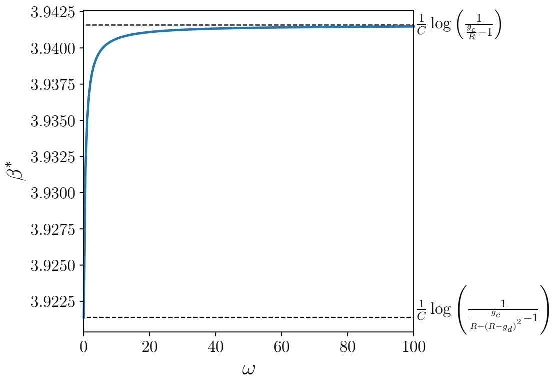


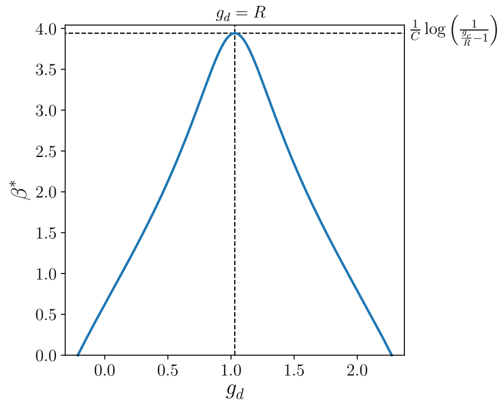

Figure 5: Values of $g_c$ and $g_d$ which are consistent with the hypotheses of Proposition 4: stability of the fundamental steady state (light grey); existence of $\beta^* \in (0, \infty)$ (dark gray). The values of the other parameters are: $D = C = \omega = 1$, $R = 1.03$.

From Fig. 6, panel (a), we see that $\beta^*$ is increasing in $\omega$ and converges to $\beta^* = \frac{1}{C} \log \left( \frac{1}{g_c/R - 1} \right)$ for $\omega \to \infty$. In this limit we recover the result of BH98 and of Sec. 3. This means that, the more the market maker is oriented against holding inventories, the more she can compensate for her own trading activity and try to keep the market anchored to the fundamental equilibrium.

Except that in this limit, an activist market maker is more likely to make the market evolve towards a non fundamental steady state than a passive market maker. Indeed, from fig. 6, panel (b), we see that $\beta^*$ is increasing in $g_d$ up to $g_d = R$ and decreasing for $g_d > R$. Thus the effect of an increasingly extrapolative chartist market maker on market stability is not monotonic: stabilizing for $g_d < R$ and destabilizing for $g_d > R$. When $g_d = R$, we recover again the same $\beta^*$ of BH98 and of Sec. 3. For any other $g_d \neq R$ we have instead that $\beta^*$ is lower than in those two models. Thus an activist market maker makes the market less stable than a passive market maker except that in this case, in which we recover the same level of stability of Sec 3 because the market maker is not trading at all (see eq. (22)). Moreover, the effect of a chartist market maker is to improve the stability of the market with respect to a fundamentalist market maker ($g_d = 0$) for any $g_d \in (0, 2R)$. From eq. (24) we see that a fundamentalist market maker lowers the price in front of excess market demand and this explains the negative effect on market stability.




(a) $g_d = 1.05$




(b) $\omega = 1$

Figure 6: Value of $\beta^*$ for different values of $\omega$ and $g_d$. The other parameters are fixed at the following values: $D = C = 1$, $R = 1.03$, $g_c = 1.05$.

