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HOW A MINIMAL SURFACE LEAVES A THIN OBSTACLE

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Abstract. We prove the optimal regularity and a detailed analysis of the free boundary of the solutions to the thin obstacle problem for nonparametric minimal surfaces with flat obstacles.

1. Introduction

The present note focuses on the analysis of the thin obstacle problem for nonparametric minimal surfaces. This is a well-known variational problem which has been extensively considered in the literature, cf. the classical works by Nitsche [45], Giusti [30, 31, 32], Kinderlehrer [36] and Frehse [24, 25]. In this respect, the vast literature on thin obstacle problems with quadratic energies, which correspond to the linearization of the area functional, has to be taken into account. Starting with the pioneering contributions by Lewy [40, 41], Richardson [46], Caffarelli [5], Kinderlehrer [37], and Ural'tseva [49, 51, 50], a renewed impulse towards a deeper understanding of the problem has started more recently with the works of Athanasopoulos and Caffarelli [1], Athanasopoulos, Caffarelli and Salsa [2], Caffarelli, Salsa and Silvestre [6] and has been then developed by many others [20, 21, 35, 26, 38] etc... we warn the readers that this is only a small excerpt from the literature on the topic. To complete the overview on the topic we also mention the parametric approach to minimal surfaces with thin obstacles, which has been started by De Giorgi (identifying the relaxation of the problem via the introduction of the nowadays called De Giorgi's measure) and developed in the monograph by De Giorgi, Colombini and Piccinini [10], and then in the papers by De Giorgi [9] and by De Acutis [7]. Very recently it has been further extended by Fernández-Real and Serra [16].

Despite the nonlinear thin obstacle problem naturally arises in several applications and has attracted the attention of distinguished mathematicians, some of the most important questions concerning the regularity of the solutions remained unsolved for many years. For partial results in this regards, aside from the quoted papers by Nitsche, Giusti, Frehse and Ural'tseva on nonlinear variational operators, we mention the more recent contributions by Milakis and Silvestre [42], Fernández-Real [15], Ros-Oton and Serra [47] in the fully nonlinear case.

Building upon the works by Frehse [25] and Ural'tseva [51] together with our previous work [23], in the present paper we give the first comprehensive analysis in the relevant geometric setting of nonparametric minimal surfaces with thin obstacles, developing an approach which can be further extended to more general nonlinear operators. For the sake of simplicity, we confine ourselves to the following elementary formulation of the thin obstacle problem for the nonparametric area functional: given $g \in C^2(\mathbb{R}^{n+1})$ satisfying $g|_{\mathbb{R}^n\times\{0\}} \geq 0$ and $g(x', x_{n+1}) = g(x', -x_{n+1})$, we consider the variational problem

$$\min_{v\in\mathscr{A}_g} \quad \int_{B_1} \sqrt{1+|\nabla v|^2}\,\mathrm{d}x \tag{1.1}$$

where $\mathscr{A}_g := \big\{v \in g|_{B_1} + W^{1,\infty}_0(B_1) : \ v|_{B_1'} \geq 0,\, v(x', x_{n+1}) = v(x', -x_{n+1})\big\}$. Here $B_1' = B_1 \cap \{x_{n+1} = 0\}$, in addition we set $B_1^+ := B_1 \cap \{x_{n+1} > 0\}$. As reported right below, the

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assumption of flat obstacles allows to solve the problem in the space of Lipschitz functions, while for non-flat obstacle the right space to work with is that of functions of bounded variation. Part of the results of this paper can be generalized to non-flat and non-zero obstacles (see, e.g., the techniques in our paper [23] on the fractional obstacle problem), but to the best of our knowledge a complete analysis in the general case is still missing.

Existence and uniqueness of a solution u in the class $g|_{B_1} + W^{1,\infty}_0(B_1)$ has been established by Giusti [30, 31, 32] (following the analysis of minimal surfaces with classical obstacles by Giaquinta and Pepe [29] – see also Giaquinta and Modica [28]), showing that u can be characterized as the weak solution to the system:

$$\begin{cases} \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = 0 & \text{in } B_1^+, \\ \partial_{n+1} u \le 0 \quad \text{and} \quad u\,\partial_{n+1} u = 0 & \text{on } B_1'. \end{cases} \tag{1.2}$$

Lipschitz continuity for u is the best possible global regularity in B_1, as simple examples show. Nevertheless, the solution is expected to be more regular on both sides of the obstacle, thus leading to the investigation of the one-sided regularity on $B_1^+ \cup B_1'$. This is a central question in understanding the qualitative properties of the solutions to variational inequalities with thin obstacles and several important results have been achieved in the last decades. The first contributions to this issue were given by H. Lewy in the two dimensional setting [40, 41]. Lately, continuity of the first derivatives of u taken along tangential directions to B_1' in any dimension and one-sided continuity (up to B_1') for the normal derivative in two dimensions (*i.e.* $n = 1$) were obtained by Frehse [24, 25] for solutions to very general variational inequalities. On the other hand, for the corresponding problem in the uniformly elliptic setting, more refined results on the one-sided regularity are available: in particular, the Hölder continuity of the derivatives, firstly established by Richardson [46] in dimension two and by Caffarelli [5] in any dimension, is shown by different proofs and in different degrees of generality, see [37, 49, 51, 50, 1, 26, 35, 38] only to mention few references.

Despite all the mentioned recent achievements, for the geometric nonlinear case of nonparametric minimal surfaces the $C^{1,\alpha}$ one-sided regularity of solutions was not known in general (except for the two dimensional case considered by Frehse [24] and more recently by Fernández-Real and Serra [16][1]). In this paper we establish the first result on the optimal $C^{1,1/2}$ regularity (to the best of our knowledge there are no other examples of optimal regularity for nonlinear variational inequalities with thin obstacles) and we provide a detailed analysis of the free boundary of the coincidence set. Our approach is based on the pioneering analysis by Frehse [24, 25], by Uralt'seva [49, 51, 50] and on our previous analysis of the Signorini problem [20, 23]. Starting from these results, we develop here a blowup analysis for the study of nonparametric minimal surfaces with thin obstacle, which can be further extended to other nonlinearities. In particular, we do not use the optimal regularity for the scalar Signorini problem established in [1], but we can actually reprove it easily adapting the arguments of the present note.

The following is the main result of the paper (actually, more refined conclusions will be shown, cf. the statement of Theorem 6.1).

Theorem 1.1. *Let u be a solution to the thin obstacle problem (1.1) and let $\Gamma(u)$ be its free boundary, namely the closure of $\{(x',0) \in B_1' : u(x',0) = 0\}$ in the relative topology of B_1'. Then,*

(i) $u \in C^{1,1/2}_{\rm loc}(B_1^+ \cup B_1')$;
(ii) $\Gamma(u)$ *has locally finite $(n-1)$-dimensional Hausdorff measure and is $\mathcal{H}^{n-1}$-rectifiable.*

More in details, concerning the proof of the results we proceed in several steps. Complementing Frehse's result [25], we establish first in Section 3 the one-sided C^1-smoothness of the normal derivative of the solution u. Then, we show the Hölder continuity of the first derivatives (one-sided for the normal one) in Section 4. In doing this we use a penalization argument together with

[1] After the appeareance of this manuscript, in the second version of the preprint [16] the authors establish the almost optimal regularity in any dimension, proving that the solutions to the parametric thin obstacle problem for Caccioppoli sets are $C^{1,1/2-\varepsilon}$ regular for every $\varepsilon > 0$. This improvement gives a different proof of the non-optimal $C^{1,1/2-\varepsilon}$ regularity provided in this note.

the celebrated De Giorgi's method to prove Hölder regularity, following the approach outlined by Ural'tseva [51] in the strongly elliptic case. Optimal regularity then follows by an interesting connection with the theory of minimal surfaces highlighted in Section 5. More precisely, we show that solutions to the thin obstacle problem for the area functional correspond to two-valued minimal graphs. Given this, we can exploit the recent results by Simon and Wickramasekera [48] to infer the optimal one-sided $C^{1,1/2}$ regularity. This association links thin obstacle problems with the program started by Krummel and Wickramasekera [39] about the regularity of multiple valued solutions to the minimal surface system. In this regards, the results in [39] are mostly concerned with the regularity of harmonic multiple valued functions (see also [12, 19, 11] for more results), while their extension to the minimal surface system are not yet known: further investigations in this direction are needed to extend the approach developed here and in [11, 17, 21] to prove the regularity of multiple valued minimal graphs.

In the last section of the paper, we consider the free boundary analysis, i.e. the study of the measure theoretic and geometric properties of the *free boundary* set $\Gamma(u)$, defined as the topological boundary in the relative topology of B_1' of the *coincidence set* $\Lambda(u) = \{(x',0) \in B_1' : u(x',0) = 0\}$. In this respect we follow our recent paper on the Signorini problem for the fractional Laplacian [21, 22, 23] and show the $\mathcal{H}^{n-1}$-rectifiability of the free boundary and the local finiteness of its Hausdorff measure (actually of its Minkowski content). In Section 6 we provide the essential key tools to follow the strategy developed in [21, 23]. In particular, we prove a quasi-monotonicity formula for the Almgren's type frequency function

$$I_u(x_0,r) := \frac{r\int \phi\big(\frac{|x-x_0|}{r}\big)\frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}}\mathrm{d}x}{-\int \phi'\Big(\frac{|x-x_0|}{r}\Big)\frac{1}{|x-x_0|}\frac{u^2}{\sqrt{1+|\nabla u|^2}}\,\mathrm{d}x} \tag{1.3}$$

for $r < 1-|x_0|$ and $x_0 \in B_1'$ (see Section 6.2 for the definition of the auxiliary function ϕ and the details).

2. Preliminaries

Throughout the paper we use the following notation: for any subset $E \subset \mathbb{R}^{n+1}$ we set

$$E^{\pm} := E \cap \{x \in \mathbb{R}^{n+1} : \pm x_{n+1} > 0\} \quad \text{and} \quad E' := E \cap \{x_{n+1} = 0\}.$$

For $x \in \mathbb{R}^{n+1}$ we write $x = (x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ and $B_r(x) \subset \mathbb{R}^{n+1}$ denotes the open ball centered at $x \in \mathbb{R}^{n+1}$ with radius $r > 0$ (we omit to write the point x if the origin and, when there is no source of ambiguity, we write x' for the point $(x', 0)$).

In what follows we shall use the terminology *solution of the thin obstacle problem* for a minimizer u of the area funtional on B_1^+ with respect to its own boundary conditions and additionally satisfying the unilateral obstacle constraint $u|_{B_1'} \geq 0$.

We recall the following two results which will be used in the sequel.

Proposition 2.1. *Let u and $v \in W^{1,\infty}(B_1)$ be two solutions to the thin obstacle problem. If $u|_{\partial B_1} \leq v|_{\partial B_1}$, then $u \leq v$ on $\bar{B}_1$.*

The proof is a direct consequence of the comparison principle for minimal surfaces (cf. [34, Chapter 1, Lemma 1.1]).

The second result we need is due to Frehse [25]. In order to state it, we introduce the following general formulation: let $F : \mathbb{R}^{n+1} \times \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function (we denote its variables by (x, z, p)) and consider the corresponding functional

$$\mathcal{F}(u) := \int_{B_1} F(x, u(x), \nabla u(x))\,\mathrm{d}x.$$

We assume that the Hessian matrix $\big(\frac{\partial^2 F}{\partial p_i \partial p_j}\big)_{i,j=1,\dots,n+1}$ of F is uniformly elliptic (*i.e.* uniformly positive definite) and bounded. The thin obstacle problem related to F is then obtained by minimizing $\mathcal{F}$ among all functions in $\mathcal{A}_g$.

Theorem 2.2 ([25]). *Under the assumptions above on F, the Lipschitz solutions u to the corresponding thin obstacle problems satisfy:*

(i) *if $n = 1$, then $u \in C^1(B_1^+ \cup B_1')$ with*

$$|\nabla u(x) - \nabla u(y)| \le \omega_0(|x-y|) \quad \forall\, x, y \in B_1^+ \cup B_1'$$

where $\omega_0(t) = C|\log t|^{-q}$ with $q \ge 0$ is any constant and $C > 0$;

(ii) *if $n \ge 2$, then the tangential derivatives $\partial_i u \in C^0(B_1^+ \cup B_1')$ for $i \in \{1, \dots, n\}$ with*

$$|\partial_i u(x) - \partial_i u(y)| \le \omega_1(|x-y|) \quad \forall\, x, y \in B_1^+ \cup B_1',$$

where $\omega_1(t) = C|\log t|^{-q(n)}$ with $q(n) \in (0, \frac{2}{(n+1)^2-2n-2})$ and $C > 0$.

3. C^1 REGULARITY

The existence, uniqueness and the Lipschitz regularity of the solutions to the variational problem (1.1) have been studied in [30, 31, 32]. In this section we show that the solutions to the thin obstacle problem have one-sided continuous derivative. In two dimension, this result is due to Frehse [25] for general nonlinear variational inequalities. In higher dimensions, this is not known in this generality and here we provide a proof for the specific case of the area functional.

Proposition 3.1. *Let $u \in W^{1,\infty}(B_1)$ be a solution to the thin obstacle problem. Then, $u \in C^1(B_1^+ \cup B_1')$.*

For the proof of the proposition we start with the following two lemmas.

Lemma 3.2. *For every $a > 0$ there exists $\varepsilon > 0$ such that the solution $w_\varepsilon : B_1 \to \mathbb{R}$ to the thin obstacle problem with boundary value $g_\varepsilon(x) = -a|x_{n+1}| + \varepsilon$ satisfies*

$$w_\varepsilon|_{B'_{3/4}} \equiv 0. \tag{3.1}$$

Proof. From the uniqueness of the solutions to the obstacle problems (1.1) and the radial symmetry of the boundary value $g_\varepsilon(x', x_{n+1}) = g_\varepsilon(y', x_{n+1})$ if $|x'| = |y'|$, we deduce that $w_\varepsilon(x) = \phi_\varepsilon(|x'|, x_{n+1})$ for some function $\phi_\varepsilon : B_1 \subset \mathbb{R}^2 \to \mathbb{R}$. Moreover, from the regularity of w_ε (see, e.g., [33, Theorem 4]) and from its variational characterization, it follows that ϕ_ε is locally Lipschitz and solves the variational problem

$$\phi_\varepsilon \in \operatorname{argmin}_{\phi \in \mathcal{C}} \int_{B_1} \sqrt{1 + |\nabla \phi(\rho, t)|^2}\, \rho^{n-1}\, \mathrm{d}\rho \mathrm{d}t \tag{3.2}$$

with

$$\mathcal{C} := \big\{ \phi|_{\partial B_1'} \ge 0 \quad \text{and} \quad \phi(\rho, t) = -a|t| + \varepsilon \quad \forall\, (\rho, t) \in \partial B_1 \big\}.$$

In particular, from Theorem 2.2 (i) it follows that where the integrand is uniformly elliptic, the solutions ϕ_ε have uniform continuity bounds on their derivatives. Thus, in particular,

$$|\nabla \phi_\varepsilon(x) - \nabla \phi_\varepsilon(y)| \le \omega_0(|x-y|) \quad \forall\, x, y \in B^+_{3/4} \setminus B^+_{1/4},$$

where ω_0 is the modulus of continuity in Theorem 2.2 (i). In particular, from Proposition 2.1 it follows that w_ε converge in $C^1(B^+_{3/4} \setminus B^+_{1/4})$ to $w_\infty(x) := -a x_{n+1}$ and

$$\lim_{\varepsilon \to 0} \|\partial_{n+1} w_\varepsilon + a\|_{L^\infty(B^+_{3/4} \setminus B^+_{1/4})} = 0. \tag{3.3}$$

We then infer that there exists $\varepsilon_0 > 0$ such that

$$\partial_{n+1} w_\varepsilon(x) \le -a/2 \quad \forall\, \varepsilon \in (0, \varepsilon_0),\ \forall\, x \in B^+_{3/4} \setminus B^+_{1/4},$$

and in view of Theorem 2.2 (i)

$$\partial_{n+1} w_\varepsilon(x', 0^+) := \lim_{t \to 0^+} \frac{w_\varepsilon(x', t) - w_\varepsilon(x', 0)}{t} \le -a/2 \tag{3.4}$$

for $\varepsilon \in (0, \varepsilon_0)$ and $x' \in B'_{3/4} \setminus B'_{1/4}$. Recalling the Euler–Lagrange equations associated to the thin obstacle problem (1.2), this implies that $B'_{3/4} \setminus B'_{1/4} \subset \Lambda(w_\varepsilon)$ for all $\varepsilon < \varepsilon_0$.

