Asymptotic expansion of solutions to the Cauchy problem for doubly degenerate parabolic equations with measurable coefficients

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Abstract

We study the asymptotic behaviour of nonnegative solutions of the Cauchy problem for doubly degenerate parabolic equations with variable coefficients. When the initial datum has a finite mass, the asymptotic expansion of the solution for a large time, uniformly in whole space, is established.

To celebrate the 70th birthday of our friend Juan Luis Vazquez

AMS Subject Classification (2010): Primary 35K65; Secondary 35K92, 35K10, 35B65.

Key Words: Asymptotic expansion, doubly degenerate parabolic equations, Cauchy problem.

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1 Introduction

We study the large time behaviour of a solution of the Cauchy problem for quasilinear degenerate parabolic equations of the form

$$(1.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)u^{m-1}\,|Du|^{p-2}\,\frac{\partial u}{\partial x_i}\right), \\ u(x,0) = u_0(x), \quad x \in \mathbf{R}^N. \end{cases}$$

Where , $(x,t) \in Q_T = \mathbf{R}^N \times (0,T), N \geq 1, \quad T > 0$. We assume that $m + p - 3 > 0, p > 1$ which means that (1.1) has a structure of the slow diffusion (see [22]). It is assumed that $a_{ij}(x,t) = a_{ji}(x,t)$ $i,j = 1,..,N$ are measurable functions and there exists $\nu > 1$ such that

$$(1.2) \qquad \nu^{-1}|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \leq \nu|\xi|^2$$

for any $\xi \in \mathbf{R}^N$, a.e. $x \in \mathbf{R}^N$, $u_0(x)$ is a non negative measurable function belonging to $L^1_{loc}(\mathbf{R}^N)$. If $a_{ij}(x,t) = \delta_{ij}, i,j = 1,..,N$, where δ_{ij} is the Kronecker symbol, then (1.1) reduces to the doubly degenerate parabolic equation:

$$(1.3) \qquad u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(u^{m-1}\,|Du|^{p-2}\,\frac{\partial u}{\partial x_i}\right).$$

In particular, if $p = 2$, then (1.3) is the porous media equation (or PME for short):

$$(1.4) \qquad u_t = \frac{1}{m}\Delta u^m$$

and, if $m = 1$, (1.3) is the nonstationary $p-$Laplacian:

$$(1.5) \qquad u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|Du|^{p-2}\,\frac{\partial u}{\partial x_i}\right).$$

Recall that (1.3) admits the one parameter family of Barenblatt's solutions which have a selfsimilar form

$$
(1.6) \qquad \begin{cases} E(x,t) = t^{-\alpha}\left(C - c(m,p,N)\,|\xi|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{m+p-3}}, \\ \text{with } \alpha = N/\beta, \beta = N(m+p-3)+p,\ \xi = xt^{-\frac{1}{\beta}}, \\ c(m,p,N) = \beta^{-1/(p-1)}(m+p-3)/p \end{cases}
$$

Here C is a free parameter. In particular, choosing C such that

$$
(1.7) \qquad \int_{\mathbf{R}^N} E(x,t)dx = M > 0.
$$

$E(x,t)$ satisfies the condition

$$
(1.8) \qquad E(x,0) = M\delta(x),
$$

where $\delta(x)$ is the Dirac measure. This is the reason why $E(x,t)$ is also called a fundamental solution (FS for short).

The function given in (1.6) suggests us the sharp bounds for the maximum, for the speed of propagation of the interface and for the admissible regularity of the solutions to degenerate parabolic equations (1.1). In what follows, we denote $E_M(x,t)$ the FS satisfying (1.7).

The purpose of this paper is to get the asymptotic expansion as $t \to \infty$ of a solution of (1.1) uniformly in the whole space when u_0 belongs to $L^1(\mathbf{R}^N)$.

Note that if define $v = u^\alpha$ where $\alpha = \dfrac{m+p-2}{m+1}$ and u is the solution of (1.1), we have that v satisfies the equation

$$
(v^{\frac{1}{\alpha}})_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\,|Dv|^{p-2}\,\frac{\partial v}{\partial x_i}\right).
$$

This equation is degenerate whenever $\dfrac{1}{\alpha} + 1 > p$ i.e. when $m+p > 3$ (for more details about this classification we refer the reader to ([37]).

Hence we have to understand (1.1)) in the weak sense. We say that $u(x,t) \geq 0$ is a weak solution of (1.1)) in Q_T, $T > 0$ if for any bounded domain Ω of $\mathbf{R}^N$

$$
(1.9) \qquad u \in C((0,T];\ L^1_{loc}(\mathbf{R}^N)) \text{ and } \int_0^T\int_\Omega u^{m-1}\,|Du|^{p-1}\,dxdt < \infty
$$

and for every test function $\varphi(x,t) \in W^{1,\infty}(0,T;L^\infty(\mathbf{R}^N)) \cap L^\infty(0,T;W_0^{1,\infty}(\mathbf{R}^N))$ vanishing on the hyperplane $\{t = T\}$

$$\iint_{Q_T} (-u\varphi_t + \sum_{i,j=1}^{N} a_{ij}(x,t) u^{m-1} |Du|^{p-2} u_{x_i} \varphi_{x_j}) dxdt = \tag{1.10}$$

$$= \int_{\mathbf{R}^N} u_0(x)\varphi(x,0)dx.$$

We call $u(x,t)$ a strong solution to (1.1) in Q_T, $T > 0$ if u is a weak solution to (1.1)) and $u_t \in L^1((0,T), L^1_{loc}(\mathbf{R}^N))$.
We say that u is a FS of (1.3) in Q_T with mass $M > 0$ if u is a nonnegative weak solution of (1.3), (1.8) in Q_T in the sense that the following identity holds for any test function φ as in (1.10)

$$\iint_{Q_T} (-u\varphi_t + \sum_{i=1}^{N} u^{m-1} |Du|^{p-2} u_{x_i} \varphi_{x_i}) dxdt = M\varphi(0,0). \tag{1.11}$$

Hence the initial conditions (1.8) is satisfied as

$$\lim_{t\to 0} \int_{\mathbf{R}^N} u(x,t)\psi(x)dx = M\psi(0) \tag{1.12}$$

for any continuous ψ with compact support in $\mathbf{R}^N$.
We have

Theorem 1.1 *Let $u(x,t)$ be a weak solution of the Cauchy problem (1.1) in Q_∞ with non negative initial datum belonging to $L^1(\mathbf{R}^N)$ and $\|u_0\|_1 = M$. Assume that $a_{ij}, i,j = 1,..,N$ satisfy (1.2) and*:

$$\lim_{\rho\to\infty} \rho^{-N-\beta} \int_0^{\rho^\beta} \int_{B_\rho(0)} |a_{ij}(y,\tau) - \delta_{ij}|^p \, dyd\tau = 0. \tag{1.13}$$

Then the following limit exists

$$\lim_{t\to\infty} t^\alpha |u(x,t) - E_M(x,t)| = 0 \tag{1.14}$$

uniformly in $x \in \mathbf{R}^N$. Where α and β are defined as in (1.6).

