Università di Firenze, Università di Perugia, INdAM consorziate nel CIAFM

DOTTORATO DI RICERCA
IN MATEMATICA, INFORMATICA, STATISTICA

CURRICULUM IN MATEMATICA
CICLO XXXII

Sede amministrativa Università degli Studi di Firenze
Coordinatore Prof. Graziano Gentili

Special Solutions for Shell Models of Energy Cascade in Turbulence

Settore Scientifico Disciplinare MAT/05

Dottorando
Dott. Carlo Metta

Tutore
Prof. Francesco Morandin

Coordinatore
Prof. Graziano Gentili

Anni 2016/2019

Contents

Introduction

The theory of turbulence deals with the non homogeneous behavior of intense fluid flow. We know that the evolution of all these phenomena can be described through the **Navier-Stokes equations**. NSE regards the nature of fully developed turbulence and is one of the most important models of mathematical physics. There are still cardinal open questions due to the nature of non-linearity present in the equation

$$\partial_t v_i + v_j \partial_j v_i = -\partial_i p + \nu \partial_{jj} v_i + f_i$$
$$\partial_i v_i = 0$$

Navier-Stokes Equations

The first equation states that the acceleration of a fluid fragment equals the sum of the forces acting on it while the continuity equation represents the conservation of mass for incompressible fluids. From these four equations is theoretically possible to determine the three components of the fluid velocity and the pressure field. However, no general solutions to the NSE are known yet apart for very simple laminar flows.

The first contribute to the theory of turbulence can be found in [60] by Richardson. He describes a fluid as formed of large eddies splitting up into smaller eddies, which again split up into yet smaller eddies until they vanish by viscosity. Energy is inserted at large scales and then it cascades into smaller scales until it disappears at the viscous scale. Richardson studies led Kolmogorov [47] to develop a more structured theory of turbulence.

In 1941 Kolmogorov [47] proposed the picture of a flow sustained by a force active on a large scales such that the flow is in a state of statistical equilibrium. The state of the flow is characterized by the mean energy dissipation ϵ due to viscosity. The velocity at a given length scale ℓ is the velocity difference $\delta v(\ell) = |v(r+l) - v(r)|$, it is characteristic of the velocity related to an eddy of size l. The effect of flow velocity on a larger scale is to move the eddy through the flow as a rigid body. Similarly, if we consider a much smaller eddy within the larger eddy, the effect of the larger eddy on the smaller is the same as the effect of the larger scale flow on the large eddy. It is common to assume that the velocity difference $\delta v(\ell)$ is a function of the scale ℓ and the mean energy dissipation ϵ. From a dimensional study the only possible relationship is the famous Kolmogorov scaling law

$$\delta v(\ell) \sim (\epsilon \, \ell)^{1/3}.$$

Kolmogorov K41 Law

K41 contains all the essence of Kolmogorov turbulence theory.

One natural approach to study small scale turbulence is to introduce the **structure functions** of the velocity field of different orders:

$$S_p(\ell) = \langle \delta v(\ell)^p \rangle$$

where the brackets stand for the statistical average among the range scale ℓ.

In 1945, Kolmogorov [48] exhibits an exact relation for the third order structure function:

$$S_3(\ell) = \frac{4}{5} \epsilon \, \ell.$$

Kolmogorov four-fifth law

This is probably the most important and exact result in fully developed turbulence. A theory to be acceptable must either satisfy the four-fifth law, or violate the assumptions made in deriving it.

The K41 scaling theory predicts the scaling

$$S_p(\ell) \sim \ell^{p/3}.$$

Although, many real experiments have observed flows that behave intermittently: calm periods are interrupted by sudden blasts of energy at small scales which are then effectively damped. There is enough experimental evidence to state that deviations to the scaling law are present in the inertial range of fully developed turbulence. The structure function still scales with length, but this time the scaling exponent $\zeta(p)$

$$S_p(\ell) \sim \ell^{\zeta(p)}$$

are different from $p/3$. The function $\zeta(p)$ is called the **anomalous scaling exponent**, it represents the nonuniform essence of intermittent flow. Calculating the anomalous scaling exponents from NSE is a major challenge in which there has been little success. Such intermittent behaviour is much easier to understand in the case of **Shell models**. This has been a major motivation for studying them.

Shell Models

It is reasonable to develop alternative models that are consistent with but simpler than the original Navier-Stokes equation. In the last decades many models have been introduced to study all different characteristics of turbulent fluids. Among these models we are interested in the family of the **Dyadic models**.

In the Fourier representation of NSE, the transfer of energy from large to small scales is described as a flux of energy from small wave numbers to large wave numbers. The idea behind shell models is to divide the space into concentric spheres with exponentially growing radius $k_n = \lambda^n$. We then call n-th **shell** the set of wave numbers contained in the n-th sphere and not contained in the $(n$-1$)$-th sphere.

Shell models investigate the energy cascade flow with a system of coupled nonlinear ordinary differential equation of the form:

$$\frac{d}{dt}u_n = k_n G_n[u, u] - \nu_n u_n + f_n,$$

where the variable u_n represents the evolution over time of the velocity over a wavelength of scale k_n. The nonlinear function $G_n[\cdot, \cdot]$ is chosen to preserve some suitable properties inherited from the original nonlinear terms of NSE. Moreover, it is common for $G_n[\cdot, \cdot]$ to couple only scales that are close to each other (for instance nearest and next-to-nearest shells).

The constraints to have local interaction, quadratic non-linearity, preserving total energy (or total helicity), and phase-space evolution do not fix in a unique way the form $G_n[\cdot, \cdot]$. Consequently, many models have been developed in recent years in order to study different aspects of turbulent fluids.

Dyadic Models

This thesis deepens the study of some dyadic models all related to the original dyadic model introduced by Obukhov [58] in 1971 and by Desnianskii and Novikov [32], [33] in 1974.

Following Katz and Pavlovic [44], we can think of a tree-like structure J, where the nodes are eddies: for every eddy $j \in J$ we denote its father by $\bar{\jmath}$ and the set of its offspring by $\mathcal{O}_j$, that corresponds to the smaller eddies produced from j by instability. Eddies belongs to **generations**: level 0 is made of the largest eddy, level n of eddies produced from eddies of level n-1. We relate every eddy j to its intensity $X_j(t)$, at time t, such that the total energy of eddy j is $X_j^2(t)$. We then couple eddies intensities by a system of ordinary differential equation which specifies that the intensity of eddy j increases thanks to a flux of energy from $\bar{\jmath}$ and decreases thanks to a flux of energy to the set $\mathcal{O}_j$.

$$\frac{d}{dt}X_j = c_j X_{\bar{\jmath}}^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k,$$

Katz and Pavlovic model

for some specific positive coefficients c_j.

We stress that in 2-dimensional fluids it has been observed a phenomenon called **vortex cannibalization** that corresponds to an **inverse cascade** dynamics from small to large eddies. This is not taken into account by Katz and Pavlovic model. On the other hand, in 3-dimensional fluids the main flux of energy is attributed only to the direct cascade, thus we will think Katz and Pavlovic model always as prototype of 3D fluids dynamics.

Later, Waleffe [63] proposed a simplified model where instead of the branching tree structure comprises a linear tree of the functions $X_j(t)$ satisfying an infinite system of ordinary differential equations

$$\frac{d}{dt}X_n = \lambda_n X_{n-1}^2 - \lambda_{n+1} X_n X_{n+1},$$

Katz and Pavlovic linear model

where $\lambda_n = \lambda^n$, and $\lambda > 0$ is a positive parameter. Also this model shows a intrinsic mechanism of transferring the energy to higher nodes. Waleffe then suggested the following different model

$$\frac{d}{dt}X_n = \lambda_n X_{n-1} X_n - \lambda_{n+1} X_{n+1}^2,$$

Obukhov linear model

This model is reminiscent of Obukhov work [57]. Unlike Katz-Pavlovic model, Obukhov model lacks of the transferring energy mechanism and presents a more subtle and thus perhaps a more realistic behaviour.

These two models constitute the two basic building blocks of all linear tree models satisfying four natural conditions: (i) **quadratic non-linearity**, (ii) **scaling properties**, (iii) **energy conservation**, (iv) and **nearest neighbor coupling**. All of these except the last one are the features derived from the NSE.

Overview and Main Results

Existence and uniqueness of solutions are the first questions to ask when dealing with shell models. Many works that address these questions can be found in literature, we mention in particular [22], [26], [25], [36], [44], [46]. In Chapter 1 we start with an overview of the main results about Katz and Pavlovic dyadic linear model, focusing our attention to the existence and uniqueness of special class of solutions, namely **stationary** and **self-similar** solutions.
These two classes of functions possess many important properties often used to explain the overall dynamics of the related model. Existence of stationary solutions for Katz and Pavlovic linear model is a classical result, on the other hand self-similar solutions are a more complex matter. In [6] it was proven, by using complex analysis argument and with the help of numerical computation, existence and uniqueness of self-similar solution. In Section 2.1.6 we present a different proof based on a pullback technique that will be useful frequently throughout all the thesis.
With similar technique we give an explicit presentation of stationary solutions for the viscous forced model. Such solutions obey to K41 scaling law as long as the exponent of the viscous friction is less than a critical value. Beyond such critical value the solution is damped out with super exponential velocity.

In Section 2.2 we focus our attention on the more general **Mixed dyadic linear model**. This model carries both Katz-Pavlovic and Obukhov dynamics, thus it gives birth to a more complex structure: even the simple uniqueness properties do not hold anymore. Because of its complex dynamics, no results were found in literature until 2019 [41], where the author shows the existence of self-similar solution for particular value of the model parameters, in addition to a local uniqueness theorem. We extend such results and give a complete spectrum of existence and uniqueness results for both stationary and self-similar solutions, for every possible choice of the model parameters.

Chapter 3 is entirely devoted to studying dyadic models with tree structure. In Section 3.1 we start by recalling essential result about Katz and Pavlovic tree model, in particular its general version recently developed in [17], where the authors show that such model possess a unique stationary solution. This solutions shows a multifractal nature as well as the spatial intermittency phenomenon. We extend this study looking for special solutions that are not time independent. In particular we prove existence and uniqueness of a self-similar solution. Furthermore, we investigate what happens to the only stationary solution in the presence of viscous friction.
In Section 3.2 we present a dyadic tree model with a non-linearity of Obukhov type.

Such model manifests an **inverse cascade** energy dynamics, thus we will think of it as toy model for studying 2D fluids dynamics. Unlike the previous model, this one admits infinitely many stationary and self-similar solutions. Among stationary solutions, just one of them shows no intermittent behaviour while, for infinite of them, one can explicitly exhibit spatial intermittency. Moreover, self-similar solutions, when they exist, have the same regularity of particular stationary solutions. We conclude this thesis with Section 3.3, where we extend previous settings to a **mixed tree model**. This model carries with it both Katz-Pavlovic and Obukhov non-linearity in a tree-like branching structure. Its complex dynamics it is far from being understood, however a recent existence result has been proposed in [55]. We contribute by proving existence of stationary solutions and, depending on the model parameters, their uniqueness.

We give here a brief overview of the entire thesis as a guide through the main results.

Katz and Pavlovic linear model

The **Tree dyadic model** is a more structured version of the so called Katz-Pavlovic **Dyadic linear model** of turbulence. The linear model is based on variables Y_n which represent a cumulative intensity of the n-th shell, $n \in \mathbb{N}$. We consider the equations for Y_n in the following general form

$$\begin{aligned} \frac{dY_n(t)}{dt} &= k_{n-1}Y_{n-1}^2(t) - k_nY_n(t)Y_{n+1}(t) - \nu k_n^\gamma Y_n, \\ Y_n(0) &= y_n, \\ Y_0(t) &= F, \qquad\qquad \forall n \geq 0, \forall t \geq 0 \end{aligned} \tag{1}$$

where the coefficients satisfy $k_n = 2^{\beta n}$ for $\beta > 0$, $\nu > 0$ and $\gamma > 0$ are respectively the viscosity coefficient and exponent, y_n are the initial conditions and $F \geq 0$ is a non negative force added to the first component.
When $F = 0$ we will address model (1) as *unforced* while when $\nu = 0$ we will refer to it as *inviscid*. Without lost of generality, from now on we will fix $\nu = 1$.
We will consider the following definitions of a solution for a system of equation like (1), extending them naturally to more complex models.

Definition. *Let $I \subset \mathbb{R}^+$ be an interval.*
A local weak solution on I is a sequence of differentiable functions $Y = (Y_n)_{n\geq 1}$ satisfying (1).
A weak solution is a sequence $Y = (Y_n)_{n\geq 1}$ of differentiable function on all the

positive line $[0,\infty)$*, satisfying (1).*
A finite energy solution is a weak solution such that $Y(t) \in \ell^2$ *for all* $t \geq 0$*.*
A Leray-Hopf solution is a finite energy solution such that $||Y(t)||_{\ell^2}$ *is a non increasing function of* t*.*

Many classical results about existence and uniqueness of solutions can be found in literature. To allow the reader to easily go through this overview, we mention in particular that, given any initial condition $y \in \ell^2$, there exists at least one Leray-Hopf solution. Uniqueness is a more complex matter. In general, uniqueness does not hold for all initial finite energy conditions. However, if the initial condition $y = (y_n)_n$ is also non negative for all $n \geq 0$, then there exists a unique weak solution.

We now introduce two special classes of solutions, namely **stationary** and **self-similar** solutions.

Definition. *A stationary solution* Y *is a sequence of real number* $(y_n)_{n\geq 1}$ *such that* $Y = (y_n)_{n\geq 1} \in \ell^2$ *is solution of the unviscid system (1).*

Such definition can be extended also to the viscous case.

Definition. *A viscous stationary solution* Y^v *is a sequence of real number* $(y_n^v)_{n\geq 1}$ *such that* $Y^v = (y_n^v)_{n\geq 1} \in \ell^2$ *is solution of the viscous system (1).*

Proposition. *If* $F = 0$*, the only stationary solution of (1) is* $Y \equiv 0$*. If* $F > 0$ *then exists only one stationary solution*

$$y_n = \sqrt{F} \cdot k_n^{-1/3}, \quad n \geq 1. \tag{2}$$

Despite their simplicity, stationary solutions are one of the most important class of solution. They retain many properties that are extremely useful in order to described the overall model dynamics. First of all we can observe from (2) the anticipated Kolmogorov K41 law; moreover, in [25], the authors showed that, given a forcing term F, the only stationary solution is an exponential **global attractor** for every finite energy solution.

The other special class of solution is represented by the **self-similar** solutions.

Definition. *A self-similar solution is a finite energy solution* Y *such that there exists a differentiable function* $\phi(t)$ *and a sequence of real numbers* $a = (a_n)_{n\geq 1}$ *such that* $Y_n(t) = a_n \cdot \phi(t)$ *for all* $n \geq 1$ *and all* $t \geq 0$*.*

Next theorems establish the existence of positive self-similar and viscous stationary solution. Furthermore, they reveal the existence of an upper limit for the viscous term γ and a threshold for the forcing term F so that the corresponding viscous stationary solutions are not regular enough and still show K41 behavior.

Theorem. *Given* $t_0 < 0$*, there exists a unique positive self-similar solution with* $a_1 \neq 0$ *of the form*

$$Y_n(t) = \frac{a_n}{t - t_0}, \qquad t \geq t_0.$$

Moreover, given $t_0 < 0$ *and* $n_0 \geq 0$*, there exists a unique positive self-similar solution with*

$$a_1 = a_2 = \cdots a_{n_0} = 0, a_{n_0+1} > 0.$$

In addition, the coefficients a_n *have the property*

$$\lim_{n\to\infty} \frac{a_n}{k_n^{-1/3}} = C_{n_0},$$

for some constant C_{n_0}*.*

This theorem shows that the Kolmogorov scaling law appears in these special solution, phenomenologically associated to decaying turbulence. But it is for us a very difficult open problem to understand whether all other solutions approach the self-similar ones and in which sense. The Theorem was originally proved in [6] by using complex analysis argument and with the help of numerical computation. We present a different proof based on a useful pullback technique.
The existence of finite energy self-similar solutions is of theoretical interest in itself, moreover the existence of such solutions has a number of implications. For instance, we will see that they realize perfectly the energy bound decay rate of a general solution. It has been conjectured that the set of all self-similar solutions attracts all other finite energy solutions. If this is the case, their the decay rate would be the true one for all solutions. Moreover, in many model self-similar solutions offer an easy example of lack of uniqueness.
We extend previous classical results by looking also for viscous stationary solution of system (1).

Theorem. *Consider the forced viscous dyadic model (1). Then*

1. *if* $\gamma < 2/3$*, the model admits a unique viscous stationary solution* $Y_n^v(t) = y_n^v$*. Moreover, it exists a threshold* $F_0 > 0$ *such that:*

   (a) *if* $F > F_0$*, then the coefficients* y_n^v *have the property*

   $$\lim_{n\to\infty} \frac{y_n^v}{k_n^{-1/3}} = C_{F,\gamma},$$

   for some constant $C_{F,\gamma}$*.*

(b) *if* $F \leq F_0$*, then exists* $z = z_F \in \mathbb{R}^+$ *such that the coefficients* y_n^v *have the property*

$$\lim_{n\to\infty} \frac{y_n^v}{2^{-k_n \cdot z-(n+2)(\gamma-1)}} = C_{\gamma,z},$$

for some constant $C_{\gamma,z}$*.*

2. *if* $\gamma \geq 2/3$*, the model admits a unique viscous stationary solution* $Y_n^v(t) = y_n^v$*. Moreover, for every* $F > 0$ *exists* $z = z_{F,\gamma} \in \mathbb{R}^+$ *such that the coefficients* a_n *have the property*

$$\lim_{n\to\infty} \frac{y_n^v}{2^{-k_n \cdot z-(n+2)(\gamma-1)}} = C_{\gamma,z},$$

for some constant $C_{\gamma,z}$*.*

If we read the original Katz-Pavlovic viscous model [44], it is not difficult to see that every solution is regularized after the critical viscosity value $\gamma^c = \frac{4}{5}$. However, nothing forbids special classes of solution to be regularized with lower dissipation values. We show that constant solutions starting with enough energy can withstand a lower critical dissipation value, precisely $\gamma = \frac{2}{3}$. After that, every stationary solution becomes more regular and decay with super exponential velocity.

Mixed linear model

Katz Pavlovic and Obukhov models constitute the two basic blocks of all linear models satisfying four characteristic features derived from NSE. It is then natural to consider the following more general (unviscid) model

$$\begin{aligned}
\frac{dY_n(t)}{dt} &= \delta_1[k_n Y_{n-1}^2(t) - k_{n+1} Y_n(t) Y_{n+1}(t)] - \delta_2[k_n Y_{n+1}^2(t) - k_{n-1} Y_n(t) Y_{n-1}(t)] \\
Y_n(0) &= y_n \\
Y_0(t) &= F, \qquad \forall n \geq 0, \forall t \geq 0
\end{aligned} \tag{3}$$

where $k_n = 2^{\beta n}$ for some $\beta > 0$, $\delta_1, \delta_2 \geq 0$ non negative parameters, $F \geq 0$ is the usual force to the first component and y_n some initial condition.

The mixed linear model reduces to (inviscid) Katz-Pavlovic and Obukhov models by setting respectively $\delta_2 = 0$ and $\delta_1 = 0$, hence we expect it to carry both Katz-Pavlovic and Obukhov dynamics giving birth to a more complex structure.

Because of its complex dynamics, no results were found in literature until 2019 [41], where the author shows the existence of self-similar solution for particular value of parameters (δ_1, δ_2), in addition to a local uniqueness theorem. Moreover, in [55], the author proved existence of weak solution of the mixed dyadic model for every initial condition $y \in \ell^2$.
We extend such results with the following theorems, giving a complete spectrum of existence and uniqueness results for both (positive) stationary and self-similar solutions, for every positive couple of parameter (δ_1, δ_2).

Theorem. *The forced mixed model* (3) *admits positive stationary solutions for every choice of coefficient* $\delta_1, \delta_2 > 0$. *In particular:*

- *if* $\frac{\delta_1}{\delta_2} < k_1^{-4/3}$, *then for every* $a_0 = F > 0$ *and every* $a_1 > 0$ *there is just one positive stationary solution* $\{a_n\}_{n\geq 0}$ *of (2.30);*

- *if* $\frac{\delta_1}{\delta_2} > k_1^{-4/3}$, *then for every* $a_0 = F > 0$ *there is just one positive stationary solution* $\{a_n\}_{n\geq 0}$ *of (2.30).*

Moreover, any such stationary solution satisfies Kolmogorov's scaling law

$$\lim_{n\to\infty} \frac{a_n}{k_n^{-1/3}} = C$$

for some positive constant $C > 0$.

We divide the positive plane in two sub-regions: above the line $\delta_1/\delta_2 = k_1^{-4/3}$ there are infinitely many finite energy stationary solution; below the same line uniqueness holds for every forcing term $F > 0$.

It easy to prove that self-similar solutions for model (3) in the unforced case $(F = 0)$, have the form

$$Y_n(t) = \frac{a_n}{t - t_0}, \quad a_0 = 0,$$

with $t > t_0$ and $t_0 < 0$.

Theorem. *Given* $t_0 < 0$, *and* $k_1^{-4} \leq \delta_1/\delta_2 \leq 1$, *there exist self-similar solutions of the unforced* $(F = 0)$ *model (3). In particular*

- *if* $k_1^{-4} \leq \delta_1/\delta_2 \leq k_1^{-4/3}$ *then for every* $a_1 > 0$ *there is just one self-similar solution* $\{a_n\}_{n\geq 0}$*;*

- *if* $k_1^{-4/3} < \delta_1/\delta_2 \leq 1$ *then there is just one self-similar solution* $\{a_n\}_{n\geq 0}$.

In addition, any such self-similar solution satisfies Kolmogorov's scaling law

$$\lim_{n\to\infty} \frac{a_n}{k_n^{-1/3}} = C$$

for some positive constant $C > 0$.

This time we divide the positive plane in four sub-regions: above the line $\delta_1/\delta_2 = k_1^{-4}$ and below $\delta_1/\delta_2 = 1$ previous theorem does not give any information about existence of self-similar solution; between the lines $\delta_1/\delta_2 = k_1^{-4}$ and $\delta_1/\delta_2 = k_1^{-4/3}$ we have existence but not uniqueness; between the lines $\delta_1/\delta_2 = k_1^{-4/3}$ and $\delta_1/\delta_2 = 1$ we have existence and uniqueness of self-similar solution. However, numerical simulation confirmed the existence of **true** bounds $L_{true} < k_1^{-4}$ and $1 < U_{true}$ such that existence holds in the wider domain $\delta_2 \cdot L_{true} \leq \delta_1 \leq \delta_2 \cdot U_{true}$. Outside of such domain no self-similar solution exists.

Katz and Pavlovic tree model

In Chapter 3 we are interested in studying a generalization of Katz and Pavlovic tree model first developed in [17], extending it to an inverse cascade model with a non-linearity of Obukhov-type as well as to a mixed model similar to (3).

The model we are interested in is described by the following system of equations:

$$X_j'(t) = c_j X_{\bar{j}}^2(t) - \sum_{k\in\mathcal{O}_j} c_k X_j(t) X_k(t), \qquad j \in J, t \geq 0 \tag{4}$$

where $c_j = d_j \cdot 2^{\beta|j|}$, for some $\beta > 0$, $d_j > 0$ for every $j \in J$, $d_\emptyset = 1$ and $X_{\bar{\emptyset}}(t) = f \geq 0$ is the forcing term on the first component.

We observe that in the case of the linear dyadic model (1), the Kolmogorov inertial range spectrum reads

$$Y_n \sim k_n^{-1/3}.$$

For the tree dyadic model (4) the Kolmogorov inertial range spectrum corresponds to

$$X_j \sim 2^{-\frac{\beta+d}{3}|j|}.$$

The generalization to coefficients d_j is the key point of model (4). It completely changes the behaviour of anomalous energy dissipation and makes the structure function ζ_p strictly concave, as it should be according to the most recent numerical simulations of realistic turbulence phenomena. Allowing d_j to be different from 1, forces spatial intermittency on the solutions.

In all tree models we always consider the quantity $|\log d_j|$ bounded, and explicit the latter limitedness by assuming the existence of $M > 0$ so that

$$\frac{1}{M} \leq d_j \leq M, \qquad \forall j \in J.$$

However, many explicit computations are possible only in the special case where the same fixed 2^d coefficients appear in every set $\{d_k \,|\, k \in \mathcal{O}_j\}$. We call this model **Repeated Coefficient Model** (RCM).

In the RCM case we set $\{d_k \,|\, k \in \mathcal{O}_j\} = \{\delta_w \,|\, w \in \Omega\}$, for some Ω of cardinality 2^d. Hence, we can introduce the log-s-norm of the coefficients. For $s \in \mathbb{R} \setminus 0$ let

$$\ell_s = \frac{1}{s} \log_2\left(\frac{1}{2^d} \sum_{w \in \Omega} \delta_w^s\right).$$

This can be extended to obtain a bounded, non-decreasing and continuous function ℓ on $[-\infty, \infty]$.
It is useful to mention here the natural spaces to study regularity of solutions:

$$H^s = \{u : J \to \mathbb{R} \mid ||u||_{H^s} = \sqrt{\sum_{j \in J} 2^{2s|j|} u_j^2} < \infty\}.$$

In particular we write $H = H^0 = \ell^2(J)$.

Compared to the linear dyadic model, this time existence and uniqueness of solutions are more subtle matters.
In [2] it has been proved that if $d_j = 1$ for every $j \in J$, then for any initial condition with non negative components there exists at least one Leray-Hopf solution. The generalization to the general model is straightforward. Uniqueness of solutions is an open problem even for the model with $d_j = 1$.
In [17] it has been proved the existence, and uniqueness in some sense, of a stationary solution by introducing a forcing term on the first component. We extend this framework in order to prove existence of solutions that are not constant in time as well as investigate the evolution of the unique stationary solution when we add a viscous friction in equation (4), discovering an interesting regularizing phenomenon due to the presence of coefficients d_j.

Theorem. *(from [17]) Suppose*

$$\sup_{j \in J} \log_2 d_j - \inf_{j \in J} \log_2 d_j = L < \infty.$$

Then, there exists a stationary weak solution X of model (4). Moreover, $X \in H^r$ for all

$$r < \frac{1}{3}(\beta - \frac{d}{2}) - L.$$

If we restrict ourselves to the RCM model, computation of many quantities simplifies enormously while still showing peculiar features that clarifies the **multifractal nature** of the stationary solution as well as the spatial intermittency phenomenon.

Theorem. *(from [17]) The RCM admits a stationary weak solution $X \in H^s$ for all $s < s_0(p)$, where $p \geq 1$ and*

$$s_0(p) = \frac{1}{3}(\beta - \frac{d}{2}) + \frac{1}{2}(\ell_{3/2} - \ell_{p/2}).$$

Such solution is unique inside any H^s and it admits an explicit form

$$u_j = f \cdot 2^{q|j|+q} \prod_{k \leq j} \sqrt{d_k}, \qquad j \in J,$$

where

$$q = \frac{1}{3}(\beta + d) - \frac{1}{2}\ell_{3/2}.$$

If $\beta > d/2$ the solution is of Leray-Hopf.

One step forward in the study of model (4) is to search for solutions that are not constant in time. One natural way is then to look for self-similar solutions. In Section 3.1 we prove the following

Theorem. *Suppose there exist constant $d < C < 2\beta - 2\log M$ and $1 \leq M < 2^{\beta/2}$ so that*

$$1/M \leq d_k \leq M, \quad \forall k \in J \setminus \emptyset$$

Then there exists one and only one positive self-similar solution $X = (X_j(t))_{j \in J}$ of model (4).

We then extend our techniques to RCM model like (4) in the presence of a viscous friction, i.e.

$$X_j'(t) = c_j X_{\bar{j}}^2(t) - \sum_{k \in \mathcal{O}_j} c_k X_j(t) X_k(t) - c_j^{\gamma} X_j(t), \quad j \in J, \quad t \geq 0, \quad \gamma \in \mathbb{R}, \tag{5}$$

as well as investigate for which values of the friction exponent γ, the only stationary solution found in [17] still show the same dynamics as in the inviscid model.
Next theorem addresses such questions.

Theorem. *It exists a critical friction value* γ^c

$$\gamma^c = \frac{1}{3\beta}(2\beta - 2\log M - C)$$

for some constant $C \geq d$*, such that for every* $\gamma < \gamma^c$*, the viscous forced model (3.26) admits a unique positive stationary finite energy solution* $Y = \{Y_j(t) = a_j\}_{j\in J}$*. Moreover, such stationary solution satisfies*

$$\frac{C_1}{2^{(\beta(2-3\gamma)-2\log M)|j|/3}} \leq a_j \leq \frac{C_1}{2^{(\beta(2-3\gamma)-2\log M-C)|j|/3}},$$

for some positive constants $C_1, C_2 > 0$*.*

Furthermore, $\gamma^c \leq \frac{4}{15}$ *when* $\beta = d/2+1$ *and* $d = 3$*.*

Inequality

$$\gamma < \gamma^c = \frac{1}{3\beta}(2\beta - 2\log M - C)$$

suggests that d_j play a key role in regularization of the stationary solution under a friction force: as the upper bound M decreases to 1, the stationary viscous solution can withstand a critical friction coefficient of

$$\gamma^c = \frac{2}{3} - \frac{C}{3\beta} \leq \frac{2}{3} - \frac{d}{3\beta}.$$

In the meaningful case where $d = 3$ and $\beta = 1 + d/2 = 5/2$ this assumes the value $\frac{4}{15}$. On the other hand, as the upper bound M increases, the stationary viscous solution can withstand a progressively lower friction value, i.e. it will be regularized to a super exponential velocity with a friction force with lower intensity.

Inverse cascade tree model

In Section 3.2 we introduce a dyadic tree model with Obukhov non-linearity in order to simulate the inverse cascade phenomenon of two-dimensional fluid vortexes. The model we are interested in is conceptually similar to models developed in previous sections.

$$X_j' = -2^{\beta|j|}d_j X_{\bar{j}} X_j + \sum_{k\in\mathcal{O}_j} 2^{\beta|k|} d_k X_k^2, \quad j \in J, \tag{6}$$

where $X_j = X_j(t)$ are differentiable real functions and $X_{\bar{\emptyset}} \equiv f \geq 0$ is a forcing on the first component, which acts as a dissipative term:

$$X_\emptyset' = -d_\emptyset f X_\emptyset + \sum_{k\in\mathcal{O}_\emptyset} 2^{\beta|k|} d_k X_k^2.$$

We observe that the non-linearity is similar to the one proposed by Obukhov in his classic linear model, which is formally conservative, thus for solutions with positive components it provokes energy flow from lower to larger scales, i.e. from higher to lower nodes. Thus, we let the energy enter from high generation nodes, let say at N-th generation, and then let $N \to \infty$.
We then look for stationary solution of model (6) possibly different from the trivial null solution. In particular there exists a special stationary solution X_f, we address as **flat**, that exhibits no intermittent behavior, as well as infinitely many other stationary solution with spatial intermittency, as explained by the following results.

Theorem. *Let $X = \{X_j\}_{j\in J}$ be a constant solution of (6). Then $X \in H^s$ for every $s < s_0$, where the exponent s_0 satisfies:*

$$\frac{1}{3} \cdot (\beta - d/2) \leq s_0 \leq \frac{1}{3} \cdot \beta.$$

Moreover $s_0 = \frac{1}{3} \cdot (\beta - d/2)$ for the flat solution X_f.

We are interested in investigating whether stationary solutions show intermittency behaviour. Next Proposition states that the **flat** stationary solution shows no intermittency and satisfies Kolmogorov K41 law.

Proposition. *The exponent ζ_p of the structure function of the stationary flat solution satisfies*

$$\zeta_p = \frac{p}{3}(\beta - \frac{d}{2}),$$

which becomes $\zeta_p = \frac{p}{3}$ when we consider $\beta = d/2 + 1$.

Next theorem tells that there are infinitely many **special** stationary solutions with non-linear scaling exponent ζ_p that show spatial intermittency.

Theorem. *For every $s \in \mathbb{R}$, there exists a stationary solution whose exponents of the structure function are given by*

$$\zeta_p = \frac{p}{3}(\beta - \frac{d}{2}) + \frac{p}{3}(s\ell_S - s\ell_{\frac{sp}{3}}) \tag{7}$$

The anomalous exponents (7) are reminiscent of the related exponents for the stationary solution of Katz and Pavlovic tree model. They retain all the information regarding the non homogeneous and multifractal essence of such special solutions.

One step forward consists to look for self-similar solutions also for this inverse cascade model. However, this time we consider a slight modification of model

(6), where we allow ourselves to choose a not stationary forcing term on the first component. In particular we consider

$$f(t) = \frac{a_{\overline{\emptyset}}}{t - t_0} \geq 0, \quad a_{\overline{\emptyset}} \in \mathbb{R}^+$$

with $t > t_0$ for some $t_0 < 0$.
In section 3.2 we prove the following theorems that state, under particular assumptions, existence of self-similar solutions, and, when they exist, they show the same asymptotic behaviour of particular stationary solutions.

Theorem. *Suppose that exist constants $d > 2\log M - 2\beta$ and $1 \leq M < 2$ so that*

$$1/M \leq d_k \leq M, \quad \forall k \in J \setminus \emptyset$$

Then there exists a positive self-similar solution $Y = (Y_j(t))_{j\in J}$.

Theorem. *Let's consider a self-similar solution $Y = \{Y_j(t)\}_{j\in J}$ of the RCM model (6). Then exists a positive stationary solution $X = \{X_j\}_{j\in J}$ of the same model such that $X \in H^r$ if and only if $Y \in H^r$ for some $r \in \mathbb{R}$.*

In particular we can infer the existence of intermittent self-similar solutions for the inverse cascade model.

Mixed tree model

In Section 3.3 we conclude this thesis by presenting a **Mixed tree dyadic model** that combines both Novikov and Obukhov non-linearity.

$$\begin{aligned}\frac{dX_j(t)}{dt} =& \delta_1\Big(2^{\beta|j|} d_j X_{\bar{j}}^2(t) - \sum_{k\in\mathcal{O}_j} 2^{\beta|k|} d_k X_j(t) X_k(t)\Big) \\ &- \delta_2\Big(-2^{\beta|j|} d_j X_j(t) X_{\bar{j}}(t) + \sum_{k\in\mathcal{O}_j} 2^{\beta|k|} d_k X_k^2(t)\Big)\end{aligned} \tag{8}$$

where $\beta > 0$, $\delta_1, \delta_2 \geq 0$ are constants and the coefficient $\{d_k\}_{k\in J}$ are bounded from above and away from zero. Model (8) generalizes the linear mixed model (3) presented in Chapter 2. It is a special case of the more general model introduced in [15], where it was proven that for every initial condition $x \in H$ it admits at least one weak solution.
We investigate existence and uniqueness of stationary solution, at first in the basic case when $d_k = 1$ for every $j \in J$, then in more general cases. We set $\delta = \delta_2/\delta_1$, and we restrict ourselves to $\delta_1, \delta_2 > 0$.

Theorem. *Consider model (8) in the case $d_k = 1$ for every $k \in J$. For every forcing term $f > 0$, every $\delta > 0$ and every $\beta > 0$ it admits a positive stationary finite energy solution $\{a_j\}_{j\in J}$.*
Moreover,

- *if $\delta > 2^{\frac{\beta+d}{3}}$ such solution satisfies*

$$\lim_{|j|\to\infty} \frac{a_j}{2^{-\frac{(\beta+d)|j|}{3}}} = C$$

  for some $C > 0$,

- *if $0 < \delta < 2^{\frac{\beta+d}{3}}$ such solution satisfies*

$$\frac{\tilde{C}_1}{2^{\frac{(\beta+C)|j|}{3}}} \leq a_j \leq \frac{\tilde{C}_2}{2^{\frac{(\beta)|j|}{3}}}$$

  for some constants $\tilde{C}_1 > 0, \tilde{C}_2 > 0$ and $C > d$,

- *if $\delta = 2^{\frac{\beta+d}{3}}$ such solution satisfies*

$$\lim_{|j|\to\infty} \frac{a_j}{2^{-2\frac{(\beta+d)|j|}{9}}} = C$$

  for some $C > 0$.

Theorem. *For every $0 < M < 2^{\beta/2}$ so that $1/M \leq d_j \leq M$, $j \in J$, and for every forcing term $f > 0$*

- *if $\delta > 2^{\frac{\beta+d}{3}}$, model (3.53) admits infinitely many positive stationary solution,*
- *if $\delta < 2^{\frac{\beta+d}{3}}$, model (3.53) admits exactly one positive stationary solution.*

Moreover, any such solution satisfies

$$\frac{\tilde{C}_1}{2^{\frac{(\beta+C+2\log M)|j|}{3}}} \leq a_j \leq \frac{\tilde{C}_2}{2^{\frac{(\beta-2\log M)|j|}{3}}} \tag{9}$$

for some constants $\tilde{C}_1 > 0, \tilde{C}_2 > 0$ and $C \geq d$.

These latter results are consistent with those proved in previous simpler models. However, the more complex structure of the dynamics still forbids to fully express its expected intermittent structure and deepen its multifractal nature.

Chapter 1

From Navier-Stokes to Shell Models

1.1 Turbulence

The concept of fully developed turbulence springs from the universal behavior of any physical situation of an intense fluid flow, be it the swirls and eddies in a fast flowing river or the wake of a ship or submarine, atmosphere and ocean currents or the blood flow in arteries. It is an accepted concept that the evolution of all these phenomena can be described through the Navier-Stokes equation with appropriate initial and boundary conditions. This governing equation is one of the most important models of mathematical physics: despite they have been a subject of passionate research for almost two centuries, there are still cardinal open questions due to the nature of non-linearity present in the equation, they mostly regard the nature of fully developed turbulence. In particular, the question of solution regularity for three-dimensional problem was appointed as one of the Millennium Problems. It is perhaps the most challenging problem in classical physics.

A first attempt to describe turbulence was conducted by Richardson [60] and later assessed by Kolmogorov in his scaling theory [47]. This last description is still valid today and has proved to be largely correct by a multitude of experiments and observations. However, there are corrections that cannot be explainable by Kolmogorov theory, such deviations emerge in the scale exponents for the scaling of correlation functions. Possible cause for these dissimilarities is the fact that, except for few results, Kolmogorov theory is not based on the Navier-Stokes equation. A definitive theory explaining such deviations should hinge on the Navier-Stokes equation.

Shell models of turbulence were first introduced by Obukhov [57] and Gledzer

[39]. They consist of a system of ordinary differential equations structurally similar to the spectral Navier-Stokes equation. These models are more accessible and numerically more manageable than the original Navier-Stokes equation. For these models a scaling theory identical to the Kolmogorov theory has been developed. Exploring the behavior of shell models might be the key for understanding the systems described by the Navier-Stokes equation. They are intrinsically constructed to obey the same conservation laws and symmetries as the Navier-Stokes equation, moreover they exhibit energy conservation as well as conservation of a second quantity which can be identified with helicity or enstrophy. This second quantity suggests whether the models are three-dimensional-like turbulence where helicity is preserved, or two-dimensional-like where enstrohpy is preserved.

In this chapter, following [34] we present a review of some of the main features of turbulence, from the inference of the Navier-Stokes equation for incompressible fluids, passing through the Kolmogorov K41 theory, up to the spectral equation that was the starting point for raising the realm of the shell model. Turbulence is the chaotic flow of a muddled fluid. Fluid can vary a lot depending on boundaries of its vessel, stirring and heating. However, as long as the length scales in the flow are small compared to the larger scales, determined by the boundaries, and large compared to the scales of the average molecular free path, all flows seem to have common features. Turbulence is this common characteristic of the flows.