We need only to show that $B'_{1/4} \subset \Lambda(w_\varepsilon)$ if ε is suitably chosen. To this aim we show that, for ε sufficiently small, we have that

$$\phi_\varepsilon(\rho,t) \le -\frac{a}{2}\,t \quad \forall\,(\rho,t) \in \partial B_{1/2}. \tag{3.5}$$

Indeed, given for granted the last inequality, the comparison principle for the solutions to the thin obstacle problem in Proposition 2.1, yields that $w_\varepsilon(x) \le -\frac{a}{2}\,|x_{n+1}|$ for every $x \in \bar B_{1/2}$, from which $B'_{1/4} \subset \Lambda(w_\varepsilon)$ readily follows. In order to show (3.5), we notice that by (3.3)

$$\phi_\varepsilon(\rho,t) \le -\frac{a}{2}\,t \qquad \forall\,\varepsilon \in (0,\varepsilon_0),\ \forall\,\rho \in (1/4,1/2) \text{ and } \forall\,t \in (0,\sqrt{5}/4),$$

where we used that $(x',t) \in B^+_{3/4} \setminus B^+_{1/4}$ if $|x'| \in (1/4,1/2)$ and $t \in (0,\sqrt{5}/4)$. Moreover, since ϕ_ε converges to $-at$ in $B^+_{3/4} \setminus B^+_{1/4}$, we also infer that there exists $\varepsilon_1 > 0$ such that

$$\phi_\varepsilon(\rho,t) \le -at + \frac{a\sqrt{5}}{8} \le -\frac{a}{2}\,t \quad \forall\,\varepsilon \in (0,\varepsilon_1),\ \ \forall\,(\rho,t) \in \partial B^+_{1/2},\ t \ge 1/4.$$

Putting the two estimates together, we deduce that (3.5) holds for every $\varepsilon < \min\{\varepsilon_0,\varepsilon_1\}$, thus concluding the lemma. □

We prove next an auxiliary result.

Lemma 3.3. *Let $u \in W^{1,\infty}(B_1)$ be a solution to the thin obstacle problem (1.1). Then, for any sequence of points $z_k \in \Gamma(u)$ and of radii $t_k \downarrow 0$ (with $t_k \le 1 - |z_k|$), the functions*

$$u_k(x) := \frac{u(z_k + t_k x)}{t_k}$$

converge to 0 uniformly on $\bar B_1$.

Proof. The functions u_k are equi-Lipschitz continuous (with $\mathrm{Lip}(u_k) \le \mathrm{Lip}(u)$) and are solutions to the thin obstacle problem with $\underline{0} \in \Gamma(u_k)$. Therefore, up to passing to a subsequence (not relabeled for convenience), u_k converges uniformly on $\bar B_1$ to a function u_∞ which is itself a solution to the thin obstacle problem. We need now to prove that $u_\infty \equiv 0$.

We start noticing that, in view of Theorem 2.2 (ii), we have

$$|\nabla' u_k(x) - \nabla' u_k(y)| = |\nabla' u(t_k x + z_k) - \nabla' u(t_k y + z_k)| \le \omega_1(t_k|x-y|), \tag{3.6}$$

where $\nabla' = (\partial_1,\dots,\partial_n)$ denotes the horizontal gradient. Thus, by (3.6) and since $\nabla' u_k(\underline{0}) = \underline{0}$, $\|\nabla' u_k\|_\infty$ converge to 0. Being $\nabla' u_k$ equi-continuous (with modulus of continuity ω_1), we then infer that $\nabla' u_k$ converges to $\nabla' u_\infty$ uniformly and $\nabla' u_\infty \equiv \underline{0}$, *i.e.* u_∞ is a function depending exclusively on the variable x_{n+1}. By direct computation one can show that the only solutions depending on one variable are the linear functions of the form

$$u_\infty(x) = -a x_{n+1} \quad \text{on } \bar B_1^+, \text{ for some } a \ge 0.$$

The thesis is then reduced to proving that $a = 0$. Assume that $a > 0$: let $\varepsilon > 0$ be the constant in Lemma 3.2 and notice that, since u_k converges to $u_\infty = -a x_{n+1}$ uniformly on $\bar B_1^+$, it must be $u_k|_{\partial B_1} \le w_\varepsilon|_{\partial B_1}$ definitively, where w_ε is the solution to the thin obstacle problem with boundary value $g_\varepsilon(x) = -a|x_{n+1}| + \varepsilon$. By the comparison principle of Proposition 2.1 $u_k|_{B_1} \le w_\varepsilon|_{B_1}$ for k sufficiently large, which in turn by Lemma 3.2 leads to $u_k|_{B'_{3/4}} \equiv 0$. This is a contradiction to $0 \in \Gamma(u_k)$, thus establishing that $a = 0$.

Finally, since we have shown that any convergent subsequence of u_k is uniformly converging to 0, we conclude that the whole sequence u_k converges uniformly to 0 on $\bar B_1$. □

Proof of Proposition 3.1. By Frehse's Theorem 2.2, we need only to prove that the normal derivative $\partial_{n+1} u$ is a continuous function in $B_1^+ \cup B'_1$. Moreover, since $\partial_{n+1} u$ is analytic in $B_1^+ \cup B'_1 \setminus \Gamma(u)$, we have only to check its continuity at points of the free boundary $\Gamma(u) \subseteq B'_1$.

Without loss of generality, we can assume that $\underline{0} \in \Gamma(u)$ and we begin with showing that u is differentiable at $\underline{0}$ with zero normal derivative:

$$\lim_{t \to 0^+} \frac{u(0,t)}{t} = 0. \tag{3.7}$$

We apply Lemma 3.3 to any sequence $(t_k)_{k\in\mathbb{N}}$ with $t_k \downarrow 0$ and $z_k = \underline{0}$ for all k: the functions $u_k(x) = t_k^{-1}u(t_k x)$ converge uniformly to 0 in $\bar{B}_1$. In particular,

$$\lim_{k\to\infty} \frac{u(0,t_k)}{t_k} = \lim_{k\to\infty} u_k(e_{n+1}) = 0.$$

From the arbitrariness of the sequence $(t_k)_{k\in\mathbb{N}}$, (3.7) in turn follows.

Next we prove the $\partial_{n+1}u$ is continuous in $\underline{0} \in \Gamma(u)$. Let $y_k \in B_1^+ \cup \big(B_1' \setminus \Gamma(u)\big)$ be a sequence of points converging to $\underline{0}$. Let $t_k := \operatorname{dist}\big(y_k, \Gamma(u)\big) = |y_k - z_k| \to 0$, with $z_k \in \Gamma(u)$. Therefore $B_{t_k}(y_k) \cap \Gamma(u) = \emptyset$, and either $B_{t_k}(y_k) \cap \Lambda(u) = \emptyset$, in which case we set $v(x) := u(x)$ for all $x \in B_{t_k}(y_k)$, or $B_{t_k}(y_k) \cap B_1' \subseteq \Lambda(u)$ and we set

$$v(x) := \begin{cases} u(x) & \text{if } x_{n+1} \geq 0, \\ -u(x) & \text{if } x_{n+1} < 0. \end{cases}$$

In both cases v is a solution to the minimal surface equation in $B_{t_k}(y_k)$ (indeed, u solves the minimal surface equation in $B_{t_k}^+(y_k)$ either with null Neumann or with null Dirichlet boundary conditions on $B_{t_k}(y_k) \cap B_1'$, respectively; therefore v is readily regognized to be a solution in both cases). Set $\tau_k := 2\,|y_k - z_k|$ and let $v_k : B_1 \to \mathbb{R}$ be given by

$$v_k(x) := \frac{v(z_k + \tau_k\, x)}{\tau_k}.$$

By Lemma 3.3, v_k is uniformly converging to 0. Moreover, by possibly passing to a further subsequence, we can assume that $p_k := \frac{y_k - z_k}{\tau_k} \to p \in \partial B_{1/2}$. Since, the functions v_k are solutions of the minimal surface equation in $B_{1/2}(p)$ and they are converging uniformly to 0, the regularity theory for the minimal surface equation implies that the convergence is in fact smooth. In particular, in both cases discussed above we get

$$\lim_{k\to\infty} \partial_{n+1} v(y_k) = \lim_{k\to\infty} \partial_{n+1} v_k(p_k) = 0,$$

thus concluding the continuity of $\partial_{n+1}u$ at $\underline{0}$. □

4. $C^{1,\alpha}$ REGULARITY

This section is devoted to show the one-sided $C^{1,\alpha}(B_1^+ \cup B_1')$ regularity. To this aim, we need to consider approximate solutions produced by the method of penalization.

4.1. **The penalized problem.** Let $g \in C^2(\mathbb{R}^n)$ be a fixed boundary value for (1.1) and let $u \in W^{1,\infty}(B_1)$ be the unique solution to the thin obstacle problem. For the rest of the section, we set $L := \operatorname{Lip}(u)$.

We start off considering the following penalized problem: let $\beta, \chi \in C^\infty(\mathbb{R})$ be such that

$$|t| - 1 \leq |\beta(t)| \leq |t| \quad \forall\, t \leq 0, \quad \beta(t) = 0 \quad \forall\, t \geq 0, \quad \beta'(t) \geq 0 \quad \forall\, t \in \mathbb{R}\,,$$

$$\chi(t) = \begin{cases} 0 & \text{for } \ t \leq L, \\ \frac{1}{2}\,(t - 2L)^2 & \text{for } \ t > 3L, \end{cases} \qquad \chi''(t) \geq 0 \quad \forall\, t \in \mathbb{R}.$$

For every $\varepsilon > 0$ set $\beta_\varepsilon(t) := \varepsilon^{-1}\beta(t/\varepsilon)$ and we introduce the energy

$$\mathscr{E}_\varepsilon(v) := \int_{B_1} \Big(\sqrt{1 + |\nabla v|^2} + \chi(|\nabla v|)\Big)\, \mathrm{d}x + \int_{B_1'} F_\varepsilon(v(x',0))\, \mathrm{d}x',$$

where $F_\varepsilon(t) := \int_0^t \beta_\varepsilon(s)\, \mathrm{d}s$. Since the energy $\mathscr{E}_\varepsilon$ is strictly convex and quadratic, there exists a unique minimizer $u_\varepsilon \in g + W_0^{1,2}(B_1)$. Moreover, from the symmetry of g, it follows that u_ε is also even symmetric with respect to x_{n+1}.

The Euler–Lagrange equation satisfied by u_ε is then given by

$$\int_{B_1^+} A(\nabla u_\varepsilon) \cdot \nabla \eta\, \mathrm{d}x + \int_{B_1'} \beta_\varepsilon(u_\varepsilon)\, \eta\, \mathrm{d}x' = 0 \qquad \forall\, \eta \in H_0^1(B_1), \tag{4.1}$$

with $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ being the vector field

$$A(p) := \Big((1+|p|^2)^{-1/2} + \chi'(|p|)\,|p|^{-1} \Big)\, p.$$

Note that for $|p| \le L$ the second addend is actually null.

The following lemma establish the connection between the solutions of the penalized problems and the solution to the thin obstacle problem.

Lemma 4.1. *Let $g \in C^2(\mathbb{R}^{n+1})$ be even symmetric with respect to x_{n+1} and $g|_{\mathbb{R}^n\times\{0\}} \ge 0$. Then, the minimizers u_ε of $\mathscr{E}_\varepsilon$ on $g + W^{1,2}_0(B_1)$ converge weakly in $W^{1,2}$ as ε goes to 0 to the solution u to the thin obstacle problem (1.1).*

Proof. From the definition of χ one readily verifies that there exists a constant $C > 0$ such that $t^2 \le C(1+\chi(t))$ for every $t \ge 0$. Thus, it follows that the approximate solutions u_ε have equi-bounded Dirichlet energy:

$$\begin{aligned}\int_{B_1} |\nabla u_\varepsilon|^2\,dx &\le C\mathcal{L}^{n+1}(B_1) + C\int_{B_1}\chi(|\nabla u_\varepsilon|)\,dx \le C\mathcal{L}^{n+1}(B_1) + C\mathscr{E}_\varepsilon(u_\varepsilon)\\ &\le C\mathcal{L}^{n+1}(B_1) + C\mathscr{E}_\varepsilon(u) = C\mathcal{L}^{n+1}(B_1) + C\int_{B_1}\sqrt{1+|\nabla u|^2}\,dx.\end{aligned}$$

Then, up to extracting a subsequence (not relabeled), there exists a function $u_0 \in g + W^{1,2}_0(B_1)$ such that u_ε converges to u_0 in $L^2(B_1)$ and the trace $u_\varepsilon|_{B_1'}$ converges to $u_0|_{B_1'}$ in $L^2(B_1')$.

We next show that $u_0|_{B_1'} \ge 0$. Recalling that F_ε is positive and monotone decreasing, we have by Chebyshev inequality

$$F_\varepsilon(-\delta)\,\mathcal{L}^n\big(\{u_\varepsilon < -\delta\} \cap B_1'\big) \le \int_{B_1'} F_\varepsilon(u_\varepsilon)\,\mathrm{d}x \le \mathscr{E}_\varepsilon(u_\varepsilon) \le \int_{B_1}\sqrt{1+|\nabla u|^2}\,\mathrm{d}x.$$

Since $F_\varepsilon(t) \uparrow \infty$ as $\varepsilon \downarrow 0$ for all $t < 0$ and $u_\varepsilon|_{B_1'} \to u_0|_{B_1'}$ in $L^2(B_1')$, we conclude that

$$\mathcal{L}^n\big(\{u_0 < -\delta\} \cap B_1'\big) = 0 \quad \forall\, \delta > 0,$$

which implies $u_0|_{B_1'} \ge 0$, *i.e.* $u_0 \in \mathcal{B}_g$ where

$$\mathcal{B}_g := \Big\{ w \in g + W^{1,2}_0(B_1) : w|_{B_1'} \ge 0 \Big\}.$$

Furthermore, u_0 is the unique minimizer in $\mathcal{B}_g$ of the energy $\mathscr{F} : W^{1,2}(B_1) \to [0,\infty)$ defined by

$$\mathscr{F}(w) := \int_{B_1} \Big(\sqrt{1+|\nabla w|^2} + \chi(|\nabla w|) \Big)\,\mathrm{d}x\,.$$

Indeed, by convexity of $\mathscr{F}$, for every $w \in \mathcal{B}_g$ we have that

$$\mathscr{F}(u_0) \le \liminf_{\varepsilon\to 0^+} \mathscr{F}(u_\varepsilon) \le \liminf_{\varepsilon\to 0^+} \mathscr{E}_\varepsilon(u_\varepsilon) \le \liminf_{\varepsilon\to 0^+} \mathscr{E}_\varepsilon(w) = \mathscr{F}(w),$$

since $\mathcal{B}_g \subset g + W^{1,2}_0(B_1)$ and $F_\varepsilon(w) = 0$ for all $w \in \mathcal{B}_g$. To conclude, we only need to notice that the unique minimizer of $\mathscr{F}$ on $\mathcal{B}_g$ is exactly the solution to the thin obstacle problem u. Indeed, $\mathcal{A}_g \subseteq \mathcal{B}_g$ and for every $w \in \mathcal{B}_g$ we have that

$$\mathscr{F}(u) = \int_{B_1}\sqrt{1+|\nabla u|^2}\,dx \le \int_{B_1}\sqrt{1+|\nabla w|^2}\,dx \le \mathscr{F}(w),$$

where we used that $\chi(|\nabla u|) \equiv 0$ and that u is a minimizer of the thin obstacle problem for the area functional among all competitors in $\mathcal{B}_g$, and not only in $\mathcal{A}_g$ (this follows from an approximation argument).