The asymptotic expansion in the parabola $P_a = \left\{|x| < at^{1/\beta}\right\}$ for the Cauchy problem of the PME when the initial datum has a finite mass is proved in ([16]). A similar result for the $p-$Laplacian equation was obtained in ([21]). The results concerning the asymptotic expansions in the whole space were treated in ([25]) for the $p-$Laplacian equation and in ([34]) for the PME. For semilinear parabolic equations we quote ([17]). We refer the reader to the monograph ([35]) for the qualitative properties of solutions to the PME. Let us quote also the interesting results proved in ([8]) (see also references therein) where the asymptotic representation of solutions in suitable $L^q(\mathbf{R}^N)$ norms has been obtained using the sharp constant in Sobolev type inequalities. In ([2]) it is proved the asymptotic expansion in parabola for the PME for various classes of initial data which are asymptotically powers.
Let us stress the fact that the results concerning asymptotic expansions deal with model equations only. Therefore one of the main motivations of this paper is to extend the results concerning asymptotic behaviour to equations with measurable coefficients. Uniqueness of FS is known for the PME ([28]) and for nonstationary $p-$Laplacian ([25]). In the case of PME the result holds for very general operator. This is not true in the case of the p-Laplacian where the result is known only for the prototype operator. The uniqueness of FS for a doubly degenerate parabolic equations (1.3) when $\beta = N(m+p-3)+p > 0$ can be proved following the approach of ([25]) as quoted in the recent paper ([1]). In the fast diffusion case (i.e. when $2 < m+p < 3$ and $p > 1$, for more details about the fast diffusion case see the monographs ([34]) and ([35])) the uniqueness result is due to ([30]). Finally, we refer the reader to ([27]) for the uniqueness of energy solutions (i.e. weak solutions with extraregularity on the initial datum) for doubly degenerate parabolic equations.
In particular, the approach of ([27]) can be adapted to (1.1) if $a_{ij}(x,t) = a_0(x,t)\delta_{ij}$ where a_0 is a measurable function such that $\nu^{-1} \le a_0(x,t) \le \nu$ a.e. $(x,t) \in Q_T$.

Note that assumption (1.13), which characterizes the behaviour of coefficients of (1.1) at infinity, is natural and comes from the linear case. It was introduced by Kamin ([23]) (see also ([15])) where similar conditions are used to get criteria of stabilization for linear parabolic equations with variable coefficients (in which case $\beta = 2$ and $p = 2$).

The proof of Theorem 1.1 is divided in two steps:
1) the asymptotic expansion in the parabola P_a

2) L^∞ estimates outside of the parabola P_a.

In the first step we use the rescaling arguments introduced by Kamin and used in several papers (see ([23]), ([24]), ([31]), ([2]), ([21]), ([15])). The asymptotic behaviour can also be studied using different approaches. For more details on this subject we refer the reader to ([34]) and ([35]).
In the second step we use essentially the energy approach introduced in ([5]) to prove a local energy estimates outside of P_a. An alternative proof for the PME can be found in ([34]) and ([25]).
Throughout the paper we consider the solution of (1.1) as a strong solution. For the weak solution of the problem it is necessary to proceed with the Steklov mollifier in time. This process is quite standard and is described for instance in ([26]). For simplicity we omit this standard procedure.
The structure of this paper is organized as follows. Section 2 deals with preliminary results. There we state some local and global estimates of solutions of (1.1) for $u_0 \in L^1(\mathbf{R}^N)$. In section 3 we adapt to our case a technique introduced by Vazquez ([34]) in the context of the PME. Finally section 4 is devoted to the proof of Theorem 1.1.
Throughout all the paper we will use symbols c, C, b and b_i for positive constants depending only on the parameters of the problem which may vary from line to line while α and β are always defined as in (1.6).

2 Preliminary results

Let $f(x) \in L^1_{loc}(\mathbf{R}^N)$ and $r > 0$. Denote

$$|||f|||_r = \sup_{R \geq r} R^{-\frac{\beta}{m+p-3}} \int\limits_{B_R(0)} |f| \, dx$$

Let us recall a result proved in ([18]).

Theorem 2.1 *Consider equation (1.1). Assume that the coefficients $a_{i,j}(x,t)$, $i, j = 1, .., N$ satisfy conditions (1.2) and that $u_0 \in L^1_{loc}(\mathbf{R}^N)$. Assume also that $|||u_0|||_r < \infty$, $r > 0$. Then there exists a time $T = T(u_0)$ and a weak solution $u(x,t)$ of (1.1) in Q_T such that $u \in C([0,T); L^1_{loc}(\mathbf{R}^N)$ and $u(x,t) \to u_0$ in $L^1_{loc}(\mathbf{R}^N)$.*

Moreover there exists a positive constant C_0 such that for any $0 < t < T_r(u_0)$, where

$$T_r(u_0) = C_0 \, |||u_0|||_r^{-(m+p-3)}$$

we have

1) $|||u|||_r \leq c_1 \; |||u_0|||_r \,,$

2) $\|u(t)\|_{\infty,B_R(0)} \leq c_2 t^{-N/\beta} R^{p/(m+p-3)} \, |||u_0|||_r^{p/\beta}$

Where the constants c_1 and c_2 depend only upon the data.

Under the above assumptions it is possible to prove the Hölder regularity of the solution, see ([19]) and ([29]). See ([9]), ([10]) when $p = 2$ or $m = 1$.

Theorem 2.2 *Assume we are in the same hypotheses of the previous Theorem. Then $u(x,t)$ is Hölder continuous in $Q_{\tau,T,R} = B_R(0) \times (\tau, T)$, $\tau > 0$ and the Hölder constant and the Hölder exponent depend only upon N, p, m, ν, τ, T, R and $|||u_0|||_r$.*

For Cauchy problem with initial datum in $L^1(\mathbf{R}^N)$ the optimal $L^1 - L^\infty$ estimates were obtained in ([36]) in the case of nonstationary $p-$Laplacian. In ([3]) and ([14]) similar results were proved if correspondingly $p = 2$ or $m = 1$. The gradient estimates and the Hölder continuity of gradient were proved in ([12]) for systems when $m = 1$. Note also that similar results are known for solutions of the Neumann problem in domains with non compact boundaries (see ([4])) and for solutions of the Cauchy problem for doubly degenerate parabolic equations with variable coefficients degenerating at infinity (the so called inhomogeneous density, see for instance ([32])).
Notice that if $u_0 \in L^1(\mathbf{R}^N)$ then $T_r(u_0) = \infty$. This clearly implies that (1.1), is globally solvable. Moreover for all $t > 0$ (see ([18]), ([4]), ([32]))

$$\|u(t)\|_\infty \leq c_2 t^{-N/\beta} \, \|u_0\|_1^{p/\beta} \quad . \tag{2.1}$$

The following result is proved in ([5]), Lemma 3.1. Such a result holds also for systems (see ([33])).

