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Chapter 1

Introduction and main results

A *valuation* on a family $\mathcal{F}$ of sets is a function $\varphi : \mathcal{F} \longrightarrow \mathbb{R}$ satisfying

$$\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B), \tag{1.0.1}$$

for every $A, B \in \mathcal{F}$ such that $A \cup B, A \cap B \in \mathcal{F}$. This generalizes the concept of measure: every measure is a valuation, but the converse is far from being true. While measures must be non-negative and countably additive, valuations are allowed to change sign and are only finitely additive in general. Moreover, measures are to be defined on σ-algebras, whereas valuations can have any family of sets as their domain. For example, the perimeter function defined on the family $\mathcal{K}^n$ of convex bodies of $\mathbb{R}^n$ (i.e., compact and convex subsets of $\mathbb{R}^n$ which are non-empty) is a valuation, but not a measure.

The theory of valuations on $\mathcal{K}^n$ is particularly important: it is connected with the solution of Hilbert's third problem and it contains several elegant results, such as the ones by Alesker, Hadwiger and McMullen recalled in Section 2.2. The state-of-the-art book [34] by Schneider gives a comprehensive report on the present state of this theory (see in particular Chapter 6).

The notion of valuation can also be extended to function spaces: for a given set X of real-valued functions, a *valuation* on X is a functional $V : X \longrightarrow \mathbb{R}$ such that

$$V(f \vee g) + V(f \wedge g) = V(f) + V(g), \tag{1.0.2}$$

for every $f, g \in X$ with $f \vee g, f \wedge g \in X$. The operators $\vee$ and $\wedge$ are the pointwise maximum and minimum, respectively. This is the natural counterpart of (1.0.1): one way to see it is by noting that, if χ_A denotes the characteristic function of the set A, we have

$$\chi_{A_1 \cup A_2} = \chi_{A_1} \vee \chi_{A_2},$$

$$\chi_{A_1 \cap A_2} = \chi_{A_1} \wedge \chi_{A_2},$$

for all sets A_1, A_2. Definition (1.0.2) can also be motivated by the fact that, for every couple of functions $f_1, f_2 : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$\mathrm{epi}(f_1 \vee f_2) = \mathrm{epi}(f_1) \cap \mathrm{epi}(f_2),$$

$$\mathrm{epi}(f_1 \wedge f_2) = \mathrm{epi}(f_1) \cup \mathrm{epi}(f_2),$$

where $\mathrm{epi}(f)$ denotes the epigraph of f, that is, the set of points lying on or above the graph of f.

When studying valuations on a function space X, often times mathematicians look for characterization results, i.e., they seek necessary and sufficient conditions for a functional $V : X \longrightarrow \mathbb{R}$ to be a valuation satisfying certain properties. Drawing inspiration from the renowned Hadwiger theorem (Theorem 2.2.2 in the thesis), these properties usually include continuity and some kind of invariance.

The theory of valuations on function spaces is quite recent. A branch of it focusses on spaces of functions related to convexity, such as convex functions (see [3, 7, 10, 11, 12]), log-concave functions (see [28, 29]) and quasi-concave functions (see [8, 9]), providing, among other results, several characterization theorems. In the papers [17, 18], Klain studied valuations defined on the family $\mathscr{S}^n$ of star-shaped sets having radial functions of class $L^n(\mathbb{S}^{n-1})$, and he gave a characterization of all the continuous and rotation invariant ones. Recall that a *star-shaped set* (with respect to the origin) is a set S containing the origin such that, for every $x \in S$, the segment joining x and the origin lies in S. The *radial function* associated to S is the map $\rho_S : \mathbb{S}^{n-1} \longrightarrow [0, \infty)$ defined by

$$\rho_S(x) = \sup\{\lambda \geq 0 : \lambda x \in S\}.$$

Note that for every star-shaped sets S_1, S_2 we have

$$\begin{aligned} \rho_{S_1 \cup S_2} &= \rho_{S_1} \vee \rho_{S_2}, \\ \rho_{S_1 \cap S_2} &= \rho_{S_1} \wedge \rho_{S_2}. \end{aligned} \tag{1.0.3}$$

The family $\mathscr{S}^n$ can be identified with $L^n_+(\mathbb{S}^{n-1})$ (non-negative functions in $L^n(\mathbb{S}^{n-1})$), and thanks to (1.0.3), to every valuation on $\mathscr{S}^n$ there corresponds a valuation on $L^n_+(\mathbb{S}^{n-1})$. Therefore, Klain's characterization can also be seen as a characterization of continuous and rotation invariant valuations on $L^n_+(\mathbb{S}^{n-1})$. This result was extended to L^p spaces, for $1 \leq p < \infty$, in [38], and to Orlicz spaces in [19] (see also [6] for the case $p = \infty$). These results were further generalized in [37], where a characterization for valuations on Banach lattices is provided. Valuations on other function spaces have been considered, and more characterization results can be found in the literature (see for instance [21, 24] for Sobolev spaces).

The papers [35, 36, 39] concern valuations defined on the family of star-shaped sets whose radial function is continuous, and they provide a characterization for continuous and rotation invariant valuations defined on such a family. By means of (1.0.3), this yields the following result for valuations defined on the space $C(\mathbb{S}^{n-1})$ of continuous functions on the unit sphere.

Theorem 1.0.1 (Tradacete, Villanueva). *A functional $V : C(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ is a continuous (with respect to uniform convergence) and rotation invariant valuation if and only if there exists a continuous function $K : \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$V(f) = \int_{\mathbb{S}^{n-1}} K(f(x))d\mathcal{H}^{n-1}(x),$$

for every $f \in C(\mathbb{S}^{n-1})$, where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

This is the typical form of a characterization theorem in the context of the theory of valuations on function spaces, in the sense that valuations are usually represented by an integral of a certain kernel K applied to the function f at which we are evaluating V.

A natural continuation of the previous results, and of Theorem 1.0.1 in particular, would be a characterization for valuations defined on $C^1(\mathbb{S}^{n-1})$ or on the space $\mathrm{Lip}(\mathbb{S}^{n-1})$ of Lipschitz continuous functions on the sphere. We chose to focus on $\mathrm{Lip}(\mathbb{S}^{n-1})$, which, being closed with respect to the operations of pointwise maximum and minimum, makes things easier. The original results presented in the thesis were obtained in collaboration with my supervisor Prof. Andrea Colesanti from the Università degli Studi di Firenze, Prof. Pedro Tradacete from the Instituto de Ciencias Matemáticas - CSIC de Madrid and Prof. Ignacio Villanueva from the Universidad Complutense de Madrid. In this next section, we are going to state our main results and briefly discuss them.

1.1 Main results

The primary novelty of the space $\mathrm{Lip}(\mathbb{S}^{n-1})$ is that, by Rademacher's theorem, its elements are a.e. differentiable with respect to the Hausdorff measure $\mathcal{H}^{n-1}$, hence we would expect for the gradient to appear in our representation formula. This leads us to the following conjecture.

Conjecture 1.1.1. *Let $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$. Then V is a continuous (with respect to an appropriate topology) and rotation invariant valuation if and only if there exists a kernel $K : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ with a suitable property (P) to be determined such that*

$$V(f) = \int_{\mathbb{S}^{n-1}} K(f(x), \|\nabla_s f(x)\|) d\mathcal{H}^{n-1}(x),$$

for $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$.

Here, $\nabla_s f$ denotes the spherical gradient of f, properly defined in Subsection 2.1.2. We were not able to prove this conjecture. However, we will provide a few characterization results for valuations on Lipschitz functions. The first theorem we present concerns continuous, rotation invariant and dot product invariant valuations on $\mathrm{Lip}(\mathbb{S}^{n-1})$ (for the topology we are using, see Subsection 2.1.3, and for the definition of the invariance properties of a valuation see Subsection 2.1.4).

Theorem 1.1.2. *A functional $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ is a continuous, rotation invariant and dot product invariant valuation if and only if there exist constants $c_0, c_1, c_2 \in \mathbb{R}$ such that*

$$V(f) = c_0 + c_1 \int_{\mathbb{S}^{n-1}} f(x) d\mathcal{H}^{n-1}(x) + c_2 \int_{\mathbb{S}^{n-1}} \left[(n-1)f(x)^2 - \|\nabla_s f(x)\|^2\right] d\mathcal{H}^{n-1}(x), \quad (1.1.1)$$

for every $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$.

The hypothesis of dot product invariance is quite strong, and it allows to give the very explicit representation formula (1.1.1).

The following result only holds in the bidimensional case, but it involves the more general polynomial valuations instead of the dot product invariant ones (see again Subsection 2.1.4 for the definition of polynomiality).

Theorem 1.1.3. *A functional $V : \mathrm{Lip}(\mathbb{S}^1) \longrightarrow \mathbb{R}$ is a continuous, rotation invariant and polynomial valuation if and only if there exists a polynomial p in two variables such that*

$$V(f) = \int_{\mathbb{S}^1} p(f(x), \|\nabla_s f(x)\|^2) d\mathcal{H}^1(x), \quad (1.1.2)$$

for every $f \in \mathrm{Lip}(\mathbb{S}^1)$.

Theorems 1.1.2 and 1.1.3 will be proved by means of Hadwiger's and Alesker's characterization results respectively (whose statements can be found in subsections 2.2.1 and 2.2.2): the idea is that of using the valuation V to define a new valuation φ on the space of convex bodies, and then apply to φ one of the aforementioned characterization results to obtain a representation formula on the space of support functions for V. Thanks to the approximation tools developed in Chapter 3, we will be able to extend such a formula to all Lipschitz functions. As an intermediate step in the proof of Theorem 1.1.2, we will also get a homogeneous decomposition for continuous and dot product invariant valuations (see Theorem 4.1.1 and its improved version Theorem 4.5.1).

When removing the hypotheses of dot product invariance and polynomiality, things become much more complicated, because we do not have a previous characterization result to use in this case. We were not able to achieve a proper characterization theorem for valuations which are only continuous and rotation invariant, but the following result gives nonetheless a representation formula for such valuations in the bidimensional case.

Theorem 1.1.4. *Let* $V : \mathrm{Lip}(\mathbb{S}^1) \longrightarrow \mathbb{R}$ *be a continuous and rotation invariant valuation. Then there exists* $K : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *such that* $K(\cdot, \gamma)$ *is a Borel function for every* $\gamma \in \mathbb{R}^+$ *and*

$$V(g) = \int_0^{2\pi} K(g(t), |g'(t)|) d\mathcal{H}^1(t), \tag{1.1.3}$$

for all $g \in \mathscr{L}(\mathbb{S}^1)$.

In particular, for every $f \in \mathrm{Lip}(\mathbb{S}^1)$ *we have*

$$V(f) = \lim_{i \to \infty} \int_0^{2\pi} K(f_i(t), |f_i'(t)|) d\mathcal{H}^1(t), \tag{1.1.4}$$

where $\{f_i\} \subseteq \mathscr{L}(\mathbb{S}^1)$ *is a sequence such that* $f_i \underset{\tau}{\rightarrow} f$ *as* $i \to \infty$.

Note that in this statement, as in the next one, we have identified functions in $\mathrm{Lip}(\mathbb{S}^1)$ with 2π-periodic functions on $\mathbb{R}$. The symbol $\mathscr{L}(\mathbb{S}^1)$ denotes the set of piecewise linear functions on $\mathbb{S}^1$.

To prove Theorem 1.1.4, we will use the valuation V to build measures ν_g, for $g \in \mathscr{L}(\mathbb{S}^1)$, which are absolutely continuous with respect to $\mathcal{H}^1$. The Radon-Nikodym theorem will then give us Radon-Nikodym derivatives $D_g = D_g(t)$ depending only on $g(t)$ and $|g'(t)|$. This will eventually lead to (1.1.3). The representation formula on the whole space $\mathrm{Lip}(\mathbb{S}^1)$ will immediately follow, since every $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$ can be τ-approximated by a sequence in $\mathscr{L}(\mathbb{S}^{n-1})$, as will be proved in Chapter 3 (see Proposition 3.0.2).

As a corollary of Theorem 1.1.4, we can actually obtain a characterization result on $\mathrm{Lip}(\mathbb{S}^1)$ under the additional assumption of uniform continuity (with respect to τ).

Theorem 1.1.5. *Let* $V : \mathrm{Lip}(\mathbb{S}^1) \longrightarrow \mathbb{R}$. *Then* V *is a uniformly continuous and rotation invariant valuation if and only if there exists a uniformly continuous function* $K : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *such that*

$$V(f) = \int_0^{2\pi} K(f(t), |f'(t)|) d\mathcal{H}^1(t), \tag{1.1.5}$$

for every $f \in \mathrm{Lip}(\mathbb{S}^1)$.

At the end of the last chapter we will show that Conjecture 1.1.1 is actually false for continuous or a.e. continuous kernels.

Chapter 2

Preliminaries

In this chapter we will introduce some basic concepts and notations, recall some known results which will be used throughout the thesis and make a few preliminary observations.

2.1 Notations and definitions

For $n \in \mathbb{N}$, $n \geq 2$, we denote by $\mathbb{S}^{n-1}$ the unit $(n-1)$-sphere (considered with the Euclidean topology inherited by $\mathbb{R}^n$), that is,

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\},$$

where $\|\cdot\|$ represents the Euclidean norm. We will use the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ on the sphere, normalized so that $\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = 1$. For the definition of Hausdorff measure, see for instance [13, Section 2.1].

Even though we will mainly be interested in what happens on $\mathbb{S}^{n-1}$, it will often be useful to reason on the whole space $\mathbb{R}^n$, where we will use the standard basis $\{e_1, \dots, e_n\}$ and the n-dimensional Lebesgue measure $\mathcal{L}^n$. The n-dimensional Hausdorff measure $\mathcal{H}^n$ on $\mathbb{R}^n$ is just a multiple of $\mathcal{L}^n$ (see [13, Section 2.2]). In particular, $\mathcal{H}^n(A) = 0$ if and only if $\mathcal{L}^n(A) = 0$, for every A. If something happens up to a set of zero measure (whether Hausdorff or Lebesgue measure), we will say that it happens *a.e.* or for *a.e.* point (short for "almost everywhere" and "almost every", respectively).

For $x \in \mathbb{R}^n$ and $r > 0$, the symbol $B_r(x)$ stands for the n-ball of radius r centered at x. The Hausdorff measure $\mathcal{H}^n(B_1(0))$ of the unit n-ball will be denoted by ω_n.

For a real-valued function $f : D \longrightarrow \mathbb{R}$, we define

$$f^+(x) = \max\{f(x), 0\},$$

$$f^-(x) = \min\{f(x), 0\}.$$

Note that $f^+, -f^- : D \longrightarrow \mathbb{R}^+$, where $\mathbb{R}^+ := [0, \infty)$, and we can always write $f = f^+ + f^-$.

The *support* $\mathrm{supp}(f)$ of a function f is the closure of the set of points x at which $f(x) \neq 0$. If $\mathrm{supp}(f) \subseteq A$, we will write $f \prec A$ for short.

2.1.1 Measures

An *algebra* Σ over a set X is a collection of subsets of X such that

- $X \in \Sigma$;
- $A^c \in \Sigma$ for every $A \in \Sigma$, where A^c denotes the complementary of A, i.e., $A^c = X \setminus A$;
- $A \cup B \in \Sigma$ for every $A, B \in \Sigma$.

If the third property is replaced by

- $\bigcup\limits_{i\in\mathbb{N}} A_i \in \Sigma$ for every sequence $\{A_i\} \subseteq \Sigma$,

then Σ is called a *σ-algebra*.

An important σ-algebra which we are going to consider in the thesis is the Borel σ-algebra of $\mathbb{S}^{n-1}$: the *Borel σ-algebra* over a set X is the smallest σ-algebra containing the open subsets of X. Its elements are called *Borel sets*. If the spaces X and Y are equipped with Borel σ-algebras Σ_1 and Σ_2 respectively, a function $f : X \longrightarrow Y$ is said to be a *Borel function* if for every $A \in \Sigma_2$ its pre-image $f^{-1}(A)$ is in Σ_1.

We recall that a *measure* μ on a σ-algebra Σ is a non-negative set function $\mu : \Sigma \longrightarrow [0, \infty]$ such that $\mu(\emptyset) = 0$ and which is countably additive, i.e.,

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i) \tag{2.1.1}$$

for every family $\{A_i\} \subseteq \Sigma$ of pairwise disjoint sets. If we do not ask for non-negativity, then μ is said to be a *signed measure*.

The concept of measure can be generalized to include set functions defined on algebras: if $\mathcal{A}$ is just an algebra, a *pre-measure* on $\mathcal{A}$ is a function $\nu : \mathcal{A} \longrightarrow [0, \infty]$ such that $\nu(\emptyset) = 0$ and (2.1.1) holds for every sequence $\{A_i\} \subseteq \Sigma$ of pairwise disjoint sets with $\bigcup_{i\in\mathbb{N}} A_i \in \Sigma$. Charathéodory's extension theorem states that a pre-measure can always be extended to a measure (it actually says something more general, but for our purposes the following version will be sufficient).

Theorem 2.1.1 (Charathéodory, [5, Theorem 3.5.2])**.** *Let ν be a bounded pre-measure on the algebra $\mathcal{A}$. Then ν can be extended to a measure on $\sigma(\mathcal{A})$, the smallest σ-algebra containing $\mathcal{A}$.*

Measures can also be obtained from outer measures. Let X be a set and let $\mathcal{P}(X)$ be the power set of X, i.e., the collection of all subsets of X. An *outer measure* on X is a non-negative set function $\mu^* : \mathcal{P}(X) \longrightarrow [0, \infty]$ such that

- $\mu^*(\emptyset) = 0$;
- μ^* is monotone increasing, that is, $\mu^*(A) \le \mu^*(B)$ for every $A, B \in X$ such that $A \subseteq B$;
- μ^* is countably subadditive, i.e.,
$$\mu^*\left(\bigcup_{i\in\mathbb{N}} A_i\right) \le \sum_{i\in\mathbb{N}} \mu^*(A_i),$$
for every sequence $\{A_i\} \subseteq \mathcal{P}(X)$.

Let μ^* be an outer measure on X. A set $A \subseteq X$ is called *μ^*-measurable* if

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c),$$

for every $B \subseteq X$, where A^c denotes the complementary of A in X.

An outer measure can be used to construct a measure, as stated in the following well-known result.

Theorem 2.1.2 (Charathéodory)**.** *Let μ^* be an outer measure on X. The set of all μ^*-measurable sets is a σ-algebra, and the restriction of μ^* to this σ-algebra is a measure.*

2.1.2 Lipschitz functions and differentiability

Let $\mathrm{Lip}(\mathbb{S}^{n-1})$ be the space of Lipschitz continuous maps defined on $\mathbb{S}^{n-1}$, i.e., the set of functions $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ for which there exists a constant $L \geq 0$ such that

$$|f(x) - f(y)| \leq L\|x - y\|,$$

for every $x, y \in \mathbb{S}^{n-1}$. The smallest constant for which this inequality holds is called the *Lipschitz constant* associated with f and is denoted by $L(f)$:

$$L(f) = \sup\left\{\frac{|f(x) - f(y)|}{\|x - y\|} : x, y \in \mathbb{S}^{n-1},\, x \neq y\right\}.$$

Note that Lipschitz functions are uniformly continuous, as it immediately follows from their definition.

For a function $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$, we can define its spherical gradient $\nabla_s f$. In order to do so, we first introduce the concept of differentiability on $\mathbb{S}^{n-1}$.

The sphere can be locally parametrized, in the sense that for every $x \in \mathbb{S}^{n-1}$ there are a neighbourhood $\mathcal{U}$ of x in $\mathbb{S}^{n-1}$, an open set $\Omega \subseteq \mathbb{R}^{n-1}$ and a diffeomorphism $\phi : \Omega \longrightarrow \mathcal{U}$, namely a differentiable and invertible function with inverse function still differentiable. We call ϕ a *local parametrization*.

We will say that a function $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ is *differentiable* at a point $x \in \mathbb{S}^{n-1}$ if for every local parametrization $\phi : \Omega \longrightarrow \mathcal{U}$ of a neighbourhood $\mathcal{U}$ of x we have that the function $f \circ \phi : \Omega \longrightarrow \mathbb{R}$ is differentiable at the point $\phi^{-1}(x)$.

Now, if $x \in \mathbb{S}^{n-1}$ is a differentiability point for $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$, we can consider the *differential* of f at x, which is a linear map $df_x : T_x(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$, where $T_x(\mathbb{S}^{n-1})$ is the tangent space to $\mathbb{S}^{n-1}$ at x. By linearity, df_x can be represented by a vector $\nabla_s f(x) \in T_x(\mathbb{S}^{n-1})$, that is, for every $v \in T_x(\mathbb{S}^{n-1})$

$$\langle \nabla_s f(x), v\rangle = df_x(v) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma_x(t)),$$

where $\gamma_x : [-1, 1] \longrightarrow \mathbb{S}^{n-1}$ is an arbitrary C^∞ curve such that $\gamma_x(0) = x$, $\gamma_x'(0) = v$, and $\langle \cdot, \cdot\rangle$ denotes the Euclidean dot product. We call $\nabla_s f(x)$ the *spherical gradient* of f at x.

Rademacher's theorem (see [13, Subsection 3.1.2]) states that every locally Lipschitz function on $\mathbb{R}^n$ is a.e. differentiable. As a corollary (see [31, Corollary 2.4.2]), we have that every $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$ is a.e. differentiable on $\mathbb{S}^{n-1}$, i.e.,

$$\mathcal{H}^{n-1}\left(\left\{x \in \mathbb{S}^{n-1} : \nabla_s f(x) \text{ exists}\right\}\right) = 1.$$

The spherical gradient of a function $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$, where it is defined, is bounded by the Lipschitz constant, that is,

$$\|\nabla_s f(x)\| \leq L(f) \tag{2.1.2}$$

for a.e. $x \in \mathbb{S}^{n-1}$. Indeed, fix a point of differentiability $x \in \mathbb{S}^{n-1}$, and for every direction $v \in T_x(\mathbb{S}^{n-1})$ consider a C^∞ curve $\gamma_x : [-1, 1] \longrightarrow \mathbb{S}^{n-1}$ such that $\gamma_x(0) = x$, $\gamma_x'(0) = v$. We can always use an arc-length parametrization of γ_x, that is, a parametrization with respect to which the equality $\|\gamma_x'(t)\| = 1$ holds for every $t \in [-1, 1]$. For every $t_1, t_2 \in [-1, 1]$, we have

$$\left|\frac{f(\gamma_x(t_1)) - f(\gamma_x(t_2))}{t_1 - t_2}\right| = \frac{|f(\gamma_x(t_1)) - f(\gamma_x(t_2))|}{\|\gamma_x(t_1) - \gamma_x(t_2)\|} \cdot \frac{\|\gamma_x(t_1) - \gamma_x(t_2)\|}{|t_1 - t_2|} \leq L(f), \tag{2.1.3}$$

by Lipschitz cointinuity and the fact that (if for instance $t_1 \leq t_2$)

$$\|\gamma_x(t_1) - \gamma_x(t_2)\| = \left\| \int_{t_1}^{t_2} \gamma_x'(t)dt \right\| \leq \int_{t_1}^{t_2} \|\gamma_x'(t)\| dt = t_2 - t_1 \leq |t_1 - t_2|.$$

Now, if $\gamma_x : [-1,1] \longrightarrow \mathbb{S}^{n-1}$ is a C^∞ curve such that $\gamma_x(0) = x$, $\gamma_x'(0) = \dfrac{\nabla_s f(x)}{\|\nabla_s f(x)\|}$, from (2.1.3) we get

$$\|\nabla_s f(x)\| = \left\langle \nabla_s f(x), \frac{\nabla_s f(x)}{\|\nabla_s f(x)\|} \right\rangle = \frac{d}{dt}\bigg|_{t=0} f(\gamma_x(t)) = \lim_{h\to 0} \frac{f(\gamma_x(h)) - f(\gamma_x(0))}{h} \leq L(f).$$

2.1.3 Topology on $\mathrm{Lip}(\mathbb{S}^{n-1})$

The space $\mathrm{Lip}(\mathbb{S}^{n-1})$ has a natural topology induced by the so-called *Lipschitz norm*, defined by

$$\|f\|_{\mathrm{Lip}} = \max\{\|f\|_\infty, L(f)\}, \quad f \in \mathrm{Lip}(\mathbb{S}^{n-1}), \tag{2.1.4}$$

where, $\|f\|_\infty := \max_{x\in\mathbb{S}^{n-1}} |f(x)|$. Such a topology, however, would not work in our case: indeed, our approach is based upon a density result, namely Proposition 3.0.2, which is not true in the topology induced by the aforementioned norm, since the space $\mathscr{L}(\mathbb{S}^{n-1})$ of piecewise linear functions is separable with respect to $\|\cdot\|_{\mathrm{Lip}}$ and $\mathrm{Lip}(\mathbb{S}^{n-1})$ is not.

Therefore, we are going to consider another topology on $\mathrm{Lip}(\mathbb{S}^{n-1})$. We need a few definitions first. A subset A of a topological space X is *sequentially open* if for every sequence $\{x_i\} \subseteq X$ converging to $x \in A$ there exists $I \in \mathbb{N}$ such that $x_i \in A$, for all $i > I$. Open sets are sequentially open, but the converse is not true in general. A *sequential space* is a topological space whose open sets are exactly the sequentially open ones, i.e., a sequential space is the most general topological space in which convergent sequences are enough to determine the topology.

Let us consider $\mathrm{Lip}(\mathbb{S}^{n-1})$ as a sequential space with the topology τ induced by the following convergence: we say that a sequence $\{f_i\} \subseteq \mathrm{Lip}(\mathbb{S}^{n-1})$ converges to $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$ with respect to τ, in symbols $f_i \underset{\tau}{\to} f$, as $i \to \infty$, if

- $f_i \to f$ uniformly on $\mathbb{S}^{n-1}$, that is, $\|f_i - f\|_\infty \to 0$;
- $\nabla_s f_i(x) \to \nabla_s f(x)$ for a.e. $x \in \mathbb{S}^{n-1}$;
- there exists a suitable constant $C \geq 0$ such that $\|\nabla_s f_i(x)\| \leq C$, for every $i \in \mathbb{N}$ and a.e. $x \in \mathbb{S}^{n-1}$.

With this topology, we will be able to build continuous and rotation invariant valuations on $\mathrm{Lip}(\mathbb{S}^{n-1})$ via a quite simple formula (see Lemma 4.3.1). Note that this topology is weaker than the one induced by $\|\cdot\|_{\mathrm{Lip}}$, in the sense that if $\|f_i - f\|_{\mathrm{Lip}} \to 0$, then $f_i \underset{\tau}{\to} f$.

2.1.4 Valuations on $\mathrm{Lip}(\mathbb{S}^{n-1})$

Let X be a family of real-valued functions. A functional $V : X \longrightarrow \mathbb{R}$ is said to be a *valuation* if

$$V(f \vee g) + V(f \wedge g) = V(f) + V(g), \tag{2.1.5}$$

for every $f, g \in X$ such that $f \vee g, f \wedge g \in X$, where $\vee$ and $\wedge$ are the pointwise maximum and pointwise minimum respectively. Since the space $\mathrm{Lip}(\mathbb{S}^{n-1})$ is closed with respect to these operations, a valuation on $\mathrm{Lip}(\mathbb{S}^{n-1})$ is a functional $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ such that (2.1.5) holds for every $f, g \in \mathrm{Lip}(\mathbb{S}^{n-1})$.

Every valuation on a lattice of real-valued functions, and in particular every valuation on $\mathrm{Lip}(\mathbb{S}^{n-1})$, satisfies the inclusion-exclusion principle, which can be proved by induction.

Proposition 2.1.3 (Inclusion-exclusion principle)**.** *Let $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ be a valuation. Then*

$$\begin{aligned} V\left(\bigvee_{j=1}^{N} f_j\right) &= \sum_{1\le j\le N} V(f_j) - \sum_{1\le j_1<j_2\le N} V(f_{j_1}\wedge f_{j_2}) + \\ &\quad + \sum_{1\le j_1<j_2<j_3\le N} V(f_{j_1}\wedge f_{j_2}\wedge f_{j_3}) - \ldots + (-1)^{N-1} V\left(\bigwedge_{j=1}^{N} f_j\right), \end{aligned}$$

for every $f_1, \ldots, f_N \in \mathrm{Lip}(\mathbb{S}^{n-1})$. The same holds exchanging the roles of $\vee$ and $\wedge$.

We will say that a valuation $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ is *continuous* if it is continuous with respect to the topology τ defined earlier, unless otherwise stated. Also, $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ will be called *uniformly continuous* if for every $\varepsilon > 0$ there exists a neighbourhood $U \subseteq \mathrm{Lip}(\mathbb{S}^{n-1})$ of the null function $\mathbb{O}$ (with respect to τ) such that

$$f_1 - f_2 \in U \Rightarrow |V(f_1) - V(f_2)| < \varepsilon.$$

Besides continuity, we will be interested in other properties, namely the rotational invariance, dot product invariance, polynomiality and homogeneity. These concepts are all defined below.

A valuation $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ is *rotation invariant* if for every $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$ and $\sigma \in \mathcal{O}(n)$ we have

$$V(f \circ \sigma) = V(f),$$

where $\mathcal{O}(n)$ is the orthogonal group of $\mathbb{R}^n$, i.e., the set of isometries of $\mathbb{R}^n$ fixing the origin.

Moreover, V is called *dot product invariant* if, for every $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$ and $x \in \mathbb{R}^n$,

$$V(f + \langle \cdot, x\rangle) = V(f),$$

where $\langle \cdot, \cdot \rangle$, again, denotes the standard scalar product in $\mathbb{R}^n$. This is to say that if $g(y) = f(y) + \langle y, x\rangle$, $y \in \mathbb{S}^{n-1}$, then $V(g) = V(f)$. In other words, V is dot product invariant if it is invariant under the addition of linear functions restricted to $\mathbb{S}^{n-1}$.

A valuation $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ is called *polynomial* if for every $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$ we have that

$$V(f + \langle \cdot, x\rangle) = p_f(x)$$

is a polynomial in $x \in \mathbb{R}^n$ (depending on f). Recall that a *polynomial* on $\mathbb{R}^n$ is a function $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$p(x) = \sum_{i=0}^{\ell} \sum_{\substack{j_1,\ldots,j_n\in\{0,\ldots,\ell\}:\\ j_1+\ldots+j_n=i}} a_{j_1\ldots j_n} x_1^{j_1} \ldots x_n^{j_n},$$

for suitable coefficients $a_{j_1\ldots j_n} \in \mathbb{R}$. Note that every dot product invariant valuation is polynomial with $p_f \equiv V(f)$.

Finally, for a natural number $0 \le i \le n$, we say that V is *i-homogeneous* if

$$V(\lambda f) = \lambda^i V(f),$$

for every $\lambda > 0$ and $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$.

2.1.5 The space $\mathcal{K}^n$

We will also work with valuations defined on the space $\mathcal{K}^n$ of *convex bodies* of $\mathbb{R}^n$, namely compact and convex subsets of $\mathbb{R}^n$ which are non-empty. We recall some definitions in this context. Our reference book for this area is [34].

The topology on $\mathcal{K}^n$ is induced by the *Hausdorff metric*, defined by

$$d_H(K,L) = \max\left\{\sup_{x\in K}\inf_{y\in L}\|x-y\|,\ \sup_{y\in L}\inf_{x\in K}\|x-y\|\right\},$$

for every $K, L \in \mathcal{K}^n$. The set

$$C^2_+ = \left\{K \in \mathcal{K}^n : \partial K \in C^2 \text{ and has strictly positive Gaussian curvature at every point}\right\}$$

is dense in $\mathcal{K}^n$ with respect to this topology, as is the set of strictly convex bodies, i.e., convex bodies whose boundary does not contain any segment (see [34, Theorem 2.7.1]).

A *valuation* on $\mathcal{K}^n$ is a function $\varphi : \mathcal{K}^n \longrightarrow \mathbb{R}$ such that

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L),$$

for every $K, L \in \mathcal{K}^n$ satisfying $K \cup L \in \mathcal{K}^n$. Such a φ is *rotation invariant* if

$$\varphi(\sigma(K)) = \varphi(K),$$

for every $K \in \mathcal{K}^n$ and $\sigma \in \mathcal{O}(n)$. It is called *translation invariant* if

$$\varphi(K + x) = \varphi(K),$$

for every $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$. The valuation φ is said to be *polynomial* if for every $K \in \mathcal{K}^n$ we have that

$$\varphi(K + x) = p_K(x)$$

is a polynomial in $x \in \mathbb{R}^n$ (depending on K). For a natural number $0 \leq i \leq n$, φ is *i-homogeneous* if

$$\varphi(\lambda K) = \lambda^i \varphi(K),$$

for every $\lambda > 0$ and $K \in \mathcal{K}^n$.

2.1.6 Support functions and piecewise linear maps

Recall the definition of support function: for every $K \in \mathcal{K}^n$, its *support function* is $h_K : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$h_K(x) = \max_{y\in K}\langle x, y\rangle,\ x \in \mathbb{R}^n.$$

Support functions are convex and 1-homogeneous, that is, $h_K(\lambda x) = \lambda h_K(x)$ for every $\lambda > 0$ and $x \in \mathbb{R}^n$. Vice versa, any 1-homogeneous convex function on $\mathbb{R}^n$ is the support function of some $K \in \mathcal{K}^n$. Moreover, for every $\alpha, \beta \geq 0$ and $K, L \in \mathcal{K}^n$ we have

$$h_{\alpha K+\beta L} = \alpha h_K + \beta h_L \tag{2.1.6}$$

and

$$\|h_K - h_L\|_\infty = d_H(K, L). \tag{2.1.7}$$

All these results can be found in [34, sections 1.7, 1.8].

The notion of piecewise linear function will also be useful. A continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be *piecewise linear* if there exist closed convex cones $C_1, \dots, C_m$ with vertex at the origin and pairwise disjoint interiors satisfying

$$\bigcup_{i=1}^{m} C_i = \mathbb{R}^n,$$

and linear functions $L_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, \dots, m$, such that $f = L_i$ on C_i, for $i = 1, \dots, m$.

We denote by $\mathscr{H}(\mathbb{S}^{n-1})$ and $\mathscr{L}(\mathbb{S}^{n-1})$ the sets of the restrictions to $\mathbb{S}^{n-1}$ of support functions and piecewise linear functions respectively. When considering these same functions defined on the whole space $\mathbb{R}^n$ we will use the symbols $\mathscr{H}(\mathbb{R}^n)$ and $\mathscr{L}(\mathbb{R}^n)$ instead. Note that, since support functions are convex, they are locally Lipschitz continuous (see [34, Theorem 1.5.3]), hence $\mathscr{H}(\mathbb{S}^{n-1}) \subseteq \mathrm{Lip}(\mathbb{S}^{n-1})$. We also have the inclusion $\mathscr{L}(\mathbb{S}^{n-1}) \subseteq \mathrm{Lip}(\mathbb{S}^{n-1})$.

We introduce one last notation, which will come in handy in the future, denoting by

$$\widehat{\mathscr{H}}(\mathbb{S}^{n-1}) = \left\{ \bigwedge_{i=1}^{m} h_{K_i} : m \in \mathbb{N},\ h_{K_i} \in \mathscr{H}(\mathbb{S}^{n-1}) \text{ for } i = 1, \dots, m \right\}$$

the space of the minima of finitely many support functions.

2.2 Preliminary results

As we have already anticipated, to prove theorems 1.1.2 and 1.1.3 we are going to switch from the valuation $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ to a new valuation $\varphi : \mathcal{K}^n \longrightarrow \mathbb{R}$, to which some known results for valuations on convex bodies can be applied. We are now going to recall such results.

2.2.1 McMullen's decomposition and Hadwiger's characterization theorem

In Section 4.1 we will prove a McMullen-type decomposition result for continuous and dot product invariant valuations on $\mathrm{Lip}(\mathbb{S}^{n-1})$ (see Proposition 4.1.1); we hereby recall the original McMullen decomposition theorem for valuations defined on $\mathcal{K}^n$, which will be used in the proof of our result in Section 4.1.

Theorem 2.2.1 (McMullen, [34, Theorem 6.3.5])**.** *Let $\varphi : \mathcal{K}^n \longrightarrow \mathbb{R}$ be a translation invariant valuation which is continuous with respect to the Hausdorff metric. Then there exist continuous and translation invariant valuations $\varphi_0, \dots, \varphi_n : \mathcal{K}^n \to \mathbb{R}$ such that φ_i is i-homogeneous, for $i = 0, \dots, n$, and*

$$\varphi(\lambda K) = \sum_{i=0}^{n} \lambda^i \varphi_i(K), \tag{2.2.1}$$

for every $K \in \mathcal{K}^n$ and $\lambda > 0$.

The famous Hadwiger characterization theorem for continuous and rigid motion invariant valuations on convex bodies, recalled below, will be of crucial importance in the proof of Theorem 1.1.2.

Theorem 2.2.2 (Hadwiger, [34, Theorem 6.4.14]). *A map $\varphi : \mathcal{K}^n \longrightarrow \mathbb{R}$ is a rotation and translation invariant valuation which is continuous with respect to the Hausdorff metric if and only if there exist constants $c_0, \ldots, c_n \in \mathbb{R}$ such that*

$$\varphi(K) = \sum_{i=0}^{n} c_i V_i(K),$$

for every $K \in \mathcal{K}^n$, where V_i denotes the i^{th} intrinsic volume.

For the definition of the intrinsic volumes, see [34, Chapter 4]. We recall here that for every $i = 0, \ldots, n$, the intrinsic volume $V_i : \mathcal{K}^n \longrightarrow \mathbb{R}^+$ is a continuous, increasing, i-homogeneous, rotation and translation invariant valuation. Whatever the dimension n is, we have that, for every $K \in \mathcal{K}^n$,

$$V_0(K) = 1 \tag{2.2.2}$$

and $V_n = C_n \mathcal{H}^n$, for some constant $C_n > 0$ (see [33, page 210]). There are also integral representations for V_1 and V_2: formulas (4.2.26) and (5.3.12) from [33] imply that

$$V_1(K) = \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} h_K \, d\mathcal{H}^{n-1}, \tag{2.2.3}$$

for every $K \in \mathcal{K}^n$, and, if $K \in C^2_+$, formulas (4.2.26), (5.3.11) and (2.5.23) from [33] give

$$V_2(K) = \frac{1}{2\omega_{n-2}} \int_{\mathbb{S}^{n-1}} h_K \big[(n-1)h_K + \Delta_s h_K\big] d\mathcal{H}^{n-1}, \tag{2.2.4}$$

$\Delta_s h_K$ being the spherical Laplacian of h_K, that is, the trace of the $(n-1) \times (n-1)$ matrix of the second covariant derivatives of h_K with respect to a local orthonormal frame on the sphere. Such formulas will be extremely useful in the proof of Theorem 1.1.2.