Finally, being the solution to the Signorini problem unique, by Urysohn property we conclude that the whole family $(u_\varepsilon)_{\varepsilon>0}$ converges to u. □

4.2. $W^{2,2}$ **estimate.** Next we show that the solution to the penalized problem, as well as the solution to the thin obstacle problem, possess second derivatives in $L^2(B_1^+)$. The proof is at all analogous to the standard L^2-theory for quasilinear equations: we report it for readers convenience.

We recall the standard notation of the difference quotient

$$\tau_{h,i} f(x) := h^{-1}\big(f(x+he_i) - f(x)\big),$$

if $x \in \{y \in B_1 : y + he_i \in B_1\}$ and $\tau_{h,i} f(x) := 0$ otherwise, where $f : B_1 \to \mathbb{R}$ is any measurable function and e_i a coordinate vector, $i \in \{1, \dots, n+1\}$.

Proposition 4.2. *The solutions u_ε to the penalized problems (4.1) for every $\varepsilon > 0$ and the solution u to the thin obstacle problem satisfy the following property: there exists a constant $C = C(n, L) > 0$ such that, if either $v = u_\varepsilon$ or $v = u$, then*

$$\int_{B_r^+(x_0)} |\nabla^2 v|^2 \, \mathrm{d}x \le \frac{C}{r^2} \int_{B_{2r}^+(x_0)} |\nabla' v|^2 \, \mathrm{d}x \quad \forall\, x_0 \in B_1^+ \cup B_1',\ \forall\, 0 < r < \tfrac{1-|x_0|}{2}\,. \tag{4.2}$$

Proof. The result is classical if $x_0 \in B_1^+$ and $B_r(x_0) \subset\subset B_1^+$. We shall prove only the case in which $x_0 \in B_1'$, and the general case follows by a covering argument. Without loss of generality we may assume $x_0 = \underline{0}$.

We provide first an estimate for the horizontal derivatives of the weak gradient of u_ε. Let $\zeta \in C_c^1(B_{2r})$, $2r < 1$, be a test function with $\zeta \equiv 1$ in B_r and $|\nabla \zeta| \le C\, r^{-1}$ for some dimensional constant $C > 0$. We test (4.1) with $\eta := \tau_{-h,i}\big(\zeta^2\, \tau_{h,i} u_\varepsilon\big)$, with $|h| < 1 - 2r$ and $i \in \{1, \dots, n\}$. For convenience, in the following computation we omit to write the index $i \in \{1, \dots, n\}$ in the notation of the difference quotients. We start off noticing that the first addend in (4.1) rewrites as

$$\int_{B_1^+} A(u_\varepsilon) \cdot \nabla \eta \, \mathrm{d}x = \int_{B_1^+} \tau_h\big(A(\nabla u_\varepsilon)\big) \cdot \nabla\big(\zeta^2\, \tau_h u_\varepsilon\big) \, \mathrm{d}x, \tag{4.3}$$

where we used the basic integration by parts formula for discrete derivatives

$$\int (\tau_h f)\, \varphi \, \mathrm{d}x = \int f\, (\tau_{-h} \varphi) \, \mathrm{d}x \qquad \forall f, \varphi \text{ measurable, } \varphi \text{ having compact support}\,.$$

We now compute as follows: set

$$\psi(t) := A\big((1-t)\nabla u_\varepsilon(x) + t \nabla u_\varepsilon(x + he_i)\big);$$

then,

$$\begin{aligned} \tau_h\big(A(\nabla u_\varepsilon)\big) &= \frac{1}{h} \int_0^1 \psi'(t) \, \mathrm{d}t = \frac{1}{h} \int_0^1 \nabla A\big((1-t)\nabla u_\varepsilon(x) + t \nabla u_\varepsilon(x+he_i)\big) \, \mathrm{d}t\, \tau_h(\nabla u_\varepsilon) \\ &=: \mathbb{A}_\varepsilon^h(x) \tau_h(\nabla u_\varepsilon). \end{aligned}$$

Note that there exist constants $0 < \lambda < \Lambda$ (depending on $L = \mathrm{Lip}(u)$) such that

$$\lambda\, \mathrm{Id}_{n+1} \le \mathbb{A}_\varepsilon^h(x) \le \Lambda\, \mathrm{Id}_{n+1} \quad \forall\, x \in B_1^+,$$

because

$$\begin{aligned} \nabla A(p) &= \nabla \left(\frac{p}{\sqrt{1+|p|^2}} + \chi'(|p|)\, \frac{p}{|p|} \right) \\ &= \frac{\mathrm{Id}}{(1+|p|^2)^{3/2}} + \big((1+|p|^2)^{-3/2} + \chi'(|p|)\, |p|^{-3}\big)\big(|p|^2\, \mathrm{Id}_{n+1} - p \otimes p\big) + \chi''(|p|)\, \frac{p \otimes p}{|p|^2} \end{aligned}$$

is uniformly elliptic and bounded. Therefore, we can rewrite (4.3) as

$$\begin{aligned} \int_{B_1^+} A(u_\varepsilon) \cdot \nabla \eta \, \mathrm{d}x &= \int_{B_1^+} \mathbb{A}_\varepsilon^h\, \tau_h(\nabla u_\varepsilon) \cdot \nabla\big(\zeta^2\, \tau_h u_\varepsilon\big) \, \mathrm{d}x \\ &= \int_{B_1^+} \Big(\zeta^2\, \mathbb{A}_\varepsilon^h\, \tau_h(\nabla u_\varepsilon) \cdot \tau_h(\nabla u_\varepsilon) \, + 2\zeta\, (\tau_h u_\varepsilon)\, \mathbb{A}_\varepsilon^h\, \tau_h(\nabla u_\varepsilon) \cdot \nabla \zeta\Big) \, \mathrm{d}x. \end{aligned}$$

On the other hand, by the monotonicity of β_ε the second addend in (4.1) is non-positive. Indeed, being β_ε increasing, we have

$$\int_{B_1'} \beta_\varepsilon(u_\varepsilon)\tau_{-h}\big(\zeta^2\,\tau_h u_\varepsilon\big)\,\mathrm{d}x' = \int_{B_1'} \tau_h\big(\beta_\varepsilon(u_\varepsilon)\big)(\tau_h u_\varepsilon)\,\zeta^2\,\mathrm{d}x'$$
$$= \int_{B_1'} \frac{\beta_\varepsilon(u_\varepsilon(x'+he_i)) - \beta_\varepsilon(u_\varepsilon(x'))}{h}\,\frac{u_\varepsilon(x'+he_i)-u_\varepsilon(x')}{h}\,\zeta^2\,\mathrm{d}x' \geq 0.$$

Thus, from (4.1) we infer that

$$\int_{B_1^+} \Big(\mathbb{A}_\varepsilon^h\,\tau_h(\nabla u_\varepsilon)\cdot\tau_h(\nabla u_\varepsilon)\,\zeta^2 + 2\zeta\,\tau_h(u_\varepsilon)\,\mathbb{A}_\varepsilon^h\,\tau_h(\nabla u_\varepsilon)\cdot\nabla\zeta\Big)\,\mathrm{d}x \leq 0.$$

Hence, in view of Cauchy-Schwarz inequality and of the ellipticity of $\mathbb{A}_\varepsilon^h$ we conclude that

$$\int_{B_1^+} |\tau_h(\nabla u_\varepsilon)|^2\,\zeta^2\,\mathrm{d}x \leq 4\frac{\Lambda}{\lambda}\int_{B_1^+} |\tau_h u_\varepsilon|^2\,|\nabla\zeta|^2\,\mathrm{d}x.$$

The latter estimate implies that ∇u_ε has weak i-th derivative in $L^2(B_r^+)$, for all $i\in\{1,\dots,n\}$, $r<1/2$, with

$$\int_{B_r^+} |\partial_i(\nabla u_\varepsilon)|^2\,\mathrm{d}x \leq \frac{C}{r^2}\int_{B_{2r}^+} |\partial_i u_\varepsilon|^2\,\mathrm{d}x, \tag{4.4}$$

for a constant $C>0$ depending only on L.

To conclude the proof for $v=u_\varepsilon$ it suffices to prove that $\partial_{n+1}u_\varepsilon$ has $(n+1)$-th weak derivative in B_1^+. Writing $A(p)=(A^1(p),\dots,A^{n+1}(p))$, we have that

$$\partial_j A^i(\nabla u_\varepsilon)\partial_{ij}u_\varepsilon = 0.$$

Moreover, $\lambda\leq\partial_{n+1}A^{n+1}(p)\leq\Lambda$ for every $p\in\mathbb{R}^{n+1}$, from which we deduce that

$$\partial_{n+1}^2 u_\varepsilon = \frac{1}{\partial_{n+1}A^{n+1}(\nabla u_\varepsilon)}\sum_{(i,j)\neq(n+1,n+1)}\partial_j A^i(\nabla u_\varepsilon)\partial_{i,j}^2 u_\varepsilon \in L^2_{\mathrm{loc}}(B_1^+)\,. \tag{4.5}$$

Hence, from (4.4) and the fact that ∇A is bounded, we get the estimate

$$\int_{B_r^+} |\partial_{n+1}(\nabla u_\varepsilon)|^2\,\mathrm{d}x \leq \frac{C}{r^2}\sum_{i=1}^n\int_{B_r}|\nabla(\partial_i u_\varepsilon)|^2\,\mathrm{d}x \leq \frac{C}{r^2}\int_{B_{2r}}|\nabla' u_\varepsilon|^2\,\mathrm{d}x\,, \tag{4.6}$$

with $C=C(n,L)>0$. Being estimates (4.4) and (4.6) uniform in ε, in view of Lemma 4.1, we can pass to the limit as $\varepsilon\downarrow 0$ and infer that the same estimates hold for u as well. □

4.3. $C^{1,\alpha}$ **estimate.** Next we prove that the minimizer u of the Signorini problem has weak derivatives in suitable De Giorgi classes on the flat part of the boundary. Here, we do follow the approach by Ural'tesva [51] in conjunction with the one-sided continuity of the derivatives shown in Proposition 3.1. In particular, the latter result is instrumental to establish the ensuing estimate (4.7) for $\pm\partial_{n+1}u$.

Proposition 4.3. *Let u be the solution to the thin obstacle problem, then for some constant $C=C(n,L)>0$ the function $v=\pm\partial_i u$, $i\in\{1,\dots,n+1\}$, satisfies for all $k\geq 0$*

$$\int_{B_r^+(x_0)\cap\{v>k\}}|\nabla v|^2\,\mathrm{d}x \leq \frac{C}{r^2}\int_{B_{2r}^+(x_0)}(v-k)_+^2\,\mathrm{d}x \quad \forall\, x_0\in B_1',\ 0<r<\tfrac{1-|x_0|}{2}. \tag{4.7}$$

Proof. We start off writing the equation satisfied by the horizontal derivatives of the solution to the penalized problem (4.1) and by testing it with $\eta=\partial_i\zeta$, $i\in\{1,\dots,n\}$, for $\zeta\in W^{2,2}(B_1)$ even symmetric with respect to x_{n+1} and $\mathrm{spt}\zeta\cap\partial B_1=\emptyset$:

$$0 = \int_{B_1^+}\partial_i\left(A(\nabla u_\varepsilon)\right)\cdot\nabla\zeta\,\mathrm{d}x + \int_{B_1'}\partial_i[\beta_\varepsilon(u_\varepsilon)]\,\zeta\,\mathrm{d}x'$$
$$= \int_{B_1^+}\nabla A(\nabla u_\varepsilon)\,\nabla(\partial_i u_\varepsilon)\cdot\nabla\zeta\,\mathrm{d}x + \int_{B_1'}\beta_\varepsilon'(u_\varepsilon)\partial_i u_\varepsilon\,\zeta\,\mathrm{d}x'. \tag{4.8}$$

Note that (4.8) makes sense as soon as $\zeta \in W^{1,2}(B_1^+)$ with $\operatorname{spt}\zeta \cap (\partial B_1)^+ = \emptyset$, thanks to the integrability estimates in Proposition 4.2. Therefore, as $u_\varepsilon \in W^{2,2}(B_1^+)$ we can choose $\zeta_\varepsilon := (\partial_i u_\varepsilon - k)_+ \phi^2$ for $k \geq 0$ and having fixed $\phi \in C_c^1(B_1)$, because $\zeta_\varepsilon \in W^{1,2}(B_1^+)$ with $\operatorname{spt}\zeta_\varepsilon \cap (\partial B_1)^+ = \emptyset$. With this choice at hand, note then that

$$\int_{B_1'} \beta_\varepsilon'(u_\varepsilon)\partial_i u_\varepsilon\, \zeta_\varepsilon \,\mathrm{d}x' = \int_{B_1'} \beta_\varepsilon'(u_\varepsilon)\partial_i u_\varepsilon\, (\partial_i u_\varepsilon - k)_+ \,\phi^2 \,\mathrm{d}x' \geq 0. \tag{4.9}$$

For what concerns the remaining terms, we recall that $\nabla\zeta_\varepsilon = \phi^2\nabla(\partial_i u_\varepsilon)\chi_{\{\partial_i u_\varepsilon > k\}} + 2\phi(\partial_i u_\varepsilon - k)_+\nabla\phi$. Therefore, we have that

$$\begin{aligned} 0 \geq &\int_{B_1^+\cap\{\partial_i u_\varepsilon > k\}} \phi^2 \nabla A(\nabla u_\varepsilon)\,\nabla(\partial_i u_\varepsilon)\cdot\nabla(\partial_i u_\varepsilon)\,\mathrm{d}x \\ &+ \int_{B_1^+} 2\phi(\partial_i u_\varepsilon - k)_+ \nabla A(\nabla u_\varepsilon)\,\nabla(\partial_i u_\varepsilon)\cdot\nabla\phi\,\mathrm{d}x. \end{aligned}$$

Then, a standard argument implies

$$\int_{B_1^+\cap\{\partial_i u_\varepsilon > k\}} \phi^2 |\nabla(\partial_i u_\varepsilon)|^2\,\mathrm{d}x \leq 4\frac{\Lambda}{\lambda}\int_{B_1^+} (\partial_i u_\varepsilon - k)_+^2\,|\nabla\phi|^2\,\mathrm{d}x$$

In particular, for every $k \geq 0$ and for every $x_0 \in B_1'$ and $0 < 2r < 1 - |x_0|$ if $\phi \in C_c^1(B_{2r}(x_0))$ and $\phi \equiv 1$ on $B_r(x_0)$ with $|\nabla\phi| \leq C/r$

$$\int_{B_r^+\cap\{\partial_i u_\varepsilon > k\}} |\nabla(\partial_i u_\varepsilon)|^2\,\mathrm{d}x \leq \frac{C}{r^2}\int_{B_{2r}^+} (\partial_i u_\varepsilon - k)_+^2\,\mathrm{d}x\,, \tag{4.10}$$

for some $C = C(L) > 0$. In exactly the same way, by testing (4.1) with $\zeta_\varepsilon := (-\partial_i u_\varepsilon - k)_+\,\eta^2$, we derive the analogous estimate

$$\int_{B_r^+\cap\{\partial_i u_\varepsilon < -k\}} |\nabla(\partial_i u_\varepsilon)|^2\,\mathrm{d}x \leq \frac{C}{r^2}\int_{B_{2r}^+} (-\partial_i u_\varepsilon - k)_+^2\,\mathrm{d}x\,, \tag{4.11}$$

for all $k \geq 0$ and $i \in \{1,\dots,n\}$. Estimate (4.7) for $\pm\partial_i u$, with $i = 1,\dots,n$, follows at once by passing to the limit as $\varepsilon \downarrow 0$ in (4.10) and (4.11), respectively.