1.2 The Navier-Stokes equation

Fluid mechanics investigates fluids that are on scales large compared to the mean free path length of the molecules constituting them. With this regard the fluid is considered as a continuum stream identified completely by a velocity field $v_i(\mathbf{x}, t)$, a temperature field $t(\mathbf{x})$, a pressure field $p(\mathbf{x})$ and finally by a density field $\rho(\mathbf{x})$. At each point x_i the fluid is then described by six field variables: three components of velocity, pressure, temperature, and density. Thus, we need a total of six equations in order to discover the behavior of these components. They derived from mass conservation, momentum conservation, energy conservation and the equation of state. However, when studying fully developed turbulence the fluid is regarded as incompressible, hence this allows to get rid of the equation defining density. Furthermore, when buoyancy is neglected the temperature variations dissociate from the momentum and continuity equations and we are finally left with a fluid pictured by the velocity and the pressure field. The dynamics of such a fluid is described by the following Navier-Stokes equation (NSE)

$$\partial_t v_i + v_j \partial_j v_i = -\partial_i p + \nu \partial_{jj} v_i + f_i \tag{1.1}$$

within the continuity equation

$$\partial_i v_i = 0. \tag{1.2}$$

for every 3-dimensional coordinate.
The equation states that the acceleration of a fluid fragment equals the sum of the forces acting on the fluid fragment (per unit mass). The left hand side is the derivative of the velocity field, where the second term is the advection. The first term on the right hand side is the pressure gradient force, the second is the viscous friction and the last term gathers all other forces (per unit mass). The continuity equation represents the conservation of mass, where the density does not appear since we consider incompressible fluids. From these four equations, together with appropriate boundary and initial conditions, is theoretically possible to determine the three components of the fluid velocity v_i and the pressure p. However, no general solutions to the NSE are known yet and a solution can be found only for very simple laminar flows (fluid particles following smooth paths in layers).

The NSE can be transformed into a dimensionless form by defining

$$x = Lx', \quad v = Vv', \quad t = \frac{L}{V}t', \tag{1.3}$$

where L (the outer scale) is interpreted as the length scale of the largest variations in the flow, it would usually be the size of the vessel for a bounded flow, while V is the velocity difference at this length scale. From L and V it is possible to build a timescale $T = \frac{L}{V}$, that is the time it takes the fluid at uniform velocity V to travel the distance L. We remark that NSE derives from Newton's second law, thus it is Galilean invariant, i.e. adding a uniform velocity does not change the equation, and therefore the global center of mass velocity is unchanged and only velocity differences really matter. By putting transformations (1.3) into (1.1) and dropping the superscript gives the NSE in a dimensionless form:

$$\partial_t v_i + v_j \partial v_i = -\partial_i p + \mathrm{Re}^{-1} \partial_{jj} v_i + f_i, \tag{1.4}$$

where we have introduced the dimensionless Reynolds number

$$\mathrm{Re} = \frac{VL}{\nu},$$

and absorbed a factor $\frac{V^2}{L}$ into the last forcing term. All terms are now of order unit expect for the viscosity that is of the order of the inverse of the Reynolds number. Hence, the Reynolds number quantifies the relative importance of the viscosity compared to the nonlinear term at the length scale L and velocity scale V. Small Reynolds number will cause the attenuation of the flow by viscosity or the viscous term will balance the external forces. The viscosity acts as a smoother of irregularities and has the form of a diffusion term. For larger Reynolds number the flow will be more and more dominated by the nonlinear term and the fluid motion becomes more and more irregular. For very high Reynolds numbers the flow is completely chaotic and seemingly random: this is what is called fully developed turbulence. It is believed that all the richness of bizarre behavior of fluids resides within the NSE. Despite many numerical simulations of NSE show some aspect of this richness, no general theory has been developed that links the NSE and the phenomenology observed in real life and experiments.

The first attempt to establish a formal theory of turbulence was taken by Richardson [60]. Richardson describes the flow as formed of large eddies splitting up into smaller eddies, which again split up into yet smaller eddies until finally the eddies are so small that they are loosen up or dissipated by viscosity. Energy is inserted into the flow at large scales and then it cascades into smaller scales until it disappears at the viscous scale. Richardson seminal work led Kolmogorov [47] to develop a more structured theory of turbulence.

1.3 Kolmogorov Theory

In 1941 Kolmogorov [47] proposed the picture of a flow powered by an vigorous force and then left alone to slowly consume from viscosity. The flow is assumed to be homogeneous (translationally invariant) and isotropic (rotationally invariant). The picture we are trying to describe is a flow sustained by a force active on a large scales of the flow, such that the flow is in a state of statistical equilibrium, i.e. the energy released by the force is evened out by the energy dissipated by viscosity. The state of the flow is then characterized by the mean energy dissipation (per unit of mass) ϵ due to viscosity. The velocity at a given length scale $\ell << L$ is the velocity difference $\delta v(\ell) = |v(r+l) - v(r)|$, it is characteristic of the velocity related to an eddy of size l. The effect of flow velocity on a larger scale is to move the eddy through the flow as a rigid body. Similarly, if we consider a much smaller eddy within the larger eddy, the effect of the larger eddy on the smaller is the same as the effect of the larger scale flow on the large eddy. Without loss of generality we assume that the flow is self-similar, i.e. when $\ell_1 < \ell_2 << L$ then $\delta v(l_2) = f(\frac{\ell_1}{\ell_2})\delta v(\ell_1)$, where f is some suitable universal function. Thus, the velocity difference $\delta v(\ell)$ can only be a function of the scale ℓ and the mean energy dissipation ϵ. From a dimensional study the only possible relationship is

$$\delta v(\ell) \sim (\epsilon\, \ell)^{1/3}. \tag{1.5}$$

Indeed, if we want to establish the relationship

$$\delta v(\ell) = f(\ell, \epsilon),$$

the dimension of the right and left hand side must be identical, thus f can only depend on the combination of l and ϵ which has the same dimension as the left hand side. Namely, the dimensions are $[\delta v] = \mathbf{m/s}$, $[\ell] = \mathbf{m}$, $[\epsilon] = \mathbf{m}^2/\mathbf{s}^3$, hence from $[\delta v] = [\ell]^s[\epsilon]^t$ we get $s = t = 1/3$, and $\delta v(\ell) = f[(\epsilon\,\ell)^{1/3}]$. By changing the velocity by a scaling factor λ, we get

$$\lambda\delta v(\ell) = \lambda f[(\epsilon\,\ell)^{1/3}] = f[\lambda(\epsilon\,\ell)^{1/3}].$$

Finally, we see that f must be a linear function and (1.5) follows immediately. The relation (1.5) contains all the essence of Kolmogorov K41 theory.

1.3.1 The four-fifth law

One natural approach to study small scale turbulence is to introduce the so called *structure functions* of the velocity field of different orders. A structure function of order p is the quantity

$$S_p(\ell) = \langle \delta v(\ell)^p \rangle$$

where the brackets stand for the statistical average among the range scale ℓ.

In his third turbulence paper, Kolmogorov [48] exhibits an exact relation for the third order structure function, deriving this result directly from NSE:

$$S_3(\ell) = \frac{4}{5}\epsilon\,\ell. \tag{1.6}$$

This is probably the most important result in fully developed turbulence, it is both exact and not trivial. It sets up a pivotal condition on theory of turbulence: a theory to be acceptable must either satisfy the four-fifth law, or explicitly violate the assumptions made in deriving it.

The K41 scaling theory predicts the scaling

$$S_p(\ell) \sim \ell^{p/3} \tag{1.7}$$

which can be determined by dimensional counting. Although, if the field is not Gaussian, we should not expect (1.7) to hold in general. High Reynolds number flow is observed to behave intermittently: calm periods are interrupted by sudden blasts of energy at small scales which are then effectively damped. There is now convincing experimental evidence that deviations to the scaling law (1.7) are present in the inertial range of fully developed turbulence. The structure function still scales with length, but this time the scaling exponent $\zeta(p)$ such that

$$S_p(\ell) \sim \ell^{\zeta(p)}$$

are different from $p/3$. The function $\zeta(p)$ is called the anomalous scaling exponent, it refers to the nonuniform essence of intermittent flow. Calculating the anomalous scaling exponents from NSE is a major challenge in which there has been little success despite a rather large effort in recent years. Since for shell models it is easy to understand how the intermittency emerges, the intermittent behavior of these models has been a major motivation for studying them.

1.4 Fourier Transform of NSE

In order to take a step closer to shell models we consider the Fourier transform of the velocity field and its inverse

$$\begin{aligned} \mathcal{F}: \quad & \hat{v}_i(\mathbf{y}) = \frac{1}{(2\pi)^3} \int e^{-i\mathbf{yx}} v_i(\mathbf{x}) d\mathbf{x} \\ \mathcal{F}^{-1}: \quad & v_i(\mathbf{x}) = \int e^{i\mathbf{yx}} v_i(\mathbf{y}) d\mathbf{y} \end{aligned} \tag{1.8}$$

Transforming the NSE (1.1) with $\mathcal{F}$ by using Fourier transform well known properties, it gives

$$\begin{aligned} \partial_t v_i(\mathbf{y}) = & - i \int v_j(\mathbf{y} - \mathbf{y}^{'}) y_j^{'} v_i(\mathbf{y}^{'}) d\mathbf{y}^{'} \\ & - i y_i p(\mathbf{y}) - \nu y_j y_j v_i(\mathbf{y}) + f_i(\mathbf{y}). \end{aligned} \tag{1.9}$$

Furthermore, pressure can be eliminated from the NSE using the continuity equation (1.2), and assuming the force to be rotational ($\partial_i f_i = 0$), we obtain a Poisson equation for the pressure by applying the divergence operator to the NSE

$$\partial_{ii} p = -\partial_i v_j \partial_j v_i. \tag{1.10}$$

Transforming also the last Poisson equation with $\mathcal{F}$ it gives

$$\begin{aligned} -y_j y_j p(\mathbf{y}) = & - \int (y_i - y_i^{'}) v_j(\mathbf{y} - \mathbf{y}^{'}) y_l^{'} v_m(\mathbf{y}^{'}) d\mathbf{y}^{'} \delta_{lj} \delta_{mi} \\ & - \int (y_j - y_j^{'}) v_l(\mathbf{y} - \mathbf{y}^{'}) y_l^{'} v_j(\mathbf{y}^{'}) d\mathbf{y}^{'} \\ & - \int y_j y_l^{'} v_l(\mathbf{y} - \mathbf{y}^{'}) v_j(\mathbf{y}^{'}) d\mathbf{y}^{'}, \end{aligned} \tag{1.11}$$

where we used the fact that incompressibility implies $y_j^{'} v_j(\mathbf{y}) = 0$. Finally, by substituting $p(\mathbf{y})$ from (1.11) to (1.9) gives the so called spectral NSE

$$\begin{aligned} \partial_t v_i(\mathbf{y}) =& - i y_j \int (\delta_{il} - \frac{y_i y_l^{'}}{y_j^2}) v_j(\mathbf{y}^{'}) v_l(\mathbf{y} - \mathbf{y}^{'}) d\mathbf{y}^{'} \\ & - \nu y_j^2 v_i(\mathbf{y}) + f_i(\mathbf{y}). \end{aligned} \tag{1.12}$$

If now we suppose the flow confined in a box of size L^3 with periodic boundary conditions, the Fourier transform is replaced by a Fourier series and the integral in (1.12) by the following sum

$$\begin{aligned} \partial_t v_i(\mathbf{n}) =& - i n_j (\frac{2\pi}{L}) \sum_{n^{'}} (\delta_{il} - \frac{n_i n_l^{'}}{n_j^2}) v_j(\mathbf{n}^{'}) v_l(\mathbf{n} - \mathbf{n}^{'}) \\ & - \nu n_j^2 v_i(\mathbf{n}) + f_i(\mathbf{n}), \end{aligned} \tag{1.13}$$

where the wave vectors are $\mathbf{y(n)} = \frac{2\pi\mathbf{n}}{L}$. This last form of the NSE is the starting point for the introduction of shell models of turbulence. The partial differential equation in (1.1) has been replaced by a system of coupled ordinary differential equations. The nonlinear terms are quadratic in the velocities. The interactions are such that only waves with wave vectors adding up to zero are meaningful. Such set of three waves is called a triad and it is possible to show that energy is exchanged within each triad since the inviscid energy conservation satisfied by the NSE is a perfect energy balance. The mathematics involved in proving this and many other relations concerning NSE is much simpler in the case of shell models. Hence, from now on we will perform many of the calculations in the simplified case of shell models.

1.4.1 Kolmogorov energy scaling

We use the Fourier transform once more in order to obtain the famous Kolmogorov energy scaling law.
The second order structure function is related to the energy density through a Fourier transform:

$$\begin{aligned} E =& \frac{1}{2} \int v(\mathbf{x})^2 d\mathbf{x} = \frac{1}{2} (2\pi)^3 \int_0^\infty v_i(\mathbf{y}) \overline{v_i(\mathbf{y})} d\mathbf{y} \\ =& \frac{1}{2} (2\pi)^3 4\pi \int_0^\infty y^2 |v(y)|^2 dy \equiv \int E(y) dy, \end{aligned} \tag{1.14}$$

where we have defined the spectral energy density as

$$E(y) = (2\pi)^4 y^2 |v(y)|^2. \tag{1.15}$$

The Fourier transform of the velocity is expressed in terms of the second order structure function

$$S_2(\ell) = \langle \delta v(\ell)^2 \rangle = 2 \int [v(\mathbf{x})^2 - v(\ell + \mathbf{x})v(\mathbf{x})] d\mathbf{x}, \tag{1.16}$$

and putting together (1.14), (1.15) and (1.16) we obtain the so called Wiener-Khinchin formula

$$E(y) = \frac{1}{2\pi} y^{-1} \int_0^\infty x \sin x S_2(x/y) dx. \tag{1.17}$$

Finally, putting the scaling relation (1.7) for the second order structure function into (1.17) we obtain

$$E(y) \sim \epsilon^{2/3} y^{-5/3}. \tag{1.18}$$

This relation could be obtained by the same dimensional counting as we used for deriving the scaling formula for the velocity increments. The scaling (1.18) has been verified in many experiments and real observations for developed 3D turbulence.

1.5 Shell Models

In the Fourier representation of the Navier-Stokes equation (1.9) the transfer of energy from large to small scales is described as a flux of energy from small wave numbers to large wave numbers. The idea behind shell models is to divide the space into concentric spheres with exponentially growing radius $k_n = \lambda^n$, for some constant $\lambda > 1$ (it is common to set $\lambda = 2$). We then call n-th *shell* the set of wave numbers contained in the n-th sphere and not contained in the $(n-1)$-th sphere. In a typical shell model only few wave numbers are maintained in each shell: the velocities corresponding to these wave numbers represent a kind of velocity averaged over the whole shell.

Shell models investigate the energy cascade flow with a set of coupled nonlinear ordinary differential equation:

$$\frac{d}{dt} u_n = k_n G_n[u, u] - \nu_n u_n + f_n, \tag{1.19}$$

where the dynamical variable u_n represents the evolution over time of the velocity over a wavelength of scale k_n. Depending on the model, the nonlinear function $G_n[\cdot,\cdot]$ is chosen to preserve total energy, helicity or volume in phase space as for the original nonlinear terms of NSE. Boundary conditions are imposed by requiring that fluctuations do not occur on scales larger than the typical scale L, i.e. $u_n = 0$ for $n < 0$. Moreover, it is common to impose locality of interactions in Fourier (shell) space by demanding that the nonlinear function $G_n[\cdot,\cdot]$ couples only scales that are close to each other (for instance nearest and next-to-nearest shells).

The principle behind this construction is clear. We need a simple model, consistent with but simpler than NSE, able to describe a dynamical evolution of a set of variables on a vast range of scales. In other words, one wants to define a model able to describe the phenomenological Richardson cascade but possessing a deterministic time evolution.

The constraints to have a short range, quadratic non-linearity preserving total energy, total helicity, and phase-space evolution do not fix in a unique way the form of $G_n[\cdot,\cdot]$ in equation (1.19). We concentrate our brief introduction to three main models: the models introduced by Obukhov [57] and Novikov [32], the GOY model [39], [67] and finally the SABRA model [51].

1.5.1 The Obukhov - Novikov shell models

A shell model was first proposed by A.M. Obukhov [57]. The model has been introduced as a simplified model of 3D Navier-Stokes evolution. It is not derived directly from an approximation of NSE, although is structurally similar, with an energy cascade in accordance with the Kolmogorov picture of turbulent cascade of energy. The model consists of a linear sequence of first order ordinary differential equation. Equations are non-linear and quadratic in the velocities u_n. Such velocities could be thought of as representative of spectral velocity components $v_i(\mathbf{y})$ within a shell of wave numbers $k_{n-1} < |\mathbf{y}| < k_n$. The governing equations are

$$\frac{d}{dt}u_n = k_{n-1}u_{n-1}u_n - k_n u_{n+1}^2 - \nu_n u_n + f\delta_{n,1}. \tag{1.20}$$

The first two terms on right hand side represent respectively the non-linear advection and pressure term, the third describes dissipation, and the fourth term is a force only active on the first wave number component. Despite the model does not derive directly from NSE, the advection and pressure terms are quadratic in the velocities and the dissipation term is linear and dominant for large wave numbers like in NSE. Energy must be injected at large scales (i.e. small wave numbers) in order to exhibit an energy cascade from large to small scales, then it flows through an inertial range and finally is dissipated at small scales (i.e. large wave numbers).

In 2D turbulent fluids, one way in which eddies interact with each other is through a process known as *vortex cannibalization*, i.e. when two adjacent eddies merge to form a single larger eddy. When cannibalization occurs energy flows out of the length scales of the initial small eddies and into the length scale of the final larger eddy. Hence, cannibalization results in the flow of energy from small to large length scales.
Many eddies are generally created at a small length scale (called the energy injection scale). The belief is that through interaction by cannibalization these small eddies gather and merge into larger eddies. These larger eddies are also expected to gather and merge to form even larger eddies and so on. Thus, the energy initially injected into the turbulence at the injection scale should gradually be moved by consecutive cannibalization processes to larger length scales. This type of energy flow constitutes an example of inverse energy cascade.

In the attempt to simulate a more realistic behavior of both $2D$ and $3D$ turbulence, in 1974 E.A. Novikov [32] proposed a shell model similar to (1.20),

$$\frac{d}{dt}u_n = k_n u_{n-1}^2 - k_{n+1}u_{n+1}u_n - \nu_n u_n + f\delta_{n,1}. \tag{1.21}$$

By neglecting external forces and viscosity, we show how both Obukhov and Novikov models and their generalizations are derived from the following general requirements: (i) quadratic nature of non-linear terms; (ii) scale invariance of dimensionless coefficients in the equation; (iii) only direct interaction between closest neighbors in the spectrum; (iv) conservation of energy (in the unviscid and unforced case). It follows from requirements (i) and (iii) that

$$\frac{d}{dt}u_n = a_1 u_{n-1}^2 + a_2 u_{n-1}u_n + a_3 u_{n-1}u_{n+1} + a_4 u_n^2 + a_5 u_n u_{n+1} + a_6 u_{n+1}^2. \tag{1.22}$$

for some coefficients $a_1, \dots, a_6$ depending on k_0. Conditions (ii) and (iv) yield

$$a_1 = \delta_1 k_n,\ a_2 = \delta_2 k_{n-1},\ a_3 = a_4 = 0,\ a_5 = -a_1 k_1,\ a_6 = -a_2 k_1,$$

for some $\delta_1, \delta_2 \in \mathbb{R}$. Finally we obtain

$$\frac{d}{dt}u_n = \delta_1[k_n u_{n-1}^2 - k_{n+1}u_{n+1}u_n] + \delta_2[k_{n-1}u_{n-1}u_n - k_n u_{n+1}^2]. \tag{1.23}$$

For $\delta_1 = 0, \delta_2 > 0$ we obtain model (1.20) and for $\delta_1 > 0, \delta_2 = 0$ model (1.21). The first model converts to the second for the reflection of scale change and of time.

The class of models (1.23) and their generalizations will be the main subject of this work and we shall describe their properties in details in following chapters.

To conclude our brief introduction to shell models we observe that by neglecting condition (iii) it becomes possible to obtain a more general class of models. To this class belong, in particular, the GOY and SABRA models described in the following sections.

1.5.2 The Gledzer - Okhitani - Yamada shell model

In the attempt to construct a shell model that also satisfied Liouville's theorem in Hamiltonian mechanics, E.B. Gledzer [39] in 1973 proposed the following set of equations:

$$\frac{d}{dt}u_n = A_n u_{n+1}u_{n+2} + B_n u_{n-1}u_{n+1} + C_n u_{n-2}u_{n-1} - \nu_n u_n + f_n \tag{1.24}$$

within lower boundary conditions $u_{-1} = u_0 = 0$ and possibly some upper boundary conditions $u_{N+1} = u_{N+2} = 0$.

With this choice of nonlinear interaction terms, coefficients A_n, B_n, C_n can be chosen such that energy E and enstrophy Z

$$E = \sum_n \frac{u_n^2}{2}, \qquad Z = \sum_n \frac{k_n^2 u_n^2}{2}$$

are inviscid invariants corresponding to 2D turbulence. Such model was later investigated by Okhitani and Yamada (1988). Their experiments showed that the model exhibits chaotic dynamics and enstrohpy cascade. After their seminal work, this model has become one of the most well-studied model and today it is known in its complex version called Gledzer-Okhitani-Yamada or GOY model [67].

The model was originally constructed such that only the energy E is an inviscid invariant. This can be seen noting that the invariant must be time independent, and from (1.24) with $\nu = f = 0$ it is possible to obtain

$$A_n = k_n\tilde{a}, \quad B_n = k_n\tilde{b}, \quad C_n = k_n\tilde{c}, \quad k_n\tilde{a} + k_{n+1}\tilde{b} + k_{n+2}\tilde{c} = 0.$$

Then, with the usual choice of wave numbers $k_n = k_0\lambda^n$, where the shell radius satisfies $\lambda > 1$ so that the spectral space covered by the shells grows exponentially with the shell number n, we get

$$k_n(\tilde{a} + \lambda\tilde{b} + \lambda^2\tilde{c}) = 0,$$

and with the further changes $a = \tilde{a}$, $b = \lambda\tilde{b}$, $c = \lambda^2\tilde{c}$ we have

$$a + b + c = 0.$$

This was the first version of the GOY model. However, later it was observed that from a computational point of view it is desirable to define the velocities to be complex numbers. The final form of the GOY model then becomes

$$\frac{d}{dt}u_n = i[k_n u_{n+1} u_{n+2} - b k_{n-1} u_{n-1} u_{n+1} + (b-1) k_{n-2} u_{n-2} u_{n-1}]^* - \nu_n u_n + f_n, \quad (1.25)$$

where $*$ stands for complex conjugate and b is left as free parameter together with the dimensional quantities k_0, ν_n, f_n and initial conditions $u_n(0)$.

Later studies [34] showed that for $b < 1$ the model is of the 3D type, for $b > 1$ the model is of 2D turbulence type. The case $b = 1$ divides the two turbulence types. This diversified behavior and many other noteworthy features have led the model to be one of the most studied and investigated in turbulence theory.

1.5.3 The SABRA shell model

We conclude our tour over shell models by mentioning the SABRA shell model. It is defined as before by a set of exponentially space wave numbers $k_n = k_0 \lambda^n$. The form of the governing equation is motivated by the demand that the momentum involved in the triad interactions must add up to zero as in NSE. Together with the usual construction of local interactions, inviscid conservation of energy, fulfillment of Liouville's theorem, gives the following equation for the (complex) shell velocities

$$\frac{d}{dt}u_n = i[k_n u^*_{n+1} u_{n+2} - b k_{n-1} u^*_{n-1} u_{n+1} + (1-b) k_{n-2} u_{n-2} u_{n-1}] - \nu_n u_n + f_n. \quad (1.26)$$

As in previous models, the force would be taken to be active only for small wave numbers, and boundary conditions can be specified with usual assignment $u_{-1} = u_0 = 0$.

This time the requirement on the triads is fulfilled if the wave numbers k_n are defined as a Fibonacci sequence $k_n = k_{n-1} + k_{n-2}$. The choice of a Fibonacci sequence leads to a model with shell spacing equal to the golden ratio $\varphi = \frac{\sqrt{5}+1}{2}$. Hence, in this formulation the shell spacing is not a free parameter of the model. However, using the definition of L'vov [51] of $k_n = \varphi^n$ being a *quasi momentum*, we can keep the shell spacing as a free parameter $k_n = \lambda^n$. If we interpret the momentum k_n as representative of the modulus of the wave vector, the triangle inequality implies $k_n + k_{n+1} \geq k_{n+2}$, so the Fibonacci sequence corresponds to the moduli of three parallel wave vectors. However, for the shell spacing $\lambda > \varphi$ (as the usual choice $\lambda = 2$) the triangle inequality is violated. Thus, we cannot interpret the usual shell model interactions as representative interactions between waves within three consecutive shells, since no such triplets of wave numbers constitute triangles.

Like the GOY model, SABRA can be studied by varying continuously a single free parameter b. In [61] the authors showed how the solutions of the SABRA model show a phase diagram with the inverse and direct cascade regimes. It is this rich phase diagram which makes the two-dimensional SABRA model an ideal candidate to study the dynamics in the interplay between turbulence cascade and fluxless solutions.

Chapter 2

Dyadic Linear Models

This chapter is entirely devoted to describe the *Dyadic* shell model and its variants recently proposed and studied by several authors in order to better understand the behaviour of solutions to Euler and Navier-Stokes equations.
Even though these models are extreme simplifications of the original problem, they retain the most important characteristic features of NSE. Moreover, we will show that these models are in a sense natural as they constitute the simplest class satisfying certain scaling and dimensional conditions.

In [44] Katz and Pavlovic proposed a model based on a wavelet expansion of a scalar function $v(x,t)$, $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$, over a set of three dimensional dyadic cube with sides lengths 2^j, $j \in \mathbb{Z}$ and vertices at the points $2^j\mathbb{Z}^3$.
Let $\mathcal{Q}$ be a cube of size 2^j, then its parent $\overline{\mathcal{Q}}$ is a cube with size 2^{j+1} that contains $\mathcal{Q}$; similarly let define $\mathcal{C}^1(\mathcal{Q})$ the offspring of $\mathcal{Q}$, i.e. the set of 8 children of $\mathcal{Q}$ with side length 2^{j-1}.
The Katz-Pavlovic model equations describing the evolution of wavelet coefficient of $v(x,t)$ corresponding to the cube $\mathcal{Q}$ are

$$\frac{dv_{\mathcal{Q}}}{dt} = 2^{\frac{5}{2}j} v_{\overline{\mathcal{Q}}}^2 - 2^{\frac{5}{2}(j+1)} v_{\mathcal{Q}} \sum_{\mathcal{Q}' \in \mathcal{C}^1(\mathcal{Q})} v_{\mathcal{Q}'}. \tag{2.1}$$

We stress that Equation (2.1) shows a quadratic non-linearity and formally conserves the energy $\sum_{\mathcal{Q}} v_{\mathcal{Q}}(t)^2$. This model has been motivated at first by [43], where the authors studied partial regularity of the weak solutions to the NSE with hyperdissipation.
In [44] Katz and Pavlovic showed that there exist some initial data $v_j(0)$ which lead a solution to blowup in a finite time. In [36] Frielander and Pavlovic considered a related vector model for which they also prove blowup in a finite time.
Later, Waleffe [63] proposed a simplified model where instead of the branching

structure comprises a linear tree of the functions $v_j(t)$ satisfying an infinite system of ordinary differential equations

$$\begin{aligned} v_j' &= \lambda^j v_{j-1}^2 - \lambda^{j+1} v_j v_{j+1}, \qquad j > j_0 \\ v_{j_0}' &= -\lambda^{j_0+1} v_{j_0} v_{j_0+1}, \end{aligned} \tag{2.2}$$

where $\lambda > 1$ is a free parameter and j_0 is an index corresponding to the largest space scale. Without loss of generality we set $j_0 = 0$ from now on.

The original Katz-Pavlovic models reduces to the system (2.2) with $\lambda = 2$ and if we assume the coefficients of all cubes of same length to be the same. Waleffe proved that there exist initial data for which the blowup occurs in any H^s, $s > 0$, where we refer to the Sobolev spaces associated to model (2.2) as

$$H^s = \{v_j \mid ||\{v_j\}||_{H^s}^2 = \sum_{j \geq j_0} \lambda^{2sj} |v_j|^2 < \infty\}.$$

In [63] it has been shown that model (2.2) is related to the inviscid Burger's equation in fluid dynamics, making blowup not surprising. This model shows a intrinsic mechanism of transferring the energy to higher nodes. Waleffe then suggested the following different model

$$\begin{aligned} v_j' &= \lambda^j v_{j-1} v_j - \lambda^{j+1} v_{j+1}^2, \qquad j > 0 \\ v_0' &= -\lambda v_1^2. \end{aligned} \tag{2.3}$$

This model is reminiscent of Obukhov work [57]. Unlike Katz-Pavlovic model, Obukhov model lacks of the transferring energy mechanism and presents a more subtle and thus perhaps a more realistic behaviour.
These two models constitute two basic building blocks of all linear tree models satisfying four natural conditions: (i) quadratic non-linearity, (ii) appropriate scaling property, (iii) energy conservation, (iv) and nearest neighbor coupling. All of these except the last one are the features derived from the NSE; the last condition is a simplification to make the problem more tractable. For the rest of the work we call, following Waleffe, model (2.1) the Katz-Pavlovic model and model (2.3) the Obukhov model.

2.1 Katz-Pavlovic linear model

The *tree* dyadic model (2.1) is a more structured version of the so called Katz-Pavlovic *dyadic model* of turbulence, that we address as *linear* or *classic* from now on. The linear model is based on variables Y_n which represent a cumulative intensity of the n-th shell, $n \in \mathbb{N}$ (shells in Fourier or wavelet space). We consider the equations for Y_n in the following general form

$$\begin{aligned} \frac{dY_n(t)}{dt} &= k_{n-1}Y_{n-1}^2(t) - k_nY_n(t)Y_{n+1}(t) - \nu k_n^\gamma Y_n, \\ Y_n(0) &= y_n, \\ Y_0(t) &= F, \qquad \forall n \geq 0, \forall t \geq 0 \end{aligned} \tag{2.4}$$

where the coefficients satisfy $k_n = 2^{\beta n}$ for $\beta > 0$, $\nu \geq 0$ and $\gamma > 0$ are respectively the viscosity coefficient and exponent, y_n are the initial conditions and $F \geq 0$ is a non negative force added to the first component.
When $F = 0$ we will address model (2.4) as *unforced* while when $\nu = 0$ we will refer to it as *inviscid*.
This model has been introduced as a simplified model of 3D Euler evolution in order to investigate a number of properties which are currently out of reach for more realistic models of fluid dynamics. We mention in particular the works [22], [26], [25], [36], [44], [46], devoted to this model and variants of it. In the following sections we present the main results regarding Katz-Pavlovic linear dyadic model that will be useful to prove later results.

2.1.1 Basic properties

Let us start introducing the natural space for the dynamics of dyadic model (2.4), $H = \ell^2(\mathbb{R})$, the Hilbert space of square summable sequences with the usual norm that we will denote by $||\cdot||$.

Although the case $F > 0$ is very interesting, it is also of interest to analyze the unforced inviscid dynamic, namely system (2.4) without any forcing or viscous term. Physically, if we accept that a dyadic model like (2.4) may describe something of turbulence, the unforced case would correspond to free decaying turbulence, a widely observed phenomenon, see for example [37]. Therefore, we start by focusing our attention to the main properties of the inviscid and unforced model, namely

$$\begin{aligned} \frac{dY_n(t)}{dt} &= k_{n-1}Y_{n-1}^2(t) - k_nY_n(t)Y_{n+1}(t), \\ Y_n(0) &= y_n, \\ Y_0(t) &= 0, \qquad \forall n \geq 0, \forall t \geq 0 \end{aligned} \tag{2.5}$$

About the choice of coefficients

If we take a closer look to the model (2.4) compared to the original Katz Pavlovic model (2.1), aside from the tree or linear structure, we notice that the coefficients $2^{\frac{5}{2}n}$ take the more general form $k_n = 2^{\beta n}$. As explained in Chapter 1, the coefficients k_n represent the speed of the energy flow from an eddy to its children. Most of the basic properties proven in latter sections hold for almost any choice of positive coefficients k_n, however the phenomenon called anomalous dissipation holds only in the smaller class of coefficients $k_n = 2^{\beta n}$ for some $\beta > 0$. There are physical reasons behind the choice. The parameter $\beta > 0$ is an approximation of the rate of this speed. In the three dimensional setting the right magnitude of k_n is the one chosen by Katz and Pavlovic

$$k_n \sim 2^{\frac{5}{2}n}.$$

This particular choice can be at least heuristically justified as follows.
Let us consider $v = \sum Y_n w_n$ the usual wavelet expansion, and the quantity $v \cdot \nabla v$ in three dimensional setting. We have

$$||v \cdot \nabla v|| \leq |v|_\infty \cdot ||\nabla v||.$$

If we consider a single wavelet w_n of unitary l^2 norm we know that its support lies in the cube $\mathcal{Q}_n$, and its l^∞ norm is therefore $2^{\frac{3}{2}n}$. Moreover, $||\nabla w_n|| \sim 2^n$, thus

$$||w_n \cdot \nabla w_n|| \lesssim 2^{\frac{3}{2}n} \cdot 2^n = 2^{\frac{5}{2}n}.$$

This choice for k_n is the one corresponding to Kolmogorov K41 scaling law. While the physical meaningful case is $\beta = \frac{5}{2}$, in this chapter we always refer to $\beta > 0$ as general positive parameter.

We propose at first few definitions to introduce the concept of solution for a system of equation like (2.5).

Definition 2.1.1. *Let $I \subset \mathbb{R}^+$ be an interval.*
A local weak solution on I is a sequence of differentiable functions $Y = (Y_n)_{n\geq 1}$ satisfying (2.5).
A weak solution is a sequence $Y = (Y_n)_{n\geq 1}$ of differentiable function on all the positive line $[0, \infty)$, satisfying (2.5).
A finite energy solution is a weak solution such that $Y(t) \in H$ for all $t \geq 0$.
A Leray-Hopf solution is a finite energy solution such that $||Y(t)||_H$ is a non increasing function of t.

Weak solutions show interesting spatial and temporal properties.

Proposition 2.1.1. ***Time Change***. *Let Y be a weak solution of (2.5) with initial condition $y \in \mathbb{R}^{\mathbb{N}}$. Let $a > 0$ and define $X(t) = a \cdot Y(at)$. Then X is a weak solution with initial condition $a \cdot y$.*

Proposition 2.1.2. ***Time Inversion***. *Let Y be a weak solution of (2.5) with initial condition $y \in \mathbb{R}^{\mathbb{N}}$. Let $t' > 0$ and define $X(t) = -Y(t' - t)$. Then X is a local weak solution on $[0, t']$ with initial condition $-X(t')$.*

Proposition 2.1.3. ***Positiveness***. *Suppose Y is a weak solution of (2.5), and $n \geq 1$, $t_0 \geq 0$. Then*

1. *If $Y_n(t_0) > 0$ then $Y_n(t) > 0$ for all $t \geq t_0$.*
2. *If $Y_n(t_0) \geq 0$ then $Y_n(t) \geq 0$ for all $t \geq t_0$.*
3. *If $Y_1(t_0) = Y_2(t_0) = \cdots = Y_n(t_0) = 0$ then $Y_1(t) = Y_2(t) = \cdots = Y_n(t) = 0$ for all $t \geq t_0$.*

Proof. By applying the variation of constants formula to system (2.5), we get that for all $n \geq 1$ and $0 \leq t_0 < t$,

$$Y_n(t) = Y_n(t_0) \cdot e^{-\int_{t_0}^t k_n Y_{n+1}(s) ds} + \int_{t_0}^t k_{n-1} Y_{n-1}^2(s) \cdot e^{-\int_s^t k_n Y_{n+1}(z) dz} ds. \tag{2.6}$$

Since $Y_{n-1}^2 \geq 0$, equation (2.6) proves the first two statements. Then notice that if $Y_n(t_0) = 0$ and $Y_{n-1} = 0$ on the interval $[t_0, t]$, then from (2.6) we deduce that $Y_n = 0$ on $[t_o, t]$, so the third statement follows by induction and the hypothesis $Y_0 = 0$. □

Proposition 2.1.4. ***Forward Shift***. *Let Y be a weak solution of (2.5) with initial condition $y \in \mathbb{R}^{\mathbb{N}}$. Let m be a positive integer and, for all $n \geq 1$, let us define*

$$X_n(t) = Y_{n-m}(k_m \cdot t), \quad \text{if} \quad n > m, \quad \text{and} \quad X_n(t) = 0 \quad \text{if} \quad n \leq m$$

and

$$Z_n(t) = k_m^{-1} Y_{n-m}(t), \quad \text{if} \quad n > m, \quad \text{and} \quad Z_n(t) = 0 \quad \text{if} \quad n \leq m.$$

Then $X = (X_n)_{n\geq 1}$ and $Z = (Z_n)_{n\geq 1}$ are weak solutions with shifted and scaled initial conditions

$$x_n = y_{n-m}, \quad \text{if} \quad n > m, \quad \text{and} \quad x_n = 0 \quad \text{if} \quad n \leq m$$

and

$$z_n = k_m^{-1} y_{n-m}, \quad \text{if} \quad n > m, \quad \text{and} \quad z_n = 0 \quad \text{if} \quad n \leq m.$$

Proposition 2.1.5. ***Backward Shift***. *Suppose $m \geq 1$ and Y is a weak solution of (2.5) with initial condition $y \in \mathbb{R}^{\mathbb{N}}$ such that $y_1 = y_2 = \cdots y_m = 0$ and $y_{m+1} \neq 0$. Define for all $n \geq 1$*

$$X_n(t) = Y_{n+m}(k_m^{-1}t), \qquad Z_n(t) = k_m Y_{n+m}(t).$$

Then $X = (X_n)_{n\geq 1}$ and $Z = (Z_n)_{n\geq 1}$ are weak solutions with shifted and scaled initial conditions $x_n = y_{n+m}$ and $z_n = k_m y_{n+m}$.

2.1.2 Existence of solutions

One of the key tools for studying shell models is the energy of a solution. We first introduce the notation for finite size blocks energy. For all $n \geq 1$ and $t \geq 0$, let

$$E_n(t) = \sum_{i \leq n} Y_i^2(t)$$

the total amount of energy that flows through the first block of n components. An easy computation shows that

$$E_n^{'} = -2k_n Y_n^2 Y_{n+1}, \tag{2.7}$$

thus, one can study variation of energy by looking at the sign of the components.

