Pluralism in Proof-Theoretic Semantics

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Contents

Introduction

Proof-theoretic semantics is a well-established inferentialist theory of meaning that develops ideas proposed by Prawitz and Dummett. The main aim of this theory is to find a foundation of logic based on some aspects of the linguistic use of the logical terms, as opposed to the regular foundation offered by a model-theoretic approach *à la* Tarski, in which the denotation of non-linguistic entities is central.

Traditionally, intuitionistic logic is considered justified in proof-theoretic semantics (although some doubts are raised regarding *ex falso quodlibet*). In the first chapter of this thesis I give a general introduction of logical inferentialism and a summary of the standard development of proof-theoretic semantics. In the first section about inferentialism I already discuss the possibility of a dependence relation between the meaning of the logical terms, which is not considered in standard proof-theoretic semantics, but only in some of its recent modifications, as we will see in the second chapter.

Even though proof-theoretic semantics has made big progresses in the last decades, it remains nonetheless controversial the existence of a justification of classical logic that suits its restraints. In the second chapter I examine various proposals that try to give such a justification: Prawitz's proposal of a classical rule of *reductio*, Milne's recent formulation via general introduction rules, Rumfitt's bilateral systems, Hacking's and Restall's proposals to base semantical investigation on sequent calculus instead of natural deduction, Boričić's and Read's proposals of a multiple-conclusion formulation of natural deduction and Milne's and Prawitz's different proposals of rejecting purity and simplicity as definitional requirements of I-rules. I conclude that only the last proposal has some chances of success, but that it needs some serious change in order to work. Indeed I argue (*contra* the mainstream opinion of some experts in proof-theoretic semantics) that a sound objection used to reject Boričić's and Read's solutions based on multiple conclusions leads inevitably to the abandonment of multiple assumptions as well. Given this position, the proposal of accepting meaning-conferring rules in which more than one connective occur has to be endorsed even in order to justify intuitionistic logic.

In the third chapter, I develop a new version of proof-theoretic semantics in order to deal with this important change. I manage to prove the following results for the single-assumption single-conclusion formulation of both classical and intuitionistic logic:

Weak separability: To prove a logical consequence $A \vdash B$ we only need to use the rules for the logical constants that occur in A and B, together with the rules for the constants on which those depend.

Harmony: For every derivation $A \vdash B$, there is an equivalent derivation in which there are no *maximal formulae*.

In order to prove the last result, I adapt the notion of *maximal formulae* to our new context and downgrade the usual requirement of normalization to the bare existence of normal form. Both changes are defended in the thesis. I then discuss the possibility of giving an explicit proof-theoretic definition of validity in this new framework. While there are no problems for intuitionistic logic, some issues are discussed regarding classical logic.

Since there seems to be no reason to exclude that more than one logic suit the *desiderata* of this revised proof-theoretic semantics, I conclude the thesis with a disquisition about logical pluralism in chapter 4, that is, the thesis that more than one logic is correct. In spite of the general antirealist approach about meaning endorsed throughout the entire thesis, I evaluate the acceptability of such a pluralist position also according to an agnostic position regarding meaning and in a realist framework. I conclude that the question cannot be settled without specifying a precise theory of meaning but that logical pluralism suits both realism and antirealism. However, while according to the realist pluralism two logics can disagree and nonetheless speak of the same logical terms, according to the antirealist version of this thesis two logics that disagree on the behaviour of a logical term speak of different entities. As a consequence, in this framework the possibility of a disagreement in logic is admitted only for applications, *à la* Carnap.

I try to isolate the proof of formal results in appendix B, in order to render the thesis more fluid. In appendix A, I give a schematic presentation of the systems developed in the thesis. The thesis focuses on natural deduction systems, and we skip to sequent calculus only to make easier some proofs. The meaning-theoretical reasons for doing so are explained in section 2.4.1.

Notation and Preliminaries

I will use capital Latin letters to denote formulae, lowercase Greek letters to denote real sentences and lowercase Latin letters to denote real atomic sentences. So, while 'A' is just used to designate the possible occurrence of a sentence, 'γ' and 'p' should be intended as real sentences, like 'it is raining' and 'it is raining and I am sad'.

I will speak of a "rule of inference" or simply of a "rule" to refer to the general schema of inference, in which only schematic sentences and formulae occur and only the logical constants that are essential to the application of the rule are displayed. As an example, *Modus Ponendo Ponens* is the rule of inference:

$$MPP \frac{A \supset B \qquad A}{B}$$

With "inference" or "application of a rule" I refer instead to the examples of a rule of inference, that is, the result of a uniform substitution of sentences or formulae in place of the schematic formulae of the rule. So an application can be fully formal, as in

$$MPP \frac{A \wedge B \supset B \qquad A \wedge B}{B}$$

or it can regard real sentences, like in

$$MPP \frac{1 \in \mathbb{N} \supset 2 \in \mathbb{N} \qquad 1 \in \mathbb{N}}{2 \in \mathbb{N}}$$

We will frequently refer to real sentences specifying only their logical structure, that is, using lowercase Greek and Latin letters. The applications of rules constructed in this way are to be considered as real inferences regarding sentences, as opposed to inferences constructed using formulae.

I will use capital Greek letters to denote, sets, multisets, and lists of formulae or sentences. I will specify each time which of these readings is the right one.

Chapter 1

Inferentialism

1.1 Different kinds of inferentialism and their problems

Inferentialism is a theory of meaning that considers the inferential role of a linguistic object as the main constitutive part of its meaning.[1] There are different grades and flavours of inferentialism, but here we are interested only in a very precise kind of inferentialist theories of meaning for the logical constants: proof-theoretic semantics. Nonetheless, we will be very general in this first chapter, in order to evaluate some alternatives generally underestimated by standard proof-theoretic semantics.