For what concerns the partial derivative in direction $n+1$, we test the equation (4.8) with $\partial_{n+1}\zeta$, for $\zeta \in W^{2,2}(B_1^+)$ with $\operatorname{spt}\zeta \cap (\partial B_1)^+ = \emptyset$:

$$\begin{aligned} 0 &= \int_{B_1^+} \partial_{n+1}\left(A(\nabla u_\varepsilon)\right)\cdot\nabla\zeta\,\mathrm{d}x + \int_{B_1'} A(\nabla u_\varepsilon)\cdot\nabla\zeta\,\mathrm{d}x' - \int_{B_1'}\beta_\varepsilon(u_\varepsilon)\,\partial_{n+1}\zeta\,\mathrm{d}x' \\ &= \int_{B_1^+} \partial_{n+1}\left(A(\nabla u_\varepsilon)\right)\cdot\nabla\zeta\,\mathrm{d}x + \int_{B_1'} A'(\nabla u_\varepsilon)\cdot\nabla'\zeta\,\mathrm{d}x', \end{aligned} \tag{4.12}$$

where we set $A'(p) := (A^1(p),\dots,A^n(p))$. The last equality holds thanks to Euler-Lagrange condition induced by (4.1):

$$\begin{cases} \operatorname{div}\big(A(\nabla u_\varepsilon)\big) = 0 & \text{in } B_1^+, \\ A^{n+1}(\nabla u_\varepsilon) = \beta_\varepsilon\big(u_\varepsilon\big) & \text{on } B_1'. \end{cases} \tag{4.13}$$

For $0 < k \leq \|\partial_{n+1}u\|_{L^\infty(B_1^+)}$ set

$$\zeta_\delta := \phi^2\gamma_\delta(-\partial_{n+1}u - k),$$

where $\delta > 0$ will be suitably chosen, $\gamma_\delta \in C^\infty(\mathbb{R})$ is an increasing function such that $\gamma_\delta(t) = 0$ for $t \leq 0$, $\gamma_\delta(t) > 0$ for $t > 0$, $\gamma_\delta(t) = t - \delta$ for $t \geq 2\delta$, $|\gamma_\delta'(t)| \leq 1$ (such a function can be easily exhibited), and $\phi \in C_c^\infty(B_{2r}(x_0))$, $\phi|_{B_r(x_0)} = 1$, $|\nabla\phi| \leq C/r$. We use $\partial_{n+1}u \in C^0(B_1^+ \cup B_1')$ (cf. Proposition 3.1) to infer that for $k > 0$ the set $B_1' \cap \{\partial_{n+1}u < -k\}$ is an open set with compact closure in $\Lambda(u)$ (recall that $\partial_{n+1}u = 0$ on $B_1' \setminus \Lambda(u)$). This implies that, if $\delta > 0$ is sufficiently small, $\zeta_\delta \in C^\infty(B_1^+)$ with $\operatorname{spt}\zeta_\delta \cap (\partial B_1)^+ = \emptyset$. Indeed, $u \in C^\infty(B_r^+(y_0))$ for all $y_0 \in B_1' \cap \operatorname{spt}\zeta_\delta$ and $r < \operatorname{dist}\big(B_1' \cap \{\partial_{n+1}u \leq -k\}, B_1' \cap \{\partial_{n+1}u = 0\}\big)$, being u itself minimum of the area problem

with null Dirichlet boundary conditions on $B'_r(y_0)$. Taking $\zeta = \zeta_\delta$ we evaluate each addend in (4.12) separately. To begin with, the first term rewrites as

$$I^{\varepsilon,\delta} := \int_{B_1^+} \partial_{n+1}\left(A(\nabla u_\varepsilon)\right)\cdot\nabla\zeta_\delta\,\mathrm{d}x = 2\int_{B_1^+}\phi\,\gamma_\delta(-\partial_{n+1}u-k)\nabla A(\nabla u_\varepsilon)\nabla(\partial_{n+1}u_\varepsilon)\cdot\nabla\phi\,\mathrm{d}x$$
$$-\int_{B_1^+}\phi^2\gamma'_\delta(-\partial_{n+1}u-k)\nabla A(\nabla u_\varepsilon)\nabla(\partial_{n+1}u_\varepsilon)\cdot\nabla(\partial_{n+1}u)\,\mathrm{d}x.$$

Taking the limits as $\varepsilon\downarrow 0$ in each term above, since ∇A is a Lipschitz function and $\nabla u_\varepsilon\to\nabla u$ in L^2 and $\nabla(\partial_{n+1}u_\varepsilon)\rightharpoonup\nabla(\partial_{n+1}u)$ in L^2, we conclude that

$$\lim_{\varepsilon\to 0} I^{\varepsilon,\delta} = 2\int_{B_1^+}\phi\,\gamma_\delta(-\partial_{n+1}u-k)\nabla A(\nabla u)\nabla(\partial_{n+1}u)\cdot\nabla\phi\,\mathrm{d}x$$
$$-\int_{B_1^+}\phi^2\gamma'_\delta(-\partial_{n+1}u-k)\nabla A(\nabla u)\nabla(\partial_{n+1}u)\cdot\nabla(\partial_{n+1}u)\,\mathrm{d}x.$$

Moreover, since $\gamma_\delta(-\partial_{n+1}u-k)\to(-\partial_{n+1}u-k)_+$ strongly in $W^{1,2}(B_1^+)$ as $\delta\downarrow 0$, then we infer

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} I^{\varepsilon,\delta} = 2\int_{B_1^+}\phi\,(-\partial_{n+1}u-k)_+\nabla A(\nabla u)\nabla(\partial_{n+1}u)\cdot\nabla\phi\,\mathrm{d}x$$
$$-\int_{B_1^+\cap\{\partial_{n+1}u\le -k\}}\phi^2\nabla A(\nabla u)\nabla(\partial_{n+1}u)\cdot\nabla(\partial_{n+1}u)\,\mathrm{d}x.$$

Similarly, to deal with the second addend in (4.12) we argue as follows: as $\nabla u_\varepsilon\to\nabla u$ strongly in $L^2_{\mathrm{loc}}(B'_1)$ by Proposition 4.2 and the compactness of the trace operator, the Lipschitz continuity of A' implies for all $\delta>0$ that

$$\lim_{\varepsilon\downarrow 0}\int_{B'_1} A'(\nabla u_\varepsilon)\cdot\nabla'\zeta_\delta\,\mathrm{d}x' = \int_{B'_1} A'(\nabla u)\cdot\nabla'\zeta_\delta = 0\,.$$

In the last equality we have used that $B'_1\cap\operatorname{spt}\zeta_\delta\subset\subset\Lambda(u)$, and being (the trace of) u in $W^{1,2}(B'_1)$ by Proposition 4.2, then $\nabla' u = 0$ $\mathcal{L}^n$ a.e. on $B'_1\cap\operatorname{spt}\zeta_\delta$, so that $A'(\nabla u)=0$ $\mathcal{L}^n$ a.e. on $B'_1\cap\operatorname{spt}\zeta_\delta$.

Hence, by using the ellipticity of ∇A we infer that for every $k>0$, by Hölder's inequality

$$\int_{B_r^+\cap\{\partial_{n+1}u<-k\}}|\nabla(\partial_{n+1}u)|^2\,\mathrm{d}x\le\frac{C}{r^2}\int_{B_{2r}^+}(-\partial_{n+1}u-k)_+^2\,\mathrm{d}x\,. \tag{4.14}$$

Clearly, (4.14) holds for $k=0$ by letting $k\downarrow 0$ in the inequality itself, and also for $k>\|\partial_{n+1}u\|_{L^\infty(B_1^+)}$ being trivial in those cases. The case of $\partial_{n+1}u$ is treated similarly. □

We are now ready to establish the claimed one-sided $C^{1,\alpha}$ regularity of u: the argument follows closely Ural'tesva [51, Lemmata 2, 3] and Giaquinta and Giusti [27].

Corollary 4.4. *Let u be the solution to the thin obstacle problem, then $u\in C^{1,\alpha}_{loc}(B_1^+\cup B'_1)$ for some $\alpha\in(0,1)$.*

Proof. By standard results in elliptic regularity we have that $u\in C^\infty(B_1^+)$. Let $x_0\in B'_1$, $\rho\in(0,1-|x_0|)$ and $\rho_j:=2^{-j}\rho$, $j\ge 0$. We start off considering the case

$$\mathcal{L}^n(\Lambda(u)\cap B'_{\rho_j}(x_0))\ge 1/2\,\mathcal{L}^n(B'_{\rho_j}(x_0)). \tag{4.15}$$

Then for all $i\in\{1,\dots,n\}$ we also get

$$\mathcal{L}^n(\{\partial_i u=0\}\cap B'_{\rho_j}(x_0))\ge 1/2\,\mathcal{L}^n(B'_{\rho_j}(x_0)).$$

Let $i\in\{1,\dots,n\}$ be fixed and set $k_j:=\frac{1}{2}(\max_{B'_{\rho_j}(x_0)}\partial_i u+\min_{B'_{\rho_j}(x_0)}\partial_i u)$. Without loss of generality, we can assume that $k_j\ge 0$ (if this is not the case, we consider $-\partial_i u$). Then,

$$\mathcal{L}^n\big(\{\partial_i u\le k_j\}\cap B'_{\rho_j}(x_0)\big)\ge 1/2\,\mathcal{L}^n\big(B'_{\rho_j}(x_0)\big).$$

By Proposition 4.2, a contradiction argument yields that the Poincaré type inequality

$$\|(\partial_i u-k)_+\|_{L^2(B_1^+)}\le C\|\nabla(\partial_i u-k)_+\|_{L^2(B_1^+)}\quad\forall\,k\ge k_j\ge 0,$$

for some constant $C = C(n) > 0$. Hence, by taking into account (4.7) in Proposition 4.3, the usual De Giorgi's argument can be run to conclude that

$$\operatorname{osc}_{B^+_{\rho_{j+1}}(x_0)}(\partial_i u) \le \kappa \operatorname{osc}_{B^+_{\rho_j}(x_0)}(\partial_i u) \tag{4.16}$$

for all $i \in \{1,\dots,n\}$, where $\kappa \in (0,1)$ depends only on L (cf. [34, Lemma 7.2]).

On the other hand, if (4.15) does not hold, then by virtue of the (ambiguous) boundary conditions in (1.2)

$$\mathcal{L}^n(\{\partial_{n+1} u = 0\} \cap B'_{\rho_j}(x_0)) \ge 1/2 \mathcal{L}^n(B'_{\rho_j}(x_0)).$$

Note that $\partial_{n+1} u|_{B'_1} \le 0$, therefore $k_{n+1} := \frac{1}{2}(\max_{B'_{\rho_j}(x_0)}(-\partial_{n+1} u) + \min_{B'_{\rho_j}(x_0)}(-\partial_{n+1} u)) \ge 0$. Thus arguing as above, in view of (4.7) we conclude that

$$\operatorname{osc}_{B^+_{\rho_{j+1}}(x_0)}(\partial_{n+1} u) \le \kappa \operatorname{osc}_{B^+_{\rho_j}(x_0)}(\partial_{n+1} u) \tag{4.17}$$

where $\kappa \in (0,1)$ depends only on L.

By means of estimates (4.16) and (4.17), we next show that for some constant $C = C(L) > 0$ and for all $r \in (0, 1 - |x_0|)$

$$\operatorname{osc}_{B^+_r(x_0)}(v) \le C\, r^\alpha\,, \tag{4.18}$$

either for $v = \partial_i u$ for all $i \in \{1,\dots,n\}$, or for $v = \partial_{n+1} u$. With this aim, fix $N \in \mathbb{N}$ and consider the radii ρ_j for $0 \le j \le 2N-1$. Clearly, we can find (at least) N radii ρ_{j_h}, $h = 1,\dots,N$, such that one between (4.16) and (4.17) holds for all such h's. In particular, we infer that for all $1 \le h \le N$

$$\operatorname{osc}_{B^+_{\rho_{j_{h+1}}}(x_0)}(v) \le \kappa \operatorname{osc}_{B^+_{\rho_{j_h}}(x_0)}(v)\,,$$

with the function v being equal either to $\partial_{n+1} u$ or to $\partial_i u$, in the latter case any $i \in \{1,\dots,n\}$ works. Thus, iteratively, we conclude that

$$\operatorname{osc}_{B^+_{\rho_{2N}}(x_0)}(v) \le \operatorname{osc}_{B^+_{\rho_{j_{N+1}}}(x_0)}(v) \le \kappa^{N+1} \operatorname{osc}_{B^+_\rho(x_0)}(v).$$

Therefore, if $r \in (0,\rho)$ let $N \in \mathbb{N}$ be such that $r \in [\rho_{2N+1}, \rho_{2N})$ we conclude then that

$$\operatorname{osc}_{B^+_r(x_0)}(v) \le \operatorname{osc}_{B^+_{\rho_{2N}}(x_0)}(v) \le \big(r/\rho\big)^{|\log_2 \kappa|/2} \operatorname{osc}_{B^+_\rho(x_0)}(v) = C\, r^\alpha.$$

Actually, the last inequality always holds true for $\partial_{n+1} u$. Indeed, considering the level $k = 0 \vee \min_{B^+_r(x_0)} v$ in Proposition 4.3, with $v = \pm\partial_i u$ and $i \in \{1,\dots,n\}$, from (4.18) we infer that

$$\int_{B^+_r(x_0)} |\nabla v|^2 \,\mathrm{d}x \le C\, r^{n-1+2\alpha}.$$

Hence, if (4.18) holds for $v = \partial_i u$ for all $i \in \{1,\dots,n\}$, then by using the estimate deriving from (4.6) as $\varepsilon \downarrow 0+$ and the latter inequality we conclude that

$$\int_{B^+_r(x_0)} |\nabla(\partial_{n+1} u)|^2 \,\mathrm{d}x \le C\, r^{n-1+2\alpha}.$$

In turn, Morrey's theorem implies that

$$\operatorname{osc}_{B^+_r(x_0)}(\partial_{n+1} u) \le C\, r^\alpha.$$

Hence, in any case we have shown that $\partial_{n+1} u \in C^{0,\alpha}_{loc}(B'_1)$. In particular, we can infer that the co-normal derivative of u is Hölder continuous in B'_1 in view of the boundary conditions in (1.2):

$$\frac{\partial_{n+1} u}{\sqrt{1+|\nabla u|^2}} = \frac{\partial_{n+1} u}{\sqrt{1+|\partial_{n+1} u|^2}} \in C^{0,\alpha}_{loc}(B'_1).$$

Note that the co-normal derivative is zero on $B'_1 \setminus \Lambda(u)$.

We next use interior regularity and boundary regularity for the Dirichlet problem for the minimal surface equation together with an ad-hoc argument to infer that $u \in C^{1,\beta}_{\rm loc}(B^+_1 \cup B'_1)$ for some $\beta = \beta(n,L) \in (0,\alpha)$, recalling that $L = \operatorname{Lip}(u)$. For the sake of simplicity we show that $u \in C^{1,\beta}\big(B^+_{3/4} \cup B'_{3/4}\big)$. Let $x_0 \in B'_{3/4}$ and $r \in (0, 1/4)$. If $B'_r(x_0) \subseteq \Lambda(u)$, we conclude by the

regularity theory for the Dirichlet problem for uniformly elliptic equations (cf. [27]) that for some $\beta > 0$

$$\Phi(x_0,\rho) := \int_{B_\rho^+(x_0)} |\nabla u - (\nabla u)_{B_\rho^+(x_0)}|^2 dx \le C\Big(\frac{\rho}{r}\Big)^{n+2\beta}\Phi(x_0,r), \tag{4.19}$$

provided that $\rho < r$, where $(v)_E := \fint_E v(x)\,\mathrm{d}x$ denotes the average of a function v in the set E. Instead, if there exists $z \in \Gamma(u) \cup (B_r'(x_0) \setminus \Lambda(u))$, then we show that

$$\Phi(x_0,\rho) \le C\Big(\frac{\rho}{r}\Big)^{n+2\beta}\Phi(x_0,r) + C[g]^2_{C^{0,\alpha}(B_{3/4})} r^{n+2\alpha}, \tag{4.20}$$

provided that $4\rho < r$. Note that (4.20) and [34, Lemma 7.3] yield for all $\rho < r \le 1/4$

$$\Phi(x_0,\rho) \le C\Big(\frac{1}{r^{n+2\beta}}\Phi(x_0,r) + [g]^2_{C^{0,\alpha}(B_{3/4})}\Big)\rho^{n+2\beta}, \tag{4.21}$$

with $C = C(n, L, \alpha, \beta) > 0$.