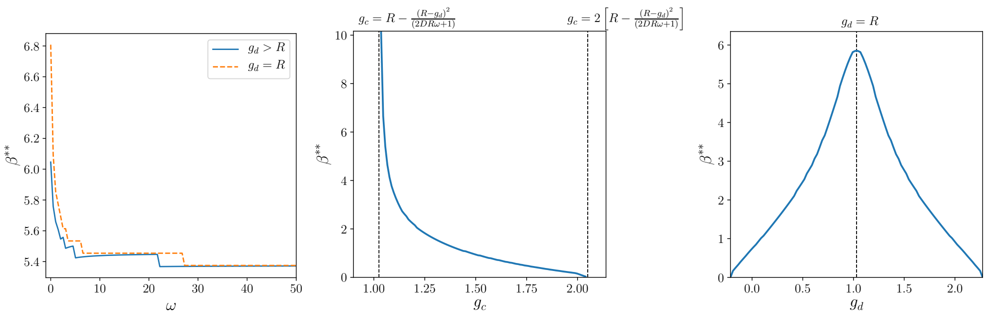
According to Proposition 2 of the previous Section there exists a critical value $\beta^{**}$ above which the two non fundamental steady states become themselves unstable. We have already underlined that this result is important because, when this condition occurs, the system might follow non trivial dynamical paths and even exhibit complex or chaotic trajectories. The equivalent of Proposition 2 for the system decribed by eq. (26) is stated as follows:

**Proposition 5.** *(Secondary Bifurcation). Let $E_1$ and $E_2$ be the non fundamental steady states as in Proposition 4. Assume $0 \leq R - \frac{(R-g_d)^2}{(2DR\omega+1)} < g_c < 2\left[R - \frac{(R-g_d)^2}{(2DR\omega+1)}\right]$, $g_d \in \left(R - \sqrt{R(2DR\omega+1)}, \quad R + \sqrt{R(2DR\omega+1)}\right)$ and $C > 0$. Let $\beta^* \geq 0$ be the pitchfork bifurcation value. Further suppose that $R \leq \frac{4}{3}$. Then there exists $\beta^{**}$ such that $E_1$ and $E_2$ are stable for $\beta^* < \beta < \beta^{**}$ and unstable for $\beta > \beta^{**}$. For $\beta = \beta^{**}$ $E_1$ and $E_2$ exhibit a Hopf bifurcation.*

*Proof.* See Appendix A. □

The numerical analysis of the discriminant of the characteristic equation at the non fundamental steady states and the numerical computation of its solutions yield essentially the same results of the previous model. There is an interval of values of $\beta$ in which the discriminant is negative and thus there exist two conjugate complex roots which cross the unit circle at a value $\beta^{**}$. At this critical value the non fundamental steady state becomes unstable and for higher values of $\beta$ we observe an oscillating behavior between the fundamental and non fundamental steady states. The resulting dynamics is qualitatively identical to the 3-D system in BH98 and to the one depicted in Fig. 3.

We remark that, using the same parameter values $\omega = D = C = 1$ and $R = 1.03$, plus $g_c = 1.05$ we obtain that the critical values $\beta^{**}$ are lower than in Sec. 3 except when $g_d = R$ and the market maker doesn't trade (Fig. 7, panels (a) and (c)). Moreover, $\beta^{**}$ is decreasing in $\omega$ and thus the same considerations apply, namely that market making is making the non fundamental steady states more likely to be stable.




(a) $g_c = 1.05$ (b) $\omega = 1$, $g_d = 1.1$ (c) $\omega = 1$, $g_c = 1.05$

Figure 7: Value of $\beta^{**}$ with $D = C = 1$, $R = 1.03$.

As final step, we replicate the results concerning homoclinic orbits and global instability of the previous Section. The equivalent of Proposition 3 of Sec. 3, which is still based on the boundedness of the unstable manifold, is stated as follows:

**Proposition 6.** *Assume* $C > 0$, $g_d \in \left(R - \sqrt{R(2DR\omega+1)},\quad R + \sqrt{R(2DR\omega+1)}\right)$ *and* $\beta \to \infty$. *Suppose that* $g_c > R - \frac{(R-g_d)^2}{(2DR\omega+1)} \geq 0$ *so that the fundamental steady state* $E_0 = (0, -1)$ *is locally unstable. Let's fix*