1.1.1 Natural Deduction

We will deal meanly with natural deduction systems, but we will also use sequent calculus as an instrument to prove some metatheorems for the systems of the first kind.

Natural deduction gives the following very elegant formulation of minimal logic:

$$\wedge\mathrm{I}\ \frac{A \qquad B}{A \wedge B} \qquad \wedge\mathrm{E}\ \frac{A \wedge B}{A} \qquad \wedge\mathrm{E}\ \frac{A \wedge B}{B} \qquad \vee\mathrm{I}\ \frac{A}{A \vee B} \qquad \vee\mathrm{I}\ \frac{B}{A \vee B}$$

$$\vee\mathrm{E}\ \frac{A \vee B \qquad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C} \qquad \supset\mathrm{I}\ \frac{\begin{matrix}[A]\\ \vdots\\ B\end{matrix}}{A \supset B} \qquad \supset\mathrm{E}\ \frac{A \supset B \qquad A}{B} \qquad \neg\mathrm{I}\ \frac{\begin{matrix}[A]\\ \vdots\\ \bot\end{matrix}}{\neg A} \qquad \neg\mathrm{E}\ \frac{\neg A \qquad A}{\bot}$$

Intuitionistic logic can be obtained by extending minimal logic with either of these rules:

$$\textit{ex falso quodlibet}\ \frac{\bot}{A} \qquad \text{disjunctive syllogism}\ \frac{\neg A \qquad A \vee B}{B}$$

We call **NJ** the system obtained extending minimal logic with *ex falso quodlibet.* Classical logic can be obtained adding either of the following rules to intuitionistic logic or either of the first two rules to minimal logic:

$$\textit{classical reductio}\ \frac{\begin{matrix}[\neg A]\\ \vdots\\ \bot\end{matrix}}{A} \qquad \neg\neg\mathrm{E}\ \frac{\neg\neg A}{A} \qquad \textit{tertium non datur}\ \frac{}{A \vee \neg A} \qquad \textit{dilemma}\ \frac{\begin{matrix}[A]\\ \vdots\\ B\end{matrix} \qquad \begin{matrix}[\neg A]\\ \vdots\\ B\end{matrix}}{B}$$

These are the standard natural deduction systems for minimal, intuitionistic and classical logic.[2] Let us notice that only one of the rules for classical logic can be easily categorised as an I or an E-rule: we will see that this fact leads to some interesting problems in the justification of this logic. Moreover, even though $\neg\neg$E is unquestionably an elimination rule, it has a quite strange structure nonetheless since several occurrences of negation are removed. On the contrary, *ex falso quodlibet* can arguably be considered as an E-rule for $\bot$. In the development of this work, we will display also other, more exotic systems for these logics.

[1] [Brandom, 2000]

[2] I take this overview from [Milne, 1994], p. 51-52.

1.1.2 Holism, molecularism and atomism

Let us now consider the inferentialist reading of natural deduction. Generally speaking, in an inferentialist approach the logical rules define the meaning of the logical constants they are about. Nonetheless, it is not necessary for all of them to define the meaning of the logical constant, it can also be defined by a proper subset of them. When this is the case, we distinguish between:

- meaning-conferring rules;
- non-meaning-conferring rules.

When the second class is not empty, we need justification for its elements. That is, if a rule is not justified because it gives meaning to a logical constant – so it is not valid by definition – it must be validated by something else. We will deal later with the problem of non-meaning-conferring rules, now let us analyse the choices we can do about meaning-conferring ones.

Meaning of sentences and meaning of terms

First of all, let us point out a preliminary observation. Since inferences deal with sentences (or judgements, or propositions - we will not distinguish between these alternatives here) but not directly with terms, every inferentialist theory of meaning treats directly the meaning of sentences and only indirectly those of terms.

Definition 1.1.1 (Meaning of a term)**.** The meaning of a term of the language $\mathcal{L}$ in a sentence is the contribution made by it to the meaning of the sentence.

So the meaning of the sentence comes first, and the meaning of the terms can be reconstructed from it.[3]

Of course, we will not search for a complete picture of the relation between the meaning of terms and the meaning of sentences. It is enough for our purposes to focus only on the logical terms, and we will be satisfied with an informal specification of the relation between the meaning of the logical sentences and the meaning of the logical terms that occur in them. Some explicit reconstruction of the relations between logical (and also non-logical) terms and sentences have been proposed by Nissim Francez and Gilad Ben-Avi for proof-theoretic semantics, but we will be happy with the bare statement that the rules that give meaning to the logically complex sentences also give meaning to the logical terms.[4]

In the language as a whole

Inferentialism, like other theories of meaning, is compatible with different positions concerning the relation between the meaning of different sentences. We will first of all focus on the dependence relation between the meanings of all the sentences (logical and non-logical) of the language. In general, we can easily detect two very extreme positions, and subsequently all the intermediate ones.

Definition 1.1.2 ((General) Meaning holism)**.** The meaning of every sentence of the language $\mathcal{L}$ depends on the meaning of every other sentence of the language.

This position has been endorsed by Wittgenstein and by Quine, and arises from the observation that it is not generally possible to isolate the meaning of a term from that of every other term.[5]

To reject this thesis, philosophers have usually endorsed an opposite position:

Definition 1.1.3 ((General) Meaning atomism)**.** The meaning of every sentence of the language $\mathcal{L}$ is independent of the meaning of every other sentence of the language, apart from its sub-sentences.

So, according to this position, the meaning of atomic sentences should be independent one from another.

