A note on the Hausdorff dimension of the singular set of solutions to elasticity type systems

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Abstract

We prove partial regularity for minimizers to elasticity type energies in the nonlinear framework with p-growth, $p > 1$, in dimension $n \geq 3$. It is an open problem in such a setting either to establish full regularity or to provide counterexamples. In particular, we give an estimate on the Hausdorff dimension of the potential singular set by proving that is strictly less than $n-(p^* \wedge 2)$, and actually $n-2$ in the autonomous case (full regularity is well-known in dimension 2).

The latter result is instrumental to establish existence for the strong formulation of Griffith type models in brittle fracture with nonlinear constitutive relations, accounting for damage and plasticity in space dimensions 2 and 3.

1 Introduction

In this paper we investigate partial regularity of local minimizers for a class of energies whose prototype is

$$\int_\Omega \frac{1}{p}\left(\left(\mathbb{C}e(u)\cdot e(u)+\mu\right)^{p/2}-\mu^{p/2}\right)dx+\kappa\int_\Omega |u(x)-g(x)|^p\,dx$$

for $u \in W^{1,p}(\Omega;\mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$ bounded and open, $p \in (1,\infty)$ (see below for the precise assumptions on the relevant quantities). In addition, we establish an estimate on the Hausdorff dimension of the related singular set.

The main motivations for our study arise from Griffith's variational approach to brittle fracture. In such a model the equilibrium state of an elastic solid body deformed by external forces is determined by the minimization of an energy in which a bulk term and a surface one are in competition (see [22, 6, 14]). The former represents the elastic stored energy in the uncracked part of the body, instead the latter is related to the energy spent to create a crack, and it is typically proportional to the measure of the crack surface itself. As a model case, for $p \in (1,\infty)$ and κ, $\mu \geq 0$ one looks for minimizers (Γ, u) of

$$E[\Gamma,u]:=\int_{\Omega\setminus\Gamma}\frac{1}{p}\left(\left(\mathbb{C}e(u)\cdot e(u)+\mu\right)^{p/2}-\mu^{p/2}\right)dx+\kappa\int_{\Omega\setminus\Gamma}|u(x)-g(x)|^p\,dx+2\beta\mathcal{H}^{n-1}(\Gamma\cap\Omega) \tag{1.1}$$

over all closed sets $\Gamma \subset \overline{\Omega}$ and all deformations $u \in C^1(\Omega\setminus\Gamma;\mathbb{R}^n)$ subject to suitable boundary and irreversibility conditions. Here $\Omega \subset \mathbb{R}^n$ is the reference configuration, the function $\kappa|\xi - g(x)|^p \in C^0(\Omega\times\mathbb{R}^n)$ represents external volume forces, $e(u) = (\nabla u+\nabla u^T)/2$ is the elastic strain, $\mathbb{C} \in \mathbb{R}^{(n\times n)\times(n\times n)}$ is the matrix of elastic coefficients, $\beta > 0$ the surface energy. More precisely, the energy in (1.1) for $p = 2$ corresponds to classical Griffith's fracture model, while densities having p-growth with $p \neq 2$ may be instrumental for a variational formulation of fracture with nonlinear constitutive relations, accounting for damage and plasticity (see for example [29, Sections 10-11] and references therein).

In their seminal work [15], De Giorgi, Carriero and Leaci have introduced a viable strategy to prove existence of minimizers for the corresponding scalar energy,

$$E_{\mathrm{MS}}[\Gamma, u] := \int_{\Omega\setminus\Gamma} \frac{1}{p}\Big(\big(|Du|^2+\mu\big)^{p/2}-\mu^{p/2}\Big)\,dx + \kappa\int_{\Omega\setminus\Gamma}|u(x)-g(x)|^p\,dx + 2\beta\mathcal{H}^{n-1}(\Gamma\cap\Omega)\,, \quad (1.2)$$

better known for $p = 2$ as the Mumford and Shah functional in image segmentation (cf. the book [2] for more details on the Mumford and Shah model and related ones). From a mechanical perspective the scalar setting matches the case of anti-plane deformations $u : \Omega\setminus\Gamma \to \mathbb{R}$. Following a customary idea in the Calculus of Variations, the functional E_{MS} is first relaxed in a wider space, so that existence of minimizers can be obtained. The appropriate functional setting in the scalar framework is provided by a suitable subspace of BV functions. Surface discontinuities in the distributional derivative of the deformation u are then allowed, they are concentrated on a $(n-1)$-dimensional (rectifiable) set S_u. Then, existence for the strong formulation is recovered by establishing a mild regularity result for minimizers u of the weak counterpart: the essential closedness of the jump set S_u, namely $\mathcal{H}^{n-1}(\Omega\cap\overline{S_u}\setminus S_u) = 0$, complemented with smoothness of u on $\Omega\setminus\overline{S_u}$. Given this, $(u, \overline{S_u})$ turns out to be a minimizing couple for (1.1).

In the approach developed by De Giorgi, Carriero and Leaci in [15], regularity issues for local minimizers of the restriction of E_{MS} in (1.2) to Sobolev functions, such as decay properties of the L^p norm of the corresponding gradient, play a key role for establishing both the essential closedness of S_u for a minimizer u of (1.1) and the smoothness of u itself on $\Omega\setminus\overline{S_u}$. Nowadays, these are standard subjects in elliptic regularity theory (cf. for instance the books [25, 28, 27]).

Following such a streamline of ideas, in a recent paper [12] we have proved existence in the two dimensional framework for the functional in (1.1) for suitably regular g (see also [11] that settles the case $p = 2$). In passing we mention that the domain of the relaxed functional is provided for the current problem by a suitable subset, SBD (actually $GSBD$), of the space BD (GBD) of functions with (generalized) bounded deformation (we omit the precise definitions since they are inessential for the purposes of the current paper and rather refer to [12, 8]). More in details, our modification of the De Giorgi, Carriero, and Leaci approach rests on three main ingredients: the compactness and the asymptotic analysis of sequences in SBD having vanishing jump energy; the approximation in energy of general $(G)SBD$ maps with more regular ones; and the decay and smoothness properties of local minimizers of the functional in (1.1) when restricted to Sobolev functions. The compactness issue is dealt with in [12] in the two dimensional case and in [8] in higher dimensions, in both papers for all $p > 1$. The asymptotic analysis is performed in [12] and holds without dimensional limitations. The approximation property holds in any dimension as well, it is established in the companion paper [13] (see also the recent work [9] for an improved version which requires no integrability assumptions on the displacements). Instead, the regularity properties of local minimizers of energies like

$$\int_\Omega \frac{1}{p}\left(\big(\mathbb{C}e(u)\cdot e(u)+\mu\big)^{p/2}-\mu^{p/2}\right)dx + \kappa\int_\Omega |u(x)-g(x)|^p\,dx \quad (1.3)$$

on $W^{1,p}(\Omega;\mathbb{R}^n)$ are the object of investigation in the current paper. More generally, we study smoothness of local minimizers of elastic-type energies

$$\mathscr{F}_{\mu,\kappa}(u) = \int_\Omega f_\mu(e(u))\,dx + \kappa\int_\Omega |u-g|^p dx, \quad (1.4)$$

on $W^{1,p}(\Omega;\mathbb{R}^n)$, $n \geq 2$, for f_μ satisfying suitable convexity, smoothness and growth conditions (see Section 2.1 for the details). We carry over the analysis in any dimension since the results of the current paper, together with the compactness property established in [8] mentioned above, imply a corresponding existence result for the minimizers of (1.1) in the physical dimension $n = 3$, for any $p > 1$, and for $\mu > 0$ (see [10] for the case of Dirichlet boundary conditions). In this respect it is essential for us to derive an estimate on the Hausdorff dimension of the (potential) singular set, and prove that it is strictly less than $n-1$. We recall that if $p=2$ the regularity properties of the aforementioned local minimizers are well-known, so that the corresponding existence result for the minimizers of (1.1) follows straightforwardly from [12] in dimension $n=2$ and from [8] in any dimension.

The starting point of our study is the equilibrium system satisfied by minimizers of (1.4) that reads as

$$-\operatorname{div}\left(\nabla f_\mu(e(u))\right)+\kappa p|u-g|^{p-2}(u-g)=0, \tag{1.5}$$

in the distributional sense on Ω. Variants of (1.5) have been largely studied in fluid dynamics (we refer to the monograph [24] for all the details). In this context the system (1.5) with $\kappa=0$ is coupled with a divergence-free constraint and represents a stationary generalized Stokes system. It describes a steady flow of a fluid when the velocity u is small and the convection can be neglected. To our knowledge all contributions present in literature and concerning (1.5) are in this framework, apart from the case $p=2$ which is classical, see for example [25, 27, 34].

Under the divergence-free constraint and $\kappa=0$, regularity of solutions has been established first for $p\geq 2$ and every $\mu\geq 0$, see [24, 23], then in the planar setting for $1<p<2$ and every $\mu>0$, see [4, 5], and for $\mu\geq 0$, see [18] (the papers [4, 5] actually deal with the more general case of integrands satisfying $p-q$ growth conditions, the latter with the case of growth in terms of N-functions). L^q estimates for solutions to (1.5) with the divergence-free constraint have been obtained in the 3-dimensional setting in [17] for every $\mu\geq 0$. Regularity up to the boundary for the second derivative of solutions is proved for $p>2$ and $\mu>0$ in [3].

We stress explicitly that we have not been able to find in literature the mentioned estimate on the Hausdorff dimension of the singular set. Moreover, we also point out that the special structure of our lower order term does not fit the usual assumptions in literature (see for instance [31, Theorem 1.2] in the case of the p-laplacian). Despite this, it is possible to extend the results of this paper to a wider class of energies, as those satisfying for instance the conditions [31, (1.1)-(1.2)] building upon the ideas and techniques developed in [31, 30, 32] (see also [33] for a complete report).

In conclusion, we provide here detailed proofs for the decay estimates (with $\kappa,\mu\geq 0$, see Proposition 3.4 and Corollary 4.3) and for full or partial regularity of solutions (the former for $n=2$, the latter for $n\geq 3$ and $\mu>0$, see Section 4). We stress that if $n\geq 3$ it is a major open problem to prove or disprove full regularity even in the non degenerate, i.e. $\mu>0$, symmetrized p-laplacian case for $p\neq 2$. In these regards, if $n\geq 3$ we provide an estimate of the Hausdorff dimension of the potential singular set that seems to have been overlooked in the literature. In particular, the potential singular set has dimension strictly less than $n-1$.

Finally, we resume briefly the structure of the paper. In Section 2 we introduce the notation and the (standard) assumptions on the class of integrands f_μ. We also recall the basic properties of the nonlinear potential V_μ, an auxiliary function commonly employed in literature for regularity results in the non quadratic case. In addition, we review the framework of shifted N-functions introduced in [16], that provides the right technical tool for deriving Caccioppoli's type inequalities for energies depending on the symmetrized gradient. Caccioppoli's inequalities are the content of Section 3.1, as a consequence of those in Section 3.2 we derive the mentioned decay properties of the L^2 norm of $V_\mu(e(u))$. We remark that the Morrey type estimates in Section 3.2 and the improvement in Corollary 4.3 are helpful for the purposes of [12, 8] only for $n\in\{2,3\}$ in view of the decay rate established there. Partial regularity with an estimate on the Hausdorff dimension of the singular set are the objects of Section 4. More precisely, the higher integrability of $V_\mu(e(u))$ is addressed in Section 4.1, from this the full regularity of local minimizers in the two dimensional case easily follows by Sobolev embedding (cf. Section 4.2). Section 4.3 deals with the autonomous case $\kappa=0$, for which we use a linearization argument in the spirit of vectorial regularity results (the needed technicalities for these purposes are collected in Appendix A). The non-autonomous case is then a consequence of a perturbative approach as in the classical paper [26] (see Section 4.4).

2 Preliminaries

With Ω we denote an open and bounded Lipschitz set in $\mathbb{R}^n$, $n \geq 2$. The Euclidean scalar product is indicated by $\langle \cdot, \cdot \rangle$. We use standard notation for Lebesgue and Sobolev spaces. By s^* we denote the Sobolev exponent of s if $s \in [1, n)$, otherwise it can be any positive number strictly bigger than n. If $w \in L^1(B; \mathbb{R}^n)$, $B \subseteq \Omega$, we set

$$(w)_B := \fint_B w(y)dy. \tag{2.1}$$

In what follows we shall use the standard notation for difference quotients

$$\triangle_{s,h} v(x) := \frac{1}{h}(v(x + h\epsilon_s) - v(x)), \quad \tau_{s,h} v(x) := h \triangle_{s,h} v(x), \tag{2.2}$$

if $x \in \Omega_{s,h} := \{x \in \Omega : x + h\epsilon_s \in \Omega\}$ and 0 otherwise in Ω, where $v : \Omega \to \mathbb{R}^n$ is any measurable map and ϵ_s is any coordinate unit vector of $\mathbb{R}^n$.

2.1 Assumptions on the integrand

For given $\mu \geq 0$ and $p > 1$ we consider a function $f_\mu : \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ satisfying

(Reg) $f_\mu \in C^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ if $p \in (1, 2)$ and $\mu > 0$ or $p \in [2, \infty)$ and $\mu \geq 0$, while $f_0 \in C^1(\mathbb{R}^{n\times n}_{\mathrm{sym}}) \cap C^2(\mathbb{R}^{n\times n}_{\mathrm{sym}} \setminus \{0\})$ if $p \in (1, 2)$;

(Conv) for all $p \in (1, \infty)$ and for all symmetric matrices ξ and $\eta \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ we have

$$\frac{1}{c}\big(\mu + |\xi|^2\big)^{p/2-1}|\eta|^2 \leq \langle \nabla^2 f_\mu(\xi)\eta, \eta\rangle \leq c\big(\mu + |\xi|^2\big)^{p/2-1}|\eta|^2, \tag{2.3}$$

with $c = c(p) > 0$, unless $\mu = |\xi| = 0$ and $p \in (1, 2)$. We further assume $f_\mu(0) = 0$ and $Df_\mu(0) = 0$.

Remark 2.1. *The prototype functions we have in mind for applications to the mentioned Griffith fracture model are defined by*

$$f_\mu(\xi) = \frac{1}{p}\left(\big(\mathbb{C}\xi \cdot \xi + \mu\big)^{p/2} - \mu^{p/2}\right), \tag{2.4}$$

for all $\mu \geq 0$ and $p \in (1, \infty)$. Clearly (Reg) is satisfied, moreover we have

$$\nabla f_\mu(\xi) = \big(\mathbb{C}\xi \cdot \xi + \mu\big)^{p/2-1}\mathbb{C}\xi$$

(with $\nabla f_0(0) = 0$), and in addition

$$\nabla^2 f_\mu(\xi) = \big(\mathbb{C}\xi \cdot \xi + \mu\big)^{p/2-2}\Big((p-2)\mathbb{C}\xi \otimes \mathbb{C}\xi + (\mathbb{C}\xi \cdot \xi + \mu)\mathbb{C}\Big) \tag{2.5}$$

(with $\nabla^2 f_0(0) = 0$ if $p \in (2, \infty)$, $\nabla^2 f_0(0) = \mathbb{C}$ if $p = 2$, $\nabla^2 f_0(0)$ undefined if $p \in (1, 2)$). The lower inequality in (2.3) is clearly satisfied for $p \in [2, \infty)$; to check it if $p \in (1, 2)$ consider the quantity

$$\alpha := (p-2)(\mathbb{C}\xi \cdot \eta)^2 + (\mathbb{C}\xi \cdot \xi + \mu)(\mathbb{C}\eta \cdot \eta).$$

Since $\mathbb{C}$ defines a scalar product on the space of symmetric matrices, Cauchy-Schwarz inequality

$$\mathbb{C}\xi \cdot \eta \leq (\mathbb{C}\xi \cdot \xi)^{1/2}(\mathbb{C}\eta \cdot \eta)^{1/2}$$

yields for $p \in (1, 2)$

$$\alpha \geq [(p-2)(\mathbb{C}\xi \cdot \xi) + (\mathbb{C}\xi \cdot \xi + \mu)](\mathbb{C}\eta \cdot \eta) \geq (p-1)(\mathbb{C}\xi \cdot \xi + \mu)(\mathbb{C}\eta \cdot \eta),$$

the other inequality in (2.3) can be proved analogously.

Note that from (Conv) we deduce the p-growth conditions

$$c^{-1}(|\xi|^2+\mu)^{p/2-1}|\xi|^2 \le f_\mu(\xi) \le c(|\xi|^2+\mu)^{p/2-1}|\xi|^2 \tag{2.6}$$

and

$$|\nabla f_\mu(\xi)| \le c(|\xi|^2+\mu)^{p/2-1}|\xi| \tag{2.7}$$

for all $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ with $c = c(p) > 0$ (see also Lemma 2.3 below). Therefore, for all κ, $\mu \ge 0$, the functional $\mathscr{F}_{\mu,\kappa} : W^{1,p}(\Omega;\mathbb{R}^n) \to \mathbb{R}$ given by

$$\mathscr{F}_{\mu,\kappa}(v) = \int_\Omega f_\mu(e(v))dx + \kappa\int_\Omega |v-g|^p dx \tag{2.8}$$

is well-defined.

2.2 The nonlinear potential V_μ

In what follows it will also be convenient to introduce the auxiliary function $V_\mu : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$,

$$V_\mu(\xi) := (\mu+|\xi|^2)^{(p-2)/4}\xi,$$

with $V_0(0) = 0$ (we do not highlight the p dependence for the sake of simplicity).

Remark 2.2. *Note that $|V_0(\xi)|^2 = |\xi|^p$ for every $\xi \in \mathbb{R}^{n\times n}$, and for all $\mu > 0$*

$$|V_\mu(\xi)|^2 \le (\mu+|\xi|^2)^{p/2} = |V_\mu(\xi)|^2 + \mu(\mu+|\xi|^2)^{p/2-1} \le c|V_\mu(\xi)|^2 + c\,\mu^{p/2} \tag{2.9}$$

with $c = c(p) > 0$.

The following two basic lemmas will be needed in this section (see [1, Lemma 2.1 and Lemma 2.2] and [28, Lemma 8.3] for more details).

Lemma 2.3. *For every $\gamma > -1/2$, $r \ge 0$, and $\mu \ge 0$ we have*

$$c_1 \le \frac{\int_0^1 \left(\mu+|\eta+t(\xi-\eta)|^2\right)^\gamma (1-t)^r dt}{(\mu+|\xi|^2+|\eta|^2)^\gamma} \le c_2, \tag{2.10}$$

for all ξ, $\eta \in \mathbb{R}^k$ such that $\mu+|\xi|^2+|\eta|^2 \ne 0$, with $c_i = c_i(\gamma, r) > 0$.

Proof. If $\gamma \ge 0$ the upper bound follows easily by $|\eta+t(\xi-\eta)|^2 \le |\eta|^2+|\xi|^2$ and the monotonicity of $(0,\infty) \ni s \mapsto (\mu+s)^\gamma$ with $c_2 = 1$. To prove the lower bound we observe that if $|\xi| \le |\eta|$ then

$$|\eta+t(\xi-\eta)| \ge |\eta| - t|\xi| - t|\eta| \ge \frac{1}{3}|\eta| \qquad \forall t \in [0, 1/3],$$

which implies the other inequality with $c_1 = c_1(\gamma)$.

The lower bound for $\gamma < 0$ is analogous to the previous upper bound. The remaining upper bound requires an explicit computation and the integrability assumption $\gamma > -1/2$, see [1, Lemma 2.1], which results in $c_2 = 8/(2\gamma+1)$. □

Lemma 2.4. *For every $\gamma > -1/2$ and $\mu \ge 0$ we have*

$$c_3|\xi-\eta| \le \frac{|(\mu+|\xi|^2)^\gamma\xi - (\mu+|\eta|^2)^\gamma\eta|}{(\mu+|\xi|^2+|\eta|^2)^\gamma} \le c_4\,|\xi-\eta|, \tag{2.11}$$

for all ξ, $\eta \in \mathbb{R}^n$ such that $\mu+|\xi|^2+|\eta|^2 \ne 0$, with $c_i = c_i(\gamma) > 0$.

Proof. Assume $\mu > 0$ and consider the smooth convex function $h(\xi) := \frac{1}{2(\gamma+1)}(\mu + |\xi|^2)^{\gamma+1}$. For all $\xi \in \mathbb{R}^n$ we have

$$\nabla h(\xi) = (\mu + |\xi|^2)^\gamma \xi, \qquad \nabla^2 h(\xi) = (\mu + |\xi|^2)^\gamma \Big(Id + 2\gamma \frac{\xi \otimes \xi}{\mu + |\xi|^2} \Big).$$

Noting that for all ξ, $\eta \in \mathbb{R}^n$ it holds

$$\big(1 \wedge (1+2\gamma)\big)|\eta|^2 \le \frac{\langle \nabla^2 h(\xi)\eta, \eta\rangle}{(\mu + |\xi|^2)^\gamma} \le \big(1 \vee (1+2\gamma)\big)|\eta|^2,$$

the conclusion follows easily from $\nabla h(\xi) - \nabla h(\eta) = \int_0^1 \nabla^2 h(\eta + t(\xi - \eta))(\xi - \eta)dt$ and Lemma 2.3 with $c_3 = \big(1 \wedge (1+2\gamma)\big)c_1$ and $c_4 = \big(1 \vee (1+2\gamma)\big)c_2$ being c_1 and c_2 the constants there.

If $\mu = 0$ we can simply pass to the limit in formula (2.11) as $\mu \downarrow 0$, since c_3 and c_4 depend only on γ. □

We collect next several properties of V_μ instrumental for the developments in what follows.

Lemma 2.5. *For all ξ, $\eta \in \mathbb{R}^{n\times n}$ and for all $\mu \ge 0$ we have*

(i) if $p \ge 2$: $c|V_\mu(\xi - \eta)| \le |V_\mu(\xi) - V_\mu(\eta)|$ for some $c = c(p) > 0$, and for all $L > 0$ there exists $c = c(\mu, L) > 0$ such that $|V_\mu(\xi) - V_\mu(\eta)| \le c|V_\mu(\xi - \eta)|$ if $|\eta| \le L$;

(ii) if $p \in (1, 2)$: $|V_\mu(\xi - \eta)| \ge c|V_\mu(\xi) - V_\mu(\eta)|$ for some $c = c(p) > 0$, and for all $L > 0$ there exists $c = c(\mu, L) > 0$ such that $|V_\mu(\xi - \eta)| \le c|V_\mu(\xi) - V_\mu(\eta)|$ if $|\eta| \le L$;

(iii) $|V_\mu(\xi + \eta)| \le c(p)(|V_\mu(\xi)| + |V_\mu(\eta)|)$ for all $p \in (1, \infty)$;

(iv) $(2(\mu \vee |\xi|^2))^{p/2-1}|\xi|^2 \le |V_\mu(\xi)|^2 \le |\xi|^p$ if $p \in (1, 2)$, $|\xi|^p \le |V_\mu(\xi)|^2 \le 2^{p/2-1}(\mu^{p/2-1}|\xi|^2 + |\xi|^p)$ if $p \ge 2$;

(v) $\xi \mapsto |V_\mu(\xi)|^2$ is convex for all $p \in [2, \infty)$; for all $p \in (1, 2)$ we have $(\mu^{(2-p)/2} + |\xi|^{2-p})^{-1}|\xi|^2 \le |V_\mu(\xi)|^2 \le c(p)(\mu^{(2-p)/2} + |\xi|^{2-p})^{-1}|\xi|^2$ and $\xi \mapsto (\mu^{(2-p)/2} + |\xi|^{2-p})^{-1}|\xi|^2$ is convex.

Proof. If $p \in [2, \infty)$, property (i) follows from Lemma 2.4, while properties (iii) and (iv) are simple consequences of the very definition of V_μ.

Instead, for the case $p \in (1, 2)$ we refer to [7, Lemma 2.1]. More precisely, item (ii) above is contained in items (v) and (vi) there, (iii) above in (iii) there, and (iv) above in (i) there.

Finally, (v) follows by a simple computation. Indeed, first note that $|V_\mu(\xi)|^2 = \phi_\mu(|\xi|^2)$, where $\phi_\mu(t) := (\mu + t)^{p/2-1}t$ for $t \ge 0$. Then $\xi \mapsto |V_\mu(\xi)|^2$ is convex if and only if for all η, $\xi \in \mathbb{R}^n$

$$2\phi'_\mu(|\xi|^2)|\eta|^2 + 4\phi''_\mu(|\xi|^2)\langle \eta, \xi\rangle^2 \ge 0.$$

Using the explicit formulas for the first and second derivatives of ϕ_μ this amounts to prove for all η, $\xi \in \mathbb{R}^n$

$$(\mu + |\xi|^2)\big(2\mu + p|\xi|^2\big)|\eta|^2 + \big(4(p-2)\mu + p(p-2)|\xi|^2\big)\langle \eta, \xi\rangle^2 \ge 0.$$

In particular the conclusion is straightforward for $p \ge 2$. Instead, for $p \in (1, 2)$ we follow [20, Section 3]. We first observe that $\|x\|_{\frac{2}{2-p}} \le \|x\|_1 \le 2^{p/2}\|x\|_{\frac{2}{2-p}}$ applied to the vector $(\mu^{(2-p)/2}, \xi^{2-p})$ gives $(\mu^{(2-p)/2} + |\xi|^{2-p})^{-1}|\xi|^2 \le |V_\mu(\xi)|^2 \le c(p)(\mu^{(2-p)/2} + |\xi|^{2-p})^{-1}|\xi|^2$. A direct computation finally shows that $t \mapsto (\mu^{(2-p)/2} + t^{2-p})^{-1}t^2$ is convex and monotone increasing on $[0, +\infty)$, and that it vanishes for $t = 0$. We conclude that $\xi \mapsto (\mu^{(2-p)/2} + |\xi|^{2-p})^{-1}|\xi|^2$ is convex. □

Finally, we state a useful property established in [17, Lemma 2.8].