With the aim of proving (4.20), let w be the solution of

$$\begin{cases} \operatorname{div}\Big(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\Big) = 0 & B_r^+(x_0), \\ \partial_{n+1} w = 0 & B_r'(x_0), \\ w = u & (\partial B_r(x_0))^+. \end{cases}$$

The existence of w is guaranteed by an even reflection across $B_r'(x_0)$ of the boundary datum and by applying classical results on the existence of minimal surfaces with given Dirichlet boundary conditions (cf. [34, Chapter 1]). By simple triangular inequalities, we have

$$\Phi(x_0,r) \le 6\int_{B_r^+(x_0)} |\nabla u - \nabla w|^2 dx + 4\int_{B_r^+(x_0)} |\nabla w - (\nabla w)_{B_r^+(x_0)}|^2 dx. \tag{4.22}$$

We estimate the right hand side in (4.22) starting with the first addend. With this aim test (1.2) with $u - w \in H_0^1(B_r(x_0))$ to deduce that

$$\int_{B_r^+(x_0)} \Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\Big)\cdot\nabla(u-w)\mathrm{d}x + \int_{B_r'(x_0)} g(x')(u(x') - w(x'))\mathrm{d}x' = 0$$

where we have set $g := \frac{\partial_{n+1} u}{\sqrt{1+|\partial_{n+1}u|^2}}$. Recall that $g \in C^{0,\alpha}_{\mathrm{loc}}(B_1')$ and $g(z) = 0$. In particular, by the Divergence theorem we get

$$\begin{aligned}
(1+L^2)^{-3/2}\int_{B_r^+(x_0)} |\nabla(u-w)|^2\mathrm{d}x &\le \int_{B_r'(x_0)} |g(x') - g(z)||u(x') - w(x')|\mathrm{d}x' \\
&\le [g]_{C^{0,\alpha}(B_{3/4}')}(2r)^\alpha \int_{B_r'(x_0)} |u(x') - w(x')|\mathrm{d}x' \\
&\le [g]_{C^{0,\alpha}(B_{3/4})}(2r)^\alpha \int_{B_r^+(x_0)} \operatorname{div}(|u-w|e_{n+1})\mathrm{d}x \\
&\le [g]_{C^{0,\alpha}(B_{3/4})}(2r)^\alpha \int_{B_r^+(x_0)} |\nabla(u-w)|\mathrm{d}x \\
&\le 2^\alpha \omega_n^{1/2} [g]_{C^{0,\alpha}(B_{3/4})} r^{\alpha+n/2} \|\nabla(u-w)\|_{L^2(B_r^+(x_0))}\,.
\end{aligned} \tag{4.23}$$

Hence, for some constant $C = C(n, L) > 0$ we deduce that

$$\int_{B_r^+(x_0)} |\nabla(u-w)|^2\mathrm{d}x \le C[g]^2_{C^{0,\alpha}(B_{3/4})} r^{n+2\alpha}. \tag{4.24}$$

For the second term, we note that $w \in W^{2,2}(B_\rho^+(x_0))$ for every $\rho < r$ by arguing as in Proposition 4.2. Moreover, if $i \in \{1,\dots,n\}$ the function $\partial_i w$ is a solution of

$$\begin{cases} \operatorname{div}(\mathbb{B}(x)\nabla(\partial_i w)) = 0 & B_r^+(x_0), \\ \partial_i w = 0 & B_r'(x_0), \end{cases} \tag{4.25}$$

where the measurable matrix field $\mathbb{B}$ is given by

$$\mathbb{B}(x) := \frac{\mathrm{Id}}{\sqrt{1+|\nabla w|^2}} - \frac{\nabla w \otimes \nabla w}{(1+|\nabla w|^2)^{3/2}}.$$

By the a priori gradient estimate for the minimal surface equation by Bombieri, De Giorgi and Miranda [4], there exists a constant $C > 0$ such that

$$\|Dw\|_{L^\infty(B_{r/2}(x_0))} \le Ce^{Cr^{-1}\|w\|_{L^\infty(B_r(x_0))}} = Ce^{Cr^{-1}\|u\|_{L^\infty(\partial B_r(x_0))}} \le Ce^{CL}.$$

In particular, the matrix field $\mathbb{B}$ is coercive and bounded in $B_{r/2}$, with bounds depending only on the Lipschitz constant L of the solution u to the thin obstacle problem. Thus, by De Giorgi's theorem [8] we have that $\partial_i w \in C^{0,\beta}_{\mathrm{loc}}(B_r^+)$ with $\beta = \beta(n, L)$ and

$$\int_{B_\rho^+(x_0)} |\partial_i w - (\partial_i w)_{B_\rho^+(x_0)}|^2 \,\mathrm{d}x \le C\Big(\frac{\rho}{r}\Big)^{n+2\beta} \int_{B_r^+(x_0)} |\partial_i w - (\partial_i w)_{B_r^+(x_0)}|^2 \,\mathrm{d}x \quad \forall\, 2\rho < r. \tag{4.26}$$

In addition, being $\partial_i w$ a solution of (4.25), it satisfies a Caccioppoli's inequality

$$\int_{B_\rho^+(x_0)} |\nabla(\partial_i w)|^2 \,\mathrm{d}x \le \frac{C}{\rho^2} \int_{B_{2\rho}^+(x_0)} |\partial_i w - (\partial_i w)_{B_{2\rho}^+(x_0)}|^2 \,\mathrm{d}x\,, \tag{4.27}$$

with $4\rho < r$. Using the equation we can bound $\partial^2_{n+1} w$ with the other derivatives (cf. (4.5) and (4.6)) as follows

$$\int_{B_\rho^+(x_0)} |\nabla(\partial_{n+1} w)|^2 \,\mathrm{d}x \le C \sum_{i=1}^n \int_{B_\rho^+(x_0)} |\nabla(\partial_i w)|^2 \,\mathrm{d}x\,, \tag{4.28}$$

with $C = C(n, L) > 0$. Then, Poincare's inequality together with (4.26), (4.27) and (4.28) give

$$\int_{B_\rho^+(x_0)} |\partial_{n+1} w - (\partial_{n+1} w)_{B_r^+}|^2 \,\mathrm{d}x \le C\Big(\frac{\rho}{r}\Big)^{n+2\beta} \sum_{i=1}^n \int_{B_r^+(x_0)} |\partial_i w - (\partial_i w)_{B_r^+(x_0)}|^2 \,\mathrm{d}x\,.$$

Estimate (4.20) then follows at once from (4.22), (4.24), (4.26) and the latter inequality.

In addition, if $x_0 \in B_1^+$ and $r \le \mathrm{dist}(x_0, B_1')$, then $B_r(x_0) \subset B_1^+$. Hence, by the standard regularity theory for uniformly elliptic equations we have for all $\rho < r$

$$\Phi(x_0, \rho) \le C\Big(\frac{\rho}{r}\Big)^{n+2\beta} \Phi(x_0, r),$$

$C = C(n, L) > 0$ (cf. [34]).

From what we have proven, we deduce that there exists $C = C(n, L, \alpha, \beta) > 0$ such that for all $x_0 \in B^+_{3/4}$ and $\rho < 1/4$

$$\Phi(x_0, \rho) \le C\rho^{n+2\beta}\,,$$

from which the conclusion $u \in C^{1,\beta}\big(B^+_{3/4} \cup B'_{3/4}\big)$ readily follows by Campanato's theorem [34]. □

5. Optimal $C^{1,1/2}$-regularity

In this section we deduce the optimal $C^{1,1/2}$-regularity of the solutions u to the thin obstacle problem from results by Simon and Wickramasekera [48] on stationary graphs of two-valued functions. We give few preliminaries on the topic. We consider pairs of real valued Lipschitz functions $U = \{u_1, u_2\}$ with the components u_i defined on an open subset $\Omega \subset \mathbb{R}^N$. The union of the graphs of u_1, u_2, namely

$$\mathrm{G}_U := \big\{(x, u_i(x)) : x \in \Omega,\ i = 1, 2\big\}$$

naturally inherits the structure of rectifiable varifold, which by a slight abuse of notation we keep denoting G_U. Note that, $\mathrm{G}_U = \mathrm{G}_{u_1} + \mathrm{G}_{u_2}$ as varifolds, where G_{u_i} denotes the varifolds associated to the graphs of the real valued functions u_i. Following [48] we say that u is a two-valued minimal graph if G_U is stationary for the area functional, *i.e.*

$$\int_{\mathrm{G}_U} \mathrm{div}_{\mathrm{G}_U} Y \,\mathrm{d}\mathcal{H}^{n+1} = 0 \quad \forall\, Y \in C^\infty_c(\Omega \times \mathbb{R}),$$

where $\mathrm{div}_{\mathrm{G}_U} Y$ denotes the tangential divergence of Y in the direction of the tangent to G_U. Clearly, if u_1 and u_2 are both solutions to the minimal surface equation, then u is a two-valued minimal graph, but the vice-versa does not hold. For more on multiple valued graphs we refer to [12, 13]. In particular, we recall the definition of the metric for two-points: $U = \{u_1, u_2\}$ and $V = \{v_1, v_2\}$,

$$\mathcal{G}(U, V) := \min\Big\{\sqrt{|u_1 - v_1|^2 + |u_2 - v_2|^2}, \sqrt{|u_1 - v_2|^2 + |u_2 - v_1|^2}\Big\}.$$

For two-valued functions the usual notion of continuity and Hölder continuity can be accordingly introduced. Moreover, a two-valued function U is C^1 if there exists a continuous two-valued function $DU = \{Du_1, Du_2\}$ with $Du_i \in \mathbb{R}^n$ such that, setting $V_x(y) = \{u_1(x) + Du_1(x)(y - x), u_2(x) + Du_2(x)(y - x)\}$, we have

$$\lim_{y \to x} \frac{\mathcal{G}(U(y), V_x(y))}{|x - y|} = 0.$$

Finally, we say that U is $C^{1,\alpha}$ if DU is Hölder continuous with exponent α.

The link between the thin obstacle problem for the area functional and the two-valued minimal graphs is given in the next proposition.

Proposition 5.1. *Let u be a solution to the thin obstacle problem (1.1). Then, the multiple-valued map $U = \{u, -u\}$ is a minimal two-valued graph.*

Proof. According to the definition of minimal two-valued graphs, we need to show that

$$\int_{\mathrm{G}_U} \mathrm{div}_{\mathrm{G}_U} Y \, \mathrm{d}\mathcal{H}^{n+1} = 0 \quad \forall\, Y \in C^1_c(B_1 \times \mathbb{R}). \tag{5.1}$$

To this aim, we set

$$\mathrm{G}_1 = \mathrm{G}_{u|_{\{x_{n+1} \geq 0\}}}, \quad \mathrm{G}_2 = \mathrm{G}_{u|_{\{x_{n+1} \leq 0\}}}, \quad \mathrm{G}_3 = \mathrm{G}_{-u|_{\{x_{n+1} \geq 0\}}}, \quad \text{and} \quad \mathrm{G}_4 = \mathrm{G}_{-u|_{\{x_{n+1} \leq 0\}}}.$$

Clearly, we have that

$$\int_{\mathrm{G}_U} \mathrm{div}_{\mathrm{G}_U} Y \, \mathrm{d}\mathcal{H}^{n+1} = \sum_{i=1}^{4} \int_{\mathrm{G}_i} \mathrm{div}_{\mathrm{G}_i} Y \, \mathrm{d}\mathcal{H}^{n+1} \quad \forall\, Y \in C^\infty_c(B_1 \times \mathbb{R}). \tag{5.2}$$

Note that $u|_{\{x_{n+1} \geq 0\}}$ and $u|_{\{x_{n+1} \leq 0\}}$ are C^1 functions (cf. Proposition 3.1), therefore, G_1, G_2, G_3, G_4 are C^1-smooth submanifolds with boundary. Let $\eta_i \in \mathbb{R}^{n+2}$ be the external co-normal to $\partial \mathrm{G}_i$ (*i.e.* $|\eta_i| = 1$, η_i is normal to $\partial \mathrm{G}_i$ and tangent to G_i, pointing outward with respect to G_i). For instance, regarding η_1 we have that for every point $(x, u(x)) \in \partial \mathrm{G}_1 \cap \{x_{n+1} = 0\}$

$$\eta_1(x, u(x)) \cdot e_{n+1} < 0, \quad \eta_1(x, u(x)) \cdot \big(-\nabla u(x), 1\big) = 0$$
$$\text{and} \quad \eta_1(x, u(x)) \cdot \big(e_i + \partial_i u(x)\, e_{n+2}\big) = 0 \quad \forall\, i = 1, \dots, n.$$

Therefore, by taking into account that $\partial_i u \cdot \partial_{n+1} u = 0$ on B_1' for $i \in \{1, \dots, n\}$, in view of (1.2) and Proposition 3.1, simple algebra yields that for every $x = (x', 0)$ we have

$$\eta_1(x, u(x)) = \left(0, -\frac{1}{\sqrt{1 + |\partial_{n+1} u(x)|^2}}, -\frac{\partial_{n+1} u(x)}{\sqrt{1 + |\partial_{n+1} u(x)|^2}}\right).$$

Similarly, we have

$$\eta_2(x, u(x)) = \left(0, \frac{1}{\sqrt{1 + |\partial_{n+1} u(x)|^2}}, \frac{-\partial_{n+1} u(x)}{\sqrt{1 + |\partial_{n+1} u(x)|^2}}\right),$$

$$\eta_3(x, -u(x)) = \left(0, -\frac{1}{\sqrt{1 + |\partial_{n+1} u(x)|^2}}, \frac{\partial_{n+1} u(x)}{\sqrt{1 + |\partial_{n+1} u(x)|^2}}\right),$$

and

$$\eta_4(x, -u(x)) = \left(0, \frac{1}{\sqrt{1 + |\partial_{n+1} u(x)|^2}}, \frac{\partial_{n+1} u(x)}{\sqrt{1 + |\partial_{n+1} u(x)|^2}}\right),$$

where $\partial_{n+1}u(x',0)=\lim_{t\downarrow 0}\frac{u(x',t)}{t}$ for every $(x',0)\in\Lambda(u)$. Hence, using Stokes' theorem we infer that

$$\begin{aligned}\sum_{i=1}^4\int_{\mathrm{G}_i}\mathrm{div}_{\mathrm{G}_i}Y\,\mathrm{d}\mathcal{H}^{n+1}&=\sum_{i=1}^4\int_{\partial\mathrm{G}_i}Y\cdot\eta_i\,\mathrm{d}\mathcal{H}^n\\&=\sum_{i=1}^4\int_{\partial\mathrm{G}_i\setminus(\Lambda(u)\times\mathbb{R})}Y\cdot\eta_i\,\mathrm{d}\mathcal{H}^n+\sum_{i=1}^4\int_{\partial\mathrm{G}_i\cap(\Lambda(u)\times\mathbb{R})}Y\cdot\eta_i\,\mathrm{d}\mathcal{H}^n.\end{aligned}$$