Proposition 2.1.6. *If the initial condition $y \in H$ has infinitely many non negative components, then every weak solution is a Leray-Hopf solution.*

Proof. Let $(n_i)_{i\geq 1}$ be an increasing sequence such that $y_{n_i} \geq 0$. By the positiveness property, $Y_{n_i}(t) \geq 0$ for all $t \geq 0$, hence, by equation (2.7), for every $i \geq 1$, E_{n_i} is a non increasing function. Moreover, since $E_{n_i} \longrightarrow ||Y||^2$ pointwise as $i \longrightarrow \infty$, then also $||Y||^2$ is non increasing. □

Theorem 2.1.7. *Given any initial condition $y \in H$, there exists at least one Leray-Hopf solution. Given $y \in \mathbb{R}^{\mathbb{N}}$ with infinitely many non negative components, there exists at least one weak solution.*

Proof. For every $N \geq 1$, let's consider the following Galerkin approximation of system (2.5):

$$\begin{aligned} &Y_n^{'}(t) = k_{n-1}Y_{n-1}^2(t) - k_n Y_n(t) Y_{n+1}(t), \\ &Y_n(0) = y_n, \qquad\qquad n = 1, 2, \dots N \\ &Y_0(t) \equiv Y_{N+1}(t) \equiv 0, \qquad t \geq 0. \end{aligned} \tag{2.8}$$

Thanks to positiveness property and equation (2.7), we have $E_N^{'} \equiv 0$ and $x_{m+1} \geq 0$ for every $m < N$ implies $E_m(t) \geq E_m(0)$ for all $t \geq 0$.
System (2.8) is a finite dimensional initial value problem with locally Lipschitz vector field, so there is uniqueness and local existence of solutions. Moreover, since E_N is constant, the solution is bounded, this ensures global existence. Let $Y^N = (Y_1^N, Y_2^N, \dots, Y_N^N)$ be this solution.
Fix $n \geq 1$ and $T > 0$. In order to apply Ascoli-Arzelá theorem to the sequence $(Y_n^N)_{N \geq n}$ on the interval $[0, T]$, we need a uniform bound.
By hypothesis $y \in H$ and E_N^N is constant, thus for every $N \geq n$ and $t \geq 0$

$$(Y_n^N(t))^2 \leq E_N^N(t) = E_N^N(0) \leq E_N(0) \leq ||y||^2 = B_n^2.$$

Likewise, if y has infinitely many non-negative components, let $m = \inf\{k \geq n \,|\, x_{k+1} \geq 0\}$, then for $N \geq n$ and $t \geq 0$

$$(Y_n^N(t))^2 \leq E_{m \wedge N}^N(t) \leq E_{m \wedge N}^N(0) = E_{m \wedge N}(0) \leq E_m(0) = B_n^2.$$

This uniform bound for the solution becomes an unniform bound on its derivative thanks to relation (2.8):

$$|(Y_n^N)^{'}| \leq k_{n-1} B_{n-1}^2 + k_n B_n B_{n+1}, \qquad N \geq n$$

finally yielding the equicontinuity.
Hence, from Ascoli-Arzelá theorem, for every n there is a sequence $(N_k^{(n)})_{k \geq 1}$ such that $Y_n^{N_k^{(n)}}$ converges uniformly to a continuous function Y_n as $k \to \infty$. By a diagonal procedure, one can modify the previous extraction and get a single sequence $(N_k)_{k \geq 1}$ such that for all $n \geq 1$, $Y_n^{N_k} \to Y_n$ uniformly as $k \to \infty$. Thus, in the equation

$$Y_n^{N_k(t)} = y_n + \int_0^t [k_{n-1}(Y_{n-1}^{N_k}(s))^2 - k_n Y_n^{N_k}(s) Y_{n+1}^{N_k}(s)] ds, \qquad n \geq 1, t \in [0, T]$$

we can pass to the limit and prove that

$$Y_n(t) = y_n + \int_0^t [k_{n-1}(Y_{n-1}(s))^2 - k_n Y_n(s) Y_{n+1}(s)] ds, \qquad n \geq 1, t \in [0, T]$$

Thus, X_n are continuously differentiable and satisfy system (2.5) on $[0, T]$. The extension from an arbitrary bounded time interval to all $t \geq 0$ is a classical procedure.
We finally prove that, if $y \in H$, then the solution is Leray-Hopf, i.e. $||Y(t)||$ is a non increasing function of t.
For all $n \geq 1$, for all k such that $N_k \geq n$ and all $t \geq 0$

$$E_n^{N_k}(t) \leq E_{N_k}^{N_k}(t) = E_{N_k}^{N_k}(0) = E_{N_k}(0) \leq ||y||^2.$$

When $k \to \infty$ we get $E_n(t) \leq ||y||^2$, while when $n \to \infty$ we get $||Y(t)|| \leq ||y||$.
Let's pick $s \in [o, t]$, $n \geq 1$ and k such that $N_k \geq n$. If $E_{N_k}(0) \leq E_{N_k}(s)$, then

$$E_n^{N_k}(t) \leq E_{N_k}(0) \leq E_{N_k}(s) \leq ||Y(s)||^2$$

so that $E_{N_k} \leq E_{N_k}(s)$ for infinitely many k, and by taking again the limit on this subsequence k_m and then in n, we get $||Y(t)|| \leq ||Y(s)||$.
On the other hand, let's suppose $E_{N_k}(0) > E_{N_k}(s)$ for $k \geq k_0$. If $E_{N_k}(0) > E_{n_k}(s)$, then the derivative of E_{n_k} must have been negative for some $t_0 \in [0, s]$, yielding $Y_{N_k+1}(t_0) > 0$ and hence, by positiveness property, $Y_{Nk+1}(u) > 0$ for all $u \in [s, t]$, and in particular $E_{N_k}(t) \leq E_{N_k}(s)$. Since the latter is true for all $k \geq k_0$, by taking the limit we find again $||Y(t)|| \leq ||Y(s)||$.

□

Theorem 2.1.8. *Given $y \in \mathbb{R}_+^{\mathbb{N}}$, any weak solution of system 2.5 with initial condition y is positive. Moreover, any such solution satisfies the following properties:*

1. *for every $n \geq 1$, $t \geq 0$ we have*

$$\frac{d}{dt} \sum_{j=1}^{n} Y_j^2(t) = -k_n Y_n^2(t) Y_{n+1}(t)$$

   and hence

$$\sum_{j=1}^{n} Y_j^2(t) \leq \sum_{j=1}^{n} y_j^2$$

2. *if $y_n > 0$ for some $n \geq 1$, then $Y_m(t) > 0$ for all $m \geq n$ and $t > 0$.*

2.1.3 Energy dissipation

One can extend the computation (2.7) to the total energy of the system. Formally we have

$$\frac{d}{dt} \sum_{i \geq 1} Y_i^2(t) = 2 \sum_{i \geq 1} Y_i(t) Y_i^{'}(t) = 2 \sum_{i \geq 1} (k_{i-1} Y_i(t) Y_{i-1}^2(t) - k_1 Y_i^2(t) Y_{i+1}(t)) = 0, \tag{2.9}$$

since the last summation is telescoping. However, the above implications can be performed rigorously if the solution is very regular.
Indeed, by Hölder inequality

$$\sum_{i \geq 1} |k_{i-1} Y_i(t) Y_{i-1}^2(t)| \leq \left[\sum_{i \geq 1} k_{i-1} |Y_i(t)|^3 \right]^{\frac{1}{3}} \left[\sum_{i \geq 1} k_{i-1} |Y_{i-1}(t)|^3 \right]^{\frac{2}{3}} = 2^{-\frac{\beta}{3}} \sum_{i \geq 1} k_i |Y_i(t)|^3,$$

hence, if $\sum_{i\geq 1} k_i|Y_i(t)|^3 < \infty$, for every $t \in [0,T]$, then the sum of derivatives in (2.9) is uniformly absolutely convergent and the above passages are rigorous. This would prove that the solution conserves energy on $[0,T]$.
It is possible to show that if the initial condition is sufficiently regular, the regularity is maintained locally and so also energy conservation [36]. Although, the natural regularity of the solutions is much lower than this and after a finite time a blow up occurs. Afterwards, the solution dissipates energy and eventually converges to zero. The intuitive mechanism is a very fast transfer of energy from small to large components.

We recall from [6] the main results of energy dissipation for positive solutions: infinite initial energy becomes finite immediately; the energy of a finite energy solution tends to zero as $t \to \infty$.

Proposition 2.1.9. *Assume $y \in \ell^\infty \cap \mathbb{R}_+^{\mathbb{N}}$ and let Y be a positive weak solution of system (2.5) with initial condition y. Then Y has finite energy for positive times.*

Proposition 2.1.10. *If Y is a positive finite energy solution, then*

$$\lim_{t\to\infty} |Y(t)|_H^2 = 0.$$

Furthermore, given $L > 0$ and $\alpha > 0$, there exists $t' > 0$ depending only on L and α such that for all positive finite energy solutions Y with $|Y(0)|_H \leq L$, we have $|Y(t')|_H^2 \leq \alpha$.

The following two propositions establish the rate of decay of energy as $t \to \infty$, which essentially say that solutions decay as t^{-1}. The results are restricted to positive weak solutions. The first result is due to a scaling argument based on the fact that the non-linearity is homogeneous of degree two.

Proposition 2.1.11. *Let Y be a positive weak solution, with initial condition $y \in \ell^\infty \cap \mathbb{R}_+^{\mathbb{N}}$. Then there exists $C > 0$ such that*

$$|Y(t)|_H^2 \leq \frac{C}{t^2}$$

for $t \leq 1$.

Proposition 2.1.12. *Let Y be a positive weak solution, with initial condition $y \in \ell^\infty \cap \mathbb{R}_+^{\mathbb{N}}$. Let $n_0 + 1$ be the minimum integer with the property $y_{n_0+1} > 0$. We know that $Y_n(t) > 0$ for all $n > n_0$ and $t > 0$. Then, for some constant $C > 0$, and for every $n > n_0$, $t \geq 1$ we have*

$$\int_1^t Y_{n+1}(s)ds \geq k_n^{-1} \log t + k_n^{-1} \log(\frac{Y_n(1)}{C}).$$

Hence, for every $n > n_0$,

$$\int_1^t Y_{n+1}(s)ds = \infty$$

and

$$\limsup_{t\to\infty} t \cdot Y_{n+1(t)} \geq k_n^{-1}$$

2.1.4 Uniqueness

Uniqueness of solutions for the dyadic model is a complex matter. In general, uniqueness does not hold for all initial conditions in H. In order to prove uniqueness for special class of initial conditions one needs regularity requirements. In this section we recall essential results about uniqueness for linear inviscid dyadic model found in literature ([6], [3], [22], [44], [46]).

We start with a first result of uniqueness for weak solutions with non negative initial condition.

Theorem 2.1.13. *Let $y \in \mathbb{R}^{\mathbb{N}}$ be an initial condition with all non negative components. Suppose that for any weak solution Y with initial condition y the following equality holds*

$$\lim_{n\to\infty} 2^{-n}k_n \int_0^t Y_n^3(s)ds = 0, \qquad t \geq 0. \tag{2.10}$$

then there exists a unique weak solution with initial condition y.

Corollary 2.1.14. *Let $y \in H$ be an initial condition so that $y_n \geq 0$ for all $n \geq 0$. For all $\beta < 1$ there exists a unique weak solution with initial condition y.*

Proof. If Y is a weak solution with initial condition y, by Proposition 2.1.6 Y is Leray-Hopf, hence we can uniformly bound $Y_n^3(s)$ by $||y||^3$. Since $\beta < 1$, we also have $2^{-n}k_n \to 0$, so condition (2.1.22) holds. □

Since we usually choose $\beta = \frac{5}{2} > 1$, we state below a more general version of Corollary (2.1.14) for every $\beta > 0$.

Theorem 2.1.15. *Let $y \in H$ with $y_n \geq 0$ for all $n \geq 0$. There exists a unique weak solution with initial condition y.*

2.1.5 Stationary and Self-Similar solutions

We now introduce two special classes of solutions, namely *stationary* and *self-similar* solutions.

Definition 2.1.2. *A stationary solution is a sequence of real numbers $(y_n)_{n\geq 1}$ such that $Y = (y_n)_{n\geq 1} \in H$ is solution of system (2.5).*

Proposition 2.1.16. *If $F = 0$ then the only stationary solution is $Y \equiv 0$. If $F > 0$ then exists only one stationary solution*

$$y_n = \sqrt{F} \cdot k_n^{-1/3}, \quad n \geq 1.$$

Proof. Let y be a stationary solution of system (2.5). We set $r_n = k_n^{1/3} y_n$, so that $F = k_1 y_1 y_2 = r_1 r_2$ and for all $n \geq 2$

$$0 = k_{n-1} y_{n-1}^2 - k_n y_n y_{n+1} = k_{n-1} r_{n-1}^2 - k_n^{2/3} k_{n+1}^{-1/3} r_n r_{n+1}.$$

By definition $k_a k_b = k_{a+b}$, thus recursion (2.5) is equivalent to

$$r_1 r_2 = F, \qquad r_n r_{n+1} = r_{n-1}^2, \quad n \geq 2.$$

It is immediate to see that if $F = 0$ then $r_n = 0$ for all n, and if$F > 0$ then $r_n > 0$ for all n. Suppose then $F > 0$, then by taking logarithms on both sides we find

$$\log r_{n+1} = -\log r_n + 2 \log r_{n-1}, \qquad n \geq 2.$$

The latter is a linear recurrence with general solution $\log r_n = a(-2)^n + b$, for some $a, b \in \mathbb{R}$. Since we require $y \in H$, the only way is to impose $a = 0$. Thus, the only solution is the constant $r_n = \sqrt{F}$, concluding the proof. □

Despite their simplicity, stationary solutions are one of the most important class of solutions. First of all we observe from Proposition 2.1.16 the anticipated Kolmogorov K41 law as well as the first example of anomalous dissipation. Moreover, in [25], the authors showed that, given a forcing term F, the only stationary solution is an exponential global attractor for every finite energy solution.
The existence of a global attractor for an inviscid system is, perhaps, surprising. However it is exactly consistent with the concept of anomalous or turbulent dissipation conjectured by Onsager [59].

The other special class of solution is represented by the *self-similar* solutions.

Definition 2.1.3. *A self-similar solution is a finite energy solution Y such that there exists a differentiable function $\phi(t)$ and a sequence of real numbers $a = (a_n)_{n\geq 1}$ such that $Y_n(t) = a_n \cdot \phi(t)$ for all $n \geq 1$ and all $t \geq 0$.*

Definition 2.1.4. *A viscous stationary solution is a sequence of real number $(y_n^v)_{n\geq 1}$ such that $Y^v = (y_n^v)_{n\geq 1} \in H$ is solution of system (2.4).*

Next theorems establish the existence of positive self-similar and viscous stationary solution. Furthermore, Theorem 2.1.18 reveals the existence of an upper limit for the viscous term γ and a threshold for the forcing term F so that the corresponding viscous stationary solutions are not regular enough and show anomalous dissipation.
Theorem 2.1.17 was originally proved in [6] by using complex analysis argument and with the help of numerical computation. In the next section we present a different proof based on a pullback technique that will be useful frequently in next chapters.
In order to lessen the notation, we state and prove Theorem 2.1.17 and 2.1.18 in the case $\beta = \nu = 1$. The general case will be a straightforward consequence.

Observation. It easy to observe that positive self-similar solutions satisfying equations (2.5) have the form

$$Y_n(t) = \frac{a_n}{t - t_0}, \tag{2.11}$$

for some $t > t_0$ and $t_0 < 0$.
Indeed, if a positive solution is of the form (2.11), then

$$\begin{aligned} -\frac{a_n}{(t-t_0)^2} = \frac{dY_n(t)}{dt} &= k_{n-1} Y_{n-1}^2(t) - k_n Y_n(t) Y_{n+1}(t) \\ &= k_{n-1} \frac{a_{n-1}^2}{(t-t_0)^2} - k_n \frac{a_n a_{n+1}}{(t-t_0)^2}, \end{aligned}$$

that leads us to the sequence $\{a_n\}_n$ satisfying

$$a_n a_{n+1} = \frac{a_{n-1}^2}{2^\beta} + 2^{-\beta n} a_n.$$

Although it is possible for the first terms $a_1, a_2, \dots, a_{n_0}$ to be zero, if $a_{n_0+1} > 0$ then all the subsequent coefficients must be positive too:

$$a_{n+1} = 2^{-\beta n} + \frac{a_{n-1}^2}{2^\beta a_n} > 0, \quad \forall n \geq n_0 + 1. \tag{2.12}$$

Theorem 2.1.17. *Given $t_0 < 0$, there exists a unique positive self-similar solution with $a_1 \neq 0$. Moreover, given $t_0 < 0$ and $n_0 \geq 0$, there exists a unique positive self-similar solution with*

$$a_1 = a_2 = \cdots a_{n_0} = 0, a_{n_0+1} > 0.$$

In addition, the coefficients a_n have the property

$$\lim_{n\to\infty} \frac{a_n}{k_n^{-1/3}} = C_{n_0},$$

for some constant C_{n_0}.

Theorem 2.1.17 shows that the Kolmogorov scaling law (1.5) appears in these special solution, phenomenologically associated to decaying turbulence. But it is for us a very difficult open problem to understand whether all other solutions approach the self-similar ones and in which sense.
The existence of finite energy self-similar solutions is of theoretical interest in itself, in comparison with analogous investigations for Euler and Navier-Stokes equations, moreover the existence of such solutions has a number of implications. For instance, they realize perfectly the decay rate t^{-1}, coherently with Theorem 2.1.11 and 2.1.12. It has been conjectured that the set of all self-similar solutions (set depending on $t_0 \in \mathbb{R}$ and $n_0 \geq 0$) attracts all other finite energy solutions. If this is the case, the decay rate t^{-1} would be the true one for all solutions. Moreover, self-similar solutions offer an easy example of lack of uniqueness as shown by next observation.

Observation. It is possible to prove that for some initial conditions in H with all negative components there exist infinitely many energy solutions.
Indeed, by Theorem 2.1.17, there exists a self-similar solution Y whose total energy is strictly decreasing. Let $T > 0$, then $X(t) = -Y(T-t)$ is a local weak solution on $[0, T]$ by the time inversion property. For any time $s \in [0, T]$, let's consider the solution X^s obtained by attaching X on $[0, s]$ to a Leray-Hopf solution on $[s, \infty)$ given by Theorem 2.1.7 with initial condition $X(s) = -Y(T-s) \in H$. The energy of this solution strictly increases on $[0, s]$ and then is non-increasing on $[s, \infty)$. Thus, to different values of s correspond finite energy solutions which are really different, but all with the same negative initial condition $-Y(T)$.

Theorem 2.1.18. *Consider the forced viscous dyadic model (2.4). Then*

1. *if $\gamma < 2/3$, system (2.4) admits a unique viscous stationary solution $Y_n^v(t) = y_n^v$. Moreover, it exists a threshold $F_0 > 0$ such that:*

   (a) *if $F > F_0$, then the coefficients y_n^v have the property*

   $$\lim_{n\to\infty} \frac{y_n^v}{k_n^{-1/3}} = C_{F,\gamma},$$

   for some constant $C_{F,\gamma}$ depending on the initial force F and viscosity γ. Such solution is not regular enough and shows anomalous dissipation.

   (b) *if $F \leq F_0$, then exists $z = z_F \in \mathbb{R}^+$ (depending only on F) such that the coefficients y_n^v have the property*

   $$\lim_{n\to\infty} \frac{y_n^v}{2^{-k_n \cdot z - (n+2)(\gamma-1)}} = C_{\gamma,z},$$

   for some constant $C_{\gamma,z}$ depending on γ and z, thus the solution is conservative and regular.

2. *if* $\gamma \geq 2/3$, *system* (2.4) *admits a unique viscous stationary solution* $Y_n^v(t) = y_n^v$. *Moreover, for every* $F > 0$ *exists* $z = z_{F,\gamma} \in \mathbb{R}^+$ *(depending only on* F *and* γ*) such that the coefficients* a_n *have the property*

$$\lim_{n\to\infty} \frac{y_n^v}{2^{-k_n \cdot z-(n+2)(\gamma-1)}} = C_{\gamma,z},$$

for some constant $C_{\gamma,z}$ *depending on* γ *and* z*, thus the solution is conservative and regular.*

Observation. If we read the Katz-Pavlovic viscous model with the right choice of coefficient $\beta = \frac{5}{2}$, it is immediate to see that every solution is regularized after the critical viscosity value $\gamma^c = \frac{4}{5}$. However, nothing forbids special classes of solution to be regularized with lower dissipation values. Theorem 2.1.18 tells that constant solutions starting with enough energy can withstand a lower critical dissipation value, precisely $\gamma = \frac{2}{3}$, showing anomalous dissipation. After that, every constant solution becomes regular and conservative.

Observation. If Y is a positive viscous stationary solution then

$$\begin{aligned} 0 = \frac{dY_n(t)}{dt} =& k_{n-1}Y_{n-1}^2(t) - k_nY_n(t)Y_{n+1}(t) \\ =& k_{n-1} \cdot a_{n-1}^2 - k_n \cdot a_n a_{n+1} - \nu k_n^\gamma \cdot a_n, \end{aligned}$$

and since we consider the forced case $F > 0$, this leads us to the sequence $\{a_n\}_n$ satisfying

$$a_n a_{n+1} = \frac{a_{n-1}^2}{2^\beta} - \nu 2^{\beta n(\gamma-1)} a_n, \quad a_0 = F > 0.$$

We observe that this time it is not possible for the first terms $a_1, a_2, \ldots, a_{n_0}$ to be zero, otherwise if $a_{n_0+1} > 0$ then the subsequent coefficient must be negative

$$a_{n_0+2} = \frac{a_{n_0}^2}{2^\beta a_{n_0+1}} - \nu 2^{\beta(n_0+1)(\gamma-1)} = -\nu 2^{\beta(n_0+1)(\gamma-1)} < 0. \tag{2.13}$$

2.1.6 Proof of Theorem 2.1.17 and 2.1.18

First of all we start by proving the following theorem.

Theorem 2.1.19. *Consider the following recursion*

$$a_{n+1} = \frac{a_{n-1}^2}{2a_n} + \epsilon_n, \tag{2.14}$$

where $\{\epsilon_n\}_{n\in\mathbb{N}}$ *is a sequence of real (**not necessarily positive**) numbers such that* $\{|\epsilon_n|\}_{n\in\mathbb{N}}$ *is a decreasing sequence and*

$$\sum_{n=0}^{\infty} |\epsilon_n| \cdot 2^{n/3} < \infty.$$

Then there is one and only one $u = \{u_n\}_{n\in\mathbb{N}}$ *positive real sequence satisfying (2.14). Moreover, such* $\{u_n\}$ *lies in* H^s *for any* $s < 1/3$.

Proof. We structure our proof in three different steps. At first we prove the existence of solution for recursion (2.14), then we show regularity of such solution, finally we prove uniqueness among positive solutions with finite energy.

Step (1): **Existence**.

We start by considering the following two definition.

Definition 2.1.5. *We call **strong self-similar** any positive sequence* $\{a_n\}_{n\in\mathbb{N}}$ *satisfying the recurrence:*

$$a_{n+1} = \frac{a_{n-1}^2}{2a_n} + \epsilon_n. \tag{2.15}$$

Definition 2.1.6. *We call **weak self-similar** any positive sequence* $\{\tilde{a}_n\}_{n\in\mathbb{N}}$ *satisfying the recurrence:*

$$\tilde{a}_{n+1} = \frac{\tilde{a}_{n-1}^2}{\tilde{a}_n} + \zeta_n, \quad \zeta_n = \epsilon_n \cdot 2^{\frac{n-2}{3}}. \tag{2.16}$$

Remark. It is easy to verify that if $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is a weak self-similar sequence then $\{u_n = \frac{\tilde{u}_n}{2^{n/3}}\}_{n\in\mathbb{N}}$ is a strong self-similar sequence. Conversely, for any strong self-similar sequence it is possible to recover the corresponding weak sequence from the equality above.

In order to prove existence of strong self-similar sequence we use a pull back technique: we first consider recursion (2.16) backwards fixing $N > 2$ and two appropriate starting values b_{N+1} and b_N, then compute b_n for lower coefficients $n < N$; finally we let $N \to \infty$ proving convergence by compactness and recovering a strong self-similar sequence from the remark above.

For any fixed $N > 2$ we are interested in the following truncated reversed recursion:

$$\begin{aligned} &(b_{n-1}^{(N)})^2 = b_n^{(N)}(b_{n+1}^{(N)} - \zeta_n), \quad n \le N, \quad \zeta_n = \epsilon_n \cdot 2^{\frac{n-2}{3}} \\ &b_{N+1}^{(N)} = b_N^{(N)} = L > 0, \\ &b_n^{(N)} = 0, \quad n > N+1. \end{aligned} \tag{2.17}$$

where the initial value L will be chosen later accordingly to our requirements.

The following proposition poses sufficient conditions for existence of weak self-similar solution.

Proposition 2.1.20. *Consider the system of equations (2.17) and suppose that coefficients ζ_n satisfy*

$$M = \sum_{i=1}^{\infty} |\zeta_i| < \infty.$$

Then any initial value $L > M$ gives rise to a well defined weak sequence $\{\tilde{u}_n^{(N)}\}$ for every $N > 2$.

Proof. Consider the two following truncated reversed recursions

$$\begin{aligned} &(b_{n-1}^{(N)_1})^2 = b_n^{(N)_1}(b_{n+1}^{(N)_1} - |\zeta_n|), \;\; n \le N, \quad \zeta_n = \epsilon_n \cdot 2^{\frac{n-2}{3}}, \\ &b_{N+1}^{(N)_1} = b_N^{(N)_1} = L > 0, \\ &b_n^{(N)_1} = 0, \;\; n > N+1, \end{aligned} \tag{2.18}$$

$$\begin{aligned} &(b_{n-1}^{(N)_2})^2 = b_n^{(N)_2}(b_{n+1}^{(N)_2} + |\zeta_n|), \;\; n \le N, \quad \zeta_n = \epsilon_n \cdot 2^{\frac{n-2}{3}}, \\ &b_{N+1}^{(N)_2} = b_N^{(N)_2} = L > 0, \\ &b_n^{(N)_2} = 0, \;\; n > N+1. \end{aligned} \tag{2.19}$$

For every fixed N, let $\{\tilde{u}_n^{(N)_1}\}$ and $\{\tilde{u}_n^{(N)_2}\}$ satisfy respectively recursion (2.18) and (2.19). It is immediate to verify that if $\{\tilde{u}_n^{(N)}\}$ is a truncated weak self-similar sequence, then

$$\tilde{u}_n^{(N)_1} \le \tilde{u}_n^{(N)} \le \tilde{u}_n^{(N)_2}, \quad \forall n \le N+1.$$

First we prove that every $\{\tilde{u}_n^{(N)_1}\}$ that satisfies (2.18), when well defined, is weakly increasing. With the same argument it will follow that every $\{\tilde{u}_n^{(N)_2}\}$ that satisfies (2.19) is weakly decreasing.

We proceed by induction on n.
First two base cases are easy to verify:

$$\tilde{u}_{N+1}^{(N)_1} = L = \tilde{u}_N^{(N)_1},$$

$$\tilde{u}_{N-1}^{(N)_1} = \sqrt{u_N^{(N)_1}(u_{N+1}^{(N)_1} - |\zeta_{n-1}|)} = \sqrt{L(L - |\zeta_{n-1}|)} < L = \tilde{u}_N^{(N)_1}.$$

For the inductive step, we consider by hypothesis

$$\tilde{u}_{n+2}^{(N)_1} \geq \tilde{u}_{n+1}^{(N)_1} \;\; (i), \;\; \tilde{u}_{n+3}^{(N)_1} \geq \tilde{u}_{n+2}^{(N)_1} \implies \tilde{u}_{n+3}^{(N)_1} - |\zeta_{n+1}| \geq \tilde{u}_{n+2}^{(N)_1} - |\zeta_n| \;\; (ii)$$

and multiplying together inequalities (i) and (ii) we get:

$$(\tilde{u}_{n+1}^{(N)_1})^2 = \tilde{u}_{n+2}^{(N)_1}(\tilde{u}_{n+3}^{(N)_1} - |\zeta_{n+1}|) \geq \tilde{u}_{n+1}^{(N)_1}(\tilde{u}_{n+2}^{(N)_1} - |\zeta_n|) = (\tilde{u}_n^{(N)_1})^2,$$

proving the claim $\tilde{u}_{n+1}^{(N)_1} \geq \tilde{u}_n^{(N)_1}$.

Let us consider again the decreasing property in the following form:

$$(\tilde{u}_{n+1}^{(N)_1})^2 \geq (\tilde{u}_n^{(N)_1})^2 = \tilde{u}_{n+1}^{(N)_1}(\tilde{u}_{n+2}^{(N)_1} - |\zeta_n|),$$

and dividing both sides for the positive term $\tilde{u}_{n+1}^{(N)_1}$ we finally get

$$\tilde{u}_{n+2}^{(N)_1} - \tilde{u}_{n+1}^{(N)_1} \leq |\zeta_n|.$$

Applying a recursive argument to the inequality above it is possible to show

$$\tilde{u}_N^{(N)_1} - \tilde{u}_1^{(N)_1} \leq \sum_{i=1}^{N-1} |\zeta_i| \leq \sum_{i=1}^{\infty} |\zeta_i| = M.$$

With a similar argument it follows also

$$\tilde{u}_1^{(N)_2} - \tilde{u}_N^{(N)_2} \leq \sum_{i=1}^{N-1} |\zeta_i| \leq \sum_{i=1}^{\infty} |\zeta_i| = M.$$

We finally deduce that any initial value L satisfying

$$0 < L - M \leq \tilde{u}_n^{(N)_1} \leq \tilde{u}_n^{(N)_1} \leq \tilde{u}_n^{(N)_2} \leq L + M$$

gives rise to a well defined truncated weak self-similar sequence. In particular it is sufficient that $L > M$, concluding the proof. □

Proposition 2.1.20 tells that for every $N > 2$, $\{\tilde{u}_n^{(N)}\}_n$ lies in the compact set $[L - M, L + M]$, thus by compactness and a diagonal extraction argument we can choose a subsequence $(N_i)_i \in \mathbb{N}$ such that $\tilde{u}_n^{(N_i)}$ converges for all $n \in \mathbb{N}$ to some number $\tilde{u}_n$. The sequence $\tilde{u} = \{\tilde{u}_n\}_n$ satisfies recursion (2.16) by construction, thus it is a weak self-similar sequence, and $u = \{u_n = \frac{\tilde{u}_n}{2^{n/3}}\}_n$ is the corresponding strong self-similar sequence.

Furthermore, we observe that the condition

$$M = \sum_{i=1}^{\infty} |\zeta_i| < \infty$$

it is equivalent to

$$\sum_{i=1}^{\infty} |\epsilon_i| \cdot 2^{n/3} < \infty,$$

as required by theorem 2.1.19.

Step (2): **Regularity**.

We are now ready to prove that any $u = \{u_n\}_{n\in\mathbb{N}}$ strong self-similar sequence has finite energy, i.e.

$$\sum_{n=1}^{\infty} u_n^2 < \infty.$$

Moreover, such $\{u_n\}$ lies in H^s for any $s < 1/3$.

The following proposition gives condition on L so that the corresponding strong self-similar sequence lies in H^s.

Proposition 2.1.21. *For every $s < 1/3$, if $L > M$ then any strong self-similar sequence built from L is well defined, it has finite energy and lies in H^s.*

Proof. During Step (1) we have already shown that if $L > M$ then any weak self-similar sequence built from L is well defined and satisfies

$$0 < L - M \leq \tilde{u}_n \leq L + M.$$

By recovering the correct expression for the related strong self-similar sequence, we derive

$$\sum_{n=1}^{\infty} 2^{2sn} \cdot u_n^2 = \sum_{n=1}^{\infty} 2^{2sn} \cdot \frac{\tilde{u}_n}{2^{2n/3}} \leq (L + M) \sum_{n=1}^{\infty} 2^{n(2s-2/3)}.$$

Finally, from the latter equation it follows that any strong self-similar sequence built from L lies in H^s for every $s < 1/3$. □

Step (3): **Uniqueness**.

We now prove uniqueness among strong self-similar sequence with finite energy, i.e. any strong self-similar sequence built as in the previous steps starting from an initial value L. This time we require a slight stronger condition for the initial value: $L \geq M + 2^{-\frac{10}{3}}$.

Proposition 2.1.22. *Let $\{v_n\}_n$ be a solution of recursion (2.13) different from $\{u_n\}_n$. Then exists $\alpha > 0$ such that for every $n \geq 3$:*

$$v_n \geq u_n \cdot 2^{\alpha n}, \quad n \text{ odd}$$
$$v_n \leq u_n \cdot 2^{-\alpha n}, \quad n \text{ even}$$

or

$$v_n \geq u_n \cdot 2^{\alpha n}, \quad n \text{ even}$$
$$v_n \leq u_n \cdot 2^{-\alpha n}, \quad n \text{ odd}$$

Proof. We prove only the first case of the proposition, the second being similar. We start by considering odd values of n and finally even values.

Case (1): n odd. By induction over n. By hypothesis $\{v_n\}_n$ is different from $\{u_n\}_n$, so without loss of generality we can suppose $v_3 > u_3$ and the existence of a real number $\alpha_1 > 0$ so that

$$v_3 \geq u_3 \cdot 2^{3\alpha_1},$$

moreover by definition 2.1.19

$$v_4 = \frac{v_2^2}{2v_3} + \epsilon_3 < \frac{u_2^2}{2u^3} + \epsilon_3 = u_4,$$

thus exist also a real number $\alpha_2 > 0$ so that

$$v_4 \leq u_4 \cdot 2^{-4\alpha_2},$$

finally by setting $\alpha = \min\{\alpha_1, \alpha_2\}$ we have proved base cases of induction.
If n is an odd number, then by hypothesis we have

$$v_{n+1} = \frac{v_{n-1}^2}{2v_n} + \epsilon_n \geq \frac{2^{2\alpha(n-1)} u_{n-1}^2}{2^{-\alpha n}(2u_n)} + \epsilon_n.$$

In what follows we will show that

$$\frac{2^{2\alpha(n-1)} u_{n-1}^2}{2^{-\alpha n}(2u_n)} + \epsilon_n \geq 2^{\alpha(n+1)}(\frac{u_{n-1}^2}{2u_n} + \epsilon_n) = 2^{\alpha(n+1)} v_{n+1},$$

concluding the proof.
Let us first rewrite the latter inequality in the more compact form

$$(u_{n+1} - \epsilon_n) \cdot (2^{3\alpha n - 2\alpha} - 2^{\alpha n + \alpha}) \geq \epsilon_n \cdot (2^{\alpha n + \alpha} - 1). \tag{2.20}$$

Then we observe that the above inequality is trivially true in the case $\epsilon_n \leq 0$, so we restrict ourself to positive value of ϵ_n. We structure the proof in two different steps.

Step (1): $u_{n+1} \geq \epsilon_{n-1}$.

Let's rewrite the claim in terms of the corresponding $\tilde{u}_n$ weak sequence:

$$u_{n+1} \geq \epsilon_{n-1} \iff \frac{\tilde{u}_{n+1}}{2^{(n+1)/3}} \geq \epsilon_{n-1} \iff \tilde{u}_{n+1} \geq 2^{\frac{-(2n-4)}{3}}.$$

By monotonic property of both sides it is enough to prove that

$$\tilde{u}_{n+1} \geq \tilde{u}_3 \geq L - M \geq 2^{-\frac{10}{3}} \geq 2^{\frac{-(2n-4)}{3}}.$$

Finally, the initial requirement of

$$L > M + 2^{-\frac{10}{3}},$$

concludes the proof.

Step (2): $(2^{3\alpha n-2\alpha} - 2^{\alpha n+\alpha}) \geq (2^{\alpha n+\alpha} - 1)$.

First we rewrite above inequality in the form

$$2^{3\alpha n-2\alpha} + 1 \geq 2^{\alpha n+\alpha+1},$$

then observing that both sides are increasing function of n and left sides grows faster than right sides, it is enough to prove the claim for the least meaningful value of odd n, i.e. $n = 5$.
By letting $n = 5$ we obtain:

$$\begin{aligned} 2^{13\alpha} - 2^{6\alpha+1} + 1 = (2^{\alpha} - 1) \cdot [(2^{12\alpha} - 2^{5\alpha}) + (2^{11\alpha} - 2^{4\alpha}) + (2^{10\alpha} - 2^{3\alpha}) + \\ (2^{9\alpha} - 2^{2\alpha}) + (2^{8\alpha} - 2^{\alpha}) + (2^{7\alpha} - 1) + 2^{6\alpha}] > 0 \end{aligned}$$

because product of positive numbers. By multiplying together inequalities in Step (1) and Step (2) one can derive (2.20).

Case (2): n even. By induction over n. In the previous case we have already shown that exists a real number $\alpha > 0$ so that

$$v_3 \geq u_3 \cdot 2^{3\alpha}, \ \ v_4 \leq u^4 \cdot 2^{-4\alpha}.$$

If n is an even number, then by hypothesis we have

$$v_{n+1} = \frac{v_{n-1}^2}{2v_n} + \epsilon_n \leq \frac{2^{-2\alpha(n+1)} \cdot u_{n-1}^2}{2^{\alpha n} u_n} + \epsilon_n.$$

We will now show that

$$\frac{2^{-2\alpha(n+1)} \cdot u_{n-1}^2}{2^{\alpha n} \cdot u_n} + \epsilon_n \leq 2^{-\alpha(n+1)}(\frac{u_{n-1}^2}{2u_n} + \epsilon_n) = 2^{-\alpha(n+1)}u_{n+1}.$$

concluding the proof.
First, we rewrite inequality above as follows:

$$(u_{n+1} - \epsilon_n)(2^{-3\alpha n-2\alpha} - 2^{-\alpha n-\alpha}) \leq \epsilon_n(2^{-\alpha n-\alpha} - 1). \tag{2.21}$$

As in the previous case, we observe that the inequality above is trivially true if $\epsilon_n \leq 0$, so we again restrict ourself to positive value of ϵ_n.
In the previous case we have already shown that $u_{n+1} \geq \epsilon_{n-1}$, thus as a fortiori argument we have the following:

$$2^n u_{n+1} \geq \epsilon_{n-1}.$$

Moreover, observing that for every $\alpha > 0$

$$(2^{-3\alpha n-2\alpha} - 2^{-\alpha n-\alpha}) < 0, \quad (2^{-\alpha n-\alpha} - 1) < 0,$$

in order to prove (2.21) it is enough to show that

$$2^{-n}(2^{-3\alpha n-2\alpha} - 2^{-\alpha n-\alpha}) \geq (2^{-\alpha n-\alpha} - 1),$$

or equivalently

$$2^{3\alpha n+2\alpha+n} + 1 \geq 2^{2\alpha n+\alpha}(2^n + 1).$$

Both sides are increasing function of n and left sides increases faster than right side, so it is enough to prove the claim for the least admissible even n, i.e. $n = 6$. Namely, we need to prove

$$2^{20\alpha+6} + 1 \geq 2^{13\alpha+6} + 2^{13\alpha}.$$

Let's consider the function

$$f(\alpha) = 2^{20\alpha+6} + 1 - 2^{13\alpha+6} - 2^{13\alpha}.$$

It is easy to prove that $f(0) = 0$ and $f(x)$ has positive derivative on the positive x-asis, namely

$$\frac{df}{dx} = 5 \cdot 2^{13} \cdot log_2(2^{7x+8} - 169) > 0, \quad x \geq 0,$$

this proves the claim. □

Proposition 2.1.22 tells that every other solution v_n different from u_n, cannot have finite energy. Moreover, any other solution except u_n cannot lie in any space H^s even for negative values of s.

□

We now observe that Theorem 2.1.17 is an immediate consequence of Theorem 2.1.19 (by letting $\epsilon_n = 2^{-n}$). In order to prove part 1.(a) of Theorem 2.1.18 one can set $\epsilon_n = -2^{n(\gamma-1)}$, but it is left to prove the existence of a threshold F_0 for the initial force.

Let us consider once more recursion (2.19), with $\epsilon_n = -2^{n(\gamma-1)}$ and $\gamma < 2/3$.

It is immediate to verify that $\tilde{u}_n^{(N)_2} > \tilde{u}_n^{(N)_3}$ for every fixed N and every $n \leq N+1$, where $u_n^{(N)_3}$ is a solution of the following recursion

$$\begin{aligned} &(b_{n-1}^{(N)_3})^2 = b_n^{(N)_3} \cdot \zeta_n, \quad n \leq N, \quad \zeta_n = \epsilon_n \cdot 2^{\frac{n-2}{3}},\ \epsilon_n = 2^{n(\gamma-1)} \\ &b_{N+1}^{(N)_3} = b_N^{(N)_3} = L > 0, \\ &b_n^{(N)_3} = 0, \quad n > N+1. \end{aligned} \tag{2.22}$$

By a direct calculation it is possible to express a solution of (2.22) explicitly as:

$$u_n^{(N)_3} = L^{\frac{1}{2^{N-n}}} \cdot 2^{(\gamma-\frac{2}{3})\sum_{i=1}^{N-n}(\frac{i}{2^i}) - \frac{2}{3}\sum_{i=1}^{N-n}(2^i)}.$$

Finally, by letting $N \to \infty$ we get

$$\tilde{u}_n > 2^{2\gamma-2}.$$

Thus, we have shown that exists a threshold F_0 for the initial force F, as required by Theorem 2.1.18. Furthermore, by observing once more that in our case $L \leq \tilde{u}_n \leq L + M$, this shows that any $F > F_0$ gives rise to a stationary solution as in Theorem 2.1.18, and the coefficients a_n have the property

$$\lim_{n\to\infty} \frac{a_n}{k_n^{-1/3}} = C_{F,\gamma},$$

for some constant $C_{F,\gamma}$ depending on the initial force F and the viscosity γ.