Lemma 2.6. *For all $\mu \ge 0$ there exists a constant $c = c(n, p, \mu) > 0$ such that for every $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ if $B_r(x_0) \subset \Omega$*

$$\int_{B_r(x_0)} \Big|V_\mu(e(u)) - \big(V_\mu(e(u))\big)_{B_r(x_0)}\Big|^2 dx \le \int_{B_r(x_0)} \Big|V_\mu(e(u)) - V_\mu\big((e(u))_{B_r(x_0)}\big)\Big|^2 dx$$
$$\le c\int_{B_r(x_0)} \Big|V_\mu(e(u)) - \big(V_\mu(e(u))\big)_{B_r(x_0)}\Big|^2 dx.$$

2.3 Shifted N-functions

We fix $p \in (1,\infty)$ and $\mu \geq 0$, and, following [16, Definition 22] for every $a \geq 0$ we consider the function $\phi_a : [0,\infty) \to \mathbb{R}$,

$$\phi_a(t) := \int_0^t \big(\mu + (a+s)^2\big)^{p/2-1} s\, ds. \tag{2.12}$$

A simple computation shows that $\phi_a'' > 0$ and, further,

$$\phi_a'(t) \leq c\phi_a''(t)t \qquad \text{for all } t \geq 0 \tag{2.13}$$

(ϕ_a turns out to be a N-function in the language of [16, Appendix]). From the definition one easily checks that for all $a, t \geq 0$ we have

$$\phi_a(t) \leq \frac{1}{p}\big(\big(\mu + (a+t)^2\big)^{p/2} - (\mu + a^2)^{p/2}\big). \tag{2.14}$$

More precisely, for every $t \geq 0$ we have

$$(\mu + (a+t)^2)^{p/2-1}\frac{t^2}{2} \leq \phi_a(t) \leq \frac{t^p}{p} \qquad \text{if } p \in (1,2), \tag{2.15}$$

$$\frac{t^p}{p} \leq \phi_a(t) \leq (\mu + (a+t)^2)^{p/2-1}\frac{t^2}{2} \qquad \text{if } p \in [2,\infty). \tag{2.16}$$

In addition, if $p \in (1,2)$, for every $t \geq 0$ we have

$$\phi_a(t) \leq (\mu + a^2)^{p/2-1}\frac{t^2}{2}. \tag{2.17}$$

A simple change of variables shows that the family $\{\phi_a\}_{a\geq 0}$ satisfies the $\triangle_2$ and ∇_2 conditions uniformly in a, that is for all $a \geq 0$

$$\lambda^{p\wedge 2}\phi_a(t) \leq \phi_a(\lambda t) \leq \lambda^{p\vee 2}\phi_a(t), \tag{2.18}$$

for all $\lambda \geq 1$ and $t \geq 0$. We define the polar of ϕ_a in the sense of convex analysis by

$$\phi_a^*(s) := \sup_{t\geq 0}\{st - \phi_a(t)\}\,. \tag{2.19}$$

By convexity and growth of ϕ_a one sees that the supremum is attained at a t such that $s = \phi_a'(t)$. For all $a \geq 0$ we have

$$\lambda^{\frac{p}{p-1}\wedge 2}\phi_a^*(s) \leq \phi_a^*(\lambda s) \leq \lambda^{\frac{p}{p-1}\vee 2}\phi_a^*(s), \tag{2.20}$$

for every $\lambda \geq 1$ and for every $t \geq 0$. In view of (2.18) and (2.20) above, Young's inequality holds uniformly in $a \geq 0$: for all $\delta \in (0,1]$ there exists $C_{\delta,p} > 0$ such that

$$s\,t \leq \delta\,\phi_a^*(s) + C_{\delta,p}\,\phi_a(t) \quad \text{ and } \quad s\,t \leq \delta\,\phi_a(t) + C_{\delta,p}\,\phi_a^*(s) \tag{2.21}$$

for every s and $t \geq 0$ and for all $a \geq 0$ (see also [16, Lemma 32]).

Convexity of ϕ_a implies

$$\frac{t}{2}\,\phi_a'\big(\frac{t}{2}\big) \leq \phi_a(t) \leq t\,\phi_a'(t) \qquad \forall t \geq 0. \tag{2.22}$$

From $\phi_a^*(\phi_a'(t)) = \phi_a'(t)t - \phi_a(t)$, (2.22), (2.21) we infer that there is a constant $c > 0$ such that for all $a \geq 0$

$$\frac{1}{c}\,\phi_a(t) \leq \phi_a^*(\phi_a'(t)) \leq c\,\phi_a(t), \tag{2.23}$$

for every $t \geq 0$ (see also [16, formula (2.3)]).

Finally, note that by the first inequality in Lemma 2.4 with exponent $\gamma = (p-2)/4 > -1/4$ we have

$$c|\xi - \eta|^2(\mu + |\xi|^2 + |\eta|^2)^{p/2-1} \le |V_\mu(\xi) - V_\mu(\eta)|^2$$

for every ξ, $\eta \in \mathbb{R}^{n\times n}$. Furthermore, by the second inequality in (2.22),

$$\phi_{|\xi|}(|\xi - \eta|) \le c|\xi - \eta|^2(\mu + |\xi|^2 + |\xi - \eta|^2)^{p/2-1},$$

and therefore

$$\phi_{|\xi|}(|\xi - \eta|) \le c|V_\mu(\xi) - V_\mu(\eta)|^2 . \tag{2.24}$$

3 Basic regularity results

In this section we prove some regularity results on local minimizers of generalized linear elasticity systems. The ensuing Propositions 3.1 and 3.3 contain the main Caccioppoli's type estimates in the super-quadratic and sub-quadratic case, respectively. In turn, those results immediately entail a higher integrability result in any dimension that will be instrumental for establishing partial regularity together with an estimate of the Hausdorff dimension of the singular set (see Propositions 4.1 and Theorem 4.7), as well as for proving $C^{1,\alpha}$ regularity for minimizers in the two dimensional case. Moreover, in the two and three dimensional setting useful decay properties that were needed in the proof of the density lower bound in [12] and [8] can be deduced from Propositions 3.1, 3.3, and 4.1 (cf. Proposition 3.4 and Corollary 4.3).

We point out that if $p \in [2, \infty)$ a more direct and standard proof can be provided that does not need the shifted N-functions ϕ_a in (2.12). Instead, those tools seem to be instrumental for the sub-quadratic case. Therefore, for simplicity, we have decided to provide a common framework for both.

In what follows we will make extensive use of the difference quotients introduced in (2.2) and of the mean values in (2.1).

3.1 Caccioppoli's inequalities

We start off dealing with the super-linear case. For future applications to higher integrability (cf. Proposition 4.1) it is convenient to set, for $p > 2$,

$$\tilde{p}(\lambda) := \frac{\lambda p(p-2)}{\lambda(p-1)-1} \tag{3.1}$$

for every $\lambda \in (\frac{1}{p-1}, 1]$. For $p > 2$, $\tilde{p}(\cdot)$ is a decreasing function on $(\frac{1}{p-1}, 1]$ with $\tilde{p}(1) = p$ and $\tilde{p} \to \infty$ as $\lambda \to \frac{1}{p-1}$. In addition, define $\lambda_0 \in (\frac{1}{p-1}, 1]$ to be such that $\tilde{p}(\lambda_0) = p^*$, being $p^* := \frac{np}{n-p}$, if $p \in (1, n)$, and $\lambda_0 = \frac{1}{p-1}$ otherwise and $\tilde{p}(\lambda_0)$ can be any positive exponent. If $p = 2$ we set $\lambda = \lambda_0 = 1$ and $\tilde{p}(\lambda_0) = p^*$. In particular, by Sobolev embedding, $u \in L^{\tilde{p}(\lambda)}(\Omega; \mathbb{R}^n)$ for all $\lambda \in [\lambda_0, 1]$.

Proposition 3.1. *Let $n \ge 2$, $p \in [2, \infty)$, κ and $\mu \ge 0$, $g \in W^{1,p}(\Omega; \mathbb{R}^n)$ and let $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ be a local minimizer of $\mathscr{F}_{\mu,\kappa}$ defined in (2.8).*

Then, $V_\mu(e(u)) \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}})$ and, in addition, $u \in W^{2,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ if $p > 2$ for $\mu > 0$, and if $p = 2$ for $\mu \ge 0$. More precisely, if $\lambda \in [\lambda_0, 1]$ there is a constant $c = c(n, p, \lambda) > 0$ such that for $B_{2r}(x_0) \subset \Omega$

$$\int_{B_r(x_0)} |\nabla\big(V_\mu(e(u))\big)|^2 dx \le c\frac{1+\kappa}{r^2} \int_{B_{2r}(x_0)} |V_\mu(e(u)) - (V_\mu(e(u)))_{B_{2r}(x_0)}|^2 dx$$
$$+ c\,\kappa r^{\frac{2}{p-1}} \int_{B_{2r}(x_0)} \big(|u-g|^{\tilde{p}(\lambda)} + |\nabla(u-g)|^{\lambda p}\big) dx. \tag{3.2}$$

Proof. We begin with showing that there is a constant $c = c(n,p) > 0$ such that if $B_{2r}(x_0) \subset \Omega$ then for any matrix $Q \in \mathbb{R}^{n\times n}$ we have

$$\int_{B_r(x_0)} |\nabla(V_\mu(e(u)))|^2 dx \leq \frac{c}{r^2}\int_{B_{2r}(x_0)\setminus B_r(x_0)} \phi_{|e(u)|}(|\nabla u - Q|)dx + \frac{\kappa}{r^2}\int_{B_{2r}(x_0)} |\nabla u - Q|^p dx + c\,\kappa\, r^{\frac{2}{p-1}} \int_{B_{2r}(x_0)} \big(|u-g|^p + |\nabla(u-g)|^p\big)dx\,. \quad (3.3)$$

In particular, on account of (2.16), we infer from (3.3) that $V_\mu(e(u)) \in W^{1,2}(B_r(x_0))$. A covering argument implies then that $V_\mu(e(u)) \in W^{1,2}_{\rm loc}(\Omega;\mathbb{R}^{n\times n}_{\rm sym})$.

Local minimality yields that u is a solution of

$$\int_\Omega \langle \nabla f_\mu(e(u)), e(\varphi)\rangle dx + \kappa\, p\int_\Omega |u-g|^{p-2}\langle u-g,\varphi\rangle dx = 0 \quad \forall \varphi \in W^{1,p}_0(\Omega;\mathbb{R}^n). \quad (3.4)$$

We can use the test field $\varphi := \triangle_{s,-h}\big(\zeta^p(\triangle_{s,h}u - Q\epsilon_s)\big)$, with $\zeta \in C^\infty_c(B_{2r}(x_0))$, $0 \leq \zeta \leq 1$, $\zeta|_{B_r(x_0)} \equiv 1$ and $|\nabla\zeta| \leq c/r$ to infer, for h sufficiently small,

$$\int_\Omega \langle \triangle_{s,h}\big(\nabla f_\mu(e(u))\big), \zeta^p \triangle_{s,h}(e(u))\rangle dx = -p\int_\Omega \langle \triangle_{s,h}\big(\nabla f_\mu(e(u))\big), \zeta^{p-1}\nabla\zeta \odot (\triangle_{s,h}u - Q\epsilon_s)\rangle dx - \kappa\, p\int_\Omega \langle \triangle_{s,h}\big(|u-g|^{p-2}(u-g)\big), \zeta^p(\triangle_{s,h}u - Q\epsilon_s)\rangle dx. \quad (3.5)$$

Recalling that $f_\mu \in C^2(\mathbb{R}^{n\times n})$ if $p \geq 2$ for all $\mu \geq 0$ we compute

$$\triangle_{s,h}\big(\nabla f_\mu(e(u))\big)(x) = \int_0^1 \nabla^2 f_\mu\big(e(u) + th\,\triangle_{s,h}(e(u))\big)\,\triangle_{s,h}(e(u))dt =: \mathbb{A}_{s,h}(x)\triangle_{s,h}(e(u))(x). \quad (3.6)$$

By taking into account (3.6), equality (3.5) rewrites as

$$\int_\Omega \zeta^p\langle \mathbb{A}_{s,h}(x)\triangle_{s,h}(e(u)), \triangle_{s,h}(e(u))\rangle dx = -\,p\int_\Omega \zeta^{p-1}\langle \mathbb{A}_{s,h}(x)\triangle_{s,h}(e(u)), \nabla\zeta\odot(\triangle_{s,h}u - Q\epsilon_s)\rangle dx - \kappa\, p\int_\Omega \langle \triangle_{s,h}\big(|u-g|^{p-2}(u-g)\big), \zeta^p(\triangle_{s,h}u - Q\epsilon_s)\rangle dx\,. \quad (3.7)$$

Setting

$$W_{s,h}(x) := \int_0^1 \big(\mu + |e(u)(x) + t\,\tau_{s,h}(e(u))(x)|^2\big)^{p/2-1}dt,$$

the estimates in (2.3) give for all $\eta \in \mathbb{R}^{n\times n}_{\rm sym}$

$$\frac{1}{c}W_{s,h}(x)|\eta|^2 \leq \langle \mathbb{A}_{s,h}(x)\eta,\eta\rangle \leq c\,W_{s,h}(x)|\eta|^2 \quad (3.8)$$

with $c = c(p) > 0$. Therefore, using (3.8) in (3.7) yields for some $c = c(p) > 0$

$$\int_\Omega \zeta^p W_{s,h}(x)|\triangle_{s,h}(e(u))|^2 dx \leq c\int_\Omega \zeta^{p-1}W_{s,h}(x)|\nabla\zeta||\triangle_{s,h}(e(u))||\triangle_{s,h}u - Q\epsilon_s|dx - \kappa\, p\int_\Omega \langle \triangle_{s,h}\big(|u-g|^{p-2}(u-g)\big), \zeta^p(\triangle_{s,h}u - Q\epsilon_s)\rangle dx\,. \quad (3.9)$$

Proceeding as in (3.6), and using $|\nabla V_\mu(\xi)| \le c\,(\mu+|\xi|^2)^{(p-2)/4}$, we obtain

$$|\triangle_{s,h}\big(V_\mu(e(u))\big)| \le c\int_0^1 \big(\mu + |e(u)(x) + t\,\tau_{s,h}(e(u))(x)|^2\big)^{(p-2)/4} dt |\triangle_{s,h}(e(u))|\,.$$

Using Jensen's inequality in this integral and then comparing with the definition of $W_{s,h}$ we infer from (3.9)

$$\int_\Omega \zeta^p|\triangle_{s,h}\big(V_\mu(e(u))\big)|^2dx \le c\int_\Omega \zeta^{p-1}W_{s,h}(x)|\nabla\zeta||\triangle_{s,h}(e(u))||\triangle_{s,h}u - Q\epsilon_s|dx$$
$$- \kappa\,p\int_\Omega \langle\triangle_{s,h}\big(|u-g|^{p-2}(u-g)\big), \zeta^p(\triangle_{s,h}u - Q\epsilon_s)\rangle dx\,.$$

In turn, from this inequality and (2.10) we get for some $c = c(p) > 0$

$$\begin{aligned}\int_{B_{2r}(x_0)} &\zeta^p|\tau_{s,h}\big(V_\mu(e(u))\big)|^2dx \\ &\le \frac{c}{r}\int_{B_{2r}(x_0)\setminus B_r(x_0)} \zeta^{p-1}\big(\mu + |e(u)|^2 + |e(u)(x+he_s)|^2\big)^{p/2-1}|\tau_{s,h}(e(u))||\tau_{s,h}u - hQ\epsilon_s|dx \\ &\quad - \kappa\,p\int_{B_{2r}(x_0)} \langle\tau_{s,h}\big(|u-g|^{p-2}(u-g)\big), \zeta^p(\tau_{s,h}u - Qh\epsilon_s)\rangle dx =: I_1 + I_2.\end{aligned} \tag{3.10}$$

By considering the functions ϕ_a, $a \ge 0$, introduced in (2.12) above, the first term on the right hand side of the last inequality can be estimated by

$$I_1 = \frac{c}{r}\int_{B_{2r}(x_0)\setminus B_r(x_0)} \zeta^{p-1}\phi'_{|e(u)|}\big(|\tau_{s,h}(e(u))|\big)|\tau_{s,h}u - hQ\epsilon_s|dx.$$

Since $\zeta \in (0,1]$, Young's inequality in (2.21) gives for every $\delta \in (0,1)$ and for some $c = c(p) > 0$

$$\begin{aligned} I_1 &\overset{(2.21)}{\le} c\,\delta\int_{B_{2r}(x_0)\setminus B_r(x_0)} \phi^*_{|e(u)|}\big(\zeta^{p-1}\phi'_{|e(u)|}(|\tau_{s,h}(e(u))|)\big)dx \\ &\quad + C_{\delta,p}\int_{B_{2r}(x_0)\setminus B_r(x_0)} \phi_{|e(u)|}\big(\frac{1}{r}|\tau_{s,h}u - hQ\epsilon_s|\big)dx \\ &\overset{(2.20)}{\le} c\,\delta\int_{B_{2r}(x_0)\setminus B_r(x_0)} \zeta^p\phi^*_{|e(u)|}\big(\phi'_{|e(u)|}(|\tau_{s,h}(e(u))|)\big)dx \\ &\quad + C_{\delta,p}\int_{B_{2r}(x_0)\setminus B_r(x_0)} \phi_{|e(u)|}\big(\frac{1}{r}|\tau_{s,h}u - hQ\epsilon_s|\big)dx \\ &\overset{(2.23)}{\le} c\,\delta\int_{B_{2r}(x_0)\setminus B_r(x_0)} \zeta^p\phi_{|e(u)|}\big(|\tau_{s,h}(e(u))|\big)dx \\ &\quad + C_{\delta,p}\int_{B_{2r}(x_0)\setminus B_r(x_0)} \phi_{|e(u)|}\big(\frac{1}{r}|\tau_{s,h}u - hQ\epsilon_s|\big)dx.\end{aligned}$$

By using estimate (2.24) in the last but one term from the latter inequality we get for some $c = c(p) > 0$

$$\begin{aligned} I_1 &\le c\,\delta\int_{B_{2r}(x_0)\setminus B_r(x_0)} \zeta^p|\tau_{s,h}\big(V_\mu(e(u))\big)|^2dx \\ &\quad + C_{\delta,p}\int_{B_{2r}(x_0)\setminus B_r(x_0)} \phi_{|e(u)|}\big(\frac{1}{r}|\tau_{s,h}u - hQ\epsilon_s|\big)dx.\end{aligned} \tag{3.11}$$

We now estimate the second term in (3.10). We preliminarily note that by Meyers-Serrin's theorem and the Chain rule formula for Sobolev functions the field $w := |u-g|^{p-2}(u-g)$ belongs to $W^{1,p'}(A,\mathbb{R}^n)$ for every Lipschitz open subset $A \subseteq \Omega$. More precisely, we have

$$\|\nabla w\|_{L^{p'}(A,\mathbb{R}^{n\times n})} \le c\|u-g\|^{p-2}_{L^{\bar p(\lambda)}(A,\mathbb{R}^n)}\|\nabla(u-g)\|_{L^{\lambda p}(A,\mathbb{R}^{n\times n})}, \tag{3.12}$$

for some constant $c = c(n, p, \lambda) > 0$, for all $\lambda \in [\lambda_0, 1]$ where $\tilde{p}$ is the function defined in (3.1) and $p' = \frac{p}{p-1}$ (we recall that if $p = 2$ then $\lambda = \lambda_0 = 1$).

Therefore, by (3.12), Hölder's and Young's inequalities we may estimate I_2 for h sufficiently small as follows

$$\begin{aligned} h^{-2} I_2 &\leq \kappa\, p \|\triangle_{s,h} u - Q\epsilon_s\|_{L^p(B_{2r}(x_0),\mathbb{R}^n)} \|\zeta^p \triangle_{s,h} w\|_{L^{p'}(B_{2r}(x_0),\mathbb{R}^n)} \\ &\leq \kappa\, p \,\|\triangle_{s,h} u - Q\epsilon_s\|_{L^p(B_{2r}(x_0),\mathbb{R}^n)} \|\nabla w\|_{L^{p'}(B_{2r}(x_0),\mathbb{R}^n)} \\ &\leq \frac{\kappa}{r^2} \int_{B_{2r}(x_0)} |\triangle_{s,h} u - Q\epsilon_s|^p dx + \kappa(p-1)\, r^{\frac{2}{p-1}} \int_{B_{2r}(x_0)} |\nabla w|^{p'} dx \\ &\leq \frac{\kappa}{r^2} \int_{B_{2r}(x_0)} |\triangle_{s,h} u - Q\epsilon_s|^p dx + c\,\kappa\, r^{\frac{2}{p-1}} \int_{B_{2r}(x_0)} \big(|u-g|^{\tilde{p}(\lambda)} + |\nabla(u-g)|^{\lambda p}\big) dx \end{aligned} \tag{3.13}$$

for some $c = c(n, p, \lambda) > 0$. Hence, from inequalities (3.10), (3.11) and (3.13) for $\delta = \delta(p) > 0$ sufficiently small we conclude that

$$\begin{aligned} \int_{B_r(x_0)} |\triangle_{s,h}\big(V_\mu(e(u))\big)|^2 dx &\leq \frac{c}{h^2} \int_{B_{2r}(x_0)\setminus B_r(x_0)} \phi_{|e(u)|}\Big(\frac{1}{r}|\tau_{s,h} u - hQ\epsilon_s|\Big) dx \\ &+ \frac{\kappa}{r^2} \int_{B_{2r}(x_0)} |\triangle_{s,h} u - Q\epsilon_s|^p dx + c\,\kappa\, r^{\frac{2}{p-1}} \int_{B_{2r}(x_0)} \big(|u-g|^{\tilde{p}(\lambda)} + |\nabla(u-g)|^{\lambda p}\big) dx, \end{aligned} \tag{3.14}$$

with $c = c(n, p, \lambda) > 0$. Finally, (2.18) and the last inequality for sufficiently small h yield for some $c = c(n, p, \lambda) > 0$

$$\begin{aligned} \int_{B_r(x_0)} |\triangle_{s,h}\big(V_\mu(e(u))\big)|^2 dx &\leq \frac{c}{r^2} \int_{B_{2r}(x_0)\setminus B_r(x_0)} \phi_{|e(u)|}\big(|\triangle_{s,h} u - Q\epsilon_s|\big) dx \\ &+ \frac{\kappa}{r^2} \int_{B_{2r}(x_0)} |\triangle_{s,h} u - Q\epsilon_s|^p dx + c\,\kappa\, r^{\frac{2}{p-1}} \int_{B_{2r}(x_0)} \big(|u-g|^{\tilde{p}(\lambda)} + |\nabla(u-g)|^{\lambda p}\big) dx. \end{aligned} \tag{3.15}$$

Hence, by summing on $s \in \{1, \ldots, n\}$ in inequality (3.15) and by letting $h \downarrow 0$ there, we conclude (3.3). Furthermore, since $|\nabla V_\mu(\xi)|^2 \geq c(n, p)\mu^{p/2-1}$ for all $\xi \in \mathbb{R}^{n\times n}$ if $p \geq 2$, the latter estimate, (3.3) and a covering argument imply $u \in W^{2,2}_{\rm loc}(\Omega; \mathbb{R}^n)$ if $\mu > 0$.

To conclude the Caccioppoli's type inequality in (3.2) first observe that

$$\int_{B_{2r}(x_0)} \phi_{|e(u)|}\big(|\nabla u - (\nabla u)_{B_{2r}(x_0)}|\big) dx \leq c \int_{B_{2r}(x_0)} \phi_{|e(u)|}\big(|e(u) - (e(u))_{B_{2r}(x_0)}|\big) dx. \tag{3.16}$$

This follows from Korn's inequality by using that if $\psi_a(t) := a^{p-2}t^2 + \mu^{p/2-1}t^2 + t^p$ then $c^{-1}\psi_a(t) \leq \phi_a(t) \leq c\,\psi_a(t)$ for all $t \geq 0$ and for some $c = c(p) > 0$. One inequality follows from (2.16), the other one is similar. Alternatively, (3.16) follows directly from Korn's inequality in Orlicz spaces for shifted N-functions (cf. [17, Lemma 2.9]).