Hadwiger's volume theorem on n-homogeneous valuations will also make an appearance in the proof of Theorem 4.5.1.

Theorem 2.2.3 (Hadwiger, [34, Theorem 6.4.8]). *Let $\varphi : \mathcal{K}^n \longrightarrow \mathbb{R}$ be a continuous and translation invariant valuation which is homogeneous of degree n. Then there exists $c \in \mathbb{R}$ such that $\varphi = cV_n$.*

2.2.2 Alesker's characterization theorem

Theorem 1.1.3 follows from a characterization of polynomial valuations on $\mathcal{K}^n$ presented by Alesker. We are going to state Alesker's result separately for $n = 2$ and $n \geq 3$.

Theorem 2.2.4 (Alesker, [1]). *Let $\varphi : \mathcal{K}^2 \longrightarrow \mathbb{R}$. Then φ is a polynomial and rotation invariant valuation which is continuous with respect to the Hausdorff metric if and only if there exist polynomials p_0, p_1 in two variables such that*

$$\varphi(K) = \sum_{i=0}^{1} \int_{\mathbb{R}^2 \times \mathbb{S}^1} p_i(\|s\|^2, \langle s, x\rangle) d\Theta_i(K; s, x), \tag{2.2.5}$$

for every $K \in \mathcal{K}^2$, where $\Theta_i(K; \cdot)$ is the i^{th} support measure of K.

For the definition of support measures, see [34, Chapter 4]. The situation in dimension $n \geq 3$ is a bit different; even though we will only apply Theorem 2.2.4 in the thesis, we state Alesker's result in higher dimension for completeness.

Theorem 2.2.5 (Alesker, [1])**.** *Let $n \geq 3$ and let $\varphi : \mathcal{K}^n \longrightarrow \mathbb{R}$ be a polynomial and $\mathcal{SO}(n)$-invariant valuation which is continuous with respect to the Hausdorff metric. Then there exist polynomials $p_0, \ldots, p_{n-1}$ in two variables such that*

$$\varphi(K) = \sum_{i=0}^{n-1} \int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} p_i(\|s\|^2, \langle s, x \rangle) d\Theta_i(K; s, x), \tag{2.2.6}$$

for every $K \in \mathcal{K}^n$. Moreover, every function φ of the form (2.2.6) *is a polynomial and $\mathcal{O}(n)$-invariant valuation which is continuous with respect to the Hausdorff metric.*

Here $\Theta_i(K; \cdot)$ still denotes the i^{th} support measure of K and $\mathcal{SO}(n)$ is the subgroup of $\mathcal{O}(n)$ consisting of proper rotations.

The next lemma, which is a variant of Lemma 4.2.2 from [34], is a very useful tool to deal with the integrals appearing in (2.2.5).

Lemma 2.2.6. *Let $K \in \mathcal{K}^n$ be a strictly convex body and $\psi : \mathbb{R}^n \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ be a continuous and Lebesgue measurable function. Then*

$$\int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} \psi(s, x) d\Theta_i(K; s, x) = \int_{\mathbb{S}^{n-1}} \psi(s_K(x), x) dS_i(K; x),$$

for every $i = 0, \ldots, n$, where $s_K(x)$ is the unique boundary point of K at which x is attained as outer normal vector and S_i is the i^{th} area measure.

For the definition of area measures, see [34, Chapter 4]. As in the case of intrinsic volumes, there are representation formulas for some of these area measures. We will be interested in the following two: there exists a constant $C > 0$ such that, for every Borel set $A \subseteq \mathbb{S}^{n-1}$,

$$S_0(K; A) = C\mathcal{H}^{n-1}(A), \tag{2.2.7}$$

and

$$S_1(K; A) = \frac{C}{n-1} \int_A \Delta h_K(x) d\mathcal{H}^{n-1}(x), \tag{2.2.8}$$

where Δ denotes the Euclidean Laplacian. Formula (2.2.7) is (4.2.22) from [33] and formula (2.2.8) follows from (4.2.20), (2.5.23) and the formula right above the latter from the same book. The presence of the constant C is due to our normalization $\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = 1$.

2.2.3 The Radon-Nikodym theorem and other results from classical analysis

As we have said before, to prove Theorem 1.1.4 (and consequently Theorem 1.1.5), we are going to need the (signed version of the) Radon-Nikodym theorem, hereby recalled.

Theorem 2.2.7 (Radon-Nikodym, [4, Theorem 2.2.1])**.** *Let (X, Σ, μ) be a measure space (i.e., X is a set, Σ a σ-algebra on X and μ a measure on Σ) with $\mu(X) < \infty$. Let ν be a signed measure on Σ such that $\nu \ll \mu$. Then there exists a function $\frac{d\nu}{d\mu}$, called the Radon-Nikodym derivative of ν with respect to μ, which is integrable with respect to μ on X and such that*

$$\nu(A) = \int_A \frac{d\nu}{d\mu}(x) d\mu(x),$$

for every $A \in \Sigma$. Moreover, the Radon-Nikodym derivative is unique up to sets of null measure.

Here, "$\nu \ll \mu$" means that ν is *absolutely continuous* with respect to μ, that is, $\nu(A) = 0$ for every $A \in \Sigma$ such that $\mu(A) = 0$. If $X = \mathbb{R}^n$, Σ is the Borel σ-algebra and $\mu = \mathcal{H}^n$, this is equivalent to saying that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \in \Sigma$ satisfies $\mathcal{H}^n(A) < \delta$, then $\nu(A) < \varepsilon$.

Lebesgue-Besicovitch's differentiation theorem will also be of crucial importance in proving Theorem 1.1.4.

Theorem 2.2.8 (Lebesgue-Besicovitch, [13, Subsection 1.7.1])**.** *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For a.e. $x \in \mathbb{R}^n$,*

$$f(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\mathcal{L}^n(B_\varepsilon(x))} \int_{B_\varepsilon(x)} f(y) d\mathcal{L}^n(y) = \lim_{\varepsilon \to 0^+} \frac{1}{\mathcal{H}^n(B_\varepsilon(x))} \int_{B_\varepsilon(x)} f(y) d\mathcal{H}^n(y).$$

We are also going to need some more well-known results from classical analysis. For instance, the dominated convergence theorem will be of tremendous importance in many of our proofs.

Theorem 2.2.9 (Dominated convergence)**.** *Let (X, Σ, μ) be a measure space. Consider a sequence $\{f_i\} \subseteq L^1(X)$ such that*

- *$f(x) = \lim_{i \to \infty} f_i(x)$ exists for μ-a.e. $x \in X$;*
- *$|f_i(x)| \leq g(x)$ for every $i \in \mathbb{N}$ and μ-a.e. $x \in X$, for a suitable function $g \in L^1(X)$.*

Then

$$\lim_{i \to \infty} \int_X |f_i(x) - f(x)| d\mu(x) = 0.$$

In particular,

$$\lim_{i \to \infty} \int_X f_i(x) d\mu(x) = \int_X f(x) d\mu(x).$$

Another important result that we are going to need is the Ascoli-Arzelà theorem, which requires a couple of definitions to be introduced. Let $X \subseteq \mathbb{R}^n$ and consider a sequence $\{f_i\} \subseteq C(X)$ of continuous functions on X. We say that $\{f_i\}$ is *uniformly bounded* if there exists a constant $C > 0$ such that $|f_i(x)| \leq C$, for every $x \in X$ and $i \in \mathbb{N}$. The sequence $\{f_i\}$ is called *equicontinuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f_i(x) - f_i(y)| < \varepsilon$ for every $i \in \mathbb{N}$ and $x, y \in X$ with $\|x - y\| < \delta$. We can now state the result (we will state it in a less general context than the one in which it actually holds).

Theorem 2.2.10 (Ascoli-Arzelà)**.** *Let X be a closed subspace of $\mathbb{R}^n$ and let $\{f_i\} \subseteq C(X)$ be a uniformly bounded and equicontinuous sequence. Then there exists a subsequence $\{f_{i_j}\} \subseteq \{f_i\}$ which converges uniformly on the compact sets $K \subseteq X$.*

At some point, we are also going to use the mean value theorem for functions of n variables. We recall it here.

Theorem 2.2.11 (Mean value)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a convex and open set, and let $f : \Omega \longrightarrow \mathbb{R}$ be a differentiable function. Then, for every $x, y \in \Omega$, there exists $\lambda \in (0, 1)$ such that*

$$f(x) - f(y) = \langle \nabla f((1 - \lambda)x + \lambda y), x - y \rangle ,$$

where ∇f denotes the Euclidean gradient of f.

To prove Lemma 3.3.1 we will use mollifiers. We recall that a *mollifier* is a family $\{\psi_\varepsilon\}_{\varepsilon > 0} \subseteq L^1(\mathbb{R}^n)$ such that

- $\int_{\mathbb{R}^n} \psi_\varepsilon(x)dx = 1$;
- $\sup_{\varepsilon>0} \int_{\mathbb{R}^n} |\psi_\varepsilon(x)|dx < \infty$;
- $\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B_\delta(0)} |\psi_\varepsilon(x)|dx = 0$ for every $\delta > 0$,

where the integrals are done with respect to the Lebesgue measure $\mathcal{L}^n$. Mollifiers can be used to approximate functions in the following sense.

Theorem 2.2.12. *Let $\{\psi_\varepsilon\}_{\varepsilon>0}$ be a mollifier. For $p \geq 1$ and $f \in L^p(\mathbb{R}^n)$, consider the convolution*

$$\psi_\varepsilon * f(x) = \int_{\mathbb{R}^n} \psi(x-y)f(y)dy = \int_{\mathbb{R}^n} \psi(y)f(x-y)dy, \quad x \in \mathbb{R}^n.$$

Then $\psi_\varepsilon * f \to f$ in $L^p(\mathbb{R}^n)$.

This actually holds on any $\mathcal{L}^n$-measurable subset $X \subseteq \mathbb{R}^n$, but we are just going to need the version stated above.

2.2.4 Extensions of functions and their gradients

The spherical gradient of a function $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$ is hard to deal with. This is why it will often be convenient to extend $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ to $\bar{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$, so that we will be able to work with the standard Euclidean gradient $\nabla \bar{f}$ instead of the spherical one.

Every extension $\bar{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$ of $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$ satisfies

$$\langle \nabla_s f(x), v\rangle = \left\langle \nabla \bar{f}(x), v\right\rangle, \tag{2.2.9}$$

for a.e. $x \in \mathbb{S}^{n-1}$ and all $v \in T_x(\mathbb{S}^{n-1})$.

In fact, let $x \in \mathbb{S}^{n-1}$ be a point of differentiability for f, $v \in T_x(\mathbb{S}^{n-1})$ and consider a C^∞ curve $\gamma_x : [-1,1] \longrightarrow \mathbb{S}^{n-1}$ such that $\gamma_x(0) = x$, $\gamma_x'(0) = v$. By definition,

$$\langle \nabla_s f(x), v\rangle = \left.\frac{d}{dt}\right|_{t=0} f(\gamma_x(t)) = \left.\frac{d}{dt}\right|_{t=0} \bar{f}(\gamma_x(t)) = \left\langle \nabla \bar{f}(\gamma_x(0)), \gamma_x'(0)\right\rangle = \left\langle \nabla \bar{f}(x), v\right\rangle.$$

One way of extending a function $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ to the whole space $\mathbb{R}^n$ is to do it 1-homogeneously. Let us denote by $\tilde{f}$ this extension:

$$\tilde{f}(x) = \begin{cases} \|x\|\, f\left(\dfrac{x}{\|x\|}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{2.2.10}$$

Euler's formula helps us to handle these 1-homogeneous extensions.

Proposition 2.2.13 (Euler)**.** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be an a.e. differentiable function. If f is k-homogeneous for some $1 \leq k \leq n$, then*

$$\langle x, \nabla f(x)\rangle = kf(x), \tag{2.2.11}$$

for a.e. $x \in \mathbb{R}^n$.

This can be proved by differentiating the equation

$$f(\lambda x) = \lambda^k f(x), \ \ \lambda > 0, \ x \in \mathbb{R}^n,$$

with respect to λ and then choosing $\lambda = 1$.

Let $x \in \mathbb{S}^{n-1}$ be a point at which f is differentiable; then its 1-homogeneous extension $\tilde{f}$ is differentiable at x as well. We can link the norms of the spherical and Euclidean gradient through the following equality:

$$\left\|\nabla \tilde{f}(x)\right\|^2 = \|\nabla_s f(x)\|^2 + f(x)^2. \tag{2.2.12}$$

To prove this, fix $x \in \mathbb{S}^{n-1}$, $v \in T_x(\mathbb{S}^{n-1})$ and consider an orthonormal basis $\{\nu_1, \dots, \nu_{n-1}\}$ of $T_x(\mathbb{S}^{n-1})$. Then $\{\nu_1, \dots, \nu_{n-1}, x\}$ is an orthonormal basis of $\mathbb{R}^n$, and we can write

$$\nabla \tilde{f}(x) = \sum_{i=1}^{n-1} \langle \nabla \tilde{f}(x), \nu_i \rangle \nu_i + \langle \nabla \tilde{f}(x), x \rangle x,$$

hence

$$\left\|\nabla \tilde{f}(x)\right\|^2 = \sum_{i=1}^{n-1} \langle \nabla \tilde{f}(x), \nu_i \rangle^2 + \langle \nabla \tilde{f}(x), x \rangle^2 = \sum_{i=1}^{n-1} \langle \nabla_s f(x), \nu_i \rangle^2 + \tilde{f}(x)^2 = \|\nabla_s f(x)\|^2 + f(x)^2,$$

where we have used (2.2.9), (2.2.11) and the fact that $\tilde{f}$ extends f. This proves (2.2.12).

The support functions introduced in Subsection 2.1.6, for instance, can be seen as 1-homogeneous extensions of functions defined on the sphere. Other than (2.2.12), the Euclidean gradient of a support function possesses the following property.

Proposition 2.2.14 ([34, Corollary 1.7.3])**.** *Let $K \in \mathcal{K}^n$. If h_K is differentiable at $x \in \mathbb{S}^{n-1}$, then $\nabla h_K(x) \in \partial K$, and $\nabla h_K(x)$ is the only point of ∂K with outer normal vector x.*

There is another way of extending Lipschitz functions from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$ which we will be interested in; it is stated in the following theorem, which was proved independently by Kirszbraun, McShane and Whitney during the same year.

Theorem 2.2.15 (Kirszbraun-McShane-Whitney, [16, 27, 42])**.** *Let $S \subseteq \mathbb{R}^n$ and $f : S \longrightarrow \mathbb{R}$ be a Lipschitz function with Lipschitz constant L. Then the map $\hat{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by*

$$\hat{f}(x) = \sup_{z \in S} \left[f(z) - L\|x - z\| \right],$$

for $x \in \mathbb{R}^n$, is still Lipschitz continuous with the same Lipschitz constant L, and its restriction to S coincides with f.

This can be used to prove the following version of Corollary 1 from [13, Section 3.1]; we are going to need it in Chapter 5.

Lemma 2.2.16. *Let $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$, $c \in \mathbb{R}$ and $Z_c = \{x \in \mathbb{S}^{n-1} : f(x) = c\}$. Then $\nabla_s f(x) = 0$ for a.e. $x \in Z_c$.*

Proof. For a given $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$, extend it to $\mathbb{R}^n$ via Theorem 2.2.15. The function $f - c : \mathbb{R}^n \longrightarrow \mathbb{R}$ is still Lipschitz continuous, hence Corollary 1 from [13, Subsection 3.1.2] implies that $\nabla f(x) = 0$ for a.e. $x \in \{y \in \mathbb{R}^n : f(y) = c\}$, and in particular for a.e. $x \in Z_c$. From (2.2.9) we get that $\nabla_s f(x) = 0$ for a.e. $x \in Z_c$ too. $\square$

We will also need an adaptation of the Kirszbraun-McShane-Whitney extension theorem.

Lemma 2.2.17. *Let $A, A_0, B \subseteq \mathbb{R}^n$ be such that $A, A_0 \subseteq B$. Let $f : A \cup A_0 \longrightarrow \mathbb{R}$, $g : B \longrightarrow \mathbb{R}$ be Lipschitz functions with Lipschitz constants $L(f), L(g) \leq L$ such that $f|_A = g$, $f|_{A_0} = 0$. Then f can be extended to a Lipschitz function $\bar{f} : B \longrightarrow \mathbb{R}$ with Lipschitz constant $L(\bar{f}) \leq L$ such that $g^- \leq \bar{f} \leq g^+$ on B and $\|\bar{f}\|_\infty \leq \|g\|_\infty$.*

Proof. Consider the Kirszbraun-McShane-Whitney extension of f to B, defined by

$$\hat{f}(x) = \sup_{y \in A \cup A_0} \big[f(y) - L\|x - y\|\big],$$

for $x \in B$ (see Theorem 2.2.15). This function is Lipschitz continuous with Lipschitz constant $L(\hat{f}) \leq L$, and $\hat{f}|_{A \cup A_0} = f$.

Let $\bar{f} = \left(\hat{f} \vee g^-\right) \wedge g^+$; then $\bar{f}$ is still Lipschitz continuous with Lipschitz constant $L(\bar{f}) \leq L$ and $\bar{f}|_{A \cup A_0} = f$. Moreover, $\bar{f} \leq g^+$ and $\bar{f} = \left(\hat{f} \wedge g^+\right) \vee g^- \geq g^-$.

It remains to be seen that $\|\bar{f}\|_\infty \leq M$, where $M = \|g\|_\infty$. To prove this, fix an arbitrary $x \in B$ and reason as follows: if $g(x) \geq 0$,

$$\bar{f}(x) = \left(\hat{f}(x) \vee 0\right) \wedge g(x) = \left(\hat{f}(x) \wedge g(x)\right) \vee 0,$$

so that on the one hand we have

$$\bar{f}(x) \leq g(x) \leq M,$$

and on the other hand

$$\bar{f}(x) \geq 0 \geq -M,$$

hence $\left|\bar{f}(x)\right| \leq M$. We proceed similarly if $g(x) \leq 0$. ☐

2.2.5 Valuations on $\mathscr{H}(\mathbb{S}^{n-1})$

In this section we collect some technical remarks concerning support functions and valuations on $\mathscr{H}(\mathbb{S}^{n-1})$. It is convenient to study the behaviour of support functions with respect to the operators $\vee$ and $\wedge$. The result hereby presented is well-known, but we include the proof for completeness.

Lemma 2.2.18. *Let $K, L \in \mathcal{K}^n$. Then $h_K \vee h_L = h_{\mathrm{conv}(K \cup L)}$, where $\mathrm{conv}(K \cup L)$ denotes the convex hull of $K \cup L$. Moreover, if $K \cup L \in \mathcal{K}^n$ we have*

$$h_K \vee h_L = h_{K \cup L}, \tag{2.2.13}$$

$$h_K \wedge h_L = h_{K \cap L}. \tag{2.2.14}$$

Proof. Recall that

$$\mathrm{conv}(K \cup L) = \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^m \lambda_i x_i,\, m \in \mathbb{N},\, \lambda_i \geq 0,\, \sum_{i=1}^m \lambda_i = 1,\, x_i \in K \cup L \right\}.$$

Clearly $\mathrm{conv}(K \cup L)$ contains K and L, thus $h_{\mathrm{conv}(K \cup L)} \geq h_K$ and $h_{\mathrm{conv}(K \cup L)} \geq h_L$. This implies the inequality

$$h_{\mathrm{conv}(K \cup L)}(x) \geq h_K \vee h_L(x),$$

for every $x \in \mathbb{R}^n$. Vice versa, if $x \in \mathbb{R}^n$, then

$$h_{\mathrm{conv}(K\cup L)}(x) = \max_{z\in \mathrm{conv}(K\cup L)} \langle x, z\rangle,$$

where the maximum will be attained in correspondence of a certain element $z = \sum_{i=1}^m \lambda_i x_i$, with $m \in \mathbb{N}$, $\lambda_i \geq 0$, $\sum_{i=1}^m \lambda_i = 1$, $x_i \in K \cup L$. Up to a reordering, we may assume that $\{x_1, \ldots, x_\ell\} \subseteq K$ and $\{x_{\ell+1}, \ldots, x_m\} \subseteq L$. Therefore, we have

$$\begin{aligned} h_{\mathrm{conv}(K\cup L)}(x) &= \left\langle x, \sum_{i=1}^m \lambda_i x_i \right\rangle = \sum_{i=1}^{\ell} \lambda_i \langle x, x_i\rangle + \sum_{i=\ell+1}^{m} \lambda_i \langle x, x_i\rangle \\ &\leq \sum_{i=1}^{\ell} \lambda_i h_K(x) + \sum_{i=\ell+1}^{m} \lambda_i h_L(x) \\ &\leq \left(\sum_{i=1}^{\ell} \lambda_i + \sum_{i=\ell+1}^{m} \lambda_i \right) h_K \vee h_L(x) = h_K \vee h_L(x). \end{aligned}$$

We now work under the hypothesis $K \cup L \in \mathcal{K}^n$. The first part of the proof immediately gives (2.2.13). As for (2.2.14), we start by proving that

$$(K \cup L) + (K \cap L) = K + L. \tag{2.2.15}$$

Note that

$$(K \cup L) + (K \cap L) \subseteq K + L.$$

Vice versa, let $x + y \in K + L$. If either $x \in K \cap L$ or $y \in K \cap L$, then we are done; suppose now $x \in K \setminus L$ and $y \in L \setminus K$. Because of this assumption, there exists $t \in (0, 1)$ such that

$$z := tx + (1 - t)y \in K \cap L,$$

since $K \cup L \in \mathcal{K}^n$. Therefore,

$$x + y = (1 - t)x + ty + z \in (K \cup L) + (K \cap L),$$

using the convexity of $K \cup L$ again. Equality (2.2.15) follows. Thus, from (2.1.6) we obtain

$$h_{K\cup L} + h_{K\cap L} = h_{(K\cup L)+(K\cap L)} = h_{K+L} = h_K + h_L = h_K \vee h_L + h_K \wedge h_L = h_{K\cup L} + h_K \wedge h_L,$$

where the last equality follows from (2.2.13). This proves (2.2.14). □

We are now going to state a topological result concerning the continuity of a functional $V : \mathscr{H}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$. By definition of τ, we have that if such a functional is continuous with respect to $\|\cdot\|_\infty$, then it is also continuous with respect to τ. The converse is also true.

Lemma 2.2.19. *Let $V : \mathscr{H}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$. Then V is continuous with respect to τ if and only if it is continuous with respect to $\|\cdot\|_\infty$.*

Proof. Consider a τ-continuous functional $V : \mathscr{H}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ and a sequence $\{h_{K_i}\} \subseteq \mathscr{H}(\mathbb{S}^{n-1})$ of support functions such that $\|h_{K_i} - h_K\|_\infty \to 0$, as $i \to \infty$, where $K \in \mathcal{K}^n$. It is enough to prove that $h_{K_i} \underset{\tau}{\to} h_K$.

Define

$$D_i = \{x \in \mathbb{S}^{n-1} : h_{K_i} \text{ is differentiable at } x\},$$

for $i \in \mathbb{N}$, and

$$D_0 = \{x \in \mathbb{S}^{n-1} : h_K \text{ is differentiable at } x\}.$$

We also set

$$D = \bigcap_{i=0}^{\infty} D_i.$$

Note that $\mathcal{H}^{n-1}(\mathbb{S}^{n-1} \setminus D) = 0$, because of Rademacher's theorem.

For every $x \in D$ we have

$$\nabla_s h_{K_i}(x) \to \nabla_s h_K(x). \tag{2.2.16}$$

Indeed, consider a subsequence $\{h_{K_{i_j}}\} \subseteq \{h_{K_i}\}$. For every $j \in \mathbb{N}$, the differentiability of $h_{K_{i_j}}$ at x gives (see [34, Theorem 1.5.12])

$$h_{K_{i_j}}(y) \geq h_{K_{i_j}}(x) + \left\langle \nabla h_{K_{i_j}}(x), y - x \right\rangle, \tag{2.2.17}$$

for every $y \in \mathbb{R}^n$. The condition $\|h_{K_i} - h_K\|_\infty \to 0$ implies that $K_i \to K$ with respect to the Hausdorff metric (thanks to (2.1.7)), hence there is a convex body K_0 such that $K_i \subseteq K_0$, for every $i \in \mathbb{N}$. From Proposition 2.2.14 we have that

$$\nabla h_{K_{i_j}}(x) \in \partial K_{i_j} \subseteq K_{i_j} \subseteq K_0,$$

thus there is a subsequence $\{h_{K_{i_{j_\ell}}}\} \subseteq \{h_{K_{i_j}}\}$ such that $\lim_{l \to \infty} \nabla h_{K_{i_{j_\ell}}}(x)$ exists, by the Bolzano-Weierstrass theorem. Writing (2.2.17) for this subsequence and letting $\ell \to \infty$ we obtain

$$h_K(y) \geq h_K(x) + \left\langle \lim_{\ell \to \infty} \nabla h_{K_{i_{j_\ell}}}(x), y - x \right\rangle,$$

for every $y \in \mathbb{R}^n$. Recalling the uniqueness of the subgradient at differentiability points for convex functions (see [34, Theorem 1.5.15]), the last inequality implies

$$\lim_{\ell \to \infty} \nabla h_{K_{i_{j_\ell}}}(x) = \nabla h_K(x).$$

This, together with relation (2.2.12) and the arbitrariness of $\{h_{K_{i_j}}\} \subseteq \{h_{K_i}\}$, proves (2.2.16).

Moreover, for every $x \in D$ and $i \in \mathbb{N}$ we have, applying (2.2.12) and Proposition 2.2.14 again,

$$\|\nabla_s h_{K_i}(x)\| \leq \|\nabla h_{K_i}(x)\| \leq \max\{\|y\| : y \in \partial K_i\} \leq \max\{\|y\| : y \in K_0\}.$$

Thus we have a uniform bound on $\|\nabla_s h_{K_i}\|$ in D. Then $h_{K_i} \underset{\tau}{\to} h_K$, as desired. $\square$

The theorems recalled in subsections 2.2.1, 2.2.2 concern valuations on convex bodies, and since we will be interested in studying valuations on support functions using these results, it would be nice to know that we can "move" valuations from $\mathscr{H}(\mathbb{S}^{n-1})$ to $\mathcal{K}^n$ without losing any property. This is stated precisely in the next result.

Lemma 2.2.20. *Let $V : \mathscr{H}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$. Define $\varphi : \mathcal{K}^n \longrightarrow \mathbb{R}$ by setting*

$$\varphi(K) = V(h_K),$$

for every $K \in \mathcal{K}^n$. Then

i) if V is a valuation, then so is φ;

ii) if V is τ-continuous, then φ is continuous with respect to the Hausdorff metric;

iii) if V is rotation invariant, then so is φ;

iv) if V is dot product invariant, then φ is translation invariant;

v) if V is polynomial, then so is φ;

vi) if V is i-homogeneous for some $i \in \{0, \dots, n\}$, then so is φ.

Proof.

i) Let V be a valuation. To prove the valuation property for φ, consider $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$. From Lemma 2.2.18 and the fact that V is a valuation we get

$$\begin{aligned}\varphi(K \cup L) + \varphi(K \cap L) &= V(h_{K \cup L}) + V(h_{K \cap L}) = V(h_K \vee h_L) + V(h_K \wedge h_L) \\ &= V(h_K) + V(h_L) = \varphi(K) + \varphi(L).\end{aligned}$$

ii) For what concerns continuity, let V be τ-continuous and consider $\{K_i\} \subseteq \mathcal{K}^n$, $K \in \mathcal{K}^n$ such that $K_i \to K$ in the Hausdorff metric. Then $\|h_{K_i} - h_K\|_\infty \to 0$, by (2.1.7), and recalling that a τ-continuous functional such as V is also continuous with respect to $\|\cdot\|_\infty$ on the space of support functions (see Lemma 2.2.19), we have

$$V(h_K) = \lim_{i \to \infty} V(h_{K_i}),$$

which means

$$\varphi(K) = \lim_{i \to \infty} \varphi(K_i).$$

iii) Let V be rotation invariant. Since the dot product is rotation invariant too, for every $K \in \mathcal{K}^n$, $\sigma \in \mathcal{O}(n)$ and $x \in \mathbb{S}^{n-1}$ we have

$$h_{\sigma K}(x) = \max_{y \in K} \langle x, \sigma(y) \rangle = \max_{y \in K} \langle \sigma^{-1}(x), y \rangle = h_K \circ \sigma^{-1}(x),$$

hence $h_{\sigma K} = h_K \circ \sigma^{-1}$. This yields

$$\varphi(\sigma K) = V(h_{\sigma K}) = V(h_K \circ \sigma^{-1}) = V(h_K) = \varphi(K),$$

because of the rotational invariance of V.

iv) If V is dot product invariant, take $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$. We have

$$\varphi(K + x) = V(h_{K+x}) = V(h_K + h_{\{x\}}) = V(h_K + \langle \cdot, x \rangle) = V(h_K) = \varphi(K).$$

v) If V is polynomial, we have that for every $K \in \mathcal{K}^n$

$$\varphi(K + x) = V(h_{K+x}) = V(h_K + \langle \cdot, x \rangle) = p_K(x)$$

is a polynomial in x.

vi) Fix $i \in \{0, \dots, n\}$ and assume V to be i-homogeneous. For every $\lambda > 0$ and $K \in \mathcal{K}^n$ we get

$$\varphi(\lambda K) = V(h_{\lambda K}) = V(\lambda h_K) = \lambda^i V(h_K) = \lambda^i \varphi(K).$$

□

Chapter 3

Approximation

To prove theorems 1.1.2 and 1.1.3 we will first need to narrow down the study of our valuations from the space $\mathrm{Lip}(\mathbb{S}^{n-1})$ (respectively, $\mathrm{Lip}(\mathbb{S}^1)$) to its subset $\mathscr{H}(\mathbb{S}^{n-1})$ ($\mathscr{H}(\mathbb{S}^1)$), which is in bijection with $\mathcal{K}^n$ ($\mathcal{K}^2$), where Hadwiger's (Alesker's) theorem can be applied. More precisely, our goal is to prove that continuous valuations on $\mathrm{Lip}(\mathbb{S}^{n-1})$ are uniquely determined by the values they attain at support functions, as stated in the following proposition.

Proposition 3.0.1. *Let $V, W : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ be continuous valuations. If $V = W$ on $\mathscr{H}(\mathbb{S}^{n-1})$, then $V = W$ on $\mathrm{Lip}(\mathbb{S}^{n-1})$.*

The proof is split into four main steps, which will be detailed in the next sections. In particular, Proposition 3.0.1 will be derived from the following density result (proved in Section 3.4), which will also be used in the proofs of theorems 1.1.4 and 1.1.5.

Proposition 3.0.2. *The space $\mathscr{L}(\mathbb{S}^{n-1})$ is τ-dense in $\mathrm{Lip}(\mathbb{S}^{n-1})$.*

3.1 $\mathscr{L}(\mathbb{S}^{n-1}) \subseteq \widehat{\mathscr{H}}(\mathbb{S}^{n-1})$

First of all, we are going to show that piecewise linear functions can be written as minima of finitely many support functions.

Lemma 3.1.1. *Let $f \in \mathscr{L}(\mathbb{R}^n)$. Then there exist $m \in \mathbb{N}$ and $h_{K_1}, \dots, h_{K_m} \in \mathscr{H}(\mathbb{R}^n)$ such that*

$$f = \bigwedge_{i=1}^{m} h_{K_i}.$$

In particular, $\mathscr{L}(\mathbb{S}^{n-1}) \subseteq \widehat{\mathscr{H}}(\mathbb{S}^{n-1})$.

Proof. For $f \in \mathscr{L}(\mathbb{R}^n)$, there are closed convex cones $C_1, \dots, C_m$ with vertex at the origin and pairwise disjoint interiors such that

$$\bigcup_{i=1}^{m} C_i = \mathbb{R}^n,$$

and $f = L_i$ is linear on C_i, for $i = 1, \dots, m$.

We will now focus on the cone C_1. Consider $\tilde{f} = f - L_1$. Let $P_{C_1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ denote the *metric projection* onto C_1: for every $x \in \mathbb{R}^n$, $P_{C_1}(x)$ is the unique point in C_1 such that

$$\|x - P_{C_1}(x)\| = \min_{z \in C_1} \|x - z\|.$$

As C_1 is closed and convex, this function is well-defined. We also define the function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$g(x) = \|x - P_{C_1}(x)\| = \min_{z \in C_1} \|x - z\|,$$

for every $x \in \mathbb{R}^n$; this is the distance function from the cone C_1.

Note that g is a 1-homogeneous and convex function. To prove the 1-homogeneity, we take $\lambda > 0$, $x \in \mathbb{R}^n$ and we evaluate

$$g(\lambda x) = \min_{z \in C_1} \|\lambda x - z\|.$$

Since C_1 is a cone, we can write

$$g(\lambda x) = \min_{z \in C_1} \|\lambda x - \lambda z\| = \min_{z \in C_1} \lambda \|x - z\| = \lambda \min_{z \in C_1} \|x - z\| = \lambda g(x),$$

using the fact that $\lambda > 0$. Regarding the convexity of g, we observe that

$$P_{C_1}(x) + P_{C_1}(y) = \frac{1}{2} \cdot 2P_{C_1}(x) + \frac{1}{2} \cdot 2P_{C_1}(y) \in C_1,$$

for every $x, y \in \mathbb{R}^n$, since C_1 is a convex cone. Therefore,

$$\begin{aligned} g(x+y) &= \min_{z \in C_1} \|x + y - z\| \leq \big\|x + y - (P_{C_1}(x) + P_{C_1}(y))\big\| \\ &\leq \|x - P_{C_1}(x)\| + \|y - P_{C_1}(y)\| = g(x) + g(y). \end{aligned}$$

This proves that g is subadditive, which, together with the 1-homogeneity, yields the convexity of g. These properties imply the existence of a convex body $K \in \mathcal{K}^n$ such that $g = h_K$, as recalled in Subsection 2.1.6.

We prove that there exists a suitable constant $c > 0$ such that

$$cg(x) \geq \tilde{f}(x), \tag{3.1.1}$$

for every $x \in \mathbb{R}^n$. Suppose this to be false; then for every $c > 0$ there exists a point $x_c \in \mathbb{R}^n$ such that $cg(x_c) < \tilde{f}(x_c)$. Choosing $c = i$, $i \in \mathbb{N}$, we build a sequence $\{x_i\} \subseteq \mathbb{R}^n$ satisfying

$$g(x_i) < \frac{1}{i} \tilde{f}(x_i), \tag{3.1.2}$$

for every $i \in \mathbb{N}$. Because $g = \tilde{f} = 0$ on C_1 and the inequality is strict, we have that $x_i \neq 0$ for every $i \in \mathbb{N}$. From the 1-homogeneity we get

$$g\left(\frac{x_i}{\|x_i\|}\right) < \frac{1}{i} \tilde{f}\left(\frac{x_i}{\|x_i\|}\right).$$

This means that in (3.1.2) we may assume that $\{x_i\} \subseteq \mathbb{S}^{n-1}$ and, up to passing to a subsequence, $x_i \to x$ as $i \to \infty$, for some $x \in \mathbb{S}^{n-1}$.

We observe that $x \in C_1$. In fact, if $x \in \mathbb{R}^n \setminus C_1$, then letting $i \to \infty$ in (3.1.2) we would have $0 < g(x) \leq 0$, a contradiction.

Let $\tilde{x}_i = P_{C_1}(x_i)$. Now, the projection onto closed convex sets is contracting (see [34, Theorem 1.2.1]), hence continuous, which implies that $\tilde{x}_i \to P_{C_1}(x) = x$ as $i \to \infty$. From (3.1.2), using the fact that $\tilde{f}(\tilde{x}_i) = 0$ (since $\tilde{x}_i \in C_1$) and setting $\widetilde{L} = L(\tilde{f})$, we get

$$\|x_i - \tilde{x}_i\| = g(x_i) < \frac{1}{i}\left[\tilde{f}(x_i) - \tilde{f}(\tilde{x}_i)\right] \leq \frac{\widetilde{L}}{i} \|x_i - \tilde{x}_i\|,$$

for every $i \in \mathbb{N}$. Since $x_i \notin C_1$ (because of the strict inequality in (3.1.2)), whereas $\tilde{x}_i \in C_1$, we have $x_i \neq \tilde{x}_i$, and so the last inequality yields $i < \widetilde{L}$, for every $i \in \mathbb{N}$; letting $i \to \infty$ we obtain a contradiction. Then there must be a constant $c > 0$ such that (3.1.1) holds for every $x \in \mathbb{R}^n$.

Therefore,

$$f_1 := cg + L_1 \geq \tilde{f} + L_1 = f$$

on $\mathbb{R}^n$, with $f_1 = L_1 = f$ on the cone C_1. Furthermore, if $L_1(x) = \langle x, a_1 \rangle$, we have that

$$f_1 = cg + L_1 = ch_K + h_{\{a_1\}} = h_{cK+a_1} =: h_{K_1}$$

is a support function.

We repeat the process for each cone C_i, $i = 2, \dots, m$, building support functions $h_{K_i} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $h_{K_i} \geq f$ on $\mathbb{R}^n$ and $h_{K_i} = f$ on C_i. Thus we can write

$$f = \bigwedge_{i=1}^{m} h_{K_i}.$$

$\square$

3.2 Approximation of C^1 functions by piecewise linear functions

Piecewise linear functions can be used to approximate C^1 functions with respect to the topology τ, as stated in the following lemma.

Lemma 3.2.1. *Let $u \in C^1(\mathbb{S}^{n-1})$. Then there exists a sequence $\{f_i\} \subseteq \mathscr{L}(\mathbb{S}^{n-1})$ such that $f_i \underset{\tau}{\rightarrow} u$.*

To prove this we will need a preliminary observation, which in turn requires some definitions: a *k-simplex* Δ is a k-dimensional polytope given by the convex hull of $k+1$ affinely independent points $v_0, \dots, v_k$, that is,

$$\Delta = \left\{ \sum_{i=0}^{k} \lambda_i v_i : \sum_{i=0}^{k} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for every } i = 0, \dots, k \right\},$$

where $v_1 - v_0, \dots, v_k - v_0$ are linearly independent. The points $v_0, \dots, v_k$ are called the *vertices* of Δ.