From now on we will focus our attention to prove part 1.(b) and part 2 of Theorem 2.1.18.

When $\gamma < 2/3$ and the initial force is under threshold F_0, or when $\gamma \geq 2/3$, Theorem 2.1.18 suggests to look for solutions with different behaviour.

Theorem 2.1.23. *Consider the following recursion*

$$a_{n+1} = \frac{a_{n-1}^2}{2a_n} - \epsilon_n, \tag{2.23}$$

where $\epsilon_n = 2^{n(\gamma-1)}$, $\gamma \geq 0$. Then for every $a_0 = F > 0$, there is one and only one $u = \{u\}_{n\in\mathbb{N}}$ positive real sequence satisfying (3.58).

We adapt proof of Theorem 2.1.19 looking for solutions with different behaviour.

We start by considering the following two definition.

Definition 2.1.7. *We call* ***strong stationary*** *any positive sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfying the recurrence:*

$$a_{n+1} = \frac{a_{n-1}^2}{2a_n} - 2^{n(\gamma-1)}. \tag{2.24}$$

Definition 2.1.8. *We call* ***weak stationary*** *any positive sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ satisfying the recurrence:*

$$b_{n+1} = (\frac{b_{n-1}^2}{b_n} - 2) \cdot 2^{2^{n+1}\cdot z + 3\gamma - 4}, \tag{2.25}$$

for some real scalar $z \in \mathbb{R}^+$.

Remark. It is easy to verify that if $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is a weak stationary sequence then

$$\{u_n = \frac{\tilde{u}_n}{2^{2^{n+1}\cdot z + (n+2)(\gamma-1)}}\}_{n\in\mathbb{N}}$$

is a strong stationary sequence.
Conversely, for any strong stationary sequence it is possible to recover the corresponding weak sequence from the equality above.

We use again the pull back technique to prove existence of strong stationary sequence. We first consider recursion (2.25) backwards fixing $N > 2$ and two appropriate starting values b_{N+1} and b_N, then compute b_n for lower indices $n < N$; finally we let $N \to \infty$ proving convergence by compactness and recovering a strong stationary sequence from remark above.
Hence, for any fixed $N > 2$ we are interested in the following truncated reversed recursion:

$$\begin{aligned} &(b_{n-1}^{(N)})^2 = b_n^{(N)}(\frac{b_{n+1}^{(N)}}{2^{2^{n+1}\cdot z + 3\gamma - 4}} + 2), \quad n \leq N, \quad z \in \mathbb{R}^+ \\ &b_{N+1}^{(N)} = b_N^{(N)} = L > 0, \\ &b_n^{(N)} = 0, \quad n > N+1. \end{aligned} \tag{2.26}$$

for some starting value $L > 0$.

Next Lemma tells how to choose the starting value L.

Lemma 2.1.24. *Let $\{\tilde{u}_n^{(N)}\}$ a truncated weak stationary sequence, with starting value L. If $L < 2$, then $\{\tilde{u}_n^{(N)}\}$ is a weak decreasing sequence and satisfies*

$$L \leq \tilde{u}_n^{(N)} \leq 2 + \delta, \quad \forall n \leq N + 1, \tag{2.27}$$

for some $\delta > 0$ (depending only on γ and z). In particular $\{\tilde{u}_n^{(N)}\}$ is a bounded sequence and both lower and upper bounds do not depend on N.

Proof. We start proving by induction that $\{\tilde{u}_n^{(N)}\}$ is a weak decreasing sequence. Then left inequality of (2.27) will follow immediately.
The first base case of induction holds by definition. Furthermore

$$\tilde{u}_N^{(N)} \leq \tilde{u}_{N-1}^{(N)} \Longleftrightarrow L \leq \sqrt{L(\frac{L}{2^{2^{N+1}\cdot k+3\gamma-4}} + 2)},$$

and this holds if and only if

$$L \leq 2 \cdot \frac{2^{2^{N+1}\cdot k+3\gamma-4}}{2^{2^{N+1}\cdot k+3\gamma-4} - 1}.$$

Our choice $L < 2$ ensures that the latter inequality holds for every sufficiently large N. For the inductive step it is enough to observe that for every $n < N$

$$(\tilde{u}_n^{(N)})^2 = \tilde{u}_{n+1}^{(N)}(\frac{\tilde{u}_{n+2}^{(N)}}{2^{2^{n+2}\cdot k+3\gamma-4}} + 2) \leq \tilde{u}_n^{(N)}(\frac{\tilde{u}_{n+1}^{(N)}}{2^{2^{n+1}\cdot k+3\gamma-4}} + 2) = (\tilde{u}_{n-1}^{(N)})^2,$$

indeed by induction $\tilde{u}_{n+1}^{(N)} \leq \tilde{u}_n^{(N)}$, $\tilde{u}_{n+2}^{(N)} \leq \tilde{u}_{n+1}^{(N)}$ and a fortiori

$$\frac{\tilde{u}_{n+2}^{(N)}}{2^{2^{n+2}\cdot k+3\gamma-4}} \leq \frac{\tilde{u}_{n+1}^{(N)}}{2^{2^{n+1}\cdot k+3\gamma-4}}.$$

It is left to prove the right side of inequality (2.27).
By monotonic property we have the following relation

$$\tilde{u}_n^{(N)} \geq \tilde{u}_{n+1}^{(N)} \Longleftrightarrow \frac{\tilde{u}_{n+2}^{(N)}}{2^{2^{n+2}\cdot z+3\gamma-4}} + 2 \geq \tilde{u}_{n+1}^{(N)},$$

and by a recursive argument we obtain

$$\begin{aligned} \tilde{u}_1^{(N)} &\leq 2 + 2 \cdot (\sum_{i=1}^{N-1} \frac{1}{2^{2^i\cdot z+3\gamma-4}}) + \frac{L}{2^{2^N\cdot z+3\gamma-4}} + \frac{L}{2^{2^{N+1}\cdot z+3\gamma-4}} \\ &\leq 2 + 2 \cdot (\sum_{i=1}^{N+1} \frac{1}{2^{2^i\cdot z+3\gamma-4}}) \leq 2 + \sum_{i=1}^{\infty} \frac{2}{2^{2^i\cdot z+3\gamma-4}}. \end{aligned} \tag{2.28}$$

The right summation in (2.28) converges to some number $\delta = \delta_{\gamma,z}$ independent of N, this concludes the proof. $\square$

Lemma 2.1.24 tells that if $L < 2$, for every $N > 2$, $z \in \mathbb{R}^+$ and $\gamma \geq 0$, $\{\tilde{u}_n^{(N)}\}$ lies in the compact set $[L, 2+\delta]$, thus by compactness and a diagonal extraction argument we can choose a subsequence $(N_i)_i \in \mathbb{N}$ such that $\tilde{u}_n^{(N_i)}$ converges for all $n \in \mathbb{N}$ to some number $\tilde{u}_n$. The sequence $\tilde{u} = \{\tilde{u}_n\}_n$ satisfies recursion (2.25) by construction, thus it is a weak stationary sequence, and $u = \{u_n = \frac{\tilde{u}_n}{2^{2^n \cdot z + (n+2)(\gamma-1)}}\}_n$ is the corresponding strong stationary sequence.

Moreover, it is possible to show that the limit sequence $\tilde{u}_n$ is independent of L. Indeed, one can consider the following truncated reversed recursion:

$$\begin{aligned} &(c_{n-1}^{(N)})^2 = c_n^{(N)} \cdot 2, \quad n \leq N, \\ &c_{N+1}^{(N)} = c_N^{(N)} = L > 0, \\ &c_n^{(N)} = 0, \quad n > N+1. \end{aligned} \tag{2.29}$$

and for any L and N recover the following explicit solution

$$\tilde{v}_n^{(N)} = L^{\frac{1}{2^{N-n}}} \cdot 2^{\sum_{i=1}^{N-n} \frac{1}{2^i}}, \quad \forall n \leq N,$$

and observe by construction that for every $N > 2$ and for every $n \leq N+1$

$$\tilde{u}_n^{(N)} \geq \tilde{v}_n^{(N)}.$$

Finally letting $N \to \infty$, this will lead us to $\tilde{u}_n \geq 2$. Thus, combining together the latter result with Lemma 2.1.24 we conclude that $2 \leq \tilde{u}_n \leq 2+\delta$ is independent of the starting point L. Hence, for every initial force $F_0 \geq F > 0$, it exists only one stationary viscous solution $Y_n(t)^v = y_n^v$ satisfying part 1.(b) of Theorem 2.1.18, and for every initial force $F > 0$, it exists only one stationary viscous solution $Y_n^v(t) = y_n^v$ satisfying part 2 of Theorem 2.1.18.

Furthermore, the following proposition shows that any $u = \{u_n\}_{n\in\mathbb{N}}$ strong stationary sequence has finite energy. Moreover, such $\{u_n\}$ lies in H^s for any $s \in \mathbb{R}$.

Proposition 2.1.25. *For every $s \in \mathbb{R}$, if the starting value L satisfies $L < 2$ then any strong stationary sequence built from L has finite energy and lies in H^s.*

Proof. We have already shown that if $L < 2$ then any weak stationary sequence built from L is well defined and satisfies

$$\tilde{u}_n \leq 2 + \delta_{\gamma,z}.$$

By recovering the correct expression for the related strong stationary sequence, we derive

$$\sum_{n=1}^{\infty} 2^{2sn} \cdot u_n^2 = \sum_{n=1}^{\infty} 2^{2sn} \cdot \frac{\tilde{u}_n}{2^{2^n \cdot z + (n+2)(\gamma-1)}} \leq (2+\delta) \sum_{n=1}^{\infty} 2^{n(2s-\gamma+1)-2^n \cdot z + 2 - 2\gamma}.$$

Finally, from the latter equation it follows that any strong stationary sequence built from $L < 2$ lies in H^s for every $s \in \mathbb{R}^+$. □

2.2 Mixed linear model

In section 1.5.1 we have already observed that Katz Pavlovic and Obukhov models constitute the two basic blocks of all linear models satisfying four characteristic features derived from NSE. It is then natural, as already observed in [32], to consider the following more general (unviscid) model

$$\begin{aligned}
\frac{dY_n(t)}{dt} &= \delta_1[k_n Y_{n-1}^2(t) - k_{n+1} Y_n(t) Y_{n+1}(t)] - \delta_2[k_n Y_{n+1}^2(t) - k_{n-1} Y_n(t) Y_{n-1}(t)] \\
Y_n(0) &= y_n \\
Y_0(t) &= F, \qquad \forall n \geq 0, \forall t \geq 0
\end{aligned} \tag{2.30}$$

where $k_n = 2^{\beta n}$ for some $\beta > 0$, $\delta_1, \delta_2 \geq 0$ non negative parameters, $F \geq 0$ is the usual force to the first component and y_n some initial condition.

Considering that model (2.30) reduces to (inviscid) models (2.5) and (2.3) by setting respectively $\delta_2 = 0$ and $\delta_1 = 0$, we expect it to carry both Katz-Pavlovic and Obukhov dynamics giving birth to a more complex structure: even the simple uniqueness properties in section 2.1.4 do not hold anymore, as we show later. From now on we refer model (2.30) as *mixed (linear) dyadic model*.

Observation. In previous sections we stated the positiveness property for weak solutions of the Katz-Pavlovic linear model. This property plays a crucial role in many cardinal results, like the exponential global attraction of finite energy solutions to the unique constant solution.

Unfortunately, the positiveness property does not hold anymore in the mixed dyadic model. Indeed, by the variation of constants formula

$$\begin{aligned}
Y_n(t) = {} & Y_n(t_0) \cdot e^{-\int_{t_0}^t [\delta_1 k_{n+1} Y_{n+1}(s) - \delta_2 k_{n-1} Y_{n-1}(s)] ds} + \\
& + \int_{t_0}^t k_n [\delta_1 Y_{n-1}^2(s) - \delta_2 Y_{n+1}^2(s)] \cdot e^{-\int_s^t [\delta_1 k_{n+1} Y_{n+1}(z) - \delta_2 k_{n-1} Y_{n-1}(z)] dz} ds.
\end{aligned} \tag{2.31}$$

This time the positiveness condition

$$\delta_1 Y_{n-1}^2(t) - \delta_2 Y_{n+1}^2(t) \geq 0 \tag{2.32}$$

does not hold in general.

Because of its complex dynamics, no results were found in literature until 2019 [41], where the author shows the existence of self-similar solution for particular value of parameters (δ_1, δ_2), in addition to a local uniqueness theorem. Moreover, in [55], the author proved the following theorem about the existence of weak solution of the mixed dyadic model for every initial condition $y \in \ell^2$, in the case $\delta_1 = \delta_2 = 1$.

Theorem 2.2.1 (From [55]). *Consider the infinite dimensional shell model*

$$\frac{d}{dt}Y_n(t) = k_n Y_{n-1}^2(t) - k_{n+1}Y_n(t)Y_{n+1}(t) - k_n Y_{n+1}^2(t) + k_{n-1}Y_n(t)Y_{n-1}(t),$$

$$y(0) = y.$$

Then, for any initial condition $y \in \ell^2$ there exists at least a solution $Y(t)$ on $[0, T]$.

Extending Theorem 2.2.1 to general parameters δ_1, δ_2 is straightforward.

In the next sections we extend such results and give a complete spectrum of existence and uniqueness results for both stationary and self-similar solutions, for every positive couple of parameter (δ_1, δ_2).

2.2.1 Stationary and Self-Similar solutions

We are interested on the existence of *stationary* and *self-similar* solutions of the mixed dyadic shell model (2.30).

Stationary solutions for the forced mixed model

We recall that a *stationary* solution Y of (2.30) is a solution that is time independent, i.e. $Y_n(t) = a_n$ for all $t \geq 0$ and some $a_n \in \mathbb{R}_{\geq 0}$. In particular, we restrict ourselves to study stationary positive finite energy solutions.

Observation. If $Y = (a_n)_{n\in\mathbb{N}}$ is a stationary solution, we observe that if we allow some terms to be zero, there are two family of particular solution, namely

$$a_0 = F > 0, \quad a_{2n+1} = 0, \quad a_{2n} = (\frac{\delta_1}{\delta_2})^{n/2} \cdot F \quad \forall n \geq 0. \tag{2.33}$$

and

$$a_1 \in \mathbb{R}^+, \quad a_{2n+1} = (\frac{\delta_1}{\delta_2})^{n/2} \cdot a_1, \quad a_{2n} = 0 \quad \forall n \geq 0. \tag{2.34}$$

Such solutions are finite energy every time $\delta_1 < \delta_2$. However, for any stationary solution different from (2.33) and (2.34) it is not allowed to have any zero term. Indeed, if Y is a stationary solution then

$$\begin{aligned}\frac{dY_n(t)}{dt} =&\delta_1(k_n Y_{n-1}^2(t) - k_{n+1}Y_n(t)Y_{n+1}(t)) - \delta_2(k_n Y_{n+1}^2(t) - k_{n-1}Y_n(t)Y_{n-1}(t))\\ =&\delta_1(k_n a_{n-1}^2 - k_{n+1}a_n a_{n+1}) - \delta_2(k_n a_{n+1}^2 - k_{n-1}a_n a_{n-1}) = 0,\end{aligned} \tag{2.35}$$

and since we consider the forced case ($F > 0$), this leads us to the sequence $\{a_n\}_n$ satisfying

$$\delta_1(a_{n-1}^2 - k_1 a_n a_{n+1}) - \delta_2(a_{n+1}^2 - k_1^{-1}a_n a_{n-1}) = 0, \quad a_0 = F > 0.$$

If a_k is the first zero term of a stationary solution, then

$$\delta_1 a_{k-2}^2 + \delta_2 k_1^{-1} a_{k-1}a_{k-2} = 0 \implies \delta_1 a_{k-2} + \delta_2 k_1^{-1} a_{k-1} = 0,$$

and the latter equation would imply $a_{k-1} < 0$ or $a_{k-2} < 0$, despite the positive assumption.

In the next section we will prove the following result:

Theorem 2.2.2. *The forced mixed model (2.30) admits positive stationary solutions for every choice of coefficient $\delta_1, \delta_2 > 0$.*

In particular:

- *if $\frac{\delta_1}{\delta_2} < k_1^{-4/3}$, then for every $a_0 = F > 0$ and every $a_1 > 0$ there is just one positive stationary solution $\{a_n\}_{n\geq 0}$ of (2.30);*
- *if $\frac{\delta_1}{\delta_2} > k_1^{-4/3}$, then for every $a_0 = F > 0$ there is just one positive stationary solution $\{a_n\}_{n\geq 0}$ of (2.30).*

Moreover, any such stationary solution satisfies Kolmogorov's scaling law

$$\lim_{n\to\infty} \frac{a_n}{k_n^{-1/3}} = C$$

for some positive constant $C > 0$.

Observation. Theorem 2.2.2 divides the positive plane in two sub-regions: above the line $\delta_1/\delta_2 = k_1^{-4/3}$ there are infinitely many finite energy solution; below the same line uniqueness holds for every forcing term $F > 0$.
Figure (2.1) explains graphically the complete spectrum of existence and uniqueness of stationary solutions.

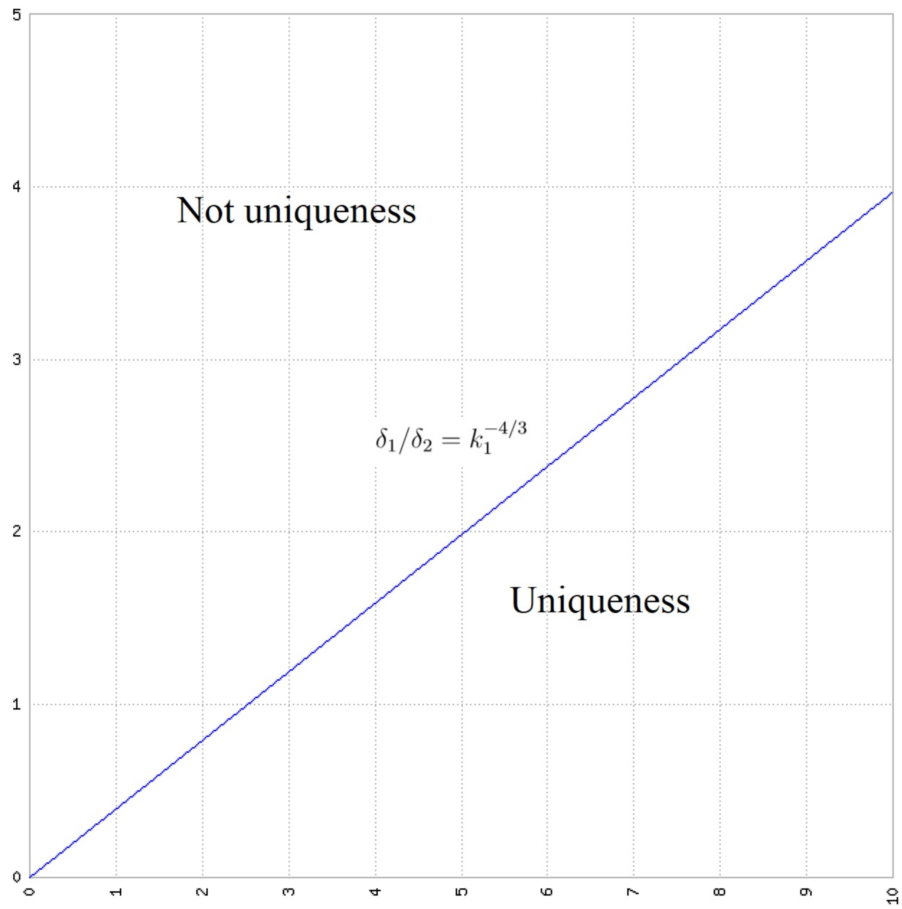

Figure 2.1: The line $\delta_1/\delta_2 = k_1^{-4/3}$ separates the uniqueness from not uniqueness domain for stationary solutions.

Self-similar solutions for the unforced mixed model

We recall that a *self-similar* is solution Y of the form $Y_n(t) = a_n \cdot \phi(t)$. As observed previously, it easy to prove that self-similar solutions satisfying equations (2.30) in the unforced case ($F = 0$), have the form

$$Y_n(t) = \frac{a_n}{t - t_0}, \quad a_0 = 0, \tag{2.36}$$

with $t > t_0$ and $t_0 < 0$.
If a positive solution is of the form (2.36), then

$$\begin{aligned} -\frac{a_n}{(t-t_0)^2} = \frac{dY_n(t)}{dt} &= \frac{\delta_1}{(t-t_0)^2}(k_n a_{n-1}^2 - k_{n+1} a_n a_{n+1}) \\ &- \frac{\delta_2}{(t-t_0)^2}(k_n a_{n+1}^2 - k_{n-1} a_n a_{n-1}), \end{aligned} \tag{2.37}$$

that leads to the sequence $\{a_n\}_{n\geq 1}$ satisfying

$$-\frac{a_n}{k_n} = \delta_1(a_{n-1}^2 - k_1 a_n a_{n+1}) - \delta_2(a_{n+1}^2 - k_1^{-1} a_n a_{n-1}).$$

Although it is possible for the first terms $a_1, a_2, \ldots, a_{n_0}$ to be zero, if $a_{n_0+1} > 0$ then all the subsequent coefficients must be not zero: indeed, from the latter relation if $a_{n_0} = 0$ then

$$-\frac{a_{n_0+1}}{k_{n_0+1}} = \delta_1(-k_1 a_{n_0+1} a_{n_0+2}) - \delta_2(a_{n_0+2}^2) \iff$$
$$\frac{a_{n_0+1}}{k_{n_0+1}} = \delta_2 a_{n_0+2}^2 + \delta_1 k_1 a_{n_0+1} a_{n_0+2}$$

and $a_{n_0+1} > 0$ implies $a_{n_0+2} \neq 0$.
Since we are interested in positive solutions, without loss of generality one can set $a_0 = 0$ and $a_n > 0$ for every $n \geq 1$.
Moreover, we are interested in positive finite energy self-similar solutions, thus we require also that

$$\sum_{n=1}^{\infty} a_n^2 < \infty.$$

In the next section we will prove the following result:

Theorem 2.2.3. *Given $t_0 < 0$, and $k_1^{-4} \leq \delta_1/\delta_2 \leq 1$, there exist self-similar solutions of the unforced ($F = 0$) model (2.30). In particular*

- *if $k_1^{-4} \leq \delta_1/\delta_2 < k_1^{-4/3}$ then for every $a_1 > 0$ there is just one self-similar solution $\{a_n\}_{n\geq 0}$ of (2.30);*
- *if $k_1^{-4/3} < \delta_1/\delta_2 \leq 1$ then there is just one self-similar solution $\{a_n\}_{n\geq 0}$ of (2.30).*

In addition, any such self-similar solution satisfies Kolmogorov's scaling law

$$\lim_{n\to\infty} \frac{a_n}{k_n^{-1/3}} = C$$

for some positive constant $C > 0$.

Theorem 2.2.3 divides the positive plane in four sub-regions: above the line $\delta_1/\delta_2 = k_1^{-4}$ and below $\delta_1/\delta_2 = 1$ Theorem 2.2.3 does not give any information about existence of self-similar solution; between the lines $\delta_1/\delta_2 = k_1^{-4}$ and $\delta_1/\delta_2 = k_1^{-4/3}$ we have existence but not uniqueness; between the lines $\delta_1/\delta_2 = k_1^{-4/3}$ and $\delta_1/\delta_2 = 1$ we have existence and uniqueness of self-similar solution.
Figure (2.2) shows graphically existence and uniqueness of self-similar solutions.

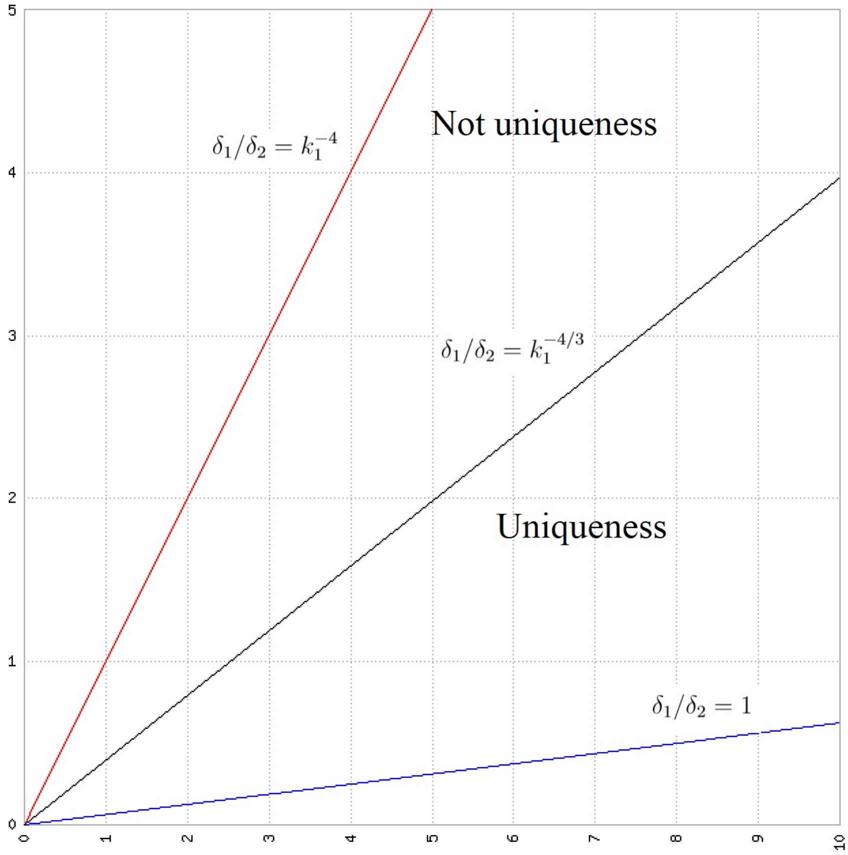

Figure 2.2: Self-similar solutions exist within the domain $k_1^{-4} \leq \delta_1/\delta_2 \leq 1$

However, as we will see later, upper and lower bounds for the ratio δ_1/δ_2 can be further refined. Numerical simulation confirmed the existence of *true* bounds $L_{true} < k_1^{-4}$ and $1 < U_{true}$ such that theorem 2.2.3 holds in the wider domain $\delta_2 \cdot L_{true} \leq \delta_1 \leq \delta_2 \cdot U_{true}$.

2.2.2 Proof of Theorems 2.2.2 and 2.2.3

This section is entirely devoted to prove theorems 2.2.2 and 2.2.3.

Let us start by considering the following recursive sequence $\{b_n\}_n$:

$$\begin{aligned} b_0 &= C > 0, \\ b_{n+1} &= \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} b_n^{-2} + 4\delta_2^2 b_n^{-1}}}{2\delta_2} \end{aligned} \tag{2.38}$$

for some positive starting value $C > 0$ and some positive coefficient δ_1, δ_2 such that $\delta_1/\delta_2 < k_1^{-4/3}$.

Lemma 2.2.4 tells useful information about the sequence $\{b_n\}_n$ and its asymptotic behaviour.

Lemma 2.2.4. *For every starting value $C > 0$, and positive coefficient δ_1, δ_2 such that $\delta_1/\delta_2 < k_1^{-4/3}$, the recursive sequence (2.38) satisfies*

$$\lim_{n\to\infty} b_n = 1.$$

Proof. Since recursion (2.38) admits 1 as unique fixed point, we first observe that if $C = 1$ then $b_n \equiv 1$ for every $n \geq 0$.

Without loss of generality let us suppose $C < 1$ (the case $C > 1$ being specular). We will prove the following properties

1. $b_{2n+1} > 1, b_{2n} < 1$, $\forall n \geq 0$;
2. $1 < b_{2n+1} < b_{2n-1}$ and $0 < b_{2n} < b_{2n+2} < 1$, $\forall n \geq 0$;
3. $\lim_{n\to\infty} b_{2n+1} = \lim_{n\to\infty} b_{2n} = 1$,

the statement will follow trivially.

We start observing that $b_1 > 1$:

$$\begin{aligned} b_1 &= \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} C^{-2} + 4\delta_2^2 C^{-1}}}{2\delta_2} \\ &> \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} + 4\delta_2^2}}{2\delta_2} = 1. \end{aligned}$$

Let's now suppose $b_{2n-1} > 1$ for some $n > 0$. Then we have

$$b_{2n+1} = \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} b_{2n}^{-2} + 4\delta_2^2 b_{2n}^{-1}}}{2\delta_2},$$

moreover, by inductive hypothesis and definition

$$\begin{aligned} b_{2n} =& \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} b_{2n-1}^{-2} + 4\delta_2^2 b_{2n-1}^{-1}}}{2\delta_2} \\ <& \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} + 4\delta_2^2}}{2\delta_2} = 1, \end{aligned}$$

hence $b_{2n}^{-1} > 1$, and finally

$$\begin{aligned} b_{2n+1} =& \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} b_{2n}^{-2} + 4\delta_2^2 b_{2n}^{-1}}}{2\delta_2} \\ &> \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} + 4\delta_2^2}}{2\delta_2} = 1, \end{aligned}$$

proving property (1).
We now focus on the first part of property (2) (the second being identical). By definition we can write

$$\begin{aligned} b_{2n+1} < b_{2n-1} &\iff \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} b_{2n}^{-2} + 4\delta_2^2 b_{2n}^{-1}}}{2\delta_2} < b_{2n-1} \\ &\iff 4\delta_1\delta_2 k_1^{4/3} b_{2n}^{-2} + 4\delta_2^2 b_{2n}^{-1} < 4b_{2n-1}^2\delta_2^2 + 4\delta_1\delta_2 b_{2n-1} k_1^{4/3} \\ &\iff \delta_1 k_1^{4/3} b_{2n}^{-2} + \delta_2 b_{2n}^{-1} < b_{2n-1}^2 \delta_1 + \delta_2 b_{2n-1} k_1^{4/3} \\ &\iff \delta_1 k_1^{4/3}(b_{2n}^{-2} - b_{2n-1}) < \delta_2(b_{2n-1}^2 - b_{2n}^{-1}). \end{aligned}$$

By hypothesis we set $\delta_1 k_1^{4/3} < \delta_2$, thus it is enough to require

$$b_{2n}^{-2} - b_{2n-1} < b_{2n-1}^2 - b_{2n}^{-1}$$

within the positive condition on the right side $0 < b_{2n-1}^2 - b_{2n}^{-1}$. We observe that the above two inequalities are both satisfied if $b_{2n}^{-1} < b_{2n-1}$, indeed:

$$b_{2n}^{-1} < b_{2n-1} \implies b_{2n}^{-1} + b_{2n}^{-2} < b_{2n-1} + b_{2n-1}^2$$

and

$$b_{2n}^{-1} < b_{2n-1} \implies b_{2n}^{-2} < b_{2n-1}^2 \implies 0 < b_{2n}^{-1} < b_{2n}^{-2} < b_{2n-1}^2,$$

the latter being true due to $b_{2n} < 1$.

We are now left to prove the sufficient condition $b_{2n}^{-1} < b_{2n-1}$ or equivalently

$$\begin{aligned} b_{2n}^{-1} = & \frac{2\delta_2}{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} b_{2n-1}^{-2} + 4\delta_2^2 b_{2n-1}^{-1}}} < b_{2n-1} \\ \iff & \frac{2\delta_2}{b_{2n-1}} + \delta_1 k_1^{4/3} < \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} b_{2n-1}^{-2} + 4\delta_2^2 b_{2n-1}^{-1}} \\ \iff & 4\delta_1^2 b_{2n-1}^{-2} + 4\delta_1\delta_2 k_1^{4/3} b_{2n-1}^{-1} < 4\delta_1\delta_2 k_1^{4/3} b_{2n-1}^{-2} + 4\delta_2^2 b_{2n-1}^{-1} \\ \iff & \delta_2 b_{2n-1}^{-2} + \delta_1 k_1^{4/3} b_{2n-1}^{-1} < \delta_1 k_1^{4/3} b_{2n-1}^{-2} + \delta_2 b_{2n-1}^{-1} \\ \iff & \delta_2 + \delta_1 k_1^{4/3} b_{2n-1} < \delta_1 k_1^{4/3} + \delta_2 b_{2n-1} \\ \iff & (\delta_2 - \delta_1 k_1^{4/3}) < (\delta_2 - \delta_1 k_1^{4/3}) \cdot b_{2n-1}, \end{aligned}$$

finally the latter inequality holds because $\delta_2 - \delta_1 k_1^{4/3} > 0$ and $b_{2n-1} > 1$.
We can now say that b_{2n+1} admits limit $\lim_{n\to\infty} b_{2n+1} = L \geq 1$. Suppose $L > 1$, then

$$L = \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} g(L)^{-2} + 4\delta_2^2 g(L)^{-1}}}{2\delta_2} \tag{2.39}$$

where

$$g(L) = \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} L^{-2} + 4\delta_2^2 L^{-1}}}{2\delta_2}$$

By a direct calculation equation (2.39) holds if and only if

$$\begin{aligned} &4\delta_2^2 L^2 + 4\delta_1\delta_2 k_1^{4/3} L = 4\delta_1\delta_2 k_1^{4/3} g(L)^{-2} + 4\delta_2^2 g(L)^{-1} \\ \iff\ &\delta_2 L^2 + \delta_1 k_1^{4/3} L = \delta_1 k_1^{4/3} g(L)^{-2} + \delta_2 g(L)^{-1} \\ \iff\ &\delta_2 (L^2 - g(L)^{-1}) = \delta_1 k_1^{4/3} (g(L)^{-2} - L). \end{aligned}$$

Let's take a closer look to the last equation. By hypothesis $\delta_2 > \delta_1 k_1^{4/3}$, so just one of the following case holds:

- $L^2 - g(L)^{-1} > 0$ and $(L^2 - g(L)^{-1}) < (g(L)^{-2} - L)$:

  from the second inequality we recover $(L^2 + L) < (g(L)^{-2} + g(L)^{-1})$. Thanks to the properties we have already proved, it is not hard from the latter to deduce $L < g(L)^{-1}$. It is now time to expand the right hand side to obtain:

  $$\begin{aligned} &L < g(L)^{-1} = \frac{2\delta_2}{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} L^{-2} + 4\delta_2^2 L^{-1}}} \\ \iff\ &4\delta_1\delta_2 k_1^{4/3} + 4\delta_2^2 L < 4\delta_2^2 + 4\delta_1 k_1^{4/2} L \\ \iff\ &L(\delta_2 - \delta_1 k_1^{4/3}) < (\delta_2 - \delta_1 k_1^{4/3}) \iff L < 1 \end{aligned}$$

  that is absurd.

- $L^2 - g(L)^{-1} < 0$ and $(L^2 - g(L)^{-1}) > (g(L)^{-2} - L)$:

  again, from our assumptions:

  $$\begin{aligned} &L^2 < g(L)^{-1} \iff L^2 < \frac{2\delta_2}{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} L^{-2} + 4\delta_2^2 L^{-1}}} \\ \iff\ &\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3} L^{-2} + 4\delta_2^2 L^{-1} < (2\delta_2/L^2 + \delta_1 k_1^{4/3})^2 \\ \iff\ &4\delta_1\delta_2 k_1^{4/3} L^{-2} + 4\delta_2^2 L^{-1} < 4\delta_2^2 L^{-4} + 4\delta_1\delta_2 k_1^{4/3} L^{-2} \\ \iff\ &L^3 < 1 \iff L < 1 \end{aligned}$$

again against our assumption.

- $L^2 = g(L)^{-1}$ and $L = g(L)^{-2}$:

  if $0 < x, y \in \mathbb{R}$ are two positive real numbers such that

$$x^2 = y, \quad x = y^2,$$

  the only solution is $x = y = 1$.

We conclude that $L = \lim b_{2n+1} = 1$. With same argument one can show $\lim_{n\to\infty} b_{2n} = 1$, concluding the proof.

□

We now state and prove an equivalent for Lemma 2.2.4 when $\delta_1/\delta_2 > k_1^{-4/3}$. First, for every $N > 1$ consider the following recursive backward sequence $\{(b_n^*)^{(N)}\}_n$:

$$(b_N^*)^{(N)} = C^* > 0,$$
$$(b_n^*)^{(N)} = \frac{-\delta_2 k_1^{-4/3} + \sqrt{\delta_2^2 k_1^{-8/3} + 4\delta_1\delta_2 k_1^{-4/3}((b_{n+1}^*)^{(N)})^{-2} + 4\delta_1^2((b_{n+1}^*)^{(N)})^{-1}}}{2\delta_1} \tag{2.40}$$

for any $0 \leq n < N$, some positive starting value $C^* > 0$ and some positive coefficient δ_1, δ_2 such that $\delta_1/\delta_2 > k_1^{-4/3}$.
Lemma 2.2.5 and Lemma 2.2.6 tell useful information about sequence $\{(b_n^*)^{(N)}\}_n$ and its asymptotic behaviour.

Lemma 2.2.5. *For every starting value $C^* > 0$, any $N > 1$ and any positive coefficients δ_1, δ_2 such that $\delta_1/\delta_2 > k_1^{-4/3}$, the recursive sequence $\{(b_n^*)^{(N)}\}_n$ defined above satisfies*

1. $0 < (b_N^*)^{(N)} < (b_n^*)^{(N)} < (b_{N-1}^*)^{(N)} < \frac{1}{C^*}$, *if* $C^* < 1$, $\forall n \geq 1$.

2. $0 < (b_{N-1}^*)^{(N)} < (b_n^*)^{(N)} < (b_N^*)^{(N)} = C^*$, *if* $C^* > 1$, $\forall n \geq 1$.

3. $(b_n^*)^{(N)} \equiv 1$, *if* $C^* = 1$, $\forall n \geq 1$.

Proof. Proof of Lemma 2.2.5 is entirely equivalent to the one we proposed for Lemma 2.2.4 by swapping δ_1 and δ_2 coefficients. The only statement left to prove is

$$(b_{N-1}^*)^{(N)} < \frac{1}{C^*}, \text{ if } C^* < 1.$$

By a direct calculation we have

$$
\begin{aligned}
(b^*_{N-1})^{(N)} < \frac{1}{C^*} &\iff \frac{-\delta_2 k_1^{-4/3} + \sqrt{\delta_2^2 k_1^{-8/3} + 4\delta_1\delta_2 k_1^{-4/3}/C^{*2} + 4\delta_1^2/C^*}}{2\delta_1} < \frac{1}{C^*} \\
&\iff \delta_2^2 k_1^{-8/3} + \frac{4\delta_1\delta_2 k_1^{-4/3}}{C^{*2}} + \frac{4\delta_1^2}{C^*} < \delta_2^2 k_1^{-8/3} + \frac{4\delta_1\delta_2 k_1^{-4/3}}{C^*} + \frac{4\delta_1^2}{C^{*2}} \\
&\iff \frac{\delta_2 k_1^{-4/3}}{C^{*2}} + \frac{\delta_1}{C^*} < \frac{\delta_2 k_1^{-4/3}}{C^*} + \frac{\delta_1}{C^{*2}} \\
&\iff C^*(\delta_1 - \delta_2 k_1^{-4/3}) < (\delta_1 - \delta_2 k_1^{-4/3}) \iff C^* < 1,
\end{aligned}
$$

due to the assumption $\delta_1/\delta_2 > k_1^{-4/3}$. □

Lemma 2.2.5 tells that for every $N > 1$, $\{(b^*_n)^{(N)}\}$ lies in the compact set $[0, C^*]$ if $C^* \geq 1$ or $[0, 1/C^*]$ if $C^* < 1$, thus by compactness and a diagonal extraction argument we can choose a subsequence $(N_i)_i \in \mathbb{N}$ such that $(b^*_n)^{(N_i)}$ converges for all $n \in \mathbb{N}$ to some number $\tilde{b}^*_n$. The sequence $\tilde{b} = \{\tilde{b}^*_n\}_n$ satisfies the following equation by construction

$$
\tilde{b}^*_n = \frac{-\delta_2 k_1^{-4/3} + \sqrt{\delta_2^2 k_1^{-8/3} + 4\delta_1\delta_2 k_1^{-4/3}(\tilde{b}^*_{n+1})^{-2} + 4\delta_1^2(\tilde{b}^*_{n+1})^{-1}}}{2\delta_1}. \tag{2.41}
$$

Lemma 2.2.6. *For every starting value $C^* > 0$, and positive coefficient δ_1, δ_2 such that $\delta_1/\delta_2 > k_1^{-4/3}$, the recursive sequence (2.41) satisfies*

$$
\lim_{n\to\infty} \tilde{b}^*_n = 1.
$$

Proof. It is equivalent to the proof of Lemma 2.2.4 by swapping δ_1 and δ_2 coefficients. □

We are now ready to prove Theorem 2.2.2.