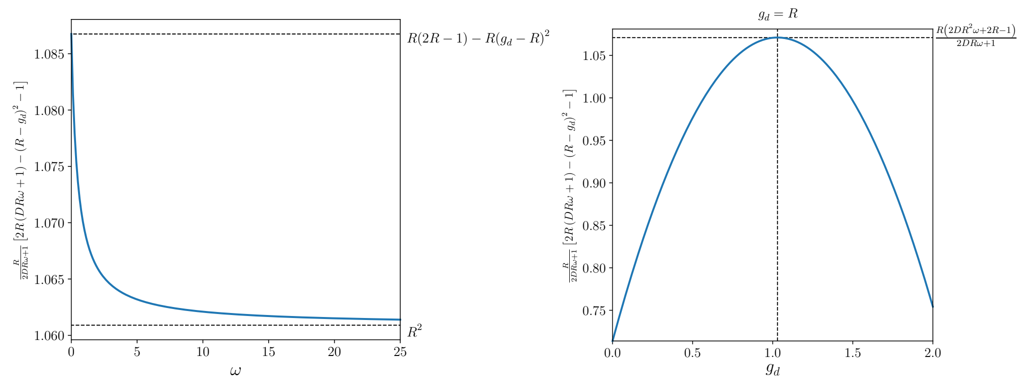
$\bar{g}_1 = \frac{R}{2DR\omega+1}\left[2R\left(DR\omega+1\right) - \left(R - g_d\right)^2 - 1\right]$. *There are two possibilities for the unstable manifold* $W^u(E_0)$*:*

1. *if* $g_c > \bar{g}_1$ *then* $W^u(E_0)$ *equals the unstable eigenvector and is unbounded*

*2. if* $R - \frac{(R-g_d)^2}{(2DR\omega+1)} < g_c < \bar{g}_1$ *then* $W^u(E_0)$ *is bounded and all orbits converge to* $E_0$

*Proof.* See Appendix A. □

From Fig. 8 we see that the effect of $\omega$ on $\bar{g}_1$ is the same of the one on $\bar{g}_0$ in previous model, thus market making still has a stabilizing effect compared to market clearing. On the other hand we see from panel (b) that the speculative activity of the market maker makes the market more likely of being globally unstable because it lowers the threshold $\bar{g}_1$ with a behavior that is equivalent to the one of fig. 6, panel (b).

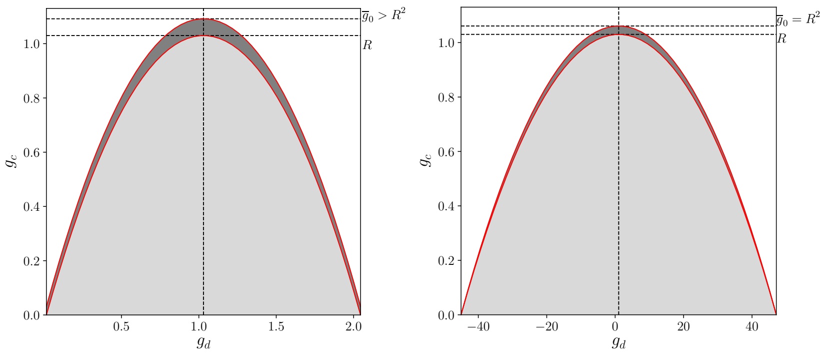



(a) $g_d = 1.1$ (b) $\omega = 1$

Figure 8: Value of $\bar{g}_1$ for different values of $\omega$ and $g_d$. The other parameters are fixed at the following values: $D = C = 1$, $R = 1.03$.

The combinations of values of the parameters $g_d$ and $g_c$ for which homoclinic orbits exist are represented by the dark grey areas of Fig. 9. We see that for $\omega \to \infty$ the dark grey area becomes thinner because the higher boundary, corresponding to $\bar{g}_1$, shifts downwards. At the same time, the threshold of $g_c$ above which the model becomes unstable is lower when compared with the thresholds of the previous model, except when $g_d = R$. Moreover, except in the same case, the range of values of $g_c$ for which homoclinic orbits exist for a given $g_d$ (i.e. the height of the dark grey area) is also smaller than in the previous model, where it is given by the fixed difference $\bar{g}_0 - R$.