We couple the different terms as follows:

$$\begin{aligned}&\int_{\partial\mathrm{G}_1\setminus(\Lambda(u)\times\mathbb{R})}Y\cdot\eta_1\,\mathrm{d}\mathcal{H}^n+\int_{\partial\mathrm{G}_2\setminus(\Lambda(u)\times\mathbb{R})}Y\cdot\eta_2\,\mathrm{d}\mathcal{H}^n\\&=\int_{B_1'\setminus\Lambda(u)}\Big[\Big(Y(x',0,u(x',0))\cdot\eta_1\big(x',0,u(x',0)\big)+Y(x',0,u\big(x',0)\big)\cdot\eta_2\big(x',0,u(x',0)\big)\Big)\cdot\\&\qquad\cdot\sqrt{1+|\nabla' u(x',0)|^2}\Big]\,\mathrm{d}x'=0,\end{aligned}$$

because $\eta_1(x',0,u(x',0))=-\eta_2(x',0,u(x',0))=-e_{n+1}$ for every $(x',0)\in B_1'\setminus\Lambda(u)$. For the same reasons

$$\int_{\partial\mathrm{G}_3\setminus(\Lambda(u)\times\mathbb{R})}Y\cdot\eta_3\,\mathrm{d}\mathcal{H}^n+\int_{\partial\mathrm{G}_4\setminus(\Lambda(u)\times\mathbb{R})}Y\cdot\eta_4\,\mathrm{d}\mathcal{H}^n=0.$$

Next we pair

$$\begin{aligned}&\int_{\partial\mathrm{G}_1\cap(\Lambda(u)\times\mathbb{R})}Y\cdot\eta_1\,\mathrm{d}\mathcal{H}^n+\int_{\partial\mathrm{G}_4\cap(\Lambda(u)\times\mathbb{R})}Y\cdot\eta_4\,\mathrm{d}\mathcal{H}^n\\&=\int_{\Lambda(u)}\Big(Y\big(x',0,0\big)\cdot\eta_1\big(x',0,0\big)+\\&\qquad Y\big(x',0,0\big)\cdot\eta_4\big(x',0,0\big)\Big)\sqrt{1+|\nabla' u(x',0)|^2}\,\mathrm{d}x'=0,\end{aligned}$$

where we used that $\eta_1\big(x',0,0\big)+\eta_4\big(x',0,0\big)=0$ for all $(x',0)\in B_1'$. With a similar argument, we also have

$$\int_{\partial\mathrm{G}_2\cap(\Lambda(u)\times\mathbb{R})}Y\cdot\eta_2\,\mathrm{d}\mathcal{H}^n+\int_{\partial\mathrm{G}_3\cap(\Lambda(u)\times\mathbb{R})}Y\cdot\eta_3\,\mathrm{d}\mathcal{H}^n=0.$$

Collecting the estimates above we conclude the proposition. □

Finally, Proposition 4.3, Proposition 5.1 and [48, Theorem 7.1] imply the optimal regularity for the solution to the thin obstacle problem.

Theorem 5.2. *Let $g\in C^2(\mathbb{R}^n)$ be even symmetric with respect to x_{n+1} with $g|_{\{x_{n+1}=0\}}\geq 0$, and let $u\in W^{1,\infty}(B_1)$ be the solution to the thin obstacle problem (1.1). Then, $u\in C^{1,1/2}_{\mathrm{loc}}(B_1^+\cup B_1')$.*

Proof. By Proposition 4.3 we have that there exists $\alpha\in(0,1)$ such that the two valued function $U=\{-u,u\}\in C^{1,\alpha}_{\mathrm{loc}}(B_1)$, because $[DU]_{C^{0,\alpha}(B_1)}\leq[Du]_{C^{0,\alpha}(B_1^+)}$ (here $[\cdot]_{C^{0,\alpha}(E)}$ denotes the Hölder seminorm of the relevant function on the set E). By Proposition 5.1 we have that the graph of U induces a two-valued minimal graph; we are in the position to apply [48, Theorem 7.1] and conclude that $U\in C^{1,1/2}_{\mathrm{loc}}(B_1)$, or equivalently $u\in C^{1,1/2}_{\mathrm{loc}}(B_1^+\cup B_1')$. □

6. The structure of the free boundary

In this section we provide a detailed analysis of the free boundary points for the thin obstacle problem for the area functional. As mentioned in the introduction we prove more refined conclusions than those contained in Theorem 1.1, recovering the analogous results shown for the Dirichlet energy in [2, 21].

To state the result we need to introduce three classes of functions Φ_m, Ψ_m and Π_m for $m \in \mathbb{N}\setminus\{0\}$, that are explicitly defined as follows:

$$\Phi_m(x_1,x_2) := \mathrm{Re}\Big[(x_1+i|x_2|)^{2m}\Big], \tag{6.1}$$

$$\Psi_m(x_1,x_2) := \mathrm{Re}\Big[(x_1+i|x_2|)^{2m-1/2}\Big], \tag{6.2}$$

$$\Pi_m(x_1,x_2) := \mathrm{Im}\Big[(x_1+i|x_2|)^{2m+1}\Big]. \tag{6.3}$$

Such families of functions exhaust the homogeneous solutions to the thin obstacle problem with null obstacle for the Dirichlet energy having top dimensional subspaces of invariances (cp. [21, Appendix A]).

Moreover, we recall that $I_u(x_0,\cdot)$, $x_0 \in B_1'$, denotes the frequency function defined in (1.3) that shall be studied in the next subsection. In particular, we shall prove that there exists finite its limit value in 0^+ denoted in what follows by $I_u(x_0,0^+)$ for all $x_0 \in \Gamma(u)$.

The following is the main theorem.

Theorem 6.1. *Let u be a solution to the thin obstacle problem (1.1). Then,*

(i) *$\Gamma(u)$ has locally finite $(n-1)$-dimensional Minkowski content, i.e. for every $K \subset\subset B_1'$ there exists a constant $C(K)>0$ such that*

$$\mathcal{L}^{n+1}\big(\mathcal{T}_r(\Gamma(u))\cap K\big) \le C(K)\, r^2 \quad \forall\, r\in(0,1), \tag{6.4}$$

where $\mathcal{T}_r(E) := \{x\in\mathbb{R}^{n+1} : \mathrm{dist}(x,E)<r\}$;

(ii) *$\Gamma(u)$ is $\mathcal{H}^{n-1}$-rectifiable, i.e. there exist at most countably many C^1-regular submanifolds $M_i\subset\mathbb{R}^n$ of dimension $n-1$ such that*

$$\mathcal{H}^{n-1}\big(\Gamma(u)\setminus\cup_{i\in\mathbb{N}}M_i\big)=0\,; \tag{6.5}$$

(iii) *$\Gamma_{3/2}(u) := \{x_0\in\Gamma(u) : I_u(x_0,0^+)=3/2\}$ is locally a $C^{1,\alpha}$ regular submanifold of dimension $n-1$ for some dimensional constant $\alpha>0$.*

Moreover, there exists a subset $\Sigma(u)\subset\Gamma(u)$ with Hausdorff dimension at most $n-2$ such that

$$I_u(x_0,0^+)\in\{2m,2m-1/2,2m+1\}_{m\in\mathbb{N}\setminus\{0\}} \quad \forall\, x_0\in\Gamma(u)\setminus\Sigma(u).$$

Theorem 6.1 generalizes to the nonlinear setting of minimal surfaces the known results for the regularity of the free boundary shown for the fractional obstacle problem. The conclusion in (iii) extends the analysis of the regular part of the free boundary done in [2] and its proof follows from [26] as a consequence of the epiperimetric inequality established in [20, 26]. While for the rest, the statements are modelled on our results in [21] and the proof is accomplished by the same arguments exploited for the Dirichlet energy in [21]; for the sake of completeness, in the following we provide the readers with the details of the needed changes.

6.1. **Obstacle problems for Lipschitz quadratic energies.** Given a solution u to (1.1), it follows from (1.2) that u minimizes the following thin obstacle problem for a specific quadratic energy:

$$\mathcal{Q}:\mathcal{A}_g\ni v\longmapsto \frac12\int_{B_1}\vartheta(x)|\nabla v(x)|^2\,dx, \qquad \text{with } \vartheta(x):=\big(1+|\nabla u(x)|^2\big)^{-1/2}. \tag{6.6}$$

Note that the above functional is coercive because

$$0<\big(1+L^2\big)^{-1/2}\le\vartheta(x)\le 1\,, \tag{6.7}$$

where as usual $L=\mathrm{Lip}(u)$. Actually, $\vartheta(x)=1$ if $x\in\Gamma(u)$. Moreover, we have that ϑ is a Lipschitz function, as proven in the following lemma.

Lemma 6.2. *Let u be a solution to the thin obstacle problem (1.1), then $\vartheta\in W^{1,\infty}(B_1)$.*

Proof. Setting $d(x) := \mathrm{dist}(x,\Gamma(u))$, $x\in B_1$, by the regularity result in Theorem 5.2 and the classical Schauder estimates we deduce that

$$|u(x)|\le C\,d^{3/2}(x), \quad |\nabla u(x)|\le C\,d^{1/2}(x) \quad\text{and}\quad |\nabla^2 u(x)|\le C\,d^{-1/2}(x),$$

for some constant $C > 0$, and therefore

$$|\nabla\vartheta| = \big(1+|\nabla u|^2\big)^{-3/2}\,|\nabla^2 u\,\nabla u| \le C. \qquad \square$$

The basic idea of exploiting the regularity of u itself to reduce the problem to quadratic energies with Lipschitz coefficients has been recently considered in the literature for the classical obstacle problem (see, *e.g.*, [17, 43, 18]).

6.2. **Frequency function.** Given the Lipschitz continuity of ϑ we can prove monotonicity of the following frequency type function at a point $x_0 \in B_1'$ defined by

$$I_u(x_0,t) := \frac{rD_u(x_0,t)}{H_u(x_0,t)} \quad \forall\, r < 1-|x_0|,$$

where

$$D_u(x_0,t) := \int \phi\big(\tfrac{|x-x_0|}{t}\big)\vartheta(x)\,|\nabla u(x)|^2 \mathrm{d}x,$$

and

$$H_u(x_0,t) := -\int \phi'\Big(\tfrac{|x-x_0|}{t}\Big)\,\vartheta(x)\tfrac{u^2(x)}{|x-x_0|}\,\mathrm{d}x.$$

Here, $\phi : [0,+\infty) \to [0,+\infty)$ is the function given by

$$\phi(t) := \begin{cases} 1 & \text{for } \ 0 \le t \le \frac12, \\ 2\,(1-t) & \text{for } \ \frac12 < t \le 1, \\ 0 & \text{for } \ 1 < t. \end{cases}$$

It is also useful to introduce

$$E_u(x_0,t) := -\int \phi'\Big(\tfrac{|x-x_0|}{t}\Big)\vartheta(x)\tfrac{|x-x_0|}{t^2}\Big(\nabla u(x)\cdot\tfrac{x-x_0}{|x-x_0|}\Big)^2\,\mathrm{d}x.$$

In what follows, we shall not highlight the dependence on the base point x_0 in the quantities above if it coincides with the origin.

By exploiting the integration by parts formulas used in [17], we show the following variant of the monotonicity formula for the frequency.

Proposition 6.3. *Let u be a solution to the thin obstacle problem (1.1) in B_1. Then, there exists a nonnegative constant $C_{6.3}$ depending on $\mathrm{Lip}(u)$, such that for all $x_0 \in B_1'$, and for $\mathcal{L}^1$ a.e. $t \in (0, 1-|x_0|)$*

$$I_u'(x_0,t) = \frac{2t}{H_u^2(x_0,t)}\big(H_u(x_0,t)E_u(x_0,t) - D_u^2(x_0,t)\big) + R_u(x_0,t), \tag{6.8}$$

with $|R_u(x_0,t)| \le C_{6.3} I_u(x_0,t)$. In particular, the function $(0,1-|x_0|) \ni t \mapsto e^{C_{6.3}t} I_u(x_0,t)$ is nondecreasing and

$$e^{C_{6.3}r_1} I_u(x_0,r_1) - e^{C_{6.3}r_0} I_u(x_0,r_0) \ge \int_{r_0}^{r_1} \frac{2\,t\,e^{C_{6.3}t}}{H_u^2(x_0,t)}\Big(H_u(x_0,t)\;E_u(x_0,t) - D_u^2(x_0,t)\Big)\,\mathrm{d}t, \tag{6.9}$$

for $0 < r_0 < r_1 < 1-|x_0|$, and the limit $I_u(x_0,0^+) = \lim_{t\downarrow 0} I_u(x_0,t)$ exists finite.

Proof. We need to estimate the derivatives of D_u and H_u: by exploiting the integration by parts formulas used in [17] one can show that for every $x_0 \in B_1'$ and for $\mathcal{L}^1$ a.e. $r \in (0,1-|x_0|)$,

$$D_u'(x_0,r) = \frac{n-1}{r} D_u(x_0,r) + 2E_u(x_0,r) + \varepsilon_D(x_0,r), \tag{6.10}$$

and

$$H_u'(x_0,r) = \frac{n}{r} H_u(x_0,r) + 2D_u(x_0,r) + \varepsilon_H(x_0,r), \tag{6.11}$$

with $|\varepsilon_D(x_0,r)| \le C\,D_u(x_0,r)$ and $|\varepsilon_H(x_0,r)| \le C H_u(x_0,r)$ for some constant $C > 0$ depending on $\mathrm{Lip}(u)$. Moreover, for all $0 < r < 1-|x_0|$,

$$D_u(x_0,r) = -\tfrac1r \textstyle\int \phi'\big(\tfrac{|x-x_0|}{r}\big)\,\vartheta(x)u(x)\nabla u(x)\cdot\tfrac{x-x_0}{|x-x_0|}\,\mathrm{d}x. \tag{6.12}$$

The details of (6.10), (6.11) and (6.12) are postponed to the appendix.

For the sake of simplicity assume $x_0 = \underline{0}$ (recall that in this case we drop the dependence on the base point in the relevant quantities). By (6.11) and (6.10), we compute the derivative of $\log I_u(t)$ as follows:

$$\frac{I_u'(t)}{I_u(t)} = \frac{1}{t} + \frac{D_u'(t)}{D_u(t)} - \frac{H_u'(t)}{H_u(t)} = 2\,\frac{E_u(t)}{D_u(t)} - 2\frac{D_u(t)}{H_u(t)} + \frac{\varepsilon_D(t)}{D_u(t)} - \frac{\varepsilon_H(t)}{H_u(t)}.$$

Hence, being $|\varepsilon_D(t)| \le C D_u(t)$ and $|\varepsilon_H(t)| \le C H_u(t)$, (6.8) readily follows. In addition,

$$I_u'(t) + C_{6.3} I_u(t) \ge \frac{2t}{H_u^2(t)}\big(H_u(t)E_u(t) - D_u^2(t)\big),$$

thus leading to inequality (6.9) by multiplying with $e^{C_{6.3}t}$ and integrating. Finally, by (6.12) and the Cauchy–Schwarz inequality, the map $t \mapsto e^{C_{6.3}t} I_u(t)$ is non-decreasing. □

We also derive additive quasi-monotonicity formula for the frequency.

Corollary 6.4. *Let u be a solution to the thin obstacle problem (1.1) in B_1. For every $A > 0$ there exists $C_{6.4} = C_{6.4}(\mathrm{Lip}(u), A) > 0$ such that for all $x_0 \in B_1'$ with $I_u(x_0, r) \le A$, $r \in (0, 1 - |x_0|)$, then*

$$(0, r] \ni t \longmapsto I_u(x_0, t) + C_{6.4}\, t \quad \textit{is nondecreasing.} \tag{6.13}$$

Proof. Proposition 6.3 implies that $I_u(x_0, t) \le e^{C_{6.3}} A$ for all $t \in (0, r]$. Therefore, from inequality (6.8) and the estimate on the rest $R_u(x_0, t)$, we deduce the conclusion with $C_{6.4} := C_{6.3}\, e^{C_{6.3}\, A}$. □

6.3. **Lower bound on the frequency and compactness.** The frequency of a solution to (1.1) at free boundary points is bounded from below by a universal constant. A preliminary lemma is needed.