Let us start by considering equation (2.35) written in the following form

$$
0 = \delta_1(a_{n-1}^2 - k_1 a_n a_{n+1}) - \delta_2(a_{n+1}^2 - k_1^{-1} a_n a_{n-1}).
$$

We already focus our interest into positive solution with any zero term, so dividing by a_n both sides and changing variable with $b_n = \frac{a_n}{a_{n-1}}$ we obtain

$$
0 = \delta_1(b_n^{-2} - k_1 b_{n+1}) - \delta_2(b_{n+1}^2 - k_1^{-1} b_n^{-1}).
$$

We now apply a further change of variable $a_n = \tilde{a}_n / k_n^{1/3}$ and consequently $b_n = \tilde{b}_n / k_1^{1/3}$ to finally get

$$
0 = \delta_1(k_1^{2/3}\tilde{b}_n^{-2} - k_1^{2/3}\tilde{b}_{n+1}) - \delta_2(k_1^{-2/3}\tilde{b}_{n+1}^2 - k_1^{-2/3}\tilde{b}_n^{-1}).
$$

We can solve the above equation of degree two restricting ourselves only to positive solution

$$\tilde{b}_0 = a_1/F > 0,$$
$$\tilde{b}_{n+1} = \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3}\tilde{b}_n^{-2} + 4\delta_2^2\tilde{b}_n^{-1}}}{2\delta_2}. \tag{2.42}$$

Lemma 2.2.4 shows that $\lim_{n\to\infty}\tilde{b}_n = 1$ every time $\delta_1/\delta_2 < k_1^{-4/3}$. Thus $\lim_{n\to\infty} b_n = k_1^{-1/3}$ and $\lim_{n\to\infty} a_n/a_{n-1} = k_1^{-1/3} < 1$, proving Theorem 2.2.2 statements in the case $\delta_1/\delta_2 < k_1^{-4/3}$.
With the same fashion, one can consider a backward change of variable $b_n = \frac{a_{n-1}}{a_n}$ and obtain

$$0 = \delta_1(b_n^2 - k_1 b_{n+1}^{-1}) - \delta_2(b_{n+1}^{-2} - k_1^{-1} b_n).$$

We now apply a further change of variable $a_n = \tilde{a}_n/k_n^{1/3}$ and consequently $b_n = \tilde{b}_n/k_1^{1/3}$ and finally get

$$0 = \delta_1(k_1^{2/3}\tilde{b}_n^2 - k_1^{2/3}\tilde{b}_{n+1}^{-1}) - \delta_2(k_1^{-2/3}\tilde{b}_{n+1}^{-2} - k_1^{-2/3}\tilde{b}_n).$$

As before we can solve the above equation of degree two restricting ourselves only to positive solution

$$\tilde{b}_0 = a_0/a_1 > 0$$
$$\tilde{b}_n = \frac{-\delta_2 k_1^{-4/3} + \sqrt{\delta_2^2 k_1^{-8/3} + 4\delta_1\delta_2 k_1^{-4/3}\tilde{b}_{n+1}^{-2} + 4\delta_1^2\tilde{b}_{n+1}^{-1}}}{2\delta_1}. \tag{2.43}$$

Lemma 2.2.5 tells $\lim_{n\to\infty}\tilde{b}_n = 1$ every time $\delta_1/\delta_2 > k_1^{-4/3}$. Thus $\lim_{n\to\infty} b_n = k_1^{-1/3}$ and $\lim_{n\to\infty} a_{n-1}/a_n = k_1^{1/3} > 1$, proving Theorem 2.2.2 statements also in the case $\delta_1/\delta_2 > k_1^{-4/3}$.

Now that we have successfully proved Theorem 2.2.2, we observe that equation (2.37) for self-similar sequences differs from equation (2.35) for stationary solution only for a perturbation term $\frac{a_n}{k_n}$. Thus, we will adapt our proof taking care of this extra term.

Indeed, without loss of generality, we can set $a_0 = 0$ and $a_n > 0$ for every $n > 0$ in equation (2.37), then dividing by a_n both sides and changing variable with $b_n = \frac{a_n}{a_{n-1}}$ we obtain

$$\delta_1(b_n^{-2} - k_1 b_{n+1}) - \delta_2(b_{n+1}^2 - k_1^{-1} b_n^{-1}) - \frac{1}{a_n k_n} = 0.$$

We now apply a further change of variable $a_n = \tilde{a}_n/k_n^{1/3}$ and consequently $b_n = \tilde{b}_n/k_1^{1/3}$ and finally get

$$\delta_1(k_1^{2/3}\tilde{b}_n^{-2} - k_1^{2/3}\tilde{b}_{n+1}) - \delta_2(k_1^{-2/3}\tilde{b}_{n+1}^2 - k_1^{-2/3}\tilde{b}_n^{-1}) - \frac{\tilde{a}_n}{k_n^{2/3}} = 0.$$

We can solve the above equation of degree two restricting ourselves only to positive solution

$$\begin{aligned} \tilde{b}_1 &= a_2/a_1 > 0, \\ \tilde{b}_{n+1} &= \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3}\tilde{b}_n^{-2} + 4\delta_2^2\tilde{b}_n^{-1} + 4\delta_2 k_1^{2/3}\epsilon_n}}{2\delta_2}. \end{aligned} \tag{2.44}$$

where $\epsilon_n = (\tilde{a}_n k_n^{2/3})^{-1} = (a_n k_n)^{-1}$.

In the next Lemma we prove that $\lim_{n\to\infty} \epsilon_n = 0$.

Lemma 2.2.7. *If $\{a_n\}_n$ is a positive self-similar sequence satisfying equation (2.36) with $k_1^{-4} \leq \delta_1/\delta_2 \leq 1$, then*

$$\lim_{n\to\infty} a_n k_n = \infty.$$

Proof. Let's consider a change of variable $a_n = c_n/k_n$ in equation (2.36) to obtain

$$-c_n = \delta_1(k_1^2 c_{n-1}^2 - c_n c_{n+1}) - \delta_2(k_1^{-2} c_{n+1}^2 - c_n c_{n-1}).$$

It is helpful to express c_{n+1} as function of previous terms

$$c_{n+1} = \frac{-\delta_1 c_n + \sqrt{\delta_1^2 c_n^2 + 4\delta_2 k_1^{-2} c_n + 4\delta_1\delta_2 c_{n-1}^2 + 4\delta_2^2 k_1^{-2} c_n c_{n-1}}}{2\delta_2 k_1^{-2}}.$$

We first prove that $c_{n+1} > c_{n-1}$ for every $n \geq 1$.

Indeed, we have

$$\begin{aligned} & c_{n+1} = \frac{-\delta_1 c_n + \sqrt{\delta_1^2 c_n^2 + 4\delta_2 k_1^{-2} c_n + 4\delta_1\delta_2 c_{n-1}^2 + 4\delta_2^2 k_1^{-2} c_n c_{n-1}}}{2\delta_2 k_1^{-2}} > c_{n-1} \\ \iff & 4\delta_2 k_1^{-2} c_n + 4\delta_1\delta_2 c_{n-1}^2 + 4\delta_2^2 k_1^{-2} c_n c_{n-1} > 4\delta_2^2 k_1^{-4} c_{n-1}^2 + 4\delta_1\delta_2 k_1^{-2} c_n c_{n-1} \\ \iff & k_1^{-2} c_n + \delta_1 c_{n-1}^2 + \delta_2 k_1^{-2} c_n c_{n-1} > \delta_2 k_1^{-4} c_{n-1}^2 + \delta_1 k_1^{-2} c_n c_{n-1} \\ \iff & k_1^{-2} c_n + c_{n-1}^2(\delta_1 - \delta_2 k_1^{-4}) + k_1^{-2} c_n c_{n-1}(\delta_2 - \delta_1) > 0, \end{aligned} \tag{2.45}$$

and last inequality holds thanks to the assumption $\delta_2 k_1^{-4} \leq \delta_1 \leq \delta_2$.

Thus, there is a positive value $D > 0$ so that $c_n \geq D$ for every $n \geq 1$.

With similar argument it is possible to prove the existence of $M > 1$ so that $c_{n+1} > c_{n-1} \cdot M$, this will conclude the proof.
Indeed,

$$c_{n+1} = \frac{-\delta_1 c_n + \sqrt{\delta_1^2 c_n^2 + 4\delta_2 k_1^{-2} c_n + 4\delta_1\delta_2 c_{n-1}^2 + 4\delta_2^2 k_1^{-2} c_n c_{n-1}}}{2\delta_2 k_1^{-2}} > c_{n-1} \cdot M$$
$$\iff k_1^{-2} c_n + \delta_1 c_{n-1}^2 + \delta_2 k_1^{-2} c_n c_{n-1} > \delta_2 k_1^{-4} M^2 c_{n-1}^2 + \delta_1 k_1^{-2} M c_n c_{n-1}.$$

Last inequality further simplifies as follows

$$k_1^{-2} c_n + c_{n-1}^2 (\delta_1 - \delta_2 k_1^{-4} M^2) + k_1^{-2} c_n c_{n-1} (\delta_2 - \delta_1 M) > 0$$
$$\iff k_1^{-2} D + D^2 (\delta_1 + k_1^{-2}\delta_2)(1 - k_1^{-2} M) > 0.$$

Finally, by hypothesis $\lambda > 1$ and $\beta > 0$, hence it is possible to choose $1 < M \leq k_1^2$ in the latter relation, so that left hand sides becomes a sum of positive term. □

Remark. We notice that upper and lower bounds on theorem 2.2.3 arise from inequality (2.45). Condition $\delta_2 k_1^{-4} \leq \delta_1 \leq \delta_2$ is sufficient in order to satisfy inequality (2.45), although the first term $k_1^{-2} c_n$ gives a small but significant positive contribute. Consequently, upper and lower bounds for the ratio δ_1/δ_2 can be further refined. Numerical simulation confirmed the existence of *true* bounds $L_{true} < k_1^{-4}$ and $1 < U_{true}$ such that theorem 2.2.3 holds in the wider domain $\delta_2 \cdot L_{true} \leq \delta_1 \leq \delta_2 \cdot U_{true}$.

Figure (2.3) shows graphically the complete behavior of self-similar solutions.

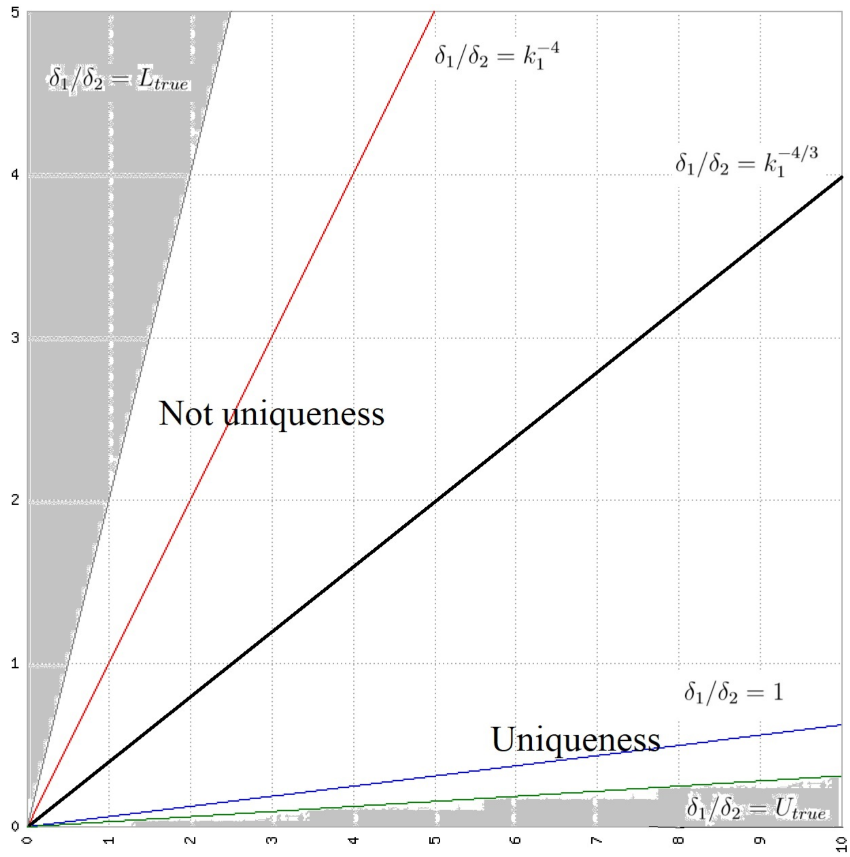

Figure 2.3: Shaded regions represent areas where no self-similar solution exists.

The following Lemma is an equivalent of Lemma 2.2.4 for the sequence (2.44).

Lemma 2.2.8. *Let's suppose* $\tilde{b}_1 = a_2/a_1 = C > 0$*, and* $k_1^{-4} \leq \delta_1/\delta_2 \leq k_1^{-4/3}$*. If* $\tilde{b}_1 \geq \tilde{b}_3$ *then*

- $C \geq \tilde{b}_3 \geq \tilde{b}_5 \geq \ldots \geq \tilde{b}_{2n+1} \geq \ldots > 1$ *for all* $n \geq 0$*;*
- $0 < \tilde{b}_2 \leq \tilde{b}_4 \leq \tilde{b}_6 \leq \ldots \tilde{b}_{2n} \leq \ldots \leq 1 + \sqrt{\epsilon_1 \frac{k_1^{2/3}}{\delta_2}}$ *for all* $n \geq 1$*.*

Otherwise, if $\tilde{b}_1 < \tilde{b}_3$ *then*

- $C \leq \tilde{b}_3 \leq \tilde{b}_5 \leq \ldots \leq \tilde{b}_{2n+1} \leq \ldots \leq 1 + \sqrt{\epsilon_2 \frac{k_1^{2/3}}{\delta_2}}$ *for all* $n \geq 0$*;*
- $\tilde{b}_2 \geq \tilde{b}_4 \geq \tilde{b}_6 \geq \ldots \tilde{b}_{2n} \geq \ldots > 1$ *for all* $n \geq 1$*.*

Moreover, $\lim_{n\to\infty} \tilde{b}_{2n+1} = \lim_{n\to\infty} \tilde{b}_{2n} = 1$*.*

Proof. We consider only the case $\tilde b_1 \geq \tilde b_3$ (the other being specular).
Let's first observe that

$$\begin{aligned}\tilde b_4 =& \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3}\tilde b_3^{-2} + 4\delta_2^2 \tilde b_3^{-1} + 4\delta_2 k_1^{2/3}\epsilon_3}}{2\delta_2} \\ \leq& \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3}\tilde b_3^{-2} + 4\delta_2^2 \tilde b_3^{-1} + 4\delta_2 k_1^{2/3}\epsilon_1}}{2\delta_2}\end{aligned}$$

thus $\tilde b_2 \leq \tilde b_4$ if and only if

$$\begin{aligned}&\sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3}\tilde b_1^{-2} + 4\delta_2^2 \tilde b_1^{-1} + 4\delta_2 k_1^{2/3}\epsilon_1} \\ &\leq \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3}\tilde b_3^{-2} + 4\delta_2^2 \tilde b_3^{-1} + 4\delta_2 k_1^{2/3}\epsilon_1} \\ &\iff 4\delta_1\delta_2 k_1^{4/3}\tilde b_1^{-2} + 4\delta_2^2\tilde b_1^{-1} \leq 4\delta_1\delta_2 k_1^{4/3}\tilde b_3^{-2} + 4\delta_2^2\tilde b_3^{-1} \\ &\iff \delta_1 k_1^{4/3}\tilde b_1^{-2} + \delta_2\tilde b_1^{-1} \leq \delta_1 k_1^{4/3}\tilde b_3^{-2} + \delta_2\tilde b_3^{-1}\end{aligned}$$

and, remembering $\delta_1/\delta_2 \leq k_1^{-4/3}$, the latter is implied by $\tilde b_1 \geq \tilde b_3$.
The following cascade of implications is then an immediate consequence

$$\begin{aligned}\tilde b_1 \geq \tilde b_3 \implies \tilde b_2 \leq \tilde b_4 \implies \tilde b_3 \geq \tilde b_5 \implies \tilde b_4 \leq \tilde b_6 \implies \dots \\ \implies \tilde b_{2n-1} \geq \tilde b_{2n+1} \implies \tilde b_{2n} \leq \tilde b_{2n+2}, \quad \forall n \geq 1.\end{aligned}$$

We now say that $\{\tilde b_{2n+1}\}_n$ admit finite limit, say L_1: with the same argument used in Lemma 2.2.4, thanks to Lemma 2.2.7 one can easily prove $L_1 = 1$.
We will now prove the upper bound

$$\tilde b_{2n} \leq 1 + \sqrt{\epsilon_1 \frac{k_1^{2/3}}{\delta_2}}, \quad \forall n \geq 1.$$

We first stress that $\tilde b_1 \geq \tilde b_3$ only if $C = \tilde b_1 > 1$, thus we can write

$$\begin{aligned}\tilde b_{2n} =& \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3}\tilde b_{2n-1}^{-2} + 4\delta_2^2 \tilde b_{2n-1}^{-1} + 4\delta_2 k_1^{2/3}\epsilon_{2n-1}}}{2\delta_2} \\ \leq& \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3}\tilde b_{2n-1}^{-2} + 4\delta_2^2 \tilde b_{2n-1}^{-1}} + \sqrt{4\delta_2 k_1^{2/3}\epsilon_{2n-1}}}{2\delta_2} \\ \leq& \frac{-\delta_1 k_1^{4/3} + \sqrt{\delta_1^2 k_1^{8/3} + 4\delta_1\delta_2 k_1^{4/3}\tilde b_{2n-1}^{-2} + 4\delta_2^2 \tilde b_{2n-1}^{-1}} + \sqrt{4\delta_2 k_1^{2/3}\epsilon_1}}{2\delta_2} \\ \leq& 1 + \sqrt{\epsilon_1 \frac{k_1^{2/3}}{\delta_2}}\end{aligned}$$

where the latter inequality is a direct consequence of $C > 1$ and $b_{2n-1} > L_1 = 1$. We know also that $\{\tilde{b}_{2n}\}_n$ admit finite limit, say L_2: again, with similar argument used in Lemma 2.2.4 and thanks to Lemma 2.2.7 we conclude $L_2 = 1$.

□

Lemma 2.2.4 and Lemma 2.2.8 show that $\lim_{n\to\infty} \tilde{b}_n = 1$ every time $k_1^{-4} \leq \delta_1/\delta_2 \leq k_1^{-4/3}$. Thus $\lim_{n\to\infty} b_n = k_1^{-1/3}$ and $\lim_{n\to\infty} a_n/a_{n-1} = k_1^{-1/3} < 1$, proving Theorem 2.2.3 statements in the case $k_1^{-4} \leq \delta_1/\delta_2 \leq k_1^{-4/3}$.

We now mimic again proof of Theorem 2.2.2 also in the case $k_1^{-4/3} < \delta_1/\delta_2 \leq 1$, by considering a backward change of variable $b_n = \frac{a_{n-1}}{a_n}$ to obtain

$$-\frac{1}{a_n k_n} = \delta_1(b_n^2 - k_1 b_{n+1}^{-1}) - \delta_2(b_{n+1}^{-2} - k_1^{-1} b_n).$$

We now apply a further change of variable $a_n = \tilde{a}_n / k_n^{1/3}$ and consequently $b_n = \tilde{b}_n / k_1^{1/3}$ and finally get

$$-\frac{1}{\tilde{a}_n k_n^{2/3}} = \delta_1(k_1^{2/3}\tilde{b}_n^2 - k_1^{2/3}\tilde{b}_{n+1}^{-1}) - \delta_2(k_1^{-2/3}\tilde{b}_{n+1}^{-2} - k_1^{-2/3}\tilde{b}_n).$$

As before we can solve the above backward equation of degree two restricting ourselves only to positive solution. For every $N > 1$ let be

$$\tilde{b}_N^{(N)} = C^* > 0,$$
$$\tilde{b}_n^{(N)} = \frac{-\delta_2 k_1^{-4/3} + \sqrt{\delta_2^2 k_1^{-8/3} + 4\delta_1\delta_2 k_1^{-4/3}(\tilde{b}_{n+1}^{(N)})^{-2} + 4\delta_1^2(\tilde{b}_{n+1}^{(N)})^{-1} - 4\delta_1\epsilon_n^*}}{2\delta_1}.$$

for any $0 \leq n < N$, with $\epsilon_n^* = \frac{1}{\tilde{a}_n^{(N)} k_n^{2/3}} = \frac{1}{a_n^{(N)} k_n}$.

Lemma 2.2.9 shows that there is $C^* > 0$ so that the sequence $\{\tilde{b}_n^{(N)}\}_n$ is well defined and lies uniformly in a compact set for every $N > 0$.

Lemma 2.2.9. *For every $k_1^{-4/3} < \delta_1/\delta_2 \leq 1$, there is $C^* > 0$ so that $\tilde{b}_n^{(N)}$ is well defined for every $0 < n \leq N$.*
Moreover, there is $M^* > 1$ and $N^* > 0$ so that the sequence $\{\tilde{b}_n^{(N)}\}_n$ satisfies

$$0 < \frac{1}{M^*} \leq \tilde{b}_n^{(N)} \leq M^*$$

for every $N > N^*$ and every $N^* < n \leq N$.

Proof. By definition $\frac{a_{N-1}^{(\tilde{N})}}{a_N^{(\tilde{N})}} = C^* > 0$, and by Lemma 2.2.7 we can choose the free parameter C^* small enough so that

$$4\delta_1\epsilon_n^* \leq 4\delta_1 \max\{\epsilon_1^*, \epsilon_2^*\} \leq \delta_2 k_1^{-8/3}$$

for every $n > 0$, this implies that the square root in the expression of sequence $\{\tilde{b}_n^{(N)}\}_n$ is well defined.

We now prove the statement by induction over $n \leq N$. Indeed, for every C^* there is $M^* > 1$ so that

$$\frac{1}{M^*} \leq C^* = \tilde{b}_N^{(N)} \leq M^*$$

Let's suppose now $\frac{1}{M^*} \leq \tilde{b}_n^{(N)} \leq M^*$ for some $n \leq N$. By definition

$$\begin{aligned}\tilde{b}_{n-1}^{(N)} \leq M^* \iff & \frac{4\delta_1\delta_2 k_1^{-4/3}}{(\tilde{b}_n^{(N)})^2} + \frac{4\delta_1^2}{\tilde{b}_n^{(N)}} \leq 4\delta_1^2 M^{*2} + 4\delta_1\delta_2 k_1^{-4/3} M^* + 4\delta_1\epsilon_n^* \\ \iff & \frac{\delta_2 k_1^{-4/3}}{(\tilde{b}_n^{(N)})^2} + \frac{\delta_1}{\tilde{b}_n^{(N)}} \leq \delta_1 M^{*2} + \delta_2 k_1^{-4/3} M^* + \epsilon_n^*.\end{aligned}$$

By hypothesis $\frac{1}{M^*} \leq \frac{1}{\tilde{b}_n^{(N)}} \leq M^*$, so it is enough to prove

$$\begin{aligned}&\delta_2 k_1^{-4/3} M^{*2} + \delta_1 M^{*2} \leq \delta_1 M^{*2} + \delta_2 k_1^{-4/3} M^* \\ &\iff \delta_2 k_1^{-4/3}(M^* - 1) \leq \delta_1(M^* - 1),\end{aligned}$$

the latter being true thanks to $M^* > 1$ and $\delta_1 > \delta_2 k_1^{-4/3}$. This proves right side of our claim.
Again, by definition

$$\begin{aligned}\tilde{b}_{n-1}^{(N)} \geq \frac{1}{M^*} \iff & \frac{4\delta_1\delta_2 k_1^{-4/3}}{(\tilde{b}_n^{(N)})^2} + \frac{4\delta_1^2}{\tilde{b}_n^{(N)}} \geq \frac{4\delta_1^2}{M^{*2}} + \frac{4\delta_1\delta_2 k_1^{-4/3}}{M^*} + 4\delta_1\epsilon_n^* \\ \iff & \frac{\delta_2 k_1^{-4/3}}{(\tilde{b}_n^{(N)})^2} + \frac{\delta_1}{\tilde{b}_n^{(N)}} \geq \frac{\delta_1}{M^{*2}} + \frac{\delta_2 k_1^{-4/3}}{M^*} + \epsilon_n^*.\end{aligned}$$

We observe that it is enough to prove

$$\begin{aligned}&\frac{\delta_2 k_1^{-4/3}}{M^{*2}} + \frac{\delta_1}{M^*} \geq \frac{\delta_1}{M^{*2}} + \frac{\delta_2 k_1^{-4/3}}{M^*} + \epsilon_n^* \\ &\iff \delta_2 k_1^{-4/3} + \delta_1 M^* \geq \delta_1 + \delta_2 k_1^{-4/3} M^* + \epsilon_n^* M^{*2} \\ &\iff (\delta_1 - \delta_2 k_1^{-4/3})\frac{M^* - 1}{M^{*2}} \geq \epsilon_n^*.\end{aligned}$$

By Lemma 2.2.7, $\lim_{n\to\infty} \epsilon_n^* = 0$, so there is $N^* > 0$ so that

$$(\delta_1 - \delta_2 k_1^{-4/3})\frac{M^* - 1}{M^{*2}} \geq \epsilon_n^*$$

for every $N^* < n \leq N$.

□

Lemma 2.2.9 shows that exists N^* such that for every $N > 0$, $\{\tilde{b}_n^{(N)}\}$ lies in a compact set, thus by compactness and a diagonal extraction argument we can choose a subsequence $(N_i)_i \in \mathbb{N}$ such that $\tilde{b}_n^{(N_i)}$ converges for all $n \in \mathbb{N}$ to some number $\tilde{b}_n^*$. The sequence $\tilde{b}^* = \{\tilde{b}_n^*\}_n$ satisfies the following equation by construction

$$\tilde{b}_n^* = \frac{-\delta_2 k_1^{-4/3} + \sqrt{\delta_2^2 k_1^{-8/3} + 4\delta_1\delta_2 k_1^{-4/3}(\tilde{b}_{n+1}^*)^{-2} + 4\delta_1^2(\tilde{b}_{n+1}^*)^{-1} - 4\delta_1\epsilon_n^*}}{2\delta_1}.$$

Finally, by the same argument used in Lemmas 2.2.5 and 2.2.7 we deduce $\lim_{n\to\infty} \tilde{b}_n^* = 1$ every time $\delta_1/\delta_2 \geq k_1^{-4/3}$. Thus $\lim_{n\to\infty} b_n = k_1^{-1/3}$ and $\lim_{n\to\infty} a_{n-1}/a_n = k_1^{1/3} > 1$, proving Theorem 2.2.3 statements also in the case $k_1^{-4/3} < \delta_1/\delta_2 \leq 1$.

Chapter 3

Dyadic Models on a Tree

In this chapter we start by presenting the main results about the *dyadic tree model* introduced by Katz and Pavlovic in [44] and studied later by Barbato et al. [2]. In Chapter 2 we have already presented Katz-Pavlovic model (2.1) in its original form. Here, we are interested in studying its generalization first developed in [17], extending it to an inverse cascade model with a non-linearity of Obukhov-type as well as to a mixed model similar to (2.30).

3.1 Direct energy cascade

Following [17], we recall the abstract tree model that simulates the direct energy cascade (non-linearity of Katz-Pavlovic type).

Let d be the dimension of the space and $\mathcal{N} = 2^d$. We consider the following set with its inherited tree structure:

$$J = \bigcup_{n=0}^{\infty} \{1, 2, \ldots, \mathcal{N}\}^n = \{\emptyset, 1, 2, \ldots, \mathcal{N}, (1,1), (1,2), \ldots\}.$$

For every pair of nodes $j = (j_1, j_2, \ldots, j_m)$, $k = (k_1, k_2, \ldots, k_n) \in J$, we define the append operator $j \circ k = (j_1, j_2, \ldots, j_m, k_1, k_2, \ldots, k_n) \in J$, the generation operator $|j| = m \in \mathbb{N}$, the partial ordering $j \leq k$ if and only if $k = j \circ h$ for some $h \in J$, the father operator $\bar{\jmath} \in J$ such that $\bar{\jmath} < j$ and $|\bar{\jmath}| = |j| - 1$ and the offspring set $\mathcal{O}_j = \{k \in J | \bar{k} = j\}$ for every $j \in J \setminus \{\emptyset\}$.

The model we are interested in is described by the following system of equations:

$$v_j'(t) = c_j v_{\bar{\jmath}}^2(t) - \sum_{k \in \mathcal{O}_j} c_k v_j(t) v_k(t), \qquad j \in J, t \geq 0 \tag{3.1}$$

where $c_j = d_j \cdot 2^{\beta|j|}$, for some $\beta > 0$, $d_j > 0$ for every $j \in J$, $d_\emptyset = 1$ and $v_{\bar{\emptyset}}(t) = f \geq 0$ is the forcing term on the first component.

Kolmogorov Spectrum

In the case of the linear dyadic model (2.4), we proved that Kolmogorov inertial range spectrum reads

$$Y_n \sim k_n^{-1/3}.$$

For the tree dyadic model (3.1), when $d = 3$ and $\beta = d/2 + 1$, the Kolmogorov inertial range spectrum corresponds to

$$X_j \sim 2^{-\frac{11}{6}|j|}. \tag{3.2}$$

This time the correct exponent is not so immediate. We will observe such behavior later for special class of solutions. In what follows we provide a simple heuristic derivation.
Kolmogorov K41 theory states that, if $v(x)$ is the velocity of the turbulent fluid at position x we have

$$\mathbb{E}[|v(x) - v(y)|^2] \sim |x - y|^{2/3},$$

where x and y are two points very close to each other. This approximately means

$$|v(x) - v(y)| \sim |x - y|^{1/3}.$$

If we think $v(x)$ written in the wavelet orthonormal basis (w_j) as

$$v(x) = \sum_j X_j w_j(x),$$

the vector field $w_j(x)$ is correlated to the velocity field of the j-th eddy. Such eddy has a support $\mathcal{Q}_j$ of the order of a cube of size $2^{-|j|}$. Given $j \in J$, let's take $x, y \in \mathcal{Q}_j$. Then we use the approximation $v(x) = X_j w_j(x)$ and $v(y) = X_j w_j(y)$. Hence,

$$|v(x) - v(y)| = |X_j||w_j(x) - w_j(y)| \sim |x - y|^{1/3}, \qquad x, y \in \mathcal{Q}_j.$$

We can consider this approximation reasonably correct when $x, y \in \mathcal{Q}_j$ have a distance of the order of $2^{-|j|}$, otherwise we should use smaller eddies.
Thus we have

$$|X_j||w_j(x) - w_j(y)| \sim 2^{-\frac{1}{3}|j|}, \quad x, y, \in \mathcal{Q}_j, |x - y| \sim 2^{-|j|}. \tag{3.3}$$

Moreover, for some points η between x and y, we have

$$|w_j(x) - w_j(y)| = |\nabla w_j(\eta)||x - y|. \tag{3.4}$$

We now recall that the size s_j of w_j inside $\mathcal{Q}_j$ can be derived from $s_j^2 2^{-3|j|} \sim 1$, since $\int w_j(x)^2 dx = 1$, hence $s_j \sim 2^{3|j|/2}$. Supposing w_j are all linear transformations of the same wavelet, we deduce that the typical values of ∇w_j in $\mathcal{Q}_j$ have the order $2^{3|j|/2}/2^{-|j|} = 2^{5|j|/2}$. Thus, from relation (3.4),

$$|w_j(x) - w_j(y)| \sim 2^{\frac{5}{2}|j|} 2^{-|j|},$$

along with (3.3), gives us

$$|X_j| 2^{\frac{5}{2}|j|} 2^{-|j|} \sim 2^{-\frac{1}{3}|j|} \implies X_j \sim 2^{-\frac{11}{6}|j|},$$

proving (3.2) on a heuristic ground. One can find a more rigorous derivation in the Appendix of [17].

About the choice of coefficients

The main novelty from the literature is that we allow the coefficients of the non-linear term to depend on the nodes of the tree and not only through their generation. Every node of the tree has $2^d = 8$ children and interacts with each one of them in the same way but for a coefficient $c_j = 2^{\beta|j|} d_j$.
The generalization to coefficients d_j is the key point of model (3.1). It completely changes the behaviour of anomalous dissipation and makes the structure function ζ_p strictly concave, as it should be according to the most recent numerical simulations of realistic turbulence phenomena. Allowing d_j to be different from 1, forces spatial intermittency on the solutions.
In our model we always consider the quantity $|\log d_j|$ bounded, and the more general result are proved in this setting. We explicit the latter limitedness by assuming the existence of $M > 0$ so that

$$\frac{1}{M} \leq d_j \leq M, \qquad \forall j \in J.$$

Although, many explicit computations are possible only in the special case that the same fixed coefficients 2^d appear in every set $\{d_k \,|\, k \in \mathcal{O}_j\}$. We call this model *Repeated Coefficient Model* (RCM).

In the RCM case we set $\{d_k \,|\, k \in \mathcal{O}_j\} = \{\delta_w \,|\, w \in \Omega\}$, for some Ω of cardinality 2^d.
In this case we introduce also the log-s-norm of the coefficients. For $s \in \mathbb{R} \setminus 0$ let

$$\ell_s = \frac{1}{s}\log_2(\frac{1}{2^d}\sum_{w\in\Omega}\delta_w^s).$$

This can be extended with

$$\ell_0 = \frac{1}{2^d}\sum_{w\in\Omega}\log_2\delta_w,$$

$$\ell_{-\infty} = \lim_{s\to-\infty}\ell_s = \min_w \delta_w, \qquad \ell_\infty = \lim_{s\to\infty}\ell_s = \max_w \delta_w,$$

to obtained a bounded, non-decreasing and continuous function ℓ on $[-\infty, \infty]$.

Similarly to the linear case, the parameter β is again left free in all statements, but from a physical point of view it is meaningful to fix $\beta = \frac{d}{2} + 1$

In [54], the author correlates Besov spaces with particular sequence from J to $\mathbb{R}$, thus it is possible to study the regularity of the velocity field by introducing the following norms on the set of functions from J to $\mathbb{R}$.

Definition 3.1.1. *For every $s \in \mathbb{R}$ we define the space H^s of the maps $u : J \to \mathbb{R}$ such that:*

$$||u||_{H^s} = \sqrt{\sum_{j\in J} 2^{2s|j|}u_j^2} < \infty.$$

In particular we write $H = H^0 = \ell^2(J)$.

Definition 3.1.2. *For every $s \in R$ and $p \geq 1$ we introduce the space $W^{s,p}$ of the maps $u : J \to \mathbb{R}$ such that:*

$$||u||_{W^{s,p}} = (\sum_{j\in J} 2^{[ps+d(\frac{p}{2}-1)]|j|}|u_j|^p)^{\frac{1}{p}} < \infty.$$

In particular $W^{s,2} = H^s$.

Definition 3.1.3. *For all $s \in (0, 1)$ we define the space C^s of the maps $u : J \to \mathbb{R}$ such that*

$$\sup_{n\geq 1}(ns + \frac{1}{2}dn + \max_{|j|=n}\log_2|u_j|) < \infty.$$

Such spaces correspond to the usual ones for the recomposed velocity field function.

Finally, in [17] it was proven that the exponents ζ_p of the structure function of a solution $u(x)$ are given by

$$\zeta_p = \min\{p\ ,\ d - \frac{p}{2}d - \limsup_{n\to\infty}\frac{1}{n}\log_2\sum_{|j|=n}|u_j|^p\}. \tag{3.5}$$

Observation. It is immediate to verify that, in the case $\beta = d/2+1$ and $d = 3$, the inertial solution $X = \{X_j \sim 2^{-\frac{11}{6}|j|}\}_{j\in J}$ satisfies

$$X \in H^s, \quad s < \frac{1}{3}, \qquad \text{and} \qquad X \in W^{s,p}, \quad p \geq 1, \;\; s < \frac{1}{3}.$$

Indeed,

$$||X||_{H^s} = \sqrt{\sum_n \sum_{|j|=n} 2^{2sn} \cdot 2^{-\frac{11}{6}n}} = \sqrt{\sum_n 2^{nd} 2^{2sn} \cdot 2^{-\frac{11}{6}n}} < \infty \iff$$
$$d + 2s - \frac{11}{6} < 0 \iff s < \frac{11}{6} - \frac{d}{2} = \frac{1}{3}.$$

Similarly,

$$||X||_{W^{s,p}} = \sqrt{\sum_n \sum_{|j|=n} 2^{[ps+d(\frac{p}{2}-1)]n} \cdot 2^{-\frac{11}{6}pn}} = \sqrt{\sum_n 2^{nd} 2^{[ps+d(\frac{p}{2}-1)]n} \cdot 2^{-\frac{11}{6}pn}} < \infty \iff$$
$$d + ps + d(\frac{p}{2} - 1) - \frac{11}{6}p < 0 \iff s < \frac{11}{6} - \frac{d}{2} = \frac{1}{3}.$$

Similarly to the linear dyadic model in the previous chapter, one can extend the following definitions to model (3.1).

Definition 3.1.4. *A weak solution is a family $v = (v_j)_{j\in J}$ of non negative differentiable functions satisfying (3.1).*
A Leray-Hopf solution is a weak solution $v \in L^\infty(\mathbb{R}^+, H)$.

Compared to the linear dyadic model, this time existence and uniqueness of solutions are more subtle matters.
In [2] it has been proved with classical Galerkin approximation that if $d_j = 1$ for every $j \in J$, then for any initial condition with non negative components there exists at least one Leray-Hopf solution. The generalization to the general model is straightforward. Uniqueness of solutions is an open problem even for the model with $d_j = 1$. In general, uniqueness does not hold if one drops the non negativity condition. One way to exploit this hypothesis is presented in [3], although it required estimates of terms of the kind

$$\int_0^t v_n^3(s)ds.$$

However, for large n they are difficult to generalize to our general setting.
In [17] it has been proved the existence, and uniqueness in some sense, of a stationary solution by introducing a forcing term on the first component. We will recall

such construction in the next section. Then we extend this framework in order to prove existence of solutions that are not constant in time. These two results are an interesting proof of anomalous dissipation.
Finally, we investigate the evolution of the unique stationary solution found in [17] in the case we add a viscous friction in equation (3.1), discovering an interesting regularizing phenomenon due to the presence of coefficients d_j.