A partition $\mathcal{P}$ of a k-dimensional set $Q \subseteq \mathbb{R}^n$ is called a *simplicial partition* if it is made up of k-simplices such that for every two of them, their intersection is either empty or a face (of any dimension between 0 and $k-1$) of both simplices. If Q is a polytope, such a partition is *symmetric* if it is symmetric with respect to all the k axes of a coordinate system which has the center o of Q as the origin and axes given by the lines passing through o and the center of two opposite facets. We also recall that a *facet* of a k-dimensional polytope is a $(k-1)$-dimensional face. We can now point out the following fact.

Remark 3.2.2. *Let $Q \subseteq \mathbb{R}^n$ be a n-cube. Then there exists a symmetric simplicial partition $\mathcal{P}$ into n-simplices of Q.*

Proof. We proceed by induction on the dimension n. If $n = 2$, then Q is a square, and we can choose $\mathcal{P}$ to be the partition made up of the four simplices obtained by connecting the center of the square to each of the four vertices.

Let $n \geq 3$. Let $\{F_1, \dots, F_{2n}\}$ be the facets of Q. Since the facets of a n-cube are $(n-1)$-cubes, we can apply the inductive hypothesis to F_1 to obtain a symmetric simplicial partition $\mathcal{P}_1$ into $(n-1)$-simplices of F_1. We replicate this same partition on each other facet of Q, and we denote by $\mathcal{P}_i$ the partition on the facet F_i. Because $\mathcal{P}_1$, hence $\mathcal{P}_i$ for every i, is symmetric, the partitions of $F_1, \dots, F_{2n}$ "glue together" well, in the sense that they give a simplicial partition $\mathcal{P}'$ into $(n-1)$-simplices of ∂Q. Connecting the center of Q to the vertices of all the simplices in $\mathcal{P}'$ we obtain a symmetric simplicial partition $\mathcal{P}$ into n-simplices of Q. $\square$

This allows us to prove the lemma stated above.

Proof of Lemma 3.2.1. Let $f \in C^1(\mathbb{S}^{n-1})$ and consider its 1-homogeneous extension $\tilde{f}$ to $\mathbb{R}^n$:

$$\tilde{f}(x) = \begin{cases} \|x\| \, f\left(\dfrac{x}{\|x\|}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then $\tilde{f} \in C^1(\mathbb{R}^n \setminus \{0\})$. In particular, $\tilde{f} \in C^1(D)$, where

$$D = \left\{x \in \mathbb{R}^n : 1 \leq \|x\| \leq \sqrt{n}\right\}.$$

Fix $\varepsilon > 0$. Since $\tilde{f}$ and its Euclidean gradient $\nabla \tilde{f}$ are uniformly continuous on D, there exists $\delta > 0$ such that for every $x, y \in D$ with $\|x - y\| \leq \delta$ we have

$$\left|\tilde{f}(x) - \tilde{f}(y)\right| < \varepsilon \tag{3.2.1}$$

and

$$\left\|\nabla \tilde{f}(x) - \nabla \tilde{f}(y)\right\| < \varepsilon. \tag{3.2.2}$$

Let $Q = [-1, 1]^n$ be the n-cube centered at the origin with edge of length 2. For each coordinate axis, we draw hyperplanes orthogonal to such axis so that Q is cut into n-cubes with edges of length $\frac{1}{N}$, where $N = \left\lceil \frac{\sqrt{n}}{\delta} \right\rceil$ ($\lceil \cdot \rceil$ denotes the ceiling function). Note that these n-cubes all have the same diameter

$$d = \left\| \left(\frac{1}{N}, \dots, \frac{1}{N}\right) - (0, \dots, 0) \right\| = \frac{\sqrt{n}}{N} \leq \delta.$$

Consider now the facets of such n-cubes (which are $(n-1)$-cubes) that are contained in the boundary ∂Q of Q. We apply Remark 3.2.2 to these facets: this determines a simplicial partition $\{\Delta_1, \dots, \Delta_m\}$ into $(n-1)$-simplices of ∂Q. For every $i = 1, \dots, m$, let

$$C_i = \{tx : t \geq 0,\, x \in \Delta_i\}.$$

Then $C_1, \dots, C_m$ are closed convex cones with pairwise disjoint interiors, and they form a partition of the whole space $\mathbb{R}^n$. Note that since the annulus D contains all the simplices Δ_i and $d \leq \delta$, formulas (3.2.1) and (3.2.2) are satisfied for every x and y belonging to the same simplex.

We consider linear maps $L_i : C_i \longrightarrow \mathbb{R}$, $i = 1, \dots, m$, such that L_i coincides with $\tilde{f}$ on each of the n vertices of Δ_i; these maps are uniquely determined. Let $g \in \mathscr{L}(\mathbb{R}^n)$ be the continuous function such that $g = L_i$ on C_i, for $i = 1, \dots, m$, and define $h = \tilde{f} - g$.

For a fixed $x \in D$, we have that $x \in C_k$ for some $k \in \{1, \dots, m\}$, and we can write $x = \lambda x'$, with $\lambda > 0$ and $x' \in \Delta_k$. Choose an arbitrary vertex v of Δ_k. Since Δ_k is compact, there is a $w \in \Delta_k$ such that

$$|L_k(v) - L_k(w)| = \max_{z \in \Delta_k} |L_k(v) - L_k(z)|.$$

Note that the function defined by $F(z) = |L_k(v) - L_k(z)|$, $z \in \Delta_k$, is convex. Since convex functions on convex polytopes attain their maximum at the vertices, w must be a vertex of Δ_k.

Given that both $\tilde{f}$ and g are 1-homogeneous, using (3.2.1) and the fact that $L_k = \tilde{f}$ on the vertices of Δ_k, we get

$$\begin{aligned} |h(x)| &= \lambda\big|\tilde{f}(x') - g(x')\big| = \lambda\big|\tilde{f}(x') - L_k(x')\big| < \lambda\varepsilon + \lambda|L_k(v) - L_k(x')| \\ &\leq \lambda\varepsilon + \lambda|L_k(v) - L_k(w)| = \lambda\varepsilon + \lambda\big|\tilde{f}(v) - \tilde{f}(w)\big| < 2\lambda\varepsilon = 2\frac{\|x\|}{\|x'\|}\varepsilon \\ &\leq 2\sqrt{n}\varepsilon. \end{aligned}$$

Therefore,

$$\max_{x \in D} |h(x)| < 2\sqrt{n}\varepsilon. \tag{3.2.3}$$

We now turn to the gradient ∇h. Fix arbitrarily $k \in \{1, \dots, m\}$ and $x \in \operatorname{relint}(\Delta_k)$, the relative interior of Δ_k (i.e., the interior of Δ_k as a subset of a $(n-1)$-dimensional affine hyperplane of $\mathbb{R}^n$). Then $\nabla h(x)$ exists. We choose a vertex of Δ_k and consider the $n-1$ edges $\ell_1, \dots, \ell_{n-1}$ incident to it; since Δ_k is a simplex, these edges lie on linearly independent directions $\nu_1, \dots, \nu_{n-1}$. Note that the restriction of h to each ℓ_i can be seen as a function of one variable which is continuous on ℓ_i, differentiable in its relative interior and satisfies $h = 0$ at the ends of ℓ_i. By Rolle's theorem, there exist points $z_i \in \ell_i$ such that

$$\frac{\partial h}{\partial \nu_i}(z_i) = 0,$$

for every $i = 1, \dots, n-1$. Using the fact that $g|_{C_k}$ is linear and the Cauchy-Schwarz inequality, for every $i = 1, \dots, n-1$ we get

$$\begin{aligned} \left|\frac{\partial h}{\partial \nu_i}(x)\right| &= \left|\frac{\partial h}{\partial \nu_i}(x) - \frac{\partial h}{\partial \nu_i}(z_i)\right| = \left|\frac{\partial \tilde{f}}{\partial \nu_i}(x) - g(\nu_i) - \frac{\partial \tilde{f}}{\partial \nu_i}(z_i) + g(\nu_i)\right| \\ &= \left|\langle \nabla\tilde{f}(x) - \nabla\tilde{f}(z_i), \nu_i\rangle\right| \leq \left\|\nabla\tilde{f}(x) - \nabla\tilde{f}(z_i)\right\| < \varepsilon, \end{aligned} \tag{3.2.4}$$

where the last inequality follows from (3.2.2).

Let H be the hyperplane passing through the n vertices of Δ_k. Since $\Delta_k \subseteq \partial Q$, the exterior unit normal vector N to H is of the form $N = \pm e_{i_k}$, for some $i_k \in \{1, \dots, n\}$; for the sake of simplicity, let us assume $N = e_n$, since the general case can be dealt with analogously. We now observe that both $\{\nu_1, \dots, \nu_{n-1}\}$ and $\{e_1, \dots, e_{n-1}\}$ are bases of H; in particular, there exist numbers $\alpha_{ij} \in \mathbb{R}$, $i, j = 1, \dots, n-1$, such that

$$e_i = \sum_{j=1}^{n-1} \alpha_{ij}\nu_j,$$

for every $i = 1, \dots, n-1$. Therefore, for $i = 1, \dots, n-1$ we have

$$\left|\frac{\partial h}{\partial x_i}(x)\right| = |\langle \nabla h(x), e_i\rangle| \leq \sum_{j=1}^{n-1} |\alpha_{ij}| \cdot \left|\frac{\partial h}{\partial \nu_j}(x)\right| < M\varepsilon, \tag{3.2.5}$$

where we have used (3.2.4) and we have set

$$M = \max \left\{ \sum_{j=1}^{n-1} |\alpha_{ij}| : i \in \{1, \ldots, n-1\} \right\}.$$

Note that the α_{ij}'s, and thus M, do not depend on ε, since they are determined by the ν_i's, which are in turn determined by the simplicial partitions of the $(n-1)$-cubes contained in ∂Q; the length $\frac{1}{N}$ of the edge of such cubes depends on ε, but the angles appearing in the aforementioned partitions, hence the ν_i's, do not.

We now write

$$\begin{aligned} \nabla h(x) &= \langle \nabla h(x), e_1 \rangle e_1 + \ldots + \langle \nabla h(x), e_{n-1} \rangle e_{n-1} + \langle \nabla h(x), e_n \rangle e_n, \\ &= \frac{\partial h}{\partial x_1}(x) e_1 + \ldots + \frac{\partial h}{\partial x_{n-1}}(x) e_{n-1} + \frac{\partial h}{\partial x_n}(x) e_n, \end{aligned}$$

which, together with (3.2.5), implies

$$\|\nabla h(x)\| < M(n-1)\varepsilon + \left| \frac{\partial h}{\partial x_n}(x) \right|. \tag{3.2.6}$$

Let us consider the radial direction $r_x = \frac{x}{\|x\|}$. We have

$$\frac{\partial h}{\partial r_x}(x) = \langle \nabla h(x), r_x \rangle = \frac{h(x)}{\|x\|},$$

thanks to Euler's formula (2.2.11). This yields

$$\left| \frac{\partial h}{\partial r_x}(x) \right| = \frac{|h(x)|}{\|x\|} \leq |h(x)| < 2\sqrt{n}\varepsilon, \tag{3.2.7}$$

because of (3.2.3). On the other hand, $r_x = \langle r_x, e_1 \rangle e_1 + \ldots + \langle r_x, e_n \rangle e_n$, hence

$$\frac{\partial h}{\partial r_x}(x) = \langle \nabla h(x), r_x \rangle = \sum_{i=1}^{n-1} \left[\langle r_x, e_i \rangle \cdot \frac{\partial h}{\partial x_i}(x) \right] + \langle r_x, e_n \rangle \cdot \frac{\partial h}{\partial x_n}(x),$$

so that

$$\begin{aligned} |\langle r_x, e_n \rangle| \cdot \left| \frac{\partial h}{\partial x_n}(x) \right| &\leq \left| \frac{\partial h}{\partial r_x}(x) \right| + \sum_{i=1}^{n-1} |\langle r_x, e_i \rangle| \cdot \left| \frac{\partial h}{\partial x_i}(x) \right| \\ &< 2\sqrt{n}\varepsilon + \sum_{i=1}^{n-1} M\varepsilon = [2\sqrt{n} + M(n-1)]\varepsilon, \end{aligned}$$

where we have used (3.2.7), (3.2.5) and the Cauchy-Schwarz inequality. But since

$$|\langle r_x, e_n \rangle| = |\langle r_x, N \rangle| = \frac{1}{\|x\|} \geq \frac{1}{\sqrt{n}},$$

we obtain

$$\left| \frac{\partial h}{\partial x_n}(x) \right| < \left[2n + M\sqrt{n}(n-1) \right] \varepsilon.$$

From (3.2.6) we conclude that
$$\|\nabla h(x)\| < C\varepsilon, \tag{3.2.8}$$
where
$$C = 2n + M(\sqrt{n} + 1)(n - 1).$$
Formula (3.2.8) holds for every $x \in \operatorname{relint}(\Delta_k)$. By 0-homogeneity of ∇h, this extends to every x in the interior of C_k. Since $k \in \{1, \dots, m\}$ was arbitrary, using (2.2.12) we have that
$$\|\nabla_s h(x)\| < C\varepsilon,$$
for a.e. $x \in \mathbb{S}^{n-1}$.

In particular, we have proved that for every $i \in \mathbb{N}$ we can find a piecewise linear function $f_i \in \mathscr{L}(\mathbb{S}^{n-1})$ such that
$$\|f - f_i\|_\infty < \frac{2\sqrt{n}}{i}$$
and
$$\|\nabla_s f(x) - \nabla_s f_i(x)\| < \frac{C}{i},$$
for a.e. $x \in \mathbb{S}^{n-1}$. Therefore, $f_i \to f$ uniformly on $\mathbb{S}^{n-1}$ and $\nabla_s f_i \to \nabla_s f$ a.e. in $\mathbb{S}^{n-1}$. Besides, for every $i \in \mathbb{N}$ and a.e. $x \in \mathbb{S}^{n-1}$ we have
$$\|\nabla_s f_i(x)\| < \frac{C}{i} + \|\nabla_s f(x)\| \le C + \max_{y \in \mathbb{S}^{n-1}} \|\nabla_s f(y)\|,$$
so that $f_i \underset{\tau}{\to} f$. $\square$

3.3 Approximation of Lipschitz functions by C^1 functions

Functions in $C^1(\mathbb{S}^{n-1})$ can in turn be used to τ-approximate Lipschitz functions defined on the sphere.

Lemma 3.3.1. *Let $f \in \operatorname{Lip}(\mathbb{S}^{n-1})$. Then there exists a sequence $\{f_i\} \subseteq C^1(\mathbb{S}^{n-1})$ such that $f_i \underset{\tau}{\to} f$.*

There exists a similar result in the literature, namely Theorem 1 from Section 6.6 in [13], which is a corollary of Whitney's extension theorem, but it is not exactly what we are going to need in the thesis.

Proof of Lemma 3.3.1. Let $f \in \operatorname{Lip}(\mathbb{S}^{n-1})$ be a Lipschitz function with Lipschitz constant L. As stated in Theorem 2.2.15, such a function can be extended to a map $\hat{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by
$$\hat{f}(x) = \max_{z \in \mathbb{S}^{n-1}} \big[f(z) - L\|x - z\| \big],$$
for $x \in \mathbb{R}^n$, which is still Lipschitz continuous with the same Lipschitz constant L. To simplify the notations, the extension will still be denoted by the same symbol f.

Consider the annuli
$$\mathcal{C}_0 = \left\{ x \in \mathbb{R}^n : \frac{1}{3} \le \|x\| \le \frac{5}{3} \right\},$$
$$\mathcal{C}_1 = \left\{ x \in \mathbb{R}^n : \frac{2}{3} \le \|x\| \le \frac{4}{3} \right\},$$

and let $\eta : \mathbb{R}^n \longrightarrow [0,1]$ be a C^∞ function with $\eta \prec \mathcal{C}_0$ such that $\eta \equiv 1$ on $\mathcal{C}_1$. For example, we can define η in the following way: take a C^∞ function $\varphi : \mathbb{R} \longrightarrow [0,1]$ with $\varphi \prec \left[\frac{1}{3}, \frac{5}{3}\right]$ and $\varphi \equiv 1$ in $\left[\frac{2}{3}, \frac{4}{3}\right]$, then set $\eta(x) = \varphi(\|x\|)$.

The function $\tilde{f} := f \cdot \eta$ is Lipschitz continuous: in fact η is, since, from Theorem 2.2.11, for every $x, y \in \mathbb{R}^n$ there exists $\lambda \in (0,1)$ such that

$$\eta(x) - \eta(y) = \langle \nabla\eta((1-\lambda)x + \lambda y), x - y \rangle ,$$

thus

$$|\eta(x) - \eta(y)| \leq L(\eta) \cdot \|x - y\|,$$

with $L(\eta) = \max\limits_{z \in \operatorname{supp}(\eta)} \|\nabla\eta(z)\|$. The product of bounded Lipschitz functions is Lipschitz continuous, hence $\tilde{f}$ is indeed a Lipschitz function; let $\widetilde{L}$ be its Lipschitz constant.

We will use a mollifier to build our approximating sequence. Consider $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ defined, for $z \in \mathbb{R}^n$, by

$$\psi(z) = \begin{cases} c \cdot e^{\frac{1}{\|z\|^2 - 1}} & \text{if } \|z\| < 1, \\ 0 & \text{if } \|z\| \geq 1, \end{cases}$$

where $c > 0$ is a constant such that

$$\int_{\mathbb{R}^n} \psi(z) dz = 1,$$

and the integral is done with respect to the Lebesgue measure $\mathcal{L}^n$. The function ψ is C^∞, and its support is the closure $\overline{B_1(0)}$ of the unit ball centered at the origin.

For $\varepsilon > 0$, we set

$$\psi_\varepsilon(z) = \frac{1}{\varepsilon^n} \psi\left(\frac{z}{\varepsilon}\right), \; z \in \mathbb{R}^n.$$

The family $\{\psi_\varepsilon\}_{\varepsilon > 0}$ is a mollifier, and for every $\varepsilon > 0$ we have that $\psi_\varepsilon \in C^\infty(\mathbb{R}^n)$ and its support is the set $\overline{B_\varepsilon(0)}$. We now consider the C^∞ functions

$$\tilde{f}_\varepsilon(x) = \psi_\varepsilon * \tilde{f}(x) = \int_{\mathbb{R}^n} \tilde{f}(x - y)\psi_\varepsilon(y) dy, \; x \in \mathbb{R}^n.$$

For $x, y \in \mathbb{R}^n$, we have

$$\left|\tilde{f}_\varepsilon(x) - \tilde{f}_\varepsilon(y)\right| = \left| \int_{\mathbb{R}^n} \psi_\varepsilon(z) \left[\tilde{f}(x - z) - \tilde{f}(y - z)\right] dz \right| \leq \widetilde{L} \, \|x - y\|.$$

This yields

$$\left\|\nabla_s \tilde{f}_\varepsilon(x)\right\| \leq \widetilde{L}, \tag{3.3.1}$$

for every $x \in \mathbb{S}^{n-1}$ and $\varepsilon > 0$, thanks to (2.1.2).

We also have that $\tilde{f}_\varepsilon \to f$ uniformly on $\mathbb{S}^{n-1}$, as $\varepsilon \to 0^+$. Indeed, since $\tilde{f}$ is uniformly continuous, for a fixed $\varepsilon' > 0$ there is a $\delta > 0$ such that

$$\left|\tilde{f}(z_1) - \tilde{f}(z_2)\right| < \frac{\varepsilon'}{2},$$

for every $z_1, z_2 \in \mathbb{R}^n$ satisfying $\|z_1 - z_2\| \le \delta$. Therefore, for $\varepsilon > 0$ and $x \in \mathbb{S}^{n-1}$ we get

$$\begin{aligned} \left|\tilde{f}_\varepsilon(x) - f(x)\right| &= \left|\int_{\mathbb{R}^n} \tilde{f}(x-y)\psi_\varepsilon(y)dy - \int_{\mathbb{R}^n} \tilde{f}(x)\psi_\varepsilon(y)dy\right| \\ &\le \int_{\mathbb{R}^n} \psi_\varepsilon(y)\left|\tilde{f}(x-y) - \tilde{f}(x)\right|dy \\ &< \frac{\varepsilon'}{2} + \int_{\mathbb{R}^n \setminus B_\delta(0)} \psi_\varepsilon(y)\left|\tilde{f}(x-y) - \tilde{f}(x)\right|dy \\ &= \frac{\varepsilon'}{2} + \int_{\overline{B_1(0)} \setminus B_\delta(0)} \psi_\varepsilon(y)\left|\tilde{f}(x-y) - \tilde{f}(x)\right|dy. \end{aligned}$$

Since

$$\lim_{\varepsilon \to 0^+} \int_{\overline{B_1(0)} \setminus B_\delta(0)} \psi_\varepsilon(y)dy = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\delta(0)} \psi_\varepsilon(y)dy = 0,$$

there exists $0 < \varepsilon_0 < 1$ such that for every $0 < \varepsilon < \varepsilon_0$ and $x \in \mathbb{S}^{n-1}$,

$$\left|\tilde{f}_\varepsilon(x) - f(x)\right| < \frac{\varepsilon'}{2} + 2M \int_{\overline{B_1(0)} \setminus B_\delta(0)} \psi_\varepsilon(y)dy < \varepsilon',$$

where

$$M = \max_{z \in \overline{B_2(0)}} \left|\tilde{f}(z)\right|.$$

This proves that $\tilde{f}_\varepsilon \to f$ uniformly on $\mathbb{S}^{n-1}$.

To show the a.e. convergence of the gradients, for an arbitrary $k \in \{1, \ldots, n\}$ we compute, for $x \in \mathbb{R}^n$,

$$\begin{aligned} \frac{\partial \tilde{f}_\varepsilon}{\partial x_k}(x) &= \lim_{h \to 0} \frac{\tilde{f}_\varepsilon(x + he_k) - \tilde{f}_\varepsilon(x)}{h} \\ &= \lim_{h \to 0} \int_{\mathbb{R}^n} \psi_\varepsilon(y) \frac{\tilde{f}(x - y + he_k) - \tilde{f}(x-y)}{h} dy \\ &= \int_{\mathbb{R}^n} \psi_\varepsilon(y) \frac{\partial \tilde{f}}{\partial x_k}(x-y)dy = \psi_\varepsilon * \frac{\partial \tilde{f}}{\partial x_k}(x), \end{aligned}$$

where we have used the dominated convergence theorem, which can be applied because of the Lipschitz continuity of $\tilde{f}$ and the Lebesgue integrability of ψ_ε. The Lipschitz continuity of $\tilde{f}$, together with the fact that η is compactly supported, also implies $\frac{\partial \tilde{f}}{\partial x_k} \in L^1(\mathbb{R}^n)$. Theorem 2.2.12 then guarantees that

$$\frac{\partial \tilde{f}_\varepsilon}{\partial x_k} = \psi_\varepsilon * \frac{\partial \tilde{f}}{\partial x_k} \to \frac{\partial \tilde{f}}{\partial x_k} \text{ in } L^1(\mathbb{R}^n), \text{ as } \varepsilon \to 0^+,$$

for $k = 1, \ldots, n$.

We now turn the family $\{\tilde{f}_\varepsilon\}_{\varepsilon > 0}$ into a sequence $\{\tilde{f}_i\}_{i \in \mathbb{N}}$ by choosing $\varepsilon = 1/i$, $i \in \mathbb{N}$, and renaming $\tilde{f}_i := \tilde{f}_{1/i}$ for the sake of simplicity. Every sequence of functions which converges in L^1 possesses a subsequence which converges a.e. to the same limit, hence there is a subsequence

$$\left\{\tilde{f}_{i_j^{(1)}}\right\}_{j \in \mathbb{N}} \subseteq \{\tilde{f}_i\}_{i \in \mathbb{N}}$$

such that

$$\frac{\partial \tilde{f}_{i_j^{(1)}}}{\partial x_1}(x) \to \frac{\partial \tilde{f}}{\partial x_1}(x) \text{ as } j \to \infty,$$

for a.e. $x \in \mathbb{R}^n$, and

$$\frac{\partial \tilde{f}_{i_j^{(1)}}}{\partial x_k} \to \frac{\partial \tilde{f}}{\partial x_k} \text{ in } L^1\left(\mathbb{R}^n\right) \text{ as } j \to \infty,$$

for $k = 2, \ldots, n$. We repeat the process for every $2 \leq j \leq n$, finding sequences

$$\left\{\tilde{f}_{i_j^{(n)}}\right\}_{j\in\mathbb{N}} \subseteq \left\{\tilde{f}_{i_j^{(n-1)}}\right\}_{j\in\mathbb{N}} \subseteq \ldots \subseteq \left\{\tilde{f}_{i_j^{(1)}}\right\}_{j\in\mathbb{N}} \subseteq \{\tilde{f}_i\}_{i\in\mathbb{N}}.$$

If we set $f_j := \tilde{f}_{i_j^{(n)}}$, $j \in \mathbb{N}$, we have that

$$\frac{\partial f_j}{\partial x_k}(x) \to \frac{\partial \tilde{f}}{\partial x_k}(x) \text{ as } j \to \infty,$$

for a.e. $x \in \mathbb{R}^n$ and for all $k = 1, \ldots, n$. This implies

$$\nabla f_j(x) \to \nabla f(x),$$

for a.e. $x \in \mathbb{R}^n$. In particular, using (2.2.9) we get that

$$\nabla_s f_j(x) \to \nabla_s f(x), \tag{3.3.2}$$

for a.e. $x \in \mathbb{S}^{n-1}$.

Recalling that $\{f_j\}_{j\in\mathbb{N}} \subseteq \left\{\tilde{f}_\varepsilon\right\}_{\varepsilon>0}$, from the fact that $\tilde{f}_\varepsilon \to f$ uniformly on $\mathbb{S}^{n-1}$ and the properties (3.3.1), (3.3.2) we conclude that $f_j \underset{\tau}{\to} f$. □

We have actually proved that $C^\infty(\mathbb{S}^{n-1})$ is τ-dense in $\mathrm{Lip}(\mathbb{S}^{n-1})$. However, for our purposes Lemma 3.3.1 will be sufficient.

3.4 Density of $\mathscr{L}(\mathbb{S}^{n-1})$ in $\mathrm{Lip}(\mathbb{S}^{n-1})$

We are now able to prove our density result.

Proof of Proposition 3.0.2. We have already noted that $\mathscr{L}(\mathbb{S}^{n-1}) \subseteq \mathrm{Lip}(\mathbb{S}^{n-1})$.

Let $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$ and let $U \subseteq \mathrm{Lip}(\mathbb{S}^{n-1})$ be an open neighbourhood of f (with respect to τ). Because of Lemma 3.3.1, U contains a function $g \in C^1(\mathbb{S}^{n-1})$. Moreover, since U is also an open neighbourhood of g, from Lemma 3.2.1 we have that there is a function $h \in \mathscr{L}(\mathbb{S}^{n-1})$ such that $h \in U$. □

The tools developed throughout this chapter allow us now to prove Proposition 3.0.1, which is one of the key ingredients for the proof of theorems 1.1.2 and 1.1.3.

Proof of Proposition 3.0.1. Let $V, W : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ be continuous valuations such that $V = W$ on $\mathscr{H}(\mathbb{S}^{n-1})$. Consider the continuous valuation $Z : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ given by $Z = V - W$. We prove that

$$Z\left(\bigwedge_{i=1}^{m} h_{K_i}\right) = 0$$

for every $h_{K_1}, \dots, h_{K_m} \in \mathscr{H}(\mathbb{S}^{n-1})$, proceeding by induction on m. If $m = 2$, using the fact that Z is a valuation and Lemma 2.2.18 we obtain

$$Z(h_{K_1} \wedge h_{K_2}) = Z(h_{K_1}) + Z(h_{K_2}) - Z\left(h_{\mathrm{conv}(K_1 \cup K_2)}\right) = 0,$$

since $Z = 0$ on $\mathscr{H}(\mathbb{S}^{n-1})$ by hypothesis. Suppose now that $Z = 0$ on the minima of $m-1$ support functions. Using the fact that Z is a valuation and $Z = 0$ on $\mathscr{H}(\mathbb{S}^{n-1})$, from the inductive hypothesis and Lemma 2.2.18 we get

$$\begin{aligned} Z\left(\bigwedge_{i=1}^{m} h_{K_i}\right) &= Z(h_{K_m}) + Z\left(\bigwedge_{i=1}^{m-1} h_{K_i}\right) - Z\left(h_{K_m} \vee \bigwedge_{i=1}^{m-1} h_{K_i}\right) \\ &= -Z\left(\bigwedge_{i=1}^{m-1} (h_{K_m} \vee h_{K_i})\right) = -Z\left(\bigwedge_{i=1}^{m-1} h_{\mathrm{conv}(K_m \cup K_i)}\right) \\ &= 0, \end{aligned}$$

where the last equality follows again from the inductive hypothesis. Then $Z = 0$ on $\widehat{\mathscr{H}}(\mathbb{S}^{n-1})$. From Lemma 3.1.1 we obtain that $Z = 0$ on $\mathscr{L}(\mathbb{S}^{n-1})$ too.

Let now $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$. From Proposition 3.0.2, there exists a sequence $\{f_i\} \subseteq \mathscr{L}(\mathbb{S}^{n-1})$ such that $f_i \underset{\tau}{\to} f$. Since Z is continuous,

$$Z(f) = \lim_{i \to \infty} Z(f_i) = 0,$$

hence the conclusion. □

3.5 Topology on $\mathrm{Val}\left(\mathrm{Lip}(\mathbb{S}^{n-1})\right)$

As a side note, we show that what we have proved so far can also be used to equip the space $\mathrm{Val}\left(\mathrm{Lip}(\mathbb{S}^{n-1})\right)$ of continuous and rotation invariant valuations on $\mathrm{Lip}(\mathbb{S}^{n-1})$ with a distance d, making it a metric space.

Let $Q = [-1, 1]^n$. For $N \in \mathbb{N}$, consider the partition Q_N of Q made up of n-cubes whose edges are parallel to the coordinate axes and have length $\frac{1}{N}$, as done in the proof of Lemma 3.2.1. Starting from Q_N, the same procedure used in the aforementioned lemma allows us to build a partition $\mathcal{P}_N = \{C_i\}_{i \in I}$ of $\mathbb{R}^n$ into closed convex cones with pairwise disjoint interiors.

Define now, for $N \in \mathbb{N}$,

$$\begin{aligned} \mathscr{L}_N = &\{f \in \mathscr{L}(\mathbb{S}^{n-1}) : f|_{C_i} \text{ is linear for every cone } C_i \in \mathcal{P}_N, \\ &\|f\|_\infty \leq N \text{ and } \|\nabla_s f\| \leq N \text{ a.e. in } \mathbb{S}^{n-1}\} \subseteq \mathrm{Lip}(\mathbb{S}^{n-1}), \end{aligned}$$

and let

$$[\![V]\!]_N = \sup\{|V(f)| : f \in \mathscr{L}_N\}. \tag{3.5.1}$$

We will prove that (3.5.1) is a well-posed definition, that is, $[\![V]\!]_N < \infty$ for every $V \in \mathrm{Val}\left(\mathrm{Lip}(\mathbb{S}^{n-1})\right)$. We are going to need a topological result first.

Lemma 3.5.1. *For every $N \in \mathbb{N}$, $\mathscr{L}_N$ is sequentially compact with respect to the topology τ.*

Proof. Fix $N \in \mathbb{N}$. Let $\{f_i\} \subseteq \mathscr{L}_N$. We look for a subsequence $\{f_{i_j}\} \subseteq \{f_i\}$ and a function $f \in \mathscr{L}_N$ such that $f_{i_j} \underset{\tau}{\to} f$ as $j \to \infty$.

Consider the set $X = \{v_1, \ldots, v_m\}$ of all the vertices of the $(n-1)$-simplices contained in ∂Q used to build $\mathcal{P}_N$, that is, the v_i's are the vertices of the Δ_j's given by $\Delta_j = \partial Q \cap C_j$. Note that $X \subseteq \partial Q \subseteq \overline{B_{\sqrt{n}}(0)} =: B$.

We extend 1-homogeneously the functions f_i from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$, as in (2.2.10), and we still use the symbol f_i to denote these extensions. For every $x \in B$ we have

$$|f_i(x)| = \|x\| \cdot \left| f_i\left(\frac{x}{\|x\|}\right)\right| \leq \sqrt{n}N,$$

being $\{f_i\} \subseteq \mathscr{L}_N$. This means that

$$\{f_i(x)\}_{i\in\mathbb{N}} \subseteq \left[-\sqrt{n}N, \sqrt{n}N\right], \tag{3.5.2}$$

for every fixed $x \in B$.

Choose $x = v_1$; since the interval $[-\sqrt{n}N, \sqrt{n}N]$ is compact, there is a subsequence

$$\{f_{i_j^1}(v_1)\}_{j\in\mathbb{N}} \subseteq \{f_i(v_1)\}_{i\in\mathbb{N}}$$

such that $f_{i_j^1}(v_1) \to \ell_1 \in \mathbb{R}$, as $j \to \infty$. It follows from (3.5.2) that

$$\{f_{i_j^1}(x)\}_{j\in\mathbb{N}} \subseteq \left[-\sqrt{n}N, \sqrt{n}N\right],$$

for every $x \in B$. Take now $x = v_2$. Again, there exists

$$\{f_{i_j^2}(v_2)\}_{j\in\mathbb{N}} \subseteq \{f_{i_j^1}(v_2)\}_{j\in\mathbb{N}}$$

such that $f_{i_j^2}(v_2) \to \ell_2 \in \mathbb{R}$, as $j \to \infty$. The sequence $\{f_{i_j^2}\}_{j\in\mathbb{N}}$ also satisfies $f_{i_j^2}(v_1) \to \ell_1$, as $j \to \infty$. Iterating this process, we build a sequence

$$\{f_{i_j}\}_{j\in\mathbb{N}} := \{f_{i_j^m}\}_{j\in\mathbb{N}} \subseteq \{f_i\}_{i\in\mathbb{N}}$$

such that

$$f_{i_j}(v_k) \to \ell_k$$

as $j \to \infty$, for every $k = 1, \ldots, m$.

Let $x \in \mathbb{R}^n$. Then x belongs to a cone $C_k \in \mathcal{P}_N$, hence we can write

$$x = \lambda_1 w_1 + \ldots + \lambda_n w_n,$$

for suitable $\lambda_1, \ldots, \lambda_n \geq 0$, where $\{w_1, \ldots, w_n\} \subseteq \{v_1, \ldots, v_m\}$ are the vertices of the $(n-1)$-simplex Δ_k used to build C_k. Since f_{i_j}'s restriction to C_k is a linear function, we have

$$f_{i_j}(x) = f_{i_j}(\lambda_1 w_1 + \ldots + \lambda_n w_n) = \lambda_1 f_{i_j}(w_1) + \ldots + \lambda_n f_{i_j}(w_n),$$

which converges as $j \to \infty$. Therefore, $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$f(x) = \lim_{j\to\infty} f_{i_j}(x), \; x \in \mathbb{R}^n,$$

is well-defined.

Fix a cone C_k in the partition $\mathcal{P}_N$. Since the f_{i_j}'s are linear on C_k, for every $x \in C_k$ we can write

$$f_{i_j}(x) = \langle x, z_{i_j}^k \rangle,$$

for suitable $z_{i_j}^k$'s. Note that $z_{i_j}^k = \nabla f_{i_j}(x)$, for any x in the interior of C_k. Now, since the gradient of a 1-homogeneous function is 0-homogeneous, from formula (2.2.12) and the fact that $\{f_{i_j}\} \subseteq \mathscr{L}_N$ we get

$$\|z_{i_j}^k\| = \|\nabla f_{i_j}(x)\| = \left\|\nabla f_{i_j}\left(\frac{x}{\|x\|}\right)\right\| = \sqrt{\left\|\nabla_s f_{i_j}\left(\frac{x}{\|x\|}\right)\right\|^2 + f_{i_j}\left(\frac{x}{\|x\|}\right)^2} \leq \sqrt{2}N. \qquad (3.5.3)$$

From the Bolzano-Weierstrass theorem, every subsequence $\{z_{i_{j_h}}^k\}_{h\in\mathbb{N}} \subseteq \{z_{i_j}^k\}_{j\in\mathbb{N}}$ possesses a further subsequence converging to some $z^k \in \mathbb{R}^n$. Therefore, the whole sequence $\{z_{i_j}^k\}$ converges to z^k as $j \to \infty$, and

$$\left|\langle x, z_{i_j}^k\rangle - \langle x, z^k\rangle\right| = \left|\langle x, z_{i_j}^k - z^k\rangle\right| \leq \|x\| \cdot \|z_{i_j}^k - z^k\| \leq \sqrt{n} \cdot \|z_{i_j}^k - z^k\| \to 0,$$

as $j \to \infty$, for every $x \in B$; this yields

$$f(x) = \lim_{j\to\infty} \langle x, z_{i_j}^k\rangle = \langle x, z^k\rangle,$$

for every $x \in C_k \cap B$. By 1-homogeneity of f, this actually holds for every $x \in C_k$. Because C_k was arbitrary, we have proved that, in particular, $f \in \mathscr{L}(\mathbb{S}^{n-1})$ with corresponding partition $\mathcal{P}_N$.

Since $\{f_{i_j}\} \subseteq \mathscr{L}_N$, the sequence $\{f_{i_j}\}$ is uniformly bounded. It is also equicontinuous, as we are about to prove: for every $x, y \in \mathbb{R}^n$, consider the segment $[x, y] = \{(1-t)x + ty : t \in [0, 1]\}$ connecting x and y and call $x = x_0, x_1, \ldots, x_{s-1}, x_s = y$ the points in $[x, y]$ such that the segment $[x_{k-1}, x_k]$ is wholly contained in some cone C_{h_k}, for every $k = 1, \ldots, s$. Note that $s = s(x, y)$ depends on x and y, but it is bounded from above by the number $|I|$ of cones in the partition $\mathcal{P}_N$. We estimate

$$\begin{aligned}
|f_{i_j}(x) - f_{i_j}(y)| &= \left|\sum_{k=1}^{s} \left[f_{i_j}(x_k) - f_{i_j}(x_{k-1})\right]\right| \leq \sum_{k=1}^{s} |f_{i_j}(x_k) - f_{i_j}(x_{k-1})| \\
&= \sum_{k=1}^{s} \left|\langle x_k - x_{k-1}, z_{i_j}^{h_k}\rangle\right| \leq \sum_{k=1}^{s} \|x_k - x_{k-1}\| \cdot \|z_{i_j}^{h_k}\| \\
&\leq \sqrt{2}N \cdot \sum_{k=1}^{s} \|x_k - x_{k-1}\| \leq \sqrt{2}N \cdot s(x, y) \cdot \|x - y\| \\
&\leq \sqrt{2}N|I| \cdot \|x - y\|,
\end{aligned}$$

where we have used (3.5.3). This gives the equicontinuity.