Lemma 6.5. *Let u be a solution to the thin obstacle problem (1.1) in B_1. Then, there exists a constant $C = C(n, [\nabla u]_{C^{0,1/2}(B^+_{3/4})}) > 0$ such that for every $x_0 \in \Gamma(u) \cap B_{1/4}$ and for every $0 < r < 1/2$*

$$\int_{\partial B_r(x_0)} |u(x)|^2 \mathrm{d}x \le Cr \int_{B_r(x_0)} |\nabla u(x)|^2 \mathrm{d}x + Cr^{n+3}. \tag{6.14}$$

Proof. By Poincaré-Wirtinger inequality we have

$$\int_{\partial B_r(x_0)} |u(x)|^2 \mathrm{d}\mathcal{H}^n \le Cr \int_{B_r(x_0)} |\nabla u(x)|^2 \mathrm{d}x + Cr^n \Big(\fint_{\partial B_r(x_0)} u(x)\, \mathrm{d}\mathcal{H}^n \Big)^2, \tag{6.15}$$

for some dimensional constant $C > 0$. To estimate the mean value of u we argue as follows. By direct calculation

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\fint_{B_r(x_0)} u(x)\, \mathrm{d}x \Big) = \fint_{B_r(x_0)} \langle \nabla u(x), \frac{x}{r} \rangle \mathrm{d}x.$$

Therefore, recalling that $\nabla u(x_0) = \underline{0}$ since $x_0 \in \Gamma(u)$, by one-sided $C^{1,1/2}$ regularity we find that

$$\left| \frac{\mathrm{d}}{\mathrm{d}r}\Big(\fint_{B_r(x_0)} u(x)\, \mathrm{d}x \Big) \right| \le \fint_{B_r(x_0)} |\nabla u(x)| \mathrm{d}x \le C r^{1/2} \tag{6.16}$$

with $C = C(n, [\nabla u]_{C^{0,1/2}(B^+_{3/4})}) > 0$. Hence, recalling that $u(x_0) = 0$, by integration we infer that

$$\left| \fint_{B_r(x_0)} u(x)\, \mathrm{d}x \right| \le C\, r^{3/2}. \tag{6.17}$$

Finally, noting that

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\fint_{B_r(x_0)} u(x)\, \mathrm{d}x \Big) = \frac{n+1}{r}\Big(\fint_{\partial B_r(x_0)} u(x)\, \mathrm{d}\mathcal{H}^n - \fint_{B_r(x_0)} u(x)\, \mathrm{d}x \Big)$$

we conclude from (6.16) and (6.17) that for some $C = C(n, [\nabla u]_{C^{0,1/2}(B^+_{3/4})}) > 0$ we have

$$\left| \fint_{\partial B_r(x_0)} u(x)\, \mathrm{d}\mathcal{H}^n \right| \le C r^{3/2}.$$

In turn, the latter inequality and (6.15) yield (6.14). □

A first rough bound from below on the frequency then easily follows.

Lemma 6.6. *Let u be a solution to the thin obstacle problem (1.1) in B_1. There exist a constant $C_{6.6} = C_{6.6}(n, L, [\nabla u]_{C^{0,1/2}(B^+_{3/4})}) > 0$ and a radius $r_{6.6} = r_{6.6}(n, L, [\nabla u]_{C^{0,1/2}(B^+_{3/4})}) \in (0, 1/2)$ such that, for every $x_0 \in \Gamma(u) \cap B_{1/4}$ we have for all $r \in (0, r_{6.6})$*

$$I_u(x_0, r) \geq C_{6.6}. \tag{6.18}$$

Proof. The co-area formula and an integration by parts give

$$H_u(x_0, r) = 2 \int_{\frac{r}{2}}^r \frac{dt}{t} \int_{\partial B_t(x_0)} \vartheta(x) |u(x)|^2 \mathrm{d}\mathcal{H}^n(x), \tag{6.19}$$

and

$$D_u(x_0, r) = \frac{2}{r} \int_{\frac{r}{2}}^r dt \int_{B_t(x_0)} \vartheta(x) |\nabla u(x)|^2 \, \mathrm{d}x \tag{6.20}$$

(cf. [21, Lemma 2.9]). Moreover, recalling that $\vartheta \in \mathrm{Lip}(B_1)$ with $\vartheta(x_0) = 1$ as $x_0 \in \Gamma(u)$ and $0 < \vartheta(x) \leq 1$ for all $x \in B_1'$, we conclude from (6.19) that

$$\begin{aligned} &\left| H_u(x_0, r) - 2 \int_{\frac{r}{2}}^r \frac{dt}{t} \int_{\partial B_t(x_0)} |u(x)|^2 \mathrm{d}\mathcal{H}^n(x) \right| \leq 2L \int_{\frac{r}{2}}^r dt \int_{\partial B_t(x_0)} |u(x)|^2 \mathrm{d}\mathcal{H}^n(x) \\ &= 2L \int_{B_r(x_0) \setminus B_{r/2}(x_0)} |u(x)|^2 \, \mathrm{d}x \overset{(A.9)}{\leq} 2L \sqrt{1 + L^2} \, e^{C_{A.2} r} r \, H_u(x_0, r). \end{aligned}$$

Hence, we find that

$$H_u(x_0, r) \leq 4 \int_{\frac{r}{2}}^r \frac{dt}{t} \int_{\partial B_t(x_0)} |u(x)|^2 \mathrm{d}\mathcal{H}^n(x) \tag{6.21}$$

provided that $0 < r \leq r_{6.6} \leq (4L\sqrt{1 + L^2} \, e^{C_{A.2}})^{-1}$.

Analogously, by taking into account (6.20) we have that

$$\left| D_u(x_0, r) - \frac{2}{r} \int_{\frac{r}{2}}^r dt \int_{B_t(x_0)} |\nabla u(x)|^2 \mathrm{d}\mathcal{H}^n(x) \right| \leq 2L \int_{\frac{r}{2}}^r dt \int_{B_t(x_0)} |\nabla u(x)|^2 \mathrm{d}\mathcal{H}^n(x),$$

from which we deduce that

$$\frac{2}{r} \int_{\frac{r}{2}}^r dt \int_{B_t(x_0)} |\nabla u(x)|^2 \mathrm{d}\mathcal{H}^n(x) \leq 4 D_u(x_0, r) \tag{6.22}$$

as $0 < r \leq r_{6.6} \leq \frac{1}{2L}$.

In particular, from the Poincaré inequality (6.14) and estimates (6.21), (6.22) we get for some constant $C = C(n, [\nabla u]_{C^{0,1/2}(B^+_{3/4})}) > 0$

$$H_u(x_0, r) \leq C \, r D_u(x_0, r) + C \, r^{n+3},$$

for all $r \in (0, r)$ provided that $r_{6.6} \leq (4L\sqrt{1 + L^2} \, e^{C_{A.2}})^{-1} \wedge 1/2$. Then, either $e^{C_{6.3} \, r} I_u(x_0, r) \geq 1$ for every $r \in (0, r_{6.6})$, from which we infer $I_u(x_0, r) \geq e^{-C_{6.3}}$ for all $r \in (0, r_{6.6})$; or $e^{C_{6.3} \, r} I_u(x_0, r) < 1$ for all $r \in (0, r_{x_0})$, $r_{x_0} < r_{6.6}$, by Proposition 6.3. In the last instance, $I_u(x_0, r) \geq e^{-C_{6.3}}$ for all $r \in [r_{x_0}, r_{6.6})$, and $I_u(x_0, r) < 1$ for every $r \in (0, r_{x_0})$. Thus, we have that $H_u(x_0, r) \geq e^{-C_{A.2}} H_u(x_0, r_{x_0}) \left(\frac{r}{r_{x_0}} \right)^{n+1}$ for all radii $r \in (0, r_{x_0})$ (cf. (A.7) in the appendix). In particular, for such radii we conclude that

$$I_u(x_0, r) \geq \frac{1}{C} - e^{C_{A.2}} \frac{r_{x_0}^{n+1}}{H_u(x_0, r_{x_0})} \, r^2,$$

and thus there exists $\rho_{x_0} \leq r_{x_0}$ such that

$$I_u(x_0, r) \geq \frac{1}{2C}$$

for all $r \in (0, \rho_{x_0})$. In turn, this and the quasi-monotonicity of the frequency in Proposition 6.3 yield that for all $r \in (0, r_{6.6})$

$$I_u(x_0, r) \geq \frac{e^{-C_{6.3}}}{2C}\,. \qquad \square$$

6.4. **Blowup profiles.** An important consequence of the quasi-monotonicity of the frequency in Proposition 6.3 and of the universal lower bound for the frequency in Lemma 6.6 is the existence of nontrivial blowup profiles. For $u : B_1 \to \mathbb{R}$ solution of (1.1) we introduce the rescalings

$$u_{x_0,r}(y) := \frac{r^{n/2}\, u(x_0 + ry)}{H^{1/2}(x_0, r)} \quad \forall\, r \in (0, 1 - |x_0|),\ \forall\, y \in B_{\frac{1-|x_0|}{r}}\,. \tag{6.23}$$

By the same arguments exploited in the blowup analysis in [21, Section 2.5], for every $x_0 \in \Gamma(u)$ and for every sequence of numbers $(r_j)_{j\in\mathbb{N}} \subset (0, 1 - |x_0|)$ with $r_j \downarrow 0$, there exist a subsequence $(r_{j_k})_{k\in\mathbb{N}}$ and a function $u_0 \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ such that

$$u_{x_0,r_{j_k}} \to u_0 \quad \text{in } W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}). \tag{6.24}$$

Moreover, u_0 is the solution to the Signorini problem for the Dirichlet energy on $\mathbb{R}^{n+1}$, *i.e.* satisfying

$$\begin{cases} \triangle u_0 = 0 & \text{in } \{x_{n+1} > 0\}, \\ \partial_{n+1} u_0 \leq 0 \quad \text{and} \quad u_0\, \partial_{n+1} u_0 = 0 & \text{on } \{x_{n+1} = 0\}, \end{cases} \tag{6.25}$$

and u_0 is $I_u(x_0, 0^+)$-homogeneous, because by rescaling $I_{u_0}(\underline{0}, r) = I_u(x_0, 0^+)$ for every $r > 0$.

In particular, the classification of the blowup profiles is the same as for the Dirichlet energy, and consists in the functions Φ_m, Ψ_m, Π_m in (6.1), (6.2) and (6.3) in case the subspace of invariant directions has maximal dimension.

6.5. **Spatial oscillation for the frequency.** Next we recall the basic estimate on the spatial oscillation of the frequency which is at the heart of the analysis in [21]. We introduce the notation: for a point $x \in B'_1$ and a radius $0 < \rho < r$, we set

$$\Delta^r_\rho(x) := I_u(x, r) + C_{6.4} r - I_u(x, \rho) - C_{6.4}\rho.$$

Note that $\Delta^r_\rho(x) \geq 0$ in view of Corollary 6.4.

The following proposition is a straightforward extension of [21, Proposition 3.3].

Proposition 6.7. *For every $A > 0$ there exists $C_{6.7}(n, \mathrm{Lip}(u), A) > 0$ such that, if $\rho > 0$, $R > 9$ and $u : B_{4R\rho}(x_0) \to \mathbb{R}$ is a solution to the thin obstacle problem (1.1) in $B_{4R\rho}(x_0)$, with $x_0 \in \Gamma(u)$ and $I_u(x_0, 4R\rho) \leq A$, then*

$$\big|I_u\big(x_1, R\rho\big) - I_u\big(x_2, R\rho\big)\big| \leq C_{6.7}\left[\Big(\Delta^{2(R+2)\rho}_{(R-4)\rho/2}(x_1)\Big)^{1/2} + \Big(\Delta^{2(R+2)\rho}_{(R-4)\rho/2}(x_2)\Big)^{1/2}\right] + C_{6.7} R\rho, \tag{6.26}$$

for every $x_1, x_2 \in B'_\rho$.

Proof. The proof is a variant of [21, Proposition 3.3]. For readers' convenience, we repeat some of the arguments with the necessary changes.

Without loss of generality, we consider $x_0 = \underline{0}$. With fixed $x_1, x_2 \in B'_\rho$, let $x_t := tx_1 + (1-t)x_2$, $t \in [0, 1]$, and consider the map $t \mapsto I_u(x_t, R\rho)$. Set $e := x_1 - x_2$, then $e \cdot e_{n+1} = 0$. Since the functions $x \mapsto H_u(x, R\rho)$ and $x \mapsto D_u(x, R\rho)$ are differentiable, we get

$$I_u(x_1, R\rho) - I_u(x_2, R\rho) = \int_0^1 \partial_e I_u(x_t, R\rho)\, \mathrm{d}t. \tag{6.27}$$

To compute the last integrand, we start off with noting that for all $\lambda \in \mathbb{R}$

$$\begin{aligned}\partial_e H_u(x_t, R\rho) &= -\int \phi'\big(\tfrac{|y|}{R\rho}\big)\,\tfrac{1}{|y|}\big(2\vartheta(y+x_t)\,u(y+x_t)\,\partial_e u(y+x_t) + \partial_e\vartheta(y+x_t)\,u^2(y+x_t)\big)\,\mathrm{d}y\\ &= -2\int \phi'\big(\tfrac{|y|}{R\rho}\big)\,\tfrac{1}{|y|}\vartheta(y+x_t)\,u(y+x_t)\big(\partial_e u(y+x_t) - \lambda u(y+x_t)\big)\,\mathrm{d}y\\ &\quad + 2\lambda H_u(x_t, R\rho) - \int \phi'\big(\tfrac{|y|}{R\rho}\big)\,\partial_e\vartheta(y+x_t)\,\tfrac{u^2(y+x_t)}{|y|}\,\mathrm{d}y, \end{aligned} \tag{6.28}$$

and by Proposition 4.2

$$\begin{aligned}\partial_e D_u(x_t, R\rho) &= \int \phi\big(\tfrac{|y|}{R\rho}\big)\big(2\vartheta(y+x_t)\nabla u(y+x_t)\cdot\nabla(\partial_e u)(y+x) \ + \partial_e\vartheta(y+x_t)|\nabla u(y+x_t)|^2\big)\,\mathrm{d}y\\ &= -\frac{2}{R\rho}\int \phi'\big(\tfrac{|y|}{R\rho}\big)\,\vartheta(y+x_t)\partial_e u(y+x_t)\,\nabla u(y+x_t)\cdot\tfrac{y}{|y|}\,\mathrm{d}y\\ &\quad + \int \phi\big(\tfrac{|y|}{R\rho}\big)\partial_e\vartheta(y+x_t)|\nabla u(y+x_t)|^2\,\mathrm{d}y\\ &\overset{(6.12)}{=} -\frac{2}{R\rho}\int \phi'\big(\tfrac{|y|}{R\rho}\big)\,\vartheta(y+x_t)\big(\partial_e u(y+x_t) - \lambda u(y+x_t)\big)\nabla u(y+x_t)\cdot\tfrac{y}{|y|}\,\mathrm{d}y\\ &\quad + 2\lambda D_u(x_t, R\rho) + \int \phi\big(\tfrac{|y|}{R\rho}\big)\partial_e\vartheta(y+x_t)|\nabla u(y+x_t)|^2\,\mathrm{d}y. \end{aligned} \tag{6.29}$$

To deduce the second equality we have applied the divergence theorem to the vector field $V(y) := \phi\big(\frac{|y|}{R\rho}\big)\,\vartheta(x+y)\,\partial_e u(y+x)\,\nabla u(y+x)$ (note that $V \in C^\infty(B_{R\rho}\setminus B'_{R\rho})$, V has zero trace on $\partial B_{R\rho}$ and the divergence of V does not concentrate on B'_1).