Proposition 2.1 *Assume we are in the same hypotheses of the previous Theorems. Let $0 < r_2 < r_1, 0 < t_2 < t_1 < t$. Define D_1 and D_2 as $D_i = (\mathbf{R}^N \backslash B_{r_i}) \times (t_i, t), i = 1, 2$. Then for all $h_1 > h_2 > 0$ and all $s > 0$*

$$\sup_{t_1<\tau<t} \int\limits_{\mathbf{R}^N\backslash B_{r_1}} (u-h_1)_+^{s+1}dx + \iint\limits_{D_1} \left| D(u-h_1)_+^{(m+p+s-2)/p} \right|^p dxd\tau$$

$$\leq c((t_1-t_2)^{-1} \iint\limits_{D_2} (u-h_2)_+^{s+1}dxd\tau$$

$$+\frac{1}{(r_1-r_2)^p} \left(\frac{h_1}{h_1-h_2}\right)^{(m-1)_+} \iint\limits_{D_2} (u-h_2)_+^{m+p+s-2}dxd\tau). \tag{2.2}$$

Where the constant c depends only on p, m and ν.

The following entropy estimates are proved in ([18]), ([3]), ([14])

Theorem 2.3 *Assume we are in the same hypotheses of the previous Theorems. Let $R > 0$ then*

$$\int\limits_0^\tau \int\limits_{B_R(0)} |Du|^{p-1} u^{m-1} dxdt \leq c\left(\|u_0\|_1, R\right)\tau^\sigma, \tag{2.3}$$

$$\int\limits_0^\tau \int\limits_{B_R(0)} udxdt \leq c\left(\|u_0\|_1, R\right)\tau^\sigma, \tag{2.4}$$

where $\sigma > 0$.

In the sequel we also need two interpolation inequalities and the celebrated Gagliardo-Nirenberg inequality (for both these results, see, for instance, Chapter 0 of [13]).

Lemma 2.1 *(see also [26])*
Let $\{Y_n\}_{n=1}^\infty$ be a sequence of equi-bounded positive numbers satisfying the recursive inequalities

$$Y_n \leq Cb^n Y_{n+1}^{1+\alpha}$$

where $C, b > 1$ and $\alpha > 0$ are given constants. If

$$Y_0 \leq C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}}$$

then $Y_n \to 0$ when $n \to +\infty$.

Lemma 2.2 *(see also [26])*
Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of equi-bounded positive numbers satisfying the recursive inequalities

$$Y_n \leq C b^n Y_{n+1}^{1-\alpha}$$

where $C, b > 1$ and $\alpha > 0$ are given constants. Then

$$Y_0 \leq \left(\frac{2C}{b^{\frac{\alpha-1}{\alpha}}} \right)^{\frac{1}{\alpha}}$$

Lemma 2.3 *Let $u \in W^{1,p}(\mathbb{R}^N)$ with $p > 1$. Let $0 < \mu < q < \frac{pN}{N-p}$ if $p < N$ and $0 < \mu < q$ otherwise*

$$\|u\|_q \leq C \|Du\|_p^a \|u\|_\mu^{1-a} \tag{2.5}$$

where C is a constant depending only by N, p, q, μ and

$$a = \Big(\frac{1}{\mu} - \frac{1}{q}\Big)\Big(\frac{1}{N} - \frac{1}{p} + \frac{1}{\mu}\Big)^{-1}.$$

Note that in general the Gagliardo-Nirenberg is stated with the assumption that $1 < \mu < q$. But the proof works also in the weaker assumption $0 < \mu < q$, even if, in general, the space $L^\mu(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$ are no longer Banach spaces.

3 Auxiliary results

Consider now the family of functions

$$w_k(x,t) = k^\alpha u(k^{1/\beta} x, kt). \tag{3.1}$$

For any k, $w_k(x,t)$ solves the problem

$$(3.2)\qquad \begin{cases} \dfrac{\partial w_k}{\partial t} = \displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}^{(k)}(x,t) w_k^{m-1} \left|Dw_k\right|^{p-2} \frac{\partial w_k}{\partial x_i} \right), \\ w_k(x,0) = k^{\alpha} u_0(k^{1/\beta} x). \end{cases}$$

with $(x,t) \in Q_T = \mathbf{R}^N \times (0,T), N \geq 1, T > 0$. Note that $a_{ij}^{(k)}(x,t) = a_{ij}(k^{1/\beta}x, kt)$ satisfy (1.2) with the same ν and $\|w_k(x,0)\|_1 = \|u_0(x,0)\|_1$. Hence by (2.1)

$$(3.3)\qquad \|w_k(t)\|_\infty \leq c_2 t^{-N/\beta} \|u_0\|_1^{p/\beta} \ .$$

Proposition 3.1 *Let $w_k(x,t)$ be a weak solution of the problem(3.2) in Q_∞ and $||u_0||_1 < \infty$, where α and β are defined in (1.6). Assume that (1.2) and (1.13) hold. Then there exists a sequence k_n and a continuous function $\widetilde{q}(x,t)$ in Q_T for any $T>0$ such that $w_{k_n} \to \widetilde{q}(x,t)$ uniformly in any compact subset of Q_T. Moreover*

$$(3.4)\qquad -\iint\limits_{Q_T} \widetilde{q}(x,t)\varphi_t dxdt + \iint\limits_{Q_T} |D\widetilde{q}|^{p-2}\, \widetilde{q}^{m-1} D\widetilde{q} D\varphi dxdt = \|u_0\|_1 \varphi(0,0).$$

for any $\varphi(x,t)$ in $C^\infty(Q_T)$ with compact support in $\mathbf{R}^N$ vanishing on the hyperplane $\{t=T\}$

Note that the proof works even if $u_0(x) \in L^1_{loc}(\mathbf{R}^N)$. In such a case,the right-hand side of (3.4) can be replaced by $\displaystyle\lim_{k_n\to\infty} \int\limits_{\mathbf{R}^N} w_{k_n}(x,0)\varphi(x,0)dx$ provided such a limit exists.

Proof.
Fix $R>0$. Let $\zeta(x)$ be a smooth cutoff function in a ball $B_R(0)$ such that $\zeta(x) = 1$ when $|x| < R/2$, $\zeta(x) = 0$ outside of $B_R(0)$ and $|D\zeta| \leq c/R$. Multiplying the equation (3.2) by $w_k^a \zeta^p$, $a>0$, and integrating by parts over $B_R(0) \times (\tau, T)$, we get

$$\frac{1}{a+1} \int\limits_{B_R(0)} w_k^{a+1}(\cdot,T)\zeta^p dx + a \int\limits_{\tau}^{T} \int\limits_{B_R(0)} \zeta^p w_k^{a-1} \left(\sum_{i,j=1}^{N} a_{ij}^{(k)}(x,t) \left|Dw_k\right|^{p-2} w_k^{m-1} w_{k,x_i} w_{k,x_j} \right) dxdt$$

$$= \frac{1}{a+1} \int\limits_{B_R(0)} w_k^{a+1}(\cdot,\tau)\zeta^p dx +$$

$$(3.5) \qquad -p \int\limits_\tau^T \int\limits_{B_R(0)} \zeta^{p-1} w_k^a \left(\sum_{i,j=1}^N a_{ij}^{(k)}(x,t) \left|Dw_k\right|^{p-2} w_k^{m-1} w_{k,x_i} \zeta_{x_j} \right) dxdt.$$