Following the same kind of arguments proposed at the end of the previous section, in this case we might summarize the results by saying, in the first place, that the trading activity of the market maker, contrary to the previous model, has an impact on the existence and stability of the fundamental steady state and that this effect occurs for any value of $g_c$, i.e. it is independent from




(a) $\omega = 0$ (b) $\omega = 1,000$

Figure 9: Combinations of $g_d$ and $g_c$ for which homoclinic orbits exist. The other parameters are fixed at the following values: $D = C = 1$, $R = 1.03$.

the aggressiveness of chartists according to Proposition 4, although the value of $\beta^*$ depends also on $g_c$. The interpretation of the other two results instead can be very similar to the previous model. Indeed from Fig. 6 we see that any value of $\beta^{**}$ that can be obtained by choosing a value of $\omega$ or $g_d$ can be also obtained by choosing a value of $g_c$. Thus, as in the previous case, what we can obtain with a specific behavior of the market maker we can obtain as well with a more aggressive chartist speculator. Instead, according to Proposition (6), the existence of the market maker impacts the global stability of the system for any given behavioral attitude of the chartist speculators. What is different is that the threshold for global instability might be lowered, and thus the system might become more likely to be globally unstable, if the market maker trades.

# 5 Conclusions

The dynamic behavior of the models presented in this paper is qualitatively identical to the one described in BH98, which is derived as a special case in this setting. This shows that the complex dynamics of that model is independent from the institutional framework of the market and in particular from the assumption of market clearing. While this result was already established, under a different setting, by Hommes *et al.* (2005), I show that the market maker has conflicting effects on the stability of the market. She acts as a stabilizer when she allows for market imbalances, and as a destabilizer when she manages aggressively her inventories or when she trades actively. In particular, the speculative behavior of the market maker becomes critical for the stability of the market because it limits the behavior of the speculators, although some thresholds, and especially those governing the global instability of the model (Propositions (3) and (6)), are independent from the behavioral parameters of the speculators. Indeed, according to the two models presented here the more stable institutional framework is one in which the market maker is inventory neutral and doesn't trade at all, but even in this scenario the typical complex behavior of BH98 occurs.

It is less straightforward to compare the results of this paper with other works which employ a dynamic optimization setting instead of the static optimization setting which is customary in models with bounded rational heterogeneous agents. For instance, Leach and Madhavan (1993) suppose that the market maker might deviate from a statically optimal price because she takes into account the effect of her current pricing decision on the future trading of her counterparts and thus on her future wealth. This strategy might lend more stability to markets (in particular avoiding that the market shuts down) at the cost of wider spreads. But these results rely on the critical assumption the some speculators are more informed than the market maker, which is exluded from the present framework. The same considerations apply to the model of Madhavan and Smidt (1991), which is somewhat similar to this paper in spirit because it considers a market maker which is both a dealer and a speculator, but again depends critically on the hypothesis of information asymmetry.

A critical assumption of this paper is that the wholesale market is always perfectly liquid for the market maker. This hypothesis is required to achieve positive profits since it allows the market maker to adjust her inventory at a favourable price. By making this assumption, we are implicitly introducing some liquidity provider of last resort in the model. Who might play this role in the FX market? One option are hedgers, like non financial corporations, who represent important actors on the FX market. The model of Evans and Lyons (2002) includes two distinct classes of agents besides dealers: the first one, represented by speculators, demands liquidity from FX dealers at the beginning of the day, the second one, represented by hedgers or "profit takers", supplies to dealers the necessary liquidity to balance the market. Empirical evidence confirms that financial customers demand liquidity while hedgers are net liquidity providers (King *et al.*, 2013). Alternatively, we should not disregard banks: given the pivotal role of the banking sector in the FX market, their highly elastic supply of liquidity accomodates the needs of the operators on the money markets for the domestic currency and, through liquidity swaps arranged by central banks, for the most traded foreign currencies. A more careful assessment of this topic is left for future research.

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# A Proofs

*Proof of Proposition 1.* Starting from eq. (18), the steady state must satisfy the following equation:

$$x^* = \frac{1}{DR\omega + 1}\left[\frac{1}{2} - g\left(D\omega + \frac{1}{2R}\right) g\,\frac{m^* - 1}{2}\right] x^* \tag{28}$$

where

$$m^* = n_0^* - n_1^* = \tanh\left[\frac{\beta}{2}\left(-C + Dg\left(R - 1\right)\left(x^*\right)^2\right)\right] \tag{29}$$

The non fundamental solution is obtained by solving the following equation

$$m^* = 1 - \frac{2R}{g} \tag{30}$$

We obtain that

$$x^* = \sqrt{\frac{C\beta + \log\left(\frac{g}{R} - 1\right)}{D\beta g\left(R - 1\right)}} \tag{31}$$

Since $x^*$ must be real, the following must hold:

$$\left(\frac{g}{R} - 1\right) e^{C\beta} \geq 1 \tag{32}$$

solving for $g$ we obtain $g \geq R\left(1 + e^{-C\beta}\right)$. Letting $\beta \to \infty$ we obtain the first claim, letting $\beta \to 0$ the second claim regarding the existence of two symmetric non fundamental steady states.