3.1.1 Stationary Leray-Hopf solution

We consider now Leray-Hopf solutions $u = (u_j)_{j \in J}$ not depending on time, what we call stationary solutions in previous chapter.
From equation (3.1) we have

$$0 = c_j u_{\bar{j}}^2 - \sum_{k \in \mathcal{O}_j} c_k u_j u_k, \qquad j \in J,$$

that leads us to the fundamental recursion for stationary solution

$$d_j u_{\bar{j}}^2 = 2^{\beta} \sum_{k \in \mathcal{O}_j} d_k u_j u_k, \qquad j \in J. \tag{3.6}$$

One could be tempted to solve (3.6) recursively, although there are some major difficulties. First of all, given u_j and $u_{\bar{j}}$, there are $2^d - 1$ degrees of freedom for choosing variables u_k for $k \in \mathcal{O}_j$. Moreover, it is not trivial to prove that any such solution belongs to H. Indeed, *a posteriori* it turns out that there exists a unique Leray-Hopf solution, hence all choices but one produce a sequences u_j satisfying recursion (3.6) but not belonging to H.
Alternatively, it is possible to replicate the pull-back technique already introduced in Section 2.1.6: let's fix arbitrarily u_j for all $j \in J$ with given large generation $|j| = n$, then compute u_k for the lower generations $|k| < n$ and then let $n \to \infty$, and finally proving convergence by compactness. In [17], the only Leray-Hopf solution of (3.1) has been shown explicitly

$$\tilde{u}_j = f \cdot 2^{\sum_{h \leq j} \tilde{q}_h} \prod_{k \leq j} \sqrt{d_k}, \qquad j \in J,$$

where

$$\tilde{q}_j = \lim_{n \to \infty} q_j^{(n)}$$

and

$$\begin{aligned}
q_j^{(n)} &= 0, && |j| > n,\\
q_j^{(n)} &= x \in \mathbb{R}, && |j| = n,\\
q_j^{(n)} &= -\frac{1}{2}\beta - \frac{1}{2}\log_2\Big(\sum_{k\in\mathcal{O}_j} d_k^{\frac{3}{2}} 2^{q_k^{(n)}}\Big) && |j| < n.
\end{aligned}$$

We now recall some results about the only stationary solution $\tilde{u}$.

Theorem 3.1.1. *Suppose*

$$\sup_{j\in J}\log_2 d_j - \inf_{j\in J}\log_2 d_j = L < \infty.$$

Then, there exists a stationary weak solution $\tilde{u}$ of (3.1) such that its coefficients $\tilde{q}_j$ are bounded.
Moreover, $\tilde{u} \in H^r$ for all

$$r < \frac{1}{3}(\beta - \frac{d}{2}) - L.$$

In particular, if $\beta > \frac{d}{2}$ and

$$\frac{\sup_{j\in J} d_j}{\inf_{j\in J} d_j} \le 2^{\frac{1}{3}(\beta - \frac{d}{2})},$$

then there exists a constant Leray-Hopf solution.

Next theorem shows that uniqueness of stationary solutions holds in the union of H^r for all $r \in \mathbb{R}$.

Theorem 3.1.2. *Under the same hypothesis of Theorem 3.1.1, for all $s \in \mathbb{R}$ there exists at most one stationary weak solution in H^s.*

Model with repeated coefficients

If we restrict ourselves to the model with repeated coefficients, computation of many quantities simplifies enormously while still showing peculiar features.

Theorem 3.1.3. *The RCM admits a stationary weak solution $u \in W^{s,p}$ for all $p \ge 1$ and $s < s_0(p)$, where*

$$s_0(p) = \frac{1}{3}(\beta - \frac{d}{2}) + \frac{1}{2}(\ell_{3/2} - \ell_{p/2}).$$

Such solution is unique inside any H^s and it admits an explicit form

$$u_j = f \cdot 2^{q|j|+q} \prod_{k \le j} \sqrt{d_k}, \qquad j \in J,$$

where

$$q = \frac{1}{3}(\beta + d) - \frac{1}{2}\ell_{3/2}.$$

If $\beta > d/2$ the solution is of Leray-Hopf.

The proof of Theorem 3.1.3 is base upon the following useful Lemma we recall from [17].

Lemma 3.1.4. *If the model is RCM, then for any real function ϕ and any positive integer n,*

$$\sum_{|j|=n} \prod_{k \le j} \phi(d_k) = (\sum_{\omega \in \Omega} \phi(\delta_\omega))^n$$

We conclude this section recalling from [17] Theorem 3.1.5 that clarifies the multifractal nature of the stationary solution as well as the spatial intermittency phenomenon. In Section 3.2 we state an equivalent of Theorem 3.1.5 for an Obukhov-type model that shows an inverse cascade of energy.

Theorem 3.1.5. *Let's consider the quantity*

$$h = \frac{1}{3}(\beta - \frac{d}{2}) - \frac{1}{2}(\ell_\infty - \ell_{3/2}),$$

And suppose $h \in (0,1)$. Then there exists a unique constant Leray-Hopf solution which lies in C^s if and only if $s \le h$. Furthermore, the exponents ζ_p of the structure function are given by

$$\zeta_p = \min\{p \ ; \ \frac{p}{3}(\beta - \frac{d}{2}) + \frac{p}{2}(\ell_{3/2} - \ell_{p/2})\}, \quad p \ge 0.$$

3.1.2 Self-Similar solutions

One step forward in the study of model (3.1) is to search for solutions that are not constant in time. One natural way is then to look for self-similar solutions like we did in section (2.1.6) for the linear dyadic model.
In this setting we call *self-similar* any solution Y of the form $Y_j(t) = a_j \cdot \phi(t)$, $j \in J$. Furthermore, we will restrict ourselves to **positive** finite energy self-similar solutions.
It is easy to prove that self-similar solutions of model (3.1) have again the form

$$Y_j(t) = \frac{a_j}{t - t_0}, \quad j \in J \tag{3.7}$$

with $t > t_0$ for some $t_0 < 0$.
Indeed, by equation (3.1) we obtain

$$\begin{aligned} a_j\phi'(t) &= 2^{\beta|j|}d_j a_j^2\phi(t)^2 - \sum_{k\in\mathcal{O}_j} 2^{\beta|k|}d_k a_j a_k \phi(t)^2 \\ &= \phi(t)^2\cdot(2^{\beta|j|}d_j a_j^2 - \sum_{k\in\mathcal{O}_j} 2^{\beta|k|}d_k a_j a_k) \end{aligned}$$

and relation (3.7) follows immediately.
Aim of this section is to prove, for small value of $M > 1$, existence and uniqueness of finite energy self-similar solutions.

Let $X = (\frac{a_j}{t-t_0})_{j\in J}$ be a self-similar solution, where $a_j > 0$ for every $j \in J$. By substituting expression (3.7) into equation (3.1) one obtain

$$\begin{aligned} X_j'(t) = -\frac{a_j}{(t-t_0)^2} &= 2^{\beta|j|}d_j\frac{a_j^2}{(t-t_0)^2} - \sum_{k\in\mathcal{O}_j} 2^{\beta|k|}d_k\frac{a_j}{t-t_0}\frac{a_k}{t-t_0} \\ &= c_j X_j^2(t) - \sum_{k\in\mathcal{O}_j} c_k X_j(t)X_k(t), \\ &\iff -a_j = 2^{\beta|j|}d_j a_j^2 - \sum_{k\in\mathcal{O}_j} 2^{\beta|k|}d_k a_j a_k \quad j\in J \end{aligned}$$

the latter leads to the following fundamental recursion for self-similar solution:

$$2^\beta a_j\sum_{k\in\mathcal{O}_j} d_k a_k = d_j a_j^2 + \frac{a_j}{2^{\beta|j|}}, \quad j\in J. \tag{3.8}$$

In order to solve recursion (3.8) we introduce the weights $(\pi_j)_{j\in J}$ such that

$$\begin{aligned} &2^\beta a_j d_k a_k = \pi_k(d_j a_j^2 + \frac{a_j}{2^{\beta|j|}}), \quad k\in\mathcal{O}_j, \quad |K|\geq 2 \\ &2^\beta a_k d_k = \pi_k, \quad |k| = 1 \\ &\sum_{k\in\mathcal{O}_j}\pi_k = 1, \quad \pi_k > 0, \quad j\in J \end{aligned} \tag{3.9}$$

For all choices of $a_\emptyset$ and π_k satisfying the condition, this recursion builds a solution to which there corresponds one weak self-similar solution of the original system.

To find a regular solution, one should control the growth of the coefficients a_k. Unfortunately, because the non linearity is the one from Katz-Pavlovic, we expect the same behaviour found in [17] for stationary solutions. Hence, we expect that only one regular solution exists, and its existence can be proved by pull-back techniques.

Hence, we fix a generation N, i.e., fix the values of a_k in a suitable way at $|k| = N$ and perhaps $|k| = N-1$, and then look for a solution in the first N generations. The backward recursion that we need to solve is given by

$$a_j^2 = \frac{a_j}{d_j}(2^\beta \sum_{k\in\mathcal{O}_j} d_k a_k - 2^{-\beta|j|}), \qquad |j| \geq 1. \tag{3.10}$$

In the case of stationary solutions, there is an algebraic rewriting of the system that makes the recursion of the first order, i.e., one can write a_j as a function of a_k for $k \in \mathcal{O}_j$. Since our recursion is of second order, we get different definitions of a_j: one for each $k \in \mathcal{O}_j$. This rises the problem to find compatibility conditions for these different equations.

More problems arise because by fixing the values of a_k at $k = N$ and also at $k = N-1$ we are given too many constraints. As seen in the forward recursion, the space of solutions for the finite system with $j \leq N$, is given by $1+2^d+\ldots+2^{Nd}$ variables with $1+2^d+\ldots+2^{(N-1)d}$ linear constraints, yielding 2^{Nd} degrees of freedom. This is exactly the number of variable a_k with $|k| = N$, so it is impossible to fix both the last and the second-last generations and expect to find compatible equations in the recursion.

Our strategy is then the following:

1. We prove that for any choice of a_k at $|k| = N$ there exist a solution of the finite system.

2. We find a way to control the a_k at $|K| = N-1$ so that we can apply Lemma 3.1.7, which controls the coefficients of the whole finite tree based on the last two generations.

There are several different ways to rewrite the system in such a way that we can expect the coefficients to be controllable for the regular solution. Inspired by the construction of stationary solution, we propose the following:

$$a_j = 2^{-\frac{\beta}{3}|j|} \prod_{i\leq j} \sqrt{d_i}\alpha_j \tag{3.11}$$

yielding

$$\begin{aligned}
\alpha_j &= \beta_j \alpha_{\bar{j}}^2, & |j| \geq 1 \\
\beta_j &= (\sum_{k \in \mathcal{O}_j} d_k^{3/2} \alpha_k - s_j)^{-1}, & |j| \geq 1 \\
s_j &= 2^{-\frac{2}{3}\beta(|j|+1)} d_j^{-1/2}, & |j| \geq 1 \\
s_\emptyset &= \sum_{|j|=1} d^{3/2} \alpha_j = 2^{-\frac{2\beta}{3}}.
\end{aligned} \tag{3.12}$$

The case $N = 2$
Suppose we are given $\alpha_k > 0$ for all $|k| = 2$. Then we immediately compute β_j for all $|j| = 1$ and by (3.12) we compute $\alpha_\emptyset$ and hence α_j for all $|j| = 1$.
In view of the second point, we write explicitly the functions involved, $\overline{\alpha} : \mathbb{R}_+^{2^d} \to \mathbb{R}_+^{2^d}$ and $\overline{\beta} : \mathbb{R}_+^{2^{2d}} \to \mathbb{R}_+^{2^d}$, that is

$$\begin{aligned}
\overline{\alpha}_j(\beta) &= \frac{\beta_j s_\emptyset}{\sum_{|j|=1} d_j^{3/2} \beta_j} \\
\overline{\beta}_j(\alpha) &= (\sum_{k \in \mathcal{O}_j} d_k^{3/2} \alpha_k - s_j)^{-1}
\end{aligned} \tag{3.13}$$

We notice that $\overline{\beta}$ is defined on a subset of $\mathbb{R}_+^{2^{2d}}$, as it is required to be positive, and that $\overline{\alpha}(\beta) \leq s_\emptyset d_{\min}^{-3/2}$. Moreover, if we take α with all equal components and large, $\overline{\alpha} \circ \overline{\beta}(\alpha)$ tends to a given point in $\mathbb{R}_+^{2^d}$ which depends on the coefficients d_k.

The case $N = 3$
Suppose we are given $\alpha_k > 0$ for all $|k| = 3$. Then we easily compute β_j for all $|j| = 2$. We would like to obtain the corresponding α_j and then apply the case $N = 2$. Let's consider the inverse map that gives β_j as functions of α_j. From α_j with $|j| = 2$ we can get the α_j with $|j| = 1$ and then $\beta_j = \alpha_j \alpha_{\bar{j}}^{-2}$. If we can invert this map, we are done (we are automatically inside the domain of β).

Let $\varphi : \mathbb{R}_+^{2^{2d}} \to \mathbb{R}_+^{2^{2d}}$ be defined by

$$\varphi_j(\alpha) = a_j \cdot [\overline{\alpha}_{\bar{j}} \circ \overline{\beta}(\alpha)]^{-2}.$$

We prove that φ is surjective. Let's first perform a logarithmic transformation as follow

$$\psi_j(x) = x_j - 2 \log \circ \overline{\alpha}_{\bar{j}} \circ \overline{\beta} \circ \exp x_*,$$

where the $_*$ denotes the componentwise versions of common functions. Then we can write

$$\psi = \mathrm{id} - 2V \log_* \circ \overline{\alpha} \circ \overline{\beta} \circ \exp_*,$$

where V is the $2^{2d} \times 2^d$ matrix given by $V_{i,j} = 1_{\bar{i},j}$. Next, we rotate the ambient space in such a way to make the column vectors of V the first canonical vectors (they are orthogonal, with norm $2^{d/2}$) to obtain

$$\overline{\psi} = \mathrm{id} - 2^{1+d/2} \sum_{|j|=1} e_j \log_* \circ \overline{\alpha}_j \circ \overline{\beta} \circ \exp_* R.$$

To study the surjectivity of $\overline{\psi}$, let's consider the equation $\overline{\psi}(x) = z \in \mathbb{R}_+^{2^{2d}}$. We can fix the components out of $|j| = 1$, as they are not changed by the map, and just study the restriction of $\overline{\psi}$ to $U_z = U + z$, where $U = \mathrm{Span}(e_j, |j| = 1)$. Such restriction can be read on U as a map of the form $\mathrm{id} + f$ with

$$f(x) = f^z(x) = -2^{1+d/2} \sum_{|j|=1} e_j \log_* \circ \overline{\alpha}_j \circ \overline{\beta} \circ \exp_*(2^{-d/2} V x + v),$$

for suitable $v \in \mathbb{R}_+^{2^{2d}}$ depending on z. By applying a further homothety, we get rid of the two factors $2^{d/2}$. Finally we are able to study this function by components:

$$\overline{\beta}_j \circ \exp_*(Vx + v) = (\sum_{k \in \mathcal{O}_j} d_k^{3/2} e^{x_j + v_k} - s_j)^{-1} = s_j^{-1}(e^{x_j - c_j} - 1)^{-1}, \quad |j| = 1. \tag{3.14}$$

This is monotone decreasing in all components separately. After applying $\overline{\alpha}_j$ we get that when x_j is increased keeping all other components constant, the corresponding component of f increases while the other components decrease.

The domain of f is $\prod_{|j|=1}(c_j, \infty)$. It is possible to choose $\tilde{x}$ near $c = (c_j)_{|j|=1}$ such that the weights $\overline{\beta}_j$ are all equal. The corresponding point $P = f(\tilde{x})$ characterizes the image of $\overline{\psi}$ as the point $\tilde{x} + P = \overline{\psi}(\tilde{x})$ arbitrarily near to $c + P$. It is in the image and represents a sort of lower bound vertex of the image of $\overline{\psi}$.

Similarly, it is possible to choose $\hat{x}$ with large components of similar value such that the weights $\overline{\beta}_j$ are all equal. Then the points $\hat{x} + P = \overline{\psi}(\hat{x})$ is also in the image.

We now have two opposed vertices of a large box, thus given the monotonicity of components, all the points in the rectangular box with those vertices are in the image, hence proving that all the open quadrant with vertex $c + P$ is inside the image. Then φ is invertible for large enough components. Finally, by writing the functions involved, it is obvious that $\log \circ \overline{\alpha}_{\bar{j}} \circ \overline{\beta} \circ \exp_*(x)$ is not much larger than x for $\psi_j(x)$ large given,

$$\begin{aligned} \overline{\overline{\alpha}}(\beta) &= \varphi^{-1}(\beta) \\ \overline{\overline{\beta}}_j(\alpha) &= (\sum_{k \in \mathcal{O}_j} d_k^{3/2} \alpha_k - s_j)^{-1} = \overline{\beta}_j(\alpha). \end{aligned} \tag{3.15}$$

The general case

The general case can be done in a similar way to the previous case $N = 3$, but using $\overline{\overline{\alpha}}$ and its properties instead of $\overline{\alpha}$.

We now continue to study recursion (3.10). In what follows we are interested in proving the following theorem.

Theorem 3.1.6. *Suppose there exist constant $0 < C < 2\beta - 2\log M$ and $1 \leq M < 2^{\beta/2}$ so that*

$$2^{-C} \leq \pi_k, \quad 1/M \leq d_k \leq M, \quad \forall k \in J \setminus \emptyset$$

Then there exists one and only one positive self-similar solution $X = (X_j(t))_{j \in J}$ satisfying recursion (3.8).

To prove Theorem 3.1.6 we will need the following Lemma.

Lemma 3.1.7. *Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a summable sequence of positive real number. Let's consider the following recursive sequence*

$$\begin{aligned} b_{n+1} &= \sqrt{b_n \cdot (b_{n-1} - \epsilon_n)}, \quad n \geq 1 \\ b_1, b_0 &\in \mathbb{R}^+. \end{aligned} \tag{3.16}$$

If $b_1 > L - \epsilon_1 > 0$ and $b_0 > L - \epsilon_0 > 0$ for some positive real number $L > \sum_{i=0}^{\infty} \epsilon_i$, then the sequence $\{b_n\}_{n\in\mathbb{N}}$ is well defined and satisfies

$$L - \sum_{i=1}^{n} \epsilon_i < b_n \leq \max\{b_0, b_1\}, \quad \forall\, n \geq 0.$$

Proof. First of all let's call $\Sigma = \sum_{n=0}^{\infty} \epsilon_n < \infty$. We observe that the sequence $\{b_n\}_n$ is well defined if and only if $b_n - \epsilon_{n+1} > 0$ for every $n \geq 0$.
We prove by induction that

$$L - \Sigma < L - \sum_{i=1}^{n} \epsilon_i \leq b_n \leq \max\{b_0, b_1\}, \quad \forall\, n \geq 0.$$

By hypothesis $b_1 > L - \epsilon_1$ and $b_0 > L - \epsilon_0$, moreover if $b_n > L - \sum_{i=0}^{n} \epsilon_i$, $b_{n-1} > L - \sum_{i=0}^{n-1} \epsilon_i$, then

$$b_{n+1} = \sqrt{b_n(b_{n-1} - \epsilon_n)} > \sqrt{(L - \sum_{i=0}^{n} \epsilon_i)^2} = L - \sum_{i=0}^{n} \epsilon_i > L - \sum_{i=0}^{n+1} \epsilon_i.$$

Likewise, $b_0, b_1 \leq \max\{b_0, b_1\}$, and if $b_n, b_{n+1} \leq \max\{b_0, b_1\}$ then

$$b_{n+1} = \sqrt{b_n(b_{n-1} - \epsilon_n)} \leq \sqrt{\max\{b_0, b_1\}(\max\{b_0, b_1\} - \epsilon_n)} < \max\{b_0, b_1\},$$

concluding the proof.

□

Proof. We can now prove Theorem 3.1.6 by starting rewriting recursion (3.10) in the following form

$$\begin{aligned} b_J^{(N)} &= 0, \qquad\qquad |j| > N \\ b_J^{(N)} &= L > 0, \quad |j| = N, N-1 \\ b_{\bar{j}}^{(N)} &= \sqrt{\frac{d_k}{d_j \pi_k} b_j^{(N)}(b_k^{(N)} - \epsilon_j)}, \end{aligned} \tag{3.17}$$

where $\epsilon_j = \frac{\pi_k}{d_k 2^{2\beta|k|/3}}$. By hypothesis the following two inequalities holds for every $j \in K$ and $k \in \mathcal{O}_j$

$$\frac{1}{M^2} \leq \frac{d_k}{d_j} \leq M^2, \quad 1 \leq \pi_k^{-1} \leq 2^C$$

for some positive constants $M \geq 1$ and $C > 0$, thus it is easy to observe that $b_j^{(N)} \geq \tilde{b_j}^{(N)}$, where $\{\tilde{b_j}^{(N)}\}_{j \in J}$ satisfies

$$\tilde{b_{\bar{j}}}^{(1)} = \sqrt{\frac{1}{M^2} \tilde{b_j}^{(1)}(\tilde{b_k}^{(1)} - \epsilon_j)}, \quad k \in \mathcal{O}_j, \;\; j \in J \tag{3.18}$$

In order to use Lemma 3.1.7 we need to reach a recursion of the form (3.16). To do so we consider the proper change of variable $\tilde{b_j}^{(N)} \to \tilde{c_j}^{(N)} \cdot 2^{(2\log M)|j|/3}$ to get the following recursion

$$\tilde{c_{\bar{j}}}^{(N)} = \sqrt{\tilde{c_j}^{(N)}(\tilde{c_k}^{(N)} - \epsilon_j^*)}, \quad k \in \mathcal{O}_j, \;\; j \in J$$

where $\epsilon_j^* = \epsilon_j \cdot 2^{-(2\log M)|j|/3 + 2\log M/3}$.
The sequence $\{\epsilon_j^*\}_{j \in J}$ is summable, thus it satisfies Lemma 3.1.7. and the recursive sequence $\{\tilde{c_j}^{(N)}\}_{j \in J}$ is well defined and lies in a compact set for initial values

$\tilde{c_j}^{(N)} = L$, $|J| = N, N-1$, larger than some positive threshold. Thus, also the sequence $\{a_j^{(N)} = \frac{\tilde{c_j}^{(N)}}{2^{(\beta-2\log M)|j|/3}}\}_{j\in J}$ is well defined for starting values larger than some positive threshold.

We can now prove compactness and finite energy property. With an argument similar to the previous one $b_j^{(N)} \le \hat{b}_j^{(N)}$, where $\{\hat{b}_j^{(N)}\}_{j\in J}$ satisfies

$$\hat{b}_{\bar{j}}^{(N)} = \sqrt{2^{2\log M + C}\hat{b}_j^{(N)}(\hat{b}_k^{(N)} - \epsilon_j)}, \quad k \in \mathcal{O}_j, \quad j \in J \tag{3.19}$$

We consider now the following change of variable $\hat{b}_j^{(N)} \to \frac{\hat{c}_j^{(N)}}{2^{(2\log M + C)|j|/3}}$, to lead ourselves to the following recursion

$$\hat{c}_{\bar{j}}^{(N)} = \sqrt{\hat{c}_j^{(N)}(\hat{c}_k^{(N)} - \hat{\epsilon}_j)}, \quad k \in \mathcal{O}_j, \quad j \in J$$

where $\hat{\epsilon}_j = \epsilon_j \cdot 2^{(2\log M + C)|j|/3 - (2\log M + C)/3}$.
Thanks to our assumptions $\{\hat{\epsilon}_j\}_{j\in J}$ is a summable sequence.
By Lemma 3.1.7, the recursive sequence $\{\hat{c}_j^{(N)}\}_{j\in J}$ is well defined and lies in a compact set for starting values $\tilde{c}_j^{(N)}$, $|j| = N, N-1$ larger than some positive threshold.

By compactness and a diagonal extraction argument we can choose subsequences $(\tilde{N}_i)_i, (\hat{N}_i)_i \in \mathbb{N}$ such that $\tilde{b}_j^{(\tilde{N}_i)}$ converges for all $j \in J$ to some number $\tilde{b}_j$, and $\hat{b}_j^{(\hat{N}_i)}$ converges for all $j \in J$ to some number $\hat{b}_j$. The sequence $\tilde{b} = \{\tilde{b}_j\}$ satisfies recursion (3.18) by construction, as well as $\hat{b} = \{\hat{b}_j\}$ satisfies recursion (3.19). Hence, we can choose another subsequence $(N_i)_i \in \mathbb{N}$ such that $a_j^{(N_i)}$ converges for all $j \in J$ to some number a_j. By the uniqueness of the limit $\{a_j\}_{j\in J}$ is the only sequence that satisfies recursion (3.17) by construction and consequently also recursion (3.9), and the following relation holds

$$\frac{C_1}{2^{(2\log M + C + \beta)|j|/3}} \le a_j \le \frac{C_2}{2^{(\beta - 2\log M)|j|/3}}, \quad j \in J$$

for some constants $C_1, C_2 > 0$. From the hypothesis $M < 2^{\beta/2}$ we deduce that $\{a_J\}_{j\in J}$ has finite energy. □

Remark. From Theorem 3.1.6 we derive the following condition in the meaningful case where $\beta = \frac{d}{2} + 1$

$$M < 2^{d/2+1-C/2}.$$

In the general settings where the coefficients d_k are not all equal to 1, in order to actually prove existence of some self-similar solution it is necessary to require $M > 1$, thus

$$d \le C < d + 2.$$

this is equivalent to

$$\pi_j \geq \frac{1}{2^{d+2}}, \quad j \in J. \tag{3.20}$$

Let's first observe that condition (3.20) is required to hold at least definitely for $|j| > n^*$ for some $n^* > 0$.

We claim that condition (3.20) is indeed satisfied for some choice of coefficients d_j, $j \in J$.
First, if $d_k = 1$ by an heuristic argument one can prove

$$\lim_{|j|\to\infty} \pi_j = 2^{-d}. \tag{3.21}$$

Indeed, from (3.9)

$$b_{\bar{j}} = \sqrt{\frac{2^\beta b_j}{\pi_k}(b_k - \frac{\pi_k}{2^{2\beta|k|}})}, \tag{3.22}$$

moreover the Kolmogorov spectrum reads

$$b_j \sim 2^{-\frac{\beta+d}{3}|j|}. \tag{3.23}$$

Let's say $|j| = n$, thus putting (3.23) into (3.22) we get

$$2^{-\frac{\beta+d}{3}(n-1)} \sim \sqrt{\frac{2^\beta 2^{-\frac{\beta+d}{3}n}}{\pi_k}(2^{-\frac{\beta+d}{3}(n+1)} - \frac{\pi_k}{2^{2\beta(n+1)}})}.$$

Latter relation can be progressively simplified as follows

$$\begin{aligned}
2^{-\frac{2}{3}(\beta+d)(n-1)} &\sim \frac{2^{\beta-\frac{\beta+d}{3}(2n+1)}}{\pi_k} - 2^{-(\beta+\frac{\beta+d}{3})n} \\
\iff \pi_k &\sim \frac{2^{\beta-\frac{\beta+d}{3}(2n+1)}}{2^{-\frac{2}{3}(\beta+d)(n-1)} + 2^{-(\beta+\frac{\beta+d}{3})n}} \\
\iff \pi_k &\sim \frac{2^{\frac{2\beta-d}{3}}}{2^{\frac{2}{3}(\beta+d)} + \nu_n},
\end{aligned} \tag{3.24}$$

where $\nu_n = 2^{\frac{d-2\beta}{3}n}$. In the meaningful case $\beta = d/2+1$, ν_n becomes infinitesimally small for $n \to \infty$, hence

$$\lim_{|j|\to\infty} \pi_j = \frac{2^{\frac{2\beta-d}{3}}}{2^{\frac{2}{3}(\beta+d)}} = 2^{-d}. \tag{3.25}$$

Thus π_j are definitely close to 2^{-d}. Thanks to a continuity argument, for every small $\epsilon > 0$ such that

$$1 - \epsilon < d_j < 1 + \epsilon, \quad j \in J$$

exists $0 < \eta < 2$ and n^* so that

$$\frac{1}{2^{d+\eta}} \leq \pi_j, \quad \forall\, |j| > n^*.$$

Thus, we can always choose coefficient d_j that are not so distant to 1 in order to satisfy condition (3.20).

Remark. We conclude by observing that the positive requirement $b_j > 0$, $j \in J\backslash\{\emptyset\}$ is actually necessary, otherwise it would be possible to find infinitely many self-similar solutions (see for example [14] to find self-similar solutions that would be counter examples to Theorem 3.1.6).

3.1.3 Viscous stationary solutions

In this section we are going to study RCM model like (3.1) in the presence of a viscous friction, i.e.

$$\begin{aligned} &v_j'(t) = c_j v_{\bar{j}}^2(t) - \sum_{k\in\mathcal{O}_j} c_k v_j(t) v_k(t) - c_j^{\gamma} v_j(t), \quad j \in J, \quad t \geq 0, \quad \gamma \in \mathbb{R}, \\ &v_{\bar{\emptyset}} = f > 0. \end{aligned} \tag{3.26}$$

as well as to investigate for which values of the friction exponent γ, the only stationary solution found in [17] still show the same dynamics as in the inviscid model.
Next theorem addresses such questions.

Theorem 3.1.8. *It exists a critical friction value* γ^c

$$\gamma^c = \frac{1}{3\beta}(2\beta - 2\log M - C)$$

for some constant $C \geq d$*, such that for every* $\gamma < \gamma^c$*, the viscous forced model (3.26) admits a unique positive stationary finite energy solution* $Y = \{Y_j(t) = a_j\}_{j\in J}$*. Moreover, such stationary solution satisfies*

$$\frac{C_1}{2^{(\beta(2-3\gamma)-2\log M)|j|/3}} \leq a_j \leq \frac{C_1}{2^{(\beta(2-3\gamma)-2\log M - C)|j|/3}},$$

for some positive constants $C_1, C_2 > 0$.

Furthermore, $\gamma^c \leq \frac{4}{15}$ *when* $\beta = d/2 + 1$ *and* $d = 3$.

Proof. Let $Y = \{Y_j(t) = a_j\}_{j\in J}$ be a positive stationary viscous solution of (3.26). Then, by introducing the set of weights $\{\pi_k\}_{k\in J}$ as in (3.9), we get the following expression

$$\pi_k d_j a_{\bar{j}}^2 = 2^\beta d_k a_j a_k + \pi_k 2^{\beta|j|(\gamma-1)} d_j^\gamma a_j,$$

and by restricting ourselves to positive solutions we obtain

$$a_{\bar{j}} = \sqrt{\frac{d_k}{d_j}\frac{2^\beta}{\pi_k} a_j (a_k + \frac{\pi_k d_j^\gamma}{d_k 2^{\alpha|j|(1-\gamma)+\beta}})}. \tag{3.27}$$

Remark. Equation (3.27) resembles recursions (3.10) obtained for self-similar solutions, with the difference that this time the perturbation term comes with positive sign, thus the recursion is well defined for every possible starting value. Moreover, similarly to the case of self-similar solutions, it is possible to show that exists just one possible set of weights $\{\pi_j\}_{j\in J}$ such that (3.27) is a well defined recursion.

From now on we refer to π_k as the only set of weights that satisfies compatibility conditions for recursion (3.27). Moreover, by limitedness there exists $C \geq d$ so that

$$\frac{1}{2^C} \leq \pi_j \leq 1, \quad j \in J.$$

As before, we start by considering the change of variables $a_j = \frac{b_j}{2^{\beta|j|/3}}$ to obtain

$$b_{\bar{j}} = \sqrt{\frac{d_k}{d_j}\frac{1}{\pi_k} b_j (b_k + \frac{\pi_k d_j^\gamma}{d_k 2^{\beta|j|(2/3-\gamma)+2\beta/3}})}. \tag{3.28}$$

It is immediate to verify that $b_j \leq c_j$, where we define

$$c_{\bar{j}} = \sqrt{M^2 2^C c_j (c_k + \frac{M^{\gamma+1}}{2^{\beta|j|(2/3-\gamma)+2\beta/3}})} \tag{3.29}$$

and remember that coefficients d_k are bounded from above and away from zero, i.e. exists $M \geq 1$ so that $1/M \leq d_k \leq M$ for every $k \in J$. Then, by changing variable $c_j = \frac{\tilde{c}_j}{2^{(C+2\log M)|j|/3}}$, we obtain

$$\tilde{c}_{\bar{j}} = \sqrt{c_j (c_k + \frac{M^{\gamma+1}}{2^{[\beta(2/3-\gamma)-C/3-2\log M/3]|j|+2(\beta-C/2-\log M)/3}})}. \tag{3.30}$$

If coefficients d_k are not all identical (i.e. $d_k = 1\ \forall k \in J$), as the tree J grows deeper and deeper the energy flow will increasingly concentrate in paths of the tree whose nodes have larger coefficients d_k. Accordingly, solutions over paths whose nodes share smaller coefficients will be regularized by smaller friction coefficient (eventually negative exponent) and will decay with super exponential velocities.

Therefore, it is sufficient to prove existence and investigate the behavior of the solution along paths whose nodes have maximum possible coefficients.

Thanks to Lemma 3.1.7 and the same diagonal extraction argument used in previous sections, a necessary and sufficient condition for the convergence of the backward sequence $\{\tilde{c}_j\}_{j\in J}$ (hence also of the sequence $\{b_j\}_{j\in J}$) is

$$\beta(2/3-\gamma)-C/3-2\log M>0 \tag{3.31}$$

that proves the claim

$$a_j \leq \frac{C_1}{2^{(\beta(2-3\gamma)-2\log M-C)|j|/3}},$$

for some positive constant $C_1>0$.
With similar argument one can prove the existence of $C_2>0$ such that

$$\frac{C_2}{2^{(\beta(2-3\gamma)-2\log M-C)|j|/3}} \leq a_j. \tag{3.32}$$

Inequality (3.31) is equivalent to

$$\gamma < \frac{1}{3}(2-\frac{\log(M^2 2^C)}{\beta}). \tag{3.33}$$

If $d_j=1$ for every $j\in J$ (hence $M=1$) we deduce

$$\gamma < \frac{1}{3}(2-\frac{C}{\beta}) \leq \frac{1}{3}(2-\frac{d}{\beta}).$$

When $\beta=d/2+1$ and $d=3$, latter inequality becomes

$$\gamma < \frac{1}{3}(2-\frac{6}{5}) = \frac{4}{15}.$$

□

Observation. Similarly to Theorem 2.1.18, we notice that $\frac{4}{15}$ is consistently less than the global critical friction value $\frac{4}{5}$ after which every solution will be certainly regularized.
Furthermore, inequality (3.33) reveals that coefficients d_j possess a interesting regularizing property: as the upper bound M increases, the solution withstands a progressively lower friction coefficient.

3.2 Inverse energy cascade

The aim of this section is to introduce a dyadic tree model with Obukhov non-linearity in order to simulate the inverse cascade phenomenon of two-dimensional fluid vortexes. The model we are interested in is conceptually similar to models developed in previous sections.

Let's consider the following infinite dimensional system

$$X_j' = -2^{\beta|j|} d_j X_{\bar{j}} X_j + \sum_{k \in \mathcal{O}_j} 2^{\beta|k|} d_k X_k^2, \quad j \in J, \tag{3.34}$$

where $X_j = X_j(t)$ are differentiable real functions and $X_{\bar{\emptyset}} \equiv f \geq 0$ is a forcing on the first component, which acts as a dissipative term:

$$X_\emptyset' = -d_\emptyset f X_\emptyset + \sum_{k \in \mathcal{O}_\emptyset} 2^{\beta|k|} d_k X_k^2. \tag{3.35}$$

We impose conditions over the parameter d_j, $j \in J$ and β identical to direct cascade model, and for the sake of simplicity we fix $f = 1$. We will look for stationary solution of model (3.34) possibly different from the trivial null solution. We observe that the non-linearity is similar to the one proposed by Obukhov in his classic linear model, which is formally conservative, thus for solutions with positive components it provokes energy flow from lower to larger scales, i.e. from higher to lower nodes. Thus, we let the energy enter from high generation nodes, let say at N-th generation, and then let $N \to \infty$. To this end we introduce the finite dimensional models with a dummy force. From $N \leq 1$ the N-model satisfies (3.34) for $|j| < N$, where X_j is given for $|j| = N$ and $X_j = 0$ for $|j| > N$.

3.2.1 Stationary Solutions

If X_j is a stationary solution of model (3.34), we have

$$d_j X_{\bar{j}} X_j = \sum_{k \in \mathcal{O}_j} 2^\beta d_k X_k^2, \quad j \in J. \tag{3.36}$$

We introduce once more the weights $(\pi_k)_{k \in J}$

$$\pi_k d_j X_{\bar{j}} X_j = 2^\beta d_k X_k^2, \quad k \in \mathcal{O}_j. \tag{3.37}$$

Remark. We recall that, by definition, every set of weights $(\pi_k)_{k \in J}$ satisfies $\sum_{k \in \mathcal{O}_j} \pi_k = 1$. We call *flat* points of $\mathbb{R}^J$ algebraic solutions of system (3.37) such that $\pi_k = 2^{-d}$ for every $k \in J$.

Let $(j_i)_i$ be the enumeration of the nodes on a path starting from $\emptyset$, so that $j_0 = \emptyset$ and $j_{i-1} = \bar{\jmath}_i$ for $i \geq 1$.

For every path $\emptyset = j_0, j_1, \ldots, j_n$, equation (3.36) leads to

$$\begin{aligned}\log X_{j_n} = &-\frac{\beta}{3}n + \sum_{0<h\leq n}(-\frac{1}{2})^{n-h+1}\log d_{j_h} + \frac{1}{3}\sum_{h=1}^{n}(1-(-\frac{1}{2})^{n-h+1})\log \pi_{j_h} \\ &-\frac{\beta}{9}(1-(-\frac{1}{2})^n) + (\frac{2+(-\frac{1}{2})^n}{3})\log X_\emptyset.\end{aligned} \tag{3.38}$$

It is useful to express equation (3.38) also in its exponential form

$$\begin{aligned}X_{j_n} = &2^{-\frac{\beta}{3}n} \cdot \prod_{0<h\leq n} d_{j_h}^{(-\frac{1}{2})^{n-h+1}} \cdot \prod_{h=1}^{n} \pi_{j_h}^{\frac{(1-(-\frac{1}{2})^{n-h+1})}{3}} \\ &\cdot 2^{-\frac{\beta}{9}(1-(-\frac{1}{2})^n)} \cdot X_\emptyset^{(\frac{2+(-\frac{1}{2})^n}{3})}.\end{aligned} \tag{3.39}$$

Equation (3.39) suggests that the regularity of constant solution X_j depends heavily on the set of weights $\{\pi_k\}_{k\in\mathcal{O}_j, j\in J}$. Next Proposition shows that regularity is minimum, for example, in the flat case ($\pi_k = 2^{-d}$ for every $k \in \mathcal{O}_j$) and can possibly increase till an upper limit depending on weights distribution over the model tree.

Proposition 3.2.1. *Let $X = \{X_j\}_{j\in J}$ be a constant solution of (3.34). Then $X \in H^s$ for every $s < s_0$, where the exponent s_0 satisfies:*

$$\frac{1}{3}\cdot(\beta - d/2) \leq s_0 \leq \frac{1}{3}\cdot\beta.$$

Moreover $s_0 = \frac{1}{3}\cdot(\beta - d/2)$ in the flat case $\pi_k = 2^{-d}$ for every $k \in J \setminus \{\emptyset\}$, and $s_0 = \frac{1}{3}\beta$ in the limit case where in every offspring exactly one weight $\pi_k = 1$ and all the other weights are null.

Proof. We recall that $X \in H^s$ if and only if

$$\sum_{n=0}^{\infty} 2^{2ns}\sum_{|j|=n}|X_j|^2 < \infty.$$

From equation (3.39) we claim that it is possible to write

$$\sum_{|j|=n}|X_j|^2 = C_n \sum_{|j|=n} 2^{-\frac{2}{3}\beta n}\cdot\prod_{i=1}^{n}\pi_{j_i}^{\frac{2+(-\frac{1}{2})^{n-i}}{3}} \tag{3.40}$$

where the summation takes into account all the 2^{dn} possible paths from the $n-$generation nodes and the tree root, and C_n is uniformly bounded by a constant in n.