From Theorem 2.2.10, there exists a subsequence of $\{f_{i_j}\}$, which we still denote $\{f_{i_j}\}$, converging uniformly on the compact B. Since we already know that $f_{i_j} \to f$ pointwise, this yields $\|f_{i_j} - f\|_\infty \to 0$.

We also know that, for every $k \in I$,

$$\nabla f_{i_j}(x) = z_{i_j}^k \to z^k = \nabla f(x)$$

as $j \to \infty$, for a.e. $x \in C_k$. From the arbitrariness of C_k, we conclude that $\nabla f_{i_j} \to \nabla f$ a.e. in $\mathbb{R}^n$, which in turn implies $\nabla_s f_{i_j} \to \nabla_s f$ a.e. in $\mathbb{S}^{n-1}$, thanks to formula (2.2.12). We also have $\|\nabla_s f_{i_j}\| \leq N$ a.e. in $\mathbb{S}^{n-1}$, since $\{f_{i_j}\} \subseteq \mathscr{L}_N$, thus $f_{i_j} \underset{\tau}{\to} f$.

We have shown above that $f \in \mathscr{L}(\mathbb{S}^{n-1})$ with corresponding partition $\mathcal{P}_N$. Actually, $f \in \mathscr{L}_N$. Indeed, from the fact that $f_{i_j} \underset{\tau}{\to} f$ we get

$$\|f\|_\infty = \lim_{j\to\infty} \|f_{i_j}\|_\infty \leq N,$$

and moreover

$$\|\nabla_s f\| = \lim_{j\to\infty} \|\nabla_s f_{i_j}\| \le N$$

a.e. in $\mathbb{S}^{n-1}$. $\square$

To prove that $[\![\cdot]\!]_N$ is well-defined, we fix $N \in \mathbb{N}$ and for $V \in \mathrm{Val}\left(\mathrm{Lip}(\mathbb{S}^{n-1})\right)$ we consider a maximizing sequence for $[\![V]\!]_N$, i.e., a sequence $\{f_i\} \subseteq \mathscr{L}_N$ such that

$$[\![V]\!]_N = \lim_{i\to\infty} |V(f_i)|.$$

Using Lemma 3.5.1, we can find a subsequence $\{f_{i_j}\} \subseteq \{f_i\}$ and $f \in \mathscr{L}_N$ such that $f_{i_j} \underset{\tau}{\to} f$, hence $V(f_{i_j}) \to V(f)$, since V is continuous. This yields

$$[\![V]\!]_N = \lim_{j\to\infty} |V(f_{i_j})| = |V(f)| < \infty,$$

thus $[\![\cdot]\!]_N$ is well-defined.

Note that $[\![\cdot]\!]_N$ is a *seminorm* for every $N \in \mathbb{N}$, that is, it satisfies the following properties:

1. $[\![V]\!]_N \ge 0$ for every $V \in \mathrm{Val}\left(\mathrm{Lip}(\mathbb{S}^{n-1})\right)$;
2. $[\![\lambda V]\!]_N = |\lambda| \cdot [\![V]\!]_N$, for every $\lambda \in \mathbb{R}$ and $V \in \mathrm{Val}\left(\mathrm{Lip}(\mathbb{S}^{n-1})\right)$;
3. $[\![V-W]\!]_N \le [\![V-Z]\!]_N + [\![Z-W]\!]_N$, for every $V, W, Z \in \mathrm{Val}\left(\mathrm{Lip}(\mathbb{S}^{n-1})\right)$.

Using a standard procedure, we can now define a metric

$$d : \mathrm{Val}\left(\mathrm{Lip}(\mathbb{S}^{n-1})\right) \times \mathrm{Val}\left(\mathrm{Lip}(\mathbb{S}^{n-1})\right) \longrightarrow \mathbb{R}$$

by setting

$$d(V,W) = \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{[\![V-W]\!]_N}{1 + [\![V-W]\!]_N}, \tag{3.5.4}$$

for every $V, W \in \mathrm{Val}\left(\mathrm{Lip}(\mathbb{S}^{n-1})\right)$. We now prove that (3.5.4) actually defines a distance.

Clearly, if $V = W$, then $d(V,W) = 0$. Vice versa, suppose $d(V,W) = 0$. This yields $[\![V-W]\!]_N = 0$, for every $N \in \mathbb{N}$, which in turn implies

$$V(f) = W(f), \quad \forall f \in \bigcup_{N\in\mathbb{N}} \mathscr{L}_N. \tag{3.5.5}$$

By rearranging a bit the proof of Lemma 3.2.1, we can show that every function in $C^1(\mathbb{S}^{n-1})$ can be τ-approximated by a sequence in $\bigcup_{N\in\mathbb{N}} \mathscr{L}_N$. Lemma 3.3.1 then implies the τ-density of $\bigcup_{N\in\mathbb{N}} \mathscr{L}_N$ in $\mathrm{Lip}(\mathbb{S}^{n-1})$, by the same argument used in the proof of Proposition 3.0.2. Since V and W are continuous, it follows from (3.5.5) that $V = W$ on the whole space $\mathrm{Lip}(\mathbb{S}^{n-1})$.

The symmetry of d is an immediate consequence of the definition (3.5.4), and the triangular inequality follows from the triangular inequality of $[\![\cdot]\!]_N$ and from the fact that the function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by $\psi(x) = \frac{x}{1+x}$ is increasing and subadditive: in fact, for every $V, W, Z \in \mathrm{Val}\left(\mathrm{Lip}(\mathbb{S}^{n-1})\right)$ we have

$$\begin{aligned} d(V,Z) + d(Z,W) &= \sum_{N=1}^{\infty} \frac{1}{2^N} \left[\frac{[\![V-Z]\!]_N}{1+[\![V-Z]\!]_N} + \frac{[\![Z-W]\!]_N}{1+[\![Z-W]\!]_N} \right] \\ &\ge \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{[\![V-Z]\!]_N + [\![Z-W]\!]_N}{1+[\![V-Z]\!]_N + [\![Z-W]\!]_N} \\ &\ge \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{[\![V-W]\!]_N}{1+[\![V-W]\!]_N} = d(V,W). \end{aligned}$$

Chapter 4

Dot product invariant and polynomial valuations

In this chapter we will prove a McMullen-type decomposition result for continuous and dot product invariant valuations on $\mathrm{Lip}(\mathbb{S}^{n-1})$. We will also show that there are no non-trivial k-homogeneous, continuous, rotation invariant and dot product invariant valuations on $\mathrm{Lip}(\mathbb{S}^{n-1})$, for any $3 \leq k \leq n$. Thanks to these results, we will be able to prove theorems 1.1.2 and 1.1.3 in sections 4.3 and 4.4 respectively.

4.1 The homogeneous decomposition

This section is devoted to proving the following homogeneous decomposition formula for continuous and dot product invariant valuations on $\mathrm{Lip}(\mathbb{S}^{n-1})$.

Theorem 4.1.1. *Let $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ be a continuous and dot product invariant valuation. Then there exist continuous and dot product invariant valuations $Z_0, \dots, Z_n : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ such that Z_i is i-homogeneous, for $i = 0, \dots, n$, and*

$$V(\lambda f) = \sum_{i=0}^{n} \lambda^i Z_i(f),$$

for every $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$ and $\lambda > 0$.

Moreover, if V is rotation invariant, the Z_i's are rotation invariant too.

Proof. Let $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ be a continuous and dot product invariant valuation. Consider the map $\varphi : \mathcal{K}^n \longrightarrow \mathbb{R}$ defined by $\varphi(K) = V(h_K)$, for $K \in \mathcal{K}^n$; because of Lemma 2.2.20, this is a valuation on $\mathcal{K}^n$ which is translation invariant and continuous with respect to the Hausdorff metric. From Theorem 2.2.1 we obtain continuous and translation invariant valuations $\varphi_0, \dots, \varphi_n : \mathcal{K}^n \longrightarrow \mathbb{R}$ such that each φ_i is i-homogeneous and (2.2.1) holds.

Define now $Z_i : \mathscr{H}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$, $i = 0, \dots, n$, by setting $Z_i(h_K) = \varphi_i(K)$, for $h_K \in \mathscr{H}(\mathbb{S}^{n-1})$. Reading McMullen's formula (2.2.1) in the support functions' setting, we have that for every $h_K \in \mathscr{H}(\mathbb{S}^{n-1})$ and $\lambda > 0$

$$V(\lambda h_K) = V(h_{\lambda K}) = \varphi(\lambda K) = \sum_{i=0}^{n} \lambda^i \varphi_i(K) = \sum_{i=0}^{n} \lambda^i Z_i(h_K). \qquad (4.1.1)$$

This is the desired decomposition formula stated for support functions; we would now like to extend it to all Lipschitz functions $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$. To do that, we must first extend each Z_i to $\mathrm{Lip}(\mathbb{S}^{n-1})$.

We write (4.1.1) for an arbitrary $h_K \in \mathscr{H}(\mathbb{S}^{n-1})$ and for $\lambda = k = 1, \dots, n+1$:

$$V(kh_K) = \sum_{i=0}^{n} k^i Z_i(h_K). \tag{4.1.2}$$

We see it as a system of $n+1$ equations in the $n+1$ unknowns $Z_0(h_K), Z_1(h_K), \dots, Z_n(h_K)$. The matrix associated with this system is

$$M = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & n & n^2 & \cdots & n^n \\ 1 & n+1 & (n+1)^2 & \cdots & (n+1)^n \end{pmatrix},$$

which is a Vandermonde matrix, hence

$$\det M = \prod_{1 \le i < j \le n+1} (j-i) \neq 0,$$

and then M is nonsingular. Therefore, the system (4.1.2) is invertible and we can find coefficients a_{ij}, $i = 0, \dots, n$, $j = 1, \dots, n+1$, such that

$$Z_i(h_K) = \sum_{j=1}^{n+1} a_{ij} V(jh_K);$$

note that the coefficients are independent of h_K. This allows us to extend the Z_i's to $\mathrm{Lip}(\mathbb{S}^{n-1})$: for $i = 0, \dots, n$ we set

$$Z_i(f) := \sum_{j=1}^{n+1} a_{ij} V(jf), \tag{4.1.3}$$

for every $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$.

We observe that, for every $j \in \{1, \dots, n+1\}$, the function defined on $\mathrm{Lip}(\mathbb{S}^{n-1})$ by $f \mapsto V(jf)$ inherits all the properties of V, i.e., it is a continuous and dot product invariant valuation on $\mathrm{Lip}(\mathbb{S}^{n-1})$ as well, hence the Z_i's defined by (4.1.3) are continuous and dot product invariant valuations too. As for the i-homogeneity of Z_i, fix $i \in \{0, \dots, n\}$ and $\lambda > 0$. The continuous valuations $Z_i^1, Z_i^2 : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ defined by

$$Z_i^1(f) = Z_i(\lambda f), \quad Z_i^2(f) = \lambda^i Z_i(f), \quad f \in \mathrm{Lip}(\mathbb{S}^{n-1}),$$

coincide on $\mathscr{H}(\mathbb{S}^{n-1})$, hence they coincide on $\mathrm{Lip}(\mathbb{S}^{n-1})$, by Proposition 3.0.1. This proves that every Z_i is i-homogeneous.

Consider now the continuous valuation

$$\widetilde{V} = \sum_{i=0}^{n} Z_i.$$

By (4.1.1), V and $\widetilde{V}$ coincide on $\mathscr{H}(\mathbb{S}^{n-1})$; hence, by Proposition 3.0.1, $V = \widetilde{V}$ on $\mathrm{Lip}(\mathbb{S}^{n-1})$.

Finally, if we go back to (4.1.3), we deduce that the Z_i's are rotation invariant if V is. $\square$

4.2 Homogeneity and valuations on $\mathrm{Lip}(\mathbb{S}^{n-1})$

The proof of Theorem 1.1.2 requires a preliminary study of continuous, rotation invariant and dot product invariant valuations on $\mathrm{Lip}(\mathbb{S}^{n-1})$ which are k-homogeneous, for $3 \leq k \leq n$. The following result states that these are all trivial.

Proposition 4.2.1. *Let $n \geq 3$ and $3 \leq k \leq n$. Let $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ be a continuous, rotation invariant, dot product invariant and k-homogeneous valuation. Then $V \equiv 0$ on $\mathrm{Lip}(\mathbb{S}^{n-1})$.*

Remark 4.2.2. *This proposition also shows that a significant number of valuations defined on the space of support functions $\mathscr{H}(\mathbb{S}^{n-1})$, namely the intrinsic volumes $h_K \mapsto V_k(K)$ with homogeneity degree greater or equal than three, cannot be extended from $\mathscr{H}(\mathbb{S}^{n-1})$ to the wider set* $\mathrm{Lip}(\mathbb{S}^{n-1})$*. In particular, the volume functional cannot be extended to* $\mathrm{Lip}(\mathbb{S}^{n-1})$*, in dimension three or higher.*

To ease the reading, we have stated some of the steps of the proof of Proposition 4.2.1 as lemmas. Their proofs are provided along the way.

Proof. Let n, k and V be as in the hypothesis. Define $\varphi : \mathcal{K}^n \longrightarrow \mathbb{R}$ by setting

$$\varphi(K) = V(h_K),$$

for $K \in \mathcal{K}^n$. The functional φ is a k-homogeneous, translation and rotation invariant valuation which is continuous with respect to the Hausdorff metric, thanks to Lemma 2.2.20. From Theorem 2.2.2, we have that there exists a constant $c \in \mathbb{R}$ such that

$$V(h_K) = \varphi(K) = cV_k(K),$$

for every $K \in \mathcal{K}^n$.

If $c = 0$, then $V = 0$ on $\mathscr{H}(\mathbb{S}^{n-1})$, and from Proposition 3.0.1 we have the assertion.

Suppose now $c \neq 0$. We will show that this leads to a contradiction. Since the functional $\frac{1}{c}V$ retains all of V's properties, up to dividing by c we can assume that

$$V(h_K) = V_k(K), \tag{4.2.1}$$

for every $K \in \mathcal{K}^n$.

For $x \in \mathbb{R}^n$ we write $x = (\xi, \eta)$, with $\xi \in \mathbb{R}^k$ and $\eta \in \mathbb{R}^{n-k}$. Fix $\overline{\xi} \in \mathbb{S}^{k-1}$ and define $f_{\overline{\xi}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ by setting

$$f_{\overline{\xi}}(x) = f_{\overline{\xi}}(\xi, \eta) = \left\| \xi - \langle \xi, \overline{\xi} \rangle \overline{\xi} \right\|,$$

for $x \in \mathbb{R}^n$. To simplify the notation, we will be using the same symbol $\| \cdot \|$ for all the Euclidean norms throughout the proof, independently of the number of components of the vectors we are applying them to (and the same goes for the standard dot product $\langle \cdot, \cdot \rangle$). Consider the $(k-1)$-dimensional disk in $\mathbb{R}^n$ defined by

$$D_{\overline{\xi}} = \left\{ (\xi, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : \langle \xi, \overline{\xi} \rangle = 0, \|\xi\| \leq 1 \right\}.$$

The map $f_{\overline{\xi}}$ is the support function of $D_{\overline{\xi}}$. In fact, up to a change of coordinate system, we may assume $\overline{\xi} = (1, 0, \dots, 0)$; from the definition of support function, for every $(\xi, \eta) \in \mathbb{R}^n$ we have

$$h_{D_{\overline{\xi}}}(\xi, \eta) = \max_{(\xi', 0) \in D_{\overline{\xi}}} \langle \xi, \xi' \rangle = \max_{(\xi', 0) \in D_{\overline{\xi}}} \langle (\xi_2, \dots, \xi_k), (\xi'_2, \dots, \xi'_k) \rangle = \|(\xi_2, \dots, \xi_k)\| = f_{\overline{\xi}}(\xi, \eta).$$

Define now $g_{\overline{\xi}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ by setting

$$g_{\overline{\xi}}(x) = g_{\overline{\xi}}(\xi, \eta) = \langle \xi, \overline{\xi} \rangle,$$

for $x \in \mathbb{R}^n$; $g_{\overline{\xi}}$ is the support function of the singleton $\{(\overline{\xi}, 0)\}$.

For $\lambda \geq 1$, consider $\psi_{\lambda, \overline{\xi}} : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ defined by

$$\psi_{\lambda, \overline{\xi}} = (\lambda f_{\overline{\xi}} - g_{\overline{\xi}}) \wedge \mathbb{O},$$

where $\mathbb{O}$ denotes the function which is identically zero on $\mathbb{S}^{n-1}$. Note that $\psi_{\lambda, \overline{\xi}} = h_{\lambda D_{\overline{\xi}} - (\overline{\xi}, 0)} \wedge \mathbb{O} \in \mathrm{Lip}(\mathbb{S}^{n-1})$, being a minimum of Lipschitz functions. Therefore, V can be evaluated at $\psi_{\lambda, \overline{\xi}}$, and we do that in the following lemma.

Lemma 4.2.3. *We have*

$$V(\psi_{\lambda, \overline{\xi}}) = -\frac{\tilde{\omega}_{k-1}}{k} \lambda^{k-1},$$

where $\tilde{\omega}_{k-1}$ denotes the Lebesgue measure of the unit ball of $\mathbb{R}^{k-1}$.

Proof. From the valuation property we get

$$V(\psi_{\lambda, \overline{\xi}}) = V((\lambda f_{\overline{\xi}} - g_{\overline{\xi}}) \wedge \mathbb{O}) = V(\lambda f_{\overline{\xi}} - g_{\overline{\xi}}) - V((\lambda f_{\overline{\xi}} - g_{\overline{\xi}}) \vee \mathbb{O}) \tag{4.2.2}$$

since $V(\mathbb{O}) = 0$, because of the homogeneity.

As we have already pointed out, $\lambda f_{\overline{\xi}} - g_{\overline{\xi}} = h_{\lambda D_{\overline{\xi}} - (\overline{\xi}, 0)}$, and remembering (4.2.1) and the properties of the intrinsic volumes we obtain

$$V(\lambda f_{\overline{\xi}} - g_{\overline{\xi}}) = V_k(\lambda D_{\overline{\xi}} - (\overline{\xi}, 0)) = V_k(\lambda D_{\overline{\xi}}) = \lambda^k V_k(D_{\overline{\xi}}) = 0, \tag{4.2.3}$$

where the last equality follows from the fact that $D_{\overline{\xi}}$ has dimension $k-1$.

Now, $(\lambda f_{\overline{\xi}} - g_{\overline{\xi}}) \vee \mathbb{O}$ is the support function of $\mathrm{conv}\left((\lambda D_{\overline{\xi}} - (\overline{\xi}, 0)) \cup \{0\}\right)$ (see Lemma 2.2.18), which is a cone with vertex at the origin, base $\lambda D_{\overline{\xi}} - (\overline{\xi}, 0)$ and height 1, since $\|\overline{\xi}\| = 1$. From (4.2.2), (4.2.3) and (4.2.1) we get

$$V(\psi_{\lambda, \overline{\xi}}) = -V((\lambda f_{\overline{\xi}} - g_{\overline{\xi}}) \vee \mathbb{O}) = -V_k\left(\mathrm{conv}\left((\lambda D_{\overline{\xi}} - (\overline{\xi}, 0)) \cup \{0\}\right)\right) = -\frac{\tilde{\omega}_{k-1}}{k} \lambda^{k-1}.$$

□

The next lemma concerns the support set $\mathrm{supp}(\psi_{\lambda, \overline{\xi}})$ of the function $\psi_{\lambda, \overline{\xi}}$.

Lemma 4.2.4. *For every $(\xi, 0) \in \mathrm{supp}(\psi_{\lambda, \overline{\xi}})$ we have*

$$\|\xi - \overline{\xi}\| < \frac{\sqrt{2}}{\lambda}.$$

Proof. Like before, we assume $\overline{\xi} = (1, 0, \dots, 0)$. Thus, for every $(\xi, \eta) \in \mathbb{S}^{n-1}$,

$$\psi_{\lambda, \overline{\xi}}(\xi, \eta) = \left(\lambda \sqrt{\xi_2^2 + \ldots + \xi_k^2} - \xi_1\right) \wedge 0.$$

If $(\xi, 0) \in \mathrm{supp}(\psi_{\lambda, \overline{\xi}})$, we have $\|\xi\| = 1$ and $\lambda \sqrt{\xi_2^2 + \ldots + \xi_k^2} - \xi_1 \leq 0$, hence

$$\sqrt{\xi_2^2 + \ldots + \xi_k^2} \leq \frac{\xi_1}{\lambda}. \tag{4.2.4}$$

In particular, this implies $\xi_1 \geq 0$.

We write $\xi = (\xi_1, \xi')$, with $\xi' \in \mathbb{R}^{k-1}$. Since $\|\xi\| = 1$ and $\xi_1 \geq 0$, we have $\xi_1 = \sqrt{1 - \|\xi'\|^2}$. Using this last equality in (4.2.4) we obtain

$$\|\xi'\| \leq \frac{\sqrt{1 - \|\xi'\|^2}}{\lambda},$$

which in turn gives

$$\|\xi'\|^2 \leq \frac{1}{1 + \lambda^2} < \frac{1}{\lambda^2}.$$

We can also estimate

$$|\xi_1 - 1| = 1 - \xi_1 = 1 - \sqrt{1 - \|\xi'\|^2} = \frac{\|\xi'\|^2}{1 + \sqrt{1 - \|\xi'\|^2}} \leq \|\xi'\|^2 < \frac{1}{\lambda^2}.$$

From these inequalities we get

$$\left\|\xi - \overline{\xi}\right\|^2 = \|\xi - (1, 0, \ldots, 0)\|^2 = |\xi_1 - 1|^2 + \|\xi'\|^2 < \frac{1}{\lambda^4} + \frac{1}{\lambda^2} \leq \frac{2}{\lambda^2},$$

since $\lambda \geq 1$. The assertion follows. $\square$

This result yields the following one.

Lemma 4.2.5. *For every $x_1, x_2 \in \mathbb{S}^{k-1}$ such that $\|x_1 - x_2\| \geq \frac{4}{\lambda}$ we have*

$$\psi_{\lambda,\, x_1} \cdot \psi_{\lambda,\, x_2} = \mathbb{O}.$$

Proof. Take x_1, x_2 as in the hypothesis. Suppose the result to be false. Then there is a point $(\widetilde{\xi}, \widetilde{\eta}) \in \mathbb{S}^{n-1}$ such that

$$\psi_{\lambda,\, x_1}(\widetilde{\xi}, \widetilde{\eta}) \cdot \psi_{\lambda,\, x_2}(\widetilde{\xi}, \widetilde{\eta}) \neq 0.$$

Note that $\psi_{\lambda,\, x_1}(0, \widetilde{\eta}) = \psi_{\lambda,\, x_2}(0, \widetilde{\eta}) = 0$, hence $\widetilde{\xi} \neq 0$.

For $i = 1, 2$, the function

$$\psi_{\lambda,\, x_i}(\xi, \eta) = [\lambda\|\xi - \langle \xi, x_i \rangle x_i\| - \langle \xi, x_i \rangle] \wedge 0$$

is 1-homogeneous with respect to ξ, and since $\psi_{\lambda,\, x_i}(\widetilde{\xi}, \widetilde{\eta}) \neq 0$, we also have $\psi_{\lambda,\, x_i}(\widehat{\xi}, \widetilde{\eta}) \neq 0$, where $\widehat{\xi} = \widetilde{\xi} / \|\widetilde{\xi}\|$. This means that $(\widehat{\xi}, \widetilde{\eta}) \in \operatorname{supp}(\psi_{\lambda,\, x_i})$, hence $(\widehat{\xi}, 0) \in \operatorname{supp}(\psi_{\lambda,\, x_i})$ too (since $\psi_{\lambda,\, x_i}$ does not depend on η), and from the previous lemma we have

$$\left\|\widehat{\xi} - x_i\right\| < \frac{\sqrt{2}}{\lambda},$$

for $i = 1, 2$. Therefore,

$$\|x_1 - x_2\| \leq \left\|x_1 - \widehat{\xi}\right\| + \left\|\widehat{\xi} - x_2\right\| < \frac{2\sqrt{2}}{\lambda},$$

which contradicts the hypothesis. $\square$

Iterating, the previous result can be extended to any finite number of points.

Corollary 4.2.6. *Let $N \in \mathbb{N}$ and $x_1, \ldots, x_N \in \mathbb{S}^{k-1}$ be such that $\|x_i - x_j\| \geq \frac{4}{\lambda}$, for every $i \neq j$. Then*

$$\psi_{\lambda,\, x_i} \cdot \psi_{\lambda,\, x_j} = \mathbb{O},$$

for every $i \neq j$.

We will need a couple more results. The first one concerns the behaviour of a general valuation on non-positive orthogonal functions.

Lemma 4.2.7. *Let $N \in \mathbb{N}$ and $f_1, \ldots, f_N \in \mathrm{Lip}(\mathbb{S}^{n-1})$. If $f_i \leq 0$ for every $i = 1, \ldots, N$ and $f_i \cdot f_j = \mathbb{O}$ for $i \neq j$, then*

$$V\left(\bigwedge_{i=1}^{N} f_i\right) = \sum_{i=1}^{N} V(f_i).$$

Proof. The hypotheses imply that $f_{j_1} \vee \ldots \vee f_{j_m} = \mathbb{O}$, for every $m \in \{1, \ldots, N\}$ and $\{j_1, \ldots, j_m\} \subseteq \{1, \ldots, N\}$. The conclusion follows from Proposition 2.1.3. □

The next well-known lemma allows us to find sufficiently many points on the unit sphere which are not too close to each other.

Lemma 4.2.8. *Let $N \in \mathbb{N}$, $N \geq 2$. For every $m \in \mathbb{N}$ there are $N_m = m^{N-1}$ points $x_1, \ldots, x_{N_m} \in \mathbb{S}^{N-1}$ such that*

$$\|x_i - x_j\| \geq \frac{1}{\sqrt{N}m},$$

for $i \neq j$.

Proof. Fix $N \in \mathbb{N}$, $N \geq 2$, and take $m \in \mathbb{N}$. For $a = (a_1, \ldots, a_{N-1})$, with $a_1, \ldots, a_{N-1} \in \{0, 1, \ldots, m-1\}$, we define

$$x'_a = \frac{1}{\sqrt{N}} \left(\frac{a_1}{m}, \ldots, \frac{a_{N-1}}{m}\right) \in \mathbb{R}^{N-1}.$$

These are m^{N-1} points, and they satisfy

$$\|x'_a - x'_b\| \geq \frac{1}{\sqrt{N}m},$$

for every $a \neq b$. Moreover, $\|x'_a\| < 1$ for every a.

Consider now

$$x_a = (x'_a, \sqrt{1 - \|x'_a\|^2}) \in \mathbb{S}^{N-1},$$

for $a = (a_1, \ldots, a_{N-1})$ with $a_1, \ldots, a_{N-1} \in \{0, 1, \ldots, m-1\}$. These are m^{N-1} points on the sphere, and we have

$$\|x_a - x_b\| \geq \|x'_a - x'_b\| \geq \frac{1}{\sqrt{N}m},$$

for every $a \neq b$. □

We will now use these results to build a sequence of Lipschitz functions which will yield the contradiction we are looking for. Choose $N = k$ in the last lemma and take $m \in \mathbb{N}$. Then we have $k_m = m^{k-1}$ points $x_1, \ldots, x_{k_m} \in \mathbb{S}^{k-1}$ such that

$$\|x_i - x_j\| \geq \frac{1}{\sqrt{k}m},$$

for every $i \neq j$. Let
$$\lambda_m = 4\sqrt{k}m;$$
note that $\lambda_m \geq 1$. Since $\frac{2k-2}{k} \geq \frac{4}{3} > 1$, we can pick a number
$$1 < p < \frac{2k-2}{k}$$
and define the function $\Psi_m : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$,
$$\Psi_m = \frac{1}{m^p} \bigwedge_{i=1}^{k_m} \psi_{\lambda_m, x_i}.$$

From the k-homogeneity of V, Lemma 4.2.7 (which can be applied because the fact that $\|x_i - x_j\| \geq \frac{1}{\sqrt{k}m} = \frac{4}{\lambda_m}$ allows us to use Corollary 4.2.6) and Lemma 4.2.3, we get
$$V(\Psi_m) = \frac{1}{m^{kp}} V\left(\bigwedge_{i=1}^{k_m} \psi_{\lambda_m, x_i}\right) = \frac{1}{m^{kp}} \sum_{i=1}^{k_m} V(\psi_{\lambda_m, x_i}) = -\frac{1}{m^{kp}} \frac{\tilde{\omega}_{k-1}}{k} \lambda_m^{k-1} k_m = -c_k m^{2k-2-kp},$$
where
$$c_k = 4^{k-1} \tilde{\omega}_{k-1} k^{\frac{k-3}{2}} > 0.$$
Given how p was chosen, $2k - 2 - kp > 0$, hence
$$V(\Psi_m) \to -\infty \tag{4.2.5}$$
as $m \to \infty$.

We would now like to prove that $\Psi_m \underset{\tau}{\to} \mathbb{O}$, as $m \to \infty$. For every $i = 1, \ldots, k_m$ and $(\xi, \eta) \in \mathbb{S}^{n-1}$, from the triangular and Cauchy-Schwarz inequalities we have
$$\begin{aligned} |\psi_{\lambda_m, x_i}(\xi, \eta)| &= \left|[\lambda_m f_{x_i}(\xi, \eta) - g_{x_i}(\xi, \eta)] \wedge 0\right| \leq \left|\lambda_m f_{x_i}(\xi, \eta) - g_{x_i}(\xi, \eta)\right| \\ &= \left|\lambda_m \|\xi - \langle \xi, x_i \rangle x_i\| - \langle \xi, x_i \rangle\right| \leq \lambda_m \left(\|\xi\| + \|\xi\| \cdot \|x_i\|^2\right) + \\ &\quad + \|\xi\| \cdot \|x_i\| = (2\lambda_m + 1)\|\xi\| \leq 2\lambda_m + 1, \end{aligned}$$
since $x_i \in \mathbb{S}^{k-1}$. This yields $\|\psi_{\lambda_m, x_i}\|_\infty \leq 2\lambda_m + 1$, for every $i = 1, \ldots, k_m$, and consequently
$$\|\Psi_m\|_\infty \leq \frac{2\lambda_m + 1}{m^p} = \frac{8\sqrt{k}}{m^{p-1}} + \frac{1}{m^p}.$$
Since $p > 1$, this implies that $\Psi_m \to \mathbb{O}$ uniformly on $\mathbb{S}^{n-1}$ as $m \to \infty$.

We now look for a uniform bound on $L(\Psi_m)$, the Lipschitz constant of Ψ_m. For $i \in \{1, \ldots, k_m\}$, consider $\widetilde{\psi}_{\lambda_m, x_i} = \lambda_m f_{x_i} - g_{x_i}$. For $(\xi, \eta), (\xi', \eta') \in \mathbb{S}^{n-1}$,
$$\begin{aligned} \left|\widetilde{\psi}_{\lambda_m, x_i}(\xi, \eta) - \widetilde{\psi}_{\lambda_m, x_i}(\xi', \eta')\right| &\leq \lambda_m |f_{x_i}(\xi, \eta) - f_{x_i}(\xi', \eta')| + |g_{x_i}(\xi, \eta) - g_{x_i}(\xi', \eta')| \\ &= \lambda_m \left|\|\xi - \langle \xi, x_i \rangle x_i\| - \|\xi' - \langle \xi', x_i \rangle x_i\|\right| + |\langle \xi - \xi', x_i \rangle| \\ &\leq \lambda_m \|\xi - \xi' - \langle \xi - \xi', x_i \rangle x_i\| + |\langle \xi - \xi', x_i \rangle| \\ &\leq \lambda_m (\|\xi - \xi'\| + \|\xi - \xi'\| \cdot \|x_i\|^2) + \|\xi - \xi'\| \cdot \|x_i\| \\ &= (2\lambda_m + 1)\|\xi - \xi'\| \\ &\leq (2\lambda_m + 1)\|(\xi, \eta) - (\xi', \eta')\|. \end{aligned}$$

Therefore, recalling that the Lipschitz constant of a minimum of finitely many functions is at most the maximum of the Lipschitz constants, we get

$$L(\psi_{\lambda_m,\,x_i}) \leq L(\widetilde{\psi}_{\lambda_m,\,x_i}) \leq 2\lambda_m + 1$$

and

$$L(\Psi_m) = L\left(\frac{1}{m^p}\bigwedge_{i=1}^{k_m}\psi_{\lambda_m,x_i}\right) \leq \frac{1}{m^p}\max\{L(\psi_{\lambda_m,x_i}) : i=1,\dots,k_m\} \leq \frac{8\sqrt{k}m+1}{m^p} \leq \frac{8\sqrt{k}+1}{m^{p-1}}.$$

This, together with (2.1.2), implies that

$$\|\nabla_s\Psi_m(x)\| \leq \frac{8\sqrt{k}+1}{m^{p-1}},$$

for every $m \in \mathbb{N}$ and a.e. $x \in \mathbb{S}^{n-1}$. The last inequality both tells us that $\nabla_s\Psi_m \to 0$ a.e. in $\mathbb{S}^{n-1}$, as $m \to \infty$, and that $\|\nabla_s\Psi_m\|$ is uniformly bounded by

$$C = 8\sqrt{k} + 1.$$

Therefore, $\Psi_m \underset{\tau}{\to} \mathbb{O}$ as $m \to \infty$. Since V is continuous, this gives $V(\Psi_m) \to V(\mathbb{O}) = 0$, which is in contradiction with (4.2.5). This concludes the proof of Proposition 4.2.1. □

4.3 Dot product invariant valuations on $\mathrm{Lip}(\mathbb{S}^{n-1})$

We are finally ready to prove Theorem 1.1.2. In doing so, we will provide a general recipe to build continuous and rotation invariant valuations on $\mathrm{Lip}(\mathbb{S}^{n-1})$; we present this as a stand-alone result.

Lemma 4.3.1. *Let $K : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a continuous function. Then the functional $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ defined by*

$$V(f) = \int_{\mathbb{S}^{n-1}} K(f(x), \|\nabla_s f(x)\|)d\mathcal{H}^{n-1}(x),$$

for $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$, is a continuous and rotation invariant valuation.

Proof. First note that V is well-defined. Indeed, by Weierstrass' theorem and (2.1.2) we have that, for every $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$, the function $\mathbb{S}^{n-1} \ni x \mapsto (f(x), \|\nabla_s f(x)\|)$ takes values in a compact set, hence

$$\mathbb{S}^{n-1} \ni x \mapsto K(f(x), \|\nabla_s f(x)\|)$$

is bounded in absolute value by a constant $C_f > 0$ depending on f, using Weierstrass' theorem again. Therefore,

$$\int_{\mathbb{S}^{n-1}} \big|K(f(x), \|\nabla_s f(x)\|)\big| d\mathcal{H}^{n-1}(x) \leq \int_{\mathbb{S}^{n-1}} C_f d\mathcal{H}^{n-1}(x) = C_f < \infty,$$

and V is well-defined.

To prove that V is a valuation, we take $f, g \in \mathrm{Lip}(\mathbb{S}^{n-1})$ and compute

$$\begin{aligned} V(f \vee g) + V(f \wedge g) \quad &= \int_{\mathbb{S}^{n-1}} \big[K(f \vee g, \|\nabla_s(f \vee g)\|) + K(f \wedge g, \|\nabla_s(f \wedge g)\|)\big] d\mathcal{H}^{n-1} \\ &= \int_F \big[K(f, \|\nabla_s(f \vee g)\|) + K(g, \|\nabla_s(f \wedge g)\|)\big] d\mathcal{H}^{n-1} + \\ &+ \int_G \big[K(g, \|\nabla_s(f \vee g)\|) + K(f, \|\nabla_s(f \wedge g)\|)\big] d\mathcal{H}^{n-1} + \\ &+ \int_E \big[K(f \vee g, \|\nabla_s(f \vee g)\|) + K(f \wedge g, \|\nabla_s(f \wedge g)\|)\big] d\mathcal{H}^{n-1}, \end{aligned} \tag{4.3.1}$$

where

$$F = \big\{x \in \mathbb{S}^{n-1} : \ f(x) > g(x)\big\}\,, \ G = \big\{x \in \mathbb{S}^{n-1} : \ f(x) < g(x)\big\}\,, \ E = \big\{x \in \mathbb{S}^{n-1} : \ f(x) = g(x)\big\}\,.$$

Let $x \in \mathbb{S}^{n-1}$ be such that f, g, $f \vee g$ and $f \wedge g$ are differentiable at x. Then

$$\nabla_s(f \vee g)(x) = \begin{cases} \nabla_s f(x) & \text{if } x \in F, \\ \nabla_s g(x) & \text{if } x \in G, \end{cases}$$

and

$$\nabla_s(f \wedge g)(x) = \begin{cases} \nabla_s g(x) & \text{if } x \in F, \\ \nabla_s f(x) & \text{if } x \in G. \end{cases}$$

On the other hand, if $f(x) = g(x)$ it is not too hard to prove (see also [31]) that

$$\nabla_s f(x) = \nabla_s g(x) = \nabla_s(f \vee g)(x) = \nabla_s(f \wedge g)(x).$$

Hence we can split and reassemble the integrals in (4.3.1) so that

$$V(f \vee g) + V(f \wedge g) = \int_{\mathbb{S}^{n-1}} K(f, \|\nabla_s f\|) d\mathcal{H}^{n-1} + \int_{\mathbb{S}^{n-1}} K(g, \|\nabla_s g\|) d\mathcal{H}^{n-1} = V(f) + V(g).$$

We now prove that V is continuous. Let $\{f_i\} \subseteq \mathrm{Lip}(\mathbb{S}^{n-1})$ be such that $f_i \underset{\tau}{\to} f \in \mathrm{Lip}(\mathbb{S}^{n-1})$. Then $\|f_i - f\|_\infty \to 0$, hence there exists $I \in \mathbb{N}$ such that $\|f_i\|_\infty < \|f\|_\infty + 1$ for every $i > I$. Set

$$M = \max\{\|f_1\|_\infty, \dots, \|f_I\|_\infty, \|f\|_\infty + 1\}.$$

Because of the τ-convergence, there is also a $C > 0$ such that

$$(f_i(x), \|\nabla_s f_i(x)\|) \in B := [-M, M] \times [0, C],$$

for every $i \in \mathbb{N}$ and a.e. $x \in \mathbb{S}^{n-1}$. Let $D = \max_B |K|$, thus $K\left(f_i, \|\nabla_s f_i\|\right) = K|_B\left(f_i, \|\nabla_s f_i\|\right)$ is dominated by the constant function D, which is integrable on $\mathbb{S}^{n-1}$ since the sphere has finite measure. From the dominated convergence theorem we get

$$V(f) = \int_{\mathbb{S}^{n-1}} K(f, \|\nabla_s f\|) d\mathcal{H}^{n-1} = \lim_{i \to \infty} \int_{\mathbb{S}^{n-1}} K(f_i, \|\nabla_s f_i\|) d\mathcal{H}^{n-1} = \lim_{i \to \infty} V(f_i).$$

For what concerns rotational invariance, we have that for every $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$ and $\sigma \in \mathcal{O}(n)$, if f is extended 1-homogeneously to $\mathbb{R}^n$,

$$\begin{aligned} \|\nabla_s(f \circ \sigma)(x)\| &= \sqrt{\|\nabla(f \circ \sigma)(x)\|^2 - [(f \circ \sigma)(x)]^2} = \sqrt{\left\| \left(D\sigma(x)\right)^T \nabla f(\sigma(x))\right\|^2 - f(\sigma(x))^2} \\ &= \sqrt{\|\nabla f(\sigma(x))\|^2 - f(\sigma(x))^2} = \|\nabla_s f(\sigma(x))\|, \end{aligned}$$

for a.e. $x \in \mathbb{S}^{n-1}$, where we have used (2.2.12) and the fact that the matrix $(D\sigma(x))^T$, being orthogonal, preserves the norm. Therefore,

$$\begin{aligned} V(f\circ\sigma) &= \int_{\mathbb{S}^{n-1}} K(f(\sigma(x)), \|\nabla_s(f\circ\sigma)(x)\|)d\mathcal{H}^{n-1}(x) \\ &= \int_{\mathbb{S}^{n-1}} K(f(\sigma(x)), \|\nabla_s f(\sigma(x))\|)d\mathcal{H}^{n-1}(x) = V(f), \end{aligned}$$

where we have applied the change of variables $y = \sigma(x)$. □

We now turn to the actual proof of our characterization result for dot product invariant valuations.