We can see that both positions are, at least in principle, acceptable in inferentialism, if we focus on the fact that we do not have any condition on the rules that define the meaning. Of course, a bare *sufficient* condition to have meaning holism in inferentialism is that every sentence of the language

[3] This strange inversion in the priority of the elements of the language is one of the great heredity of Frege. The meaning of a sentence can depend on the meaning of a term, but this must depend on the meaning of another sentence. In other words, the chain of dependences has to be rooted in sentences and not in terms (to see how strong is the heredity of this intuition in the analytical tradition, consider that also Kripke's theory of 'initial baptism' for the "meaning" of the names makes sentences more primitive than names).

[4] [Francez and Ben-Avi, 2011].

[5] [Quine, 1951].

occurs in a meaning-conferring rule of each given sentence. Since the meaning of terms is given by the meaning of sentences, this circularity will involve also the terms.

On the contrary, a *necessary* condition to have meaning atomism is that there are no meaning-conferring rules that have non-schematic occurrences of atomic formulae. This, of course, is possible only if every meaning-conferring rule of an atom is a non-linguistic act, as a pure act of observation or something similar.[6] Another *necessary* condition for atomism is that only sub-sentences of the conclusion can occur in the meaning-conferring rules for non-atomic sentences.

There are reasons why none of these alternatives is really satisfying for a general theory of meaning. One of the main results of this work will be the conclusion that also for the logical fragment of the language, none of these alternatives is acceptable.

I think a more satisfying approach is that of molecularism:

Definition 1.1.4 ((General) Meaning molecularism)**.** The meaning of every sentence of the language $\mathcal{L}$ is independent of the meaning of some other sentences.

This position has been endorsed explicitly by Dummett and Tennant and is now the standard position in proof-theoretic semantics.[7] If expressed in this way, molecularity is a very vague requirement for a theory of meaning – it says nothing about which sentences determine the meaning of a given sentence –, but it is not easy to find good proposals for a specification of this position. Whilst, on the one hand, it is clear that we want to restrict the dependence of the meaning of a sentence from that of only related and less complex sentences, on the other hand, we need a clear specification of this intuitive requirement. Dummett seems to point at a system of different semantic clusters that should explain how the meanings of sentences are related each other but, in our case, we are happy with a very weak formulation of this position:

Definition 1.1.5 ((General) (Minimal) Meaning molecularism)**.** The dependence relation between the sentences of the language $\mathcal{L}$ is not circular.

The idea behind this reformulation is that the problem with holism is that we could never learn a language the meaning of which is holistic, since in order to understand a sentence we should already know the meaning of all the other sentences. A similar impossibility arises when there is a circular dependence of meaning. Indeed it is obvious that if the meaning of a language is holistic, then, given two sentences α and β, in order to understand the meaning of α we need to know the meaning of β and in order to understand the meaning of β we need to know the meaning of α, so circularity of meaning follows from holism. As a consequence, the rejection of circularity of meaning entails (at least a weak version of) molecularism.

Someone could argue that, since molecularism does not entail non-circularity, we should not neglect the possibility of a molecular and circular language.[8] As an example, we could have a language with atomistic meanings for all its sentences apart from γ and δ, with the meaning of γ depending on the meaning of δ and *vice versa*. While it is technically true that there is such a possibility, it seems that the same philosophical reasons why we want a molecular meaning prevent the acceptability of its circularity.[9] So this possibility suits the letter but not the spirit of molecularism, and we will not consider it here.

Although we will deal exhaustively only with the logical vocabulary, we still need to take some positions regarding meaning in general, at least in order to specify what is the relation between the meaning of logical and non-logical terms. For example, a further reason to reject meaning holism regarding the language as a whole is the commonly accepted thesis that logic is both autonomous and innocent with respect to the meaning of non-logical terms: to grasp the meaning of a logical term we do not need to grasp that of a non logical one, and *vice-versa*. This is the first consequence of meaning holism that we can reject concerning logic. We could take position also regarding another kind of holism, that deals only with the meaning of logical terms, and asks: is the meaning of a logical constant dependent from that of every other logical constant? We will see in section 1.1.3 that our position regarding these two issues imposes severe restrictions on the systems, when considered together with the analyticity of logic.

[6] Although the naiï̈f idea of a pure (that is not inferential) act of observation is something we should dismantle, according to a lot of contemporary epistemology, we have to consider the fact that here we are interested only to *linguistic* inferences.

[7] [Tennant, 1987] and [Dummett, 1991].

[8] [Steinberger, 2011a], p. 631 poses a similar criticism.

[9] Dummett too explicitly rejects circularity of meaning, as Steinberger recognises. See [Dummett, 1991], p. 257, that we will use in detail in chapter 2.

In the logical language

Regarding pure logical language, meaning holism is usually rejected, and it is commonly believed that at least for intuitionistic logic we can devise an atomistic theory of meaning: the meaning of every logical constant is independent of the meaning of the others. The standard formulation **NJ** in natural deduction seems to achieve such a result, since in it every logical constant has its set of rules, and there is no intersection between them. Given this property, *a fortiori* in a meaning-conferring rule for, as an example, conjunction only conjunction will occur.

Nonetheless, we will see in chapter 2 that meaning atomism is only apparent for standard natural deduction formulation of intuitionistic logic, and that an atomistic theory of meaning can lead us only to a much weaker logic. In order to save intuitionistic logic and classical logic, we need a more permissive position:

Definition 1.1.6 ((Special) Meaning molecularism)**.** The dependence relation between the meaning of the logical constants is not circular.

We can make this position more precise by defining a relation $\prec$ of dependence of meanings:[10]

Definition 1.1.7 (dependence of meaning)**.** $\ominus \prec \oplus$ (the meaning of $\oplus$ depends on the meaning of $\ominus$) iff there is a sequence of logical terms $\circ_1, \ldots, \circ_n$ such that $\circ_1 = \ominus$, $\circ_n = \oplus$ and for every $1 \leqslant i < n$ $\circ_i$ occurs in the premisses or in the discharged assumptions of a meaning-conferring rule for $\circ_{i+1}$.