Moreover, since for $p \geq 2$ by the very definition of V_μ and Lemma 2.4

$$|\xi - \eta|^p \leq |V_\mu(\xi - \eta)|^2 \leq c(p)|V_\mu(\xi) - V_\mu(\eta)|^2 \qquad \forall\, \xi, \eta \in \mathbb{R}^{n\times n},$$

the standard Korn's inequality implies for some $c = c(n, p) > 0$

$$\begin{aligned} &\int_{B_{2r}(x_0)} |\nabla u - (\nabla u)_{B_{2r}(x_0)}|^p dx \leq c \int_{B_{2r}(x_0)} |e(u) - (e(u))_{B_{2r}(x_0)}|^p dx \\ &\leq c \int_{B_{2r}(x_0)} |V_\mu\big(e(u) - (e(u))_{B_{2r}(x_0)}\big)|^2 dx \leq c \int_{B_{2r}(x_0)} |V_\mu(e(u)) - V_\mu(e(u)_{B_{2r}(x_0)})|^2 dx. \end{aligned} \tag{3.17}$$

Thus, by combining (3.16) and (3.17) with (3.3), with $Q := (\nabla u)_{B_{2r}(x_0)}$, we deduce

$$\int_{B_r(x_0)} |\nabla\big(V_\mu(e(u))\big)|^2 dx \le \frac{c}{r^2}\int_{B_{2r}(x_0)} \phi_{|e(u)|}\big(|e(u)-(e(u))_{B_{2r}(x_0)}|\big)dx$$
$$+\kappa\frac{c}{r^2}\int_{B_{2r}(x_0)} |V_\mu(e(u))-V_\mu((e(u))_{B_{2r}(x_0)})|^2 dx + c\,\kappa\, r^{\frac{2}{p-1}}\int_{B_{2r}(x_0)}\big(|u-g|^{\tilde p(\lambda)}+|\nabla(u-g)|^{\lambda p}\big)dx, \tag{3.18}$$

for some constant $c = c(n,p,\lambda) > 0$. Hence, by (2.24) we get from (3.18)

$$\begin{aligned}
&\int_{B_r(x_0)} |\nabla\big(V_\mu(e(u))\big)|^2 dx\\
&\le c\frac{1+\kappa}{r^2}\int_{B_{2r}(x_0)} |V_\mu(e(u))-V_\mu((e(u))_{B_{2r}(x_0)})|^2 dx + c\,\kappa\, r^{\frac{2}{p-1}}\int_{B_{2r}(x_0)}\big(|u-g|^{\tilde p(\lambda)}+|\nabla(u-g)|^{\lambda p}\big)dx\\
&\le c\frac{1+\kappa}{r^2}\int_{B_{2r}(x_0)} |V_\mu(e(u))-(V_\mu(e(u)))_{B_{2r}(x_0)}|^2 dx + c\,\kappa\, r^{\frac{2}{p-1}}\int_{B_{2r}(x_0)}\big(|u-g|^{\tilde p(\lambda)}+|\nabla(u-g)|^{\lambda p}\big)dx.
\end{aligned}$$

The last inequality follows from Lemma 2.6. □

In the sub-quadratic case we use a regularization argument following [17, Theorem 3.2]. Indeed, even setting $\kappa = 0$, the same arguments as in Proposition 3.1 lead only to a Besov type estimate. More precisely, the first part of the argument in Proposition 3.1 up to (3.11) included, holds for all $p \in (1,\infty)$ (one only has to use ζ^2 instead of ζ^p as a cutoff function). Thus, in case $p \in (1,2)$, arguing similarly to Proposition 3.1 one deduces the ensuing estimate

$$[V_\mu(e(u))]^2_{B^{p/2,2,\infty}(B_r(x_0))} \le \frac{c}{r^p}\int_{B_{2r}(x_0)}\big(\mu + |\nabla u - (\nabla u)_{B_{2r}(x_0)}|^2\big)^{p/2} dx,$$

for some $c = c(n,p) > 0$, which is not sufficient for our purposes. Recall that the Besov space $B^{p/2,2,\infty}(A)$, $A \subset \mathbb{R}^n$ open, is the space of maps $v \in L^2(A;\mathbb{R}^{n\times n})$ such that

$$[v]_{B^{p/2,2,\infty}(A)} := \sup_h |h|^{-p/2}\sum_{s=1}^n \|\tau_{s,h} v\|_{L^2(A;\mathbb{R}^{n\times n})} < \infty.$$

Finally, we point out that the argument we use below requires only minimal assumptions on g, namely L^p summability. We start off with establishing a technical result.

Lemma 3.2. *Let $n \ge 2$, $p \in (1,2]$, κ and $\mu \ge 0$, $g \in L^p(B_{2r},\mathbb{R}^n)$, and $w \in C^\infty(\overline{B}_{2r};\mathbb{R}^n)$. Let u be the minimizer of*

$$F_L(v) := \int_{B_{2r}} f_\mu(e(v))dx + \kappa\int_{B_{2r}} |v-g|^p dx + \frac{1}{2L}\int_{B_{2r}} |\nabla^2 v|^2 dx \tag{3.19}$$

over the set of $w + W^{2,2}_0(B_{2r},\mathbb{R}^n)$. Then, $u \in W^{3,2}_{\mathrm{loc}}(B_{2r},\mathbb{R}^n)$ and there is a constant $c = c(n,p) > 0$ such that for all $\lambda \ge 0$

$$\begin{aligned}
\frac{1}{L}\int_{B_r} |\nabla\Delta u|^2 dx + \int_{B_r} |\nabla(V_\mu(e(u)))|^2 dx &\le \frac{c}{Lr^4}\int_{B_{2r}\setminus B_r} |\nabla u - (\nabla u)_{B_{2r}}|^2 dx\\
&+\frac{c}{Lr^2}\int_{B_{2r}\setminus B_r} |\nabla^2 u|^2 dx + c\frac{1+\kappa}{r^2}\int_{B_{2r}} |V_\mu(e(u)) - (V_\mu(e(u)))_{B_{2r}}|^2 dx\\
&+\kappa\, r^{\frac{\lambda p}{p-1}}\int_{B_{2r}} |u-g|^p dx + c\frac{\kappa}{r^{\frac{2\lambda p}{2-p}}}\Big(\int_{B_{2r}} |V_\mu(e(u))|^2 dx + \mu^{p/2} r^n\Big).
\end{aligned} \tag{3.20}$$

Proof. We first prove that $u \in W^{3,2}_{\rm loc}(B_{2r},\mathbb{R}^n)$. Given $V \subset\subset B_{2r}$, we set $d := \min\{1, \mathrm{dist}(V, \partial B_{2r})\}$ and take $h \leq d/2$. For $\rho \in (0, d/2)$ we consider the function

$$g(\rho) := \sup\left\{ \frac{1}{L}\int_{B_\rho(y)} |\triangle_{s,h}\nabla^2 u|^2 dx : \, y \in V \right\}.$$

Next we prove that there exists a constant $c > 0$ independent from h (but possibly depending on L) such that

$$g(\rho) \leq \frac{g(\rho')}{2} + \frac{c}{(\rho'-\rho)^4} + \frac{c\,\kappa}{\rho'-\rho}, \tag{3.21}$$

for $\rho, \rho' \in (0, d/2)$, $\rho < \rho'$.

Fix ρ, ρ' as above, $y \in V$, and consider $\zeta \in C^\infty_c(B_{\rho'}(y))$, with $\zeta = 1$ on $B_\rho(y)$ and $|\nabla^2\zeta| \leq c/(\rho'-\rho)^2$. We now test

$$\frac{1}{L}\int_{B_{2r}} \langle \nabla^2 u, \nabla^2\varphi\rangle dx + \int_{B_{2r}} \langle \nabla f_\mu(e(u)), e(\varphi)\rangle dx + \kappa\, p\int_{B_{2r}} |u-g|^{p-2}\langle u-g, \varphi\rangle\, dx = 0\,, \tag{3.22}$$

holding for every $\varphi \in C^\infty_c(B_{2r};\mathbb{R}^n)$, with the test function $\varphi := \triangle_{s,-h}(\zeta\triangle_{s,h}u)$ and we estimate each appearing term.

First note that

$$\int_{B_{\rho'}(y)} \langle \nabla^2 u, \nabla^2 \varphi\rangle dx = -\int_{B_{\rho'}(y)} \langle \triangle_{s,h}\nabla^2 u, \zeta\nabla^2\triangle_{s,h}u + z\rangle dx,$$

where the function z satisfies

$$\|z\|_{L^2(B_{\rho'}(y))} \leq \frac{c}{(\rho'-\rho)^2}\|u\|_{W^{2,2}(B_{2r})}.$$

Therefore by Young's inequality we obtain

$$-\int_{B_{\rho'}(y)} \langle \triangle_{s,h}\nabla^2 u, z\rangle dx \leq \frac{1}{2}\int_{B_{\rho'}(y)} |\triangle_{s,h}\nabla^2 u|^2 dx + \frac{c}{(\rho'-\rho)^4}\|u\|^2_{W^{2,2}(B_{2r})}. \tag{3.23}$$

Moreover we have

$$\int_{B_{\rho'}(y)} \langle \triangle_{s,h}\nabla^2 u, \zeta\nabla^2\triangle_{s,h}u\rangle dx \geq \int_{B_\rho(y)} |\triangle_{s,h}\nabla^2 u|^2 dx,$$

so that by (3.23)

$$\int_{B_{\rho'}(y)} \langle \nabla^2 u, \nabla^2\varphi\rangle dx \leq \frac{1}{2}\int_{B_{\rho'}(y)} |\triangle_{s,h}\nabla^2 u|^2 dx + \frac{c}{(\rho'-\rho)^4}\|u\|^2_{W^{2,2}(B_{2r})} - \int_{B_\rho(y)} |\triangle_{s,h}\nabla^2 u|^2 dx. \tag{3.24}$$

On the set $(B_{2r})_{s,h}$ (recall the notation introduced right after (2.2)) we define

$$\alpha^s(x) := \int_0^1 \nabla f_\mu(e(u(x+th\epsilon_s)))dt$$

and observe that (for $\mu > 0$)

$$\begin{aligned}\triangle_{s,h}\big(\nabla f_\mu(e(u))\big) &= \frac{1}{h}\int_0^h \frac{d}{dt}\big(\nabla f_\mu(e(u(x+t\epsilon_s)))\big)dt \\ &= \frac{1}{h}\int_0^h \partial_s\big(\nabla f_\mu(e(u(x+t\epsilon_s)))\big)dt = \frac{1}{h}\partial_s\int_0^h \big(\nabla f_\mu(e(u(x+t\epsilon_s)))\big)dt = \partial_s\alpha^s\,. \end{aligned}\tag{3.25}$$

By continuity one obtains $\triangle_{s,h}\big(\nabla f_\mu(e(u))\big) = \partial_s \alpha^s$ also for $\mu = 0$. Therefore we estimate

$$\int_{B_{\rho'}(y)} \langle \nabla f_\mu(e(u)), e(\varphi)\rangle dx = -\int_{B_{\rho'}(y)} \langle \triangle_{s,h}\big(\nabla f_\mu(e(u))\big), \zeta \triangle_{s,h} e(u)\rangle dx$$
$$- \int_{B_{\rho'}(y)} \langle \triangle_{s,h}\big(\nabla f_\mu(e(u))\big), \nabla\zeta \odot \triangle_{s,h} u\rangle dx$$
$$\leq \int_{B_{\rho'}(y)} \langle \alpha^s, \partial_s \nabla\zeta \odot \triangle_{s,h} u\rangle dx + \int_{B_{\rho'}(y)} \langle \alpha^s, \nabla\zeta \odot \triangle_{s,h}\partial_s u\rangle dx\,, \quad (3.26)$$

where we have used (3.6), (3.8) and (3.25). Since $u \in W^{2,2}(B_{2r}(y), \mathbb{R}^n)$ we conclude with (2.7)

$$\int_{B_{\rho'}(y)} \langle \nabla f_\mu(e(u)), e(\varphi)\rangle dx \leq \frac{c}{(\rho'-\rho)^2}\|u\|_{W^{1,p}(B_{2r})}\|\alpha^s\|_{L^{p'}(B_{\rho'}(y))}$$
$$+ \frac{c}{\rho'-\rho}\|u\|_{W^{2,p}(B_{2r})}\|\alpha^s\|_{L^{p'}(B_{\rho'}(y))} \leq \frac{c}{(\rho'-\rho)^2}\|u\|_{W^{2,p}(B_{2r})}\|\mu^{1/2} + |e(u)|\|^{p/p'}_{L^p(B_{2r})}. \quad (3.27)$$

Eventually, by Hölder's inequality and the standard properties of difference quotients we can estimate the last term on the left hand side of (3.22) as follows:

$$\int_{B_{2r}} |u-g|^{p-2}\langle u-g, \varphi\rangle\, dx \;\leq\; \|u-g\|^{p-1}_{L^p(B_{2r})}\Big(\frac{1}{\rho'-\rho}\|\nabla u\|_{L^p(B_{2r})} + \|\nabla^2 u\|_{L^p(B_{2r})}\Big). \quad (3.28)$$

Estimates (3.22), (3.24), (3.27) and (3.28) yield

$$\int_{B_\rho(y)} |\triangle_{s,h}\nabla^2 u|^2 dx \leq \frac{1}{2}\int_{B_{\rho'}(y)} |\triangle_{s,h}\nabla^2 u|^2 dx + \frac{c}{(\rho'-\rho)^4} + \frac{c\,\kappa}{\rho'-\rho},$$

for a constant $c > 0$ depending on n, p, k, L, the $W^{2,2}$ norm of u and the L^p norm of g. Then, (3.21) follows at once since $y \in V$ is arbitrary. By this, [28, Lemma 6.1], and the compactness of $\overline{V}$ we finally infer

$$\int_V |\triangle_{s,h}\nabla^2 u|^2 dx \leq c,$$

with c independent from h, and therefore $u \in W^{3,2}_{\mathrm{loc}}(B_{2r}, \mathbb{R}^n)$.

Let us now prove (3.20). Using $u \in W^{3,2}_{\mathrm{loc}}(B_{2r}, \mathbb{R}^n)$ and the fact that $e(\varphi)$ has average zero for every $\varphi \in W^{1,2}_0(B_{2r}, \mathbb{R}^n)$, we can rewrite (3.22) as

$$\frac{1}{L}\int_{B_{2r}} \langle \nabla\Delta u, \nabla\varphi\rangle dx$$
$$= \int_{B_{2r}} \langle \nabla f_\mu(e(u)) - \nabla f_\mu(e(Qx)), e(\varphi)\rangle dx + \kappa\, p\int_{B_{2r}} |u-g|^{p-2}\langle u-g, \varphi\rangle dx\,, \quad (3.29)$$

for any $Q \in \mathbb{R}^{n\times n}$.

Let now $\psi := \sum_{s=1}^n \partial_s(\zeta^q \partial_s(u - Qx))$, where $q \geq 4$, let $\zeta \in C^\infty_c(B_{3r/2}; [0,1])$ obey $\zeta = 1$ on B_r and $|\nabla\zeta| \leq c/r$. Since $u \in W^{3,2}_{\mathrm{loc}}(B_{2r}, \mathbb{R}^n)$, ψ can be strongly approximated in $W^{1,2}(B_{2r}, \mathbb{R}^n)$ by smooth functions supported in $B_{3r/2}$; therefore we can use $\varphi = \psi$ as a trial function in (3.29).

We now estimate the three terms in (3.29). We start from the second one, which we write as

$$I_2 := \int_{B_{2r}} \langle B, e(\psi)\rangle dx$$

where

$$B(x) := \nabla f_\mu(e(u)) - \nabla f_\mu(e(Qx)) = \int_0^1 \nabla^2 f_\mu(e(Qx) + te(\tilde{u})(x))e(\tilde{u})(x)dt$$

and $\tilde{u}(x) := u(x) - Qx$. We estimate, using (2.3) and Lemma 2.3,

$$
\begin{aligned}
|B| &\le \int_0^1 (\mu + |e(Qx) + te(\tilde{u})|^2)^{p/2-1}|e(\tilde{u})|dt \\
&\le c(\mu + |e(Qx)|^2 + |e(u)|^2)^{p/2-1}|e(\tilde{u})| \\
&\le c(\mu + (|e(Qx)| + |e(\tilde{u})|)^2)^{p/2-1}|e(\tilde{u})| = c\,\phi'_{|e(Qx)|}(|e(\tilde{u})|)
\end{aligned}
$$

for $\mu > 0$. By continuity, $|B| \le c\,\phi'_{|e(Qx)|}(|e(\tilde{u})|)$ holds also for $\mu = 0$. We compute

$$
e(\psi) = \sum_{s=1}^{n} (\partial_s \nabla\zeta^q) \odot \partial_s \tilde{u} + \nabla\zeta^q \odot \partial_s^2 \tilde{u} + \partial_s(\zeta^q \partial_s e(\tilde{u}))\,.
$$

We estimate the three contributions to I_2 separately. Recalling the estimate for B, we obtain

$$
\begin{aligned}
|I_{2,1}| \le& \int_{B_{2r}\setminus B_r} |B|\frac{cq^2}{r^2}|\nabla\tilde{u}|dx \le \frac{cq^2}{r^2}\int_{B_{2r}\setminus B_r} \phi'_{|e(Qx)|}(|e(\tilde{u})|)|\nabla\tilde{u}|dx \\
\le& \frac{cq^2}{r^2}\int_{B_{2r}\setminus B_r} \phi'_{|e(Qx)|}(|\nabla\tilde{u}|)|\nabla\tilde{u}|dx \le \frac{cq^2}{r^2}\int_{B_{2r}\setminus B_r} \phi_{|e(Qx)|}(|\nabla\tilde{u}|)dx
\end{aligned}
$$

where we used monotonicity of ϕ'_a and (2.22). Using Korn's inequality for shifted N-functions (cf. [17, Lemma 2.9] or (3.16)) and choosing $Q := (\nabla u)_{B_{2r}}$ we conclude

$$
|I_{2,1}| \le \frac{cq^2}{r^2}\int_{B_{2r}} \phi_{|e(Qx)|}(|e(\tilde{u})|)dx \le \frac{cq^2}{r^2}\int_{B_{2r}} |V_\mu(e(u)) - V_\mu(e(Qx))|^2 dx
$$

where in the last step we used (2.24).

For the second one, we use that for any function v in $W^{2,2}_{\text{loc}}$ one has

$$
\partial_s^2 v_j = 2\partial_s([e(v)]_{sj}) - \partial_j([e(v)]_{ss}), \tag{3.30}
$$

here $[e(v)]_{hk}$ denotes the entry of position (h,k) of the matrix $e(v)$, to obtain

$$
\begin{aligned}
|I_{2,2}| \le& \frac{cq}{r}\int_{B_{2r}\setminus B_r} |B|\zeta^{q-1}|\Delta\tilde{u}|dx \\
\le& \frac{cq}{r}\int_{B_{2r}\setminus B_r} \phi'_{|e(Qx)|}(|e(\tilde{u})|)\zeta^{q-1}|\nabla e(\tilde{u})|dx.
\end{aligned}
$$

Recalling (2.13), choosing $q \ge 2$ and since $0 \le \zeta \le 1$, we deduce by Young's inequality

$$
\begin{aligned}
|I_{2,2}| \le& \frac{cq}{r}\int_{B_{2r}\setminus B_r} \phi''_{|e(Qx)|}(|e(\tilde{u})|)\zeta^{q-1}|e(\tilde{u})|\,|\nabla e(\tilde{u})|dx \\
\le& \delta\int_{B_{2r}\setminus B_r} \zeta^q\phi''_{|e(Qx)|}(|e(\tilde{u})|)|\nabla e(\tilde{u})|^2dx \\
&+ \frac{c}{r^2}\int_{B_{2r}\setminus B_r} \phi''_{|e(Qx)|}(|e(\tilde{u})|)|e(\tilde{u})|^2dx\,,
\end{aligned}
$$

with $c = c(\delta, q) > 0$ and $\delta \in (0,1)$ to be chosen below. Hence, recalling $|\nabla V_\mu(\xi)|^2 \ge c(\mu+|\xi|^2)^{p/2-1}$ and $\phi''_a(|t-a|)|t-a|^2 \le c|V_\mu(t) - V_\mu(a)|^2$ (see Lemma 2.4 and the definition of ϕ_a), we infer

$$
|I_{2,2}| \le \delta\int_{B_{2r}\setminus B_r} \zeta^q|\nabla(V_\mu(e(u)))|^2dx + \frac{c}{r^2}\int_{B_{2r}\setminus B_r} |V_\mu(e(u)) - V_\mu(e(Qx))|^2dx\,.
$$

Finally, to deal with the last term $I_{2,3}$ we integrate by parts. Since $\partial_s B = \nabla^2 f_\mu(e(u))\partial_s e(u)$, recalling (2.3) and the definition of V_μ

$$
\begin{aligned}
-I_{2,3} =& \int_{B_{2r}} \zeta^q \sum_{s=1}^n \langle \nabla^2 f_\mu(e(u))\partial_s e(u), \partial_s e(u)\rangle dx \\
\geq & c\int_{B_{2r}} \zeta^q (\mu + |e(u)|^2)^{p/2-1} |\nabla e(u)|^2 dx \geq c\int_{B_{2r}} \zeta^q |\nabla(V_\mu(e(u)))|^2 dx\,,
\end{aligned}
$$

with $c = c(p) > 0$.

We now turn to the first term in (3.29),

$$
I_1 := \int_{B_{2r}} \langle \nabla\Delta u, \nabla\psi\rangle dx\,.
$$

Again we consider separately the contributions of the different components of $\nabla\psi$,

$$
\nabla\psi = \sum_{s=1}^n \partial_s\tilde{u} \otimes (\partial_s \nabla\zeta^q) + \partial_s^2 \tilde{u} \otimes \nabla\zeta^q + (\partial_s\zeta^q)\partial_s\nabla\tilde{u} + \zeta^q \partial_s^2 \nabla\tilde{u}\,.
$$

The first term is controlled by

$$
\begin{aligned}
|I_{1,1}| \leq & \frac{c}{r^2}\int_{B_{2r}\setminus B_r} |\nabla\Delta u|\zeta^{q-2}|\nabla\tilde{u}| dx \\
\leq & \delta \int_{B_{2r}\setminus B_r} \zeta^q |\nabla\Delta u|^2 dx + \frac{c}{r^4}\int_{B_{2r}\setminus B_r} |\nabla\tilde{u}|^2 dx
\end{aligned}
$$

for some $c = c(q,\delta) > 0$, provided that $q-2 \geq q/2$, namely $q \geq 4$. The second and the third terms are controlled, for some $c = c(q,\delta) > 0$, by

$$
\begin{aligned}
|I_{1,2} + I_{1,3}| \leq & \frac{c}{r}\int_{B_{2r}\setminus B_r} |\nabla\Delta u|\zeta^{q-1}|\nabla^2\tilde{u}| dx \\
\leq & \delta \int_{B_{2r}\setminus B_r} \zeta^q |\nabla\Delta u|^2 dx + \frac{c}{r^2}\int_{B_{2r}\setminus B_r} |\nabla^2\tilde{u}|^2 dx\,.
\end{aligned}
$$

The fourth summand in I_1 is

$$
I_{1,4} := \int_{B_{2r}} \zeta^q \nabla\Delta u \cdot \nabla\Delta u dx\,.
$$

We deal with the remaining term in (3.29)

$$
I_3 := \kappa\, p \int_{B_{2r}} |u-g|^{p-2} \langle u-g, \sum_{s=1}^n \partial_s(\zeta^q \partial_s\tilde{u})\rangle dx.
$$

Hölder's and Young's inequalities together with (3.30) yield for some constant $c = c(p,q) > 0$

$$
\kappa^{-1} I_3 \leq r^{\frac{\lambda p}{p-1}} \int_{B_{2r}} |u-g|^p dx + \frac{c}{r^{(\lambda+1)p}}\int_{B_{2r}\setminus B_r} |\nabla\tilde{u}|^p dx + \frac{c}{r^{\lambda p}}\int_{B_{2r}} \zeta^q |\nabla e(u)|^p dx.
$$

Recalling that we have chosen $Q = (\nabla u)_{B_{2r}}$, apply Korn's inequality to obtain

$$
\kappa^{-1} I_{3,2} \leq \frac{c}{r^{(\lambda+1)p}}\int_{B_{2r}} |e(u) - e(Qx)|^p dx.
$$

From Lemma 2.4 we obtain

$$
|\xi - \eta|^p \leq c|V(\xi) - V(\eta)|^p (\mu + |\xi|^2 + |\eta|^2)^{p(2-p)/4},
$$

using Young's inequality and Remark 2.2 we conclude that

$$\kappa^{-1} I_{3,2} \le \frac{1}{r^2}\int_{B_{2r}} |V_\mu(e(u)) - V_\mu(e(Qx))|^2 dx + \frac{c}{r^{\frac{2\lambda p}{2-p}}}\int_{B_{2r}} (\mu + |e(u)|^2)^{p/2} dx$$
$$\le \frac{1}{r^2}\int_{B_{2r}} |V_\mu(e(u)) - V_\mu(e(Qx))|^2 dx + \frac{c}{r^{\frac{2\lambda p}{2-p}}}\Big(\int_{B_{2r}} |V_\mu(e(u))|^2 dx + \mu^{p/2} r^n\Big).$$

with $c = c(p) > 0$. Furthermore, again by Lemma 2.4, Young's inequality and Remark 2.2 we have that

$$\kappa^{-1} I_{3,3} \le \delta\int_{B_{2r}} \zeta^q |\nabla\big(V_\mu(e(u))\big)|^2 dx + \frac{c}{r^{\frac{2\lambda p}{2-p}}}\Big(\int_{B_{2r}} |V_\mu(e(u))|^2 dx + \mu^{p/2} r^n\Big).$$

for some $c = c(\delta, p) > 0$. Therefore, we deduce that

$$\kappa^{-1} I_3 \le r^{\frac{\lambda p}{p-1}}\int_{B_{2r}} |u-g|^p dx + \frac{1}{r^2}\int_{B_{2r}} |V_\mu(e(u)) - V_\mu(e(Qx))|^2 dx +$$
$$+ \delta\int_{B_{2r}} \zeta^q |\nabla\big(V_\mu(e(u))\big)|^2 dx + \frac{c}{r^{\frac{2\lambda p}{2-p}}}\Big(\int_{B_{2r}} |V_\mu(e(u))|^2 dx + \mu^{p/2} r^n\Big).$$

Finally, we rewrite (3.29) as

$$\frac{1}{L} I_{1,4} - I_{2,3} \le \frac{1}{L}|I_{1,1}| + \frac{1}{L}|I_{1,2} + I_{1,3}| + I_{2,1} + I_{2,2} + I_3\,.$$

Choosing $q \ge 4$ and $\delta \in (0, 1/4]$, for some constant $c = c(n,p) > 0$ we have that

$$\frac{1}{L}\int_{B_{2r}} \zeta^q |\nabla\Delta u|^2 dx + \int_{B_{2r}} \zeta^q |\nabla\big(V_\mu(e(u))\big)|^2 dx$$
$$\le \frac{c}{Lr^4}\int_{B_{2r}\setminus B_r} |\nabla\tilde{u}|^2 dx + \frac{c}{Lr^2}\int_{B_{2r}\setminus B_r} |\nabla^2\tilde{u}|^2 dx$$
$$+ c\,\frac{1+\kappa}{r^2}\int_{B_{2r}} |V_\mu(e(u)) - V_\mu(e(Qx))|^2 dx$$
$$+ \kappa\, r^{\frac{\lambda p}{p-1}}\int_{B_{2r}} |u-g|^p dx + c\,\frac{\kappa}{r^{\frac{2\lambda p}{2-p}}}\Big(\int_{B_{2r}} |V_\mu(e(u))|^2 dx + \mu^{p/2} r^n\Big),$$

and (3.20) follows at once from Lemma 2.6. □

We are now ready to prove the Caccioppoli's inequality in the sub-quadratic case.