By (1.2), the second term in the left-hand side of (3.5) is bounded from below by

$$a\nu^{-1} \int\limits_\tau^T \int\limits_{B_R(0)} \zeta^p w_k^{m+a-2} \left|Dw_k\right|^p dxdt = a\nu^{-1} I_1,$$

while the second term in the right-hand side of (3.5) is bounded from above by

$$p\nu \int\limits_\tau^T \int\limits_{B_R(0)} \zeta^{p-1} w_k^{m+a-1} \left|Dw_k\right|^{p-1} \left|D\zeta\right| dxdt = p\nu I_2.$$

By Young's inequality

$$p\nu I_2 \le a\nu^{-1} I_1/2 + c(p,\nu) \int\limits_\tau^T \int\limits_{B_R(0)} \left|D\zeta\right|^p w_k^{m+p+a-2} dxdt.$$

Hence from (3.3) and (3.5) :

$$(3.6) \qquad \int\limits_\tau^T \int\limits_{B_R(0)} \zeta^p w_k^{m+a-2} \left|Dw_k\right|^p dxdt \le c\left(R, \left\|u_0\right\|_1\right) \tau^{-N(m+p+a-2)/\beta}.$$

Let be $\varphi(x,t)$ any $C^1(Q_{T,R})$,with compact support in $\mathbf{R}^N$ vanishing on the hyperplane $\{t = T\}$. Multiply both sides of the equation (3.2) by $\varphi(x,t)$ and integrate by parts to get

$$(3.7) \qquad \iint\limits_{Q_{T,R}} (-w_k \varphi_t + \sum_{i,j=1}^N a_{ij}^{(k)}(x,t) w_k^{m-1} \left|Dw_k\right|^{p-2} w_{k,x_i} \varphi_{x_j} dxdt =$$

$$= \int\limits_{B_R(0)} w_k(x,0)\varphi(x,0)dx.$$

Thanks to (3.3) and the regularity result of Theorem 2.2, $w_k(x,t)$ is uniformly bounded with respect to k and $w_k(x,t)$ is Hölder continuous in $Q_{\tau,T,R} = (\tau,T)\times B_R(0)$ for any $T>\tau>0$ with exponent and constant independent of k. Therefore there exists a sequence k_n still denoted by k and a continuous function $\widetilde{q}(x,t)$ such that $w_k \to \widetilde{q}(x,t)$ uniformly in any compact subset of $Q_{T,R}$.
In the next step, we will prove that $\widetilde{q}(x,t)$ is a weak solution of the limit problem; i.e. (3.4) holds.

In order to identify the limit equation for $\widetilde{q}(x,t)$, we split the second term in the left-hand side of (3.7) in the following way:

$$\iint\limits_{Q_{\tau,T,R}} \sum_{i,j=1}^{N}\left(a_{ij}^{(k)}(x,t)-\delta_{ij}\right)|Dw_k|^{p-2} w_k^{m-1} w_{k,x_i}\varphi_{x_j}dxdt + \iint\limits_{Q_{\tau,R,T}} |Dw_k|^{p-2} w_k^{m-1} Dw_k D\varphi dxdt +$$

$$+\int\limits_0^{\tau}\int\limits_{B_R(0)}\sum_{i,j=1}^{N} a_{ij}^{(k)}(x,t)\,|Dw_k|^{p-2} w_k^{m-1} w_{k,x_i}\varphi_{x_j}dxdt = A_{1,k}+A_{2,k}+A_{3,k}. \tag{3.8}$$

Apply Hölder's inequality to get

$$|A_{1,k}| \le c\left\{\int\limits_0^T\int\limits_{B_R(0)}\sum_{i,j=1}^{N}\left|a_{ij}^{(k)}(x,t)-\delta_{ij}\right|^p dxdt\right\}^{1/p} \tag{3.9}$$

$$\times\left\{\int\limits_{\tau}^{T}\int\limits_{B_R(0)} w_k^{(m-1)p/(p-1)}\,|Dw_k|^p\,|D\varphi|^{p/(p-1)}\,dxdt\right\}^{(p-1)/p} = cI_k^{1/p}J_k^{(p-1)/p}.$$

Going from $a_{ij}^{(k)}$ to a_{ij}, we have

$$I_k = k^{-\frac{N}{\beta-1}} \int\limits_0^{kT} \int\limits_{B_{\tilde{R}}(0)} \sum_{i,j=1}^{N} |a_{ij}(y,s) - \delta_{ij}|^p \, dyds.$$

where $\tilde{R} = k^{\frac{1}{\beta}} R$

Let $\gamma = \max \left\{ R^{-1} T^{1/\beta}, 1 \right\}$ and $\rho = \gamma k^{1/\beta} R$, then

$$I_k \leq (\gamma R)^{N+\beta} \rho^{-N-\beta} \int\limits_0^{\rho^\beta} \int\limits_{B_\rho(0)} \sum_{i,j=1}^{N} |a_{ij}(y,s) - \delta_{ij}|^p \, dyds.$$

Due to the assumption (1.13) $I_k \to 0$ as $k \to \infty$. On the other hand J_k is bounded uniformly with respect to k by (3.6) choosing $a = \dfrac{m+p-2}{p-1}$. Therefore $A_{1,k} \to 0$ as $k \to \infty$.

Using now the uniform convergence of w_k and Minty's monotonicity trick, we prove that

$$A_{2,k} \to \iint\limits_{Q_{\tau,T,R}} \widetilde{q}^{m-1} |D\widetilde{q}|^{p-2} D\widetilde{q} D\varphi dxdt \tag{3.10}$$

as $k \to \infty$.

First of all, note that by the L^∞ estimates (3.3), the pointwise convergence of w_k and the Lebesgue dominated convergence theorem we have

$$\int\limits_\tau^T \int\limits_{B_R(0)} -w_k \varphi_t dxdt \to \int\limits_\tau^T \int\limits_{B_R(0)} -\widetilde{q} \varphi_t dxdt$$

as $k \to \infty$.

Moreover by (3.6) with $a = \sigma = (m+p-2)/(p-1)$ it follows that $Dw_k^\sigma \to D\widetilde{q}^\sigma$ weakly in $L^p(Q_{\tau,T,R})$ and $|Dw_k^\sigma|^{p-2} Dw_k^\sigma \to \chi$ weakly in $L^{\frac{p}{p-1}}(Q_{\tau,T,R})$.