Using this definition, we can offer a precise formulation of the positions regarding relations between meanings:

Definition 1.1.8 ((Special) Meaning positions)**.** A logical system can give meaning to its constants according to one of the following positions:

atomism: for every logical constant $\oplus$, there is no distinct logical constant $\ominus$ such that $\ominus \prec \oplus$;

holism: for every pair of distinct logical constants $\oplus$ and $\ominus$, it holds that $\ominus \prec \oplus$;

molecularism: for every pair of distinct logical constants $\oplus$ and $\ominus$, it does not hold that both $\ominus \prec \oplus$ and $\oplus \prec \ominus$.

From its definition, it follows that $\prec$ is a transitive relation (for every triple of logical constants $\odot$, $\oslash$, $\otimes$ if $\odot \prec \oslash$ and $\oslash \prec \otimes$ then $\odot \prec \otimes$). Nonetheless, it is important to notice that it is not anti-symmetric (for every pair of logical constants $\odot$, $\oslash$ if $\odot \prec \oslash$ and $\oslash \prec \odot$ then $\odot = \oslash$); indeed we want to be able to distinguish between different logical terms also in a holistic theory, in which they all depends from each other.[11] That is we want to be able to treat languages in which there are pairs of connectives $\circ$ and $\bullet$ such that $\circ \prec \bullet$ and $\bullet \prec \circ$, but nonetheless $\bullet \neq \circ$. The reason for this is the following: also in a language with holistic meaning or in which the meaning of a logical term depends on the logical term itself, logical constants are not identical each other; as an example, although we found out that the only good theory of meaning for classical logic is holistic, $p \vee \neg p$ still remains a classical logical law, while $p \wedge \neg p$ still remains a the negation of a theorem. Even though we consider holistic and circular meaning as a defect for a language, they are not equivalent to triviality.

An interesting example of this possibility is the following:

$$\frac{\begin{array}{c}[A]\\ \vdots \\ \dagger\end{array}}{\frown A}\frown\mathrm{I} \qquad\qquad \frac{A \qquad \frown A}{\dagger}\dagger\mathrm{I}$$

If these are all and the only meaning-conferring rules for $\frown$ and $\dagger$, then the meaning of each of them depends on that of the other, that is $\frown \prec \dagger$ and $\dagger \prec \frown$. By transitivity, we can "close the circle" and derive $\frown \prec \frown$ and $\dagger \prec \dagger$, so the meaning is not molecular. Nonetheless, $\frown$ and $\dagger$ can not be identical each other, since $\dagger$ is zeroary while $\frown$ is unary. Moreover, these two rules have the same structure of $\neg$I and $\neg$E, just the second is interpreted as an introduction rule for absurdity. Of course I do not want to defend this interpretation of the rules for negation, nonetheless, it is important to notice that

[10] Cozzo investigates something similar, even though for non-logical terms ([Cozzo, 1994a], pp. 246-250, [Cozzo, 2002], pp. 32-34 and [Cozzo, 2008b], p. 305), while Milne and Prawitz develop independently the same idea for logical terms ([Milne, 2002] and [Prawitz, 2015a]). As we will see, Dummett evaluated this possibility as well, but without developing the details ([Dummett, 1991], pp. 256-258).

[11] Neither it can be symmetric (for every pair of logical constants $\odot$, $\oslash$ if $\odot \prec \oslash$ then $\oslash \prec \odot$), since the meaning of $\oplus$ can depend on the meaning of $\ominus$ although this last constant is atomistic in meaning.

the identity of $\neg$ and $\bot$ is not a consequence of it. Indeed, we have different theorems for these two terms, whatever interpretation we give of their rules.

Also the fact that $\prec$ is neither reflexive (for every logical constant $\odot$, $\odot \prec \odot$), nor anti-reflexive (for every logical constant $\odot$, $\odot \not\prec \odot$) is not inessential, since we want to distinguish between a logical constant that depends on itself and a logical constant that does not depends on itself. That is, the dependence of a logical term from itself should neither be obvious – such that also constants that do not depend on any other constants still depend on themselves – nor impossible – such that no constants can depend on itself –, but simply possible. As an example of the difference between those two situations, let us consider the following rules:[12]

$$\star\text{I}\ \frac{A}{A \star B} \qquad\qquad \ast\text{I}\ \frac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix} \qquad \begin{matrix}[A] \\ \vdots \\ \ast B\end{matrix}}{\ast A}$$

Of course, we assume that these are all the meaning-conferring rules for $\star$ and $\ast$, so they are both atomistic in meaning. The first rule poses no problem of circularity since $\star$ does not occur in any assumption or premise of $\star$I. So $\star$ does not depend on itself, that is $\star \not\prec \star$. The second represents instead the degenerate case in which a constant depends on itself since $\ast$ occur in the right premise of $\ast$I and so $\ast \prec \ast$. The distinction between these two situations is made possible by the rejection of reflexivity for $\prec$ (indeed $\star$ does not suit reflexivity) and anti-reflexivity (indeed $\ast$ does not suit anti-reflexivity).

We saw that the definition of $\prec$ does not prevent self-dependence of meaning and that we can have direct self-dependence when a term occurs in a premise or in an assumption of its I-rule, or an indirect self-dependence, when we have a chain of terms $\circ_1, \ldots, \circ_n$ such that $\circ_1 = \circ_n$ and such that $\circ_i$ occurs in a premise or in an assumption of $\circ_{i+1}$I. Now we have to deal with acceptability in general of logical constants that depend on themselves. That is, although they are not excluded by the definition of $\prec$, they raise nonetheless the same meaning-theoretical issues that we saw when we were defining general meaning holism, so it could be unclear whether we should reject $\ominus \prec \ominus$ or not. The three definitions do not speak of this particular kind of self-dependence, so technically speaking we could have atomistic and molecular languages in which this phenomenon occurs. It is also obvious that in a holistic language, every logical constant depends on itself, since for every pair of constants $\ominus \prec \oplus$ and $\oplus \prec \ominus$, so by transitivity we have both $\ominus \prec \ominus$ and $\oplus \prec \oplus$.