The positive eigenvalue at the fundamental steady state is

$$\lambda = \frac{2R + g\left(2DR\omega + 1\right)\left[\tanh\left(\frac{\beta C}{2}\right) + 1\right]}{4R\left(DR\omega + 1\right)} \tag{33}$$

Solving for $g$ the inequality $\lambda < 1$ we obtain that the fundamental steady state is locally stable if $0 < g < R\left(1 + e^{-C\beta}\right)$. Letting $\beta \to \infty$ we obtain the first claim, letting $\beta \to 0$ the second claim regarding the stability of the fundamental steady state.

Solving for $\beta$ the equality $\lambda = 1$ we obtain the critical value

$$\beta^* = \frac{1}{C}\log\left(\frac{1}{g/R - 1}\right) \tag{34}$$

We see that for $R < g < 2R$ we have that $\beta^* \in (0, \infty)$. The last claim is proved by substituting $\beta^*$ into eq. (31) since in this case we obtain $x^* = 0$. □

*Proof of Proposition 2.* The characteristic equation for the stability of the non fundamental steady states is

$$P(\lambda, K) = \lambda^3 - \lambda^2 \left( \frac{K}{R} + 1 \right) + K\lambda + K \left( 1 - \frac{1}{R} \right) = 0 \tag{35}$$

where

$$K = -\frac{g_c C}{8\,(R-1)\,(DR\omega + 1)}\,(\beta - \beta^*)\,(2DR\omega + 1)\left(\overline{m}_{nf}^2 - 1\right) \tag{36}$$

$\overline{m}_{nf} \equiv 1 - \frac{2R}{g}$ and $\beta^*$ is defined as in eq. (34). Under the hypotheses we have $\beta > \beta^*$ and $-1 < \overline{m}_{nf} < 0$, thus $K > 0$.

The critical points of $P(\lambda, K)$ are

$$x_{critical} = \frac{1}{3R}\left( K + R \pm \sqrt{(K+R)^2 - 3KR^2} \right) \tag{37}$$

We see that for $K > 0$ and $(K+R)^2 - 3KR^2 \geq 0$, the two critical points are real and positive. Since the derivative of $G(K) \equiv (K+R)^2 - 3KR^2$ is increasing in $K$, while the second derivative is strictly positive, $G(K)$ has a global minimum at $K^* = \frac{R}{2}(3R-2) > 0$. If we substitute back $K^*$ into $G$ we obtain that the minimum of $G$ is $G(K^*) = -\frac{9R^4}{4} + 3R^3$ which is nonnegative if the following holds

$$1 < R \leq \frac{4}{3} \tag{38}$$

which we assume is true from now on. From the fact that $K$ is strictly positive we derive

$$P(-1, K) = -\frac{2}{R}(K+R) < 0$$
$$P(0, K) = \frac{K}{R}(R-1) > 0$$
$$P(1, K) = \frac{2K}{R}(R-1) > 0$$

Thus the smallest of the three roots is always real and comprised between -1 and 0. The two other roots instead have always a positive real part. In fact the positive larger critical point lies to the left of the largest root if the latter is real, or otherwise it coincides with the real part of the two complex conjugate roots. On the other hand, the positive smaller critical point lies to the left of the smaller of the two real roots or to the left of the real part of the two complex conjugate roots.