Hence, we get the following identity as $k \to \infty$

$$\int\limits_\tau^T \int\limits_{\mathbf{R}^N} (\widetilde{q}\varphi_t - \sigma^{-(p-1)} \chi D\varphi) dxdt = \tag{3.11}$$

$$= \int\limits_{\mathbf{R}^N} \left(\widetilde{q}(x,T)\varphi(x,T) - \widetilde{q}(x,\tau)\varphi(x,\tau) \right) dx$$

for any $\tau > 0$ and any smooth φ such that $\varphi(.,t) \in C_0^1(\mathbf{R}^N)$ for any $t \geq 0$. Thus it remains to prove that

$$\int\limits_{\tau}^{T} \int\limits_{\mathbf{R}^N} \left(\chi - \left| D\widetilde{q}^{\sigma} \right|^{p-2} D\widetilde{q}^{\sigma} \right) D\varphi dx dt = 0. \tag{3.12}$$

Notice that we have the monotonicity property of the diffusion part, that is for any $v \in L^p_{loc}(0,T; W^{1,p}(\mathbf{R}^N))$ and for every nonnegative test function φ, we have

$$\int\limits_{\tau}^{T} \int\limits_{\mathbf{R}^N} \varphi \left(\left| Dw_k^{\sigma} \right|^{p-2} Dw_k^{\sigma} - \left| Dv^{\sigma} \right|^{p-2} Dv^{\sigma} \right) \left(Dw_k^{\sigma} - Dv^{\sigma} \right) dx dt \geq 0. \tag{3.13}$$

Multiply both sides of (3.2), by $\varphi = \psi(x) w_k^{\sigma}$ where $\psi \in C_0^1(\mathbf{R}^N)$, $0 \leq \psi \leq 1$. Then integrate by parts over $\mathbf{R}^N \times (\tau, T)$ and get

$$\frac{1}{\sigma+1} \int\limits_{\mathbf{R}^N} w_k^{\sigma+1}(x,T)\psi dx - \frac{1}{\sigma+1} \int\limits_{\mathbf{R}^N} w_k^{\sigma+1}(x,\tau)\psi dx +$$

$$+ \int\limits_{\tau}^{T} \int\limits_{\mathbf{R}^N} \sigma^{-(p-1)} \psi \left| Dw_k^{\sigma} \right|^{p-2} Dw_k^{\sigma} Dw_k^{\sigma} dx dt +$$

$$+ \int\limits_{\tau}^{T} \int\limits_{\mathbf{R}^N} \sigma^{-(p-1)} w_k^{\sigma} \left| Dw_k^{\sigma} \right|^{p-2} Dw_k^{\sigma} D\psi dx dt + \gamma_k(T,\tau) = 0 \tag{3.14}$$

where $\gamma_k(T,\tau) \to 0$ as $k \to \infty$. Applying (3.13) to the third term in (3.14) and letting $k \to \infty$, we get

$$\frac{1}{\sigma+1} \int\limits_{\mathbf{R}^N} \widetilde{q}^{\sigma+1}(x,T)\psi dx - \frac{1}{\sigma+1} \int\limits_{\mathbf{R}^N} \widetilde{q}^{\sigma+1}(x,\tau)\psi dx$$

$$+\int_\tau^T\int_{\mathbf{R}^N}\sigma^{-(p-1)}\widetilde{q}^\sigma\chi D\psi dxdt+\int_\tau^T\int_{\mathbf{R}^N}\sigma^{-(p-1)}\psi\chi Dv^\sigma dxdt$$

$$+\int_\tau^T\int_{\mathbf{R}^N}\sigma^{-(p-1)}\psi\left|Dv^\sigma\right|^{p-2}Dv^\sigma(D\widetilde{q}^\sigma-Dv^\sigma)dxdt\le 0. \tag{3.15}$$

Now plug $\varphi=\psi(x)\widetilde{q}^\sigma$ in (3.11) to get

$$\frac{1}{\sigma+1}\int_{\mathbf{R}^N}\widetilde{q}^{\sigma+1}(x,T)\psi dx-\frac{1}{\sigma+1}\int_{\mathbf{R}^N}\widetilde{q}^{\sigma+1}(x,\tau)\psi dx$$

$$+\int_\tau^T\int_{\mathbf{R}^N}\sigma^{-(p-1)}\widetilde{q}^\sigma\chi D\psi dxdt+\int_\tau^T\int_{\mathbf{R}^N}\sigma^{-(p-1)}D\widetilde{q}^\sigma\chi\psi dxdt=0.$$

Therefore from (3.15) we deduce

$$\int_\tau^T\int_{\mathbf{R}^N}\psi(\left|Dv^\sigma\right|^{p-2}Dv^\sigma-\chi)(D\widetilde{q}^\sigma-Dv^\sigma)dxdt\le 0$$

for any $v^\sigma\in L^p_{loc}(0,T;W^{1,p}(\mathbf{R}^N))$. Let $v^\sigma=\widetilde{q}^\sigma-\theta$, $\forall\theta\in C^1_0(\mathbf{R}^N)$. Thus from the last inequality we conclude

$$\int_\tau^T\int_{\mathbf{R}^N}\psi(\left|Dv^\sigma\right|^{p-2}Dv^\sigma-\chi)D\theta dxdt\le 0.$$

Finally, noting that this inequality also holds for $-\theta$, we deduce that the distributional derivative of $\psi(\left|Dv^\sigma\right|^{p-2}Dv^\sigma-\chi)$ is equal to zero and this implies (3.12).

To estimate $A_{3,k}$ we apply Theorem 2.3. By estimates (2.3) and (2.4), we get

$$\int_0^\tau\int_{B_R(0)}\left|Dw_k\right|^{p-1}w_k^{m-1}dxdt\le c\left(\left\|u_0\right\|_1,R\right)\tau^\sigma, \tag{3.16}$$

$$(3.17) \qquad \int_0^\tau \int_{B_R(0)} w_k dxdt \le c\left(\|u_0\|_1, R\right)\tau^\sigma,$$

where $\sigma > 0$ and the constant c is independent on k. Therefore letting $k \to \infty$ and $\tau \to 0$ in (3.8) we have that also $A_{3,k}$ tends to zero. Hence passing to the limit with respect to k and τ in (3.7) we deduce (3.4).
Therefore Proposition 3.1 is proved. ■

4 Proof of Theorem1.1

The proof of Theorem1.1 contains two steps. In the first step we prove the asymptotic expansion in a parabola P_R using Proposition 3.1, the uniqueness of the FS and the equicontinuity property of $w_k(x,t)$. In the second step we prove the $L^1 - L^\infty$ bound of $w_k(x,t)$ outside of P_R.

The first step.
From (3.4) we obtain

$$-\iint_{Q_T} \widetilde{q}(x,t)\varphi_t dxdt + \iint_{Q_T} \widetilde{q}^{m-1}\left|D\widetilde{q}\right|^{p-2} D\widetilde{q}D\varphi dxdt = \varphi(0,0)\int_{\mathbf{R}^N} u_0(x)dx.$$

This identity means that $\widetilde{q}(x,t)$ is FS of (1.3). Since $E_M(x,t)$ is the unique FS of (1.3) we have $\widetilde{q}(x,t) = E_M(x,t)$ and $w_k \to E_M(x,t)$ for any sequence k. Due to the Hölder continuity result of ([19]) or ([29]), we deduce that $\{w_k(x,t)\}$ is relatively compact for any $t > \tau > 0$. Thus

$$|w_k(x,t) - E_M(x,t)| \to 0$$

uniformly on $B_R(0)$ as $k \to \infty$, for all $t \in [\varepsilon, T]$, $\varepsilon > 0$.
Therefore

$$\left|k^\alpha u(k^{\frac{1}{\beta}}x,t) - E_M(x,t)\right| \to 0 \text{ uniformly on } B_R(0) \text{ as } k \to \infty.$$

Set $t = 1$ and $k = t$, to get $\left|t^\alpha u(t^{\frac{1}{\beta}}x,t) - E_M(x,1)\right| \to 0$ as $t \to \infty$. Hence, defining $y = t^{\frac{1}{\beta}}x$, we have

$$t^{\alpha}\left|u(y,t)-E_M(y,t)\right|\to 0,\ \ |y|<Rt^{1/\beta} \tag{4.1}$$

for any $R>0$.