In order to decide on the acceptability of self-dependent constants, let us consider more closely its relation with holism. As we saw in the previous section, what strikes us about holism is that if the meaning of every logical constant depends on that of each other, there is no place where we can start to investigate in order to search for the meaning of one such constant. The problem is obvious when we speak of teaching or learning a logical constant, but it is really more fundamental: it is not even clear how the rules can give meaning to the constants. For this reason, if we consider constants like $\ast$ as atomistic – since their meaning does not depends on that of other constants –, we lose the main reason for the distinction between atomistic/molecularistic/holistic theories of meaning.

In our discussion about meaning in the whole language, we decided to identify holism with circularity, since this was enough accurate for our main topic. In this case, we can not follow the same path, if we want to give a precise analysis of the meaning of logical terms. So we leave the definitions without a clause for self-dependent constants, but we define another semi-independent requirement:

Definition 1.1.9 (non-circularity)**.** For every logical constant $\ominus$, $\ominus \not\prec \ominus$.

As we have already established a non-circular language can not be holistic, since holism entails circularity. Nonetheless, the inverse entailment does not work, since if we extend a holistic language with some fresh atomistic rules for a new constant then we obtain a non-holistic language (since the new constant does not depend on each other) in which nonetheless we have circularity of meaning. Anyway, since the reason for accepting non-circularity requirement is the same as for accepting molecularity (and so *a fortiori* atomicity) of meaning, we will always pair these requirements.[13]

[12] The first one is obviously the standard rule of introduction of the disjunction, while the second is the rule proposed on p. 89 of [Dummett, 2000] for introducing intuitionistic negation. We will shortly discuss the strange status of this rule.

[13] This situation is not limited to self-dependent constants: in general, the three cases of atomistic/molecularistic/holistic languages are not exhaustive since we can have fusions of them. The fact that the extra requirement of non-circularity is almost independent of these three positions makes the situation still more complex. Nonetheless, we will deals only with some kinds of languages, avoiding the 'monsters' like circular atomistic languages fused with a holistic one.

In this thesis, we will consider as well-defined all molecular and non-circular languages. So we will assume the restriction: for every pair of (distinct or not) logical constants $\oplus$ and $\ominus$, it does not hold that both $\ominus \prec \oplus$ and $\oplus \prec \ominus$. Despite this very precise specification, we will sometimes use "molecular" and "non-holistic" to refer to languages that are both molecular and non-circular, now that our position has been pointed out.

Circularity in rules and in inferences In this paragraph, we want to point out one of the main differences between our non-circularity requirement and other similar principles that can be found in the literature. As already remarked, the example of I$*$ violates our requirement, but it has the same structure of the meaning-conferring rule proposed in [Dummett, 2000] for the intuitionistic negation. So Dummett completely disregards problems of circularity in definition? Of course, he does not, but he imposes another requirement, focused on inferences instead of rules:[14]

> "[...] the minimal demand we should make on an introduction rule intended to be self-justifying is that its form be such as to guarantee that, in any application of it, the conclusion will be of higher logical complexity than any of the premisses and than any discharged hypothesis."

A self-justifying rule is the same of a meaning-conferring rule. Apart from other differences, that are natural since our two frameworks are not the same, this requirement is not enough to exclude I$*$ from the class of meaning-conferring rules. Indeed Dummett focuses on *applications of rules* (that is *inferences*), and not on *rules* when he asks for non-circularity.

We could think that Dummett's requirement is just stronger than our requirement, since an application of the rule in which the conclusion is less complex than some assumptions or some premises is enough to reject an I-rule, but this is not completely right. Indeed if we focus on applications of rules, it is natural to consider acceptable all inferences in which the conclusion is more complex than both premises and assumptions, regardless of the form of the rule from which we obtain the inferences. In this way we can accept applications of I$*$

$$\dfrac{\begin{matrix}[\alpha] \\ \vdots \\ \beta\end{matrix} \qquad \begin{matrix}[\alpha] \\ \vdots \\ *\beta\end{matrix}}{*\alpha}$$

in which α is more complex than β. Of course, Dummett adds that this must hold "in any application of it" (the I-rule), but this does not lead necessarily to our restriction of non-circularity. Indeed the applications of a rule that violate Dummett's requirement could be useless, that is it could be possible to restrict the application of the rule in such a way to save Dummett's requirement without affecting the provable results. In this way, we have a rule that does not suit our requirement, but that suits Dummett's one, and so our principle is more demanding, at least in this case. As an example, for Dummett's $*$-like negation it can be proved that it is always possible to rearrange a proof in such a way to satisfy his requirement, so we could restrict this rule in such a way to suit Dummett's *criterion*.[15]

Nonetheless, I think that Dummett's requirement is too weak in this regard because, although this pair of rules gives a well-founded dependence relation between complex sentences in which the same logical constants occur, it does not deal satisfactorily with the definition of logical terms. Sure enough, if a meaning-conferring rule for a logical constant uses the same constant out of the conclusion, we will always need to already know its meaning in order to define it. And this holds true from the point of view both of rules and of inferences.[16] This is the reason why I decided to impose a non-circularity requirement already for the rules. Later, we will consider other restrictions on the complexity of the sentences that can occur in meaning-conferring inferences.