Then, by formulas (6.28) and (6.29), we have that

$$\begin{aligned}\partial_e I_u(x_t, R\rho) &= I_u(x_t, R\rho)\left(\frac{\partial_e D_u(x_t, R\rho)}{D_u(x_t, R\rho)} - \frac{\partial_e H_u(x_t, R\rho)}{H_u(x_t, R\rho)}\right)\\ &= \tfrac{2}{H_u(x_t,R\rho)}\int \phi'\big(\tfrac{|z-x_t|}{R\rho}\big)\,\tfrac{\vartheta(z)}{|z-x_t|}\big(\partial_e u(z) - \lambda u(z)\big)\big(\nabla u(z)\cdot(z-x_t) - I_u(x_t,R\rho)u(z)\big)\,\mathrm{d}z\\ &\quad + \tfrac{I_u(x_t,R\rho)}{D_u(x_t,R\rho)}\int \phi\big(\tfrac{|z-x_t|}{R\rho}\big)\partial_e\vartheta(z)|\nabla u(z)|^2\,\mathrm{d}z - \tfrac{I_u(x_t,R\rho)}{H_u(x_t,R\rho)}\int \phi'\big(\tfrac{|z-x_t|}{R\rho}\big)\,\partial_e\vartheta(z)\,\tfrac{u^2(z)}{|z-x_t|}\,\mathrm{d}z\\ &=: J_t^{(1)} + J_t^{(2)} + J_t^{(3)}. \end{aligned} \tag{6.30}$$

The estimate of $J_t^{(1)}$ is at all analogous to the estimate in [21, Proposition 3.3] and yields

$$J_t^{(1)} \le C\big(\triangle^{2(R+2)\rho}_{R\rho/2-2\rho}(x_1)\big)^{1/2} + C\big(\triangle^{2(R+2)\rho}_{R\rho/2-2\rho}(x_2)\big)^{1/2} \tag{6.31}$$

Recalling that ϑ is Lipschitz continuous, for $J_t^{(2)}$ and $J_t^{(3)}$ we get that there exists a constant $C = C(\mathrm{Lip}(u), A) > 0$ such that

$$|J_t^{(2)} + J_t^{(3)}| \le C|x_1 - x_2|. \tag{6.32}$$

By collecting (6.27), (6.30), (6.32) and (6.31) we conclude. □

6.6. **Proof of Theorem 6.1.** For the proof of the main theorem is now a straightforward adaptation of the arguments in [21]. We omit it the details and only recall the main steps of the proof.

6.6.1. *Mean-flatness.* Using the estimate on the spatial oscillation of the frequency in Proposition 6.7, one can easily prove the analog of [21, Proposition 4.2]: for every $A > 0$ and $R > 6$ there exists a constant $C > 0$ such that if u is a solution to (1.1) in $B_{(4R+10)r}(x_0)$, with $x_0 \in \Gamma(u)$ and with $I_u(x_0, (4R+10)r) \le A$, then for every μ finite Borel measure with $\mathrm{spt}\,(\mu) \subseteq \Gamma(u)$ and for all $p \in \Gamma(u) \cap B'_r(x_0)$ we have

$$\beta^2_\mu(p, r) \le \frac{C}{r^{n-1}}\Big(\int_{B_r(p)} \Delta^{(2R+4)\,r}_{(R-5)\,r/2}(x)\,\mathrm{d}\mu(x) + r^2\mu(B_r(p))\Big), \tag{6.33}$$

where the *mean flatness* of μ is defined by

$$\beta_\mu(x,r) := \inf_{\mathcal{L}} \Big(r^{-n-1} \int_{B_r(x)} \operatorname{dist}^2(y,\mathcal{L}) \mathrm{d}\mu(y) \Big)^{1/2}, \tag{6.34}$$

the infimum being taken among all affine $(n-1)$-dimensional planes $\mathcal{L} \subset \mathbb{R}^{n+1}$.

6.6.2. *Rigidity of homogeneous solutions.* We set for $x_0 \in B_1'$ and $t < 1 - |x_0|$

$$J_u(x_0,t) := e^{C_{6.3}t} I_u(x_0,t)$$

and given $\eta, r > 0$, $4r < 1 - |x_0|$, we say that a solution $u : B_{4r}(x_0) \to \mathbb{R}$, $x_0 \in \{x_{n+1} = 0\}$, to the thin obstacle problem (1.1) is *η-almost homogeneous in $B_{4r}(x_0)$* if

$$J_u(x_0,r) - J_u\big(x_0, r/2\big) \le \eta.$$

Then, by the compactness argument in [21, Proposition 5.6], the following rigidity property holds: for every $\tau, A > 0$ there exist $\eta > 0$ and $r_0 > 0$ such that, if $r < r_0$ and $u : B_{4r}(x_0) \to \mathbb{R}$, with $x_0 \in \{x_{n+1} = 0\}$, is a η-almost homogeneous solution in $B_{4r}(x_0)$ of the thin obstacle problem (1.1) with $x_0 \in \Gamma(u)$ and $J_u(x_0, 4r) \le A$, then

(i) either for every point $x \in \Gamma(u) \cap B_{2r}(x_0)$ we have

$$|J_u(x,r) - J_u(x_0,r)| \le \tau, \tag{6.35}$$

(ii) or there exists a linear subspace $V \subset \mathbb{R}^n \times \{0\}$ of dimension $n-2$ such that

$$\begin{cases} y \in \Gamma(u) \cap B_{2r}(x_0), \\ J_u(y,r) - J_u(y, r/2) \le \eta \end{cases} \quad \Longrightarrow \quad \operatorname{dist}(y,V) < \tau r. \tag{6.36}$$

6.6.3. *Proof of Theorem 6.1.* Finally, the main results can be obtained by following verbatim [21, Sections 6–8] (see also [22]). Indeed, [21, Proposition 6.1], that leads to the local finiteness of the Minkowskii content in item (i) of Theorem 6.1, is based on a covering argument that exploits the lower bound on the frequency in Lemma 6.6, the rigidity of almost homogeneous solutions in Subsection 6.6.2, the control of the mean oscillation via the frequency in Subsection 6.6.1 and the discrete Reifenberg theorem by Naber & Valtorta [44, Theorem 3.4].

Similarly, the $\mathcal{H}^{n-1}$-rectifiability of $\Gamma(u)$ in Theorem 6.1 (ii) is a consequence of the rectifiability criterion by Azzam & Tolsa [3, Theorem 1.1] and Naber & Valtorta [44, Theorem 3.4] together with the estimate in Subsection 6.6.1 and item (i) of Theorem 6.1 itself.

The $C^{1,\alpha}$-regularity of $\Gamma_{3/2}(u)$ follows from the approach via an epiperimetric inequality [26] being ϑ Lipschitz continuous (see also [20] for the proof of the epiperimetric inequality).

Finally, the classification of blow-up limits is exactly that stated in [21, Theorem 1.3], and proved in [21, Section 8] (see also [22]).

Appendix A. Variation formulas

In this section we show the computations for the monotonicity of the frequency based on the integration formulas exploited in [17] for the classical obstacle problem.

Proposition A.1. *Let u be a solution to the thin obstacle problem (1.1) in B_1. There exists a non negative constant $C_{A.1}$ depending on $\operatorname{Lip}(u)$, such that for every $x_0 \in B_1'$ and for $\mathcal{L}^1$ a.e. $r \in (0, 1 - |x_0|)$,*

$$D_u'(x_0,r) = \frac{n-1}{r} D_u(x_0,r) + 2E_u(x_0,r) + \varepsilon_D(x_0,r), \tag{A.1}$$

with $|\varepsilon_D(x_0,r)| \le C_{A.1}\, D_u(x_0,r)$.

Moreover, for all $0 < r < 1 - |x_0|$,

$$D_u(x_0,r) = -\tfrac{1}{r} \textstyle\int \phi'\big(\frac{|x-x_0|}{r}\big)\, \vartheta(x) u(x) \nabla u(x) \cdot \frac{x-x_0}{|x-x_0|}\, \mathrm{d}x. \tag{A.2}$$

Proof. Without loss of generality we may assume $x_0 = \underline{0}$. By direct differentiation we have

$$D'_u(r) = -\int \phi'\Big(\tfrac{|x|}{r}\Big)\tfrac{|x|}{r^2}\vartheta(x)\,|\nabla u(x)|^2\,\mathrm{d}x. \tag{A.3}$$

Consider the vector field $W \in C^\infty(B_r \setminus B'_r, \mathbb{R}^{n+1})$ defined by

$$W(x) := \phi\Big(\tfrac{|x|}{r}\Big)\vartheta(x)\,\Big(\tfrac{|\nabla u|^2}{2}x - (\nabla u\cdot x)\nabla u\Big),$$

and note that $W \in C^{0,1/2}_{\mathrm{loc}} \cap W^{1,2}_{\mathrm{loc}}(B_1^\pm \cup B'_1, \mathbb{R}^{n+1})$ by the regularity of u (cf. Proposition 4.2, Theorem 5.2 and Lemma 6.2). Then, the distributional divergence of W is a measure that might have a singular part concentrated on B'_r by the trace theorem in $W^{1,2}$. On the other hand, recalling that $u\,\partial_{n+1}u = 0$ on B'_1 we find $W(x',0^\pm)\cdot e_{n+1} = 0$ for all $(x',0) \in B'_r$. Therefore, since u minimizes (6.6), the distributional divergence of W is the $L^1(B_r)$ function given by

$$\operatorname{div} W(x) = \phi'\big(\tfrac{|x|}{r}\big)\cdot\tfrac{x}{r\,|x|}\vartheta(x)\Big(\tfrac{|\nabla u|^2}{2}x - (\nabla u\cdot x)\nabla u\Big) + \phi\big(\tfrac{|x|}{r}\big)\Big((n-1)\vartheta(x) + (\nabla\vartheta\cdot x)\Big)\tfrac{|\nabla u(x)|^2}{2}.$$

Being W with zero trace on ∂B_r we conclude that

$$\begin{aligned} 0 = \int \operatorname{div} W(x)\,\mathrm{d}x &= \int \phi'\Big(\tfrac{|x|}{r}\Big)\tfrac{|x|}{2\,r}\vartheta(x)\,|\nabla u(x)|^2\,\mathrm{d}x \\ &\quad + rE_u(r) + \frac{n-1}{2}\,D_u(r) + \int \phi\big(\tfrac{|x|}{r}\big)(\nabla\vartheta\cdot x)\tfrac{|\nabla u(x)|^2}{2}\mathrm{d}x. \end{aligned} \tag{A.4}$$

Equation (A.1) follows thanks to the equalities (A.3), (A.4), and the Lipschitz continuity of ϑ (cf. Lemma 6.2).

Next, we establish (A.2) with a similar argument. To this aim, consider the vector field $V(x) := \phi\big(\tfrac{|x|}{r}\big)\vartheta(x)\,u(x)\,\nabla u(x)$. Clearly, $V \in C^\infty(B_1 \setminus B'_1, \mathbb{R}^{n+1})$, with

$$V(x)\cdot e_{n+1} = \phi\big(\tfrac{|x|}{t}\big)\,\vartheta(x)\,u(x)\,\partial_{n+1}u(x).$$

Note that, $V \in C^{0,1/2}_{\mathrm{loc}} \cap W^{1,2}_{\mathrm{loc}}(B_1^\pm \cup B'_1, \mathbb{R}^{n+1})$ by the regularity of u, so that $V(x',0)\cdot e_{n+1} = 0$ on B'_1 recalling that $u\,\partial_{n+1}u = 0$ on B'_1. Thus, by taking into account that V has zero trace on ∂B_r and that u minimizes (6.6), the distributional divergence of V is the $L^1(B_1)$ function given by

$$\operatorname{div} V(x) = \phi'\big(\tfrac{|x|}{r}\big)\vartheta(x)\,u(x)\nabla u(x)\cdot\tfrac{x}{r\,|x|} + \phi\big(\tfrac{|x|}{r}\big)\vartheta(x)|\nabla u(x)|^2.$$

In conclusion, (A.2) follows at once from the divergence theorem. □

Let us now deal with the derivative of H_u.

Proposition A.2. *Let u be a solution to the thin obstacle problem (1.1) in B_1. There exists a non negative constant $C_{A.2}$ depending on $\mathrm{Lip}(u)$ such that for every $x_0 \in B'_1$ and for $\mathcal{L}^1$ a.e. $r \in (0, 1-|x_0|)$,*

$$H'_u(x_0,r) = \frac{n}{r}H_u(x_0,r) + 2D_u(x_0,r) + \varepsilon_H(x_0,r), \tag{A.5}$$

where $|\varepsilon_H(x_0,r)| \le C_{A.2}H_u(x_0,r)$.

Proof. As usual we assume $x_0 = \underline{0}$. Equality (A.5) is a consequence of (A.2) and the direct computation

$$\begin{aligned} H'_u(r) &= \tfrac{\mathrm{d}}{\mathrm{d}r}\Big(-r^n\textstyle\int\phi'(|y|)\,\vartheta(r\,y)\,\frac{u^2(r\,y)}{|y|}\,\mathrm{d}y\Big) \\ &= \tfrac{n}{r}\,H_u(r) - r^n\textstyle\int\phi'(|y|)\,\Big(\nabla\vartheta(r\,y)u^2(r\,y) + 2\vartheta(r\,y)u(r\,y)\,\nabla u(r\,y)\Big)\cdot\frac{y}{|y|}\,\mathrm{d}y \\ &\overset{(A.2)}{=} \tfrac{n}{r}\,H_u(r) + 2\,D_u(r) + \varepsilon_H(r). \end{aligned}$$

where $|\varepsilon_H(r)| \le C_{A.2}\,H_u(r)$ in view of the Lipschitz continuity of ϑ and (6.7). □

From Proposition A.2 we immediately deduce a monotonicity formula for H_u.

Corollary A.3. *Let u be a solution to the thin obstacle problem (1.1) in B_1. Then, for all $x_0 \in B_1'$ and $0 < r_0 < r_1 < 1 - |x_0|$, we have*

$$\frac{H_u(x_0, r_1)}{r_1^n} = \frac{H_u(x_0, r_0)}{r_0^n} \exp\left(\int_{r_0}^{r_1} \Big(2\frac{I_u(x_0,t)}{t} + \frac{\varepsilon_H(x_0,t)}{H_u(x_0,t)} \Big) \mathrm{d}t \right). \tag{A.6}$$

In particular, if $A_1 \le I(x_0, t) \le A_2$ for every $t \in (r_0, r_1)$, then

$$(r_0, r_1) \ni r \mapsto e^{-C_{A.2} r} \frac{H_u(x_0, r)}{r^{n+2A_2}} \quad \textit{is monotone decreasing,} \tag{A.7}$$

$$(r_0, r_1) \ni r \mapsto e^{C_{A.2} r} \frac{H_u(x_0, r)}{r^{n+2A_1}} \quad \textit{is monotone increasing.} \tag{A.8}$$

Moreover, for all $0 < r < 1 - |x_0|$

$$\frac{r}{4} H_u(x_0, r) \le \int_{B_r(x_0)} |u|^2 \, \mathrm{d}x \le 2\sqrt{1 + L^2}\, e^{C_{A.2} r} r\, H_u(x_0, r). \tag{A.9}$$

Proof. The proof of (A.6) (and hence of (A.7) and (A.8)) follows immediately from the differential equation (A.5).

The proof of the second inequality in (A.9) is now a direct consequence of (6.7) as follows

$$\begin{aligned} \int_{B_r(x_0)} |u|^2 \, \mathrm{d}x &= \sum_{k \in \mathbb{N}} \int_{B_{r/2^k} \setminus B_{r/2^{k+1}}(x_0)} |u|^2 \, \mathrm{d}x \\ &\le \sqrt{1 + L^2} \sum_{k \in \mathbb{N}} \frac{r}{2^k} H_u\big(x_0, r/2^k\big) \le 2\sqrt{1 + L^2}\, e^{C_{A.2} r} r\, H_u(x_0, r), \end{aligned}$$

where in the last inequality we used that $e^{C_{A.2} s} H_u(x_0, s) \le e^{C_{A.2} r} H_u(x_0, r)$ for $s \le r$ by (A.8). The opposite inequality is elementary in view of (6.7). □

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