In order to show that the largest root crosses the unit circle for $\beta \to \infty$ it suffices to show that the largest critical point is increasing in $K$, since we know that $K$ is increasing in $\beta$ and since the largest critical point is a lower bound for the absolute value of the largest root in absolute terms. Now we differentiate the largest critical point wrt $K$:

$$\frac{\partial x_{critical}}{\partial K} = \frac{1}{3R}\left(1 + \frac{K - \frac{3R^2}{2} + R}{\sqrt{-3KR^2 + (K+R)^2}}\right) \tag{39}$$

We consider the following lower bound:

$$\frac{1}{3R}\left(1 + \frac{-\frac{3R^2}{2} + R}{K+R}\right)$$

which is increasing in $K$ for $R \geq 1$. We need to prove that it is nonnegative. Solving the inequality for $K$ we obtain

$$K \geq \frac{R}{2}(3R-4) \tag{40}$$

which is automatically satisfied since $K > 0$ as soon as $R \leq 4/3$.

In order to prove that we have a Hopf bifurcation we observe that if the largest real root would become equal to unity while $K$ increases, we would necessarily have that $P(1, K) = 0$ for some value of $K$, something which contradicts our previous statements.

□

*Proof of Proposition 3.* Following BH98, in order to prove the claim we need to show that, when the fundamental steady state is unstable, the system returns to the fundamental steady state for some $T > 0$ if and only if $g < \frac{R\left(2DR^2\omega+2R-1\right)}{2DR\omega+1}$.

When $\beta \to \infty$ we have that $\overline{m}_f \equiv \tanh\left(-\beta\frac{C}{2}\right) \to -1$ and the fundamental steady state is $E_0 = (0, -1)$. The eigenvalues at the fundamental steady state are $(0, 0, \lambda_\infty)$ with

$$\lambda_\infty = \frac{R + (2DR\omega + 1)g}{2R(DR\omega + 1)} \tag{41}$$

We suppose that $g > R$, thus the fundamental steady state is unstable since $\lambda_\infty > 1$. The eigenvector associated with $\lambda_\infty$ is

$$\begin{bmatrix} \frac{(2DRg\omega+R+g)^2}{4R^2(DR\omega+1)^2} \\ \frac{2DRg\omega+R+g}{2R(DR\omega+1)} \\ 1 \end{bmatrix} \tag{42}$$

We know that the system evolves according to eqs. (18) and (19), which we reproduce here for convenience of the reader:

$$x_t = \frac{2R - g\,(2DR\omega + 1)\,(m_t - 1)}{4R\,(DR\omega + 1)} x_{t-1} \tag{18}$$

$$m_t = \tanh\left(\frac{\beta}{2}\left(Dg_c\left(Rx_{t-2} - x_{t-1}\right)x_{t-3} - C\right)\right) \tag{19}$$

Let's consider the following expression

$$C_t \equiv Dg\left(Rx_{t-2} - x_t\right)x_{t-3} \tag{43}$$

we see that for $\beta \to \infty$

$$m_t = \begin{cases} +1 & \text{if} \quad C_t > C \\ -1 & \text{if} \quad C_t < C \end{cases} \tag{44}$$

Let's suppose that, starting from the fundamental steady state, a small shock occurs at $t = -2$ and is propagated until $t = 0$. Then we have

$$\begin{aligned} x_{-2} &= \epsilon \\ x_{-1} &= \frac{2R - g\,(2DR\omega + 1)\,(m_{-1} - 1)}{4R\,(DR\omega + 1)}\epsilon \\ x_0 &= \left[\frac{2R - g\,(2DR\omega + 1)\,(m_0 - 1)}{4R\,(DR\omega + 1)}\right]\left[\frac{2R - g\,(2DR\omega + 1)\,(m_{-1} - 1)}{4R\,(DR\omega + 1)}\right]\epsilon \end{aligned}$$

By hypothesis we know that $m_{-2} = -1$. Furthermore $C_{-1} = C_0 = 0$ thus $m_{-1} = m_0 = -1$. As a consequence we obtain the following:

$$\begin{aligned} x_{-1} &= \lambda_\infty \epsilon \\ x_0 &= \lambda_\infty^2 \epsilon \end{aligned}$$

From our hypotheses we see that the system is on an explosive path. Any trajectory starting in a neighborhood of the fundamental steady state will move along the unstable eigenvector until $m_t = -1$ and thus will diverge to infinity unless $m_T = 1$ for some $T > 0$. In fact in this case we have that

$$x_{T+1} = \frac{x_T}{2(DR\omega + 1)} \tag{45}$$

and thus $x_{T+t} \to 0$ for $t \to \infty$ as long as $C_{T+t} > C$. This eventuality depends on the evolution of the value of $C_t$ over time. In particular, if $C_t \to \infty$ monotonically for $t \to \infty$ then for some $T > 0$ the conclusion follows.