Proof of Theorem 1.1.2. Assume the functional $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ to be defined by (1.1.1) for some constants $c_0, c_1, c_2 \in \mathbb{R}$. The kernel $K : \mathbb{R}\times\mathbb{R}^+ \longrightarrow \mathbb{R}$ given by

$$K(x, y) = c_0 + c_1 x + c_2[(n-1)x^2 - y^2],$$

for $(x, y) \in \mathbb{R}\times\mathbb{R}^+$, is a C^∞ function, hence V is a continuous and rotation invariant valuation, by the previous lemma.

It remains to be seen that V is dot product invariant. This can be proved with a direct computation, but it is easier to show it via a trick which also gives us the chance to derive a new integral representation for the intrinsic volume V_2, something that will be useful during the second part of the proof too. From (2.2.4), for every $K \in C^2_+$ we get

$$\begin{aligned} V_2(K) &= \frac{1}{2\omega_{n-2}} \int_{\mathbb{S}^{n-1}} \big[(n-1)h_K^2 + h_K \mathrm{div}_s\left(\nabla_s h_K\right)\big] d\mathcal{H}^{n-1} \\ &= \frac{1}{2\omega_{n-2}} \int_{\mathbb{S}^{n-1}} \big[(n-1)h_K^2 - \|\nabla_s h_K\|^2\big] d\mathcal{H}^{n-1}, \end{aligned} \tag{4.3.2}$$

where the last equality follows from the divergence theorem (here div_s denotes the spherical divergence). Therefore, (2.2.2) and (2.2.3) imply

$$V(h_K) = c_0V_0(K) + c_1\omega_{n-1}V_1(K) + 2c_2\omega_{n-2}V_2(K), \tag{4.3.3}$$

for all $K \in C^2_+$.

For $x \in \mathbb{R}^n$, consider the functional $V_x : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ defined by $V_x(f) = V(f + \langle\cdot, x\rangle)$, for $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$. This is still a continuous valuation on $\mathrm{Lip}(\mathbb{S}^{n-1})$ and, because of (4.3.3), it satisfies

$$\begin{aligned} V_x(h_K) &= V(h_K + \langle\cdot, x\rangle) = V(h_{K+x}) = c_0V_0(K + x) + c_1\omega_{n-1}V_1(K + x) + \\ &\quad +2c_2\omega_{n-2}V_2(K + x) = c_0V_0(K) + c_1\omega_{n-1}V_1(K) + 2c_2\omega_{n-2}V_2(K) = V(h_K), \end{aligned}$$

for every $K \in C^2_+$, since the intrinsic volumes are translation invariant.

Now, the integral in (2.2.4) just makes sense for support functions of C^2_+ bodies (since support functions of C^2_+ bodies are of class C^2), but its rewritten form (4.3.2) is well-defined for every support function $h_K \in \mathscr{H}(\mathbb{S}^{n-1})$. Since C^2_+ bodies are dense in $\mathcal{K}^n$ with respect to the Hausdorff metric, for an arbitrary $h_K \in \mathscr{H}(\mathbb{S}^{n-1})$ we can find a sequence $\{h_{K_i}\} \subseteq \mathscr{H}(\mathbb{S}^{n-1})$ with $\{K_i\} \subseteq C^2_+$ such that $\|h_{K_i} - h_K\|_\infty \to 0$, thanks to (2.1.7). Then we also have $h_{K_i} \underset{\tau}{\to} h_K$ (see the proof of Lemma 2.2.19), and since V_x and V are continuous with respect to τ we get $V_x(h_K) = V(h_K)$. From Proposition 3.0.1 it follows that they coincide on the whole space $\mathrm{Lip}(\mathbb{S}^{n-1})$, hence V is dot product invariant.

Vice versa, let $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ be a continuous, rotation invariant and dot product invariant valuation. As we previously did, let us consider $\varphi : \mathcal{K}^n \longrightarrow \mathbb{R}$ defined by

$$\varphi(K) = V(h_K),$$

for $K \in \mathcal{K}^n$, which is a translation and rotation invariant valuation that is continuous with respect to the Hausdorff metric, because of Lemma 2.2.20. From Theorem 2.2.2, there are real constants $c_0, c_1, \ldots, c_n$ such that

$$V(h_K) = \varphi(K) = \sum_{i=0}^{n} c_i V_i(K), \tag{4.3.4}$$

for every $K \in \mathcal{K}^n$.

From Theorem 4.1.1, there exist continuous, rotation invariant and dot product invariant valuations

$$Z_0, Z_1, \ldots, Z_n : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$$

such that Z_i is i-homogeneous, for $i = 0, 1, \ldots, n$, and

$$V(\lambda f) = \sum_{i=0}^{n} \lambda^i Z_i(f),$$

for every $\lambda > 0$ and $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$. Applying Proposition 4.2.1 to Z_i, for $i = 3, \ldots, n$, we get

$$V(\lambda f) = Z_0(f) + \lambda Z_1(f) + \lambda^2 Z_2(f), \tag{4.3.5}$$

for every $\lambda > 0$ and $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$.

Combining (4.3.4) and (4.3.5) we have that, for every $\lambda > 0$ and $K \in \mathcal{K}^n$,

$$Z_0(h_K) + \lambda Z_1(h_K) + \lambda^2 Z_2(h_K) = V(\lambda h_K) = V(h_{\lambda K}) = \sum_{i=0}^{n} c_i V_i(\lambda K) = \sum_{i=0}^{n} c_i \lambda^i V_i(K),$$

where the last equality follows from the i-homogeneity of the i^{th} intrinsic volume. This implies $Z_0(h_K) = c_0 V_0(K)$, $Z_1(h_K) = c_1 V_1(K)$, $Z_2(h_K) = c_2 V_2(K)$ and $c_3 = \ldots = c_n = 0$. Therefore, taking $\lambda = 1$, $f = h_K$ in (4.3.5) and remembering (2.2.2), (2.2.3), (4.3.2), we find

$$V(h_K) = c_0 + c_1 \int_{\mathbb{S}^{n-1}} h_K d\mathcal{H}^{n-1} + c_2 \int_{\mathbb{S}^{n-1}} \left[(n-1)h_K^2 - \|\nabla_s h_K\|^2\right] d\mathcal{H}^{n-1},$$

for every $K \in C^2_+$, where we have renamed $c_1 := c_1/\omega_{n-1}$, $c_2 := c_2/2\omega_{n-2}$. From the first part of the proof, the functional $\widetilde{V}$: $\mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ defined by

$$\widetilde{V}(f) = c_0 + c_1 \int_{\mathbb{S}^{n-1}} f d\mathcal{H}^{n-1} + c_2 \int_{\mathbb{S}^{n-1}} \left[(n-1)f^2 - \|\nabla_s f\|^2\right] d\mathcal{H}^{n-1},$$

for $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$, is a continuous valuation like V, and they coincide on the set of support functions of C^2_+ bodies, hence on $\mathscr{H}(\mathbb{S}^{n-1})$, by density. We conclude from Proposition 3.0.1. $\square$

4.4 Polynomial valuations on $\mathrm{Lip}(\mathbb{S}^1)$

This section is devoted to the proof of Theorem 1.1.3.

Proof of Theorem 1.1.3. Let V be defined by (1.1.2). The fact that V is a continuous and rotation invariant valuation follows from Lemma 4.3.1 (with kernel $K(x,y) = p(x,y^2)$). As for the polynomiality, by (2.2.12) and (2.2.7) we have that

$$V(h_K) = \int_{\mathbb{S}^1} p\left(h_K(x), \|\nabla h_K(x)\|^2 - h_K(x)^2\right) d\mathcal{H}^1(x) = \int_{\mathbb{S}^1} q\left(h_K(x), \|\nabla h_K(x)\|^2\right) dS_0(K;x),$$

for a suitable polynomial q and for all $K \in \mathcal{K}^2$. Now, if $K \in \mathcal{K}^2$ is strictly convex, from (2.2.11), Proposition 2.2.14 and Lemma 2.2.6 we get

$$\begin{aligned} V(h_K) &= \int_{\mathbb{S}^1} q\left(\langle x, \nabla h_K(x)\rangle, \|\nabla h_K(x)\|^2\right) dS_0(K;x) \\ &= \int_{\mathbb{S}^1} q\left(\langle x, s_K(x)\rangle, \|s_K(x)\|^2\right) dS_0(K;x) \\ &= \int_{\mathbb{R}^2\times\mathbb{S}^1} q\left(\langle x, s\rangle, \|s\|^2\right) d\Theta_0(K;s,x) \\ &= \int_{\mathbb{R}^2\times\mathbb{S}^1} p_0\left(\|s\|^2, \langle s, x\rangle\right) d\Theta_0(K;s,x), \end{aligned}$$

for a polynomial p_0. By density of strictly convex bodies in $\mathcal{K}^2$, continuity of V with respect to τ and Lemma 2.2.19, this extends to every $K \in \mathcal{K}^2$. The set function $\varphi : \mathcal{K}^2 \longrightarrow \mathbb{R}$ defined by $\varphi(K) = V(h_K)$, for $K \in \mathcal{K}^2$, is a rotation invariant and polynomial valuation which is continuous with respect to the Hausdorff metric, by Theorem 2.2.4. Therefore,

$$V(h_K + \langle\cdot, x\rangle) = V(h_{K+x}) = \varphi(K+x) = p_K(x)$$

is a polynomial in $x \in \mathbb{R}^2$, for every $K \in \mathcal{K}^2$. This proves that V is polynomial on $\mathscr{H}(\mathbb{S}^1)$, but since, for every fixed $x \in \mathbb{R}^2$, the functional $V_x : \mathrm{Lip}(\mathbb{S}^1) \longrightarrow \mathbb{R}$ defined by $V_x(f) = V(f + \langle\cdot, x\rangle)$ is still a continuous valuation, we conclude from Proposition 3.0.1 that $V = V_x$ on the whole space $\mathrm{Lip}(\mathbb{S}^1)$, hence proving that V is polynomial.

Vice versa, let $V : \mathrm{Lip}(\mathbb{S}^1) \longrightarrow \mathbb{R}$ be a continuous, rotation invariant and polynomial valuation. Define $\varphi : \mathcal{K}^2 \longrightarrow \mathbb{R}$ by setting

$$\varphi(K) = V(h_K),$$

for every $K \in \mathcal{K}^2$. Using Lemma 2.2.20 again, the map φ inherits all of V's properties, i.e., φ is a rotation invariant and polynomial valuation which is continuous (with respect to the Hausdorff metric). We can then apply Theorem 2.2.4 to φ to get two polynomials p_0, p_1 in two variables such that (2.2.5) holds for every $K \in \mathcal{K}^2$.

Let us consider $K \in \mathcal{K}^2$ which is C^2_+. Then K is also strictly convex, and from Lemma 2.2.6 and Proposition 2.2.14 we obtain

$$\begin{aligned} V(h_K) = \varphi(K) &= \sum_{i=0}^{1} \int_{\mathbb{S}^1} p_i\left(\|s_K(x)\|^2, \langle s_K(x), x\rangle\right) dS_i(K;x) = \\ &= \sum_{i=0}^{1} \int_{\mathbb{S}^1} p_i\left(\|\nabla h_K(x)\|^2, \langle \nabla h_K(x), x\rangle\right) dS_i(K;x) = \\ &= \sum_{i=0}^{1} \int_{\mathbb{S}^1} p_i\left(\|\nabla h_K(x)\|^2, h_K(x)\right) dS_i(K;x), \end{aligned} \tag{4.4.1}$$

where the last equality follows from the 1-homogeneity of h_K and Euler's formula (2.2.11).

It is now convenient to move the integrals from $\mathbb{S}^1$ to the interval $(0, 2\pi]$; to do that, consider the map $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$h(t) = h_K(\cos t, \sin t),$$

for every $t \in \mathbb{R}$. Note that h is a 2π-periodic function.

Let us determine the connection between ∇h_K and h'. For a.e. $t \in (0, 2\pi]$,

$$(x(t), y(t)) := \nabla h_K(\cos t, \sin t)$$

is the only point of ∂K such that the outer normal vector to ∂K at $(x(t), y(t))$ is $(\cos t, \sin t)$, because of Proposition 2.2.14. Applying once again Euler's formula (2.2.11), we get

$$h(t) = \langle (x(t), y(t)), (\cos t, \sin t) \rangle = x(t) \cos t + y(t) \sin t, \tag{4.4.2}$$

so that

$$h'(t) = x'(t) \cos t - x(t) \sin t + y'(t) \sin t + y(t) \cos t.$$

Since the tangent vector $(x'(t), y'(t))$ and the normal vector $(\cos t, \sin t)$ to K at t are orthogonal, this implies

$$h'(t) = -x(t) \sin t + y(t) \cos t. \tag{4.4.3}$$

Relations (4.4.2) and (4.4.3) yield the following equalities:

$$\begin{cases} h(t) \cos t = x(t) \cos^2 t + y(t) \sin t \cos t, \\ -h'(t) \sin t = x(t) \sin^2 t - y(t) \sin t \cos t. \end{cases}$$

Adding the two equations, we get

$$x(t) = h(t) \cos t - h'(t) \sin t. \tag{4.4.4}$$

Similarly, we have

$$y(t) = h(t) \sin t + h'(t) \cos t. \tag{4.4.5}$$

From (4.4.4) and (4.4.5) we finally obtain

$$\|\nabla h_K(\cos t, \sin t)\|^2 = x(t)^2 + y(t)^2 = h(t)^2 + h'(t)^2. \tag{4.4.6}$$

Moreover, remembering the bidimensional formula for the Laplacian expressed in polar coordinates and using (4.4.4), (4.4.5) we find

$$\Delta h_K(\cos t, \sin t) = h''(t), \tag{4.4.7}$$

for a.e. $t \in (0, 2\pi]$.

Using the definition of line integral and formulas (4.4.6), (2.2.7), (2.2.8), (4.4.7) in (4.4.1), we get

$$V(h_K) = \int_0^{2\pi} p_0(h(t)^2 + h'(t)^2, h(t)) d\mathcal{H}^1(t) + \int_0^{2\pi} p_1(h(t)^2 + h'(t)^2, h(t)) h''(t) d\mathcal{H}^1(t),$$

up to moltiplicative constants which can be incorporated into the polynomials. Then there exists a polynomial q_0 such that

$$V(h_K) = \int_0^{2\pi} q_0(h(t), h'(t)^2) d\mathcal{H}^1(t) + \int_0^{2\pi} p_1(h(t)^2 + h'(t)^2, h(t)) h''(t) d\mathcal{H}^1(t). \tag{4.4.8}$$

If we write

$$p_1(x,y) = \sum_{\substack{i,j=0,\dots,m,\\ i+j\le m}} a_{ij}x^i y^j,$$

with $a_{ij} \in \mathbb{R}$ for $i,j = 0,\dots,m$, the second integral becomes

$$\begin{aligned}\int_0^{2\pi} p_1(h(t)^2+h'(t)^2, h(t))h''(t)d\mathcal{H}^1(t) &= \sum_{\substack{i,j=0,\dots,m,\\ i+j\le m}} a_{ij}\int_0^{2\pi}\left[h(t)^2+h'(t)^2\right]^i h(t)^j h''(t)d\mathcal{H}^1(t)\\ &= \sum_{\substack{i,j=0,\dots,m,\\ i+j\le m}}\sum_{k=0}^{i}\binom{i}{k} a_{ij}H_{ijk}, \qquad (4.4.9)\end{aligned}$$

where we have set

$$H_{ijk} := \int_0^{2\pi} h(t)^{2i-2k+j}h'(t)^{2k}h''(t)d\mathcal{H}^1(t).$$

For fixed $i,j \in \{0,\dots,m\}$ such that $i+j\le m$ and $k\in\{0,\dots,i\}$, we focus on the integral H_{ijk}. Since

$$\left[\frac{1}{2k+1}(h')^{2k+1}\right]'(t) = h'(t)^{2k}h''(t),$$

we can rewrite H_{ijk} as

$$H_{ijk} = \int_0^{2\pi} h(t)^{2i-2k+j}\left[\frac{1}{2k+1}h'(t)^{2k+1}\right]' d\mathcal{H}^1(t).$$

Note that $2i-2k+j\ge 0$, being $k\le i$ and $j\ge 0$.

If $2i-2k+j=0$, then

$$H_{ijk} = \int_0^{2\pi}\left[\frac{1}{2k+1}(h')^{2k+1}\right]'(t)d\mathcal{H}^1(t) = \frac{(h'(2\pi))^{2k+1}-(h'(0))^{2k+1}}{2k+1} = 0,$$

since h, hence h', is 2π-periodic.

If $2i-2k+j>0$, integrating by parts we obtain

$$\begin{aligned}H_{ijk} &= \left[\frac{1}{2k+1}h'(t)^{2k+1}h(t)^{2i-2k+j}\right]_0^{2\pi} - \frac{2i-2k+j}{2k+1}\int_0^{2\pi} h(t)^{2i-2k+j-1}h'(t)^{2(k+1)}d\mathcal{H}^1(t)\\ &= -\frac{2i-2k+j}{2k+1}\int_0^{2\pi} h(t)^{2i-2k+j-1}h'(t)^{2(k+1)}d\mathcal{H}^1(t).\end{aligned}$$

Therefore, from (4.4.9) we have that there exists a polynomial q_1 such that

$$\int_0^{2\pi} p_1(h(t)^2+h'(t)^2, h(t))h''(t)d\mathcal{H}^1(t) = \int_0^{2\pi} q_1(h(t),h'(t)^2)d\mathcal{H}^1(t),$$

hence (4.4.8) implies

$$V(h_K) = \int_0^{2\pi} p(h(t),h'(t)^2)d\mathcal{H}^1(t),$$

where $p = q_0 + q_1$.

Recalling (4.4.6) and (2.2.12), for a.e. $t \in (0, 2\pi]$ we have

$$\begin{aligned} h(t)^2 + h'(t)^2 &= \|\nabla h_K(\cos t, \sin t)\|^2 = \|\nabla_s h_K(\cos t, \sin t)\|^2 + \\ &+ [h_K(\cos t, \sin t)]^2 = \|\nabla_s h_K(\cos t, \sin t)\|^2 + h(t)^2, \end{aligned}$$

so that

$$h'(t)^2 = \|\nabla_s h_K(\cos t, \sin t)\|^2.$$

This gives

$$V(h_K) = \int_{\mathbb{S}^1} p(h_K(x), \|\nabla_s h_K(x)\|^2) d\mathcal{H}^1(x), \tag{4.4.10}$$

which holds for every C^2_+ body K.

Recalling that C^2_+ bodies are dense in $\mathcal{K}^2$, and reasoning as in the proof of Theorem 1.1.2, we extend representation formula (4.4.10) to the space $\mathrm{Lip}(\mathbb{S}^1)$. □

In higher dimension we would have Theorem 2.2.5 to rely upon: the main problem in extending Theorem 1.1.3 to the case $n \geq 3$ is that we are missing a tool, like the combination of Theorem 4.1.1 and Proposition 4.2.1 was, for polynomial valuations.

4.5 An improved homogeneous decomposition

To conclude this chapter, we show that, using Proposition 4.2.1, we can now refine Theorem 4.1.1 as follows.

Theorem 4.5.1. *Let $n \geq 3$ and $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ be a continuous and dot product invariant valuation. Then there exist continuous and dot product invariant valuations $Z_0, \dots, Z_{n-1} : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ such that Z_i is i-homogeneous, for $i = 0, \dots, n-1$, and*

$$V(\lambda f) = \sum_{i=0}^{n-1} \lambda^i Z_i(f),$$

for every $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$ and $\lambda > 0$.

Proof. We use the notations introduced in the proof of Theorem 4.1.1; by the latter result, we only need to prove that $Z_n \equiv 0$. By Theorem 2.2.3, there exists $c \in \mathbb{R}$ such that $\varphi_n(K) = cV_n(K)$, for every $K \in \mathcal{K}^n$. In particular, φ_n is rotation invariant, hence Z_n is rotation invariant on $\mathscr{H}(\mathbb{S}^{n-1})$.

Let us prove that Z_n is rotation invariant on the whole space $\mathrm{Lip}(\mathbb{S}^{n-1})$. For a fixed $\sigma \in \mathcal{O}(n)$, consider $Z_n^\sigma : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ defined by

$$Z_n^\sigma(f) = Z_n(f \circ \sigma) - Z_n(f),$$

for $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$. Such functional is a continuous valuation on $\mathrm{Lip}(\mathbb{S}^{n-1})$; because Z_n is rotation invariant on $\mathscr{H}(\mathbb{S}^{n-1})$, $Z_n^\sigma = 0$ on $\mathscr{H}(\mathbb{S}^{n-1})$. From Proposition 3.0.1, $Z_n^\sigma = 0$ on $\mathrm{Lip}(\mathbb{S}^{n-1})$, so that $Z_n(f \circ \sigma) = Z_n(f)$, for every $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$ and $\sigma \in \mathcal{O}(n)$. Therefore, Z_n is a continuous, rotation invariant, dot product invariant and n-homogeneous valuation on $\mathrm{Lip}(\mathbb{S}^{n-1})$, hence $Z_n \equiv 0$, thanks to Proposition 4.2.1. □

We have avoided on purpose to rewrite the last sentence of Theorem 4.1.1 in the statement of Theorem 4.5.1, since the case in which V is rotation invariant is already described in more detail by Theorem 1.1.2.

Remark 4.5.2. *It would be interesting to know if also* $Z_3 = \ldots = Z_{n-1} \equiv 0$ *for an arbitrary continuous and dot product invariant valuation* V*. We know from Theorem 1.1.2 that this is the case under the additional assumption of rotational invariance.*

Chapter 5

General continuous and rotation invariant valuations on $\mathrm{Lip}(\mathbb{S}^1)$

This chapter is devoted to the proof of theorems 1.1.4 and 1.1.5. The first results we will obtain work in arbitrary dimension n, hence we will present them in this more general context, restricting to the case $n = 2$ when needed. Many of the ideas for this part come from [35].

5.1 Boundedness on $\|\cdot\|_{\mathrm{Lip}}$-bounded sets

Consider on $\mathrm{Lip}(\mathbb{S}^{n-1})$ the *Lipschitz norm*, defined by (2.1.4). We will start by proving that valuations which are continuous (with respect to τ) are bounded on $\|\cdot\|_{\mathrm{Lip}}$-bounded sets.

Note: the notation $\|\cdot\|_{\mathrm{Lip}}$ will be useful for the statement of the next result, but we will keep using the topology τ on $\mathrm{Lip}(\mathbb{S}^{n-1})$, and not the topology induced by this norm.

Lemma 5.1.1. *Let $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ be a continuous valuation (with respect to τ) and let $A \subseteq \mathrm{Lip}(\mathbb{S}^{n-1})$, $L > 0$ be such that $\|f\|_{\mathrm{Lip}} \leq L$ for every $f \in A$. Then there exists $C > 0$ such that*

$$|V(f)| \leq C,$$

for every $f \in A$.

Proof. We reason by contradiction: if this were not true, there would exist $L > 0$ and a sequence $\{f_i\} \subseteq \mathrm{Lip}(\mathbb{S}^{n-1})$ with $\|f_i\|_{\mathrm{Lip}} \leq L$, for every $i \in \mathbb{N}$, such that $|V(f_i)| \to \infty$ as $i \to \infty$.

Consider the function $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\theta(c) = V(c\mathbb{1}),$$

for $c \in \mathbb{R}$, where $\mathbb{1}$ denotes the constant function equal to one; θ is continuous because V is. Therefore, θ is uniformly continuous on $[-L, L]$ and thus bounded, that is, there exists $C > 0$ such that, for every $c \in [-L, L]$,

$$|V(c\mathbb{1})| = |\theta(c)| \leq C.$$

We define inductively two sequences $\{a_j\}$, $\{b_j\} \subseteq \mathbb{R}$. Set $a_0 = -L$, $b_0 = L$ and let $c_0 = \frac{a_0+b_0}{2}$. Note that

$$V(f_i \vee c_0\mathbb{1}) + V(f_i \wedge c_0\mathbb{1}) = V(f_i) + V(c_0\mathbb{1}).$$

Since $|V(c_0\mathbb{1})| \le C$ and $|V(f_i)| \to \infty$, there exists an infinite set $M_1 \subseteq \mathbb{N}$ such that for $i \in M_1$ either $|V(f_i \vee c_0\mathbb{1})| \to \infty$ or $|V(f_i \wedge c_0\mathbb{1})| \to \infty$, as $i \to \infty$. If we are in the first case, we set $a_1 = c_0$, $b_1 = L$ and $f_i^1 = f_i \vee c_0\mathbb{1}$. If the second case occurs, we define instead $a_1 = -L$, $b_1 = c_0$ and $f_i^1 = f_i \wedge c_0\mathbb{1}$. Note that in either case we have $\|f_i^1\|_{\mathrm{Lip}} \le L$, for every $i \in M_1$. We now define $c_1 = \frac{a_1+b_1}{2}$ and proceed similarly.

Inductively, we construct two sequences $\{a_j\}$, $\{b_j\} \subseteq \mathbb{R}$, a decreasing sequence $\{M_j\}$ of infinite subsets of $\mathbb{N}$, and sequences $\{f_i^j\}_{i\in M_j} \subseteq \mathrm{Lip}(\mathbb{S}^{n-1})$ such that, for every $j \in \mathbb{N}$,

$$|a_j - b_j| = \frac{L}{2^{j-1}},$$

$$\lim_{i\to\infty} |V(f_i^j)| = \infty,$$

and for every $i \in M_j$, $t \in \mathbb{S}^{n-1}$,

$$a_j \le f_i^j(t) \le b_j.$$

Up to passing to a further subsequence, we may assume that

$$\lim_{i\to\infty} |V(f_i^i)| = \infty.$$

Let $\lambda = \lim_{j\to\infty} a_j$ (the limit exists because $\{a_j\}$ is bounded and monotone) and $g_i = f_i^i$, for $i \in \mathbb{N}$. The sequence $\{g_i\} \subseteq \mathrm{Lip}(\mathbb{S}^{n-1})$ satisfies $a_i \le g_i \le b_i$, $\|g_i\|_{\mathrm{Lip}} \le L$, $\|g_i - \lambda\mathbb{1}\|_\infty \to 0$, and $\lim_{i\to\infty} |V(g_i)| = \infty$.

We now need a second inductive step to obtain the a.e. convergence of the gradients. We define a new double indexed sequence $\{m_i^j\}_{i,j\in\mathbb{N}}$. For the first step, consider the number $m_i^1 := m(g_i)$, for every $i \in \mathbb{N}$, where $m(g_i)$ is a median of g_i, that is, $m(g_i)$ is a number in $[-L, L]$ which satisfies

$$\begin{aligned} \mathcal{H}^{n-1}\left(\{x \in \mathbb{S}^{n-1} : g_i(x) \ge m(g_i)\}\right) &\ge \frac{1}{2}\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = \frac{1}{2}, \\ \mathcal{H}^{n-1}\left(\{x \in \mathbb{S}^{n-1} : g_i(x) \le m(g_i)\}\right) &\ge \frac{1}{2}\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = \frac{1}{2}. \end{aligned} \tag{5.1.1}$$

A median always exists.

The valuation property implies that

$$V(g_i \vee m_i^1\mathbb{1}) + V(g_i \wedge m_i^1\mathbb{1}) = V(g_i) + V(m_i^1\mathbb{1}).$$

Since $|V(m_i^1\mathbb{1})| \le C$ and $|V(g_i)| \to \infty$, there has to be an infinite set, which with a little abuse of notation will still be denoted by $M_1 \subseteq \mathbb{N}$, such that for $i \in M_1$ either $|V(g_i \vee m_i^1\mathbb{1})| \to \infty$ or $|V(g_i \wedge m_i^1\mathbb{1})| \to \infty$, as $i \to \infty$. In the first case, we set $g_i^1 = g_i \vee m_i^1\mathbb{1}$, whereas in the second case we define $g_i^1 = g_i \wedge m_i^1\mathbb{1}$. Either way we get that $\|g_i^1\|_{\mathrm{Lip}} \le L$, for every $i \in M_1$.

Since $\mathcal{H}^{n-1}((g_i^1)^{-1}(\{m_i^1\})) \ge \frac{1}{2}$ and $\nabla_S g_i^1(x) = 0$ for a.e. $x \in (g_i^1)^{-1}(\{m_i^1\})$, because of Lemma 2.2.16, we have that $\nabla_S g_i^1 = 0$ in a set of measure larger than or equal to $\frac{1}{2}$.

For every $i \in M_1$, consider the set

$$A_i^1 = \{x \in \mathbb{S}^{n-1} : g_i^1(x) \ne m_i^1\}.$$

Note that $\mathcal{H}^{n-1}(A_i^1) \le \frac{1}{2}$. Indeed, since $g_i^1 = g_i \vee m_i^1$ or $g_i^1 = g_i \wedge m_i^1$, we have that $g_i^1 = m_i^1$ if and only if $g_i \le m_i^1$ or $g_i \ge m_i^1$ respectively. Then, from (5.1.1) we get

$$\mathcal{H}^{n-1}(A_i^1) = 1 - \mathcal{H}^{n-1}(\{g_i^1 = m_i^1\}) \le 1 - \frac{1}{2} = \frac{1}{2}.$$

Now, for every $i \in M_1$ consider the median m_i^2 in A_i^1, i.e., the number verifying

$$\mathcal{H}^{n-1}\left(\{x \in A_i^1 : g_i^1(x) \geq m_i^2\}\right) \geq \frac{1}{2}\mathcal{H}^{n-1}(A_i^1),$$

$$\mathcal{H}^{n-1}\left(\{x \in A_i^1 : g_i^1(x) \leq m_i^2\}\right) \geq \frac{1}{2}\mathcal{H}^{n-1}(A_i^1).$$

Again, this median surely exists.

We proceed as before, noting that the valuation property implies

$$V(g_i^1 \vee m_i^2\mathbb{1}) + V(g_i^1 \wedge m_i^2\mathbb{1}) = V(g_i^1) + V(m_i^2\mathbb{1}).$$

Since $|V(m_i^2\mathbb{1})| \leq C$ and $|V(g_i^1)| \to \infty$, there is an infinite set $M_2 \subseteq M_1$ such that for $i \in M_2$ either $|V(g_i^1 \vee m_i^2\mathbb{1})| \to \infty$ or $|V(g_i^1 \wedge m_i^2\mathbb{1})| \to \infty$, as $i \in M_2$ goes to ∞. In the first case, we set $g_i^2 = g_i^1 \vee m_i^2\mathbb{1}$, and in the second case we set $g_i^2 = g_i^1 \wedge m_i^2\mathbb{1}$. Either way, $\|g_i^2\|_{\mathrm{Lip}} \leq L$ for every $i \in M_2$. Assume $g_i^2 = g_i^1 \vee m_i^2\mathbb{1}$ (the other case is analogous).

If $m_i^1 > m_i^2$,

$$\begin{aligned}
\mathcal{H}^{n-1}((g_i^2)^{-1}(\{m_i^1, m_i^2\})) &= \mathcal{H}^{n-1}(\{g_i^1 \vee m_i^2\mathbb{1} = m_i^1\}) + \mathcal{H}^{n-1}(\{g_i^1 \leq m_i^2\}) \\
&= \mathcal{H}^{n-1}\left((A_i^1)^c\right) + \mathcal{H}^{n-1}(A_i^1 \cap \{g_i^1 \leq m_i^2\}) \\
&\geq \mathcal{H}^{n-1}\left((A_i^1)^c\right) + \frac{1}{2}\mathcal{H}^{n-1}(A_i^1) \\
&= 1 - \frac{1}{2}\mathcal{H}^{n-1}(A_i^1) \geq \frac{3}{4}.
\end{aligned}$$

If $m_i^1 < m_i^2$ instead,

$$\mathcal{H}^{n-1}(\{g_i^2 = m_i^1\}) = \mathcal{H}^{n-1}(\{g_i^1 \vee m_i^2\mathbb{1} = m_i^1\}) = \mathcal{H}^{n-1}(\emptyset) = 0,$$

and then

$$\begin{aligned}
\mathcal{H}^{n-1}((g_i^2)^{-1}(\{m_i^1, m_i^2\})) &= \mathcal{H}^{n-1}(\{g_i^2 = m_i^2\}) = \mathcal{H}^{n-1}(\{g_i^1 \leq m_i^2\}) \\
&= \mathcal{H}^{n-1}\left((A_i^1)^c\right) + \mathcal{H}^{n-1}(A_i^1 \cap \{g_i^1 \leq m_i^2\}) \\
&\geq \mathcal{H}^{n-1}\left((A_i^1)^c\right) + \frac{1}{2}\mathcal{H}^{n-1}(A_i^1) \\
&= 1 - \frac{1}{2}\mathcal{H}^{n-1}(A_i^1) \geq \frac{3}{4}.
\end{aligned}$$

Finally, if $m_i^1 = m_i^2$ we have

$$\mathcal{H}^{n-1}((g_i^2)^{-1}(\{m_i^1, m_i^2\})) = \mathcal{H}^{n-1}((g_i^2)^{-1}(\{m_i^2\})) = \mathcal{H}^{n-1}(\{g_i^1 \leq m_i^2\}),$$

and we conclude as in the case $m_i^1 < m_i^2$. Whatever happens, we get

$$\mathcal{H}^{n-1}((g_i^2)^{-1}(\{m_i^1, m_i^2\})) \geq \frac{3}{4}.$$

It follows again from Lemma 2.2.16 that $\nabla_s g_i^2(x) = 0$ for a.e. $x \in (g_i^2)^{-1}(\{m_i^1, m_i^2\})$. Since $\mathcal{H}^{n-1}((g_i^2)^{-1}(\{m_i^1, m_i^2\}) \geq \frac{3}{4}$, we have that $\nabla_s g_i^2 = 0$ in a set of measure larger than or equal to $\frac{3}{4}$.

By induction, we obtain a decreasing sequence of infinite subsets $M_j \subseteq \mathbb{N}$ and sequences $\{g_i^j\}_{i\in M_j} \subseteq \mathrm{Lip}(\mathbb{S}^{n-1})$, $j \in \mathbb{N}$, such that, for every $j \in \mathbb{N}$ and $i \in M_j$, $a_i \leq g_i^j \leq b_i$, $\|\nabla_s g_i^j\| \leq L$ a.e., $\lim_{i\to\infty} |V(g_i^j)| = \infty$ and

$$\mathcal{H}^{n-1}\left(\{x \in \mathbb{S}^{n-1} :\ \nabla_s g_i^j(x) \neq 0\}\right) < \frac{1}{2^j}.$$

Passing to a further subsequence if needed, we may assume that the sequence $\{g_i^i\}$ verifies $\lim_{i\to\infty} \|g_i^i - \lambda \mathbb{1}\|_\infty = 0$, $\|\nabla_s g_i^i\| \leq L$ a.e., $\lim_{i\to\infty} |V(g_i^i)| = \infty$ and

$$\mathcal{H}^{n-1}\left(\{x \in \mathbb{S}^{n-1} :\ \nabla_s g_i^i(x) \neq 0\}\right) \leq \frac{1}{2^i}.$$

Therefore, $g_i^i \underset{\tau}{\to} \lambda\mathbb{1}$, but $|V(g_i^i)| \to \infty$, a contradiction with the continuity of the valuation. $\square$

5.2 Rims around sets

We need to introduce a couple more notations. For a given valuation $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ and a number $\lambda \in \mathbb{R}$, we consider the functional $V_\lambda : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ defined by

$$V_\lambda(f) = V(f + \lambda) - V(\lambda),$$

for $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$, which is still a valuation. Moreover, V_λ is continuous and/or rotation invariant if V is continuous and/or rotation invariant respectively.

For a set $A \subseteq \mathbb{S}^{n-1}$ and $\omega > 0$, the *outer parallel band* or *rim* around A is the set

$$A^\omega = \{x \in \mathbb{S}^{n-1} : 0 < d(x, A) < \omega\},$$

with the convention that $\emptyset^\omega = \emptyset$.

The next lemma allows us to control continuous valuations on rims.