Proposition 3.3. *Let $n \ge 2$, $p \in (1,2]$, κ and $\mu \ge 0$ and $g \in L^p(\Omega;\mathbb{R}^n)$. Let $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ be a local minimizer of $\mathscr{F}_{\mu,\kappa}$ defined in (2.8).*

Then, $V_\mu(e(u)) \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and $u \in W^{2,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$. More precisely, there is a constant $c = c(n,p) > 0$ such that if $B_{2r}(x_0) \subset \Omega$ and $\lambda \ge 0$

$$\int_{B_r(x_0)} |\nabla\big(V_\mu(e(u))\big)|^2 dx \le c\,\frac{1+\kappa}{r^2}\int_{B_{2r}(x_0)} |V_\mu(e(u)) - (V_\mu(e(u)))_{B_{2r}(x_0)}|^2 dx$$
$$+ \kappa\, r^{\frac{\lambda p}{p-1}}\int_{B_{2r}(x_0)} |u-g|^p dx + c\,\frac{\kappa}{r^{\frac{2\lambda p}{2-p}}}\Big(\int_{B_{2r}(x_0)} |V_\mu(e(u))|^2 dx + \mu^{p/2} r^n\Big), \quad (3.31)$$

and

$$\int_{B_r(x_0)} |\nabla(e(u))|^p dx \le c\Big(\int_{B_r(x_0)} |\nabla(V_\mu(e(u)))|^2 dx\Big)^{\frac{p}{2}}\Big(\int_{B_r(x_0)} (\mu + |e(u)|^2)^{\frac{p}{2}} dx\Big)^{1-\frac{p}{2}}. \quad (3.32)$$

Proof. By a simple translation argument we can assume $x_0 = 0$ without loss of generality. We consider the functionals F_L defined in (3.19) and correspondingly we define

$$\mathscr{F}_\infty(v) := \int_{B_{2r}} f_\mu(e(v))dx + \kappa \int_{B_{2r}} |v-g|^p dx\,.$$

Fix a sequence $u_l \in C^\infty(\overline{B}_{2r};\mathbb{R}^n)$ which converges strongly in $W^{1,p}$ to u, and let $u_{l,L}$ be the minimizer of F_L over the set of $W^{1,p}(B_{2r};\mathbb{R}^n)$ functions which coincide with u_l on the boundary, correspondingly u_l^* for $\mathscr{F}_\infty$.

For a fixed l, let v be a smooth approximation to u_l^* with the same boundary data. Then $F_L(v) \to \mathscr{F}_\infty(v)$ as $L\uparrow\infty$. Since $F_L \geq \mathscr{F}_\infty$, this implies that $F_L(u_{l,L}) \to \mathscr{F}_\infty(u_l^*)$ as $L\uparrow\infty$. In particular, the sequence $u_{l,L}$ is a minimizing sequence for $\mathscr{F}_\infty$, and since this functional is strictly convex it converges strongly in $W^{1,p}$ to the unique minimizer u_l^* of $\mathscr{F}_\infty$. Further, $L^{-1}\int_{B_{2r}}|\nabla^2 u_{l,L}|^2dx \to 0$.

Using Lemma 3.2 with $w = u_l$ and taking the limit $L\uparrow\infty$ in (3.20) we obtain

$$\begin{aligned}\int_{B_r}|\nabla(V_\mu(e(u_l^*)))|^2dx \leq c\,\frac{1+\kappa}{r^2}\int_{B_{2r}}|V_\mu(e(u_l^*)) - (V_\mu(e(u_l^*)))_{B_{2r}}|^2dx\\ +\kappa\, r^{\frac{\lambda p}{p-1}}\int_{B_{2r}}|u_l^*-g|^pdx + c\,\frac{\kappa}{r^{\frac{2\lambda p}{2-p}}}\Big(\int_{B_{2r}}|V_\mu(e(u_l^*))|^2dx + \mu^{p/2}r^n\Big).\end{aligned}$$

Finally, since $u_l \to u$ strongly the sequence $u_l^* + u - u_l$ is also a minimizing sequence for $\mathscr{F}_\infty$, and by strict convexity it converges strongly to the unique minimizer u.

We deduce that $u_l^* \to u$ strongly in $W^{1,p}(B_{2r};\mathbb{R}^n)$ and in the limit as $l\uparrow\infty$ we conclude the proof of (3.31). Eventually, (3.32) follows by Hölder's inequality and Lemma 2.4. □

3.2 Decay Estimates

As a first corollary of Propositions 3.1 and 3.3 we establish a decay property of the L^2-norm of $V_\mu(e(u))$ needed to prove the density lower bound inequality in [12] in the two dimensional setting. The result shall be improved as a consequence of the higher integrability property in the next section (cf. Corollary 4.3).

Proposition 3.4. *Let $n\geq 2$, $p\in(1,\infty)$, κ and $\mu\geq 0$. Let $u\in W^{1,p}(\Omega;\mathbb{R}^n)$ be a local minimizer of $\mathscr{F}_{\mu,\kappa}$ defined in (2.8) with $g\in L^p(\Omega;\mathbb{R}^n)$ if $p\in(1,2]$ and $g\in W^{1,p}(\Omega;\mathbb{R}^n)$ if $p\in(2,\infty)$.*

Then, for all $\gamma\in(0,2)$ there is a constant $c = c(\gamma,p,n,\kappa) > 0$ such that if $B_{R_0}(x_0) \subset\subset \Omega$, then for all $\rho < R \leq R_0 \leq 1$ if $p\geq 2$ it holds

$$\int_{B_\rho(x_0)}|V_\mu(e(u))|^2dx \leq c\rho^\gamma\Big(\frac{1}{R^\gamma}\int_{B_R(x_0)}|V_\mu(e(u))|^2dx + c\kappa\|u-g\|^p_{W^{1,p}(\Omega;\mathbb{R}^n)}\Big), \tag{3.33}$$

and if $p\in(1,2)$ it holds

$$\int_{B_\rho(x_0)}|V_\mu(e(u))|^2dx \leq c\rho^\gamma\Big(\frac{1}{R^\gamma}\int_{B_R(x_0)}|V_\mu(e(u))|^2dx + c\kappa\|u-g\|^p_{L^p(\Omega;\mathbb{R}^n)} + c\kappa\mu^{p/2}\Big), \tag{3.34}$$

Proof. Let $4r\leq R_0$, and $\zeta\in C_c^\infty(B_{2r}(x_0))$ be such that $0\leq\zeta\leq 1$, $\zeta|_{B_r(x_0)}\equiv 1$, $|\nabla\zeta|\leq 2/r$. Note that $\zeta^2 V_\mu(e(u)) \in W_0^{1,2}(B_{2r}(x_0),\mathbb{R}^{n\times n}_{\mathrm{sym}})$, therefore

$$\begin{aligned}&\int_{B_{2r}(x_0)}|\nabla\big(\zeta^2V_\mu(e(u))\big)|^2dx\\ &\qquad\leq 2\int_{B_{2r}(x_0)}\zeta^4|\nabla V_\mu(e(u))|^2dx + 8\int_{B_{2r}(x_0)}\zeta^2|\nabla\zeta|^2|V_\mu(e(u))|^2dx\\ &\qquad\leq 2\int_{B_{2r}(x_0)}|\nabla V_\mu(e(u))|^2dx + \frac{32}{r^2}\int_{B_{2r}(x_0)\setminus B_r(x_0)}|V_\mu(e(u))|^2dx.\end{aligned} \tag{3.35}$$

If $p \geq 2$ by means of Proposition 3.1 with $\lambda = 1$ we further estimate as follows

$$\begin{aligned}\int_{B_{2r}(x_0)} |\nabla(\zeta^2 V_\mu(e(u)))|^2 dx \leq \frac{c(1+\kappa)}{r^2} \int_{B_{4r}(x_0)} |V_\mu(e(u)) - (V_\mu(e(u)))_{B_{4r}(x_0)}|^2 dx \\ + c\kappa r^{\frac{2}{p-1}} \int_{B_{4r}(x_0)} \big(|u-g|^{\tilde{p}(1)} + |\nabla(u-g)|^p\big) dx + \frac{c}{r^2} \int_{B_{2r}(x_0)\setminus B_r(x_0)} |V_\mu(e(u))|^2 dx \\ \leq \frac{c(1+\kappa)}{r^2} \int_{B_{4r}(x_0)} |V_\mu(e(u))|^2 dx + c\kappa r^{\frac{2}{p-1}} \|u-g\|^p_{W^{1,p}(\Omega;\mathbb{R}^n)}, \end{aligned} \tag{3.36}$$

with $c = c(p, n) > 0$. Therefore, in view of Poincaré inequality and (3.36) we get for any $\tau \in (0, 1)$ and any $q \in (2, 2^*)$, with $2^* = 2n/(n-2)$ if $n > 2$, $2^* = \infty$ if $n = 2$,

$$\begin{aligned}\int_{B_{\tau r}(x_0)} |V_\mu(e(u))|^2 dx \leq c\,(\tau\, r)^{n(1-2/q)} \left(\int_{B_{2r}(x_0)} |\zeta^2 V_\mu(e(u))|^q dx\right)^{2/q} \\ \leq c\,\tau^{n(1-2/q)} r^2 \int_{B_{2r}(x_0)} |\nabla(\zeta^2 V_\mu(e(u)))|^2 dx \\ \leq c\,(1+\kappa)\tau^{n(1-2/q)} \Big(\int_{B_{4r}(x_0)} |V_\mu(e(u))|^2 dx + c\kappa r^{\frac{2p}{p-1}} \|u-g\|^p_{W^{1,p}(\Omega;\mathbb{R}^n)}\Big), \end{aligned} \tag{3.37}$$

with $c = c(p, q, n) > 0$. We choose $q \in (2, 2^*)$ such that $n(1-2/q) > \frac{2+\gamma}{2}$, which is the same as $q \in (\frac{4n}{2n-2-\gamma}, 2^*)$. This is possible since $\gamma \in (0, 2)$. Then, for sufficiently small τ, and for $\theta = \tau/4$

$$\int_{B_{4\theta r}(x_0)} |V_\mu(e(u))|^2 dx \leq \theta^{\frac{2+\gamma}{2}} \int_{B_{4r}(x_0)} |V_\mu(e(u))|^2 dx + c\kappa r^\gamma \|u-g\|^p_{W^{1,p}(\Omega;\mathbb{R}^n)}\,. \tag{3.38}$$

The decay formula (3.33) then follows from [28, Lemma 7.3].

Instead, if $p \in (1, 2)$ by Proposition 3.3 choosing $\lambda = \frac{2}{p} - 1 > 0$ we estimate (3.35) as follows

$$\begin{aligned}\int_{B_{2r}(x_0)} |\nabla(\zeta^2 V_\mu(e(u)))|^2 dx \leq \frac{c(\kappa+1)}{r^2} \int_{B_{4r}(x_0)} |V_\mu(e(u)) - (V_\mu(e(u)))_{B_{4r}(x_0)}|^2 dx \\ + c\kappa r^{\frac{2-p}{p-1}} \int_{B_{4r}(x_0)} |u-g|^p dx + c\kappa\mu^{p/2} r^{n-2} + c\frac{\kappa+1}{r^2} \int_{B_{4r}(x_0)} |V_\mu(e(u))|^2 dx \\ \leq c\frac{\kappa+1}{r^2} \int_{B_{4r}(x_0)} |V_\mu(e(u))|^2 dx + c\,\kappa\, r^{\frac{2-p}{p-1}} \|u-g\|^p_{L^p(\Omega;\mathbb{R}^n)} + c\,\kappa\mu^{p/2} r^{n-2}, \end{aligned}$$

with $c = c(n, p) > 0$. Then, arguing as to deduce (3.37) we conclude that

$$\begin{aligned}\int_{B_{\tau r}(x_0)} |V_\mu(e(u))|^2 dx \leq c\,\tau^{n(1-2/q)} \cdot \\ \cdot \Big((\kappa+1) \int_{B_{4r}(x_0)} |V_\mu(e(u))|^2 dx + c\,\kappa\, r^{\frac{p}{p-1}} \|u-g\|^p_{L^p(\Omega;\mathbb{R}^n)} + c\kappa\mu^{p/2} r^n\Big), \end{aligned}$$

with $c = c(p, q, n) > 0$. By choosing $q \in (2, 2^*)$ such that $n(1-2/q) > \frac{2+\gamma}{2}$, for sufficiently small τ, and for $\theta = \tau/4$

$$\int_{B_{4\theta r}(x_0)} |V_\mu(e(u))|^2 dx \leq \theta^{\frac{2+\gamma}{2}} \int_{B_{4r}(x_0)} |V_\mu(e(u))|^2 dx + c\kappa r^\gamma \big(\|u-g\|^p_{L^p(\Omega;\mathbb{R}^n)} + \mu^{p/2}\big)\,. \tag{3.39}$$

The decay formula (3.33) then follows from [28, Lemma 7.3]. □

4 Partial regularity results

In the quadratic case $p = 2$ it is well-known that the minimizer u is $C^2(\Omega;\mathbb{R}^n)$ in any dimension if $g \in C^1$ (see for instance [28, Theorem 10.14] or [27, Theorem 5.13, Corollary 5.14]).

Below we establish $C^{1,\alpha}$ regularity in the two dimensional setting and partial regularity in n dimensions together with an estimate on the Hausdorff dimension of the corresponding singular set. To our knowledge it is a major open problem in elliptic regularity to prove or disprove everywhere regularity for elasticity type systems in the nonlinear case if $n \geq 3$ and $p \neq 2$.

4.1 Higher integrability

In this subsection we prove the first main ingredient for establishing both $C^{1,\alpha}$ regularity if $n = 2$ and partial regularity if $n \geq 3$ with an estimate of the Hausdorff dimension of the singular set: the higher integrability for the gradient of $V_\mu(e(u))$, $\mu \geq 0$.

Proposition 4.1. *Let $n \geq 2$, $p \in (1,\infty)$, κ and $\mu \geq 0$. Let $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ be a local minimizer of $\mathscr{F}_{\mu,\kappa}$ defined in (2.8) with $g \in L^s(\Omega;\mathbb{R}^n)$, $s > p$, if $p \in (1,2]$ and $g \in W^{1,p}(\Omega;\mathbb{R}^n)$ if $p \in (2,\infty)$.*

Then, $V_\mu(e(u)) \in W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ for some $q > 2$. More precisely, there exist $q = q(n,p,\kappa) > 2$ and $c = c(n,p,\kappa) > 0$ such that if $B_{2r}(x_0) \subset\subset \Omega$ and $p > 2$

$$\Big(\fint_{B_r(x_0)} |\nabla\big(V_\mu(e(u))\big)|^q dx\Big)^{1/q} \leq c\Big(\fint_{B_{2r}(x_0)} |\nabla\big(V_\mu(e(u))\big)|^2 dx\Big)^{1/2} + c\left(\kappa \fint_{B_{2r}(x_0)} \big(|u-g|^{\tilde{p}(\frac{1+\lambda_0}{2})} + |\nabla(u-g)|^{\frac{1+\lambda_0}{2}p}\big)^{\frac{q}{2}} dx\right)^{1/q}, \quad (4.1)$$

with the exponent $\tilde{p}$ and $\lambda_0 \in [\frac{1}{p-1},1)$ introduced in Section 3.1, and if $p \in (1,2]$

$$\Big(\fint_{B_r(x_0)} |\nabla\big(V_\mu(e(u))\big)|^q dx\Big)^{1/q} \leq c\Big(\fint_{B_{2r}(x_0)} |\nabla\big(V_\mu(e(u))\big)|^2 dx\Big)^{1/2} + c\left(\kappa \fint_{B_{2r}(x_0)} \big(|u-g|^p + |V_\mu(e(u))|^2 + \mu^{p/2}\big)^{\frac{q}{2}} dx\right)^{1/q} \quad (4.2)$$

with $q = q(n,p,\kappa,s) > 2$ and $c = c(n,p,\kappa,s) > 0$.

Proof. Recalling that 2 is the Sobolev exponent of $\frac{2n}{n+2}$, we may use the Caccioppoli's type estimates (3.2) and (3.31), the former if $p > 2$ (with $\lambda = \frac{1+\lambda_0}{2}$) and the latter if $p \in (1,2]$ (with $\lambda = 0$), to deduce by Poincaré-Wirtinger inequality for some $c = c(n,p) > 0$

$$\fint_{B_r(x_0)} |\nabla\big(V_\mu(e(u))\big)|^2 dx \leq c(1+\kappa)\Big(\fint_{B_{2r}(x_0)} |\nabla\big(V_\mu(e(u))\big)|^{\frac{2n}{n+2}} dx\Big)^{\frac{n+2}{n}} + c\,\kappa r^{\frac{2}{p-1}} \fint_{B_{2r}(x_0)} \big(|u-g|^{\tilde{p}(\frac{1+\lambda_0}{2})} + |\nabla(u-g)|^{\frac{1+\lambda_0}{2}p}\big)dx,$$

if $p > 2$, and

$$\fint_{B_r(x_0)} |\nabla\big(V_\mu(e(u))\big)|^2 dx \leq c(1+\kappa)\Big(\fint_{B_{2r}(x_0)} |\nabla\big(V_\mu(e(u))\big)|^{\frac{2n}{n+2}} dx\Big)^{\frac{n+2}{n}} + c\,\kappa \fint_{B_{2r}(x_0)} \big(|u-g|^p + |V_\mu(e(u))|^2 + \mu^{p/2}\big)dx$$

if $p \in (1,2]$. By taking into account that $u \in W^{1,p}(\Omega;\mathbb{R}^n)$, $\lambda_0 \in (\frac{1}{p-1},1)$ and $\tilde{p}(\frac{1+\lambda_0}{2}) < p^*$ and that $V_\mu(e(u)) \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ (cf. Propositions 3.1 and 3.3), Gehring's lemma with increasing support (see for instance [28, Theorem 6.6]) yields higher integrability together with estimates (4.1) and (4.2). A covering argument provides the conclusion. □

Remark 4.2. *To apply Gehring's lemma with increasing support in order to deduce higher integrability in case $p > 2$ it is instrumental that we may choose $\lambda \in (\lambda_0, 1)$ and the corresponding exponent $\tilde{p}(\lambda) \in (p, p^*)$ in (3.2) (cf. the definition of $\tilde{p}(\cdot)$ in (3.1)).*

We improve next the decay estimates in Proposition 3.4. This version is useful to prove the density lower bound in [12] in the three dimensional setting. We do not provide the details since the proof is the same of Proposition 3.4 and only takes further advantage of Proposition 4.1.

Corollary 4.3. *Let $n \geq 3$, $p \in (1, \infty)$, κ and $\mu \geq 0$. Let $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ be a local minimizer of $\mathscr{F}_{\mu,\kappa}$ defined in (2.8) with $g \in L^s(\Omega; \mathbb{R}^n)$ with $s > p$ if $p \in (1, 2]$ and $g \in W^{1,p}(\Omega; \mathbb{R}^n)$ if $p \in (2, \infty)$.*

Then, there exists $\gamma_0 = \gamma_0(n, p, \kappa)$, with $\gamma_0 > 2$, such that for all $\gamma \in (0, \gamma_0]$ there is a constant $c = c(\gamma, p, n) > 0$ such that if $B_{R_0}(x_0) \subset\subset \Omega$, then for all $\rho < R \leq R_0 \leq 1$

$$\int_{B_\rho(x_0)} |V_\mu(e(u))|^2 dx \leq c\rho^\gamma \Big(\frac{1}{R^\gamma} \int_{B_R(x_0)} |V_\mu(e(u))|^2 dx + c\kappa \|u - g\|^p_{W^{1,p}(\Omega;\mathbb{R}^n)} \Big)$$

if $p > 2$, and if $p \in (1, 2]$,

$$\int_{B_\rho(x_0)} |V_\mu(e(u))|^2 dx \leq c\rho^\gamma \Big(\frac{1}{R^\gamma} \int_{B_R(x_0)} |V_\mu(e(u))|^2 dx + c\kappa \|u - g\|^p_{L^p(\Omega;\mathbb{R}^n)} + c\kappa\mu^{p/2} \Big)$$

with $c = c(\gamma, p, n, s) > 0$.

4.2 The 2-dimensional case

$C^{1,\alpha}$ regularity in 2d readily follows from Proposition 4.1 (see also Remark 4.5).

Proposition 4.4. *Let $n = 2$, $p \in (1, \infty)$, κ and $\mu \geq 0$. Let $u \in W^{1,p}(\Omega; \mathbb{R}^2)$ be a local minimizer of $\mathscr{F}_{\mu,\kappa}$ defined in (2.8) with $g \in L^s(\Omega; \mathbb{R}^2)$, $s > p$, if $p \in (1, 2]$ and $g \in W^{1,p}(\Omega; \mathbb{R}^2)$ if $p \in (2, \infty)$.*

Then, $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ for all $\alpha \in (0, 1)$ if $1 < p < 2$ and $\mu > 0$ or if $p \geq 2$, and for some $\alpha(p) \in (0, 1)$ if $1 < p < 2$ and $\mu = 0$.

Proof. We recall that $V_\mu(e(u)) \in W^{1,q}_{\mathrm{loc}}(\Omega; \mathbb{R}^{2\times2}_{\mathrm{sym}})$ for some $q > 2$ in view of Proposition 4.1. Therefore, by Morrey's theorem $V_\mu(e(u)) \in C^{0,1-\frac{2}{q}}_{\mathrm{loc}}(\Omega; \mathbb{R}^{2\times2}_{\mathrm{sym}})$.

Furthermore, being V_μ an homeomorphism with inverse of class $C^1(\mathbb{R}^{2\times2}; \mathbb{R}^{2\times2})$ if $p \in (1, 2]$ and $\mu \geq 0$ or if $p > 2$ and $\mu > 0$, and of class $C^{0,\frac{2}{p}}_{\mathrm{loc}}(\mathbb{R}^{2\times2}; \mathbb{R}^{2\times2})$ if $p > 2$ and $\mu = 0$, we conclude by Korn's inequality that $u \in C^{1,\alpha_p}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ for some $\alpha_p = \alpha(p) \in (0, 1)$.

To conclude the claimed $C^{1,\alpha}$ regularity for all $\alpha \in (0, 1)$, we recall first that $u \in W^{2,p\wedge2}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ (cf. Propositions 3.1 and 3.3). Actually, in the 2-dimensional setting $u \in W^{2,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ in case $p \in (1, 2)$, as well. Indeed, $|e(u)| \in L^\infty_{\mathrm{loc}}(\Omega)$ by Corollary 4.3, therefore we conclude at once from Lemma 2.4 (cf. the argument leading to (3.32)). Hence, since $f_\mu \in C^2(\mathbb{R}^{2\times2}_{\mathrm{sym}})$ if $1 < p < 2$ and $\mu > 0$ or if $p \geq 2$, one can differentiate (3.4) and deduce that the weak gradient of $e(u)$ is a weak solution to a linear uniformly elliptic system with continuous coefficients. Schauder's theory provides the conclusion (cf. [27, Theorem 5.6 and 5.15]). □

Remark 4.5. *Actually, $u \in C^k(\Omega; \mathbb{R}^2)$ if $g \in C^k(\Omega; \mathbb{R}^2)$ and $\mu > 0$ bootstrapping the previous argument.*

4.3 Partial regularity in the non-degenerate autonomous case

In this section we deal with the non-degenerate autonomous case, corresponding to $\mu > 0$ and $\kappa = 0$, by following the so called indirect methods for proving partial regularity (see [25]). Therefore, the other main ingredient besides higher integrability of the gradient, is the following excess decay lemma. We introduce the notation

$$\mathscr{E}_v(x, r) := \fint_{B_r(x)} \Big| V_\mu\Big(e(v) - \big(e(v)\big)_{B_r(x)} \Big) \Big|^2 dy \tag{4.3}$$

for the excess of any $v \in W^{1,p}(\Omega;\mathbb{R}^n)$. Recall that $\big(e(v)\big)_{B_r(x)} = \fint_{B_r(x)} e(v)dy$.