We showed that there are rules that suit Dummett's complexity principle but not our non-circularity principle, so we could think that our principle is stronger than Dummett's one. To be precise, we can also show some rules that suit our non-circularity principle but that nonetheless do not suit Dummett's complexity principle. Indeed let us consider the following rule:

[14] [Dummett, 1991], p. 258.

[15] It should be a corollary to normalization theorem and so to the division of the proof-tree in an E-part and an I-part. See [Read, 2015] for a precise analysis of this rule.

[16] This is not all Dummett's rule fault. Peter Milne points out that a similar problem always shows up with negation: [Milne, 1994], p. 61. Nonetheless, I think that Prawitz's proposal of dealing with negation as a defined connective is more promising also in this case since the logical status of absurdity is more flexible.

$$\bowtie\text{I}\ \frac{C \qquad \begin{matrix}[C]\\ \vdots \\ A\end{matrix} \qquad \begin{matrix}[C]\\ \vdots \\ B\end{matrix}}{A \bowtie B}$$

This is a completely acceptable I-rule from our point of view, since the only logical term that occurs in it is in the conclusion. Nonetheless, from the point of view of its applications, we do not have any restrictions, so the assumptions and premises used in place of C could be more complex than the conclusion and, as a consequence, it does not suit Dummett's complexity requirement. In conclusion, Dummett's complexity principle and our non-circularity principle are incomparable with each other.

1.1.3 Separability

An important remark is that general atomism/molecularism/holism can be completely independent of special atomism/molecularism/holism. Indeed while the first thesis regards the relations between sentences (and so at most real inferences), the second regards the relations between terms in rules. For example, we can accept general holism, by assuming that the meaning of each atomic sentence depends on the meaning of every other atomic sentence, and still accept special atomism, if meaning-conferring rules are purely schematic and just one logical term occurs in each of them. In this way, the meaning of every, possibly logically complex, sentence φ depends on that of every other sentence ψ, but the meaning of each logical term $\oplus$ is independent of that of every other logical term $\otimes$. As an example, according to a theory of meaning of this kind the meaning of "the ship is on the sea and the tree is in the wood" depends on the meaning of "the dog is on the armchair or the Moon is in the sky", while the meaning of "and" and "or" are independent each other.

To keep apart positions regarding general and special meaning relations, we have to accept autonomy and innocence of logic, so that the meaning of logical terms is independent on those of non-logical terms, and *vice-versa*. So it is a general molecularist position that firmly separates general and special positions regarding meaning. We will accept the autonomy of logic in this thesis, by accepting only parametric formulae in rules,[17] and innocence of the logic by assuming that no logical term is used to characterise the meaning of an atomic sentence. These assumptions are necessary if we want to make sense of our previous definition of molecularity. Indeed without them, we could have a circular dependence of meaning without violating non-circularity requirement, since the dependence of a logical term on another one can pass through a sentence that occurs non-schematically in their meaning-conferring rules. As an example, let us consider the following rules:

$$\triangledown\text{I}\ \frac{A \qquad \varphi}{A \triangledown B} \qquad \varphi\text{Rule}\ \frac{\alpha \vartriangle \beta}{\varphi} \qquad \vartriangle\text{I}\ \frac{A \triangledown C \qquad B}{A \vartriangle B}$$

Here A and B are schematic sentences, while α, β and φ are real sentences of our non-logical language. Our non-circularity requirement is ineffective to prevent this kind of situations if we let sentences occur in a rule in a non-schematic way. Indeed $\prec$ deals only with logical terms that occur in the meaning-conferring logical rules, and from this point of view $\triangledown \prec \vartriangle$ and $\vartriangle \nprec \triangledown$, since I$\triangledown$ is a purely atomistic meaning-conferring rule and $\vartriangle$I uses $\triangledown$ in one of its premises. Nonetheless, if φRule is a meaning-conferring rule for the atomic sentence φ (violating innocence of logic), the meaning of $\triangledown$ depends on that of φ (violating autonomy of logic) and the meaning of φ depends on that of $\vartriangle$. So we have both $\triangledown \prec \vartriangle$, and $\triangledown$ that depends on $\vartriangle$, that is a kind of circularity that escape from our non-circularity requirement regarding $\prec$. Of course, we want to avoid it and so we endorse innocence and autonomy of logic.[18]

There is a general lesson that we should learn from this episode: handling meaning-conferring rules we can decide to define the meaning of a logical term from some other terms or not, but in order to have a language that follows our decision we have to be able to prove some structural properties of the language. The autonomy of logic and special meaning molecularism are theses about what gives meaning to the logical terms, and I think that together they give ground to a promising framework for inferentialism. Nonetheless, these two positions can not be enough if we want to recognise:

Observation 1.1.1 (Analyticity of logic)**.** *Every logical consequence is valid in virtue of the meaning of the logical constants that occur in it.*

---

[17]The difference between a parametric occurrence is obvious if we consider the difference between an occurrence of '3' and an occurrence of 'x' in an arithmetical equation.

[18]$\bot$ has to be considered as a logical term - if we want to accept the autonomy of logic -, since it occurs non-schematically in the rules for negation. We will discuss this topic later.

First of all, since we decided to discern between meaning-conferring rules and non-meaning-conferring rules, we will need some kind of justification of the latter if we want logical consequences to be analytic. Indeed it is obvious that we will use also these rules in order to prove the validity of logical consequences and, in general, these applications will be unavoidable.

Once we have justified the non-meaning-conferring rules, in order to gain the analyticity of logic, we have to establish two formal results regarding our systems: weak separability property and weak subformula property.

Definition 1.1.10 (Weak separability)**.** To prove a logical consequence $A \vdash B$ we only need to use the rules for the logical constants that occur in A and B, together with the rules for the constants on which those depend. That is in order to prove a logical consequence $A \vdash B$, it is enough to use the rules for the constants $\circ_1, \ldots, \circ_n$ such that for every $1 \leqslant i \leqslant n$, $\circ_i$ occurs in A or B, or for some $j \neq i$ such that $1 \leqslant j \leqslant n$, $\circ_j$ occurs in A or B and $\circ_i \prec \circ_j$.