Let's suppose that $C_{t-k} < C$ for all $0 \leq k \leq t$ (otherwise the conclusion already follows) so that $m_{t-k} = -1$. Substituting in $C_t$ the iterated values obtained from an initial shock $\epsilon$ occurred at $t = -2$ we obtain

$$C_t = D\epsilon^2 g_c (\lambda^\infty)^{2t+1} (R - \lambda^\infty) \tag{46}$$

which will diverge to $\pm\infty$ depending on the sign of the rightmost term. Thus the conclusion follows solving the condition $R - \lambda^\infty > 0$ for $g$:

$$g < \frac{R(2DR^2\omega + 2R - 1)}{2DR\omega + 1} \tag{47}$$

We recover the result of BH98 considering the limit $\omega \to \infty$ where we obtain that $g < R^2$.

□

*Proof of Proposition 4.* The steady state solutions must satisfy the following equation:

$$x^* = \frac{R - (2DR\omega + 1) g_c \dfrac{m^* - 1}{2}}{2R(DR\omega + 1) - (R - g_d)^2} x^* \tag{48}$$

where $m^*$ is defined as in eq. (29). The non fundamental solution is obtained by solving the following equation:

$$m^* = \overline{m}_{nf} \tag{49}$$

where

$$\overline{m}_{nf} \equiv 1 - \frac{2R}{g_c} + \frac{2(R - g_d)^2}{g_c(2DR\omega + 1)} \tag{50}$$

We obtain that

$$x^* = \sqrt{\frac{C\beta + \log\left(\frac{1+\overline{m}_{nf}}{1-\overline{m}_{nf}}\right)}{D\beta g_c (R - 1)}} \tag{51}$$

Since $x^*$ must be real, the following must hold:

$$e^{\beta C} \frac{1+\overline{m}_{nf}}{1-\overline{m}_{nf}} \geq 1 \tag{52}$$

This expression is increasing in $\overline{m}_{nf}$ and the reader can easily check that $\overline{m}_{nf}$ is increasing in $g_c$ under the hypothesis we make on $g_d$. Thus, solving the condition for $g_c$ we obtain

$$g_c \geq \left(R - \frac{(R-g_d)^2}{2DR\omega+1}\right)\left(1+e^{-\beta C}\right) \tag{53}$$

Letting $\beta \to \infty$ we obtain the first claim, letting $\beta \to 0$ the second claim regarding the existence of two symmetric non fundamental steady states.

The non trivial eigenvalue at the fundamental steady state is

$$\lambda = \frac{2R + g_c\left(2DR\omega+1\right)\left[\tanh\left(\frac{\beta C}{2}\right)+1\right]}{4R\left(DR\omega+1\right) - 2\left(R-g_d\right)^2} \tag{54}$$

The reader can check that $\lambda$ is positive and finite under the hypothesis made on $g_d$. We see that $\lambda$ is increasing in $g_c$. Solving the inequality $\lambda < 1$ for $g_c$ we obtain that the fundamental steady state is locally stable if the following condition holds

$$0 < g_c < \left(R - \frac{(R-g_d)^2}{2DR\omega+1}\right)\left(1+e^{-\beta C}\right) \tag{55}$$

Letting $\beta \to \infty$ we obtain the first claim, letting $\beta \to 0$ the second claim regarding the stability of the fundamental steady state.

Solving the equality $\lambda = 1$ for $\beta$ we obtain the critical value $\beta^*$:

$$\beta^* = \frac{1}{C}\log\left(\frac{1-\overline{m}_{nf}}{1+\overline{m}_{nf}}\right) \tag{56}$$

It's easy to check that, under the hypothesis made on $g_d$, we have that $\beta^* \in (0,\infty)$ for $0 \leq R - \frac{(R-g_d)^2}{(2DR\omega+1)} < g_c < 2\left[R - \frac{(R-g_d)^2}{(2DR\omega+1)}\right]$ . The last claim of the proposition is proved by substituting $\beta^*$ into eq. (51) since in this case we obtain $x^* = 0$.