Technical tools exploited in the proof of the excess decay are postponed to the Appendix A. For a linearization argument there, the assumption $\mu > 0$ is crucial (cf. Theorem A.2).

Proposition 4.6. *Let $n \geq 2$, $p \in (1,\infty)$ and $\mu > 0$. Let $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ be a local minimizer of $\mathscr{F}_{\mu,0}$ defined in (2.8).*

Then, for every $L > 0$ there exists $C = C(L) > 0$ such that for every $\tau \in (0, 1/4)$ there exists $\varepsilon = \varepsilon(\tau, L) > 0$ such that if $B_r(x) \subseteq \Omega$,

$$\Big|\big(e(u)\big)_{B_r(x)}\Big| \leq L \qquad \text{and} \qquad \mathscr{E}_u(x,r) \leq \varepsilon\,,$$

then

$$\mathscr{E}_u(x,\tau\, r) \leq C\,\tau^2 \mathscr{E}_u(x,r). \tag{4.4}$$

Proof. Suppose by contradiction that there is $L > 0$ such that for all constants $C > 0$ we can find $\tau \in (0, 1/4)$ for which there exist $B_{r_h}(x_h) \subset \Omega$ such that

$$\Big|\big(e(u)\big)_{B_{r_h}(x_h)}\Big| \leq L, \quad \mathscr{E}_u(x_h, r_h) = \lambda_h^2 \downarrow 0,$$

and

$$\mathscr{E}_u(x_h, \tau\, r_h) > C\,\tau^2 \mathscr{E}_u(x_h, r_h). \tag{4.5}$$

We shall conveniently fix the value of C at the end of the proof to reach a contradiction.

Consider the field $u_h : B_1 \to \mathbb{R}^n$ defined by

$$u_h(y) := \frac{1}{\lambda_h r_h}\Big(u(x_h + r_h y) - (u)_{B_{r_h}(x_h)} - r_h\big(\nabla u\big)_{B_{r_h}(x_h)} \cdot y\Big),$$

and set $\mathbb{A}_h := \big(e(u)\big)_{B_{r_h}(x_h)}$. Then, up to a subsequence we may assume that $\mathbb{A}_h \to \mathbb{A}_\infty$ and

$$\fint_{B_1} |V_\mu(\lambda_h e(u_h))|^2 dx = \fint_{B_{r_h}(x_h)} |V_\mu(e(u) - \mathbb{A}_h)|^2 dx = \mathscr{E}_u(x_h, r_h) = \lambda_h^2. \tag{4.6}$$

Being u a local minimizer of $\mathscr{F}_{\mu,0}$ defined in (2.8), u_h is in turn a local minimizer of

$$\mathscr{F}_h(v) = \int_{B_1} F_h(e(v))dx,$$

with integrand

$$F_h(\xi) := \lambda_h^{-2}\big(f_\mu(\mathbb{A}_h + \lambda_h \xi) - f_\mu(\mathbb{A}_h) - \lambda_h \langle \nabla f_\mu(\mathbb{A}_h), \xi\rangle\big).$$

Note that $\mathscr{F}_h(u_h) \leq c\,\mathcal{L}^n(B_1)$ by (iii) in Lemma A.1 and (4.6), thus by Theorem A.2, $(u_h)_h$ converges weakly to some function $u_\infty \in W^{1,2}(B_1, \mathbb{R}^n)$ in $W^{1,p\wedge 2}(B_1, \mathbb{R}^n)$, and actually, by Corollary A.4 we have for all $r \in (0,1)$

$$\lim_{h\uparrow\infty} \int_{B_r} \lambda_h^{-2} |V_\mu(\lambda_h e(u_h - u_\infty))|^2 dx = 0. \tag{4.7}$$

Therefore, item (iii) in Lemma 2.5 and a scaling argument give for some constant $c = c(p) > 0$

$$\begin{aligned}
\lambda_h^{-2}\mathscr{E}_u(x_h, \tau r_h) &= \lambda_h^{-2} \fint_{B_\tau} \Big|V_\mu\Big(\lambda_h (e(u_h) - \big(e(u_h)\big)_{B_\tau})\Big)\Big|^2 dx \\
&\leq c\,\lambda_h^{-2} \fint_{B_\tau} \Big|V_\mu\Big(\lambda_h e(u_h - u_\infty)\Big)\Big|^2 dx + c\,\lambda_h^{-2} \fint_{B_\tau} \Big|V_\mu\Big(\lambda_h (e(u_\infty) - \big(e(u_\infty)\big)_{B_\tau})\Big)\Big|^2 dx \\
&\quad + c\,\lambda_h^{-2} \fint_{B_\tau} \Big|V_\mu\Big(\lambda_h \big(\big(e(u_h)\big)_{B_\tau} - \big(e(u_\infty)\big)_{B_\tau}\big)\Big)\Big|^2 dx.
\end{aligned}$$

The very definition of V_μ, item (v) in Lemma 2.5 and (4.7) yield

$$\limsup_{h\uparrow\infty} \lambda_h^{-2}\mathscr{E}_u(x_h,\tau r_h) \le c\mu^{p/2-1} \fint_{B_\tau} \Big|e(u_\infty) - \big(e(u_\infty)\big)_{B_\tau}\Big|^2\,dx.$$

In particular,

$$\limsup_{h\uparrow\infty} \lambda_h^{-2}\mathscr{E}_u(x_h,\tau r_h) \le \widetilde{C}\tau^2\,,$$

as u_∞ is the solution of a linear elliptic system (cf. Corollary A.3). Thus, by taking the constant $C > \widetilde{C}$, we reach a contradiction to (4.5). □

We are finally ready to establish partial regularity and an estimate on the Hausdorff dimension of the singular set in the non-degenerate autonomous case. The degenerate case, namely $\mu = 0$, corresponding to the symmetrized p-laplacian, $p \neq 2$, is not included in our results. The non-autonomous case will be treated next via a perturbation argument. We recall that in case $p = 2$ the solutions are actually smooth.

Before proceeding with the proof, we introduce some notation: for $v \in W^{1,p}(\Omega;\mathbb{R}^n)$ let

$$\Sigma_v^{(1)} := \left\{x\in\Omega:\ \liminf_{r\downarrow 0} \fint_{B_r(x)} \Big|V_\mu(e(v(y))) - \big(V_\mu(e(v))\big)_{B_r(x)}\Big|^2 dy > 0\right\}, \tag{4.8}$$

and

$$\Sigma_v^{(2)} := \left\{x\in\Omega:\ \limsup_{r\downarrow 0} \Big|\big(V_\mu(e(v))\big)_{B_r(x)}\Big| = \infty\right\}. \tag{4.9}$$

Theorem 4.7. *Let $n \ge 3$, $p \in (1,\infty)$ and $\mu > 0$. Let $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ be a local minimizer of $\mathscr{F}_{\mu,0}$ defined in (2.8).*

Then, there exists an open set $\Omega_u \subseteq \Omega$ such that $u \in C^{1,\alpha}_{loc}(\Omega_u;\mathbb{R}^n)$ for all $\alpha \in (0,1)$. Moreover,

$$\dim_{\mathcal{H}}(\Omega\setminus\Omega_u) \le (n-q)\vee 0,$$

where $q > 2$ is the exponent in Proposition 4.1.

Proof. We shall show in what follows that under the standing assumptions the singular and regular sets are given respectively by

$$\Sigma_u := \Sigma_u^{(1)} \cup \Sigma_u^{(2)}, \qquad \Omega_u := \Omega\setminus\Sigma_u. \tag{4.10}$$

By the higher integrability property established in Proposition 4.1, we know that $V_\mu(e(u)) \in W^{1,q}_{\rm loc}(\Omega;\mathbb{R}^{n\times n}_{\rm sym})$ for some $q > 2$. Therefore, $\Sigma_u = \emptyset$ if $q > n$ by Morrey's theorem. Otherwise, if $B_r(x_0) \subseteq \Omega$, by Poincarè's inequality for all $r \in (0, \mathrm{dist}(x_0,\partial\Omega))$

$$\fint_{B_r(x_0)} \Big|V_\mu(e(u)) - \big(V_\mu(e(u))\big)_{B_r(x_0)}\Big|^2 dx \le c\Big(r^{q-n}\int_{B_r(x_0)} |\nabla V_\mu(e(u))|^q dx\Big)^{2/q}.$$

Therefore, $\mathcal{H}^{n-q}\big(\Sigma_u^{(1)}\big) = 0$ by standard density estimates (cf. [28, Proposition 2.7] or [2, Theorem 2.56]), and $\dim_{\mathcal{H}}\Sigma_u^{(2)} \le n-q$ by standard properties of Sobolev functions (cf. [28, Theorem 3.22]). In conclusion, $\dim_{\mathcal{H}}\Sigma_u \le n-q$.

Let us prove that Ω_u is open and that $u \in C^{1,\alpha}(\Omega_u;\mathbb{R}^n)$ for all $\alpha \in (0,1)$. Let $x_0 \in \Omega_u$. First note that $\sup_r |\big(V_\mu(e(u))\big)_{B_r(x_0)}| < \infty$ being $x_0 \in \Omega\setminus\Sigma_u^{(2)}$. Additionally, since

$$\fint_{B_\rho(x_0)} |V_\mu(e(u))|^2 dy \le c(p)\fint_{B_\rho(x_0)} \Big|V_\mu(e(u)) - \big(V_\mu(e(u))\big)_{B_\rho(x)}\Big|^2 dy + c(p)\Big|\big(V_\mu(e(u))\big)_{B_\rho(x)}\Big|^2,$$

being $x_0 \in \Omega \setminus \Sigma_u^{(1)}$ we conclude

$$\liminf_{\rho\downarrow 0} \fint_{B_\rho(x_0)} |V_\mu(e(u))|^2 dy < \infty. \tag{4.11}$$

The last inequality and item (v) in Lemma 2.5 yield for some $L > 0$

$$\liminf_{\rho\downarrow 0} \left| \big(e(u)\big)_{B_\rho(x_0)} \right| < L.$$

In view of this, Lemma 2.5 (item (i) if $p \geq 2$ and item (ii) if $p \in (1,2)$, respectively) and Lemma 2.6 yield that $\liminf_{\rho\downarrow 0} \mathscr{E}_u(x_0,\rho) = 0$. Therefore, for all $\eta > 0$, x_0 belongs to the set

$$\Omega_u^{L,\eta} := \left\{ x \in \Omega : \left| \big(e(u)\big)_{B_r(x)} \right| < L, \quad \mathscr{E}_u(x,r) < \eta \quad \text{for some } r \in (0, \operatorname{dist}(x,\partial\Omega)) \right\}.$$

In particular, $\Omega_u \subseteq \cup_{L\in\mathbb{N}} \Omega_u^{L,\eta(L)}$, for every $\eta(L) > 0$, and clearly each $\Omega_u^{L,\eta(L)} \subseteq \Omega$ is open. We claim that actually $\Omega_u = \cup_{L\in\mathbb{N}} \Omega_u^{L,\overline{\eta}(L)}$ for some $\overline{\eta}(L) = \overline{\eta}(L,n,p,\alpha)$ conveniently defined in what follows. To this aim we distinguish the super-quadratic and sub-quadratic cases.

We start with the range of exponents $p \geq 2$. To check the claim fix any $L \in \mathbb{N}$ and $x_0 \in \Omega_u^{L,\eta}$, with corresponding radius r, then we have for all $\tau \in (0, 1/4)$

$$\begin{aligned} \left| \big(e(u)\big)_{B_{\tau r}(x_0)} \right| &\leq \left| \big(e(u)\big)_{B_r(x_0)} \right| + \left| \big(e(u)\big)_{B_{\tau r}(x_0)} - \big(e(u)\big)_{B_r(x_0)} \right| \\ &\leq \left| \big(e(u)\big)_{B_r(x_0)} \right| + \tau^{-n} \fint_{B_r(x_0)} \left| e(u) - \big(e(u)\big)_{B_r(x_0)} \right| dy \\ &\leq \left| \big(e(u)\big)_{B_r(x_0)} \right| + \tau^{-n} (\mathscr{E}_u(x_0,r))^{1/p}, \end{aligned} \tag{4.12}$$

where for the last inequality we have used item (iv) of Lemma 2.5 for $p \geq 2$. Moreover, if $\varepsilon(\tau, L)$ is the parameter provided by Proposition 4.6, and $0 < \eta \leq \varepsilon(\tau, L)$ we infer that

$$\mathscr{E}_u(x_0, \tau r) \leq C\tau^2 \mathscr{E}_u(x_0, r). \tag{4.13}$$

Having fixed any $\alpha \in (0,1)$ we choose $\tau = \tau(\alpha, L) \in (0, 1/4)$ such that $C\tau^{2\alpha} < 1$, with $C = C(L) > 0$ the constant in (4.4). Therefore, choosing $0 < \eta \leq \varepsilon(\tau, L) \wedge \tau^{np}$ we infer from (4.12) and (4.13)

$$|(e(u))_{B_{\tau r}(x_0)}| < L + 1, \quad \mathscr{E}_u(x_0, \tau r) < \tau^{2(1-\alpha)} \mathscr{E}_u(x_0, r).$$

The latter is the basic step of an induction argument leading to

$$|(e(u))_{B_{\tau^j r}(x_0)}| < L + 1, \quad \mathscr{E}_u(x_0, \tau^j r) < \tau^{2(1-\alpha)j} \mathscr{E}_u(x_0, r) \tag{4.14}$$

for all $j \in \mathbb{N}$. Note that from the last two inequalities we conclude readily that $x_0 \in \Omega_u$.

Hence we are left with showing (4.14). To this aim fix $j \in \mathbb{N}$, $j \geq 2$, and assume (4.14) true for all $0 \leq k \leq j-1$ (as noticed the first induction step corresponding to $j = 1$ has already been established above). Then, by (4.12) we get

$$|(e(u))_{B_{\tau^j r}(x_0)}| \leq |(e(u))_{B_r(x_0)}| + \tau^{-n} \sum_{k=0}^{j-1} \big(\mathscr{E}_u(x_0, \tau^k r)\big)^{1/p} < L + \frac{\tau^{-n}}{1 - \tau^{2/p(1-\alpha)}} (\mathscr{E}_u(x_0, r))^{1/p}.$$

We get the first estimate in (4.14) provided $0 < \eta \leq \varepsilon(\tau, L) \wedge \tau^{np}(1 - \tau^{2/p(1-\alpha)})^p$. Finally, to get the second inequality in (4.14) it suffices to assume in addition $0 < \eta < \varepsilon(\tau, L+1)$ and apply Proposition 4.6. In conclusion, we set

$$\overline{\eta}(L) := \varepsilon(\tau, L+1) \wedge \varepsilon(\tau, L) \wedge \tau^{np}(1 - \tau^{2/p(1-\alpha)})^p \tag{4.15}$$

(recall that $\tau = \tau(\alpha, L)$).

If $p \in (1,2)$ we only highlight the needed changes since the strategy of proof is completely analogous. We start off noting that we have for some $c = c(p) > 0$ (which may vary from line to line)

$$\begin{aligned}
&\fint_{B_\rho(x_0)} |e(u) - (e(u))_{B_\rho(x_0)}|^p dy \leq c \fint_{B_\rho(x_0)} \frac{\left|V_\mu(e(u)) - V_\mu\big((e(u))_{B_\rho(x_0)}\big)\right|^p}{\left(\mu + |e(u)|^2 + |(e(u))_{B_\rho(x_0)}|^2\right)^{\frac{p(p-2)}{4}}} dy \\
&\leq c\Big(\fint_{B_\rho(x_0)} \left|V_\mu(e(u)) - V_\mu\big((e(u))_{B_\rho(x_0)}\big)\right|^2 dy\Big)^{\frac{p}{2}} \Big(\fint_{B_\rho(x_0)} \left(\mu + |e(u)|^2 + |(e(u))_{B_\rho(x_0)}|^2\right)^{\frac{p}{2}} dy\Big)^{1-\frac{p}{2}} \\
&\leq c(\mathscr{E}_u(x_0,\rho))^{\frac{p}{2}} \Big(\mu^{p/2} + |(e(u))_{B_\rho(x_0)}|^p + \fint_{B_\rho(x_0)} |e(u) - (e(u))_{B_\rho(x_0)}|^p dy\Big)^{1-\frac{p}{2}} \\
&\leq c\big(\mu^{p/2} + |\big(e(u)\big)_{B_\rho(x_0)}|^p\big)^{1-\frac{p}{2}} (\mathscr{E}_u(x_0,\rho))^{\frac{p}{2}} + c\,\mathscr{E}_u(x_0,\rho) + \frac{1}{2}\fint_{B_\rho(x_0)} |e(u) - (e(u))_{B_\rho(x_0)}|^p dy,
\end{aligned} \tag{4.16}$$

where we have used Lemma 2.4 in the first inequality, Hölder's inequality in the second, item (ii) of Lemma 2.5 in the third, and Young's inequality in the fourth. Therefore, we get

$$\fint_{B_\rho(x_0)} |e(u) - (e(u))_{B_\rho(x_0)}| dy \leq c\big(\mu^{p/2} + |(e(u))_{B_\rho(x_0)}|^p + \mathscr{E}_u(x_0,\rho)\big)^{\frac{1}{p}-\frac{1}{2}} (\mathscr{E}_u(x_0,\rho))^{\frac{1}{2}}$$

for some constant $c = c(p) > 0$. In turn, with fixed $L \in \mathbb{N}$ and $x_0 \in \Omega_u^{L,\eta}$, for all $\tau \in (0, 1/4)$ we have instead of (4.12)

$$|(e(u))_{B_{\tau r}(x_0)}| \leq |(e(u))_{B_r(x_0)}| + c\tau^{-n}\big(\mu^{p/2} + L^p + \mathscr{E}_u(x_0,r)\big)^{\frac{1}{p}-\frac{1}{2}} (\mathscr{E}_u(x_0,r))^{\frac{1}{2}}. \tag{4.17}$$

Having fixed any $\alpha \in (0,1)$ and choosing $\tau = \tau(\alpha, L) \in (0, 1/4)$ such that $C\tau^{2\alpha} < 1$, with $C = C(L) > 0$ the constant in (4.4), we can establish inductively (4.14) provided we choose

$$\overline{\eta}(L) := \varepsilon(\tau, L+1) \wedge \varepsilon(\tau, L) \wedge 1 \wedge c\tau^{2n}\big(\mu^{p/2} + L^p + 1\big)^{1-\frac{2}{p}} (1 - \tau^{1-\alpha})^2, \tag{4.18}$$

with $c = c(p) > 0$.

Eventually, for any $p \in (1,\infty)$, $V_\mu(e(u)) \in C^{0,1-\alpha}_{loc}(\Omega_u^{L,\eta(L)}; \mathbb{R}^{n\times n}_{\rm sym})$ for all $\alpha \in (0,1)$ by Campanato's theorem and (4.14). The conclusion for $e(u)$ then follows at once from the fact that V_μ is an homeomorphism with inverse of class $C^1(\mathbb{R}^{n\times n}; \mathbb{R}^{n\times n})$ if $p > 2$ and $\mu > 0$ or if $p \in (1,2]$. □

4.4 Partial regularity in the non-degenerate case

In this section we prove partial regularity in the general non-degenerate case by following the so called direct methods for regularity. To this aim, with given κ, $\mu > 0$ and a local minimizer u on $W^{1,p}(\Omega; \mathbb{R}^n)$ of the energy $\mathscr{F}_{\mu,\kappa}(\cdot)$, with fixed $B_r(x_0) \subseteq \Omega$, we consider the minimizer w of the corresponding autonomous functional (on the ball $B_r(x_0)$)

$$\mathscr{F}_{\mu,0}\big(v, B_r(x_0)\big) := \int_{B_r(x_0)} f_\mu(e(v))dx \tag{4.19}$$

on $u + W^{1,p}_0(B_r(x_0), \mathbb{R}^n)$. This implies $\big(e(u)\big)_{B_r(x_0)} = \big(e(w)\big)_{B_r(x_0)}$.

Lemma 4.8. *Let $n \geq 3$, $p \in (1,\infty)$, κ and $\mu > 0$, $B_r(x_0) \subseteq \Omega$. Let u be a local minimizer of $\mathscr{F}_{\mu,\kappa}$ in (2.8) and w be defined as above.*

Then, there exists a constant $c = c(n,p) > 0$ such that for all symmetric matrices $\xi \in \mathbb{R}^{n\times n}_{\rm sym}$

$$\int_{B_r(x_0)} |V_\mu(e(w)) - V_\mu(\xi)|^2 dx \leq c\int_{B_r(x_0)} |V_\mu(e(u)) - V_\mu(\xi)|^2 dx, \tag{4.20}$$

and

$$\int_{B_r(x_0)} |V_\mu(e(u)) - V_\mu(e(w))|^2 dx \le c\big(\mathscr{F}_{\mu,0}(u, B_r(x_0)) - \mathscr{F}_{\mu,0}(w, B_r(x_0))\big). \tag{4.21}$$

Moreover, we have

$$\mathscr{E}_w(x_0, r) \le c_0\, \mathscr{E}_u(x_0, r) \tag{4.22}$$

for some constant $c_0 = c_0\big(n, p, \mu, M\big) > 0$*, provided that* $|(e(u))_{B_r(x_0)}| \le M$*.*

Proof. Note that for all symmetric matrices ξ, $\eta \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$

$$f_\mu(\eta) - f_\mu(\xi) - \langle \nabla f_\mu(\xi), \eta - \xi\rangle = \int_0^1 \langle \nabla^2 f_\mu(\xi + t(\eta - \xi))(\eta - \xi), \eta - \xi\rangle (1 - t) dt.$$

Therefore, from (2.3) and Lemmata 2.3, 2.4 we infer for some constant $c = c(p) > 0$

$$c^{-1}|V_\mu(\eta) - V_\mu(\xi)|^2 \le f_\mu(\eta) - f_\mu(\xi) - \langle \nabla f_\mu(\xi), \eta - \xi\rangle \le c|V_\mu(\eta) - V_\mu(\xi)|^2. \tag{4.23}$$

Since for all $\varphi \in W^{1,p}_0(B_r(x_0); \mathbb{R}^n)$

$$\int_{B_r(x_0)} \langle \nabla f_\mu(\xi), e(\varphi)\rangle dx = 0,$$

from the minimality of w for $\mathscr{F}_{\mu,0}(\cdot, B_r(x_0))$ and since $u - w \in W^{1,p}_0(\Omega; \mathbb{R}^n)$ we get that

$$\int_{B_r(x_0)} \big(f_\mu(e(w)) - f_\mu(\xi) - \langle \nabla f_\mu(\xi), e(w) - \xi\rangle\big) dx \\ \le \int_{B_r(x_0)} \big(f_\mu(e(u)) - f_\mu(\xi) - \langle \nabla f_\mu(\xi), e(u) - \xi\rangle\big) dx,$$

and (4.20) follows at once from (4.23).

For (4.21) we argue analogously: we use the minimality of w and the condition $u - w \in W^{1,p}_0(\Omega; \mathbb{R}^n)$, to infer for all $\varphi \in W^{1,p}_0(B_r(x_0); \mathbb{R}^n)$

$$\int_{B_r(x_0)} \langle \nabla f_\mu(e(w)), e(\varphi)\rangle dx = 0.$$

The conclusion follows at once by (4.23).

Finally, to prove (4.22) we use Lemma 2.5 (item (i) if $p \ge 2$, item (ii) if $p \in (1, 2)$) and (4.20) with $\xi = (e(u))_{B_r(x_0)} = (e(w))_{B_r(x_0)}$ to conclude that

$$\mathscr{E}_w(x_0, r) \le c \fint_{B_r(x_0)} |V_\mu(e(w)) - V_\mu(\xi)|^2 dx \le c \fint_{B_r(x_0)} |V_\mu(e(u)) - V_\mu(\xi)|^2 dx \le c\, \mathscr{E}_u(x_0, r).$$

for some constant $c = c\big(n, p, \mu, |\big(e(u)\big)_{B_r(x_0)}|\big) > 0$. □

We are now ready to extend the result of Section 4.3 to the non-autonomous case. Besides the sets $\Sigma^{(1)}_v$ introduced in (4.8) and $\Sigma^{(2)}_v$ in (4.9), in the framework under examination it is necessary to consider additionally the sets

$$\Sigma^{(3)}_v := \left\{x \in \Omega :\ \limsup_{r\downarrow 0} \fint_{B_r(x)} |v(y) - (v)_{B_r(x)}|^p dy > 0\right\} \\ \cup \left\{x \in \Omega :\ \limsup_{r\downarrow 0} |(v)_{B_r(x)}| = \infty\right\}, \tag{4.24}$$

and

$$\Sigma_v^{(4)} := \left\{ x \in \Omega : \limsup_{r \downarrow 0} |(\nabla v)_{B_r(x)}| = \infty \right\} \tag{4.25}$$

for all $v \in W^{1,p}(\Omega;\mathbb{R}^n)$. Note that $\Sigma_v^{(3)}$ is actually empty for exponents $p > n$. More generally we shall carefully estimate the Hausdorff dimension of such a set using Sobolev embedding and the results in Propositions 3.1 and 3.3.