Indeed, by the expression (3.39) of a stationary solution we obtain

$$\begin{aligned}\sum_{|j|=n}|X_j|^2 &= \sum_{|j|=n}[2^{-\frac{2\beta}{3}n}\cdot\prod_{0<h\le n}d_{j_h}^{-(-\frac{1}{2})^{n-h}}\cdot\prod_{h=1}^{n}\pi_{j_h}^{\frac{(2+(-\frac{1}{2})^{n-h})}{3}}\\ &\quad\cdot 2^{-\frac{\beta}{9}(2+(-\frac{1}{2})^{n-1})}\cdot X_{\emptyset}^{(\frac{4-(-\frac{1}{2})^{n-1}}{3})}]\\ &= \sum_{|j|=n}\{[2^{-\frac{2}{3}\beta n}\cdot\prod_{i=1}^{n}\pi_{j_i}^{\frac{2+(-\frac{1}{2})^{n-i}}{3}}][\prod_{h=1}^{n}d_{j_h}^{-(-\frac{1}{2})^{n-h}}\\ &\quad\cdot 2^{-\frac{\beta}{9}(2+(-\frac{1}{2})^{n-1})}\cdot X_{\emptyset}^{(\frac{4-(-\frac{1}{2})^{n-1}}{3})}]\}\end{aligned}$$

Finally, thanks to assumptions on coefficients d_k, we observe that the term

$$\prod_{h=1}^{n}d_{j_h}^{-(-\frac{1}{2})^{n-h}}\cdot 2^{-\frac{\beta}{9}(2+(-\frac{1}{2})^{n-1})}\cdot X_{\emptyset}^{(\frac{4-(-\frac{1}{2})^{n-1}}{3})}$$

is uniformly bounded by a constant in n, leading to equation (3.40).

From expression (3.40) it is possible to derive the following inequality

$$\sum_{|j|=n}2^{-\frac{2}{3}\beta|j|}\prod_{i=1}^{n}\pi_{j_i}^{\frac{2+(-\frac{1}{2})^{n-i}}{3}}\le 2^{\frac{d-2\beta}{3}n+\frac{d}{9}((-\frac{1}{2})^n-4)}.$$

Indeed, we have

$$\begin{aligned}\sum_{|j|=n}2^{-\frac{2}{3}\beta|j|}\prod_{i=1}^{n}\pi_{j_i}^{\frac{2+(-\frac{1}{2})^{n-i}}{3}} &= \sum_{|j|=n}2^{-\frac{2}{3}\beta|j|}\pi_{j_n}^{\frac{2+1}{3}}\pi_{j_{n-1}}^{\frac{2-\frac{1}{2}}{3}}\cdots\pi_{j_1}^{\frac{2+(-\frac{1}{2})^{n-1}}{3}}\\ &= 2^{-\frac{2}{3}\beta n}\sum_{|j|=n-1}\pi_{j_{n-1}}^{\frac{1}{2}}\pi_{j_{n-2}}^{\frac{3}{4}}\cdots\pi_{j_1}^{\frac{2+(-\frac{1}{2})^{n-1}}{3}}\cdot\sum_{k\in\mathcal{O}_{j_{n-1}}}\pi_k\\ &= 2^{-\frac{2}{3}\beta n}\sum_{|j|=n-1}\pi_{j_{n-1}}^{\frac{1}{2}}\pi_{j_{n-2}}^{\frac{3}{4}}\cdots\pi_{j_1}^{\frac{2+(-\frac{1}{2})^{n-1}}{3}}\\ &= 2^{-\frac{2}{3}\beta n}\sum_{|j|=n-2}\pi_{j_{n-2}}^{\frac{3}{4}}\pi_{j_{n-3}}^{\frac{5}{8}}\cdots\pi_{j_1}^{\frac{2+(-\frac{1}{2})^{n-1}}{3}}\cdot\sum_{k\in\mathcal{O}_{j_{n-2}}}\pi_k^{\frac{1}{2}}\end{aligned}\tag{3.41}$$

we now we use that, for every $i = 1, 2, \dots n$, the arithmetic mean of $\pi_k^{\frac{2+(-\frac{1}{2})^i}{3}}$ is less or equal their $[\frac{3}{2+(-\frac{1}{2})^i}]$-mean and the hypothesis $\sum_{k\in\mathcal{O}_{j_{n-1}}} \pi_k = 1$, and finally iterate previous steps n times to get

$$\begin{aligned}
&2^{-\frac{2}{3}\beta n} \sum_{|j|=n-2} \pi_{j_{n-2}}^{\frac{3}{4}} \pi_{j_{n-3}}^{\frac{5}{8}} \cdots \pi_{j_1}^{\frac{2+(-\frac{1}{2})^{n-1}}{3}} \cdot \sum_{k\in\mathcal{O}_{j_{n-2}}} \pi_k^{\frac{1}{2}} \\
\leq\ & 2^{-\frac{2}{3}\beta n} \sum_{|j|=n-2} (\pi_{j_{n-2}}^{\frac{3}{4}} \pi_{j_{n-3}}^{\frac{5}{8}} \cdots \pi_{j_1}^{\frac{2+(-\frac{1}{2})^{n-1}}{3}}) \cdot 2^d (\frac{1}{2^d} \sum_{k\in\mathcal{O}_{j_{n-2}}} \pi_k)^{1/2} \\
=&2^{-\frac{2}{3}\beta n} \sum_{|j|=n-2} 2^{d/2} (\pi_{j_{n-2}}^{\frac{3}{4}} \pi_{j_{n-3}}^{\frac{5}{8}} \cdots \pi_{j_1}^{\frac{2+(-\frac{1}{2})^{n-1}}{3}}) \\
\leq&\dots \leq 2^{-\frac{2}{3}\beta n} \cdot 2^{\frac{d}{3}(\sum_{i=1}^{n-1}(\frac{2+(-\frac{1}{2})^i}{3}))} = 2^{\frac{d-2\beta}{3}n + \frac{d}{9}((-\frac{1}{2})^n - 4)}.
\end{aligned} \tag{3.42}$$

By the property of means, equality holds if $\pi_k = 2^{-d}$ for every k. By letting $n \to \infty$ in relation (3.42) we have shown left inequality of the claim.
In order to prove right part we recall that $\pi_k \leq 1$ for every $k \in J$, thus

$$\sum_{k\in\mathcal{O}_j} \pi_k^{\frac{2+(-\frac{1}{2})^n}{3}} \geq \sum_{k\in\mathcal{O}_j} \pi_k = 1, \quad j \in J,$$

since $0 < \frac{2+(-\frac{1}{2})^n}{3} \leq 1$.
Again, from (3.40) we obtain

$$\begin{aligned}
&\sum_{|j|=n} 2^{-\frac{2}{3}\beta|j|} \prod_{i=1}^{n} \pi_{j_i}^{\frac{2+(-\frac{1}{2})^{n-i}}{3}} = \sum_{|j|=n} 2^{-\frac{2}{3}\beta|j|} \pi_{j_n}^{\frac{2+1}{3}} \pi_{j_{n-1}}^{\frac{2-\frac{1}{2}}{3}} \cdots \pi_{j_1}^{\frac{2+(-\frac{1}{2})^{n-1}}{3}} \\
&= 2^{-\frac{2}{3}\beta n} \sum_{|j|=n-1} \pi_{j_{n-1}}^{\frac{1}{2}} \pi_{j_{n-2}}^{\frac{3}{4}} \cdots \pi_{j_1}^{\frac{2+(-\frac{1}{2})^{n-1}}{3}} \cdot \sum_{k\in\mathcal{O}_{j_{n-1}}} \pi_k \\
&= 2^{-\frac{2}{3}\beta n} \sum_{|j|=n-1} \pi_{j_{n-1}}^{\frac{1}{2}} \pi_{j_{n-2}}^{\frac{3}{4}} \cdots \pi_{j_1}^{\frac{2+(-\frac{1}{2})^{n-1}}{3}} \\
&= 2^{-\frac{2}{3}\beta n} \sum_{|j|=n-2} \pi_{j_{n-2}}^{\frac{3}{4}} \pi_{j_{n-3}}^{\frac{5}{8}} \cdots \pi_{j_1}^{\frac{2+(-\frac{1}{2})^{n-1}}{3}} \cdot \sum_{k\in\mathcal{O}_{j_{n-2}}} \pi_k^{\frac{1}{2}} \\
&\geq\ 2^{-\frac{2}{3}\beta n} \sum_{|j|=n-2} \pi_{j_{n-2}}^{\frac{3}{4}} \pi_{j_{n-3}}^{\frac{5}{8}} \cdots \pi_{j_1}^{\frac{2+(-\frac{1}{2})^{n-1}}{3}} \\
&\geq \dots \geq 2^{-\frac{2}{3}\beta n},
\end{aligned}$$

where the equality is achieved in the limit case where in every offspring exactly one weight $\pi_k = 1$ and all the other weights are null.

□

Structure Function

We are interested in investigating whether stationary solutions show intermittency behaviour. Next Proposition states that the *flat* stationary solution shows no intermittency and satisfies Kolmogorov K41 law.

Proposition 3.2.2. *Assume that all coefficients d_j are uniformly bounded away from 0 and uniformly bounded from above, i.e. there exists $M > 0$ such that*

$$\frac{1}{M} \leq d_j \leq M, \qquad j \in J.$$

Then, for the flat set of weights $\pi_k = 2^{-d}$, then the exponent ζ_p of the structure function of the stationary solution associated to the set of flat weights satisfies

$$\zeta_p = \frac{p}{3}(\beta - \frac{d}{2}),$$

which becomes $\zeta_p = \frac{p}{3}$ when we consider $\beta = d/2 + 1$.

Proof. From [17], we recover the exact form of the exponent ζ_p of the structure function S_p

$$\zeta_p = d - \frac{p}{2}d - \limsup_{n\to\infty}(\frac{1}{n}\log_2(\sum_{|j|=n}|u_j|^p)).$$

Let's consider a path starting from $\emptyset$ to the n-th generation: $j_0 = \emptyset, j_1, \ldots, j_n = j$, $|j| = n$. From relation (3.38) we set x_{j_n} and observe that the argument of the lim sup is $\frac{1}{n}\log_2\sum_{|j|=n}2^{px_j}$. By expanding the summation $\sum_{|j|=n}2^{px_j}$ we obtain:

$$\sum_{|j|=n} 2^{px_j} = 2^{p[-\frac{\beta n}{3} - \frac{\beta}{9}(1-(-\frac{1}{2})^n) + \frac{x_\emptyset}{3}(2+(-\frac{1}{2})^n) - \frac{d}{3}\sum_{h=1}^{n}(1-(-\frac{1}{2})^{n-h+1})]} \cdot \sum_{|j|=n} 2^{p\sum_{h=1}^{n}(-\frac{1}{2})^{n-h+1}\log_2 d_{j_h}}. \tag{3.43}$$

We then consider

$$\frac{1}{n}\log_2\sum_{|j|=n}2^{px_j} = \frac{p}{n}[-\frac{\beta n}{3} - \frac{\beta}{9}(1-(-\frac{1}{2})^n) + \frac{x_\emptyset}{3}(2+(-\frac{1}{2})^n) - \frac{d}{3}\sum_{h=1}^{n}(1-(-\frac{1}{2})^{n-h+1})]$$
$$+\frac{1}{n}\log_2\sum_{|j|=n}2^{p\sum_{h=1}^{n}(-\frac{1}{2})^{n-h+1}\log_2 d_{j_h}}.$$

We deal with the two terms separately. The first one reduces to

$$-\frac{\beta+d}{3}p+\frac{p}{n}[\frac{\beta}{9}(1-(-\frac{1}{2})^n)+\frac{x_\emptyset}{3}(2+(-\frac{1}{2})^n)-\frac{d}{3}(1-(-\frac{1}{2})^n)],$$

where the parenthesis converges to a constant, and hence disappears in the limit $n \to \infty$.
We can now bound the second term from above (and below by replacing M with M^{-1}) as follows

$$\begin{aligned}&\frac{1}{n}\log_2\sum_{|j|=n}\prod_{h=1}^{n} d_{j_h}^{p(-\frac{1}{2})^{n-h+1}} \leq \frac{1}{n}\log_2\sum_{|j|=n}\prod_{h=1}^{n} M^{p(-\frac{1}{2})^{n-h+1}}\\ &= \frac{1}{n}\log_2(2^{nd}\prod_{h=1}^{n} M^{p(-\frac{1}{2})^{n-h+1}}) = d+\frac{1}{n}p(1-2^{-n})\log_2 M.\end{aligned}$$

Finally, taking the limit for $n \to \infty$, the latter term tends to d. Putting together all the pieces we obtain:

$$\zeta_p = d-\frac{p}{2}d+\frac{p}{3}(\beta+d)-d=\frac{p}{3}(\beta-\frac{d}{2}),$$

proving the claim. □

Proposition 3.2.2 suggests to restrict our study to set of weights that are bounded away from zero, i.e.

$$\frac{1}{2^C} \leq \pi_j, \qquad j \in J, \tag{3.44}$$

for some positive constant $C \geq d$, as clarified by the following Corollary.

Corollary 3.2.3. *If the set of weight $\{\pi_j\}_{j\in J}$ satisfies (3.44), then ζ_p is always finite and bounded by*

$$\zeta_p \leq \frac{p}{3}(\beta+C-\frac{3}{2}d).$$

Proof. It is enough to bound each π_{j_h} from below by 2^{-C} in equation (3.43) and repeat the proof of Proposition 3.2.2 to prove the claim. □

We stress the fact that condition (3.44) is also necessary in order to get a locally bounded exponent for the structure function. Indeed, without loss of generality let's suppose that exists a path in J such that

$$\pi_{j_h} = \frac{1}{2^{f(|j_h|)}},$$

where $f : \mathbb{N} \to \mathbb{R}_+$ is a non decreasing function such that $\lim_{n\to\infty} f(n) = \infty$. Then, in expression (3.43) we focus our attention to the single contribution

$$\frac{1}{n}\log_2 2^{p[\frac{1}{3}\sum_{h=1}^{n}(1-(-\frac{1}{2})^{n-h+1})\log_2 \pi_{j_h}]},$$

all the other being finite by Proposition 3.2.2. Thanks to our assumption we can write

$$\begin{aligned}
&-\lim_{n\to\infty}\frac{1}{n}\log_2 2^{p[\frac{1}{3}\sum_{h=1}^{n}(1-(-\frac{1}{2})^{n-h+1})\log_2 \pi_{j_h}]} = \lim_{|j_h|\to\infty}\frac{p}{3n}[f(|j_h|)(n-(1-\frac{1}{2})^{|j_h|})] \\
&= \lim_{|j_h|\to\infty}\frac{p}{3}f(|j_h|)(1+\frac{1}{n}(1-(-\frac{1}{2})^{|j_h|})) = \lim_{|j_h|\to\infty}\frac{p}{3}f(|j_h|)(1+\frac{1}{n}) \\
&= \lim_{|j_h|\to\infty}\frac{p}{3}f(|j_h|) = \infty,
\end{aligned}$$

the latter relation would cause the exponent of the structure function to be ∞ for every $p > 0$.

Special Stationary solutions

Let's now consider the RCM, i.e. $\{d_j\}_{j\in J} = \{d_\omega\}_{\omega\in\Omega}$ for some set $|\Omega| = 2^d$. We call *special* any stationary solution associated to the set of weight

$$\pi_j = \frac{d_j^s}{2^{d+s\ell_s}} = \frac{d_j^s}{2^{\log_2\sum_\omega d_\omega^s}} = \frac{d_j^s}{\sum_\omega d_\omega^s}, \qquad j \in J$$

for some $s \in \mathbb{R}$.
Next Proposition tells that there are Special stationary solutions with non-linear scaling exponent ζ_p that show spatial intermittency.

Proposition 3.2.4. *For every $s \in \mathbb{R}$. Let's consider the only stationary solution associated to the set of weights*

$$\pi_j = \frac{d_j^s}{2^{d+s\ell_s}}, \qquad j \in J. \tag{3.45}$$

The exponents of the structure function associated to this solutions are given by

$$\zeta_p = \frac{p}{3}(\beta - \frac{d}{2}) + \frac{p}{3}(s\ell_S - s\ell_{\frac{sp}{3}})$$

Proof. From Proposition 3.2.2, we are left to calculate the contribute given to the exponent ζ_p from

$$-\limsup_{n\to\infty}\frac{1}{n}\log_2\sum_{|j|=n}2^{p(\frac{1}{3}\sum_{h=1}^{n}(1-(-\frac{1}{2})^{n-h+1})\log_2\pi_{j_h})}.$$

The latter contribute can be further simplified as follows:

$$\limsup_{n\to\infty}\frac{1}{n}\log_2\sum_{|j|=n}\prod_{h=1}^{n}\pi_{j_h}^{-\frac{p}{3}(1-(-\frac{1}{2})^{n-h+1})}=\limsup_{n\to\infty}\frac{1}{n}\log_2\sum_{|j|=n}\prod_{h=1}^{n}\frac{d_{j_h}^{-\frac{sp}{3}(1-(-\frac{1}{2})^{n-h+1})}}{2^{-(d+s\ell_s)\frac{p}{3}}}=$$
$$=\limsup_{n\to\infty}\frac{1}{n}\log_2(\frac{2^{nd}}{2^{-\frac{np}{3}(d+s\ell_s)}}\sum_{|j|=n}\prod_{h=1}^{n}d_{j_h}^{-\frac{sp}{3}(1-(-\frac{1}{2})^{n-h+1})})=\limsup_{n\to\infty}\frac{1}{n}\log_2(\frac{2^{nd}}{2^{-\frac{np}{3}(d+s\ell_s)}}\sum_{|j|=n}\prod_{h=1}^{n}d_{j_h}^{-\frac{sp}{3}})$$

We now use the RCM property and Lemma 3.1.4 to obtain the more simple form

$$\limsup_{n\to\infty}\frac{1}{n}\log_2(\frac{2^{nd}}{2^{-\frac{np}{3}(d+s\ell_s)}}\sum_{|j|=n}\prod_{h=1}^{n}d_{j_h}^{-\frac{sp}{3}})=\limsup_{n\to\infty}\frac{1}{n}\log_2(\frac{2^{nd}}{2^{-\frac{np}{3}(d+s\ell_s)}}(\sum_{\omega\in\Omega}d_\omega^{-\frac{sp}{3}})^n)$$
$$=d+\frac{p}{3}(d+s\ell_s)-\log_2(\sum_{\omega\in\Omega}d_\omega^{\frac{sp}{3}})=\frac{p}{3}(d+s\ell_s)-\frac{p}{3}s\ell_{\frac{sp}{3}}.$$

Finally, by summing together all contributes from Proposition 3.2.2 we prove the claim. ∎

Observation. We now study more closely the exponent

$$\zeta_p=\frac{p}{3}(\beta-\frac{d}{2})+\frac{p}{3}(s\ell_S-s\ell_{\frac{sp}{3}}).$$

It is worth noting that, when $\beta=\frac{d}{2}+1$

$$\zeta_0=0,\qquad\zeta_3=1,$$

since these are physical requirements of turbulence theory and $\beta=\frac{d}{2}+1$ is the physically meaningful value.

Secondly, from the properties of log-s-norm one can easily observe that

$$\zeta_p=\frac{p}{3}(\beta-\frac{d}{2})\iff d_j=1\quad\forall j\in J\text{ or }s=0,$$

so we have actually shown that if d_k are not all equal, then the *flat* stationary solution related to the flat set of weights $\pi_j=2^{-d}$ satisfies Kolmogorov inertial spectrum, while all the other stationary solutions that stem from sets of weights

like (3.45) with $s \neq 0$, show spatial intermittency.
For example, for $s = 3/2$ we get

$$\zeta_p = \frac{p}{3}(\beta - \frac{d}{2}) + \frac{p}{2}(\ell_{3/2} - \ell_{p/2}),$$

the same exponents for the intermittent stationary solution of the Katz-Pavlovic tree model (3.1).

3.2.2 Self-Similar Solutions

We consider now a slight modification of model (3.34), where we allow ourselves to choose a not stationary forcing term on the first component. In particular we consider

$$f(t) = \frac{a_{\bar{\emptyset}}}{t - t_0} \geq 0, \quad a_{\bar{\emptyset}} \in \mathbb{R}^+ \tag{3.46}$$

with $t > t_0$ for some $t_0 < 0$.
We then call *self-similar* any solution Y of model (3.46) of the form $Y_j(t) = a_j \phi(t)$, $j \in J$. It is easy to prove that such self-similar solutions of (3.34) have the form

$$Y_j(t) = \frac{a_j}{t - t_0}, \quad j \in J. \tag{3.47}$$

Indeed, by definition of model (3.34) we obtain

$$\begin{aligned} a_j \phi^{'}(t) &= -2^{\beta|j|} d_j a_{\bar{j}} a_j \phi(t)^2 + \sum_{k \in \mathcal{O}_j} 2^{\beta|k|} d_k a_k \phi(t)^2 \\ &= \phi(t)^2 \cdot (-2^{\beta|j|} d_j a_{\bar{j}} a_j + \sum_{k \in \mathcal{O}_j} 2^{\beta|k|} d_k a_k) \end{aligned}$$

and relation (3.47) follows immediately.

We will prove that, under particular assumptions, self-similar solutions show the same asymptotic behaviour of stationary solutions.

Let now $Y_j(t) = \frac{a_j}{t - t_0}$, $j \in J$ be a self-similar solution. We restrict ourselves to consider only positive solutions $a_j > 0$ for every $j \in J$.

By introducing a collection of positive weights π_k, this time equation (3.37) becomes

$$-a_j = -2^{\beta|j|} d_j a_{\bar{j}} a_j + \sum_{k \in \mathcal{O}_j} 2^{\beta|k|} d_k a_k$$

and finally

$$\pi_k d_j a_j(a_{\bar{j}} - \epsilon_j) = 2^{\beta} d_k a_k^2, \quad k \in \mathcal{O}_j, \tag{3.48}$$

with $\epsilon_j = \frac{1}{d_j 2^{\beta|j|}}$.

The following proposition poses sufficient condition for existence of self-similar positive solutions for small value of the upper bound M.

Proposition 3.2.5. *Suppose that exist constants $C > 2\log M - 2\beta$ and $1 \le M < 2$ so that*

$$2^{-C} \le \pi_k, \quad 1/M \le d_k \le M, \quad \forall k \in J \setminus \emptyset$$

Then there exists a positive self-similar solution $Y = (Y_j(t))_{j\in J}$ associated with the set of weights $\{\pi_k\}_{k\in\mathcal{O}_j}$.

Proof. Since we restrict ourselves to consider positive self-similar solution, let us rewrite equation (3.48) as follows

$$a_k = \sqrt{\frac{\pi_k d_j}{2^{\beta} d_k} a_j(a_{\bar{j}} - \epsilon_j)}, \quad k \in \mathcal{O}_j, \quad j \in J, \tag{3.49}$$

where $\epsilon_j = \frac{1}{d_j 2^{\beta|j|}}$.
We start by proving existence of solutions for recursion (3.49). By hypothesis the following two inequalities holds for every $j \in K$ and $k \in \mathcal{O}_j$

$$\frac{1}{M^2} \le \frac{d_j}{d_k} \le M^2, \quad 2^{-C} \le \pi_k \le 1$$

for some positive constant $M \ge 1$, thus it is easy to observe that $a_j \ge a_j^{(1)}$, where $\{a_j^{(1)}\}_{j\in J}$ satisfies

$$\begin{aligned} a_k^{(1)} &= \sqrt{\frac{1}{2^{\beta+C+2\log M}} a_j^{(1)}(a_{\bar{j}}^{(1)} - \epsilon_j)}, \quad k \in \mathcal{O}_j, \quad j \in J \\ a_{j_0}^{(1)} &= a_{j_0}, a_{j_1}^{(1)} = a_{j_1}, \quad j_1 \in \mathcal{O}_{j_0}. \end{aligned} \tag{3.50}$$

Our goal is to use Lemma 3.1.7 by reaching a recursion of the form (3.16). To do so we consider the proper change of variable $a_j^{(1)} \to \frac{a_j^{(2)}}{2^{(\beta+C+2\log M)|j|/3}}$ to get the following recursion

$$\begin{aligned} a_k^{(2)} &= \sqrt{a_j^{(2)}(a_{\bar{j}}^{(2)} - \epsilon_j^*)}, \quad k \in \mathcal{O}_j, \quad j \in J \\ a_{j_0}^{(2)} &= a_{j_0}^{(1)}, a_{j_1}^{(2)} = a_{j_1}^{(1)} \cdot 2^{(\beta+C+2\log M)/3}, \quad j_1 \in \mathcal{O}_{j_0}. \end{aligned}$$

where $\epsilon_j^* = \frac{1}{d_j 2^{|J|(2\beta-2\log M+C)/3+(\beta+2\log M-C)/3}}$.
Thank to the hypothesis $C > 2\log M - 2\beta$ we deduce that $\{\epsilon_j^*\}_{j\in J}$ is summable, Lemma 3.1.7 tells that the recursive sequence $\{a_j^{(2)}\}_{j\in J}$ is well defined and lies in a compact set for values $a_{j_0}^{(2)}, a_{j_1}^{(2)}$ larger than some positive threshold. Thus, also the sequence $\{a_j^{(1)} = \frac{a_j^{(2)}}{2^{(\beta+C+2\log M)|j|/3}}\}_{j\in J}$ is well defined and summable for some proper starting values. Thus, the square root in recursion (3.50) is always well defined, i.e. $a_{\bar{j}}^{(1)} - \epsilon_j > 0$, and finally

$$a_{\bar{j}} - \epsilon_j \geq a_{\bar{j}}^{(1)} - \epsilon_j > 0,$$

hence also $\{a_j\}_{j\in J}$ is well defined for some proper starting values.

We can now prove finite energy property. With an argument similar to the previous one $a_j \leq \tilde{a}_j^{(1)}$, where $\{\tilde{a}_j^{(1)}\}_{j\in J}$ satisfies

$$\begin{aligned} \tilde{a}_k^{(1)} &= \sqrt{\frac{M^2}{2^\beta}\tilde{a}_j^{(1)}(\tilde{a}_{\bar{j}}^{(1)} - \epsilon_j)}, \quad k \in \mathcal{O}_j, \quad j \in J \\ \tilde{a}_{j_0}^{(1)} &= a_{j_0}, \tilde{a}_{j_1}^{(1)} = a_{j_1}, \quad j_1 \in \mathcal{O}_{j_0}. \end{aligned}$$

We consider now the following change of variable $\tilde{a}_j^{(1)} \to \frac{\tilde{a}_j^{(2)}}{2^{(\beta+2\log(1/M)|j|/3}}$, to led ourselves to the following recursion

$$\begin{aligned} \tilde{a}_k^{(2)} &= \sqrt{\tilde{a}_j^{(2)}(\tilde{a}_{\bar{j}}^{(2)} - \tilde{\epsilon}_j^*)}, \quad k \in \mathcal{O}_j, \quad j \in J \\ \tilde{a}_{j_0}^{(2)} &= \tilde{a}_{j_0}^{(1)}, \tilde{a}_{j_1}^{(2)} = \tilde{a}_{j_1}^{(1)} \cdot 2^{\beta/3+(\log(1/M^2))/3}, \quad j_1 \in \mathcal{O}_{j_0}. \end{aligned}$$

where $\tilde{\epsilon}_j^* = \frac{1}{d_j 2^{|j|\frac{2\beta+2\log M}{3}+\frac{\beta-2\log(M)}{3}}}$.
Thanks to our assumptions $2\beta + 2\log M \geq 0$, hence $\{\tilde{\epsilon}_j^*\}_{j\in J}$ is a summable sequence.
By Lemma 3.1.7, the recursive sequence $\{\tilde{a}_j^{(2)}\}_{j\in J}$ is well defined and lies in a compact set for values $\tilde{a}_{j_0}^{(2)}, \tilde{a}_{j_1}^{(2)}$ larger than some positive threshold. By hypothesis $M < 2^{\beta/2}$, thus $\beta + 2\log(1/M^2) > 0$ and the sequence $\{\tilde{a}_j^{(1)} = \frac{\tilde{a}_j^{(2)}}{2^{(\beta+2\log(1/M)|j|/3}}\}_{j\in J}$ is summable for some proper starting values.

Finally, also $\{a_j\}_{j\in J}$ is summable for choices of starting values a_{j_0}, a_{j_1} larger than some positive threshold. □

3.2.3 Equivalence between constant and self-similar solutions

Proposition 3.2.5 it is useful to build special self-similar solutions. In particular any set of weights $\{\pi_k\}_{k\in J}$ greater than some positive constant generates a positive self-similar solution with finite energy.

Example 3.2.1. *Flat self-similar solutions.*

We call *flat* any self-similar solution $Y(t)^{flat}$ built from the flat set of weights $\pi_k = 2^{-d}$ for every $k \in J$.

Example 3.2.2. *Special self-similar solutions.*

We now restrict ourselves to the model with repeated coefficients.

For every $s \in \mathbb{R}$ let's consider the set $\pi^s = \{\pi_k^{(s)} = \frac{d_k^s}{2^{d+s\cdot\ell_s}}\}_{k\in J}$.

It is easy to observe that π^s is a set of weight in the sense of (3.37): indeed for every $s \in \mathbb{R}$:

$$\pi_k^{(s)} = \frac{d_k^s}{2^{d+s\cdot\ell_s}} = \frac{d_k^s}{\sum_{k\in\mathcal{O}_j} d_k^s} < 1$$

$$\sum_{k\in\mathcal{O}_j} \pi_k^{(s)} = \sum_{k\in\mathcal{O}_j} \frac{d_k^s}{2^{d+s\cdot\ell_s}} = \sum_{k\in\mathcal{O}_j} \frac{d_k^s}{\sum_{k\in\mathcal{O}_j} d_k^s} = 1,$$

where in the latter equality we use the property of RCM.

We then call *special* any self-similar solution $Y(t)^s$ built from the set of weights $\pi_k^{(s)} = \frac{d_k^s}{2^{d+s\cdot\ell_s}}$ for every $k \in J$ and some $s \in \mathbb{R}$.

We are now ready to prove that, under certain assumptions, self-similar solutions share the same asymptotic behaviour with appropriate constant solutions.

Proposition 3.2.6. *Let's consider a self-similar solution $Y = \{Y_j(t) = \frac{a_j}{t-t_0}\}_{j\in J}$ of model (3.34) related to the set of weights $\{\pi_k\}_{k\in\mathcal{O}_j, j\in J}$, and to the set of coefficients $(d_j)_{j\in J}$ globally bounded from above and away from zero. Let $X = \{X_j\}_{j\in J}$ a positive stationary solution related to the same collection of weights and coefficients.*
Suppose that the set of weights $\{\pi_k\}_{k\in J}$ satisfies

$$\pi_k \geq 2^{-C}, \tag{3.51}$$

for some $0 < C < 2\beta$. Then $X \in H^r$ if and only if $Y \in H^r$ for some $r \in \mathbb{R}$.

Remark. We first take a closer look to condition (3.51) in the case of Example 3.2.1 and 3.2.2. In the flat case we have

$$2^{-d} \geq 2^{-C} \iff C > d,$$

and by assumptions $d < 2\beta - 2 < 2\beta$, so condition (3.51) holds for every $d < C < 2\beta$. Thus, by Proposition 3.2.1 $Y(t)^{flat} \in H^r$ for every $r < \frac{\beta - d/2}{3}$.

In the RCM example we have instead

$$\pi_k^s = \frac{d_k^s}{2^{d+s\cdot\ell_s}} \geq 2^{-C} \iff d + s \cdot \ell_s < C + \log_2(d_k^s),$$

thus time condition (3.51) holds if and only if

$$\log_2(\frac{\sum_{k\in\mathcal{O}_j} d_k^s}{d_k^s}) < C < 2\beta.$$

In the meaningful case where $\beta = 1 + d/2$, last inequality is satisfied for example if $s < \log_2(\frac{1}{M}) < 0$, indeed:

$$d_k^s < M^s \implies \log_2(\frac{\sum_{k\in\mathcal{O}_j} d_k^s}{d_k^s}) < \log_2(2^d M^{2s})$$

$$= d + 2s\log_2(M) < (2\beta - 2) + 2\log_2(\frac{1}{M})\log_2(M) = 2\beta.$$

Proof. By hypothesis

$$c_{j_{n+1}} = \frac{a_{j_{n+1}}}{X_{j_{n+1}}} = \frac{\sqrt{\frac{\pi_{j_{n+1}} d_{j_n}}{2^\beta d_{j_{n+1}}} a_{j_n}(a_{j_{n-1}} - \epsilon_{j_n})}}{\sqrt{\frac{\pi_{j_{n+1}} d_{j_n}}{2^\beta d_{j_{n+1}}} X_{j_n} X_{j_{n-1}}}} = \sqrt{c_{j_n}(c_{j_{n-1}} - \delta_{j_n})}, \tag{3.52}$$

where $\delta_{j_n} = \epsilon_{j_n}/X_{j_{n-1}}$.
Recursion (3.52) recalls immediately Lemma 3.1.7, hence we are left to prove that $\{\delta_{j_n}\}_{n\in\mathbb{N}}$ is summable and its sum is uniformly bounded from above and below in the set of all possible paths $P = \{\emptyset = j_0, j_1, \ldots, j_n, \ldots\}$ in J.
Indeed this will let us deduce that, along every path P, the ratio between self-similar solution Y and stationary solution X is a uniformly bounded constant, thus $X \in H^r$ if and only if $Y \in H^r$ for some $r \in \mathbb{R}$.

We start by recalling the correct expression of a general component X_{j_n} of a constant solution (3.39) within the limitedness of weights π_k and coefficients d_k, to conclude that exists an uniformly bounded constant D_n such that

$$D_n = [\prod_{h=1}^{n} (\frac{d_{j_h}^3}{\pi_{j_h}})^{\frac{(-\frac{1}{2})^{n-h+1}}{3}} \cdot 2^{-\frac{\beta}{9}(1-(-\frac{1}{2})^n)} \cdot X_\emptyset^{(\frac{2+(-\frac{1}{2})^n}{3})}]$$

and

$$\begin{aligned}\delta_{j_n} = \epsilon_{j_n}/X_{j_{n-1}} &= \frac{1}{d_{j_n}2^{\beta n}} \cdot \frac{2^{\beta n/3}}{D_n \prod_{i=1}^n \pi_{j_i}^{1/3}} \\ &= \frac{1}{D_n d_{j_n}} \frac{1}{2^{\frac{2\beta n}{3}} \prod_{i=1}^n \pi_{j_i}^{\frac{1}{3}}} \leq \frac{1}{D_n d_{j_n}} \frac{2^{\frac{Cn}{3}}}{2^{\frac{2\beta n}{3}}}.\end{aligned}$$

and similarly

$$\begin{aligned}\delta_{j_n} = \epsilon_{j_n}/X_{j_{n-1}} &= \frac{1}{d_{j_n}2^{\beta n}} \cdot \frac{2^{\beta n/3}}{D_n \prod_{i=1}^n \pi_{j_i}^{1/3}} \\ &= \frac{1}{D_n d_{j_n}} \frac{1}{2^{\frac{2\beta n}{3}} \prod_{i=1}^n \pi_{j_i}^{\frac{1}{3}}} \geq \frac{1}{D_n d_{j_n}} \frac{1}{2^{\frac{2\beta n}{3}}}.\end{aligned}$$

The claim follows immediately from the hypothesis $C < 2\beta$ and $\beta > 0$.

□

3.3 Mixed dyadic model on a Tree

Following the same scheme presented in Chapter 2, in this section we present a *mixed tree dyadic model* that combines both Novikov and Obukhov non-linearity.

$$\begin{aligned}\frac{dX_j(t)}{dt} =& \delta_1(2^{\beta|j|}d_j X_{\bar{j}}^2(t) - \sum_{k\in\mathcal{O}_j} 2^{\beta|k|}d_k X_j(t)X_k(t)) \\ &- \delta_2(-2^{\beta|j|}d_j X_j(t)X_{\bar{j}}(t) + \sum_{k\in\mathcal{O}_j} 2^{\beta|k|}d_k X_k^2(t))\end{aligned} \tag{3.53}$$

for every $j \in J \setminus \bar{0}$, and a boundary condition $X_{\bar{0}} = f > 0$, a forcing term on the first component, where $\beta > 0$, $\delta_1, \delta_2 \geq 0$ are constants and the coefficient $\{d_k\}_{k\in J}$ are bounded from above and away from zero, i.e. it exists $M \geq 1$ so that $1/M \leq d_k \leq M$ for every $k \in J$.
Model (3.53) generalizes the linear mixed model (2.30) presented in Chapter 2. It is a special case of the more general model introduced in [15], where it was proven the following theorem.

Theorem 3.3.1 (From [15]). *Consider the following dynamic system on the tree J.*

$$\frac{d}{dt}X_j = \alpha(c_j X_{\bar{j}}^2 - X_j \sum_{k\in\mathcal{O}_j} c_k X_k) + \beta(\tilde{c}_j X_{\bar{j}} X_j - \sum_{k\in\mathcal{O}_j} \tilde{c}_k X_k^2) +$$
$$+\gamma(X_{\bar{j}} \sum_{l\neq j, l\in\mathcal{O}_{\bar{j}}} \hat{c}_{j,l} X_l - \sum_{k_1\neq k_2, k_i\in\mathcal{O}_j} \hat{c}_{k_1,k_2} X_{k_1} X_{k_2}),$$

$$\frac{d}{dt}X_\emptyset = f(t, X_\emptyset) - \alpha X_\emptyset \sum_{k\in\mathcal{O}_\emptyset} c_k X_k - \beta \sum_{k\in\mathcal{O}_\emptyset} \tilde{c}_k X_k^2 - \gamma \sum_{k_1\neq k_2, k_i\in\mathcal{O}_\emptyset} \hat{c}_{k_1,k_2} X_{k_1} X_{k_2},$$

$$X(0) = x,$$

where $f(t, x) \leq c(t) + g(t)|x|$, with $c(t)$ and $g(t)$ positive continuous functions, and $\alpha, \beta, \gamma \geq 0$ non negative parameters.
Then, if $x \in \ell^2$, there exists at least a solution $X(t)$ on $[0, T]$.

By setting $\gamma = 0$ in Theorem 3.3.1 we reduce to model (3.53).

In the following section we investigate existence and uniqueness of stationary solution.

3.3.1 Stationary Solutions

Let $X_j(t)$ be a stationary solution over the whole tree J, i.e. $X_j(t) = a_j$, $j \in J$. We restrict our interest to positive stationary solutions that have also finite energy. From now on we set $\delta = \delta_2/\delta_1$, therefore we restrict ourselves to $\delta_1, \delta_2 > 0$. From (3.53) it follows immediately

$$\begin{aligned} 0 =& (2^{\beta|j|} d_j a_{\bar{j}}^2(t) - \sum_{k\in\mathcal{O}_j} 2^{\beta|k|} d_k a_j(t) X_a(t)) \\ &- \delta(-2^{\beta|j|} d_j a_j(t) a_{\bar{j}}(t) + \sum_{k\in\mathcal{O}_j} 2^{\beta|k|} d_k a_k^2(t)) \iff \\ & d_j a_{\bar{j}}^2 + \delta d_j a_{\bar{j}} a_j = 2^\beta \sum_{k\in\mathcal{O}_j} (d_k a_j a_k + \delta d_k a_k^2) \end{aligned}$$

Let's introduce the set of weights $\{\pi_k\}_{k\in J}$:

$$\begin{aligned} & 0 \leq \pi_k \leq 1, \\ & \sum_{k\in\mathcal{O}_j} \pi_k = 1, \quad j \in J \setminus \bar{0} \end{aligned} \tag{3.54}$$

in order to obtain

$$\pi_k(d_j a_{\bar{j}}^2 + \delta d_j a_{\bar{j}} a_j) = 2^\beta (d_k a_j a_k + \delta d_k a_k^2) \tag{3.55}$$

In what follows we are going to solve recursion (3.55), hence to prove existence of positive stationary finite energy solutions of model (3.53), at first in the basic case when $d_k = 1$ for every $k \in J$, then in more general cases.