In this way the logical truth depends only from the logical terms we are referring to in our statement. Unfortunately, weak separability is not a trivial consequence of which logical terms occur in the meaning-conferring rules, so it is not a trivial consequence of meaning molecularism.

As an example, let us consider the following formulation of classical logic, and in particular the rules for negation and implication:

$$\supset\text{I}\ \frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B} \qquad \supset\text{E}\ \frac{A \supset B \qquad A}{B} \qquad \neg\neg\text{E}\ \frac{\neg\neg A}{A} \qquad \bot_i\ \frac{\bot}{A} \qquad \neg\text{I}\ \frac{\begin{matrix}[A]\\ \vdots \\ \bot\end{matrix}}{\neg A} \qquad \neg\text{E}\ \frac{\neg A \qquad A}{B}$$

Where the introduction of implication and of negation are considered the only meaning-conferring rules, and the other rules are justified by them.[19] From the point of view of meaning dependence $\prec$, there seems to be no circularity in the relation of dependence between the logical constants, since $\supset$ is completely atomistic and $\neg$ depends only on $\bot$.

Nonetheless, if we consider that all the rules except $\neg\neg$E are accepted also in intuitionistic logic, it is obvious that Peirce's law $(A \supset B) \supset A \vdash A$ must be derived using it, since it is a purely classical law. So in order to prove a logical consequence that only regards $\supset$, we need to use a rule for $\neg$ and probably also a rule for $\bot$, even though $\bot \not\prec \supset$ and $\neg \not\prec \supset$. So this is a case in which there is no circularity between the meaning-conferring rules and they are molecular, but, nonetheless, weak separability is violated. The reason for this possibility rests on the fact that, while molecularity deals only with meaning-conferring rules, weak separability deals also with non-meaning-conferring rules. So if we want to have separability as a property that follows from molecularity, we need to impose some restrictions on the procedure of justification for the non-meaning-conferring rules. For now, we will just accept separability as an extra requirement, but we will see that in proof-theoretic semantics it become a provable property of some logical systems.

Another good example of this phenomenon is **NJ** plus Prior's connective *tonk*, in which each rule expresses an atomistic behaviour, since only a connective occurs in each of them, but the system violates weak separability:[20]

$$tonk\text{I}\ \frac{A}{A tonk B} \qquad tonk\text{E}\ \frac{A tonk B}{B}$$

For example, if we have $A = C \vee D$ and $B = C \wedge D$, we obtain $C \vee D \vdash C \wedge D$, that can not be proved using only rules for $\wedge$ and $\vee$. So we have only two choices: we can reject the analyticity of logic and accept *tonk* and $C \vee D \vdash C \wedge D$, or we can accept analyticity of logic and reject *tonk* together with $C \vee D \vdash C \wedge D$. Of course, our choice is to reject *tonk* and accept weak separability and analyticity of logic, but to have a precise analysis of what is wrong with *tonk* we have to wait until the next chapter.

tonk is also dangerous for the innocence of logic since it can be used to prove $\alpha \vdash \beta$ for every couple of atomic formulae of the language. So innocence of logic is like molecularity: if we accept it from the point of view of the theory of meaning, we have to make sure that the language we develop really follows this restriction. And in order to do this, just checking the meaning-conferring rules is not enough; we have to investigate the structural properties of all the language.

[19] Obviously the possibility of this justification should be evaluated but, in this case, we are only interested in the relation between molecularity and separability in general. Indeed one of the main reason behind proof-theoretic semantics is to show a justification procedure for non-meaning-conferring rules that naturally takes to separability. But nothing of this has yet been established for now.

[20] [Prior, 1960]

On the other side, while we are sure that the meaning of logical terms should be independent of that of non-logical terms and *vice versa*, it is not completely clear that we want logic to be ineffective to prove new non-logical truths. Obviously, we do not want the non-logical language to be useful to prove new logical truths (that is, we still want autonomy of logic) since we believe that logical truths are analytic, nonetheless standard non-logical and non-analytic truths could be reachable only using logic. That is the extension of the restriction from innocence to separability of logic (that we will call "ineffectiveness of logic") has to be justified somehow.

Dummett's position regarding this problem is controversial because he explicitly rejects the idea that logic is ineffective for proving atomic sentences but he never uses this position in the inferentialist part of his work. That is he probably accepts the innocence of logic, but maybe rejects separability of non-logical truths from logic:[21]

> "The conservative extension criterion is not, however, to be applied to more than a single logical constant at a time. If we so apply it, we allow for the prior existence, in the practice of using the language, of deductive inference, since there are a number of logical constants. Unless, perhaps, 'and' is an exception, the addition of just one logical constant to a language devoid of them, or, more generally, the insertion of deductive inference into a linguistic practice previously innocent of it, cannot yield a conservative extension."

This point has been frequently ignored in the literature.[22] Anyway, we will conclude that at least we do not want that our logical system proves clearly false non-logical sentences. As an example, we do not want a logical system that enables the derivation of every sentence.

Let us now consider the last requirement:

Definition 1.1.11 (Weak subformula)**.** To prove a logical consequence $A \vdash B$ we only need to use the non-logical language in A and B.

In this way, the logical truth depends only on logical terms, and not on some other kind of terms. With weak subformula, we have autonomy of logic from the meaning of non-logical terms: to establish logical truths we do not need any non-logical vocabulary.[23] So, while separability deals with molecularity and could save ineffectiveness of logic if we decide to impose it outside logic, subformula deals with the autonomy of logic. Surely we want to endorse these two principles in their role of generalizations of special molecularism and autonomy of logic that regard all the logical derivations and not just the meaning conferring rules. To ask for them is not surprising since logical truths are generally considered analytic truths *par excellence*.