The second step.
This step is based on two lemmata.

Lemma 4.1 *Let $w_k(x,t)$ be a weak solution of problem (3.2) in Q_∞. Then under the conditions of Proposition 3.1 for any $t>0$*

$$\begin{aligned}&\|w_k(\cdot,t)\|_{\infty,|x|>R}\\ &\le c(N,m,p,\nu)\delta^{-b}\left[\sup_{0<\tau<t}\int\limits_{|x|>(1-\delta)R} w_k(\cdot,\tau)dx\right]^{\frac{p}{\beta}} t^{-\alpha},\end{aligned} \tag{4.2}$$

$0<\delta<1/4,\ b=b(N,m,p)$ *provided*

$$R\ge C^*(N,m,p,\nu)\left\|u_0\right\|_1^{(m+p-3)/\beta}t^{1/\beta}. \tag{4.3}$$

Lemma 4.2 *Under the assumptions of Lemma 4.1 with C^*- large enough and $0<t\le T<2$ the following estimate holds*

$$\sup_{0<\tau<T}\int\limits_{|x|>R} w_k(\cdot,\tau)dx\le 2\int\limits_{|x|>R/2} w_{0k}(x)dx. \tag{4.4}$$

Note that

$$\int\limits_{|x|>R/2} w_{0k}(x)dx=\int\limits_{|x|>k^{\frac{1}{\beta}}R/2} u_0(x)dx\to 0\ \ \text{as } k\to\infty.$$

Then, taking $t=1$ and $k=t$, it follows from (4.2) and (4.4) that

$$t^{\alpha}\left\|u(y,t)\right\|_{\infty,|y|>Ct^{1/\beta}}\to 0 \text{ as } t\to\infty.$$

Now taking into account that supp $E_M(x,t)\subset B_{R(t)}(0)$, where $R(t)=Ct^{1/\beta}$, we have

$$t^{\alpha}\left\|u(y,t)-E_M(y,t)\right\|_{\infty,|y|>Ct^{1/\beta}}\to 0 \tag{4.5}$$

as $t \to \infty$. Therefore, Theorem 1.1 is a consequence of (4.1) and (4.5).
So, in order to complete the proof of Theorem1.1 it remains to prove Lemmata 4.1 and 4.2.

Proof of Lemma 4.1
Let $0 < r_2 < r_1, 0 < t_2 < t_1 < t$. Define D_1 and D_2 as $D_i = (\mathbf{R}^N \backslash B_{r_i}) \times (t_i, t), i = 1, 2$. By Proposition 2.1, for all $h_1 > h_2 > 0$ and all $s > 0$

$$\sup_{t_1<\tau<t} \int\limits_{\mathbf{R}^N\backslash B_{r_1}} (w_k - h_1)_+^{s+1} dx + \iint\limits_{D_1} \left| D(w_k - h_1)_+^{(m+p+s-2)/p} \right|^p dxd\tau$$

$$\leq c((t_1 - t_2)^{-1} \iint\limits_{D_2} (w_k - h_2)_+^{s+1} dxd\tau$$

$$+\frac{1}{(r_1 - r_2)^p} \left(\frac{h_1}{h_1 - h_2} \right)^{(m-1)_+} \iint\limits_{D_2} (w_k - h_2)_+^{m+p+s-2} dxd\tau). \tag{4.6}$$

Let $0 < \sigma_1 < \sigma_2 < \frac{1}{2}$ which will be chosen later. Introduce the sequences :

$t_i := \dfrac{t}{2}(1 - \sigma_2) + t2^{-i}(\dfrac{\sigma_2 - \sigma_1}{2})$,
$R_i = R(1 - \sigma_2) + R2^{-i}(\sigma_2 - \sigma_1)$,
$l_i = l(1 - \sigma_2) + l2^{-i}(\sigma_2 - \sigma_1)$.

Note that
$t_0 = \frac{t}{2}(1 - \sigma_1)$ and $t_\infty = \frac{t}{2}(1 - \sigma_2)$,
$R_0 = R(1 - \sigma_1)$, $R_\infty = R(1 - \sigma_2)$,
$l_0 = l(1 - \sigma_1)$, $l_\infty = l(1 - \sigma_2)$.

Let $B_i := B_{R_i}$, $D_i = B_i \times (t_i, t)$.

Then from (4.6) with $t_1 = t_i$, $t_2 = t_{i+1}$, $r_1 = R_i$, $r_2 = R_{i+1}$, $h_1 = \bar{l}_i$, $h_2 = \bar{l}_{i+1}$ where $\bar{l}_i = (l_i + l_{i+1})/2$, we deduce

$$\sup_{t_i<\tau<t} \int\limits_{\mathbf{R}^N\backslash B_i} (w_k - \bar{l}_i)_+^{s+1} dx + \iint\limits_{D_i} \left| D(w_k - \bar{l}_i)_+^{\frac{m+p+s-2}{p}} \right|^p dxd\tau$$

$$\leq c(\sigma_2-\sigma_1)^{-p} b_1^i \Big(t^{-1} \iint_{D_{i+1}} (w_k-\bar{l}_{i+1})_+^{s+1} dx d\tau$$

$$+R^{-p} \iint_{D_{i+1}} (w_k-\bar{l}_{i+1})_+^{p+m+s-2} dx d\tau\Big) \tag{4.7}$$

Let η_i be a smooth function such that $0 \leq \eta_i \leq 1$, $\eta_i = 1$ in D_i and $\eta_i = 0$ outside of D_{i+1}, $|D\eta_i| \leq c2^i(\sigma_2-\sigma_1)^{-1}R^{-1}$, $0 \leq \eta_{it} \leq c2^i(\sigma_2-\sigma_1)^{-1}t^{-1}$. Plugging η_i in the previous inequality, we have

$$\iint_{D_i} \left| D\left((w_k-\bar{l}_i)_+^{\frac{m+p+s-2}{p}} \eta_{i-1} \right) \right|^p dx d\tau \tag{4.8}$$

$$\leq 2^{p-1} \iint_{D_i} \left| D(w_k-\bar{l}_i)_+^{\frac{m+p+s-2}{p}} \right|^p dx d\tau + 2^{p-1} \iint_{D_i} |D\eta_{i-1}|^p (w_k-\bar{l}_i)_+^{p+m+s-2} dx d\tau$$

$$\leq c(\sigma_2-\sigma_1)^{-p} b_1^i \left(t^{-1} \iint_{D_{i+1}} (w_k-\bar{l}_{i+1})_+^{s+1} dx d\tau + R^{-p} \iint_{D_{i+1}} (w_k-\bar{l}_{i+1})_+^{p+m+s-2} dx d\tau \right)$$