Theorem 4.9. *Let $n \geq 3$, $p \in (1,\infty)$, κ and $\mu > 0$, $g \in W^{1,p} \cap L^\infty(\Omega;\mathbb{R}^n)$ if $p \in (2,\infty)$ and $g \in L^\infty(\Omega;\mathbb{R}^n)$ if $p \in (1,2]$. Let u be a local minimizer on $W^{1,p}(\Omega;\mathbb{R}^n)$ of $\mathscr{F}_{\mu,\kappa}$ in (2.8).*

Then, there exists an open set $\Omega_u \subseteq \Omega$ such that $u \in C^{1,\beta}_{\mathrm{loc}}(\Omega_u;\mathbb{R}^n)$ for all $\beta \in (0, 1/2)$. Moreover,

$$\dim_{\mathcal{H}}(\Omega \setminus \Omega_u) \leq (n - \widetilde{q}) \vee 0,$$

where $\widetilde{q} := q \wedge p^* \wedge 2^*$, $q > 2$ being the exponent in Proposition 4.1.

Proof of Theorem 4.9. In the current setting the singular and regular sets are defined respectively by $\Sigma_u := \Sigma_u^{(1)} \cup \Sigma_u^{(2)} \cup \Sigma_u^{(3)} \cup \Sigma_u^{(4)}$ and $\Omega_u := \Omega \setminus \Sigma_u$.

For the details of the estimation of the Hausdorff measures of the sets $\Sigma_u^{(i)}$'s, $i \in \{1,2\}$, we refer to the discussion in Theorem 4.7. Here we simply recall that by taking into account that $V_\mu(e(u)) \in W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ for some $q > 2$ (cf. Proposition 4.1), we get $\dim_{\mathcal{H}}(\Sigma_u^{(1)} \cup \Sigma_u^{(2)}) \leq (n-q) \vee 0$. Instead, for what concerns $\Sigma_u^{(i)}$'s, $i \in \{3,4\}$, as $u \in W^{2,p\wedge 2}(\Omega;\mathbb{R}^n)$ (see Propositions 3.1 and 3.3), by Sobolev embedding $u \in W^{1,p^*\wedge 2^*}(\Omega;\mathbb{R}^n)$, and then we deduce that $\dim_{\mathcal{H}}(\Sigma_u^{(3)} \cup \Sigma_u^{(4)}) \leq \big(n - (p^* \wedge 2^*)\big) \vee 0$ (cf. [28, Theorem 3.22]). In conclusion, the inequality $\dim_{\mathcal{H}}(\Omega \setminus \Omega_u) \leq (n - \widetilde{q}) \vee 0$ follows.

Next, we claim that the set Ω_u is open and that $u \in C^{1,\beta}_{\mathrm{loc}}(\Omega_u;\mathbb{R}^n)$ for all $\beta \in (0, 1/2)$. Let $x_0 \in \Omega_u$, then we may find an infinitesimal sequence of radii r_i and $M > 0$ such that

$$\limsup_{i \uparrow \infty} \left(|(\nabla u)_{B_{r_i}(x_0)}| \vee \left(\fint_{B_{r_i}(x_0)} |u|^p \, dy \right)^{1/p} \vee \left(\fint_{B_{r_i}(x_0)} |V_\mu(e(u))|^2 \, dy \right)^{1/2} \right) < M < \infty, \tag{4.26}$$

and that

$$\liminf_{i \uparrow \infty} \mathscr{E}_u(x_0, r_i) = 0. \tag{4.27}$$

Given $j \in \mathbb{N}$, ε, $\rho \in (0,1)$, and setting

$$\Omega_u^{j,\varepsilon,\rho} := \Big\{ x \in \Omega : |(\nabla u)_{B_r(x)}| \vee \left(\fint_{B_r(x)} |u|^p dy \right)^{1/p} \vee \left(\fint_{B_r(x)} |V_\mu(e(u))|^2 dy \right)^{1/2} < j, \\ \mathscr{E}_u(x,r) < \varepsilon \quad \text{for some } r \in (0, \rho \wedge \mathrm{dist}(x, \partial\Omega)) \Big\},$$

we conclude that $x_0 \in \Omega_u^{M,\varepsilon,\rho}$ for all choices of ε and ρ as above. Clearly, each $\Omega_u^{j,\varepsilon,\rho}$ is open and $\Omega_u \subseteq \cup_{j\in\mathbb{N}} \Omega_u^{j,\varepsilon(j),\rho(j)}$ for every choice of $\varepsilon(j)$, $\rho(j) \in (0,1)$. The rest of the proof is devoted to establish that $\cup_{j\in\mathbb{N}} \Omega_u^{j,\overline{\varepsilon}(j),\overline{\rho}(j)} \subseteq \Omega_u$, for suitable values of $\overline{\varepsilon}(j)$ and $\overline{\rho}(j)$ to be defined in what follows, and the claimed regularity for u on Ω_u.

To this aim let $x_0 \in \Omega_u^{M,\varepsilon,\rho}$, for some $M \in \mathbb{N}$, ε, $\rho \in (0,1)$, and $r \in (0, \rho \wedge \mathrm{dist}(x, \partial\Omega))$ be a radius corresponding to x_0 in the definition of $\Omega_u^{M,\varepsilon,\rho}$, i.e. such that

$$|(\nabla u)_{B_r(x_0)}| \vee \left(\fint_{B_r(x_0)} |u|^p dy \right)^{1/p} \vee \left(\fint_{B_r(x_0)} |V_\mu(e(u))|^2 dy \right)^{1/2} < M, \quad \mathscr{E}_u(x_0, r) < \varepsilon. \tag{4.28}$$

Consider the minimizer w of $\mathscr{F}_{\mu,0}(\cdot, B_r(x_0))$ on $u + W_0^{1,p}(B_r(x_0);\mathbb{R}^n)$. Since $(e(w))_{B_r(x_0)} = (e(u))_{B_r(x_0)}$, we get that

$$|(e(w))_{B_r(x_0)}| = |(e(u))_{B_r(x_0)}| < |(\nabla u)_{B_r(x_0)}| < M. \tag{4.29}$$

Moreover, from the proof of Theorem 4.7 we know that there exists $\eta(M)>0$ for which if

$$\mathscr{E}_w(x_0,r)<\eta(M)\,, \tag{4.30}$$

then $V_\mu(e(w))\in C^{0,\alpha}(B_r(x_0);\mathbb{R}^{n\times n}_{\rm sym})$ for every $\alpha\in(0,1)$, with

$$\mathscr{E}_w(x_0,\rho)\le c_1\Big(\frac{\rho}{r}\Big)^{2\alpha}\mathscr{E}_w(x_0,r) \tag{4.31}$$

for every $\rho\in(0,r)$, with $c_1=c_1(p,\alpha,M)>0$. Denoting by $c_0=c_0(n,\mu,p,M)>0$ the constant in (4.22) we first choose $\varepsilon<\frac{1}{c_0}(\eta(M)\wedge\eta(M+1))$.

Let us first check that for any $\alpha\in(0,1)$ there exist a constant $c=c(n,p,\alpha,\mu,M,\|g\|_{L^\infty(\Omega;\mathbb{R}^n)})>0$, and a radius $\rho_0=\rho_0(n,p)\in(0,1)$ satisfying the following: if $r\in(0,\rho_0\wedge{\rm dist}(x_0,\partial\Omega))$ we have for all $\tau>0$

$$\mathscr{E}_u(x_0,\tau r)\le c\,\tau^{2\alpha}\mathscr{E}_u(x_0,r)+c\tau^{-n}r \tag{4.32}$$

provided that $\varepsilon<\varepsilon_0=\varepsilon_0(p,\mu,\tau,M)\le\frac{1}{c_0}(\eta(M)\wedge\eta(M+1))$ (actually $\varepsilon_0:=\frac{1}{c_0}(\eta(M)\wedge\eta(M+1))$ for $p\ge 2$). Note that from (4.28) and from the choice $\varepsilon<\varepsilon_0$, inequalities (4.30) and (4.31) hold.

We divide the proof in different steps for ease of readability. We shall always distinguish the case $p\ge 2$ from $p\in(1,2)$.

Step 1. Proof of (4.32) for $p\ge 2$.

If $p\ge 2$, by item (iii) in Lemma 2.5 we obtain

$$\begin{aligned}|V_\mu\big(e(u)-(e(u))_{B_{\tau r}(x_0)}\big)|\le& c|V_\mu\big(e(w)-(e(w))_{B_{\tau r}(x_0)}\big)|\\&+c|V_\mu(e(u)-e(w))|+c|V_\mu\big((e(u))_{B_{\tau r}(x_0)}-(e(w))_{B_{\tau r}(x_0)}\big)|\end{aligned}$$

for some $c=c(p)>0$. Thus, by items (i) and (v) in Lemma 2.5 we infer

$$\begin{aligned}\mathscr{E}_u(x_0,\tau r)&\le c\,\mathscr{E}_w(x_0,\tau r)+c\fint_{B_{\tau r}(x_0)}|V_\mu(e(u-w))|^2dx\\&\overset{(4.31)}{\le}c\,c_1\,\tau^{2\alpha}\mathscr{E}_w(x_0,r)+c\fint_{B_{\tau r}(x_0)}|V_\mu(e(u-w))|^2dx\\&\overset{(4.22)}{\le}c_2\,\tau^{2\alpha}\mathscr{E}_u(x_0,r)+c_2\,\tau^{-n}\fint_{B_r(x_0)}|V_\mu(e(u-w))|^2dx\,,\end{aligned} \tag{4.33}$$

with $c_2=c_2(n,p,\mu,M)>0$. To estimate the last term we use (4.21) and the local minimality of u for $\mathscr{F}_{\mu,\kappa}$ to find for some $c_3=c_3(n,p)>0$ that

$$\begin{aligned}\int_{B_r(x_0)}&|V_\mu(e(u))-V_\mu(e(w))|^2dx\le c_3\big(\mathscr{F}_{\mu,0}(u,B_r(x_0))-\mathscr{F}_{\mu,0}(w,B_r(x_0))\big)\\&=c_3\big(\mathscr{F}_{\mu,\kappa}(u,B_r(x_0))-\mathscr{F}_{\mu,\kappa}(w,B_r(x_0))\big)+c_3\int_{B_r(x_0)}\big(|w-g|^p-|u-g|^p\big)dx\\&\le c_3\int_{B_r(x_0)}\big(|w-g|^p-|u-g|^p\big)dx.\end{aligned} \tag{4.34}$$

In view of the elementary inequality

$$\big||z_1|^p-|z_2|^p\big|\le p(|z_1|^{p-1}+|z_2|^{p-1})|z_1-z_2| \tag{4.35}$$

for all $z_i\in\mathbb{R}^n$, together with Hölder's, Korn's and Young's inequalities, we may proceed as follows (in all the $L^p(B_\rho(x_0);\mathbb{R}^k)$ norms in the ensuing formula $k\in\{n,n\times n\}$, for the sake of notational simplicity we write only L^p):

$$\begin{aligned}
&\int_{B_r(x_0)} \big(|w-g|^p - |u-g|^p\big)dx \\
&\qquad \le c(p) \int_{B_r(x_0)} \big(|w|^{p-1} + |u|^{p-1} + |g|^{p-1}\big)|u-w|dx \\
&\qquad \le c(p)\big(\|w-u\|_{L^p}^{p-1} + \|u\|_{L^p}^{p-1} + \|g\|_{L^p}^{p-1}\big)\|u-w\|_{L^p} \\
&\qquad \le c(p)\, c_{\mathrm{Korn}}\, r\big(\|u\|_{L^p}^{p-1} + \|g\|_{L^p}^{p-1}\big)\|e(u-w)\|_{L^p} + c(p)c_{\mathrm{Korn}} r^p \|e(u-w)\|_{L^p}^p \\
&\qquad \le c(p)\, c_{\mathrm{Korn}} \Big(r^{n+1}\big(\fint_{B_r(x_0)} |u|^p dy + \|g\|_{L^\infty}^p\big) + r\|e(u-w)\|_{L^p}^p + r^p\|e(u-w)\|_{L^p}^p\Big) \\
&\qquad \le c_4\, c_{\mathrm{Korn}}\big(r^{n+1}M^p + r\|e(u-w)\|_{L^p}^p\big) && (4.36)
\end{aligned}$$

where $c_4 = c_4(p) > 0$, and we assumed without loss of generality that $M \ge \|g\|_{L^\infty}^p$ (recall that $r < 1$). Here $c_{\mathrm{Korn}} = c_{\mathrm{Korn}}(n,p) > 0$ is the best constant in the first Korn's inequality on the unit ball. Then from (4.34) and (4.36) we find

$$\int_{B_r(x_0)} |V_\mu(e(u)) - V_\mu(e(w))|^2 dx \le c_3 c_4\, c_{\mathrm{Korn}}\big(r^{n+1}M^p + r\|e(u-w)\|_{L^p}^p\big)\,. \tag{4.37}$$

Next, recalling that $p \ge 2$, by item (iv) in Lemma 2.5 we have

$$\int_{B_r(x_0)} |e(u-w)|^p dx \le \int_{B_r(x_0)} |V_\mu\big(e(u) - e(w)\big)|^2 dx\,, \tag{4.38}$$

and moreover by item (i) in the same Lemma 2.5

$$\int_{B_r(x_0)} |V_\mu\big(e(u) - e(w)\big)|^2 dx \le c_5 \int_{B_r(x_0)} |V_\mu(e(u)) - V_\mu(e(w))|^2 dx$$

for some constant $c_5 = c_5(p) > 0$. Hence, from the latter inequality, (4.37) and (4.38), if $r \le \rho_0 \le (2c_3c_4c_5c_{\mathrm{Korn}})^{-1}$, we find

$$\fint_{B_r(x_0)} |V_\mu\big(e(u) - e(w)\big)|^2 dx \le \frac{2}{\omega_n} c_3c_4c_5c_{\mathrm{Korn}} M^p r\,.$$

In turn, from this and (4.33) we get

$$\mathscr{E}_u(x_0, \tau r) \le c_2\, \tau^{2\alpha} \mathscr{E}_u(x_0, r) + \frac{2}{\omega_n} c_2c_3c_4c_5c_{\mathrm{Korn}} \tau^{-n} M^p r, \tag{4.39}$$

for every $\tau \in (0,1)$, provided $\varepsilon < \frac{1}{c_0}(\eta(M) \wedge \eta(M+1))$. Inequality (4.32) then follows at once.

Step 2. Proof of (4.32) for $p \in (1,2)$.

First, we have for some constant $c = c(p)$ (cf. (4.17))

$$|(e(u))_{B_{\tau r}(x_0)}| \le |(e(u))_{B_r(x_0)}| + c\tau^{-n}\big(\mu^{p/2} + |(e(u))_{B_r(x_0)}|^p + \mathscr{E}_u(x_0,r)\big)^{\frac{1}{p}-\frac{1}{2}}\big(\mathscr{E}_u(x_0,r)\big)^{\frac{1}{2}}. \tag{4.40}$$

Thus if $\varepsilon < \varepsilon_0 := 1 \wedge \frac{1}{c^2}\tau^{2n}(\mu^{p/2} + M^p + 1)^{1-2/p} \wedge \frac{1}{c_0}(\eta(M) \wedge \eta(M+1))$ we conclude that

$$|(e(u))_{B_{\tau r}(x_0)}| < M + 1.$$

Hence, we may use item (ii) in Lemma 2.5 to get for some constant $c_6 = c_6(p, M) > 0$

$$\mathscr{E}_u(x_0, \tau r) \le c_6 \fint_{B_{\tau r}(x_0)} |V_\mu(e(u)) - V_\mu((e(u))_{B_{\tau r}(x_0)})|^2 dx.$$

Thus, by item (ii) in Lemma 2.5 and by Lemma 2.6, denoting by $c_7 = c_7(n,p,\mu) > 0$ the constant there, we infer (recall that since $\varepsilon < \frac{1}{c_0}\eta(M)$ inequalities (4.30) and (4.31) hold true)

$$\begin{aligned}
\mathscr{E}_u(x_0,\tau r) &\le c_6c_7 \fint_{B_{\tau r}(x_0)} |V_\mu(e(u)) - \big(V_\mu(e(u))\big)_{B_{\tau r}(x_0)}|^2 dx \\
&\le 3c_6c_7 \fint_{B_{\tau r}(x_0)} |V_\mu(e(w)) - \big(V_\mu(e(w))\big)_{B_{\tau r}(x_0)}|^2 dx + 6c_6c_7 \fint_{B_{\tau r}(x_0)} |V_\mu(e(u)) - V_\mu(e(w))|^2 dx \\
&\le 3c_6c_7 \fint_{B_{\tau r}(x_0)} |V_\mu(e(w)) - V_\mu((e(w))_{B_{\tau r}(x_0)}))|^2 dx + 6c_6c_7 \fint_{B_{\tau r}(x_0)} |V_\mu(e(u)) - V_\mu(e(w))|^2 dx \\
&\le 3c(p)c_6c_7 \fint_{B_{\tau r}(x_0)} |V_\mu\big(e(w) - (e(w))_{B_{\tau r}(x_0)}\big)|^2 dx + 6c_6c_7 \fint_{B_{\tau r}(x_0)} |V_\mu(e(u)) - V_\mu(e(w))|^2 dx \\
&= 3c(p)c_6c_7\mathscr{E}_w(x_0,\tau r) + 6c_6c_7 \fint_{B_{\tau r}(x_0)} |V_\mu(e(u)) - V_\mu(e(w))|^2 dx \\
&\overset{(4.31)}{\le} 3c(p)c_6c_7c_1\,\tau^{2\alpha}\mathscr{E}_w(x_0,r) + 6c_6c_7 \fint_{B_{\tau r}(x_0)} |V_\mu(e(u)) - V_\mu(e(w))|^2 dx \\
&\overset{(4.22)}{\le} c_8\,\tau^{2\alpha}\mathscr{E}_u(x_0,r) + c_8\tau^{-n} \fint_{B_r(x_0)} |V_\mu(e(u)) - V_\mu(e(w))|^2 dx\,,
\end{aligned} \tag{4.41}$$

with $c_8 = c_8(n,p,\mu,M) > 0$. The last term is bounded arguing exactly as in the superquadratic case: from (4.34) and (4.36) we get (4.37) (recalling that $\|g\|_{L^\infty} < M$), i.e.,

$$\int_{B_r(x_0)} |V_\mu(e(u)) - V_\mu(e(w))|^2 dx \le c_3c_4c_{\mathrm{Korn}}\big(r^{1+n}M^p + r\|e(u-w)\|^p_{L^p}\big)\,. \tag{4.42}$$

Next, Lemma 2.3, Hölder's and Young's inequalities imply for all $p\in(1,2)$

$$\begin{aligned}
\int_{B_r(x_0)} &|e(u-w)|^p dx \\
&\le \int_{B_r(x_0)} |V_\mu(e(u)) - V_\mu(e(w))|^2 dx + c_9 \int_{B_r(x_0)} \big(\mu + |e(u)|^2 + |e(w)|^2\big)^{p/2} dx\,,
\end{aligned}$$

where $c_9 = c_9(p) > 0$. Hence from the latter inequality and (4.42) we find for $r \le \rho_0(n,p) := (2c_3c_4c_{\mathrm{Korn}})^{-1}$

$$\begin{aligned}
\int_{B_r(x_0)} &|V_\mu\big(e(u)\big) - V_\mu\big(e(w)\big)|^2 dx \\
&\le 2c_3c_4c_{\mathrm{Korn}}\Big(r^{1+n}M^p + c_9 r \fint_{B_r(x_0)} \big(\mu + |e(u)|^2 + |e(w)|^2\big)^{p/2} dx\Big)\,.
\end{aligned} \tag{4.43}$$

Being u admissible to test the minimality of w, by (2.6) we have for some $c_{10} = c_{10}(\mu) > 0$

$$c_{10}^{-1}\int_{B_r(x_0)} (|e(w)|^p - 1)dx \le \mathscr{F}_{\mu,0}(w,B_r(x_0)) \le \mathscr{F}_{\mu,0}(u,B_r(x_0)) \le c_{10}\int_{B_r(x_0)} (|e(u)|^p + 1)dx\,.$$

Since, if $p\in(1,2)$, item (iv) in Lemma 2.5 yields for some $c_{11} = c_{11}(p) > 0$

$$\int_{B_r(x_0)} |e(u)|^p dx \le \int_{B_r(x_0)} |V_\mu(e(u))|^2 dx + c_{11}\mu^{p/2}r^n$$

we infer for some $c_{12} = c_{12}(n,p,\mu) > 0$

$$\int_{B_r(x_0)} \big(\mu + |e(u)|^2 + |e(w)|^2\big)^{p/2} dx \le c_{12}\Big(r^n + \int_{B_r(x_0)} |V_\mu(e(u))|^2 dx\Big) \le c_{12}r^n\big(1 + M^2\big)\,.$$

From this, (4.41) and (4.43) we get

$$\mathscr{E}_u(x_0,\tau r)\le c_8\,\tau^{2\alpha}\mathscr{E}_u(x_0,r)+\frac{2}{\omega_n}c_3c_4c_8c_{\mathrm{Korn}}\tau^{-n}(M^p+c_9c_{12}(1+M^2))r$$

provided $r<\rho_0\wedge 1\wedge \mathrm{dist}(x_0,\partial\Omega)$ with $\rho_0(n,p):=(2c_3c_4c_{\mathrm{Korn}})^{-1}$. Inequality (4.32) then follows at once.

Having established (4.32) for every $p\in(1,\infty)$, we proceed as follows. Fix $\alpha>1/2$, and let $0<\delta<1/2<\alpha$. Choose $\overline{\tau}=\overline{\tau}(\overline{c},\alpha)\in(0,1)$ such that $\overline{c}\,\overline{\tau}^{2\alpha-1}\le 1$, where $\overline{c}$ denotes the maximum of the constants in (4.32) for the bounds M and $M+1$ on the means. Thus, we have for all $\tau\in(0,\overline{\tau})$

$$\mathscr{E}_u(x_0,\tau r)\le \tau\mathscr{E}_u(x_0,r)+\overline{c}\tau^{-n}r. \tag{4.44}$$

We show next by induction that, with $\overline{\tau}$ as above, it is in fact possible to choose, in order, $\overline{\varepsilon}(M)$ and $\overline{\rho}(M)$ (here we highlight only the M dependence, for more details see Steps 3 and 4) such that for every $j\in\mathbb{N}$ we have

$$|(\nabla u)_{B_{\tau^jr}(x_0)}|\vee\Big(\fint_{B_{\tau^jr}(x_0)}|u|^pdy\Big)^{1/p}\vee\Big(\fint_{B_{\tau^jr}(x_0)}|V_\mu(e(u))|^2dy\Big)^{1/2}<M+1, \tag{4.45}$$

and

$$\mathscr{E}_u(x_0,\tau^jr)<\tau^j\mathscr{E}_u(x_0,r)+\overline{c}\tau^{-n}(\tau^{j-1}r)^{2\delta}\sum_{k=0}^{j-1}\tau^{(1-2\delta)k}, \tag{4.46}$$

provided that $\varepsilon\le\overline{\varepsilon}$, $\tau\le\overline{\tau}$ and $r<\rho\le\overline{\rho}$.

Given the latter inequalities for granted we conclude the proof. Indeed, by (4.45) and (4.46) it follows that $x_0\in\Omega_u$, so that $\cup_{j\in\mathbb{N}}\Omega_u^{j,\overline{\varepsilon}(j),\overline{\rho}(j)}\subseteq\Omega_u$. Moreover, items (iii) and (v) in Lemma 2.5, (4.46) and an elementary argument yield that

$$\mathscr{E}_u(x_0,t)\le\frac{c(p)}{\tau^n}\Big(\frac{\mathscr{E}_u(x_0,r)}{\tau\,r}\,t+\frac{\overline{c}}{\tau^{n+4\delta}(1-\tau^{1-2\delta})}\,t^{2\delta}\Big)\le c\,t^{2\delta}\,,$$

for all $t\in(0,r)$, since $\delta<1/2$ and $r<1$, with $c=c(p,\tau,r,\overline{c},\delta,\overline{\varepsilon})>0$. In addition, since by continuity (4.44) holds for all points in a ball $B_\lambda(x_0)$ with the same constants if $t\in(0,r\wedge\frac{1}{2}\mathrm{dist}(x_0,\partial\Omega))$, we deduce that $u\in C^{1,\beta}(B_\lambda(x_0);\mathbb{R}^n)$ for all $\beta\in(0,1/2)$. The result is thus proved.

Hence, to conclude we are left with showing the validity of (4.45) and of (4.46). As before we distinguish the superquadratic from the subquadratic case.

Step 3. Proof of (4.45) and (4.46).