□

*Proof of Proposition 5.* The characteristic equation for the stability of the non fundamental steady states is

$$P(\lambda, K) = \lambda^3 - \lambda^2 \left( \frac{K}{R} + 1 \right) + K\lambda + K \left( 1 - \frac{1}{R} \right) = 0 \tag{57}$$

where

$$K = -\frac{R g_c C}{(R-1)\left[8R\,(DR\omega + 1) - 4\,(R - g_d)^2\right]} \left(\beta - \beta^*\right)\left(2DR\omega + 1\right)\left(\overline{m}_{nf}^2 - 1\right) \tag{58}$$

and $\overline{m}_{nf}$ and $\beta^*$ are defined as in eqs. (50) and (56) respectively. Under the hypotheses we have $\beta > \beta^*$ and $-1 < \overline{m}_{nf} < 0$, thus the sign of $K$ depends on the sign of $F(K) \equiv 8R\,(DR\omega + 1) - 4\,(R - g_d)^2$ which is increasing in $\omega$. We have that $F(K) > 0$ if the following condition holds:

$$\omega > \frac{1}{DR^2} \left[ \frac{1}{2} (R - g_d)^2 - R \right] \tag{59}$$

Since $g_d \in \left( R - \sqrt{R(2DR\omega + 1)}, \quad R + \sqrt{R(2DR\omega + 1)} \right)$ we can derive the following inequality:

$$\omega > \frac{1}{DR} \left( \frac{(2DR\omega + 1)}{2} - 1 \right) = \omega - \frac{1}{2DR} \tag{60}$$

which is always satisfied given our hypotheses on the parameters. Thus the condition (59) is satisfied by our assumptions and we obtain that $K > 0$. Following the same arguments of Proposition 2, we obtain the conclusion.

□

*Proof of Proposition 6.* The argument follows the same lines of the proof of Proposition 2. In particular, the eigenvalues at the fundamental steady state for $\beta \to \infty$ are $(0, 0, \lambda_\infty)$ with

$$\lambda_\infty = \frac{(2DR\omega + 1)g_c + R}{2R(DR\omega + 1) - (R - g_d)^2} \tag{61}$$

The reader can check that, under the hypotheses, $\lambda_\infty$ is finite and greater than unity. The system evolves according to eq. (26) which we reproduce for convenience of the reader:

$$x_t = \frac{R - g_c\,(2DR\omega + 1)\,\dfrac{m_t - 1}{2}}{2R\,(DR\omega + 1) - (R - g_d)^2} x_{t-1} \tag{26}$$

where $m_t$ is given by eq. (19). Introducing a shock at $t = -2$ and repeating the steps of Proposition 3 we obtain that $x_0 = \lambda_\infty^2 \epsilon$ and $m_0 = -1$. As long as $m_t = -1$ for $t > 0$, we obtain that $x_t = \lambda_\infty^{2+t} \epsilon$ and the system is on an explosive path. Instead if $m_T = 1$ for some $T > 0$ we have that

$$x_{T+1} = \frac{R}{2R\left(DR\omega + 1\right) - \left(R - g_d\right)^2} x_T \tag{62}$$

The reader can check that under the hypotheses made on $g_d$ the coefficient on the RHS is positive and smaller than unity, then $x_{T+t} \to 0$ for $t \to \infty$ as long as $C_{T+t} > C$. The expression for $C_t$ is the same of eq. (46). Thus the conclusion follows from the following condition:

$$g_c < \frac{R}{2DR\omega + 1} \left[2R\left(DR\omega + 1\right) - \left(R - g_d\right)^2 - 1\right] \tag{63}$$

We recover the result of BH98 considering the limit $\omega \to \infty$ where we obtain $g_c < R^2$. In the limit $\omega \to 0$ we obtain instead the following upper bound:

$$g_c < R\left[2R - 1 - \left(R - g_d\right)^2\right] \tag{64}$$

□

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