Define $v_i := (w_k-\bar{l}_i)_+^{\frac{m+p+s-2}{p}} \eta_{i-1}$. Since (3.3) holds, i.e. $\|w_k(t)\|_\infty \leq ct^{-\alpha} \|u_0\|_1^{p/\beta}$ then using (4.4) and assumption (4.3), from the previous inequality, we get

$$\sup_{t_i<\tau<t} \int_{\mathbf{R}^N\setminus B_i} v_i^q dx + \iint_{D_i} |Dv_i|^p dx d\tau \tag{4.9}$$

$$\leq c b_2^i t^{-1} \iint_{D_{i+1}} v_{i+1}^q dx d\tau$$

where $q = (s+1)p/(p+m+s-2)$

Next, apply the Gagliardo- Nirenberg inequality (2.5) with $\mu = \dfrac{p}{p+m+s-2}$ and

$$A = \frac{aq}{p} = \frac{Ns}{\beta+Ns}, \ \frac{(1-a)q}{\mu} = \frac{\beta+sp}{\beta+Ns},$$

integrate in time and apply Hölder inequality, to get

$$ct^{-1}b_2^i \iint\limits_{D_{i+1}} v_{i+1}^q dx d\tau \leq cb_2^i t^{-A} \left(\iint\limits_{D_{i+1}} |Dv_{i+1}|^p \, dx d\tau \right)^A \left(\sup_{t_{i+1}<\tau<t} \int\limits_{\mathbf{R}^N \setminus B_{i+1}} v_{i+1}^{\mu} dx \right)^{\frac{\beta+sp}{\beta+Ns}}$$

Let

$$Y_i := \sup_{t_i<\tau<t} \int\limits_{\mathbf{R}^N \setminus B_i} v_i^q dx + \iint\limits_{D_i} |Dv_i|^p \, dx d\tau$$

With this notation, we have

$$\begin{aligned} Y_i &\leq cb_2^i t^{-A} \left(\sup_{t_{i+1}<\tau<t} \int\limits_{\mathbf{R}^N \setminus B_{i+1}} v_{i+1}^{\mu} dx \right)^{\frac{\beta+sp}{\beta+Ns}} Y_{i+1}^A \\ &\leq ct^{-A} \left(\sup_{t_\infty<\tau<t} \int\limits_{\mathbf{R}^N \setminus B_\infty} v_\infty^{\mu} dx \right)^{\frac{\beta+sp}{\beta+Ns}} b_2^i Y_{i+1}^A \end{aligned}$$

Then using the iterative Lemma 2.2 with $b = b_2, \alpha = 1-A$, and

$$C = ct^{-A} \left(\sup_{t_\infty<\tau<t} \int\limits_{\mathbf{R}^N \setminus B_\infty} v_\infty^{\mu} dx \right)^{\frac{\beta+sp}{\beta+Ns}},$$

we obtain

$$Y_0 \leq ct^{-s\frac{N}{\beta}} \left(\sup_{t_\infty<\tau<t} \int\limits_{\mathbf{R}^N \setminus B_\infty} v_\infty^{\mu} dx \right)^{1+\frac{sp}{\beta}}$$

From this inequality it follows that

$$\sup_{t_0<\tau<t} \int\limits_{\mathbf{R}^N\setminus B_0} (w_k - \bar{l}_0)_+^{1+s} dx \tag{4.10}$$

$$\leq ct^{-s\frac{N}{\beta}} \left(\sup_{t_\infty<\tau<t} \int\limits_{\mathbf{R}^N\setminus B_\infty} (w_k - l_\infty)_+ dx \right)^{1+\frac{sp}{\beta}}$$

where $t_0 = \frac{t}{2}(1-\sigma_1)$, $t_\infty = \frac{t}{2}(1-\sigma_2)$, $\bar{l}_0 = l(1-\sigma_2)+\frac{3}{2}l(\sigma_2-\sigma_1)$, $l_\infty = l(1-\sigma_2)$, $R_0 = R(1-\sigma_1)$, $R_\infty = R(1-\sigma_2)$.
From the previous inequality, we deduce that

$$\sup_{t_0<\tau<t} \int\limits_{\mathbf{R}^N\setminus B_0} (w_k - l_0)_+ dx \leq \frac{4}{(\sigma_2-\sigma_1)^s\, l^s} \sup_{t_0<\tau<t} \int\limits_{\mathbf{R}^N\setminus B_0} (w_k - \bar{l}_0)_+^{1+s} dx \tag{4.11}$$

$$\leq c\frac{4}{(\sigma_2-\sigma_1)^s\, l^s} t^{-s\frac{N}{\beta}} \left(\sup_{t_\infty<\tau<t} \int\limits_{\mathbf{R}^N\setminus B_\infty} (w_k - l_\infty)_+ dx \right)^{1+\frac{sp}{\beta}}.$$

In order to realize a second iteration, choose $\sigma_1 = \delta 2^{-n-1}$, $\sigma_2 = \delta 2^{-n}$. With this choice $t_0 \to \frac{t}{2}(1-\delta 2^{-n-1})$, $t_\infty \to \frac{t}{2}(1-\delta 2^{-n})$, $R_0 \to R(1-\delta 2^{-n-1})$, $R_\infty \to R(1-\delta 2^{-n})$, $l_0 \to l(1-\delta 2^{-n-1})$, $l_\infty \to l(1-\delta 2^{-n})$. So set $t_n = \frac{t}{2}(1-\delta 2^{-n})$, $R_n = R(1-\delta 2^{-n})$, $l_n = l(1-\delta 2^{-n})$ to have

$$M_{n+1} := \sup_{t_{n+1}<\tau<t} \int\limits_{\mathbf{R}^N\setminus B_{n+1}} (w_k - l_{n+1})_+ dx \leq c\delta^{-s} b^n l^{-s} t^{-s\frac{N}{\beta}} M_n^{1+\frac{sp}{\beta}}$$

By the iterative Lemma 2.1 , we have $M_n \to 0$ as $n \to \infty$ if $\delta^{-1} l^{-1} t^{-\frac{N}{\beta}} M_0^{\frac{p}{\beta}} \leq c_0$ where c_0 is a sufficiently small constant depending only upon the data of the problem. Therefore $w_k \leq l$ and hence, Lemma 4.1 is proved choosing $l = 2c_0^{-1}\delta^{-1}t^{-\alpha}M_0^{\frac{p}{\beta}}$. ■