Theorem 3.3.2. *Consider model (3.53) in the case $d_k = 1$ for every $k \in J$. For every forcing term $f > 0$, every $\delta > 0$ and every $\beta > 0$ it admits a positive stationary finite energy solution $\{a_j\}_{j \in J}$.*
Moreover,

- *if $\delta > 2^{\frac{\beta+d}{3}}$ such solution satisfies*

$$\lim_{|j|\to\infty} \frac{a_j}{2^{-\frac{(\beta+d)|j|}{3}}} = C$$

for some $C > 0$,

- *if $0 < \delta < 2^{\frac{\beta+d}{3}}$ such solution satisfies*

$$\frac{\tilde{C}_1}{2^{\frac{(\beta+C)|j|}{3}}} \leq a_j \leq \frac{\tilde{C}_2}{2^{\frac{(\beta)|j|}{3}}}$$

for some constants $\tilde{C}_1 > 0, \tilde{C}_2 > 0$ and $C > d$,

- *if $\delta = 2^{\frac{\beta+d}{3}}$ such solution satisfies*

$$\lim_{|j|\to\infty} \frac{a_j}{2^{-2\frac{(\beta+d)|j|}{9}}} = C$$

for some $C > 0$.

In order to prove Theorem 3.3.2 we use the following technical lemma.

Lemma 3.3.3. *Let $\delta > 0$, $\Delta > 0$ be positive constants, then*

1. *the recursion*

$$b_{n+1} = -\frac{1}{2\delta} + \sqrt{\frac{1}{4\delta^2} + \frac{1}{\Delta}(\frac{1}{\delta b_n^2} + \frac{1}{b_n})} \tag{3.56}$$

   - *converges to $\Delta^{-1/3}$ if $\delta > \Delta^{1/3}$, for every starting value $b_0 > 0$;*
   - *consists of two stationary subsequences $b_{2n} \equiv b_0, b_{2n+1} \equiv b_1$ for every $n \in \mathbb{N}$, if $\delta = \Delta^{1/3}$.*

2. *the recursion*

$$b_{n+1} = -\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + \Delta(\frac{\delta}{b_n^2} + \frac{1}{b_n})}, \tag{3.57}$$

- *converges to* $\Delta^{1/3}$ *if* $\delta < \Delta^{1/3}$, *for every starting value* $b_0 > 0$;
- *consists of two stationary subsequences* $b_{2n} \equiv b_0, b_{2n+1} \equiv b_1$ *for every* $n \in \mathbb{N}$, *if* $\delta = \Delta^{1/3}$.

Proof. We are going to prove just the claims on recursion (3.57) in the case $b_0 \geq \Delta^{1/3}$, all the others being similar.

Let's suppose $b_0 \geq \Delta^{1/3}$ in recursion (3.57). Then

$$\begin{aligned} b_0 \geq b_2 &\iff b_0 \geq -\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + \Delta(\frac{\delta}{b_1^2} + \frac{1}{b_1})} \\ &\iff (b_0 + \frac{\delta}{2})^2 \geq \frac{\delta^2}{4} + \Delta(\frac{\delta}{b_1^2} + \frac{1}{b_1}) \\ &\iff b_0^2 + b_0\delta \geq \frac{\Delta}{b_1}(\frac{\delta}{b_1} + 1) \\ &\iff b_1^2 b_0(b_0 + \delta) \geq \Delta(\delta + b_1). \end{aligned}$$

We can now expand the term b_1, as in recursion (3.56), to obtain that the inequality $b_1^2 b_0(b_0 + \delta) \geq \Delta(\delta + b_1)$ is equivalent to

$$\begin{aligned} &\frac{\delta^2}{2} + \Delta(\frac{\delta}{b_0^2} + \frac{1}{b_0} - \delta\sqrt{\frac{\delta^2}{4} + \Delta(\frac{\delta}{b_0^2} + \frac{1}{b_0})}) \cdot b_0(b_0 + \delta) \geq \Delta(\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + \Delta(\frac{\delta}{b_0^2} + \frac{1}{b_0})}) \\ \iff &(\frac{\delta^2}{2} + \frac{\Delta\delta}{b_0^2} + \frac{\Delta}{b_0})(b_0^2 + b_0\delta) - \frac{\Delta\delta}{2} \geq \sqrt{\frac{\delta^2}{4} + \Delta(\frac{\delta}{b_0^2} + \frac{1}{b_0})}(\Delta + \delta b_0^2 + \delta^2 b_0) \\ \iff &\frac{\delta^2}{2}b_0^2 + \frac{\delta^3}{2}b_0 + \Delta\delta + \frac{\Delta\delta^2}{b_0} + \Delta b_0 + \Delta\delta - \frac{\Delta}{2}\delta \geq \sqrt{\frac{\delta^2}{4} + \Delta(\frac{\delta}{b_0^2} + \frac{1}{b_0})}(\Delta + \delta b_0^2 + \delta^2 b_0) \\ \iff &(\delta^3 b_0 + \delta^2 b_0^3 + 3b_0\delta\Delta + 2\delta^2\Delta + 2b_0^2\Delta)^2 \geq (\frac{\delta^2}{4} + \Delta(\frac{\delta}{b_0^2} + \frac{1}{b_0}))(\Delta + \delta b_0^2 + \delta^2 b_0)^2 4b_0^2 \end{aligned}$$

by expanding last inequality we obtain

$\delta^6 b_0^4 + \delta^4 b_0^6 + 9\Delta^2\delta^2 b_0^2 + 4\Delta^2\delta^4 + 4\Delta^2 b_0^4 + 2\delta^5 b_0^5 + 6\Delta\delta^4 b_0^3 + 4\Delta\delta^5 b_0^2 + 4\Delta\delta^3 b_0^4 + 6\Delta\delta^3 b_0^4 +$
$4\delta\delta^4 b_0^3 + 4\Delta\delta^2 b_0^5 + 12\Delta^2\delta^3 b_0 + 12\Delta^2\delta b_0^3 + 8\Delta^2\delta^2 b_0^2 \quad \geq \quad \Delta^2\delta^2 b_0^2 + 4\Delta^3\delta + 4\Delta^3 b_0 + \delta^4 b_0^6 +$
$4\Delta\delta^3 b_0^4 + 4\Delta\delta^2 b_0^5 + \delta^6 b_0^4 + 4\Delta\delta^5 b_0^2 + 4\Delta\delta^4 b_0^3 + 2\Delta\delta^3 b_0^4 + 8\Delta^2\delta^2 b_0^2 + 8\Delta^2\delta b_0^3 + 2\Delta\delta^4 b_0^3 +$
$8\Delta^2\delta^3 b_0 + 8\Delta^2\delta^2 b_0^2 + 2\delta^5 b_0^5 + 8\Delta\delta^4 b_0^3 + 8\Delta\delta^3 b_0^4$

that further simplifies into

$$4\Delta^2\delta^3 b_0 + 4\Delta^2 + \delta b_0^3 + 4\Delta^2\delta^4 + 4\Delta^2\delta^4 + 4\Delta^2 b_0^4 \geq 4\Delta^3\delta + 4\Delta^3 b_0 + 4\Delta\delta^3 b_0^4 + 4\Delta\delta^4 b_0^3$$
$$\iff \Delta(\delta + b_0)(\delta^3 + b_0^3) \geq (\Delta^2 + \delta^3 b_0^3)(\delta + b_0) \iff \Delta(\delta^3 + b_0^3) \geq \Delta^2 + \delta^3 b_0^3$$
$$\iff \Delta(b_0^3 - \Delta) \geq \delta^3(b_0^3 - \Delta).$$

Thanks to the last implication, we conclude that

- if $\delta = \Delta^{1/3}$ then $b_0 = b_2$, and with similar argument $b_{2n} = b_0$ for every $n \in \mathbb{N}$
- if $b_0 \geq \Delta^{1/3}$ and $\Delta^{1/3} > \delta$, then $b_0 > b_2$, and with similar argument $b_{2n} > b_{2n+2}$ for every $n \in \mathbb{N}$

Last sentence tells that $\{b_{2n}\}_{n\in\mathbb{N}}$ is monotone and decreasing, thus it admits limit $L = \lim_{n\to\infty} b_{2n}$. It is left to prove that $L = \Delta^{1/3}$.

Let's start by proving that $b_{2n} \geq \Delta^{1/3}$ and $b_{2n+1} \leq \Delta^{1/3}$, by induction over n. By hypothesis $b_0 \geq \Delta^{1/3}$, thus

$$b_1 = -\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + \Delta(\frac{\delta}{b_0^2} + \frac{1}{b_0})} \leq -\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + \Delta(\frac{\delta}{\Delta^{2/3}} + \frac{1}{\Delta^{1/3}})}$$

and finally

$$-\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + \Delta(\frac{\delta}{\Delta^{2/3}} + \frac{1}{\Delta^{1/3}})} \leq \Delta^{1/3} \iff \sqrt{(\Delta^{1/3} + \frac{\delta}{2})^2} \leq (\Delta^{1/3} + \frac{\delta}{2})$$

proving that $b_1 \leq \Delta^{1/3}$. We notice that recursion (3.57) does not depend on n except for the previous term, thus with similar argument it is possible to prove the following cascade of implications

$$b_0 \geq \Delta^{1/3} \implies b_1 \leq \Delta^{1/3} \implies b_2 \geq \Delta^{1/3} \ldots \implies b_{2n-1} \leq \Delta^{1/3} \implies b_{2n} \geq \Delta^{1/3}.$$

We immediately deduce $L = \lim_{n\to\infty} b_{2n} \geq \Delta^{1/3}$. Moreover, by the very definition of terms b_{2n}

$$b_{2n+2} = -\frac{\delta}{2}+\sqrt{\frac{\delta^2}{4}+\Delta(\frac{\delta}{(-\frac{\delta}{2}+\sqrt{\frac{\delta^2}{4}+\Delta(\frac{\delta}{b_{2n}^2}+\frac{1}{b_{2n}})})^2}+\frac{1}{-\frac{\delta}{2}+\sqrt{\frac{\delta^2}{4}+\Delta(\frac{\delta}{b_{2n}^2}+\frac{1}{b_{2n}})}})}$$

and by taking limit on both sides we obtain

$$L = -\frac{\delta}{2}+\sqrt{\frac{\delta^2}{4}+\Delta(\frac{\delta}{(-\frac{\delta}{2}+\sqrt{\frac{\delta^2}{4}+\Delta(\frac{\delta}{L^2}+\frac{1}{L})})^2}+\frac{1}{-\frac{\delta}{2}+\sqrt{\frac{\delta^2}{4}+\Delta(\frac{\delta}{L^2}+\frac{1}{L})}})}.$$

We can progressively simplify the latter equation. At first by expanding the outer square root on the right hand side

$$L^2+L\delta = \Delta(\frac{\delta}{\frac{\delta^2}{2}+\Delta\frac{\delta^2}{L^2}+\frac{\Delta}{L}-\delta\sqrt{\frac{\delta^2}{4}+\Delta(\frac{\delta}{L^2}+\frac{1}{L}}}+\frac{\sqrt{\frac{\delta^2}{4}+\Delta(\frac{\delta}{L^2}+\frac{1}{L}}+\frac{\delta}{2}}{\Delta(\frac{\delta}{L^2}+\frac{1}{L})}),$$

that it simplifies as

$$\Delta\delta L^4(\frac{\frac{\delta^2}{2}+\Delta\frac{\delta^2}{L^2}+\frac{\Delta}{L}+\delta\sqrt{\frac{\delta^2}{4}+\Delta(\frac{\delta}{L^2}+\frac{1}{L}}}{\Delta^2\delta^2+\Delta^2L^2+2\Delta^2\delta L})+\Delta L^2(\frac{\sqrt{\frac{\delta^2}{4}+\Delta(\frac{\delta}{L^2}+\frac{1}{L}}+\frac{\delta}{2}}{\Delta\delta+\Delta\delta L}) =$$
$$\frac{\Delta\delta}{2}(\frac{\delta^2L^4+2\Delta\delta^2L^2+2\Delta L^3+2\delta L^4\sqrt{\frac{\delta^2}{4}+\Delta(\frac{\delta}{L^2}+\frac{1}{L}}}{\Delta^2(\delta+L)^2})+\frac{L^2}{2}(2\frac{\sqrt{\frac{\delta^2}{4}+\Delta(\frac{\delta}{L^2}+\frac{1}{L}}+\delta}{\Delta(\delta+L)}).$$

The latter equivalence can be further simplified in the more compact form

$$2\Delta(L+\delta)^3-\delta(\delta^2L^3+2\Delta\delta L+2\Delta L^2)-\Delta\delta L(L+\delta) =$$
$$2L\sqrt{\frac{\delta^2}{4}+\Delta(\frac{\delta}{L^2}+\frac{1}{L})}(\delta^2L^2+\Delta(L+\delta)))$$

and by expanding the square root on the right hand side we get

$$(3\Delta L\delta^2+3\Delta L^2\delta+2\Delta L^3+2\Delta\delta^3-L^3\delta^3)^2 =$$
$$(\frac{\delta^2}{4}+\Delta(\frac{\delta}{L^2}+\frac{1}{L}))(4L^6\delta^4+4\Delta^2L^2(L^2+\delta^2+2L\delta)+8\Delta L^4\delta^4(L+\delta)).$$

The last equation can be fully expanded as follows

$9\Delta^2L^2\delta^4 + 9\Delta^2L^4\delta^2 + 4\Delta^2L^6 + 4\Delta^2\delta^6 + \delta^6L^6 + 18\Delta^2L^3\delta^3 + 12\Delta^2L^4\delta^2 + 12\Delta^2L\delta^5 -$
$6\Delta L^4\delta^5 + 12\Delta^2L^5\delta + 12\Delta^2L^2\delta^4 - 6\Delta L^5\delta^4 + 8\Delta^2L^3\delta^3 - 4\Delta L^6\delta^3 - 4\Delta L^3\delta^6 =$
$L^6\delta^6 + \Delta^2L^4\delta^2 + \Delta^2L^2\delta^4 + 2\Delta^2L^3\delta^3 + 2\Delta L^5\delta^4 + 2\Delta L^4\delta^5 + 4\Delta L^4\delta^5 + 4\Delta^3L^2\delta +$
$4\Delta^3\delta^3 + 8\Delta^3L\delta^2 + 8\Delta^2L^3\delta^3 + 8\Delta^2L^2\delta^4 + 4\Delta L^5\delta^4 + 4\Delta^3L^3 + 4\Delta^3L\delta^2 + 8\Delta^3L^2\delta +$
$8\Delta^2L^4\delta^2 + 8\Delta^2L^3\delta^3,$

and it finally results in the following

$$3(L^3 - \Delta)\delta L(L+\delta)(\Delta - \delta^3) = (L^3 - \Delta)(\delta^3 - \Delta)(L^3 + \delta^3). \tag{3.58}$$

Let's now take a look at equality (3.58). If $L > \Delta^{1/3}$ then we can simplify on both sides to obtain

$$3\delta L(L+\delta)(\Delta - \delta^3) = (\delta^3 - \Delta)(L^3 + \delta^3) \tag{3.59}$$

By hypothesis $\delta < \Delta^{1/3}$ so left and right hand sides would have opposite signs, that would lead to an absurd. We conclude that $L = \Delta^{1/3}$.

With specular argument it is possible to show that also $\lim_{n\to\infty} b_{2n+1} = L = \Delta^{1/3}$ and conclude the proof.

□

We now prove Theorem 3.3.2.

Proof. From recursion (3.55) in the basic case $d_k = 1$ we recover

$$\pi_k(a_{\bar{j}}^2 + \delta a_{\bar{j}} a_j) = 2^\beta(a_j a_k + \delta a_k^2)$$

Since we are interested in positive stationary solution, let's divide both sides for a_j^2:

$$\pi_k((\frac{a_{\bar{j}}}{a_j})^2 + \delta\frac{a_{\bar{j}}}{a_j}) = 2^\beta(\frac{a_k}{a_j} + \delta(\frac{a_k}{a_j})^2)$$

and finally we consider two different change of variable $\frac{a_{\bar{j}}}{a_j} \to b_j$ and $\frac{a_j}{a_{\bar{j}}} \to c_j$ to obtain the following forward and backward recursion

1. *Backward recursion.* $\pi_k(b_j^2 + \delta b_j) - 2^\beta(b_k^{-1} + \delta b_k^{-2}) = 0$
   and since we focus on positive solutions: $b_j = -\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + \frac{2^\beta}{\pi_k}(\frac{\delta}{b_k^2} + \frac{1}{b_k})}$
2. *Forward recursion.* $\pi_k(c_j^{-2} + \delta c_j^{-1}) - 2^\beta(c_k + \delta c_k^2) = 0$
   for the same reason we deduce: $c_k = -\frac{1}{2\delta} + \sqrt{\frac{1}{4\delta^2} + \frac{\pi_k}{2^\beta}(\frac{1}{\delta c_j^2} + \frac{1}{c_j})}$

If we consider, for example, the *flat* set of weights $d_k = 2^{-d}$ for every $k \in J$, thanks to Lemma 3.3.3 the forward recursion converges to $\Delta^{-\frac{1}{3}} = \frac{1}{2^{\frac{\beta+d}{3}}} < 1$ if $\delta > \Delta^{1/3}$, thus also the sequence $\{a_j\}_{j\in J}$ converges and satisfies

$$\lim_{|j|\to\infty} \frac{a_j}{2^{-\frac{(\beta+d)|j|}{3}}} = C$$

for some $C > 0$, proving the first part of the claim. We stress that different sets of weights could lead to different solutions, thus, no uniqueness result is guaranteed.

On the other hand, for the backward recursion we need some extra technique. For every $N > 0$ we define a truncated version of the recursion

$$\begin{aligned} b_j^{(N)} &= -\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + \frac{2^\beta}{\pi_k}\left(\frac{\delta}{(b_k^{(N)})^2} + \frac{1}{b_k^{(N)}}\right)} \qquad \forall |j| < N \\ b_j^{(N)} &= T \qquad \forall |j| = N \\ b_j^{(N)} &= 0, \qquad \forall |j| > N \end{aligned}$$

for some positive starting value $T > 0$ set at higher generations.

Compatibility condition

Similarly to what we did in Section 3.1, we have to check compatibility conditions of the solution over the tree J. Such conditions read as follows:

$$-\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + \frac{2^\beta}{\pi_{k_1}}\left(\frac{\delta}{(b_{k_1}^{(N)})^2} + \frac{1}{b_{k_1}^{(N)}}\right)} = -\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + \frac{2^\beta}{\pi_{k_2}}\left(\frac{\delta}{(b_{k_2}^{(N)})^2} + \frac{1}{b_{k_2}^{(N)}}\right)}, \tag{3.60}$$

for every pair $k_1, k_2 \in \mathcal{O}_j$. The latter relation it further simplifies to

$$\begin{aligned} &\frac{1}{\pi_{k_1}}\left(\frac{\delta}{(b_{k_1}^{(N)})^2} + \frac{1}{b_{k_1}^{(N)}}\right) = \frac{1}{\pi_{k_2}}\left(\frac{\delta}{(b_{k_2}^{(N)})^2} + \frac{1}{b_{k_2}^{(N)}}\right) \\ \iff &\frac{\pi_{k_2}}{\pi_{k_1}} = \frac{\delta + b_{k_2}^{(N)}}{\delta + b_{k_1}^{(N)}} \cdot \frac{b_{k_1}^{(N)}}{b_{k_2}^{(N)}}. \end{aligned} \tag{3.61}$$

We have recursively defined each component $b_{k_1}^{(N)}, b_{k_2}^{(N)}$, so when we look for existence of weights π_{k_1}, π_{k_2} we can treat both $b_{k_1}^{(N)}, b_{k_2}^{(N)}$ as positive limited scalars. In particular, since we have already fixed to 1 the sum of weights in each offspring

$\mathcal{O}_j$, from (3.61) we deduce the existence of a unique set of weights $\{\pi_{k_i}\}_{k_i\in\mathcal{O}_j}$ that satisfies compatibility conditions. Since each set of weights depends on the N-th truncation, in order to lessen the notation we still denote such unique N-th set of weights with the same symbols without any superscript.

By limitedness property, it exists a positive constant $C > 0$ such that

$$\frac{1}{2^C} \leq \pi_k < 1.$$

Thus, by a similar procedure already used in previous results, there exist $\{b_j^{(N)_1}\}_{j\in J}$ and $\{b_j^{(N)_2}\}_{j\in J}$ so that

$$b_j^{(N)_1} \leq b_j^{(N)} \leq b_j^{(N)_2}, \qquad \forall N > 0, j \in J,$$

where we have defined

$$\begin{aligned}
b_j^{(N)_1} &= -\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + 2^{\beta}(\frac{\delta}{(b_k^{(N)_1})^2} + \frac{1}{b_k^{(N)_1}})} \qquad \forall |j| < N \\
b_j^{(N_1)} &= T \qquad \forall |j| = N \\
b_j^{(N_1)} &= 0, \qquad \forall |j| > N
\end{aligned}$$

and

$$\begin{aligned}
b_j^{(N)_2} &= -\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + 2^{\beta+C}(\frac{\delta}{(b_k^{(N)_2})^2} + \frac{1}{b_k^{(N)_2}})} \qquad \forall |j| < N \\
b_j^{(N_2)} &= T \qquad \forall |j| = N \\
b_j^{(N_2)} &= 0, \qquad \forall |j| > N
\end{aligned}$$

If $\delta < \Delta^{1/3}$, by Lemma 3.3.3, for every starting value $T > 0$, the sequence $\{b_j^{(N_1)}\}_{j\in J}$ accumulates around $2^{\frac{\beta}{3}} > 1$, while $\{b_j^{(N_2)}\}_{j\in J}$ accumulates around $2^{\frac{\beta+C}{3}} > 1$. Thus, by compactness and a diagonal extraction argument we can choose a subsequence $(N_i)_i \subset \mathbb{N}$ such that $b_j^{(N_i)}$ converges to some number $\tilde{b}_j$ for every $j \in J$, and the N_i-th set of weight converges to the unique limit set of weight $\{\tilde{\pi}_k\}_{k\in\mathcal{O}_j}$. Finally the limit sequence $\{\tilde{b}_j\}_{j\in J}$ satisfies recursion (1) by definition, hence $\{a_j\}_{j\in J}$ converges and satisfies

$$\frac{\tilde{C}_1}{2^{\frac{(\beta+C)|j|}{3}}} \leq a_j \leq \frac{\tilde{C}_2}{2^{\frac{(\beta)|j|}{3}}}$$

for some constants $\tilde{C}_1, \tilde{C}_2$.

We are left to prove what happens in the case $\delta = \Delta^{1/3}$. Since in this case both recursions fluctuate between their first two starting value we consider two cases:

- If $1 \leq b_0 < \Delta^{2/3}$, then in the backward recursion we have $1 < b_1 \leq \Delta^{2/3}$, thus the sequence of ratio $\{b_j\}_{j\in J}$ is definitely strictly greater than 1, so the original sequence $\{a_j\}_{j\in J}$ converges, although this time its behaviour is different: namely the sequence takes the form $\{a_{2n} = \frac{a_0}{\Delta^{2n/3}}, a_{2n+1} = \frac{a_0}{b_0\Delta^{2n/3}} = \frac{a_1}{\Delta^{2n/3}}\}$.
- If $\Delta^{-2/3} < b_0 \leq 1$, then in the forward recursion we have $\Delta^{-2/3} \leq b_0 < 1$, thus the sequence of ratio $\{b_j\}_{j\in J}$ is definitely strictly less than 1, so the original sequence $\{a_j\}_{j\in J}$ converges, also this time its behaviour is the following: $\{a_{2n} = \frac{a_0}{\Delta^{2n/3}}, a_{2n+1} = \frac{a_0 b_0}{\Delta^{2n/3}} = \frac{a_1}{\Delta^{2n/3}}\}$.

□

Corollary 3.3.4. *Consider model (3.53) in the case $d_k = 1$ for every $k \in J$. For every forcing term $f > 0$, every $\beta > 0$, if $\delta > 2^{\frac{\beta+d}{3}}$ the model admits infinitely many positive stationary finite energy solution.*
Moreover, any such solution satisfies

$$\frac{\tilde{C}_1}{2^{\frac{(\beta+C)|j|}{3}}} \leq a_j \leq \frac{\tilde{C}_2}{2^{\frac{(\beta)|j|}{3}}}$$

for some constants $\tilde{C}_1 > 0, \tilde{C}_2 > 0$ and $C > d$.

Proof. We stress the fact that we actually proved a stronger version of Theorem 3.3.2: if $\delta > 2^{\frac{\beta+d}{3}}$, we showed the existence of a stationary solution associated to the flat set of weight $\pi_k = 2^{-d}$ for every $k \in J$. This choice gives an exact asymptotic behaviour, however nothing forbids to choose a different set of weights as long as the following condition holds

$$\frac{1}{2^C} \leq \pi_j < 1, \qquad j \in J,$$

for some $C > 0$, showing the existence of infinitely many solutions with the same initial condition. Of course any such solution shows a different asymptotic behaviour depending on its associated set of weights. From what we have already proved, such behaviour reads as

$$\frac{\tilde{C}_1}{2^{\frac{(\beta+C)|j|}{3}}} \leq a_j \leq \frac{\tilde{C}_2}{2^{\frac{(\beta)|j|}{3}}}$$

for some constants $\tilde{C}_1 > 0, \tilde{C}_2 > 0$ and $C \geq d$. □

We are now ready to prove an equivalent form Theorem 3.3.2 with more general coefficients d_k.

Theorem 3.3.5. *For every $0 < M < 2^{\beta/2}$ so that $1/M \le d_j \le M$, $j \in J$, and for every forcing term $f > 0$*

- *if $\delta > 2^{\frac{\beta+d}{3}}$, model (3.53) admits infinitely many positive stationary solution,*
- *if $\delta < 2^{\frac{\beta+d}{3}}$, model (3.53) admits exactly one positive stationary solution.*

Moreover, any such solution satisfies

$$\frac{\tilde{C}_1}{2^{\frac{(\beta+C+2\log M)|j|}{3}}} \le a_j \le \frac{\tilde{C}_2}{2^{\frac{(\beta-2\log M)|j|}{3}}} \tag{3.62}$$

for some constants $\tilde{C}_1 > 0, \tilde{C}_2 > 0$ and $C \ge d$.

Proof. It is enough to slightly modify the proof of Theorem 3.3.2 and Corollary 3.3.4.

By hypothesis d_j are globally bounded and away from zero, so we can restrict ourselves to the *minimal* (resp. *maximal*) path over the tree J, namely the path along which the coefficients are constantly equal to the minimum (resp. maximum) value.
In the minimal path, backward and forward recursions read as follows

$$b_j = -\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + \frac{2^\beta d_k}{\pi_k d_j}(\frac{\delta}{b_k^2} + \frac{1}{b_k})}$$

$$c_k = -\frac{1}{2\delta} + \sqrt{\frac{1}{4\delta^2} + \frac{\pi_k d_j}{2^\beta d_k}(\frac{1}{\delta c_j^2} + \frac{1}{c_j})}$$

By hypothesis

$$\frac{1}{M^2} \le \frac{d_k}{d_J} \le M^2, \quad j \in J, k \in \mathcal{O}_j \tag{3.63}$$

and, thanks to previous results, we can find $C \ge d$ so that there is a unique set of weight that satisfies compatibility condition over the backward recursion and

$$\frac{1}{2^C} \le \pi_j \le 1, \quad j \in J. \tag{3.64}$$

By combining together (3.63) and (3.64) we obtain

$$2^{\beta-2\log M} \le \frac{2^\beta d_k}{\pi_k d_j} \le 2^{\beta+C+2\log M}. \tag{3.65}$$

Thanks to Theorem 3.3.2 we can now find infinitely many set of weights that satisfy (3.64) such that if $\delta > 2^{\frac{\beta+d}{3}}$, model (3.53) admits a positive stationary solution. Conversely if $\delta < 2^{\frac{\beta+d}{3}}$, there is a unique set of weights so that model (3.53) admits a positive stationary solution. In consequence of (3.65), any such solution satisfies (3.62). Concluding the proof. □

Observe that, in model (3.53), one could have taken different sets of coefficients d_j for the Novikov-type and Obukhov-type of non-linearity. It is immediate to deduce that, as long as each set of coefficients is bounded and away from zero, one can extend Theorem 3.3.5 to different sets of $\{d_j\}_{j\in J}$ and $\{\tilde{d}_j\}_{j\in J}$, like in Theorem 3.3.1.

Bibliography

[1] Barbato, D., Barsanti, M., Bessaih, H., Flandoli, F. 2006. *Some rigorous results on a stochastic GOY model*. Journ. Stat. Phys. 125 (3), 677-715.

[2] Barbato, D, L.A. Bianchi, Flandoli, F., Morandin, F. 2013. *A dyadic model on a tree*. Jour. of Math. Physics.

[3] Barbato, D, Flandoli, F., Morandin, F. 2010. *A theorem of uniqueness for an inviscid dyadic model*. C. R. Math. Acad. Sci. Paris, vol. 348, (9-10), 525528.

[4] Barbato, D., Flandoli, F., Morandin, F. 2010. *Uniqueness for a stochastic inviscid dyadic model*. Proc. Amer. Math. Soc. 138 (7), 2607-2617.

[5] Barbato, D., Flandoli, F., Morandin, F. 2011. *Anomalous dissipation in a stochastic inviscid dyadic model*. Ann. Appl. Prob. 32 (6), 2424-2446.

[6] Barbato, D., Flandoli, F., Morandin, F. 2011. *Energy dissipation and self-similar solutions for an unforced inviscid dyadic model*. Trans. Amer. Math. Soc. 363 (4).

[7] Barbato, D., Morandin, F. 2012. *Positive and non-positive solutions for an inviscid dyadic model: well-posedness and regularity*. Nonlinear Diff. Eq. Appl. 20 (3).

[8] Barbato, D., Morandin, F., Romito M., 2011. *Smooth solutions for the dyadic model*. Nonlinear Diff. Eq. Appl. 24, 3083-3097.

[9] Benzi, R., Biferale, L., Ciliberto, S., Struglia, M., Tripiccione, R., 1996. *Generalized scaling in fully developed turbulence*. Physica D. Nonlinear Phenomena, 96 (1), 162181.

[10] Benzi, R., Biferale, L., Parisi, G., 1993. *On intermittency in a cascade model for turbulence*. Physica D. Nonlinear Phenomena, 65 (1-2):163171.

[11] Benzi, R., Biferale, L., Tripiccione, R., Trovatore, E., 1997. (1 + 1)-*dimensional turbulence*. Phys. Fluids, 9, 2355.

[12] Benzi, R., Paladin, G., Parisi, G., Vulpiani, A., 1984. *On the multifractal nature of fully developed turbulence and chaotic systems*. Jour. of Phys. A: Math. and General, 17 (18), 3521.

[13] Bessaih, H., Flandoli, F., 2010. *Stochastic attractors for shell phenomenological models of turbulence*. Journ. Stat. Phys. 140 - 688-717.

[14] Bianchi, L.A., 2013. *Uniqueness for an inviscid stochastic dyadic model on a tree*. Elect. Commun. Prob., 18, 112.

[15] Bianchi, L.A., Flandoli, F., Metta, C., Montagnani, A., Morandin, F. *Inverse cascade as dyadic tree*. In preparation.

[16] Bianchi, L.A., Metta, C., Morandin, F. *Tree dyadic self-similar solutions*. In preparation.

[17] Bianchi, L.A., Morandin, F. 2017. *Structure Function and Fractal Dissipation for an Intermittent Inviscid Dyadic Model*. Comm. in Math. Physics, Vol. 356, (1), 231260.

[18] Biferale, L. 2003. *Shell models of energy cascade in turbulence*. Ann. Rev. Fluid Mech. 35, 441-468.

[19] Biferale, L., Blank, M., Frisch, U. 1994. *Chaotic cascades with Kolmogorov 1941 scaling*. Journ. Stat. Phys. 75 (5/6).

[20] Bowman, J.C., Doering, C.R., Eckhardt, B., Davoudi, J., Roberts, M., Schumacher, J. 2006. *Links between dissipation, intermittency, and helicity in the GOY model revisited*. Physica D 218 (1-10).

[21] Chen, G.Q., Glimm, J. 2012. *Kolmogorovs theory of turbulence and inviscid limit of the Navier-Stokes Equations in* $\mathbb{R}^3$. Commun. Math. Phys. 310, 267-283.

[22] Cheskidov, A. 2008. *Blow-up in finite time for the dyadic model of the Navier-Stokes equations*. Trans. Amer. Math. Soc. 360, n10, 5101-5120

[23] Cheskidov, A., Constantin, P., Friedlander, S., Shvydkoy, R. 2008. *Energy conservation and Onsagers conjecture for the Euler equations*. Nonlinearity, IOP, 21, 1233-1252.

[24] Cheskidov, A., Friedlander, S. 2009. *The vanishing viscosity limit for a dyadic model*. Physica D, 238 (8), 783-787.

[25] Cheskidov, A., Friedlander, S., Pavlovic, N. 2006 *An inviscid dyadic model of turbulence: the global attractor*. Disc. Contin. Dyn. Syst. 26, n3

[26] Cheskidov, A., Friedlander, S., Pavlovic, N. 2007. *Inviscid dyadic model of turbulence: the fixed point and Onsagers conjecture*. J. Math. Phys. 48, n6.

[27] Cheskidov, A., Shvydkoy, R., Friedlander, S. 2011. *A continuous model for turbulent energy cascade*. Math. Asp. Fluid Dynam., 52-69.

[28] Cheskidov, A., Zaya, K. 2015. *Regularizing effect of the forward energy cascade in the inviscid dyadic model*. Proc. Amer. Math. Soc., 144 (1), 73-85.

[29] Constantin, P., Levant, B., Titi, E.S. 2006. *Analytic study of shell models of turbulence*. Physica D. 219, 120-141.

[30] Constantin, P., Levant, B., Titi, E.S. 2007. *Regularity of inviscid shell models of turbulence*. Phys. Review. 75, 120-141.

[31] Constantin, P., Weinan, E., Titi, E.S. 1994. *Onsager's conjecture on the energy conservation for solutions of Euler's equation*. Commun. Math. Phys. 165, 207-209.

[32] Desnyansky, V.N., Novikov, E.A., 1974. *The evolution of turbulence spectra to the similarity regime*. Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana, 10, 127-136.

[33] Desnyansky, V.N., Novikov, E.A., 1974. *Simulation of cascade process in turbulent flows*. PMM, vol 38 (3), 507-513.

[34] Ditlevsen, P.D. 2011. *Turbulence and Shell Models*. Cambridge, Cambridge University Press

[35] Dombre, T., Gilson, J.L. 1998. *Intermittency, chaos and singular fluctuations in the mixed Obukhov-Novikov shell model of turbulence*. Physica D., 111, 265-287.

[36] Friedlander, S., Pavlovic, N., 2004. *Blow up in a three-dimensional vector model for the Euler equations*. Comm. Pure Appl. Math 57, 705-725.

[37] Frish, U. 1995. *Turbulence*. Cambridge Univ. Press.

[38] Frish, U., Sulem, P.L., Nelkin, M., 1978. *A simple dynamical model of intermittent fully developed turbulence*. Jour. Fluid Mech., 87 (4), 719736.

[39] Gledzer, E.B. 1973. *System of hydrodynamics type admitting two quadratic integrals of motions*. Sov. Phys. Dokl. SSSR, 18, 216-217

[40] Jensen, M.H., Paladin, G., Vulpiani, A. 1991 *Intermittency in a cascade model for three-dimensional turbulence*. Phys. Review, 43 (2), 798-805.

[41] Jeong, I.J., 2019. *Self-similar solutions for dyadic models of the Euler equations*. Journ. Diff. Eq. 266, n11, 7197-7204

[42] Kadanoff, L., Lohse, D., Wang, J., Benzi, R., 1994 *Scaling and dissipation in the GOY shell model*. Phys. Fluids, 7, 617.

[43] Katz, N., Pavlovic, N., 2002. *A cheap Caffarelli-Kohn-Nirenberg inequality for the Navier-Stokes equation with hyper-dissipation*. Geom. Funct. Anal. 12, 355-379

[44] Katz, N., Pavlovic, N., 2005. *Finite time blow up for a dyadic model of the Euler equations*. Trans. Amer. Math. Soc. 357, 695-708.

[45] Kerr, R.M., Di Siggia, E. 1978 *Cascade model of fully developed turbulence*. Journ. Stat. Phys. 19 (5), 543-552.

[46] Kiselev, A., Zlatos, A., 2005. *On Discrete Models of the Euler Equation*. Intern. Math. Res. Not. 38.

[47] Kolmogorov, A.N. 1941. *Local structure of turbulence in an incompressible liquid for very large Reynolds numbers*. C.R. (Dokl.) Acad. Sci. SSSR, 30, 299.

[48] Kolmogorov, A.N. 1941. *Dissipation of energy in locally isotropic turbulence*. C.R. (Dokl.) Acad. Sci. SSSR, 32, 16-18.

[49] Kolmogorov, A.N., 1962. *A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number*. Journ. Fluid Mech., 13, 8285.

[50] Lieb, E.H., 1984 *On characteristic exponent in turbulence*. Commun. Math. Phys. 92, 473-480.

[51] L'vov, V.S., Podivilov, E., Pomyalov, A., Procaccia, I. Vandembroueq, D. 1998. *Improved shell model of turbulence*. Phys. Rev. E, 58, 1811-1822.

[52] Mandelbrot, B.B., 1974. *Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier*. Journ. Fluid Mech., 62 (2), 331358.

[53] Metta, C. *Special solutions for mixed dyadic model*. In preparation.

[54] Meyer, Y., 1992. *Wavelets and operators*. Cambridge University Press, vol. 37.

[55] Montagnani, A. *A stationary solution for the mixed turbulence shell model.* In preparation.

[56] Obukhov, A.M., 1962. *Some specific features of atmospheric turbulence.* Journ. Geoph. Res., 67 (8), 30113014.

[57] Obukhov, A.M. 1971. *On some general characteristics of the equation of the dynamics of the atmosphere*. Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana, 7, 695-704.

[58] Obukhov, A.M., 1971 *Turbulence in an atmosphere with a non-uniform temperature*. Boundary-Layer Meteorology, 2 (1), 7-29.

[59] Onsager, L., 1949. *Statistical hydrodynamics*. Il Nuovo Cimento, 6, 279287.

[60] Richardson, L.F. 1922. *Weather Prediction by Numerical Process*. Cambridge, Cambridge University Press

[61] Rithwik, T., Samriddhi, S.R., 2018. *Revisiting the SABRA model: Statics and dynamics*. EPL (Europhysics Letters), Volume 120, Number 3

[62] Romito, M., 2013 *Uniqueness and blow-up for a stochastic viscous dyadic model.* Probab. Theory Relat. Fields,

[63] Waleffe, F. 2004, *On some dyadic models of the Euler equations*. Proc. Amer. Math. Soc. 134, 10.

[64] Tao, T., 2009 *Global regularity for a logarithmically supercritical hyperdissipative Navier-Stokes equation*. Anal. PDE, 2 (3), 361-366.

[65] Tao, T., 2014 *Finite time blowup for an averaged three-dimensional Navier-Stokes equation*. Journ. Amer. Math. Soc., 29 (3), 601-674.

[66] Yamada, M., Okhitani, K. 1987. *Lyapunov spectrum of chaotic model of three-dimensional turbulence*. Journ. Phys. Soc. Japan, 56 (12), 4210-4213.

[67] Yamada, M., Okhitani, K. 1988. *Lyapunov spectrum of a model of two-dimensional turbulence*. Phys. Rev. Lett., 60, 983-986.