Let us first prove the case $p\ge 2$. We start off deriving some useful estimates on the different means in (4.45). Let $j\in\mathbb{N}$, $j\ge 1$, then by Korn's inequality (denoting by $c_K=c_K(n,p)>0$ the best constant in such an inequality)

$$\begin{aligned}|(\nabla u)_{B_{\tau^jr}(x_0)}|&\le|(\nabla u)_{B_{\tau^{j-1}r}(x_0)}|+\big(\tau^{-n}\fint_{B_{\tau^{j-1}r}(x_0)}|\nabla u-(\nabla u)_{B_{\tau^{j-1}r}(x_0)}|^pdx\big)^{1/p}\\&\le|(\nabla u)_{B_{\tau^{j-1}r}(x_0)}|+\Big(c_K\tau^{-n}\fint_{B_{\tau^{j-1}r}(x_0)}|e(u)-(e(u))_{B_{\tau^{j-1}r}(x_0)}|^pdx\Big)^{1/p}\\&\le|(\nabla u)_{B_{\tau^{j-1}r}(x_0)}|+\big(c_K\tau^{-n}\mathscr{E}(x_0,\tau^{j-1}r)\big)^{1/p}.\end{aligned}$$

Therefore by a simple induction argument we conclude that

$$|(\nabla u)_{B_{\tau^jr}(x_0)}|\le|(\nabla u)_{B_r(x_0)}|+\sum_{k=0}^{j-1}\big(c_K\tau^{-n}\mathscr{E}(x_0,\tau^kr)\big)^{1/p}. \tag{4.47}$$

Analogously, by using Lemma 2.5 (i), we have

$$
\begin{aligned}
&\Big(\fint_{B_{\tau^j r}(x_0)} |V_\mu(e(u))|^2 dx\Big)^{1/2} \\
&\qquad \le |V_\mu\big((e(u))_{B_{\tau^{j-1}r}(x_0)}\big)| + \Big(\tau^{-n}\fint_{B_{\tau^{j-1}r}(x_0)} |V_\mu(e(u)) - V_\mu\big((e(u))_{B_{\tau^{j-1}r}(x_0)}\big)|^2 dx\Big)^{1/2} \\
&\qquad \le |V_\mu\big((e(u))_{B_{\tau^{j-1}r}(x_0)}\big)| + \big(c(\mu,K)\tau^{-n}\mathscr{E}_u(x_0,\tau^{j-1}r)\big)^{1/2},
\end{aligned}
$$

provided that $|(e(u))_{B_{\tau^{j-1}r}(x_0)}| \le K$. Therefore, using Lemma 2.5 (v) by induction

$$
\Big(\fint_{B_{\tau^j r}(x_0)} |V_\mu(e(u))|^2 dx\Big)^{1/2} \le |V_\mu\big((e(u))_{B_r(x_0)}\big)| + \sum_{k=0}^{j-1}\big(c(\mu,K)\tau^{-n}\mathscr{E}_u(x_0,\tau^k r)\big)^{1/2}, \tag{4.48}
$$

provided that $|(e(u))_{B_{\tau^k r}(x_0)}| \le K$ for all $0 \le k \le j-1$. Moreover, by Poincaré's and by Korn's inequalities we obtain for a constant $c_{KP} = c_{KP}(n,p) > 0$

$$
\begin{aligned}
\Big(\fint_{B_{\tau^j r}(x_0)} |u|^p dx\Big)^{1/p} &\le \Big(\fint_{B_{\tau^j r}(x_0)} |u - (u)_{B_{\tau^{j-1}r}(x_0)} - (\nabla u)_{B_{\tau^{j-1}r}(x_0)}\cdot(x-x_0)|^p dx\Big)^{1/p} \\
&\quad + \Big(\fint_{B_{\tau^j r}(x_0)} |(u)_{B_{\tau^{j-1}r}(x_0)} + (\nabla u)_{B_{\tau^{j-1}r}(x_0)}\cdot(x-x_0)|^p dx\Big)^{1/p} \\
&\le \tau^{j-1}r\Big(c_{KP}\tau^{-n}\fint_{B_{\tau^{j-1}r}(x_0)} |\nabla u - (\nabla u)_{B_{\tau^{j-1}r}(x_0)}|^p dx\Big)^{1/p} + |(u)_{\tau^{j-1}r(x_0)}| + \tau^j r|(\nabla u)_{B_{\tau^{j-1}r}(x_0)}| \\
&\le \tau^{j-1}r\Big(c_{KP}^2\tau^{-n}\fint_{B_{\tau^{j-1}r}(x_0)} |e(u) - (e(u))_{B_{\tau^{j-1}r}(x_0)}|^p dx\Big)^{1/p} \\
&\quad + \Big(\fint_{B_{\tau^{j-1}r}(x_0)} |u|^p dx\Big)^{1/p} + \tau^j r|(\nabla u)_{B_{\tau^{j-1}r}(x_0)}| \\
&\le \tau^{j-1}r(c_{KP}^2\tau^{-n}\mathscr{E}_u(x_0,\tau^{j-1}r))^{1/p} + \Big(\fint_{B_{\tau^{j-1}r}(x_0)} |u|^p dx\Big)^{1/p} + \tau^j r|(\nabla u)_{B_{\tau^{j-1}r}(x_0)}|.
\end{aligned}
$$

Hence, by induction we conclude that

$$
\begin{aligned}
\Big(\fint_{B_{\tau^j r}(x_0)} |u|^p dx\Big)^{1/p} &\le \Big(\fint_{B_r(x_0)} |u|^p dx\Big)^{1/p} \\
&\quad + r\sum_{k=0}^{j-1}\tau^k\big(c_{KP}^2\tau^{-n}\mathscr{E}_u(x_0,\tau^k r)\big)^{1/p} + r\sum_{k=0}^{j-1}\tau^{k+1}|(\nabla u)_{B_{\tau^k r}(x_0)}|.
\end{aligned} \tag{4.49}
$$

Let us then check the basic induction step $j = 1$ for (4.45). Indeed, note that for (4.46) it has been established in Step 2 (see (4.32) and (4.44)). From (4.47) we find

$$
|(\nabla u)_{B_{\tau r}(x_0)}| \le |(\nabla u)_{B_r(x_0)}| + (c_K\tau^{-n}\mathscr{E}_u(x_0,r))^{1/p} \le M+1
$$

provided that $\varepsilon < c_K^{-1}\tau^n$. Moreover, from (4.48) we have

$$
\Big(\fint_{B_{\tau r}(x_0)} |V_\mu(e(u))|^2 dx\Big)^{1/2} \le |V_\mu\big((e(u))_{B_r(x_0)}\big)| + \big(c(\mu,M)\tau^{-n}\mathscr{E}_u(x_0,r)\big)^{1/2} < M+1,
$$

provided that $\varepsilon < c^{-1}(\mu,M)\tau^n$. In addition, from (4.49)

$$
\begin{aligned}
\Big(\fint_{B_{\tau r}(x_0)} |u|^p dx\Big)^{1/p} &\le \big(c_{KP}^2\tau^{-n}\mathscr{E}_u(x_0,r)\big)^{1/p} + \Big(\fint_{B_r(x_0)} |u|^p dx\Big)^{1/p} + \tau r|(\nabla u)_{B_r(x_0)}| \\
&\le \big(c_{KP}^2\tau^{-n}\mathscr{E}_u(x_0,r)\big)^{1/p} + M + \tau r M < M+1,
\end{aligned}
$$

by choosing $\varepsilon < 2^{-p}c_{KP}^2\tau^n$ and $r < (2M)^{-1}$. In conclusion, (4.45) is established for $j = 1$ and $\tau < \overline{\tau}(M,\alpha)$, if $\varepsilon < \varepsilon_1 := \varepsilon_0 \wedge c_K^{-1}\tau^n \wedge c^{-1}(\mu, M)\tau^n \wedge 2^{-p}c_{KP}^{-2}\tau^n$ and $\rho < \rho_1 := \rho_0 \wedge (2M)^{-1}$ (ε_0 and ρ_0 have been defined in Step 1).

Let now $j \in \mathbb{N}$, $j \geq 2$, and assume by induction that (4.45) and (4.46) hold for all $0 \leq k \leq j-1$. Then for such values of k we have

$$\mathscr{E}_u(x_0, \tau^k r) < \tau^k \mathscr{E}_u(x_0, r) + \frac{\overline{c}\tau^{-n}}{1-\tau^{1-2\delta}}(\tau^{k-1}r)^{2\delta} \tag{4.50}$$

and then

$$\sum_{k=0}^{j-1}\big(\mathscr{E}_u(x_0,\tau^k r)\big)^{1/p} < \frac{(\mathscr{E}_u(x_0,r))^{1/p}}{1-\tau^{1/p}} + \Big(\frac{\overline{c}\tau^{-n}}{1-\tau^{1-2\delta}}\Big)^{1/p}\frac{(\tau^{-1}r)^{2\delta/p}}{1-\tau^{2\delta/p}}. \tag{4.51}$$

Hence, having fixed $\tau \in (0, \overline{\tau}]$, we may choose $\varepsilon_2 = \varepsilon_2(\varepsilon_1, p, \tau) < \varepsilon_1$ and $\rho_2 = \rho_2(\varepsilon_1, \overline{c}, p, \delta) < \rho_1$ such that if $\widetilde{C} := c_K \vee c(\mu, M) \vee c_{KP}^2 \vee 1$ and $\rho < \rho_2$, $\varepsilon < \varepsilon_2$ we find

$$(\widetilde{C}\tau^{-n})^{1/p}\sum_{k=0}^{j-1}\big(\mathscr{E}_u(x_0,\tau^k r)\big)^{1/p} < \varepsilon_1. \tag{4.52}$$

In particular, the inductive hypothesis on (4.45), (4.47) and (4.52) yield

$$|(\nabla u)_{B_{\tau^j r}(x_0)}| \leq M + (c_K\tau^{-n})^{1/p}\sum_{k=0}^{j-1}\big(\mathscr{E}_u(x_0,\tau^k r)\big)^{1/p} < M+1. \tag{4.53}$$

In turn, by the inductive assumption $|(e(u))_{B_{\tau^k r}(x_0)}| \leq M+1$ for all $0 \leq k \leq j-1$, so that thanks to (4.48) and (4.52), as $1/p \wedge 1/2 = 1/p$, we infer

$$\Big(\fint_{B_{\tau^j r}(x_0)} |V_\mu(e(u))|^2 dx\Big)^{1/2} \leq M + (c(\mu,M)\tau^{-n})^{1/2}\sum_{k=0}^{j-1}\big(\mathscr{E}_u(x_0,\tau^k r)\big)^{1/2} < M+1. \tag{4.54}$$

Finally, in view of (4.49) and (4.52) for $\beta = 1/p$ we get

$$\Big(\fint_{B_{\tau^j r}(x_0)} |u|^p dx\Big)^{1/p} \leq M + r\,(c_{KP}^2\tau^{-n})^{1/p}\sum_{k=0}^{j-1}\big(\mathscr{E}_u(x_0,\tau^k r)\big)^{1/p} + \frac{rM}{1-\tau} < M + r\big(\varepsilon_1 + \frac{M}{1-\overline{\tau}}\big). \tag{4.55}$$

Thus we have concluded (4.45) for the index j provided that $\varepsilon < \varepsilon_2$ and $\rho < \rho_2 \wedge (\varepsilon_1 + \frac{M}{1-\overline{\tau}})^{-1}$.

Finally, we prove (4.46) for the index j as follows. From (4.51) we have $\mathscr{E}_u(x_0, \tau^{j-1}r) < \varepsilon$, so that by the inductive hypothesis on the means it turns out that $x_0 \in \Omega_u^{M+1,\varepsilon,\rho}$ with corresponding radius $\tau^{j-1}r$. Moreover, the choice $\varepsilon < \frac{1}{c_0}\eta(M+1)$ and the definition of $\overline{\tau}$ (cf. the paragraph right before (4.44)) imply that (4.44) itself hold with the radii $\tau^{j-1}r$, $\tau^j r$ in place of r, τr respectively. Thus, using the inductive assumption on (4.46) for $j-1$ we conclude

$$\begin{aligned}
\mathscr{E}_u(x_0,\tau^j r) &\leq \tau\mathscr{E}_u(x_0,\tau^{j-1}r) + \overline{c}\tau^{-n}\tau^{j-1}r \\
&\leq \tau^j\mathscr{E}_u(x_0,r) + \overline{c}\tau^{-n}(\tau^{j-1}r)^{2\delta}\sum_{k=1}^{j-1}\tau^{(1-2\delta)k} + \overline{c}\tau^{-n}\tau^{j-1}r \\
&\leq \tau^j\mathscr{E}_u(x_0,r) + \overline{c}\tau^{-n}(\tau^{j-1}r)^{2\delta}\sum_{k=0}^{j-1}\tau^{(1-2\delta)k},
\end{aligned}$$

since $\delta < 1/2$.

The proof of (4.45) and (4.46) in the case $p \in (1,2)$ is quite similar. Hence, we will highlight only the main differences. First, arguing as in (4.40) (cf. (4.16), (4.17)) and using Korn's inequality we have for some constant $c_K = c_K(n,p)$

$$
\begin{aligned}
|(\nabla u)_{B_{\tau^j r}(x_0)}| \leq & \, |(\nabla u)_{B_{\tau^{j-1} r}(x_0)}| \\
& + c_K \tau^{-n} \Big(\mu^{p/2} + |(\nabla u)_{B_{\tau^{j-1} r}(x_0)}|^p + \mathscr{E}_u(x_0, \tau^{j-1} r) \Big)^{\frac{1}{p} - \frac{1}{2}} (\mathscr{E}_u(x_0, \tau^{j-1} r))^{\frac{1}{2}}.
\end{aligned}
$$

Thus, by induction we infer that

$$
\begin{aligned}
|(\nabla u)_{B_{\tau^j r}(x_0)}| \leq & \, |(\nabla u)_{B_r(x_0)}| \\
& + c_K \tau^{-n} \sum_{k=0}^{j-1} \big(\mu^{p/2} + |(\nabla u)_{B_{\tau^k r}(x_0)}|^p + \mathscr{E}_u(x_0, \tau^k r) \big)^{\frac{1}{p} - \frac{1}{2}} (\mathscr{E}_u(x_0, \tau^k r))^{\frac{1}{2}}. \qquad (4.56)
\end{aligned}
$$

Analogously to the derivation of (4.48), by Lemma 2.5 (v) and (ii) we find

$$
\Big(⨍_{B_{\tau^j r}(x_0)} |V_\mu(e(u))|^2 dx \Big)^{1/2} \leq |V_\mu\big((e(u))_{B_r(x_0)}\big)| + (c(\mu, M)\tau^{-n})^{1/2} \sum_{k=0}^{j-1} \big(\mathscr{E}_u(x_0, \tau^k r) \big)^{1/2}. \qquad (4.57)
$$

Again, by Poincaré and Korn's inequalities we find for a constant $c_{KP} = c_{KP}(n,p) > 0$ (cf. the derivation of (4.49) and (4.16))

$$
\begin{aligned}
\Big(⨍_{B_{\tau^j r}(x_0)} |u|^p dx \Big)^{1/p} \leq & \, \tau^{j-1} r \Big(c_{KP}^2 \tau^{-n} ⨍_{B_{\tau^{j-1} r}(x_0)} |e(u) - (e(u))_{B_{\tau^{j-1} r}(x_0)}|^p dx \Big)^{1/p} \\
& + \Big(⨍_{B_{\tau^{j-1} r}(x_0)} |u|^p dx \Big)^{1/p} + \tau^j r |(\nabla u)_{B_{\tau^{j-1} r}(x_0)}| \\
\leq & \, \tau^{j-1} r \, (c_{KP}^2 \tau^{-n})^{1/p} \big(\mu^{p/2} + |(e(u))_{B_{\tau^{j-1} r}(x_0)}|^p + \mathscr{E}_u(x_0, \tau^{j-1} r) \big)^{\frac{1}{p} - \frac{1}{2}} (\mathscr{E}_u(x_0, \tau^{j-1} r))^{\frac{1}{2}} \\
& + \Big(⨍_{B_{\tau^{j-1} r}(x_0)} |u|^p dx \Big)^{1/p} + \tau^j r |(\nabla u)_{B_{\tau^{j-1} r}(x_0)}|.
\end{aligned}
$$

Therefore, by induction we conclude that

$$
\begin{aligned}
& \Big(⨍_{B_{\tau^j r}(x_0)} |u|^p dx \Big)^{1/p} \leq \Big(⨍_{B_r(x_0)} |u|^p dx \Big)^{1/p} + r \sum_{k=0}^{j-1} \tau^{k+1} |(\nabla u)_{B_{\tau^k r}(x_0)}| \\
& + r \, (c_{KP}^2 \tau^{-n})^{1/p} \sum_{k=0}^{j-1} \tau^k \big(\mu^{p/2} + |(e(u))_{B_{\tau^k r}(x_0)}|^p + \mathscr{E}_u(x_0, \tau^k r) \big)^{\frac{1}{p} - \frac{1}{2}} (\mathscr{E}_u(x_0, \tau^k r))^{\frac{1}{2}}. \qquad (4.58)
\end{aligned}
$$

From (4.56)-(4.58) we easily deduce the basic induction step for (4.45), provided that we choose $\varepsilon < \varepsilon_0 \wedge (c_K^{-1} \tau^n)^2 (\mu^{p/2} + M^p + 1)^{1-2/p} \wedge c^{-1}(\mu, M)\tau^n \wedge 2^{-2}(c_{KP}^{-2} \tau^n)^{2/p} (\mu^{p/2} + M^p + 1)^{1-2/p}$ and $\rho < \rho_0 \wedge (2M)^{-1}$ (ε_0 and ρ_0 have been defined in Step 2). The general induction step $j \in \mathbb{N}$, $j \geq 2$, is now completely similar to the case $p \geq 2$. □

A Technical results

In this section we collect several technical tools we have used to settle partial regularity in the autonomous case. We recall that for sequences of scalars $\lambda_h \downarrow 0$ and of matrices $\mathbb{A}_h \to \mathbb{A}$ we set

$$
F_h(\xi) := \lambda_h^{-2} \big(f_\mu(\mathbb{A}_h + \lambda_h \xi) - f_\mu(\mathbb{A}_h) - \lambda_h \langle \nabla f_\mu(\mathbb{A}_h), \xi \rangle \big).
$$

Let us prove some properties of F_h.

Lemma A.1. *Let $p \in (1,\infty)$ and $\mu > 0$, then*

(i) $F_h \to F_\infty$ *in* $L^\infty_{\mathrm{loc}}(\mathbb{R}^{n\times n})$ *as* $h \uparrow \infty$, *where* $F_\infty(\xi) := \frac{1}{2}\langle \nabla^2 f_\mu(\mathbb{A})\xi, \xi\rangle$;

(ii) there exists $\omega : (0,+\infty) \to (0,+\infty)$ *non-decreasing such that* $\omega(t) \downarrow 0$ *as* $t \downarrow 0$ *and for every* $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ *with* $\lambda_h|\xi| \le 1$ *one has*

$$F_h(\xi) \ge F_\infty(\xi) - \omega(\lambda_h|\xi| + |\mathbb{A}_h - \mathbb{A}|)|\xi|^2;$$

(iii) there exists a constant $c = c(\mu, M) > 1$, *with* $M \ge \sup_h |\mathbb{A}_h|$, *such that for all* $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$

$$\frac{c^{-1}}{\lambda_h^2}|V_\mu(\lambda_h\xi)|^2 \le F_h(\xi) \le \frac{c}{\lambda_h^2}|V_\mu(\lambda_h\xi)|^2;$$

(iv) there exists a constant $c(p,\mu) > 0$ *such that for all* ξ, $\eta \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$

$$\begin{aligned} F_h(\xi) - F_h(\eta) \ge & \frac{c}{\lambda_h^2}|V_\mu(\mathbb{A}_h + \lambda_h\xi) - V_\mu(\mathbb{A}_h + \lambda_h\eta)|^2 \\ & + \frac{1}{\lambda_h}\langle \nabla f_\mu(\mathbb{A}_h + \lambda_h\eta) - \nabla f_\mu(\mathbb{A}_h), (\xi - \eta)\rangle; \end{aligned}$$

If, additionally, $\lambda_h|\eta| \le \mu$ *then for some constant* $c(p,\mu,M) > 0$, *with* $M \ge \sup_h |\mathbb{A}_h|$, *we have*

$$F_h(\xi) - F_h(\eta) \ge \frac{c}{\lambda_h^2}|V_\mu(\lambda_h(\xi-\eta))|^2 + \frac{1}{\lambda_h}\langle \nabla f_\mu(\mathbb{A}_h + \lambda_h\eta) - \nabla f_\mu(\mathbb{A}_h), (\xi - \eta)\rangle.$$

Proof. It suffices to take into account the representation formula

$$F_h(\xi) = \int_0^1 \langle \nabla^2 f_\mu(\mathbb{A}_h + t\lambda_h\xi)\xi, \xi\rangle (1-t)dt \tag{A.1}$$

to establish items (i) and (ii).

To prove (iii) first we notice the basic inequalities:

$$\frac{\mu}{2(\mu + M^2)}(\mu + |s\,\xi|^2) \le \mu + |\mathbb{A}_h + s\,\xi|^2 \le 2\frac{\mu + M^2}{\mu}(\mu + |s\,\xi|^2), \tag{A.2}$$

where $M \ge \sup_h |\mathbb{A}_h|$ and $s \in \mathbb{R}$. Thus, from (2.3), (A.1) and (A.2) we deduce if $p \in (2,\infty)$

$$\frac{1}{c}\Big(\frac{\mu}{2(\mu + M^2)}\Big)^{\frac{p}{2}-1}\lambda_h^{-2}|V_\mu(\lambda_h\xi)|^2 \le F_h(\xi) \le c\Big(2\frac{\mu + M^2}{\mu}\Big)^{\frac{p}{2}-1}\lambda_h^{-2}|V_\mu(\lambda_h\xi)|^2$$

for some constant $c > 0$. The inequality on the left hand side follows by arguing as in Lemma 2.3. Analogously, the case with $p \in (1,2)$ holds with opposite inequalities. Instead, if $p = 2$ (iii) is trivial.

To prove (iv) a simple computation yields

$$\begin{aligned} F_h(\xi) - F_h(\eta) = & \int_0^1 \langle \nabla^2 f_\mu(\mathbb{A}_h + t\lambda_h(\xi - \eta) + \lambda_h\eta)(\xi - \eta), (\xi - \eta)\rangle (1-t)dt \\ & + \lambda_h^{-1}\langle \nabla f_\mu(\mathbb{A}_h + \lambda_h\eta) - \nabla f_\mu(\mathbb{A}_h), (\xi - \eta)\rangle. \end{aligned}$$

Therefore, the first inequality follows from (2.3) and Lemmas 2.3, 2.4. Instead, the second inequality follows by estimating the first term on the right hand side as for (iii). □

Consider $\mathscr{F}_h : L^p(B_1;\mathbb{R}^n) \times \mathcal{A}(B_1) \to [0,+\infty]$ defined by

$$\mathscr{F}_h(u,A) = \int_A F_h(e(u))dx \tag{A.3}$$

if $u \in W^{1,p}(B_1;\mathbb{R}^n)$, and $+\infty$ otherwise. Above, $\mathcal{A}(B_1)$ is the class of all open subsets of B_1. We shall write shortly $\mathscr{F}_h(u)$ for $\mathscr{F}_h(u,B_1)$. Let u_h be a local minimizer of $\mathscr{F}_h$, that is $\mathscr{F}_h(u_h) = \inf_{u_h+W_0^{1,p}(B_1;\mathbb{R}^n)} \mathscr{F}_h$, and moreover assume that

$$\fint_{B_1} u_h dx = 0, \ \fint_{B_1} \nabla u_h dx = 0, \ \text{and} \quad \sup_h \int_{B_1} \lambda_h^{-2}|V_\mu(\lambda_h e(u_h))|^2 dx < +\infty. \tag{A.4}$$

In view of (A.4) and item (v) in Lemma 2.5, it follows from Korn's inequality for N-functions in [17, Lemma 2.9] applied to $|V_\mu(\cdot)|^2$ (cf. item (v) in Lemma 2.5) that

$$\sup_h \int_{B_1} \lambda_h^{-2}|V_\mu(\lambda_h \nabla u_h)|^2 dx < +\infty. \tag{A.5}$$

Moreover, by item (v) in Lemma 2.5 an application of Poincaré's inequality for N-functions ([19, Theorem 6.5]) yields

$$\sup_h \int_{B_1} \lambda_h^{-2}|V_\mu(\lambda_h u_h)|^2 dx < +\infty. \tag{A.6}$$

where with abuse of notation we define $V_\mu : \mathbb{R}^n \to \mathbb{R}^n$ by the same formula used for matrices.

The ensuing result is instrumental to prove that actually $(u_h)_h$ converges to u_∞ strongly in $W^{1,p\wedge 2}_{\mathrm{loc}}(B_1;\mathbb{R}^n)$.

Theorem A.2. *Let $\mathscr{F}_\infty : L^2(B_1;\mathbb{R}^n) \times \mathcal{A}(B_1) \to [0,+\infty]$ be given by*

$$\mathscr{F}_\infty(u;A) = \frac{1}{2}\int_A \langle \nabla^2 f_\mu(\mathbb{A})e(u), e(u)\rangle \, dx \tag{A.7}$$

if $u \in W^{1,2}(B_1;\mathbb{R}^n)$ and $+\infty$ otherwise.