Proof of Lemma 4.2

Let $R_n = R(1-2^{-n-1})$, $\overline{R}_n = (R_n + R_{n+1})/2$ and $\zeta_n(x)$ be a smooth cutoff function $\zeta_n(x) = 1$, $|x| > R_{n+1}$, $\zeta_n(x) = 0$ outside of $|x| > \overline{R}_n$, $|D\zeta_n| \le c2^n R^{-1}$. Multiplying both sides of the equation (3.2) by $\zeta_n(x)^p$ and integrating by parts, we get

$$\int\limits_{\mathbf{R}^N\setminus\overline{B}_n} \zeta_n^p(x) w_k(x,t)dx - \int\limits_{\mathbf{R}^N\setminus\overline{B}_n} \zeta_n^p(x) w_k(x,0)dx$$

$$= -p \iint\limits_{Q_t} \zeta_n^{p-1}(x) w_k^{m-1} |Dw_k|^{p-2} \sum_{i,j=1}^N a_{ij}^k(x,t) w_{kx_i} \zeta_{nx_j} dxd\tau. \tag{4.12}$$

By using Hölder inequality we bound the right-hand side of (4.12) by

$$p\nu \left(\iint\limits_{Q_t} \zeta_n(x)^p \tau^\mu w_k^{-\theta} w_k^{m-1} |Dw_k|^p \, dxd\tau \right)^{(p-1)/p}$$

$$\times \left(\iint\limits_{Q_t} |D\zeta|^p \tau^{-(p-1)\mu} w_k^{(p-1)\theta} w_k^{m-1} dxd\tau \right)^{1/p} = J_1^{(p-1)/p} J_2^{1/p} \tag{4.13}$$

where $\theta = \frac{2-m}{p-1}$ and μ is a constant which satisfies the condition: $\frac{N}{\beta} < \mu < \frac{1}{p-1}$
To estimate J_1, consider the equation

$$\frac{\partial w_k}{\partial t} = \sum_{i,j=1}^N \frac{\partial}{\partial x_j} \left(a_{ij}^{(k)}(x,t) w_k^{m-1} |Dw_k|^{p-2} \frac{\partial w_k}{\partial x_i} \right),$$

choose as test function $\zeta_n^p(x)\tau^\mu w_k^{1-\theta}$ and integrate by parts, to get

$$\frac{1}{2-\theta} \int\limits_{\mathbf{R}^N\setminus\overline{B}_n} \zeta_n^p(x)\tau^\mu w_k^{2-\theta}(x,\tau)dx + (1-\theta) \iint\limits_{Q_t} \zeta_n^p(x)\tau^\mu w_k^{m-1-\theta} |Dw_k|^{p-2} \sum_{i,j=1}^N a_{ij}^k(x,\tau) w_{kx_i} w_{kx_j} dxd\tau$$

$$= \frac{\mu}{2-\theta} \iint\limits_{Q_t} \zeta_n^p(x)\tau^{\mu-1} w_k^{2-\theta}(x,\tau)dxdt - p \iint\limits_{Q_t} \zeta_n^{p-1}(x)\tau^\mu w_k^{m-\theta} |Dw_k|^{p-2} \sum_{i,j=1}^N a_{ij}^k(x,\tau) w_{kx_i} \zeta_{nx_j} dxd\tau$$

Then using the ellipticity of the coefficients and dropping the first term from the left-hand side, we obtain

$$(1-\theta)v^{-1}J_1 \leq \frac{\mu}{2-\theta}\iint\limits_{Q_t}\zeta_n^p(x)\tau^{\mu-1}w_k^{2-\theta}(x,\tau)dxd\tau + p\nu\iint\limits_{Q_t}\zeta_n^{p-1}(x)\tau^{\mu}w_k^{m-\theta}\left|Dw_k\right|^{p-1}\left|D\zeta_n\right|dxd\tau$$

On the other hand by Young inequality we have

$$p\nu\iint\limits_{Q_t}\zeta_n^{p-1}(x)\tau^{\mu}w_k^{m-\theta}\left|Dw_k\right|^{p-1}\left|D\zeta_n\right|dxdt \leq (p-1)\nu\epsilon^{\frac{p}{p-1}}J_1$$

$$+\nu\epsilon^{-p}\iint\limits_{Q_t}\tau^{\mu}w_k^{p+m-\theta-1}\left|D\zeta_n\right|^p dxdt.$$

Choosing ϵ such that: $(p-1)\nu\epsilon^{\frac{p}{p-1}} = p\nu/2$, we get

$$J_1 \leq c\iint\limits_{Q_t}\zeta_n^p(x)\tau^{\mu-1}w_k^{2-\theta}(x,\tau)dxd\tau + c\iint\limits_{Q_t}\tau^{\mu}w_k^{p+m-\theta-1}\left|D\zeta_n\right|^p dxd\tau \tag{4.14}$$

By Lemma 4.1, plugging in (4.2) $\delta = 1/(2^{n+2}-3)$ and $R = \overline{R}_n$ we have

$$\|w_k\|_{\infty,|x|>\overline{R}_n} \leq cb^n t^{-\frac{N}{\beta}}\left[\sup_{0<\tau<t}\int\limits_{|x|>R_n} w_k(x,\tau)dx\right]^{p/\beta}. \tag{4.15}$$

Therefore from (4.12), (4.14) (4.15) we get

$$J_1 \leq c\left[\sup_{0<\tau<t}\int\limits_{|x|>R_n} w_k dx\right]$$

$$\times\left(\int_0^t \|w_k\|_{\infty,|x|>\overline{R}_n}^{1-\theta}\tau^{\mu-1}d\tau + 2^{np}R^{-p}\int_0^t \|w_k\|_{\infty,|x|>\overline{R}_n}^{1-\theta+m+p-3}\tau^{\mu}d\tau\right),$$

and

$$J_2 \le b^n R^{-p} \left(\sup_{0<\tau<t} \int\limits_{|x|>R_n} w_k dx \right) \left(\int\limits_0^t \tau^{-(p-1)\mu} d\tau \right),$$

Hence, if we define

$$E_{n+1}(T) = \sup_{0<\tau<T} \int\limits_{|x|>R_{n+1}} w_k(\cdot,\tau) dx$$

by the previous estimates we deduce

$$E_{n+1}(T) \le \int\limits_{|x|>R/2} w_{0k} dx + b^n T^{1/\beta} R^{-1} E_n^{1+(m+p-3)/\beta}(T). \tag{4.16}$$

where $b = b(\mu, \theta, N, m, p) > 1$.
If

$$E_\infty \le 2 \int\limits_{|x|>R/2} w_{0k} dx$$

we have nothing to prove. If not, we have

$$E_\infty > 2 \int\limits_{|x|>R/2} w_{0k} dx$$

and this implies

$$E_{n+1}(T) \le 2 b^n T^{1/\beta} R^{-1} E_n^{1+(m+p-3)/\beta}(T).$$

Hence the interpolation Lemma 2.1 implies that $E_n \to 0$ as $n \to \infty$ provided

$$T^{1/\beta} R^{-1} E_0^{(m+p-3)/\beta} \le \epsilon(N, m, p).$$

Choosing now in (4.3) C^* large enough, we deduce that $E_\infty = 0$ that, in turn, implies the statement of Lemma 4.2. ■

Acknowledgement. The authors have been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).
We would like to express our gratitudine for the important remarks and comments made by Professor S. Kamin.

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