Lemma 5.2.1. *Let $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ be a continuous valuation. Take two Borel sets A, $B \subseteq \mathbb{S}^{n-1}$ and let $\lambda \in \mathbb{R}$, $\gamma \in \mathbb{R}^+$. Then*

$$\lim_{\omega\to 0^+} \sup\{|V_\lambda(f)| : f \prec A^\omega \cup B^\omega,\ L(f) \leq \gamma\} = 0. \tag{5.2.1}$$

In particular,

$$\lim_{\omega\to 0^+} \sup\{|V_\lambda(f)| : f \prec A^\omega,\ L(f) \leq \gamma\} = 0$$

for every Borel set $A \subseteq \mathbb{S}^{n-1}$.

To prove this, we are going to need this technical result first.

Lemma 5.2.2. *Let $A \subseteq \mathbb{S}^{n-1}$ be a Borel set. Then*

$$\lim_{\omega\to 0^+} \mathcal{H}^{n-1}(A^\omega) = 0. \tag{5.2.2}$$

Proof. If $A = \emptyset$, then $\mathcal{H}^{n-1}(A^\omega) = 0$ for every $\omega > 0$, and we are done. Suppose now $A \neq \emptyset$.

If (5.2.2) were not true, we would have a number $\varepsilon > 0$ and a sequence $\omega_i \searrow 0$ such that $\mathcal{H}^{n-1}(A^{\omega_i}) > \varepsilon$ for every $i \in \mathbb{N}$.

If $x \in \bigcap_{i\in\mathbb{N}} A^{\omega_i}$, then

$$0 < d(x, A) < \omega_i,$$

for every $i \in \mathbb{N}$; passing to the limit in the second inequality we have a contradiction. Therefore $\bigcap_{i\in\mathbb{N}} A^{\omega_i} = \emptyset$, hence

$$0 = \mathcal{H}^{n-1}\left(\bigcap_{i\in\mathbb{N}} A^{\omega_i}\right) = \lim_{i\to\infty} \mathcal{H}^{n-1}(A^{\omega_i}) > \varepsilon,$$

which is false. □

Proof of Lemma 5.2.1. If $A = B = \emptyset$, there is nothing to prove. Assume now $A \cup B \neq \emptyset$.

We reason by contradiction: if the limit in (5.2.1) is strictly positive, there exist $\varepsilon > 0$ and a strictly decreasing sequence $\omega_i \searrow 0$ such that

$$\sup\{|V_\lambda(f)| : f \prec A^{\omega_i} \cup B^{\omega_i},\ L(f) \leq \gamma\} \geq \varepsilon,$$

for all $i \in \mathbb{N}$. By definition of supremum, for every $i \in \mathbb{N}$ there is a Lipschitz function f_i with $f_i \prec A^{\omega_i} \cup B^{\omega_i}$ and $L(f_i) \leq \gamma$ such that

$$|V_\lambda(f_i)| > \sup\{|V_\lambda(f)| : f \prec A^{\omega_i} \cup B^{\omega_i},\ L(f) \leq \gamma\} - \frac{\varepsilon}{2} \geq \frac{\varepsilon}{2}. \tag{5.2.3}$$

Since $K_i = \operatorname{supp}(f_i)$ is compact, for every $i \in \mathbb{N}$ we can write $\|f_i\|_\infty = |f_i(x_i)|$, for some $x_i \in K_i$. Note that, for every $i \in \mathbb{N}$, $K_i \subseteq A^{\omega_i} \cup B^{\omega_i} \subseteq \overline{A^{\omega_1} \cup B^{\omega_1}}$, which is compact; then there exists $\{x_{i_j}\} \subseteq \{x_i\}$ such that $x_{i_j} \to x$ as $j \to \infty$, for some $x \in \overline{A^{\omega_1} \cup B^{\omega_1}}$.

We actually have $x \in \partial A \cup \partial B$. Indeed, if

$$x \notin \partial A \cup \partial B = \bigcap_{i\in\mathbb{N}} \overline{A^{\omega_i}} \cup \bigcap_{i\in\mathbb{N}} \overline{B^{\omega_i}} = \bigcap_{i\in\mathbb{N}} \overline{A^{\omega_i}} \cup \overline{B^{\omega_i}} = \bigcap_{i\in\mathbb{N}} \overline{A^{\omega_i} \cup B^{\omega_i}},$$

since $x \in \overline{A^{\omega_1} \cup B^{\omega_1}}$ there must be a number $I \in \mathbb{N}$ such that

$$x \in \overline{A^{\omega_I} \cup B^{\omega_I}} \setminus \overline{A^{\omega_{I+1}} \cup B^{\omega_{I+1}}}.$$

Now, $\{x_i\}_{i\geq I+1} \subseteq A^{\omega_{I+1}} \cup B^{\omega_{I+1}}$, which implies $x = \lim_{i\to\infty} x_i \in \overline{A^{\omega_{I+1}} \cup B^{\omega_{I+1}}}$, a contradiction. Therefore, $x \in \partial A \cup \partial B$. Without loss of generality, assume $x \in \partial A$.

Let us prove that there exists $J \in \mathbb{N}$ such that $f_{i_j}(x) = 0$ for every $j > J$. If this was not the case, there would be a sequence $\{f_{i_{j_\ell}}\} \subseteq \{f_{i_j}\}$ such that $f_{i_{j_\ell}}(x) \neq 0$, for every $\ell \in \mathbb{N}$. This would imply $x \in \operatorname{supp}(f_{i_{j_\ell}}) \subseteq A^{\omega_{i_{j_\ell}}} \cup B^{\omega_{i_{j_\ell}}}$, but since $x \in \partial A$ we would actually have $x \in B^{\omega_{i_{j_\ell}}}$, for every $\ell \in \mathbb{N}$. If $B = \emptyset$, this is already a contradiction. If $B \neq \emptyset$, we have $d = d(x, B) > 0$, and then there would exist $h \in \mathbb{N}$ such that $\omega_{i_{j_h}} < d$, hence $x \notin B^{\omega_{i_{j_h}}}$, a contradiction.

By Lipschitz continuity, we get that for sufficiently large j

$$\|f_{i_j}\|_\infty = |f_{i_j}(x_{i_j})| = |f_{i_j}(x_{i_j}) - f_{i_j}(x)| \leq \gamma \|x_{i_j} - x\| \to 0.$$

Moreover, $\|\nabla_s f_{i_j}\| \leq \gamma$ a.e., from (2.1.2), and since $\mathcal{H}^{n-1}(K_{i_j}) \leq \mathcal{H}^{n-1}(A^{\omega_{i_j}} \cup B^{\omega_{i_j}}) \to 0$ (because of Lemma 5.2.2), $\mathcal{H}^{n-1}(\{f_{i_j} = 0\}) \to 1$. From Lemma 2.2.16 we conclude that $\nabla_s f_{i_j} \to 0$ a.e. in $\mathbb{S}^{n-1}$. Therefore, $f_{i_j} \underset{\tau}{\to} \mathbb{O}$, where $\mathbb{O}$ again denotes the identically null function. This is a contradiction with (5.2.3) and the continuity of V. □

5.3 The control measure

We fix a continuous and rotation invariant valuation $V : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$. Define $V_{flat} : \mathrm{Lip}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}$ by setting

$$V_{flat}(f) = \int_{\mathbb{S}^{n-1}} V(f(x) \cdot \mathbb{1}) d\mathcal{H}^{n-1}(x),$$

for $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$. Since $\eta : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\eta(\lambda) = V(\lambda\mathbb{1})$ is continuous, this V_{flat} is still a continuous and rotation invariant valuation, from Theorem 1.0.1 or Lemma 4.3.1.

Note that $V_{slope} := V - V_{flat}$ satisfies $V_{slope}(\lambda \cdot \mathbb{1}) = V(\lambda \cdot \mathbb{1}) - V_{flat}(\lambda \cdot \mathbb{1}) = 0$, for every $\lambda \in \mathbb{R}$ (remember that, with our normalization, $\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = 1$). Since V_{slope} is again a continuous and rotation invariant valuation, up to replacing V by V_{slope} we can assume V to be null on constant functions.

For $\lambda \in \mathbb{R}$, $\gamma \in \mathbb{R}^+$, we will build a measure $\mu_{\lambda,\gamma}$ which"controls" (in a sense yet to be defined) the valuation V_λ. We will separately build its positive and negative part. We start by constructing the positive one. To do so, we first build an outer measure $\mu^*_{\lambda,\gamma}$ on $\mathbb{S}^{n-1}$.

5.3.1 Definition of $\mu^*_{\lambda,\gamma}$ on open sets

Fix $\lambda \in \mathbb{R}$, $\gamma \in \mathbb{R}^+$. We begin with the definition of $\mu^*_{\lambda,\gamma}$ on open sets: for an open set $G \subseteq \mathbb{S}^{n-1}$,

$$\mu^*_{\lambda,\gamma}(G) := \lim_{\ell \to 0^+} \sup\{V_\lambda(f) : f \prec G,\, \|f\|_\infty \le \ell,\, L(f) \le \gamma\}, \tag{5.3.1}$$

where the functions f over which we are taking the supremum are in $\mathrm{Lip}(\mathbb{S}^{n-1})$. Note that the mapping

$$\ell \mapsto \sup\{V_\lambda(f) : f \prec G,\, \|f\|_\infty \le \ell,\, L(f) \le \gamma\}$$

decreases as $\ell \searrow 0$ and it is lower bounded by $V_\lambda(\mathbb{O}) = 0$. Therefore, the limit exists and $\mu^*_{\lambda,\gamma}$ is well-defined. Moreover, $\mu^*_{\lambda,\gamma}(G) < \infty$ for every open set $G \subseteq \mathbb{S}^{n-1}$, by Lemma 5.1.1. Let us prove that $\mu^*_{\lambda,\gamma}$ is finitely subadditive on open sets.

Lemma 5.3.1. *If $G_1, G_2 \subseteq \mathbb{S}^{n-1}$ are open sets, then*

$$\mu^*_{\lambda,\gamma}(G_1 \cup G_2) \le \mu^*_{\lambda,\gamma}(G_1) + \mu^*_{\lambda,\gamma}(G_2).$$

Proof. Let $G_1, G_2 \subseteq \mathbb{S}^{n-1}$ be open sets. In the following reasoning, for every set $A \subseteq \mathbb{S}^{n-1}$ the symbol A^c will denote its complementary in $G_1 \cup G_2$, i.e., $A^c = (G_1 \cup G_2) \setminus A$.

For $\omega > 0$ and $A_1, A_2 \subseteq \mathbb{S}^{n-1}$, consider the sets

$$A_1^\omega = \{x \in G_1 \cup G_2 : 0 < d(x, A_1) < \omega\},$$

$$A_2^\omega = \{x \in G_1 \cup G_2 : 0 < d(x, A_2) < \omega\},$$

$$G_1(\omega) = \{x \in G_1 : d(x, G_1^c) \ge \omega\},$$

$$G_2(\omega) = \{x \in G_2 : d(x, G_2^c) \ge \omega\}.$$

With a little abuse of notation, we are using the symbols A_i^ω, $i = 1, 2$, to denote $A_i^\omega \cap (G_1 \cup G_2)$.

Note that, for every $\omega > 0$,

$$G_1 \cup G_2 = G_1(\omega) \cup G_2(\omega) \cup \left[G_1(\omega)^{2\omega} \cap G_2(\omega)^{2\omega}\right].$$

Fix now $\varepsilon > 0$. Lemma 5.2.1 implies the existence of an $\omega > 0$ such that

$$\sup\left\{|V_\lambda(f)| : f \prec (G_i^c)^{\frac{3}{2}\omega},\ L(f) \le \gamma\right\} < \varepsilon \tag{5.3.2}$$

for $i = 1, 2$, and

$$\sup\left\{|V_\lambda(f)| : f \prec (G_1^c)^{\frac{3}{2}\omega} \cup (G_2^c)^{\frac{3}{2}\omega},\ L(f) \le \gamma\right\} < \varepsilon. \tag{5.3.3}$$

Given this ω, there exists $0 < \ell < \frac{\omega}{2}\gamma$ such that

$$\mu^*_{\lambda,\gamma}(G_1 \cup G_2) < \sup\{V_\lambda(f) : f \prec G_1 \cup G_2,\ \|f\|_\infty \le \ell,\ L(f) \le \gamma\} + \frac{\varepsilon}{2}, \tag{5.3.4}$$

$$\sup\{V_\lambda(f) : f \prec G_1,\ \|f\|_\infty \le \ell,\ L(f) \le \gamma\} < \mu^*_{\lambda,\gamma}(G_1) + \varepsilon, \tag{5.3.5}$$

$$\sup\{V_\lambda(f) : f \prec G_2,\ \|f\|_\infty \le \ell,\ L(f) \le \gamma\} < \mu^*_{\lambda,\gamma}(G_2) + \varepsilon, \tag{5.3.6}$$

by definition of $\mu^*_{\lambda,\gamma}$. From (5.3.4), there also exists a Lipschitz function $h \prec G_1 \cup G_2$ with $\|h\|_\infty \le \ell$, $L(h) \le \gamma$, such that

$$\mu^*_{\lambda,\gamma}(G_1 \cup G_2) < V_\lambda(h) + \varepsilon.$$

Let $\tilde{h}_i : G_i(\omega) \cup G_i\left(\frac{\omega}{2}\right)^c \longrightarrow \mathbb{R}$,

$$\tilde{h}_i(x) = \begin{cases} h(x) & x \in G_i(\omega), \\ 0 & x \in G_i\left(\frac{\omega}{2}\right)^c, \end{cases}$$

for $i = 1, 2$. Note that $\tilde{h}_1$ and $\tilde{h}_2$ are Lipschitz continuous on their respective domains with Lipschitz constants $L(\tilde{h}_1), L(\tilde{h}_2) \le \gamma$. Indeed, for $i = 1, 2$, if $x, y \in G_i\left(\frac{\omega}{2}\right)^c$ then $\left|\tilde{h}_i(x) - \tilde{h}_i(y)\right| = 0$, if $x, y \in G_i(\omega)$ then $\left|\tilde{h}_i(x) - \tilde{h}_i(y)\right| \le \gamma\|x - y\|$ and if $x \in G_i(\omega)$, $y \in G_i\left(\frac{\omega}{2}\right)^c$ then

$$\frac{\left|\tilde{h}_i(x) - \tilde{h}_i(y)\right|}{\|x - y\|} \le \frac{\ell}{\|x - y\|} \le \frac{2\ell}{\omega} < \gamma.$$

We can now use Lemma 2.2.17 to extend $\tilde{h}_i$, $i = 1, 2$, to a Lipschitz function $h_i : G_1 \cup G_2 \longrightarrow \mathbb{R}$ such that $h^- \le h_i \le h^+$, $L(h_i) \le \gamma$ and $\|h_i\|_\infty \le \ell$.

Define $\tilde{h}_0 : \left[G_1(\omega)^{2\omega} \cap G_2(\omega)^{2\omega}\right] \cup G_1\left(\frac{3}{2}\omega\right) \cup G_2\left(\frac{3}{2}\omega\right) \longrightarrow \mathbb{R}$,

$$\tilde{h}_0(x) = \begin{cases} h(x) & x \in G_1(\omega)^{2\omega} \cap G_2(\omega)^{2\omega}, \\ 0 & x \in G_1\left(\frac{3}{2}\omega\right) \cup G_2\left(\frac{3}{2}\omega\right), \end{cases}$$

and again use Lemma 2.2.17 to extend this to $h_0 : G_1 \cup G_2 \longrightarrow \mathbb{R}$ such that $h^- \le h_0 \le h^+$, $L(h_0) \le \gamma$ and $\|h_0\|_\infty \le \ell$.

Write $h = h^+ + h^-$ and note that

$$h^+ = h_0^+ \vee h_1^+ \vee h_2^+,$$

$$h^- = h_0^- \wedge h_1^- \wedge h_2^-.$$

From the valuation property and the inclusion-exclusion principle, we now get

$$\begin{aligned} V_\lambda(h) &= V_\lambda(h) + V_\lambda(\mathbb{O}) = V_\lambda(h^+) + V_\lambda(h^-) = V_\lambda(h_0^+ \vee h_1^+ \vee h_2^+) + V_\lambda(h_0^- \wedge h_1^- \wedge h_2^-) \\ &= V_\lambda(h_0^+) + V_\lambda(h_1^+) + V_\lambda(h_2^+) - V_\lambda(h_0^+ \wedge h_1^+) - V_\lambda(h_1^+ \wedge h_2^+) - V_\lambda(h_0^+ \wedge h_2^+) + \\ &\quad + V_\lambda(h_0^+ \wedge h_1^+ \wedge h_2^+) + V_\lambda(h_0^-) + V_\lambda(h_1^-) + V_\lambda(h_2^-) - V_\lambda(h_0^- \vee h_1^-) - \\ &\quad - V_\lambda(h_1^- \vee h_2^-) - V_\lambda(h_0^- \vee h_2^-) + V_\lambda(h_0^- \vee h_1^- \vee h_2^-). \end{aligned}$$

Since $h_0^{\pm}=0$ on $G_1\left(\frac{3}{2}\omega\right)\cup G_2\left(\frac{3}{2}\omega\right)$, we have that $h_0^{\pm}\prec (G_1^c)^{\frac{3}{2}\omega}\cup(G_2^c)^{\frac{3}{2}\omega}$; moreover, $L(h_0^{\pm})\leq\gamma$, and from (5.3.3) we get

$$|V_\lambda(h_0^{\pm})|<\varepsilon.$$

Similarly, since on $G_1\left(\frac{3}{2}\omega\right)\cup G_2\left(\frac{3}{2}\omega\right)$ we have $h_0^+\wedge h_1^+=0\wedge h_1^+=0$, and analogously $h_0^+\wedge h_2^+=h_0^-\vee h_1^-=h_0^-\vee h_2^-=h_0^+\wedge h_1^+\wedge h_2^+=h_0^-\vee h_1^-\vee h_2^-=0$, we obtain

$$\begin{aligned}
|V_\lambda(h_0^+\wedge h_1^+)| &< \varepsilon,\\
|V_\lambda(h_0^+\wedge h_2^+)| &< \varepsilon,\\
|V_\lambda(h_0^-\vee h_1^-)| &< \varepsilon,\\
|V_\lambda(h_0^-\vee h_2^-)| &< \varepsilon,\\
|V_\lambda(h_0^+\wedge h_1^+\wedge h_2^+)| &< \varepsilon,\\
|V_\lambda(h_0^-\vee h_1^-\vee h_2^-)| &< \varepsilon.
\end{aligned}$$

Therefore,

$$\begin{aligned}
V_\lambda(h) &< V_\lambda(h_1^+)+V_\lambda(h_2^+)-V_\lambda(h_1^+\wedge h_2^+)+V_\lambda(h_1^-)+V_\lambda(h_2^-)-V_\lambda(h_1^-\vee h_2^-)+8\varepsilon\\
&= V_\lambda(h_1)+V_\lambda(h_2)-V_\lambda(h_1^+\wedge h_2^+)-V_\lambda(h_1^-\vee h_2^-)+8\varepsilon, \qquad (5.3.7)
\end{aligned}$$

from the valuation property.

Now, for $i=1,2$, we have that $h_i\prec G_i$, $\|h_i\|_\infty\leq\ell$ and $L(h_i)\leq\gamma$, hence (5.3.5) and (5.3.6) imply $V_\lambda(h_i)<\mu^*_{\lambda,\gamma}(G_i)+\varepsilon$, so that from (5.3.7) we get

$$V_\lambda(h)<\mu^*_{\lambda,\gamma}(G_1)+\mu^*_{\lambda,\gamma}(G_2)-V_\lambda(h_1^+\wedge h_2^+)-V_\lambda(h_1^-\vee h_2^-)+10\varepsilon. \qquad (5.3.8)$$

Suppose by contradiction

$$V_\lambda(h_1^+\wedge h_2^+)\leq -6\varepsilon, \qquad (5.3.9)$$

define

$$g_1(x)=\begin{cases} h_1^+(x) & \text{if } x\in G_2(\omega)^c,\\ 0 & \text{if } x\in G_2\left(\frac{3}{2}\omega\right),\end{cases}$$

and extend g_1 to $G_1\cup G_2$ using Lemma 2.2.17. We do the same for

$$g_2(x)=\begin{cases} h_2^+(x) & \text{if } x\in G_1(\omega)^c,\\ 0 & \text{if } x\in G_1\left(\frac{3}{2}\omega\right).\end{cases}$$

Note that $g_1,g_2\geq 0$ on $G_1\cup G_2$.

We also have that, for $i=1,2$,

$$g_i\vee(h_1^+\wedge h_2^+)=h_i^+. \qquad (5.3.10)$$

Indeed, for $x\in G_2(\omega)^c$ we have

$$g_1(x)\vee[h_1^+(x)\wedge h_2^+(x)]=h_1^+(x)\vee[h_1^+(x)\wedge h_2^+(x)]=h_1^+(x),$$

and for $x\in G_2(\omega)$

$$g_1(x)\vee[h_1^+(x)\wedge h_2^+(x)]=g_1(x)\vee[h_1^+(x)\wedge h^+(x)]=g_1\vee h_1^+(x)=h_1^+(x).$$

Analogously for $i=2$.

Let $g : G_1 \cup G_2 \longrightarrow \mathbb{R}$ be the Lipschitz function defined by $g = g_1 \vee g_2$; since $g_1, g_2 \geq 0$, we have that $g \geq 0$ on $G_1 \cup G_2$. From the valuation property and (5.3.10) we get

$$\begin{aligned} V_\lambda(g) \ &= \ V_\lambda(g_1) + V_\lambda(g_2) - V_\lambda(g_1 \wedge g_2) = V_\lambda(h_1^+) + V_\lambda(g_1 \wedge h_1^+ \wedge h_2^+) - V_\lambda(h_1^+ \wedge h_2^+) + \\ &\quad + V_\lambda(h_2^+) + V_\lambda(g_2 \wedge h_1^+ \wedge h_2^+) - V_\lambda(h_1^+ \wedge h_2^+) - V_\lambda(g_1 \wedge g_2) = V_\lambda(h_1^+ \vee h_2^+) - \\ &\quad - V_\lambda(h_1^+ \wedge h_2^+) + V_\lambda(g_1 \wedge h_1^+ \wedge h_2^+) + V_\lambda(g_2 \wedge h_1^+ \wedge h_2^+) - V_\lambda(g_1 \wedge g_2). \end{aligned}$$

Now,

$$g_1 \wedge h_1^+ \wedge h_2^+ \prec (G_2^c)^{\frac{3}{2}\omega},$$

$$g_2 \wedge h_1^+ \wedge h_2^+ \prec (G_1^c)^{\frac{3}{2}\omega},$$

$$g_1 \wedge g_2 \prec (G_1^c)^{\frac{3}{2}\omega} \cup (G_2^c)^{\frac{3}{2}\omega},$$

so that (5.3.2) and (5.3.3) imply, for $i = 1, 2$,

$$|V_\lambda(g_i \wedge h_1^+ \wedge h_2^+)| < \varepsilon,$$

$$|V_\lambda(g_1 \wedge g_2)| < \varepsilon.$$

Moreover, from the valuation property,

$$V_\lambda(h_1^+ \vee h_2^+) = V_\lambda(h^+) + V_\lambda(h_0^+ \wedge (h_1^+ \vee h_2^+)) - V_\lambda(h_0^+),$$

where $h_0^+ \wedge (h_1^+ \vee h_2^+) \prec (G_1^c)^{\frac{3}{2}\omega} \cup (G_2^c)^{\frac{3}{2}\omega}$, hence

$$|V_\lambda(h_0^+ \wedge (h_1^+ \vee h_2^+))| < \varepsilon.$$

Putting things together, from assumption (5.3.9) we obtain

$$V_\lambda(g) > V_\lambda(h^+) + \varepsilon.$$

The function $\tilde{g} = g + h^-$ satisfies $\tilde{g} \prec G_1 \cup G_2$, $\|\tilde{g}\|_\infty \leq \ell$ (being $\tilde{g} \leq g \leq \ell$ and $\tilde{g} \geq h^- \geq -\ell$), $L(\tilde{g}) \leq \gamma$, and recalling that $g \geq 0$ on $G_1 \cup G_2$ we find

$$\begin{aligned} V_\lambda(\tilde{g}) \ &= \ V_\lambda(\tilde{g}^+) + V_\lambda(\tilde{g}^-) = V_\lambda(g) + V_\lambda(h^-) \\ &> \ V_\lambda(h^+) + V_\lambda(h^-) + \varepsilon = V_\lambda(h) + \varepsilon \\ &> \ \mu^*_{\lambda,\gamma}(G_1 \cup G_2), \end{aligned}$$

a contradiction with the definition of $\mu^*_{\lambda,\gamma}(G_1 \cup G_2)$.

This proves that

$$V_\lambda(h_1^+ \wedge h_2^+) > -6\varepsilon,$$

and similarly

$$V_\lambda(h_1^- \vee h_2^-) > -6\varepsilon.$$

From (5.3.8) we conclude that

$$\mu^*_{\lambda,\gamma}(G_1 \cup G_2) < \mu^*_{\lambda,\gamma}(G_1) + \mu^*_{\lambda,\gamma}(G_2) + 23\varepsilon.$$

□

5.3.2 Extension of $\mu^*_{\lambda,\gamma}$

Now, for every $A \subseteq \mathbb{S}^{n-1}$, we define

$$\mu^*_{\lambda,\gamma}(A) = \inf\{\mu^*_{\lambda,\gamma}(G) : \, A \subseteq G, \, G \text{ open}\}. \tag{5.3.11}$$

This clearly coincides with definition (5.3.1) on open sets. We have the following.

Lemma 5.3.2. *The function $\mu^*_{\lambda,\gamma}$ defined by* (5.3.11) *is an outer measure on $\mathbb{S}^{n-1}$.*

Proof. Note that $\mu^*_{\lambda,\gamma}$ is monotone increasing and satisfies $\mu^*_{\lambda,\gamma}(\emptyset) = 0$. To check that this is indeed an outer measure we have to prove countable subadditivity.

Let $\{A_i\}$ be a sequence of subsets of $\mathbb{S}^{n-1}$ and take $\varepsilon > 0$. For every $i \in \mathbb{N}$, choose an open set G'_i such that $A_i \subseteq G'_i$ and

$$\mu^*_{\lambda,\gamma}(A_i) > \mu^*_{\lambda,\gamma}(G'_i) - \frac{\varepsilon}{2^i}.$$

Also, consider an open set $G' \supseteq \bigcup_{i\in\mathbb{N}} A_i$ such that

$$\mu^*_{\lambda,\gamma}\left(\bigcup_{i\in\mathbb{N}} A_i\right) > \mu^*_{\lambda,\gamma}(G') - \varepsilon.$$

Define now $G_i = G'_i \cap G'$, for every $i \in \mathbb{N}$, and $G = \bigcup_{i\in\mathbb{N}} G_i$. By monotonicity of $\mu^*_{\lambda,\gamma}$, these sets still verify

$$\mu^*_{\lambda,\gamma}(A_i) > \mu^*_{\lambda,\gamma}(G_i) - \frac{\varepsilon}{2^i}$$

and

$$\mu^*_{\lambda,\gamma}\left(\bigcup_{i\in\mathbb{N}} A_i\right) > \mu^*_{\lambda,\gamma}(G) - \varepsilon.$$

For the previously chosen ε, Lemma 5.2.1 guarantees the existence of $\omega > 0$ such that, for every $f \prec (G^c)^\omega$ with $L(f) \leq \gamma$, $|V_\lambda(f)| < \varepsilon$. This implies $\mu^*_{\lambda,\gamma}((G^c)^\omega) \leq \varepsilon$, recalling definition (5.3.1).

Consider now

$$G(\omega) = \{x \in G : d(x, G^c) \geq \omega\};$$

the set $G(\omega) \subseteq G = \bigcup_{i\in\mathbb{N}} G_i$ is compact (being bounded and closed), thus there exist $N \in \mathbb{N}$ and $i_1, \ldots, i_N \in \mathbb{N}$ such that $G(\omega) \subseteq G^N := \bigcup_{j=1}^N G_{i_j}$. Then, $G = G^N \cup (G^c)^\omega$.

Finite subadditivity on open sets implies that

$$\mu^*_{\lambda,\gamma}(G) \leq \mu^*_{\lambda,\gamma}(G^N) + \mu^*_{\lambda,\gamma}((G^c)^\omega) \leq \sum_{j=1}^N \mu^*_{\lambda,\gamma}(G_{i_j}) + \varepsilon \leq \sum_{i\in\mathbb{N}} \mu^*_{\lambda,\gamma}(G_i) + \varepsilon.$$

Since $\bigcup_{i\in\mathbb{N}} A_i \subseteq G' \cap \bigcup_{i\in\mathbb{N}} G'_i = G$, it follows that

$$\mu^*_{\lambda,\gamma}\left(\bigcup_{i\in\mathbb{N}} A_i\right) \leq \mu^*_{\lambda,\gamma}(G) \leq \sum_{i\in\mathbb{N}} \mu^*_{\lambda,\gamma}(G_i) + \varepsilon \leq \sum_{i\in\mathbb{N}} \mu^*_{\lambda,\gamma}(A_i) + 2\varepsilon.$$

We conclude from the arbitrariness of ε. $\square$

5.3.3 The measure $\mu_{\lambda,\gamma}$

The outer measure $\mu^*_{\lambda,\gamma}$ gives rise to a measure on the Borel σ-algebra of $\mathbb{S}^{n-1}$.

Proposition 5.3.3. *The Borel σ-algebra Σ of $\mathbb{S}^{n-1}$ is $\mu^*_{\lambda,\gamma}$-measurable, and the set function $\mu^+_{\lambda,\gamma}$ defined as the restriction of $\mu^*_{\lambda,\gamma}$ to Σ is a measure.*

Proof. By Theorem 2.1.2, it is enough to show that every open set $G \subseteq \mathbb{S}^{n-1}$ is $\mu^*_{\lambda,\gamma}$-measurable. Because of Lemma 5.3.1, it suffices to check that, for a fixed open set $G \subseteq \mathbb{S}^{n-1}$,

$$\mu^*_{\lambda,\gamma}(A) \geq \mu^*_{\lambda,\gamma}(A \cap G) + \mu^*_{\lambda,\gamma}(A \cap G^c),$$

for every $A \subseteq \mathbb{S}^{n-1}$.

Fix $A \subseteq \mathbb{S}^{n-1}$ and $\varepsilon > 0$. It follows from the definition (5.3.11) of $\mu^*_{\lambda,\gamma}$ that there exists an open set $U \supseteq A$ such that

$$\mu^*_{\lambda,\gamma}(U) < \mu^*_{\lambda,\gamma}(A) + \varepsilon.$$

Recalling (5.3.1), there also exists $\ell_0 > 0$ such that

$$\sup\{V_\lambda(f) : f \prec U,\, \|f\|_\infty \leq \ell_0,\, L(f) \leq \gamma\} < \mu^*_{\lambda,\gamma}(U) + \varepsilon,$$

Now, $U \cap G$ is an open set, hence we can choose $0 < \ell_1 \leq \ell_0$ and $f_1 \prec U \cap G$ with $\|f_1\|_\infty \leq \ell_1 \leq \ell_0$, $L(f_1) \leq \gamma$, such that

$$\mu^*_{\lambda,\gamma}(U \cap G) < V_\lambda(f_1) + \varepsilon.$$

We consider the compact set $K = \mathrm{supp}(f_1) \subseteq U \cap G$. Then $U \cap G^c \subseteq U \cap K^c$, and this last set is open. Choose now $0 < \ell_2 \leq \ell_0$ and $f_2 \prec U \cap K^c$ with $\|f_2\|_\infty \leq \ell_2 \leq \ell_0$, $L(f_2) \leq \gamma$, such that

$$\mu^*_{\lambda,\gamma}(U \cap K^c) < V_\lambda(f_2) + \varepsilon.$$

Note that f_1 and f_2 have disjoint supports (since $\mathrm{supp}(f_2) \subseteq K^c = (\mathrm{supp}(f_1))^c$), both of them contained in U. Therefore, the function $g = f_1 + f_2$ satisfies $g \prec U$, $\|g\|_\infty \leq \ell_0$ and $L(g) \leq \gamma$. Moreover,

$$f_1^+ \wedge f_2^+ = f_1^- \vee f_2^- = \mathbb{O}$$

and

$$f_1^+ \vee f_2^+ = (f_1 + f_2)^+,$$
$$f_1^- \wedge f_2^- = (f_1 + f_2)^-,$$

hence

$$\begin{aligned} V_\lambda(f_1) + V_\lambda(f_2) &= V_\lambda(f_1^+) + V_\lambda(f_1^-) + V_\lambda(f_2^+) + V_\lambda(f_2^-) = V_\lambda(f_1^+ \vee f_2^+) + V_\lambda(f_1^- \wedge f_2^-) \\ &= V_\lambda((f_1 + f_2)^+) + V_\lambda((f_1 + f_2)^-) = V_\lambda(f_1 + f_2). \end{aligned}$$

This implies

$$\begin{aligned} \mu^*_{\lambda,\gamma}(A) &> \mu^*_{\lambda,\gamma}(U) - \varepsilon > \sup\{V_\lambda(f) : f \prec U,\, \|f\|_\infty \leq \ell_0,\, L(f) \leq \gamma\} - 2\varepsilon \\ &\geq V_\lambda(f_1 + f_2) - 2\varepsilon = V_\lambda(f_1) + V_\lambda(f_2) - 2\varepsilon \\ &> \mu^*_{\lambda,\gamma}(U \cap G) + \mu^*_{\lambda,\gamma}(U \cap K^c) - 4\varepsilon \\ &\geq \mu^*_{\lambda,\gamma}(U \cap G) + \mu^*_{\lambda,\gamma}(U \cap G^c) - 4\varepsilon \\ &\geq \mu^*_{\lambda,\gamma}(A \cap G) + \mu^*_{\lambda,\gamma}(A \cap G^c) - 4\varepsilon, \end{aligned}$$

where we have used the fact that $\mu^*_{\lambda,\gamma}$ is monotone increasing. □

We now have the measure $\mu^+_{\lambda,\gamma}$. In a similar fashion, we can define a measure $\mu^-_{\lambda,\gamma}$ on Σ which on open sets would be given by

$$\begin{aligned}\mu^-_{\lambda,\gamma}(G) &= \lim_{\ell\to 0^+} \sup\{-V_\lambda(f) : f \prec G,\ \|f\|_\infty \le \ell,\ L(f) \le \gamma\}\\ &= \lim_{\ell\to 0^+} -\inf\{V_\lambda(f) : f \prec G,\ \|f\|_\infty \le \ell,\ L(f) \le \gamma\}.\end{aligned}$$

The following remarks will be crucial in our reasoning.

Remark 5.3.4. *The measure $\mu_{\lambda,\gamma} := \mu^+_{\lambda,\gamma} + \mu^-_{\lambda,\gamma}$ controls the absolute value of the valuation on open sets, in the sense that for every open set $G \subseteq \mathbb{S}^{n-1}$ we have*

$$\lim_{\ell\to 0^+} \sup\{|V_\lambda(f)| : f \prec G,\ \|f\|_\infty \le \ell,\ L(f) \le \gamma\} \le \mu_{\lambda,\gamma}(G).$$

Indeed, if $G_{\ell,\gamma}$ denotes the set of Lipschitz functions f such that $f \prec G$, $\|f\|_\infty \le \ell$ and $L(f) \le \gamma$,

$$\begin{aligned}\sup_{f\in G_{\ell,\gamma}} |V_\lambda(f)| &= \sup_{f\in G_{\ell,\gamma}} \{\max\{V_\lambda(f), -V_\lambda(f)\}\} \le \max\left\{\sup_{f\in G_{\ell,\gamma}} V_\lambda(f),\ \sup_{f\in G_{\ell,\gamma}} [-V_\lambda(f)]\right\}\\ &\le \sup_{f\in G_{\ell,\gamma}} V_\lambda(f) + \sup_{f\in G_{\ell,\gamma}} [-V_\lambda(f)],\end{aligned}$$

where the last inequality follows from the fact that both the suprema are non-negative. Therefore,

$$\lim_{\ell\to 0^+} \sup_{f\in G_{\ell,\gamma}} |V_\lambda(f)| \le \mu^+_{\lambda,\gamma}(G) + \mu^-_{\lambda,\gamma}(G) = \mu_{\lambda,\gamma}(G).$$

Remark 5.3.5. *For every $\lambda \in \mathbb{R}$, $\gamma \in \mathbb{R}^+$, the measure $\mu_{\lambda,\gamma}$ is rotation invariant (since V is) and finite (from Lemma 5.1.1), hence (see for instance [25]) there exists a number $\vartheta(\lambda,\gamma) \in \mathbb{R}^+$ such that*

$$\mu_{\lambda,\gamma} = \vartheta(\lambda,\gamma)\mathcal{H}^{n-1}.$$

Gathering these observations, we get the following.

Proposition 5.3.6. *Let $\lambda \in \mathbb{R}$, $\gamma \in \mathbb{R}^+$. Then there exists a number $\vartheta(\lambda,\gamma) \in \mathbb{R}^+$ such that*

$$\lim_{\ell\to 0^+} \sup\{|V_\lambda(f)| : f \prec G,\ \|f\|_\infty \le \ell,\ L(f) \le \gamma\} \le \vartheta(\lambda,\gamma)\mathcal{H}^{n-1}(G),$$

for every open set $G \subseteq \mathbb{S}^{n-1}$.

The function $\vartheta : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is bounded on bounded sets, in the following sense.

Lemma 5.3.7. *Let $\lambda_0, \gamma_0 \in \mathbb{R}^+$. Then*

$$\Theta := \sup\{\vartheta(\lambda,\gamma) : |\lambda| \le \lambda_0, 0 < \gamma \le \gamma_0\} < \infty.$$

Proof. If this is not the case, for every $M \in \mathbb{R}$ there exist λ_M, γ_M with $|\lambda_M| \le \lambda_0$, $0 < \gamma_M \le \gamma_0$, such that

$$\vartheta(\lambda_M, \gamma_M) > M.$$

Remembering our normalization $\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = 1$ and Remark 5.3.5, this implies

$$\mu_{\lambda_M,\gamma_M}(\mathbb{S}^{n-1}) \ge M,$$

for every $M \in \mathbb{R}$.