If $\mathscr{F}_h$ are the functionals in (A.3) and $(u_h)_h$ is the sequence in (A.4), then after extracting a subsequence $(u_h)_h$ converges weakly in $W^{1,p\wedge 2}(B_1;\mathbb{R}^n)$ to some function $u_\infty \in W^{1,2}(B_1;\mathbb{R}^n)$,

$$\liminf_{h\uparrow\infty} \mathscr{F}_h(u_h,B_r) \geq \mathscr{F}_\infty(u_\infty,B_r) \quad \text{for all } r \in (0,1], \tag{A.8}$$

and

$$\limsup_{h\uparrow\infty} \mathscr{F}_h(u_h,B_r) \leq \mathscr{F}_\infty(u_\infty,B_r) \quad \text{for } \mathcal{L}^1 \text{ a.e. } r \in (0,1). \tag{A.9}$$

Proof. First, we notice that, up to the extraction of a subsequence not relabeled for convenience, there exists $u_\infty \in W^{1,p\wedge 2}(B_1;\mathbb{R}^n)$ such that $(u_h)_h$ converges weakly in $W^{1,p\wedge 2}(B_1;\mathbb{R}^n)$ to u_∞ with

$$\fint_{B_1} u_\infty dx = \fint_{B_1} \nabla u_\infty dx = 0. \tag{A.10}$$

Indeed, for $p \geq 2$ from (A.4) we deduce that $\sup_h \|e(u_h)\|_{L^2(B_1;\mathbb{R}^n)} \leq c\,\mu^{1-p/2}$, thus the Korn's inequality, Poincarè inequality, and the fact that u_h and its gradient have null mean value (cf. (A.4)) provide the conclusion. We observe that (A.4) also implies that $e(\lambda_h^{1-2/p} u_h)$ is bounded in $L^p(B_1;\mathbb{R}^{n\times n})$; hence, possibly after extracting a further subsequence, we can assume that $\lambda_h^{1-2/p} u_h$ converges weakly in $W^{1,p}(B_1;\mathbb{R}^n)$ and pointwise almost everywhere to some function z. Since u_h converges pointwise to u_∞, we deduce that $z = 0$, and in particular $\lambda_h^{1-2/p} u_h \to 0$ in $L^p(B_1;\mathbb{R}^n)$.

Instead, in case $p \in (1,2)$, we first note that as $\lambda_h \in (0,1)$ we have

$$\int_{B_1} |V_\mu(e(u_h))|^2 dx \leq \int_{B_1} \lambda_h^{-2}|V_\mu(\lambda_h e(u_h))|^2 dx\,,$$

so that (A.4) implies $\sup_h \|e(u_h)\|_{L^p(B_1;\mathbb{R}^n)} < +\infty$. Arguing as in the previous case we establish the claimed result.

Next, we prove separately (A.8) and (A.9) in the super-quadratic and in the sub-quadratic case.

The super-quadratic case $p > 2$. We first prove the lower bound inequality for $r \in (0,1]$. Set $E_h := \{\lambda_h^{1/2}|e(u_h)| \geq 1\}$, then $\mathcal{L}^n(E_h) \downarrow 0$ and $e(u_h)\chi_{E_h^c} \rightharpoonup e(u_\infty)$ weakly in $L^2(B_1;\mathbb{R}^{n\times n})$ as $h \uparrow \infty$. Therefore, by (ii) in Lemma A.1

$$\begin{aligned}\mathscr{F}_h(u_h, B_r) \geq \int_{B_r\cap E_h^c} F_h(e(u_h))dx &\geq \int_{B_r\cap E_h^c} \big(F_\infty(e(u_h)) - \omega(\lambda_h^{1/2} + |\mathbb{A}_h - \mathbb{A}|)|e(u_h)|^2\big)dx \\ &\geq \int_{B_r} F_\infty(e(u_h)\chi_{E_h^c})dx - \omega(\lambda_h^{1/2} + |\mathbb{A}_h - \mathbb{A}|)\int_{B_1}|e(u_h)|^2dx,\end{aligned}$$

and thus by L^2 weak lower semicontinuity of $\mathscr{F}_\infty(\cdot, B_r)$ we conclude (A.8).

To prove the upper bound for all but countably many $r \in (0,1)$, we note that by Urysohn's property it suffices to show that for every subsequence $h_k \uparrow \infty$ we can extract $h_{k_j} \uparrow \infty$ such that

$$\limsup_{j\uparrow\infty} \mathscr{F}_{h_{k_j}}(u_{h_{k_j}}, B_r) \leq \mathscr{F}_\infty(u_\infty, B_r).$$

By Friederich's theorem there exists $z_j \in C^\infty(\overline{B_1};\mathbb{R}^n)$ such that $z_j \to u_\infty$ in $W^{1,2}(B_1;\mathbb{R}^n)$. Hence, given $h_k \uparrow \infty$ we can extract h_{k_j} such that

$$\lim_{j\uparrow\infty} \lambda_{h_{k_j}}^{p-2}\int_{B_1}\big(|\nabla z_j|^p + |z_j|^p\big)dx = 0,$$

and the measures $\nu_j := \lambda_{h_{k_j}}^{-2}|V_\mu\big(\lambda_{h_{k_j}}e(u_{h_{k_j}})\big)|^2\mathcal{L}^n \llcorner B_1$ converge weakly* in B_1 to some finite measure ν.

Let now $\rho \in (0,r)$ be fixed, let $\varphi \in \mathrm{Lip}\cap C_c(B_r;[0,1])$ be such that $\varphi|_{B_\rho} = 1$ and $\|\nabla\varphi\|_{L^\infty(B_1;\mathbb{R}^n)} \leq 2(r-\rho)^{-1}$ and set

$$w_j := \varphi z_j + (1-\varphi)u_{h_{k_j}}.$$

Then, $w_j \in u_{h_{k_j}} + W_0^{1,2}(B_1;\mathbb{R}^n)$ with $w_j \to u_\infty$ in $L^2(B_1;\mathbb{R}^n)$. Therefore, by local minimality of $u_{h_{k_j}}$ we get

$$\mathscr{F}_{h_{k_j}}(u_{h_{k_j}}, B_r) \leq \mathscr{F}_{h_{k_j}}(w_j, B_r) = \int_{B_\rho} F_{h_{k_j}}(e(z_j))dx + \int_{B_r\setminus B_\rho} F_{h_{k_j}}(e(w_j))dx.$$

Clearly, by generalized Lebesgue dominated convergence theorem

$$\limsup_{j\uparrow\infty}\int_{B_\rho} F_{h_{k_j}}(e(z_j))dx \leq \int_{B_\rho} F_\infty(e(u_\infty))dx,$$

and by items (ii) and (iii) in Lemma 2.5

$$\begin{aligned}\int_{B_r\setminus B_\rho} F_{h_{k_j}}(e(w_j))dx &\leq \frac{c}{\lambda_{h_{k_j}}^2}\int_{B_r\setminus B_\rho}|V_\mu(\lambda_{h_{k_j}}e(w_j))|^2dx \\ &\leq \frac{c}{\lambda_{h_{k_j}}^2}\int_{B_r\setminus B_\rho}\big(|V_\mu(\lambda_{h_{k_j}}e(u_{h_{k_j}}))|^2 + |V_\mu(\lambda_{h_{k_j}}e(z_j))|^2 + |V_\mu(\lambda_{h_{k_j}}\nabla\varphi\odot(u_{h_{k_j}} - z_j))|^2\big)dx \\ &\leq c\,\nu_j(B_r\setminus\overline{B_\rho}) + c\int_{B_r\setminus B_\rho}\big(|e(z_j)|^2 + \lambda_{h_{k_j}}^{p-2}|e(z_j)|^p\big)dx \\ &\quad + \frac{c}{(r-\rho)^p}\int_{B_r\setminus B_\rho}\big(|u_{h_{k_j}} - z_j|^2 + \lambda_{h_{k_j}}^{p-2}|u_{h_{k_j}} - z_j|^p\big)dx.\end{aligned}$$

Summarizing, if $r \in (0,1)$ and $\rho \in (0,r)$ are chosen such that $\nu(\partial B_r) = \nu(\partial B_\rho) = 0$, recalling that $u_h \to u$, $z_j \to u$ in $L^2(B_1;\mathbb{R}^n)$, and that $\lambda_h^{1-2/p} u_h \to 0$, $\lambda_{h_{k_j}}^{1-2/p} w_j \to 0$ in $L^p(B_1;\mathbb{R}^n)$, we have

$$\limsup_{j\uparrow\infty} \int_{B_r \setminus B_\rho} F_{h_{k_j}}(e(w_j))dx \le c\,\nu(B_r \setminus \overline{B_\rho}) + c\int_{B_r\setminus B_\rho} |e(u_\infty)|^2 dx.$$

Thus, if $\rho_l \uparrow r$, we conclude at once by an easy diagonalization argument.

The sub-quadratic case $p \le 2$. We first prove that $u_\infty \in W^{1,2}(B_1;\mathbb{R}^n)$. Set $E_h := \{\lambda_h^{1/2}|e(u_h)| \ge 1\}$, then $\mathcal{L}^n(E_h) \downarrow 0$ as $h \uparrow \infty$ and

$$(\mu+1)^{p/2-1}\int_{E_h^c} |e(u_h)|^2 dx \le \|\lambda_h^{-1} V_\mu(\lambda_h e(u_h))\|^2_{L^2(B_1;\mathbb{R}^n)}.$$

Therefore, up to a subsequence not relabeled, $(e(u_h)\chi_{E_h^c})_h$ converges weakly in $L^2(B_1;\mathbb{R}^{n\times n})$ to some function ϑ. Moreover, as for all $\varphi \in L^{\frac{p}{p-1}}(B_1;\mathbb{R}^{n\times n})$, $\varphi\chi_{E_h^c} \to \varphi$ in $L^{\frac{p}{p-1}}(B_1;\mathbb{R}^{n\times n})$, from the weak convergence of $(e(u_h))_h$ to $e(u_\infty)$ in $L^p(B_1;\mathbb{R}^{n\times n})$ we conclude

$$\int_{B_1}\langle\vartheta,\varphi\rangle dx = \lim_{h\uparrow\infty}\int_{B_1}\langle e(u_h)\chi_{E_h^c},\varphi\rangle dx = \lim_{h\uparrow\infty}\int_{B_1}\langle e(u_h),\varphi\chi_{E_h^c}\rangle dx = \int_{B_1}\langle e(u_\infty),\varphi\rangle dx,$$

in turn implying $\vartheta = e(u_\infty)$ $\mathcal{L}^n$ a.e. in B_1. Thus, by (A.10), Korn's inequality yields that $u_\infty \in W^{1,2}(B_1;\mathbb{R}^n)$.

The lower bound inequality in (A.8) for $r \in (0,1]$ follows by arguing exactly as to derive it in case $p \ge 2$.

If $p \in (1,2)$ the proof of (A.9) is similar to the super-quadratic case, though some additional difficulties arise. With fixed $r \in (0,1)$, by Urysohn's property it is sufficient to show that for every subsequence $h_k \uparrow \infty$ we can extract $h_{k_j} \uparrow \infty$ such that

$$\limsup_{j\uparrow\infty} \mathscr{F}_{h_{k_j}}(u_{h_{k_j}}, B_r) \le \mathscr{F}_\infty(u_\infty, B_r).$$

Given a sequence $h_k \uparrow \infty$ we can find a subsequence h_{k_j} and some finite measure ν, such that the measures $\nu_j := \lambda_{h_{k_j}}^{-2}|V_\mu(\lambda_{h_{k_j}} e(u_{h_{k_j}}))|^2\mathcal{L}^n \llcorner B_1$ converge weakly* on B_1 to ν.

Let now $\rho \in (0,r)$ and $\varphi \in \mathrm{Lip}\cap C_c(B_r;[0,1])$ be such that $\varphi|_{B_\rho} = 1$ and $\|\nabla\varphi\|_{L^\infty(B_r;\mathbb{R}^n)} \le 2(r-\rho)^{-1}$ and set

$$w_j := \varphi u_\infty + (1-\varphi)u_{h_{k_j}}.$$

Then, $w_j \in u_{h_{k_j}} + W_0^{1,2}(B_1;\mathbb{R}^n)$ with $w_j \to u_\infty$ in $L^p(B_1;\mathbb{R}^n)$. Moreover,

$$\mathscr{F}_{h_{k_j}}(u_{h_{k_j}}, B_r) \le \mathscr{F}_{h_{k_j}}(w_j, B_r) = \int_{B_\rho} F_{h_{k_j}}(e(u_\infty))dx + \int_{B_r\setminus B_\rho} F_{h_{k_j}}(e(w_j))dx.$$

Clearly, by Lebesgue dominated convergence theorem

$$\limsup_{j\uparrow\infty}\int_{B_\rho} F_{h_{k_j}}(e(u_\infty))dx \le \int_{B_\rho} F_\infty(e(u_\infty))dx,$$

and by item (iii) both in Lemma 2.5 and in Lemma A.1

$$\begin{aligned}\int_{B_r\setminus B_\rho} F_{h_{k_j}}(e(w_j))dx &\le \frac{c}{\lambda^2_{h_{k_j}}}\int_{B_r\setminus B_\rho}|V_\mu(\lambda_{h_{k_j}}e(w_j))|^2dx\\ &\le \frac{c}{\lambda^2_{h_{k_j}}}\int_{B_r\setminus B_\rho}|V_\mu(\lambda_{h_{k_j}}e(u_{h_{k_j}}))|^2dx+\frac{c}{\lambda^2_{h_{k_j}}}\int_{B_r\setminus B_\rho}|V_\mu(\lambda_{h_{k_j}}e(u_\infty))|^2dx\\ &\qquad+\frac{c}{(r-\rho)^2\lambda^2_{h_{k_j}}}\underbrace{\int_{B_r\setminus B_\rho}|V_\mu(\lambda_{h_{k_j}}(u_{h_{k_j}}-u_\infty))|^2dx}_{I_j:=}\\ &\le c\,\nu_j(B_r\setminus\overline{B_\rho})+c\int_{B_r\setminus B_\rho}|e(u_\infty)|^2dx+\frac{c}{(r-\rho)^2}\lambda^{-2}_{h_{k_j}}I_j.\end{aligned}$$

In order to estimate the last term we use a Lipschitz truncation in order to use Rellich's theorem separately on the part with quadratic growth and on the one with p-growth. Precisely, let $E_j := \{\lambda_{h_{k_j}}|\nabla(u_{h_{k_j}}-u_\infty)|>1\}$. Then there is a set F_j with $E_j\subset F_j\subset B_1$ such that $\lambda_{h_{k_j}}(u_{h_{k_j}}-u_\infty)$ is c-Lipschitz in $B_1\setminus F_j$ and [21, Theorem 3, Section 6.6.3]

$$|F_j|\le c\lambda^p_{h_{k_j}}\int_{E_j}|\nabla(u_{h_{k_j}}-u_\infty)|^pdx\le c\int_{E_j}|V_\mu(\lambda_{h_{k_j}}\nabla(u_{h_{k_j}}-u_\infty))|^2dx\le c\lambda^2_{h_{k_j}}.$$

Let w_j be a $c\lambda^{-1}_{h_{k_j}}$-Lipschitz extension of $u_{h_{k_j}}-u_\infty|_{B_1\setminus F_j}$. We estimate

$$\int_{B_1}|\nabla w_j|^2dx\le c\lambda^{-2}_{h_{k_j}}|F_j|+\int_{B_1\setminus F_j}\lambda^{-2}_{h_{k_j}}|V_\mu(\lambda_{h_{k_j}}\nabla(u_{h_{k_j}}-u_\infty))|^2dx\le c.$$

Therefore $(w_j)_j$ is bounded in $W^{1,2}(B_1;\mathbb{R}^n)$, and, since it converges (up to a subsequence) pointwise almost everywhere to zero, it converges also strongly in $L^2(B_1;\mathbb{R}^n)$ to zero. Consider now the difference $d_j = u_{h_{k_j}}-u_\infty-w_j$. We estimate

$$\begin{aligned}\int_{B_1}|\nabla d_j|^pdx&\le c\int_{E_j}|\nabla(u_{h_{k_j}}-u_\infty)|^pdx+c|F_j|\lambda^{-p}_{h_{k_j}}\\ &\le c\int_{E_j}\lambda^{-p}_{h_{k_j}}|V_\mu(\lambda_{h_{k_j}}\nabla(u_{h_{k_j}}-u_\infty))|^2dx+c\lambda^{2-p}_{h_{k_j}}\le c\lambda^{2-p}_{h_{k_j}}.\end{aligned}$$

Therefore $(\lambda^{1-2/p}_{h_{k_j}}d_j)_j$ is bounded in $W^{1,p}(B_1;\mathbb{R}^n)$ and converges in measure to zero, hence it converges also strongly in $L^p(B_1;\mathbb{R}^n)$ to zero. We finally estimate, recalling that for $p\le 2$ we have $|V_\mu(\xi)|^2\le c(|\xi|^2\wedge|\xi|^p)$,

$$\begin{aligned}\lambda^{-2}_{h_{k_j}}\int_{B_1}|V_\mu(\lambda_{h_{k_j}}(u_{h_{k_j}}-u_\infty))|^2dx&\le c\lambda^{-2}_{h_{k_j}}\int_{B_1}|V_\mu(\lambda_{h_{k_j}}w_{h_{k_j}})|^2dx\\ &+c\lambda^{-2}_{h_{k_j}}\int_{F_j}|V_\mu(\lambda_{h_{k_j}}d_{h_{k_j}})|^2dx\le c\int_{B_1}|w_{h_{k_j}}|^2dx+c\int_{B_1}\lambda^{p-2}_{h_{k_j}}|d_{h_{k_j}}|^pdx\end{aligned}$$

and see that each term in the right-hand side converges to zero.

Therefore, we deduce that

$$\limsup_{j\uparrow\infty}\lambda^{-2}_{h_{k_j}}I_j=0.$$

Thus, in conclusion provided $r\in(0,1)$ and $\rho\in(0,r)$ are such that $\nu(\partial B_r)=\nu(\partial B_\rho)=0$ we have

$$\limsup_{j\uparrow\infty}\int_{B_r\setminus B_\rho}F_{h_{k_j}}(e(w_j))dx\le c\,\nu(B_r\setminus\overline{B_\rho})+c\int_{B_r\setminus B_\rho}|e(u_\infty)|^2dx.$$

Thus, if $\rho_l\uparrow r$, we conclude by an easy diagonalization argument. □

We next deduce that u_∞ is actually the solution of a linear elliptic system.

Corollary A.3. *The limit function $u_\infty \in W^{1,2}(B_1;\mathbb{R}^n)$ satisfies*

$$\int_{B_1} \langle \nabla^2 f_\mu(\mathbb{A}_\infty) e(u_\infty), e(\varphi)\rangle dx = 0 \tag{A.11}$$

for all $\varphi \in C_c^\infty(B_1;\mathbb{R}^n)$.

Proof. Being u_h a local minimizer of $\mathscr{F}_h$, for all $\varphi \in C_c^\infty(B_1;\mathbb{R}^n)$ it holds

$$\lambda_h^{-1} \int_{B_1} \langle \nabla f_\mu(\mathbb{A}_h + \lambda_h e(u_h)) - \nabla f_\mu(\mathbb{A}_h), e(\varphi)\rangle dx = 0.$$

Consider the sets

$$E_h^+ := \{x \in B_1 : |\lambda_h e(u_h)| \geq \sqrt{\mu}\}, \quad E_h^- := \{x \in B_1 : |\lambda_h e(u_h)| < \sqrt{\mu}\}.$$

By the weak convergence of $(u_h)_h$ to u_∞ in $W^{1,p\wedge 2}(B_1;\mathbb{R}^n)$, we get

$$\mathcal{L}^n(E_h^+) \leq \mu^{-p/2\wedge 1} \int_{B_1} |\lambda_h e(u_h)|^{p\wedge 2} dx \leq c\lambda_h^{p\wedge 2}, \tag{A.12}$$

so that $\mathcal{L}^n(E_h^+) = o(\lambda_h)$ as $h \uparrow \infty$. Hence, we deduce that

$$\begin{aligned}
&\limsup_{h\uparrow\infty} \left| \lambda_h^{-1} \int_{E_h^+} \langle \nabla f_\mu(\mathbb{A}_h + \lambda_h e(u_h)) - \nabla f_\mu(\mathbb{A}_h), e(\varphi)\rangle dx \right| \\
&\overset{(2.7)}{\leq} \limsup_{h\uparrow\infty} \Big(c\,\frac{\mathcal{L}^n(E_h^+)}{\lambda_h} + c\lambda_h^{p-2} \int_{E_h^+} |e(u_h)|^{p-1} dx \Big) \\
&\leq c \limsup_{h\uparrow\infty} \lambda_h^{p-2} (\mathcal{L}^n(E_h^+))^{1/p} \Big(\int_{E_h^+} |e(u_h)|^p dx \Big)^{(p-1)/p} \\
&\leq c \limsup_{h\uparrow\infty} \lambda_h^{1-\frac{2}{p}} (\mathcal{L}^n(E_h^+))^{1/p} \Big(\int_{E_h^+} \lambda_h^{-2} |V_\mu(\lambda_h e(u_h))|^2 dx \Big)^{(p-1)/p} \\
&\overset{(A.4),(A.12)}{\leq} c \limsup_{h\uparrow\infty} \lambda_h^{(2-\frac{2}{p})\wedge 1} = 0,
\end{aligned}$$

by taking into account item (iv) in Lemma 2.5 to infer the last but one inequality. Finally, note that

$$\begin{aligned}
&\lambda_h^{-1} \int_{E_h^-} \langle \nabla f_\mu(\mathbb{A}_h + \lambda_h e(u_h)) - \nabla f_\mu(\mathbb{A}_h), e(\varphi)\rangle dx \\
&\qquad = \int_{E_h^-} \langle \Big(\int_0^1 \nabla^2 f_\mu(\mathbb{A}_h + t\lambda_h e(u_h)) dt \Big) e(u_h), e(\varphi)\rangle dx,
\end{aligned}$$

then as $(u_h)_h$ converges weakly to u_∞ in $W^{1,p\wedge 2}(B_1;\mathbb{R}^n)$, $\lambda_h e(u_h) \to 0$ $\mathcal{L}^n$ a.e. on B_1, and as $f_\mu \in C^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ if $\mu > 0$, by the dominated convergence theorem we get

$$\lim_{h\uparrow\infty} \lambda_h^{-1} \int_{E_h^-} \langle \nabla f_\mu(\mathbb{A}_h + \lambda_h e(u_h)) - \nabla f_\mu(\mathbb{A}_h), e(\varphi)\rangle dx = \int_{B_1} \langle \nabla^2 f_\mu(\mathbb{A}_\infty) e(u_\infty), e(\varphi)\rangle dx.$$

□

In turn, this last result provides the claimed local strong convergence.

Corollary A.4. *Let $(u_h)_h$ be the sequence in (A.4) converging weakly in $W^{1,p\wedge 2}(B_1;\mathbb{R}^n)$ to the function $u_\infty \in W^{1,2}(B_1;\mathbb{R}^n)$. Then, for all $r \in (0,1)$*

$$\lim_{h\uparrow\infty}\int_{B_r}\lambda_h^{-2}|V_\mu(\lambda_h e(u_h-u_\infty)|^2dx = 0.$$

In particular, $(u_h)_h$ converges to u_∞ in $W^{1,p\wedge 2}_{\mathrm{loc}}(B_1;\mathbb{R}^n)$.

Proof. It is sufficient to show the conclusion for all those $r \in (0,1)$ for which both inequalities (A.8) and (A.9) in Theorem A.2 hold true. In such a case, we have

$$\lim_{h\uparrow\infty}\mathscr{F}_h(u_h,B_r) = \mathscr{F}_\infty(u_\infty,B_r).$$

We observe that $u_\infty \in C^\infty(B_1;\mathbb{R}^n)$ by Corollary A.3 and the regularity theory for linear elliptic systems. Therefore for h sufficiently large we have $\lambda_h|e(u_\infty)| < \mu$ uniformly on B_r. By item (iv) in Lemma A.1 we get

$$\begin{aligned}\mathscr{F}_h(u_h,B_r)-\mathscr{F}_h(u_\infty,B_r) \geq & c\int_{B_r}\lambda_h^{-2}|V_\mu(\lambda_h e(u_h-u_\infty)|^2dx \\ & + \frac{1}{\lambda_h}\int_{B_r}\langle\nabla f_\mu(\mathbb{A}_h+\lambda_h e(u_\infty))-\nabla f_\mu(\mathbb{A}_h),e(u_h-u_\infty)\rangle dx \\ = & c\int_{B_r}\lambda_h^{-2}|V_\mu(\lambda_h e(u_h-u_\infty)|^2dx \\ & + \int_{B_r}\langle\Big(\int_0^1\nabla^2 f_\mu(\mathbb{A}_h+t\lambda_h e(u_\infty))dt\Big)e(u_\infty),e(u_h-u_\infty)\rangle dx.\end{aligned}$$

Since $\mathscr{F}_h(u_\infty,B_r)\to\mathscr{F}_\infty(u_\infty;B_r)$ as $h\uparrow\infty$, and

$$\int_0^1\nabla^2 f_\mu(\mathbb{A}_h+t\lambda_h e(u_\infty))e(u_\infty)dt \to \int_0^1\nabla^2 f_\mu(\mathbb{A}_\infty)e(u_\infty)dt$$

in $L^\infty_{\mathrm{loc}}(B_1;\mathbb{R}^{n\times n})$, we conclude by the weak convergence of $(u_h)_h$ to u_∞ in $W^{1,p\wedge 2}(B_1;\mathbb{R}^{n\times n})$. □

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