Fix $M \in \mathbb{R}$. By definition of $\mu^+_{\lambda_M,\gamma_M}$, $\mu^-_{\lambda_M,\gamma_M}$, we have that for every $\varepsilon > 0$ there exists $0 < \ell < 1$ such that

$$\left|\mu^+_{\lambda_M,\gamma_M}(\mathbb{S}^{n-1}) - \sup\left\{V_{\lambda_M}(f) : \|f\|_\infty \leq \ell,\, L(f) \leq \gamma_M\right\}\right| < \frac{\varepsilon}{4},$$

$$\left|\mu^-_{\lambda_M,\gamma_M}(\mathbb{S}^{n-1}) + \inf\left\{V_{\lambda_M}(f) : \|f\|_\infty \leq \ell,\, L(f) \leq \gamma_M\right\}\right| < \frac{\varepsilon}{4}.$$

From μ_{λ_M,γ_M}'s definition and the triangular inequality we get

$$\left|\mu_{\lambda_M,\gamma_M}(\mathbb{S}^{n-1}) - \sup_f V_{\lambda_M}(f) + \inf_f V_{\lambda_M}(f)\right| < \frac{\varepsilon}{2}.$$

In particular,

$$M \leq \mu_{\lambda_M,\gamma_M}(\mathbb{S}^{n-1}) < \sup_f V_{\lambda_M}(f) - \inf_f V_{\lambda_M}(f) + \frac{\varepsilon}{2}.$$

There exist $f_M, g_M \in \mathrm{Lip}(\mathbb{S}^{n-1})$ such that $\|f_M\|_\infty, \|g_M\|_\infty \leq \ell$, $L(f_M), L(g_M) \leq \gamma_M$ and

$$\sup_f V_{\lambda_M}(f) < V_{\lambda_M}(f_M) + \frac{\varepsilon}{4},$$

$$\inf_f V_{\lambda_M}(f) > V_{\lambda_M}(g_M) - \frac{\varepsilon}{4},$$

which in turn implies

$$M < V_{\lambda_M}(f_M) - V_{\lambda_M}(g_M) + \varepsilon. \tag{5.3.12}$$

The functions $f_M + \lambda_M, g_M + \lambda_M \in \mathrm{Lip}(\mathbb{S}^{n-1})$ satisfy

$$\|f_M + \lambda_M\|_\infty, \|g_M + \lambda_M\|_\infty \leq \ell + |\lambda_M| < 1 + \lambda_0,$$

$$L(f_M + \lambda_M) = L(f_M) \leq \gamma_M \leq \gamma_0,$$

$$L(g_M + \lambda_M) = L(g_M) \leq \gamma_M \leq \gamma_0,$$

so that

$$\|f_M + \lambda_M\|_{\mathrm{Lip}}, \|g_M + \lambda_M\|_{\mathrm{Lip}} \leq \Lambda := \max\{1 + \lambda_0, \gamma_0\}.$$

From Lemma 5.1.1, there exists $C > 0$ such that

$$|V_{\lambda_M}(f_M)|, |V_{\lambda_M}(g_M)| \leq C.$$

Inequality (5.3.12) then implies

$$M < 2C + \varepsilon;$$

since M is arbitrary, this is a contradiction. □

5.4 The representing measure

We are now going to build another measure, which will in some sense “represent” our valuation. In order to do so, we need to stick to the bidimensional case, i.e., for the rest of the chapter we will focus on the case $n = 2$.

As we did in the case of polynomial valuations, we can identify functions $f \in \mathrm{Lip}(\mathbb{S}^1)$ with 2π-periodic functions on $\mathbb{R}$. With this convention, we will work with integrals over $(0, 2\pi]$ instead of $\mathbb{S}^1$ and with derivatives f' instead of gradients $\nabla_s f$. Having already described how the identification works in the proof of Theorem 1.1.3, we will not be detailing it again here.

Note that, with this identification, the rotation invariant valuation V becomes translation invariant, in the sense that

$$V(f \circ \mathcal{T}_{t_0}) = V(f),$$

for every $f \in \mathrm{Lip}(\mathbb{S}^1)$ and $t_0 \in \mathbb{R}$, where $\mathcal{T}_{t_0}$ is defined by $\mathcal{T}_{t_0}(t) = t + t_0$, for $t \in (0, 2\pi]$. Moreover, it follows again from the rotational invariance that if f_R denotes the reflection of the function $f \in \mathrm{Lip}(\mathbb{S}^1)$ with respect to the axis $x = \pi$, that is, $f_R(x) = f(2\pi - x)$, $x \in \mathbb{R}$, then $V(f) = V(f_R)$ for every $f \in \mathrm{Lip}(\mathbb{S}^1)$.

Consider the algebra $\mathcal{A}_1$ defined by

$$\mathcal{A}_1 = \left\{ \bigcup_{j=1}^{m} I_j : m \in \mathbb{N},\, I_j = (a_j, b_j] \subseteq (0, \pi],\, I_j \cap I_k = \emptyset \text{ for } j \neq k \right\}.$$

We say that $g \in \mathscr{L}(\mathbb{S}^1)$ is *symmetric* if, on the interval $(0, 2\pi]$, it is symmetric with respect to the axis $x = \pi$. For every symmetric $g \in \mathscr{L}(\mathbb{S}^1)$, we define a set function

$$\nu_g : \mathcal{A}_1 \longrightarrow \mathbb{R}$$

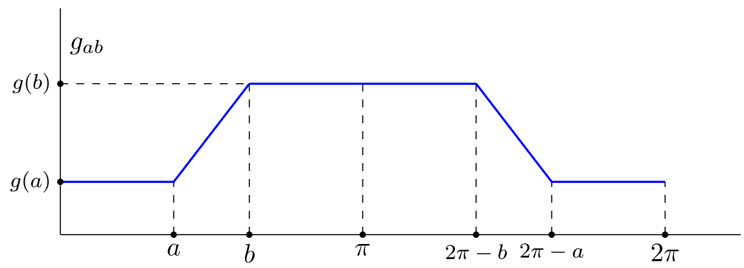
which on intervals $(a, b] \subseteq (0, \pi]$ is given by

$$\nu_g((a, b]) = \frac{1}{2} V(g_{ab}),$$

where

$$g_{ab}(x) = \begin{cases} g(a) & \text{in } (0, a] \cup (2\pi - a, 2\pi], \\ g(x) & \text{in } (a, b] \cup (2\pi - b, 2\pi - a], \\ g(b) & \text{in } (b, 2\pi - b], \end{cases}$$

for $x \in (0, 2\pi]$, and g_{ab} is extended 2π-periodically to $\mathbb{R}$. To justify this definition, recall that we are assuming V to be null on constant functions and g to be symmetric.



Let us prove that for consecutive intervals $I=(a,b], J=(b,c]\subseteq(0,\pi]$ we have

$$\nu_g(I)+\nu_g(J)=\nu_g(I\cup J),$$

that is,

$$V(g_{ab})+V(g_{bc})=V(g_{ac}).$$

If $\lambda=g(b)$, the functions $g_{ab}^\lambda=g_{ab}-\lambda$ and $g_{bc}^\lambda=g_{bc}-\lambda$ have disjoint supports, and then satisfy

$$\begin{aligned}(g_{ab}^\lambda)^+\vee(g_{bc}^\lambda)^+&=(g_{ab}^\lambda+g_{bc}^\lambda)^+,\\ (g_{ab}^\lambda)^-\wedge(g_{bc}^\lambda)^-&=(g_{ab}^\lambda+g_{bc}^\lambda)^-,\\ (g_{ab}^\lambda)^+\wedge(g_{bc}^\lambda)^+&=\mathbb{O},\\ (g_{ab}^\lambda)^-\vee(g_{bc}^\lambda)^-&=\mathbb{O}.\end{aligned}$$

Therefore,

$$\begin{aligned}V(g_{ab})+V(g_{bc}) &= V_\lambda(g_{ab}^\lambda)+V_\lambda(g_{bc}^\lambda)=V_\lambda((g_{ab}^\lambda)^+)+V_\lambda((g_{ab}^\lambda)^-)+V_\lambda((g_{bc}^\lambda)^+)+V_\lambda((g_{bc}^\lambda)^-)\\ &= V_\lambda((g_{ab}^\lambda)^+\vee(g_{bc}^\lambda)^+)+V_\lambda((g_{ab}^\lambda)^-\wedge(g_{bc}^\lambda)^-)=V_\lambda((g_{ab}^\lambda+g_{bc}^\lambda)^+)+\\ &\quad +V_\lambda((g_{ab}^\lambda+g_{bc}^\lambda)^-)=V_\lambda(g_{ab}^\lambda+g_{bc}^\lambda)=V(g_{ab}+g_{bc}^\lambda)=V(g_{ac}),\end{aligned}$$

as desired.

This implies the finite additivity of ν_g on intersecting intervals: indeed, if $I=(a,b]$ and $J=(c,d]$ with $0\le a<c<b<d\le\pi$, from what we have just proved we get that

$$\begin{aligned}\nu_g(I)+\nu_g(J) &= \nu_g((a,b])+\nu_g((c,d])=\nu_g((a,b])+\nu_g((c,b])+\nu_g((b,d])\\ &= \nu_g((a,d])+\nu_g((c,b])=\nu_g(I\cup J)+\nu_g(I\cap J).\end{aligned}$$

If we set

$$\nu_g\left(\bigcup_{j=1}^m I_j\right):=\sum_{j=1}^m\nu_g(I_j),$$

for every pairwise disjoint and semi-open intervals $I_1,\dots,I_m\subseteq(0,\pi]$, the finite additivity on intersecting intervals ensures that this is a well-posed definition and that ν_g is finitely additive.

The following technical lemma will soon be useful.

Lemma 5.4.1. *Let $\lambda\in\mathbb{R}$, $\gamma\in\mathbb{R}^+$. Take $\varepsilon>0$ and $G\subseteq(0,2\pi]$ an open interval. Let $\ell>0$ be such that*

$$|V_\lambda(f)|\le(\vartheta(\lambda,\gamma)+\varepsilon)\mathcal{H}^1(G) \tag{5.4.1}$$

for every $f\in\mathrm{Lip}(\mathbb{S}^1)$ with $f\prec G$, $\|f\|_\infty\le\ell$ and $L(f)\le\gamma$.

Then, for every open interval $G'\subseteq G$ such that $\mathcal{H}^1(G)=k\mathcal{H}^1(G')$ for some $k\in\mathbb{N}$, we have

$$|V_\lambda(f)|\le(\vartheta(\lambda,\gamma)+\varepsilon)\mathcal{H}^1(G'),$$

for every $f\in\mathrm{Lip}(\mathbb{S}^1)$ with $f\prec G'$, $\|f\|_\infty\le\ell$ and $L(f)\le\gamma$.

Proof. First of all, note that an ℓ such that (5.4.1) holds always exists, thanks to Proposition 5.3.6.

We choose an open interval $G'\subseteq G=(a,b)$ with $\mathcal{H}^1(G)=k\mathcal{H}^1(G')$ and reason by contradiction: suppose there exists a function $f\in\mathrm{Lip}(\mathbb{S}^1)$ with $f\prec G'$, $\|f\|_\infty\le\ell$ and $L(f)\le\gamma$ such that $|V_\lambda(f)|>(\vartheta(\lambda,\gamma)+\varepsilon)\mathcal{H}^1(G')$. We can write

$$|V_\lambda(f)|=(\vartheta(\lambda,\gamma)+\varepsilon)\mathcal{H}^1(G')+\rho,$$

for a suitable $\rho > 0$.

Because of rotational invariance, we may assume that the left ends of G and G' coincide. Since $\mathcal{H}^1(G) = k\mathcal{H}^1(G')$, we can divide G into intervals $G_i = (\lambda_{i-1}, \lambda_i)$, $i = 1, \ldots, k$, where $a = \lambda_0 < \lambda_1 < \ldots < \lambda_k = b$, $G_1 = G'$ and $\mathcal{H}^1(G_i) = \mathcal{H}^1(G')$, for every $i = 1, \ldots, k$. We have

$$\overline{G} = \bigcup_{i=1}^{k} \overline{G_i},$$

where, for every $i = 1, \ldots, k$, $G_i = \sigma_i^{-1}(G')$ and σ_i is the rotation (translation as a function on $\mathbb{R}$) bringing λ_{i-1} and λ_i in λ_0 and λ_1 respectively.

Define the function $g \in \mathrm{Lip}(\mathbb{S}^1)$ by setting

$$g(x) = \sum_{i=1}^{k} f(\sigma_i(x)), \ x \in \mathbb{R}.$$

Since $f \circ \sigma_i \prec G_i$ for every $i = 1, \ldots, k$, the supports of the $f \circ \sigma_i$'s are pairwise disjoint, hence

$$g \vee \mathbb{O} = \bigvee_{i=1}^{k} (f \circ \sigma_i) \vee \mathbb{O},$$

$$g \wedge \mathbb{O} = \bigwedge_{i=1}^{k} (f \circ \sigma_i) \wedge \mathbb{O}.$$

Now, the $f \circ \sigma_i \vee \mathbb{O}$'s still have pairwise disjoint supports, which implies

$$V_\lambda(g \vee \mathbb{O}) = \sum_{i=1}^{k} V_\lambda((f \circ \sigma_i) \vee \mathbb{O}),$$

from the inclusion-exclusion principle. Analogously,

$$V_\lambda(g \wedge \mathbb{O}) = \sum_{i=1}^{k} V_\lambda((f \circ \sigma_i) \wedge \mathbb{O}).$$

Moreover, $g \prec G$, $\|g\|_\infty \leq \ell$ and $L(g) \leq \gamma$.

From the valuation property and the rotational invariance we get

$$\begin{aligned} |V_\lambda(g)| &= |V_\lambda(g \vee \mathbb{O}) + V_\lambda(g \wedge \mathbb{O})| = \left| \sum_{i=1}^{k} V_\lambda((f \circ \sigma_i) \vee \mathbb{O}) + \sum_{i=1}^{k} V_\lambda((f \circ \sigma_i) \wedge \mathbb{O}) \right| \\ &= \left| \sum_{i=1}^{k} V_\lambda(f \circ \sigma_i) \right| = k|V_\lambda(f)| = k(\vartheta(\lambda, \gamma) + \varepsilon)\mathcal{H}^1(G') + k\rho \\ &= (\vartheta(\lambda, \gamma) + \varepsilon)\mathcal{H}^1(G) + k\rho, \end{aligned}$$

a contradiction with the hypothesis. □

We can now prove that ν_g is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^1$ on the algebra $\mathcal{A}_1$.

Lemma 5.4.2. *Let $g \in \mathscr{L}(\mathbb{S}^1)$ be symmetric and let $\lambda_0 = \|g\|_\infty$, $\gamma_0 = L(g)$. Then, for every pairwise disjoint semi-open intervals $I_1, \ldots, I_m \subseteq (0, \pi]$, we have*

$$\left| \nu_g \left(\bigcup_{j=1}^m I_j \right) \right| \leq (\Theta + 1)\mathcal{H}^1 \left(\bigcup_{j=1}^m I_j \right), \tag{5.4.2}$$

where Θ is defined as in Lemma 5.3.7. Therefore, $\nu_g \ll \mathcal{H}^1$ on $\mathcal{A}_1$, and in particular ν_g is bounded on $\mathcal{A}_1$.

Proof. We preliminarily prove the result for $m = 1$, i.e.,

$$|\nu_g(I)| \leq (\Theta + 1)\mathcal{H}^1(I), \tag{5.4.3}$$

for $I = (a, b] \subseteq (0, \pi]$.

Define

$$t_{\max} = \sup \left\{ t \in [a, b] : |\nu_g((a, t])| \leq (\Theta + 1)\mathcal{H}^1((a, t]) \right\}.$$

This set contains a (since V is null on constant functions), hence it is not empty and the definition is well-posed.

We claim that

$$|\nu_g((a, t_{\max}])| = (\Theta + 1)\mathcal{H}^1((a, t_{\max}]). \tag{5.4.4}$$

If $t_{\max} = a$, this is clearly true. Suppose $t_{\max} > a$. To prove (5.4.4), note that $g_{at} \underset{\tau}{\to} g_{at_{\max}}$, as $t \to t_{\max}$. Indeed, if $a \leq t \leq t_{\max}$ we have

$$\begin{aligned} \sup_{s\in(0,\,2\pi]} |g_{at}(s) - g_{at_{\max}}(s)| &= \sup_{s\in(t,\,\pi]} |g_{at}(s) - g_{at_{\max}}(s)| \\ &\leq \sup_{s\in(t,\,t_{\max}]} |g(t) - g(s)| + \sup_{s\in(t_{\max},\,\pi]} |g(t) - g(t_{\max})| \\ &\leq 2\gamma_0 \|t - t_{\max}\|, \end{aligned}$$

and if $t_{\max} < t \leq \pi$

$$\begin{aligned} \sup_{s\in(0,\,2\pi]} |g_{at}(s) - g_{at_{\max}}(s)| &= \sup_{s\in(t_{\max},\,\pi]} |g_{at}(s) - g_{at_{\max}}(s)| \\ &\leq \sup_{s\in(t_{\max},\,t]} |g(s) - g(t_{\max})| + \sup_{s\in(t,\,\pi]} |g(t) - g(t_{\max})| \\ &\leq 2\gamma_0 \|t - t_{\max}\|, \end{aligned}$$

hence $g_{at} \to g_{at_{\max}}$ uniformly on $(0, 2\pi]$. Let

$$J_t = \begin{cases} [t, t_{\max}] & \text{if } t < t_{\max}, \\ [t_{\max}, t] & \text{if } t > t_{\max}. \end{cases}$$

Then $g'_{at}(s) = 0 = g'_{at_{\max}}(s)$ for a.e. $s \in (0, 2\pi] \setminus J_t$, and $\mathcal{H}^1(J_t) \to 0$ as $t \to t_{\max}$, so that $g'_{at} \to g'_{at_{\max}}$ a.e. for $t \to t_{\max}$. Finally, $|g'_{at}| \leq \gamma_0$ a.e. in $(0, 2\pi]$. Therefore, $g_{at} \underset{\tau}{\to} g_{at_{\max}}$ as $t \to t_{\max}$.

So, letting $t \to t^-_{\max}$ in $|\nu_g((a, t])| \leq (\Theta + 1)\mathcal{H}^1((a, t])$, by the continuity of V we get

$$|\nu_g((a, t_{\max}])| \leq (\Theta + 1)\mathcal{H}^1((a, t_{\max}]).$$

For the other inequality, note that for every $t > t_{\max}$ we have

$$|\nu_g((a,t])| > (\Theta + 1)\mathcal{H}^1((a, t_{\max}]),$$

and use continuity again. This proves (5.4.4).

We would like to show that $t_{\max} = b$. Suppose this is not the case, i.e., $t_{\max} < b$. Consider the open interval $G = (t_{\max}, 2\pi - t_{\max})$ and let $\lambda_{\max} = g(t_{\max})$. Fix $0 < \varepsilon < 1$. From Proposition 5.3.6, there exists $\ell_0 > 0$ such that for every $f \in \mathrm{Lip}(\mathbb{S}^1)$ with $f \prec G$, $\|f\|_\infty \leq \ell_0$ and $L(f) \leq \gamma_0$, we have

$$|V_{\lambda_{\max}}(f)| \leq (\vartheta(\lambda_{\max}, \gamma_0) + \varepsilon)\, \mathcal{H}^1(G).$$

Since $t_{\max} < b$ and g is piecewise linear, we can choose $t_0 \in (0, \pi)$, $\alpha > 0$ such that $t_{\max} < t_0 < t_0 + \alpha < b$, $g|_{[t_{\max},\, t_0]}$ is monotone and satisfies $|g(t) - \lambda_{\max}| \leq \ell_0$ for every $t \in [t_{\max}, t_0]$, and

$$\mathcal{H}^1(G) = k\mathcal{H}^1(G'),$$

for some $k \in \mathbb{N}$, where $G' = (t_{\max}, 2t_0 + \alpha - t_{\max})$ (note that t_0 is independent of α). Suppose g to be increasing in $[t_{\max}, t_0]$; if $g|_{[t_{\max},\, t_0]}$ is decreasing we can argue similarly.

Note: the fact that g is piecewise linear allows us to choose t_0 such that $g|_{[t_{\max},\, t_0]}$ is monotone. This cannot be done for an arbitrary Lipschitz function f, and this is the only thing that prevents us from defining ν_f for a symmetric $f \in \mathrm{Lip}(\mathbb{S}^1)$.

From Lemma 5.4.1 we get that

$$|V_{\lambda_{\max}}(f)| \leq (\vartheta(\lambda_{\max}, \gamma_0) + \varepsilon)\mathcal{H}^1(G'), \tag{5.4.5}$$

for every $f \in \mathrm{Lip}(\mathbb{S}^1)$ such that $f \prec G'$, $\|f\|_\infty \leq \ell_0$ and $L(f) \leq \gamma_0$.

Consider the function $h = g_{t_{\max}\, t_0} - \lambda_{\max}$ and let σ be the rotation defined by

$$\sigma(t) = t - 2(\pi - t_0),$$

for $t \in (0, 2\pi]$. Note that $\sigma(2\pi - t_0) = t_0$. From the rotational invariance and the valuation property, we have that

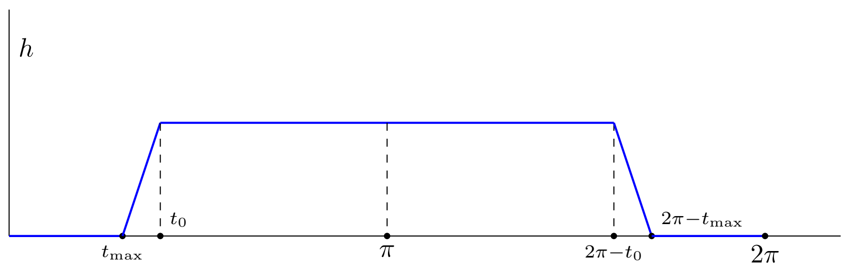
$$2V_{\lambda_{\max}}(h) = V_{\lambda_{\max}}(h) + V_{\lambda_{\max}}(h \circ \sigma) = V_{\lambda_{\max}}(h \vee (h \circ \sigma)) + V_{\lambda_{\max}}(h \wedge (h \circ \sigma)). \tag{5.4.6}$$

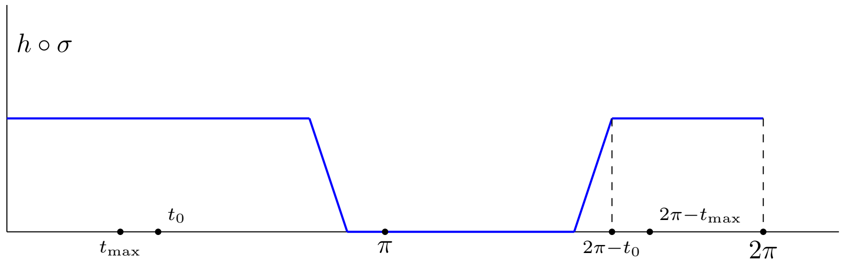
For $f \in \mathrm{Lip}(\mathbb{S}^1)$, define

$$\rho(f) = \mathcal{L}^1\left(\{t \in (0, 2\pi] : f(t) = g(t_0) - \lambda_{\max}\}\right),$$

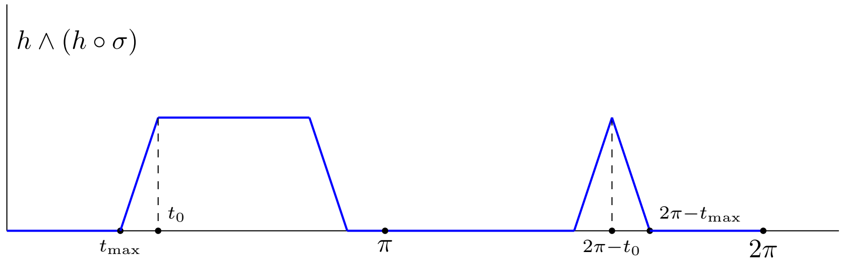
$$\rho_0(f) = \mathcal{L}^1\left(\{t \in (0, 2\pi] : f(t) = 0\}\right).$$

Note that $\rho(h) = 2(\pi - t_0)$, $\rho_0(h) = 2t_{\max}$. If, for instance, $\rho(h) > \rho_0(h)$, the graphs of our functions are of the following forms:





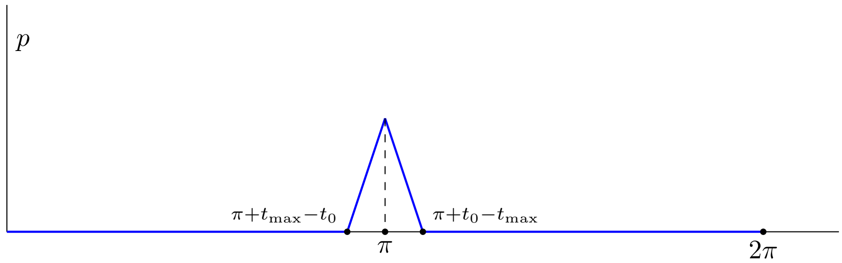
so that $h \vee (h \circ \sigma)$ is a constant function and $h \wedge (h \circ \sigma)$ is of the form

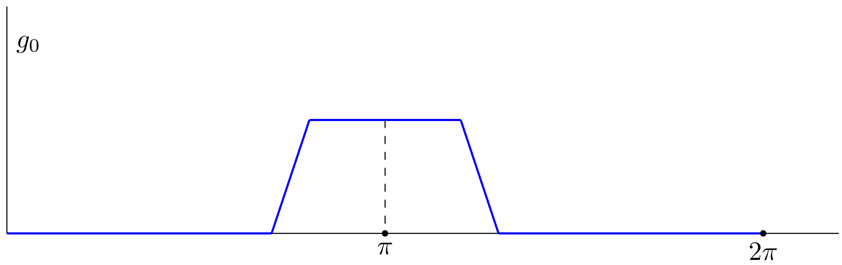


Since $V_{\lambda_{\max}}$ is null on constant functions, from (5.4.6), using the valuation property and the rotational invariance, we get

$$2V_{\lambda_{\max}}(h) = V_{\lambda_{\max}}(h \wedge (h \circ \sigma)) = V_{\lambda_{\max}}(p) + V_{\lambda_{\max}}(g_0), \tag{5.4.7}$$

where

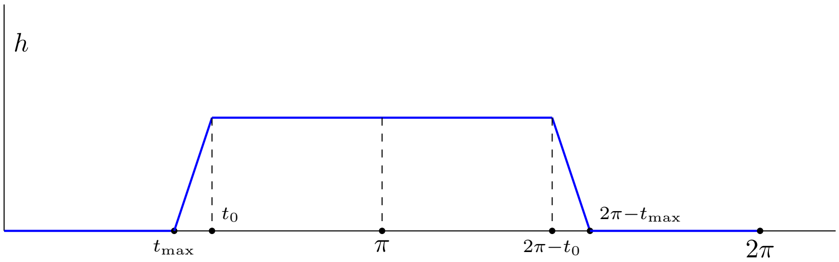




The function g_0 has the same form as h; we call these *pudding functions*. Also, note that $\rho(g_0) < \rho(h)$, $\rho_0(g_0) > \rho_0(h)$.

If $\rho(h) < \rho_0(h)$, we reason similarly, obtaining (5.4.7) again, with a different g_0 such that $\rho(g_0) > \rho(h)$, $\rho_0(g_0) < \rho_0(h)$.

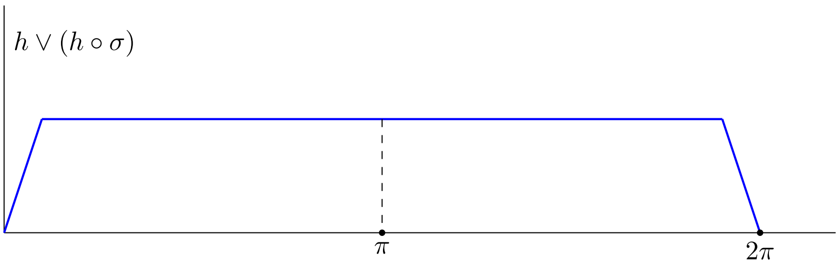
Finally, if $\rho(h) = \rho_0(h)$ we have



and we get

$$2V_{\lambda_{\max}}(h) = V_{\lambda_{\max}}(h \vee (h \circ \sigma)) + V_{\lambda_{\max}}(h \wedge (h \circ \sigma)),$$

where, up to translations, $h \wedge (h \circ \sigma) = p$ and



Setting $g_0 = h \vee (h \circ \sigma)$, which is still a pudding function, we find again

$$2V_{\lambda_{\max}}(h) = V_{\lambda_{\max}}(p) + V_{\lambda_{\max}}(g_0).$$

Since g_0 has, apart from its lengths ρ, ρ_0, the same exact structure as h, we can repeat the argument with g_0 replacing h. We obtain

$$2V_{\lambda_{\max}}(g_0) = V_{\lambda_{\max}}(p) + V_{\lambda_{\max}}(g_1),$$

where g_1 is a pudding function. Then

$$2V_{\lambda_{\max}}(h) = V_{\lambda_{\max}}(p) + V_{\lambda_{\max}}(g_0) = \frac{3}{2}V_{\lambda_{\max}}(p) + \frac{1}{2}V_{\lambda_{\max}}(g_1).$$

By induction, we have that for every $m \in \mathbb{N}$

$$2V_{\lambda_{\max}}(h) = \sum_{j=0}^{m} \frac{1}{2^j} V_{\lambda_{\max}}(p) + \frac{1}{2^m} V_{\lambda_{\max}}(g_m),$$

that is,

$$V_{\lambda_{\max}}(h) = \sum_{j=1}^{m+1} \frac{1}{2^j} V_{\lambda_{\max}}(p) + \frac{1}{2^{m+1}} V_{\lambda_{\max}}(g_m), \tag{5.4.8}$$

where g_m is a pudding function.

Since, by Lemma 5.1.1, $V_{\lambda_{\max}}$ is bounded on $\|\cdot\|_{\mathrm{Lip}}$-bounded sets, and

$$\|g_m\|_{\mathrm{Lip}} \leq \max\{\lambda_0, \gamma_0\}$$

for every $m \in \mathbb{N}$, we have that the second term in the right-hand side of (5.4.8) goes to zero as $m \to \infty$. Therefore, passing to the limit in (5.4.8) we find

$$V_{\lambda_{\max}}(h) = V_{\lambda_{\max}}(p),$$

where $p \in \mathrm{Lip}(\mathbb{S}^1)$ satisfies $\|p\|_\infty \leq \ell_0$ and $L(p) \leq \gamma_0$. Moreover, the support of p has measure smaller than $\mathcal{H}^1(G')$, so that $p \prec G'$, up to a rotation. From (5.4.5) we obtain

$$\begin{aligned} |\nu_g((t_{\max}, t_0])| &= \frac{1}{2}|V(g_{t_{\max}\, t_0})| = \frac{1}{2}|V_{\lambda_{\max}}(h)| = \frac{1}{2}|V_{\lambda_{\max}}(p)| \\ &\leq (\vartheta(\lambda_{\max}, \gamma_0) + \varepsilon) \cdot \frac{1}{2}\mathcal{H}^1((t_{\max}, 2t_0 + \alpha - t_{\max})) \\ &< (\Theta + 1)\mathcal{H}^1\left(\left(t_{\max}, t_0 + \frac{\alpha}{2}\right]\right). \end{aligned} \tag{5.4.9}$$

From the finite additivity of ν_g on consecutive intervals, (5.4.4), (5.4.9) and the finite additivity of $\mathcal{H}^1$ we have

$$\begin{aligned} |\nu_g((a, t_0])| &\leq (\Theta + 1)\mathcal{H}^1((a, t_{\max}]) + |\nu_g((t_{\max}, t_0])| \\ &< (\Theta + 1)\mathcal{H}^1((a, t_{\max}]) + (\Theta + 1)\mathcal{H}^1\left(\left(t_{\max}, t_0 + \frac{\alpha}{2}\right]\right) \\ &= (\Theta + 1)\mathcal{H}^1\left(\left(a, t_0 + \frac{\alpha}{2}\right]\right). \end{aligned}$$

Letting $\alpha \to 0^+$ we get a contradiction with the definition of $t_{\max}$. Thus $t_{\max} = b$, and from (5.4.4) we get (5.4.3) (with equality).

For the general case, note that for every pairwise disjoint semi-open intervals $I_1, \ldots, I_m$ we have, because of (5.4.3),

$$\left|\nu_g\left(\bigcup_{j=1}^m I_j\right)\right| = \left|\sum_{j=1}^m \nu_g(I_j)\right| \leq (\Theta + 1)\sum_{j=1}^m \mathcal{H}^1(I_j) = (\Theta + 1)\mathcal{H}^1\left(\bigcup_{j=1}^m I_j\right).$$

□

If we now consider the algebra

$$\mathcal{A}_2 = \left\{\bigcup_{j=1}^m I_j : m \in \mathbb{N},\, I_j = (a_j, b_j] \subseteq (\pi, 2\pi],\, I_j \cap I_k = \emptyset \text{ for } j \neq k\right\},$$

for a symmetric $g \in \mathscr{L}(\mathbb{S}^1)$ we can analogously build a finitely additive function ν_g on $\mathcal{A}_2$ such that (5.4.2) holds for every pairwise disjoint semi-open intervals $I_1, \ldots, I_m \subseteq (\pi, 2\pi]$.

For an arbitrary $g \in \mathscr{L}(\mathbb{S}^1)$, we can now define a function ν_g on the algebra

$$\mathcal{A} = \left\{\bigcup_{j=1}^m I_j : m \in \mathbb{N},\, I_j = (a_j, b_j] \subseteq (0, 2\pi],\, I_j \cap I_k = \emptyset \text{ for } j \neq k\right\},$$

which coincides with the algebra generated by the semi-open intervals in $(0, 2\pi]$, by setting

$$\nu_g\left(\bigcup_{j=1}^m I_j\right) := \nu_{g_1}\left(\bigcup_{j=1}^m (I_j \cap (0, \pi])\right) + \nu_{g_2}\left(\bigcup_{j=1}^m (I_j \cap (\pi, 2\pi])\right),$$

for every pairwise disjoint and semi-open intervals $I_1, \dots, I_m \subseteq (0, 2\pi]$, where g_1, g_2 are the symmetric extensions to $(0, 2\pi]$ of $g|_{(0,\pi]}$ and $g|_{(\pi,2\pi]}$ respectively. We have that ν_g is finitely additive and satisfies

$$|\nu_g| \leq (\Theta + 1)\mathcal{H}^1 \tag{5.4.10}$$

on $\mathcal{A}$. This allows us to extend ν_g to a signed measure.

Lemma 5.4.3. *For every $g \in \mathscr{L}(\mathbb{S}^1)$, ν_g can be extended to a signed measure on the σ-algebra*

$$\Sigma = \sigma\left(\{(a, b] : 0 \leq a \leq b \leq 2\pi\}\right)$$

generated by the semi-open intervals of $(0, 2\pi]$, which coincides with the Borel σ-algebra of $(0, 2\pi]$, and $\nu_g \ll \mathcal{H}^1$ on Σ.

Proof. Fix $g \in \mathscr{L}(\mathbb{S}^1)$. Inequality (5.4.10) implies the boundedness of ν_g on $\mathcal{A}$. From [5, Theorem 2.5.3, (1)-(9)], if we define

$$\nu_g^+(A) = \sup\{\nu_g(B) : B \subseteq A,\, B \in \mathcal{A}\},$$

$$\nu_g^-(A) = \sup\{-\nu_g(B) : B \subseteq A,\, B \in \mathcal{A}\},$$

for $A \in \mathcal{A}$, then ν_g^+, ν_g^- are non-negative and bounded charges (i.e., set functions which are null on the empty set and finitely additive on disjoint sets) such that $\nu_g = \nu_g^+ - \nu_g^-$. Note that $\nu_g^\pm(A) \leq (\Theta + 1)\mathcal{H}^1(A)$ for every $A \in \mathcal{A}$, hence $\nu_g^\pm \ll \mathcal{H}^1$ on $\mathcal{A}$.

Let us prove that ν_g^+ and ν_g^- are (bounded) pre-measures on the algebra $\mathcal{A}$: for pairwise disjoint sets $\{A_i\} \subseteq \mathcal{A}$ such that $A = \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}$, we have to show that

$$\nu_g^\pm\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \nu_g^\pm(A_i).$$

Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $\mathcal{H}^1(B) < \delta$ implies $\nu_g^\pm(B) < \varepsilon$, for every $B \in \mathcal{A}$. Since

$$\sum_{i \in \mathbb{N}} \mathcal{H}^1(A_i) = \mathcal{H}^1(A) < \infty,$$

there is a number $M \in \mathbb{N}$ such that

$$\mathcal{H}^1\left(\bigcup_{i=m}^{\infty} A_i\right) = \sum_{i=m}^{\infty} \mathcal{H}^1(A_i) < \delta$$

for every $m \geq M$. Now,

$$\bigcup_{i=m}^{\infty} A_i = \left(\bigcup_{i \in \mathbb{N}} A_i\right) \cap \left(\bigcup_{i=1}^{m-1} A_i\right)^c \in \mathcal{A},$$

hence

$$\nu_g^\pm\left(\bigcup_{i=m}^{\infty} A_i\right) < \varepsilon,$$

for $m \geq M$, that is,

$$\lim_{m \to \infty} \nu_g^\pm\left(\bigcup_{i=m}^{\infty} A_i\right) = 0.$$

From the finite additivity of $\nu_g^{\pm}$ on the algebra, we have

$$\nu_g^{\pm}\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i=1}^{m-1} \nu_g^{\pm}(A_i) + \nu_g^{\pm}\left(\bigcup_{i=m}^{\infty} A_i\right).$$

Letting $m \to \infty$ we conclude.

Thus ν_g^+, ν_g^- are pre-measures on the algebra $\mathcal{A}$. Theorem 2.1.1 then implies that they can be extended to measures on $\sigma(\mathcal{A})$, hence on Σ; this allows us to extend ν_g to a signed measure on Σ.

Let us now prove that $\nu_g \ll \mathcal{H}^1$ on Σ. It is enough to show that $\nu_g^{\pm} \ll \mathcal{H}^1$. Fix $\varepsilon > 0$. From the absolute continuity of $\nu_g^{\pm}$ on the algebra, we have that there exists a $\delta > 0$ such that, for every $B \in \mathcal{A}$, if $\mathcal{H}^1(B) < \delta$ then $\nu_g^{\pm}(B) < \varepsilon/2$.

Pick now $A \in \Sigma$ such that $\mathcal{H}^1(A) < \delta_0 := \delta/2$. By regularity of the Hausdorff measure, there exists an open set $U \supseteq A$ such that $\mathcal{H}^1(U \setminus A) < \delta/2$. Then

$$\mathcal{H}^1(U) = \mathcal{H}^1(A) + \mathcal{H}^1(U \setminus A) < \delta.$$

We can write $U = \bigcup_{j\in\mathbb{N}} I_j$, where the I_j's are pairwise disjoint open intervals. Note that

$$\sum_{j\in\mathbb{N}} \nu_g^{\pm}(I_j) = \nu_g^{\pm}(U) < \infty,$$

hence there exists $m \in \mathbb{N}$ such that

$$\sum_{j=m}^{\infty} \nu_g^{\pm}(I_j) < \frac{\varepsilon}{2}.$$

Now, if $I_j = (a_j, b_j)$ for every $j \in \mathbb{N}$, we have

$$\mathcal{H}^1\left(\bigcup_{j=1}^{m-1} (a_j, b_j]\right) = \mathcal{H}^1\left(\bigcup_{j=1}^{m-1} I_j\right) \le \mathcal{H}^1(U) < \delta,$$

with $\bigcup_{j=1}^{m-1} (a_j, b_j] \in \mathcal{A}$. By monotonicity and additivity of $\nu_g^{\pm}$, and using the fact that $\nu_g^{\pm}$ is null at singletons, we get

$$\nu_g^{\pm}(A) \le \nu_g^{\pm}(U) = \nu_g^{\pm}\left(\bigcup_{j=1}^{m-1} I_j\right) + \sum_{j=m}^{\infty} \nu_g^{\pm}(I_j) = \nu_g^{\pm}\left(\bigcup_{j=1}^{m-1} (a_j, b_j]\right) + \sum_{j=m}^{\infty} \nu_g^{\pm}(I_j) < \varepsilon.$$

□

This last lemma allows us to use Theorem 2.2.7, which yields the following.

Proposition 5.4.4. *For every $g \in \mathscr{L}(\mathbb{S}^1)$, there exists a function $D_g = \frac{d\nu_g}{d\mathcal{H}^1} \in L^1(\mathbb{S}^1)$ such that*

$$\nu_g(A) = \int_A D_g(t)\, d\mathcal{H}^1(t),$$

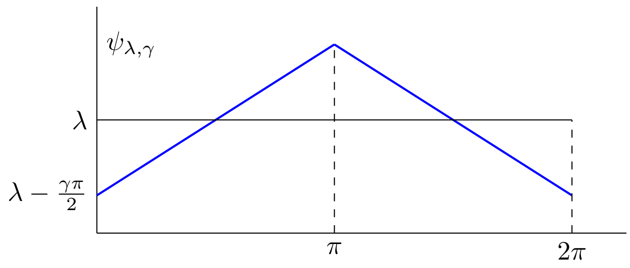
for every $A \in \Sigma$.

We will call this ν_g the *representing measure*, since it is the one which will give us the representation formula for the valuation on piecewise linear functions, as shown in the next section.

5.5 Representation formula on $\mathscr{L}(\mathbb{S}^1)$

For every $\lambda \in \mathbb{R}$, $\gamma \in \mathbb{R}^+$, we define the function

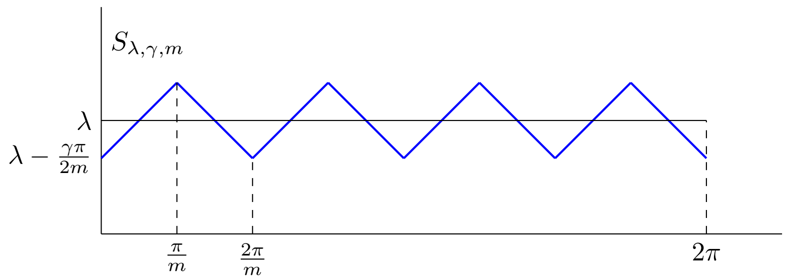
$$\psi_{\lambda,\gamma}(t) = \begin{cases} \lambda + \gamma(t - \frac{\pi}{2}) & \text{if } t \in [0, \pi], \\ \lambda + \gamma(\frac{3\pi}{2} - t) & \text{if } t \in [\pi, 2\pi]. \end{cases}$$



For fixed $\lambda \in \mathbb{R}$, $\gamma \in \mathbb{R}^+$ and $m \in \mathbb{N}$, let us consider the *saw function* $S_{\lambda,\gamma,m}$ obtained by joining m shrinked and translated copies of $\psi_{\lambda m,\gamma}$ as follows:

$$S_{\lambda,\gamma,m}(t) = \frac{1}{m} \sum_{j=1}^{m} \psi_{\lambda m,\gamma}(mt - 2(j-1)\pi)\chi_{\left(\frac{2(j-1)\pi}{m}, \frac{2j\pi}{m}\right]}(t),$$

for $t \in (0, 2\pi]$, and extend $S_{\lambda,\gamma,m}$ 2π-periodically to $\mathbb{R}$.



Note that, for every $m \in \mathbb{N}$, $|S'_{\lambda,\gamma,m}(t)| = \gamma$ for a.e. $t \in (0, 2\pi]$, and

$$\|S_{\lambda,\gamma,m}\|_\infty = \frac{1}{m} \max_{t \in \left(0, \frac{2\pi}{m}\right]} |\psi_{\lambda m,\gamma}(mt)| = \frac{1}{m} \max \left\{ |\psi_{\lambda m,\gamma}(0)|, \left|\psi_{\lambda m,\gamma}\left(\frac{\pi}{m}\right)\right| \right\} \leq |\lambda| + \frac{\gamma\pi}{2},$$

so that

$$\|S_{\lambda,\gamma,m}\|_{\mathrm{Lip}} \leq |\lambda| + \frac{\gamma\pi}{2}.$$

Thus, it follows from Lemma 5.1.1 that $\sup_{m \in \mathbb{N}} |V(S_{\lambda,\gamma,m})| < \infty$. We can then define

$$K(\lambda, \gamma) := C_0 \limsup_{m \to \infty} V(S_{\lambda,\gamma,m}), \tag{5.5.1}$$

where $C_0 > 0$ is the constant such that

$$\mathcal{H}^1 = \frac{1}{8\pi C_0} \mathcal{L}^1.$$

This kernel is very closely related to our Radon-Nikodym derivative, in the sense stated in the following lemma.

Lemma 5.5.1. *Let $\gamma \in \mathbb{R}^+$ and $0 \leq a < b \leq 2\pi$. If $g \in \mathscr{L}(\mathbb{S}^1)$ is such that $|g'(t)| = \gamma$ for a.e. $t \in (a, b]$, then*

$$K(g(t), \gamma) = D_g(t) \tag{5.5.2}$$

for a.e. $t \in (a, b]$.

Proof. Let $g \in \mathscr{L}(\mathbb{S}^1)$ be as in the hypothesis. Consider $(c, d) \subseteq (a, b]$ such that $g'(t) = \gamma$ for a.e. $t \in (c, d)$. By Theorem 2.2.8, we have that for a.e. $t \in (c, d)$

$$D_g(t) = \lim_{\varepsilon \to 0} \frac{1}{\mathcal{H}^1(t - \varepsilon, t + \varepsilon)} \int_{t-\varepsilon}^{t+\varepsilon} D_g(s) d\mathcal{H}^1(s).$$

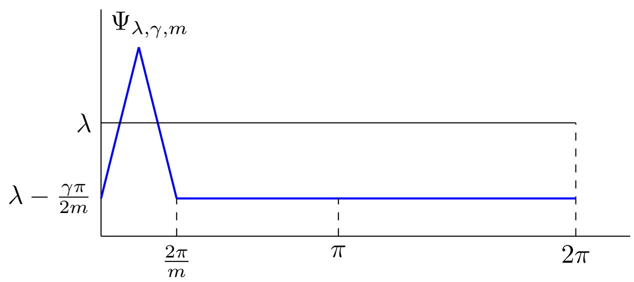
Take $t \in (c, d)$, $t \neq \pi$, such that this holds, and set $\lambda = g(t)$. By the inclusion-exclusion principle, remembering that V is null on constant functions and using the rotational invariance, for every $m \in \mathbb{N}$ we get

$$\begin{aligned}
V(S_{\lambda,\gamma,m}) &= V\left(\frac{1}{m}\sum_{j=1}^{m} \psi_{\lambda m,\gamma}(m \cdot -2(j-1)\pi)\chi_{\left(\frac{2(j-1)\pi}{m}, \frac{2j\pi}{m}\right]}\right) \\
&= V\left(\bigvee_{j=1}^{m}\left(\frac{1}{m}\psi_{\lambda m,\gamma}(m \cdot -2(j-1)\pi)\chi_{\left(\frac{2(j-1)\pi}{m}, \frac{2j\pi}{m}\right]} + \left(\lambda - \frac{\gamma\pi}{2m}\right)\chi_{\left(0, \frac{2(j-1)\pi}{m}\right] \cup \left(\frac{2j\pi}{m}, 2\pi\right]}\right)\right) \\
&= \sum_{j=1}^{m} V\left(\frac{1}{m}\psi_{\lambda m,\gamma}(m \cdot -2(j-1)\pi)\chi_{\left(\frac{2(j-1)\pi}{m}, \frac{2j\pi}{m}\right]} + \left(\lambda - \frac{\gamma\pi}{2m}\right)\chi_{\left(0, \frac{2(j-1)\pi}{m}\right] \cup \left(\frac{2j\pi}{m}, 2\pi\right]}\right) \\
&= mV\left(\frac{1}{m}\psi_{\lambda m,\gamma}(m \cdot)\chi_{\left(0, \frac{2\pi}{m}\right]} + \left(\lambda - \frac{\gamma\pi}{2m}\right)\chi_{\left(\frac{2\pi}{m}, 2\pi\right]}\right).
\end{aligned} \tag{5.5.3}$$

Now, let

$$\Psi_{\lambda,\gamma,m}(s) = \frac{1}{m}\psi_{\lambda m,\gamma}(ms)\chi_{\left(0, \frac{2\pi}{m}\right]}(s) + \left(\lambda - \frac{\gamma\pi}{2m}\right)\chi_{\left(\frac{2\pi}{m}, 2\pi\right]}(s), \quad s \in (0, 2\pi],$$

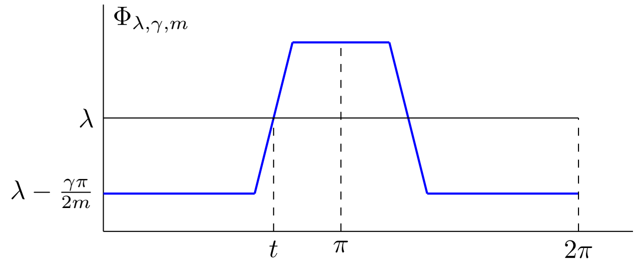
with $\Psi_{\lambda,\gamma,m}$ extended 2π-periodically as always.



If $t < \pi$, using rotational invariance and arguing as in the proof of Lemma 5.4.2, we have that the number $V(\Psi_{\lambda,\gamma,m})$ coincides with the value attained by the valuation at the function

$$\Phi_{\lambda,\gamma,m}(s) = \begin{cases} \lambda - \frac{\gamma\pi}{2m} & \text{in } (0, t - \frac{\pi}{2m}] \cup (2\pi - t + \frac{\pi}{2m}, 2\pi], \\ g(s) & \text{in } (t - \frac{\pi}{2m}, t + \frac{\pi}{2m}], \\ g(2\pi - s) & \text{in } (2\pi - t - \frac{\pi}{2m}, 2\pi - t + \frac{\pi}{2m}], \\ \lambda + \frac{\gamma\pi}{2m} & \text{in } (t + \frac{\pi}{2m}, 2\pi - t + \frac{\pi}{2m}], \end{cases}$$

whose graph is given by



if m is big enough so that $t + \frac{\pi}{2m} < \pi$. We proceed similarly in the case $t > \pi$.

From (5.5.3), using the definition of the representing measure and Proposition 5.4.4 we find

$$\begin{aligned} V(S_{\lambda,\gamma,m}) &= mV(\Psi_{\lambda,\gamma,m}) = mV(\Phi_{\lambda,\gamma,m}) = 2m\nu_g\left(\left(t - \frac{\pi}{2m}, t + \frac{\pi}{2m}\right]\right) \\ &= \frac{1}{C_0\mathcal{H}^1\left(\left(t - \frac{2\pi}{m}, t + \frac{2\pi}{m}\right]\right)} \int_{t-\frac{2\pi}{m}}^{t+\frac{2\pi}{m}} D_g(s)d\mathcal{H}^1(s), \end{aligned}$$

since

$$\mathcal{H}^1\left(\left(t - \frac{2\pi}{m}, t + \frac{2\pi}{m}\right]\right) = \frac{1}{8\pi C_0}\mathcal{L}^1\left(\left(t - \frac{2\pi}{m}, t + \frac{2\pi}{m}\right]\right) = \frac{1}{8\pi C_0} \cdot \frac{4\pi}{m} = \frac{1}{2mC_0}.$$

Thus, taking the limit superior for $m \to \infty$ we obtain (5.5.2) for a.e. $t \in (c, d)$. The case $g'(t) = -\gamma$ for a.e. $t \in (c, d)$ is analogous. $\square$

This allows us to prove the Borel measurability of $K(\cdot, \gamma)$, for every $\gamma \in \mathbb{R}^+$.

Remark 5.5.2. *Fix $\gamma \in \mathbb{R}^+$. For every $m \in \mathbb{Z}$, $\psi_{\gamma\pi m,\gamma}(0) = \frac{\gamma\pi(2m-1)}{2}$ and $\psi_{\gamma\pi m,\gamma}(\pi) = \frac{\gamma\pi(2m+1)}{2}$. By the intermediate value theorem, if $\lambda \in \left(\frac{\gamma\pi(2m-1)}{2}, \frac{\gamma\pi(2m+1)}{2}\right]$ then there exists $t \in [0, \pi]$ such that $\psi_{\gamma\pi m,\gamma}(t) = \lambda$. From Lemma 5.5.1,*

$$K(\lambda, \gamma) = \sum_{m\in\mathbb{Z}} D_{\psi_{\gamma\pi m,\gamma}}\left(\psi^{-1}_{\gamma\pi m,\gamma}(\lambda)\right) \chi_{\left(\frac{\gamma\pi(2m-1)}{2}, \frac{\gamma\pi(2m+1)}{2}\right]}(\lambda),$$

for every $\lambda \in \mathbb{R}$. As a consequence, we have that for every $\gamma \in \mathbb{R}^+$, $K(\cdot, \gamma)$ is a Borel function on $\mathbb{R}$ (and it is in fact integrable on every bounded interval).

We can finally prove Theorem 1.1.4.

Proof of Theorem 1.1.4. Let V be as in the hypothesis and let K be defined by (5.5.1). We preliminarily prove the representation formula for symmetric functions $g \in \mathscr{L}(\mathbb{S}^1)$. Fix such a g. Then $|g'|$ is piecewise constant, that is, there exists a partition

$$(0, 2\pi] = \bigcup_{i=1}^{\ell}(t_{i-1}, t_i],$$

with $t_0 = 0$, $t_\ell = 2\pi$, such that $|g'| = \gamma_i \in \mathbb{R}^+$ a.e. in $(t_{i-1}, t_i]$, for $i = 1, \ldots, \ell$. Consider the quantity $\nu_g((0, 2\pi])$. On the one hand, from Proposition 5.4.4 and Lemma 5.5.1 we have

$$\begin{aligned} \nu_g((0, 2\pi]) &= \int_0^{2\pi} D_g(t)d\mathcal{H}^1(t) = \sum_{i=1}^{\ell}\int_{t_{i-1}}^{t_i} D_g(t)d\mathcal{H}^1(t) = \sum_{i=1}^{\ell}\int_{t_{i-1}}^{t_i} K(g(t), |g'(t)|)d\mathcal{H}^1(t) \\ &= \int_0^{2\pi} K(g(t), |g'(t)|)d\mathcal{H}^1(t), \end{aligned}$$

and on the other hand, since g is symmetric with respect to $x = \pi$,

$$\nu_g((0, 2\pi]) = \nu_g((0, \pi]) + \nu_g((\pi, 2\pi]) = \frac{1}{2}V(g) + \frac{1}{2}V(g) = V(g).$$

This proves (1.1.3) for piecewise linear and symmetric functions.

Take now an arbitrary $g \in \mathscr{L}(\mathbb{S}^1)$. Recalling what we said at the beginning of Section 5.4, and denoting by g_R the reflection of g with respect to $x = \pi$, from the valuation property we get that

$$V(g) = \frac{1}{2}\big[V(g) + V(g_R)\big] = \frac{1}{2}\big[V(g \vee g_R) + V(g \wedge g_R)\big].$$

Now, $g \vee g_R$ and $g \wedge g_R$ are piecewise linear and symmetric. What we have seen in the first part of the proof then implies

$$\begin{aligned}
V(g) &= \frac{1}{2}\int_0^{2\pi} \big[K(g \vee g_R(t), |(g \vee g_R)'(t)|) + K(g \wedge g_R(t), |(g \wedge g_R)'(t)|)\big] d\mathcal{H}^1(t) \\
&= \frac{1}{2}\int_0^{2\pi} \big[K(g(t), |g'(t)|) + K(g_R(t), |g_R'(t)|)\big] d\mathcal{H}^1(t) \\
&= \frac{1}{2}\int_0^{2\pi} \big[K(g(t), |g'(t)|) + K(g(2\pi - t), |g'(2\pi - t)|)\big] d\mathcal{H}^1(t) \\
&= \frac{1}{2}\int_0^{2\pi} \big[K(g(t), |g'(t)|) + K(g(-t), |g'(-t)|)\big] d\mathcal{H}^1(t), \qquad (5.5.4)
\end{aligned}$$

where the last equality follows from the fact that g and g' are 2π-periodic. With the change of variable $t = 2\pi - s$ we get

$$\begin{aligned}
\int_0^{2\pi} K(g(-t), |g'(-t)|) d\mathcal{H}^1(t) &= -\int_{2\pi}^{0} K(g(s - 2\pi), |g'(s - 2\pi)|) d\mathcal{H}^1(s) \\
&= \int_0^{2\pi} K(g(s), |g'(s)|) d\mathcal{H}^1(s),
\end{aligned}$$

hence (5.5.4) gives

$$V(g) = \int_0^{2\pi} K(g(t), |g'(t)|) d\mathcal{H}^1(t),$$

as desired.

Formula (1.1.4) follows immediately from Proposition 3.0.2. $\square$

5.6 Uniformly continuous valuations on $\mathrm{Lip}(\mathbb{S}^1)$

If we now ask for uniform continuity (with respect to τ), we are able to prove a characterization result, namely Theorem 1.1.5. In order to do so, we start off by proving the following inequality.

Lemma 5.6.1. *Let $\{a_\ell\}, \{b_\ell\} \subseteq \mathbb{R}$ be such that $\sup_\ell a_\ell, \sup_\ell b_\ell < \infty$, where ℓ ranges in a countable set. Then*

$$\Big|\sup_\ell a_\ell - \sup_\ell b_\ell\Big| \leq \sup_\ell |a_\ell - b_\ell|.$$

Proof. For every i, we have

$$a_i = a_i - b_i + b_i \leq |a_i - b_i| + b_i \leq \sup_\ell |a_\ell - b_\ell| + \sup_\ell b_\ell,$$

and taking the supremum in i we get

$$\sup_\ell a_\ell - \sup_\ell b_\ell \leq \sup_\ell |a_\ell - b_\ell|.$$

Swapping the roles of a_i, b_i and repeating the same reasoning we also find

$$\sup_\ell b_\ell - \sup_\ell a_\ell \leq \sup_\ell |a_\ell - b_\ell|,$$

and then we are done. □

We can now prove the uniform continuity of our kernel K, under the assumption of uniform continuity on the valuation.

Lemma 5.6.2. *If $V : \mathrm{Lip}(\mathbb{S}^1) \longrightarrow \mathbb{R}$ is a uniformly continuous and rotation invariant valuation, then the kernel $K : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by* (5.5.1) *is uniformly continuous.*

Proof. By contradiction, suppose that there exists a number $\varepsilon > 0$ such that for every $\delta > 0$ there are $(\lambda, \gamma), (\widetilde{\lambda}, \widetilde{\gamma}) \in \mathbb{R} \times \mathbb{R}^+$ satisfying

$$\sqrt{\left(\lambda - \widetilde{\lambda}\right)^2 + \left(\gamma - \widetilde{\gamma}\right)^2} < \delta$$

and

$$\left|K(\lambda, \gamma) - K(\widetilde{\lambda}, \widetilde{\gamma})\right| > \varepsilon.$$

We can then build two sequences $\{(\lambda_i, \gamma_i)\}, \{(\widetilde{\lambda}_i, \widetilde{\gamma}_i)\} \subseteq \mathbb{R} \times \mathbb{R}^+$ such that $\lambda_i - \widetilde{\lambda}_i \to 0$, $\gamma_i - \widetilde{\gamma}_i \to 0$ as $i \to \infty$, and

$$\left|K(\lambda_i, \gamma_i) - K(\widetilde{\lambda}_i, \widetilde{\gamma}_i)\right| > \varepsilon$$

for every $i \in \mathbb{N}$. By definition of K and Lemma 5.6.1, we get

$$\frac{\varepsilon}{C_0} < \lim_{m \to \infty} \left| \sup_{\ell \geq m} V(S_{\lambda_i, \gamma_i, \ell}) - \sup_{\ell \geq m} V\left(S_{\widetilde{\lambda}_i, \widetilde{\gamma}_i, \ell}\right) \right| \leq \limsup_{m \to \infty} \left| V(S_{\lambda_i, \gamma_i, m}) - V\left(S_{\widetilde{\lambda}_i, \widetilde{\gamma}_i, m}\right) \right|. \tag{5.6.1}$$

We claim that, for a fixed $m \in \mathbb{N}$,

$$S_{\lambda_i, \gamma_i, m} - S_{\widetilde{\lambda}_i, \widetilde{\gamma}_i, m} \underset{\tau}{\to} \mathbb{O}, \tag{5.6.2}$$

as $i \to \infty$. To prove this we estimate, for $i \in \mathbb{N}$ and $t \in (0, 2\pi]$,

$$\begin{aligned}
\left|S_{\lambda_i, \gamma_i, m}(t) - S_{\widetilde{\lambda}_i, \widetilde{\gamma}_i, m}(t)\right| &\leq \frac{1}{m} \max_{s \in [0, 2\pi]} \left|\psi_{\lambda_i m, \gamma_i}(s) - \psi_{\widetilde{\lambda}_i m, \widetilde{\gamma}_i}(s)\right| \cdot \sum_{j=1}^m \chi_{\left(\frac{2(j-1)\pi}{m}, \frac{2j\pi}{m}\right]}(t) \\
&= \frac{1}{m} \max_{s \in [0, \pi]} \left|\psi_{\lambda_i m, \gamma_i}(s) - \psi_{\widetilde{\lambda}_i m, \widetilde{\gamma}_i}(s)\right| \\
&= \frac{1}{m} \left|\psi_{\lambda_i m, \gamma_i}(s_{i,m}) - \psi_{\widetilde{\lambda}_i m, \widetilde{\gamma}_i}(s_{i,m})\right|,
\end{aligned}$$

for suitable $s_{i,m}$'s in $[0,\pi]$, where

$$\begin{aligned}\left|\psi_{\lambda_i m,\gamma_i}(s_{i,m}) - \psi_{\widetilde{\lambda}_i m,\widetilde{\gamma}_i}(s_{i,m})\right| &= \left|\lambda_i m + \gamma_i\left(s_{i,m} - \frac{\pi}{2}\right) - \widetilde{\lambda}_i\, m - \widetilde{\gamma}_i\left(s_{i,m} - \frac{\pi}{2}\right)\right| \\ &\leq m\left|\lambda_i - \widetilde{\lambda}_i\right| + \frac{\pi}{2}\left|\gamma_i - \widetilde{\gamma}_i\right|.\end{aligned}$$

This implies

$$\left|S_{\lambda_i,\gamma_i,m}(t) - S_{\widetilde{\lambda}_i,\widetilde{\gamma}_i,m}(t)\right| \leq \left|\lambda_i - \widetilde{\lambda}_i\right| + \frac{\pi}{2}\left|\gamma_i - \widetilde{\gamma}_i\right|,$$

so that

$$S_{\lambda_i,\gamma_i,m} - S_{\widetilde{\lambda}_i,\widetilde{\gamma}_i,m} \to \mathbb{O}$$

uniformly on $(0,2\pi]$, as $i\to\infty$.

As for the a.e. convergence of the derivatives, note that the set

$$B = \left\{t \in (0,2\pi] : S'_{\lambda_i,\gamma_i,m}(t),\ S'_{\widetilde{\lambda}_i,\widetilde{\gamma}_i,m}(t) \text{ exist for every } i,\, m\in\mathbb{N}\right\}$$

has full measure, since the functions $S_{\lambda_i,\gamma_i,m}$, $S_{\widetilde{\lambda}_i,\widetilde{\gamma}_i,m}$ are differentiable on the set $(0,2\pi]\setminus\left\{\frac{j\pi}{m} : j = 1,\dots,2m\right\}$, for every $i,m\in\mathbb{N}$. For $t\in B$,

$$\left|S'_{\lambda_i,\gamma_i,m}(t) - S'_{\widetilde{\lambda}_i,\widetilde{\gamma}_i,m}(t)\right| = \left|\gamma_i - \widetilde{\gamma}_i\right| \to 0$$

as $i\to\infty$, and

$$\left|S'_{\lambda_i,\gamma_i,m}(t) - S'_{\widetilde{\lambda}_i,\widetilde{\gamma}_i,m}(t)\right| \leq \max\{\gamma_1 - \widetilde{\gamma}_1,\dots,\gamma_J - \widetilde{\gamma}_J, 1\},$$

where J is such that $|\gamma_i - \widetilde{\gamma}_i\,| \leq 1$ for every $i > J$. This proves (5.6.2).

Since V is uniformly continuous, there exists a neighbourhood $U\subseteq \mathrm{Lip}(\mathbb{S}^1)$ of $\mathbb{O}$ such that

$$f_1 - f_2 \in U \Rightarrow |V(f_1) - V(f_2)| < \frac{\varepsilon}{C_0}.$$

From what we have just proved, we have that there exists $I\in\mathbb{N}$ independent of m such that $S_{\lambda_i,\gamma_i,m} - S_{\widetilde{\lambda}_i,\widetilde{\gamma}_i,m} \in U$ for $i > I$ and every $m\in\mathbb{N}$, hence

$$\left|V(S_{\lambda_i,\gamma_i,m}) - V\left(S_{\widetilde{\lambda}_i,\widetilde{\gamma}_i,m}\right)\right| < \frac{\varepsilon}{C_0}$$

for every $i > I$ and $m\in\mathbb{N}$. Inequality (5.6.1) yields a contradiction. □

The kernel is also bounded on compact sets.

Lemma 5.6.3. *For every $\Lambda\in\mathbb{R}$, $C>0$, there exists $M>0$ such that $|K(\lambda,\gamma)|\leq M$, for every $(\lambda,\gamma)\in[-\Lambda,\Lambda]\times[0,C]$.*

Proof. For $(\lambda,\gamma)\in[-\Lambda,\Lambda]\times[0,C]$ we have

$$|K(\lambda,\gamma)| = C_0 \lim_{m\to\infty}\left|\sup_{\ell\geq m} V(S_{\lambda,\gamma,\ell})\right| \leq C_0 \lim_{m\to\infty}\sup_{\ell\geq m}|V(S_{\lambda,\gamma,\ell})|,$$

where, for every $\ell\geq m$,

$$\|S_{\lambda,\gamma,\ell}\|_{\mathrm{Lip}} \leq \Lambda + \frac{\pi C}{2}.$$

By Lemma 5.1.1, there exists a constant $M > 0$ (depending on Λ and C) such that

$$|V(S_{\lambda,\gamma,\ell})| \leq \frac{M}{C_0},$$

for every $\ell \geq m$, hence

$$|K(\lambda, \gamma)| \leq M.$$

□

We can now prove our last characterization result.

Proof of Theorem 1.1.5. A kernel which is uniformly continuous always gives rise to a uniformly continuous and rotation invariant valuation. To see this, thanks to Lemma 4.3.1, we just have to prove that if K is uniformly continuous and $V : \mathrm{Lip}(\mathbb{S}^1) \to \mathbb{R}$ is defined by

$$V(f) = \int_0^{2\pi} K(f(t), |f'(t)|) d\mathcal{H}^1(t),$$

for $f \in \mathrm{Lip}(\mathbb{S}^1)$, then V is uniformly continuous.

Because K is uniformly continuous, for a fixed $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\big|K(\lambda_1, \gamma_1) - K(\lambda_2, \gamma_2)\big| < \frac{\varepsilon}{\mathcal{H}^1((0, 2\pi])},$$

for every $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\gamma_1, \gamma_2 \in \mathbb{R}^+$ such that

$$\sqrt{(\lambda_1 - \lambda_2)^2 + (\gamma_1 - \gamma_2)^2} < \delta.$$

Let

$$U = \left\{ f \in \mathrm{Lip}(\mathbb{S}^1) : \sqrt{f(t)^2 + |f'(t)|^2} < \delta \text{ for a.e. } t \in (0, 2\pi] \right\}.$$

This set is a neighbourhood of $\mathbb{O}$ with respect to τ. Take $f_1, f_2 \in \mathrm{Lip}(\mathbb{S}^1)$ such that $f_1 - f_2 \in U$. Then

$$|V(f_1) - V(f_2)| \leq \int_0^{2\pi} \big|K(f_1(t), |f_1'(t)|) - K(f_2(t), |f_2'(t)|)\big| d\mathcal{H}^1(t) < \varepsilon,$$

and we are done.

On the other hand, let $V : \mathrm{Lip}(\mathbb{S}^1) \longrightarrow \mathbb{R}$ be a uniformly continuous and rotation invariant valuation. By Theorem 1.1.4, for every $f \in \mathrm{Lip}(\mathbb{S}^1)$ we can write

$$V(f) = \lim_{i \to \infty} \int_0^{2\pi} K(f_i(t), |f_i'(t)|) d\mathcal{H}^1(t), \tag{5.6.3}$$

where $\{f_i\} \subseteq \mathscr{L}(\mathbb{S}^1)$ is such that $f_i \underset{\tau}{\to} f$ as $i \to \infty$.

Noting that the pair $(f_i(t), |f_i'(t)|)$ belongs to a compact set $[-\Lambda, \Lambda] \times [0, C]$ for every $i \in \mathbb{N}$ and a.e. $t \in (0, 2\pi]$ (because of τ-convergence), we can use Lemma 5.6.3 to find that the integrand in (5.6.3) is dominated by an integrable function (in fact, a constant one). Thanks to Lemma 5.6.2, we can apply the dominated convergence theorem in (5.6.3) to obtain (1.1.5). □

5.7 Counterexamples to Conjecture 1.1.1

Theorem 1.1.5 shows that if the kernel is uniformly continuous, then the valuation defined by it is uniformly continuous, and vice versa. We also know (see Lemma 4.3.1) that a continuous kernel gives rise to a continuous valuation. However, the continuity of the valuation is not enough to guarantee the continuity of the kernel, as the following example shows.

Example 5.7.1. *Consider the kernel $K : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by*

$$K(\lambda, \gamma) = |\gamma| \cdot \chi_{[1,\infty)}(\lambda),$$

for $\lambda \in \mathbb{R}$, $\gamma \in \mathbb{R}^+$. Let $V : \mathrm{Lip}(\mathbb{S}^1) \longrightarrow \mathbb{R}$ be the valuation associated with this kernel, that is,

$$V(f) = \int_0^{2\pi} |f'(t)|\chi_{[1,\infty)}(f(t))d\mathcal{H}^1(t), \tag{5.7.1}$$

for $f \in \mathrm{Lip}(\mathbb{S}^1)$.

Take $\{f_i\} \subseteq \mathrm{Lip}(\mathbb{S}^1)$ such that $f_i \underset{\tau}{\rightarrow} f \in \mathrm{Lip}(\mathbb{S}^1)$. We have

$$\lim_{i\to\infty} V(f_i) = \lim_{i\to\infty} \left[\int_{\{f\neq 1\}} |f_i'(t)|\chi_{[1,\infty)}(f_i(t))d\mathcal{H}^1(t) + \int_{\{f=1\}} |f_i'(t)|\chi_{[1,\infty)}(f_i(t))d\mathcal{H}^1(t) \right].$$

If $t \in (0, 2\pi]$ is such that $f(t) < 1$, assume by contradiction that $\chi_{[1,\infty)}(f_i(t)) \not\to \chi_{[1,\infty)}(f(t)) = 0$ as $i \to \infty$. Then, for every $j \in \mathbb{N}$, there exists $f_{i_j} \in \mathrm{Lip}(\mathbb{S}^1)$ such that

$$\left|\chi_{[1,\infty)}(f_{i_j}(t))\right| > \frac{1}{j}.$$

This implies $\chi_{[1,\infty)}(f_{i_j}(t)) = 1$, for every $j \in \mathbb{N}$, i.e., $f_{i_j}(t) \geq 1$. Letting $j \to \infty$ we get a contradiction. We can similarly prove that $\chi_{[1,\infty)}(f_i(t)) \to \chi_{[1,\infty)}(f(t))$ if $f(t) > 1$. From the τ-convergence and the dominated convergence theorem we obtain

$$\begin{aligned} \lim_{i\to\infty} V(f_i) &= \int_{\{f\neq 1\}} |f'(t)|\chi_{[1,\infty)}(f(t))d\mathcal{H}^1(t) + \lim_{i\to\infty} \int_{\{f=1\}} |f_i'(t)|\chi_{[1,\infty)}(f_i(t))d\mathcal{H}^1(t) \\ &= V(f) + \lim_{i\to\infty} \int_{\{f=1\}} |f_i'(t)|\chi_{[1,\infty)}(f_i(t))d\mathcal{H}^1(t), \end{aligned}$$

where the last equality is due to Corollary 1 from [13, Section 3.1]. Now,

$$|f_i'(t)|\chi_{[1,\infty)}(f_i(t)) \leq |f_i'(t)|,$$

where, as $i \to \infty$, $|f_i'(t)| \to |f'(t)| = 0$, for a.e. t such that $f(t) = 1$, again by the aforementioned Corollary. So, $|f_i'(t)|\chi_{[1,\infty)}(f_i(t)) \to 0$ for a.e. t such that $f(t) = 1$, and from the dominated convergence theorem we have that the last limit in the previous chain of equalities is null, hence V is continuous.

The kernel defined in the example above is such that $K(\lambda, \cdot)$ is continuous for every $\lambda \in \mathbb{R}$ and $K(\cdot, \gamma)$ is only a.e. continuous for every $\gamma \in \mathbb{R}^+$. Nonetheless, the valuation defined by (5.7.1) is continuous: this might lead us to think that Lemma 4.3.1 is improvable, and that it would still hold for a kernel $K : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ which is continuous in one variable and only a.e. continuous in the other one. Unfortunately, this is not the case, as shown in the next example. We will actually prove something more, and discuss it afterwards.

Example 5.7.2. *Consider $K : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by*

$$K(\lambda, \gamma) = \chi_{(0,\infty)}(\lambda),$$

for $\lambda \in \mathbb{R}$, $\gamma \in \mathbb{R}^+$. This kernel is clearly continuous in γ (for fixed λ) and a.e. continuous in λ (for fixed γ). The valuation V defined by

$$V(f) = \int_0^{2\pi} K(f(t), |f'(t)|) d\mathcal{H}^1(t) = \int_0^{2\pi} \chi_{(0,\infty)}(f(t)) d\mathcal{H}^1(t),$$

for $f \in \mathrm{Lip}(\mathbb{S}^1)$, is not continuous, since the sequence $f_i \equiv \frac{1}{i}$ converges to $\mathbb{O}$ with respect to τ and

$$\lim_{i\to\infty} V(f_i) = \lim_{i\to\infty} \int_0^{2\pi} \chi_{(0,\infty)} \left(\frac{1}{i}\right) d\mathcal{H}^1(t) = \mathcal{H}^1((0, 2\pi]) \neq 0 = V(\mathbb{O}).$$

Take now $K : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by

$$K(\lambda, \gamma) = \begin{cases} \dfrac{1}{\gamma} \cdot \chi_{(0,\infty)}(\gamma) & \text{if } \gamma > 0, \\ \\ 0 & \text{if } \gamma = 0. \end{cases}$$

The function $K(\cdot, \gamma)$ is obviously continuous, for every fixed $\gamma \in \mathbb{R}^+$. Note that, for every fixed $\lambda \in \mathbb{R}$, $K(\lambda, \cdot)$ is a.e. continuous. Indeed, $K(\lambda, \cdot)$ is continuous on $(0, \infty)$ but it is not continuous at $\gamma = 0$, since

$$K\left(\lambda, \frac{1}{i}\right) = i \to \infty \neq 0 = K(\lambda, 0),$$

as $i \to \infty$. Let $V : \mathrm{Lip}(\mathbb{S}^1) \longrightarrow \mathbb{R}$ be the valuation given by

$$V(f) = \int_0^{2\pi} K(f(t), |f'(t)|) d\mathcal{H}^1(t),$$

for $f \in \mathrm{Lip}(\mathbb{S}^1)$. The sequence $f_i = \psi_{1,1/i}$, where $\psi_{\lambda,\gamma}$ is defined as in Section 5.5, satisfies $f_i \underset{\tau}{\to} \mathbb{1}$, but

$$\begin{aligned} \lim_{i\to\infty} V(f_i) &= \lim_{i\to\infty} \int_0^{2\pi} K\left(f_i, \frac{1}{i}\right) d\mathcal{H}^1(t) = \lim_{i\to\infty} \int_0^{2\pi} i \cdot \chi_{(0,\infty)} \left(\frac{1}{i}\right) d\mathcal{H}^1(t) \\ &= \lim_{i\to\infty} i \cdot \mathcal{H}^1((0, 2\pi]) = \infty \neq 0 = \int_0^{2\pi} K(1, 0) d\mathcal{H}^1(t) = V(\mathbb{1}). \end{aligned}$$

This proves that V is not continuous.

Example 5.7.1 showed that Conjecture 1.1.1 does not hold for K continuous (i.e., if $(P) =$ continuity). The kernels defined in Example 5.7.2 are a.e. continuous in one of the two variables and even constant in the other one, but this is not enough to guarantee the continuity of the valuation $V : \mathrm{Lip}(\mathbb{S}^1) \longrightarrow \mathbb{R}$ defined by

$$V(f) = \int_0^{2\pi} K(f(t), |f'(t)|) d\mathcal{H}^1(t), \quad f \in \mathrm{Lip}(\mathbb{S}^1).$$

This makes it harder to find a suitable property (P) for the conjecture to hold.

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