As an example of a result that does not suit weak subformula property, let us consider $\exists x \exists y(x \neq y)$.[24] This is not a logical theorem of any well-known system but, nonetheless, it is an obvious consequence of the arithmetical theorem $1 \neq 0$. So although it is a true sentence, it can not be analytically true and, for this reason, we do not consider it as a logical truth, by appealing to weak subformula requirement: in order to prove it, we need to use some arithmetical terminology that does not occur in it.[25]

1.2 Proof-theoretic semantics

Proof-theoretic semantics is a special kind of inferentialism, and it is based on two concepts:[26]

- The property of *harmony* that justifies non-meaning-conferring rules;
- A distinction between canonical derivations and non-canonical derivations used to define inductively which derivations are valid.

We will devote a section to each of these two aspects.

[21] [Dummett, 1991] p. 220.

[22] Indeed Steinberger attributes to Dummett the opposite thesis: [Steinberger, 2011a], p. 619 (N.B. Steinberger uses the term 'innocence of logic' to refer to separability of non-logical results from logical rules, that is to the ineffectiveness of logic.). A counterexample to this frequent mischaracterization is [Milne, 1994], p. 87.

[23] Analyticity of logic in an atomistic theory of meaning asks for strong separability and strong subformula property.

[24] In this work, apart from this brief exception, we will deal only with propositional logic.

[25] That this sentence raises problems for analyticity of logic is well know, and it is especially problematic for logicists, since they consider analytically true not only logic but also mathematics: see [Shapiro, 1998a].

[26] To be precise, there is some controversy about the real relation between inferentialism and proof-theoretic semantics. Prawitz's rejects this label for his theory of meaning ([Prawitz, 2015b], p. 60), while Murzi and Steinberger agree with my categorization ([Murzi and Steinberger, 2017]). I suspect, nonetheless, that the issue is controversial only for the non-logical fragment of the language, that Prawitz's theory of meaning investigate with a broadly verificationist (but not inferentialist) approach.

1.2.1 Harmony

The standard view in proof-theoretic semantics is that I-rules are the meaning-conferring rule for logical constants:

Definition 1.2.1 (Meaning of a logical term)**.** The meaning of a logical term is given by its rules of introduction.

While the *application* of an I-rule defines the meaning of the conclusion (a sentence), the schematic *rule* which is applied define the meaning of the logical term. Indeed introduction rules defines the contribution of the connective to the meaning of the conclusion, as asked by definition 1.1.1.

There are reasons why for some connectives it seems easier to consider E-rules as meaning-conferring. For example, we will see that $\supset$I poses some problems of circularity for the definition of valid inferences. The possibility of considering $\supset$E as meaning-constitutive can be a solution to them.[27] [Dummett, 1991] proposed a double-sided justification of logical rules, although it seems that a verificationist approach (connected with the priority of I-rules) preserves the central role in his work. Nonetheless, we will see that most of the problems raised by $\supset$I can be solved using the distinction between canonical and non-canonical derivations. For this reason, in this work we will consider I-rules as meaning-conferring and E-rules as justified rules, even though further generalizations about this assumption could be very interesting.

Already [Gentzen, 1969b] had a similar position (inspired by the constructivist **BHK** interpretation of the meaning of logical constants), and later Prawitz developed this idea in a series of works, starting from [Prawitz, 1965]. As a consequence, there has to be a justification procedure for E-rules, and Prawitz proposes his Inversion Principle for this purpose:[28]

> "Let α be an application of an elimination rule that has B as consequence. Then, deductions that satisfy the sufficient condition $[\cdots]$ for deriving the major premiss of α, when combined with deductions of the minor premisses of α (if any), already "contain" a deduction of B; the deduction of B is thus obtainable directly from the given deductions without the addition of α."

To show that a pair of rules for a logical constant observes Inversion Principle, Prawitz uses the availability of reduction steps. We can justify an E-rule if, when its major premise is derived using an I-rule, we already have a derivation of the conclusion of the E-rule in the derivation of the premise. To make this intuition more precise, Prawitz introduces the notion of maximal formula:

Definition 1.2.2 (Maximal formulae (Prawitz))**.** Given a derivation $\mathfrak{D}$, a maximal formula in it is a formula that is the conclusion of an I-rule and the major premise of an E-rule.

Inversion Principle holds for a pair of I and E-rules iff we can remove the maximal formulae they generate. The procedure of removal of these maximal formulae is called 'reduction' in [Prawitz, 1965] and justification in [Prawitz, 1973]. These procedures are defined for every maximal formula.

As an example, let us consider the following justification of $\wedge$E, given standard $\wedge$I:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ A\end{matrix} \quad \begin{matrix}\vdots \\ B\end{matrix}}{A \wedge B}\,{\scriptstyle \wedge I}}{A}\,{\scriptstyle \wedge E} \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ A\end{matrix}$$

In order to have harmony for a pair of rules, we have to guarantee that not only there is a reduction step for every maximal formula, but that there is a way of applying these steps that erases every occurrence of maximal formulae in the derivation.[29] So Inversion Principle is only a necessary (but not sufficient) condition for harmony.

We also have to consider a generalization of maximal formulae due to the presence of rules in which the conclusion has the same form of one of the premise, like $\vee$E:

Definition 1.2.3 (Maximal sequence (Prawitz))**.** Given a derivation $\mathfrak{D}$, a maximal sequence in it is a list of formulae $C_1, \cdots, C_n$ such that:

- C_1 is the conclusion of an I-rule.

[27] [Rumfitt, 2000], p. 790.
[28] [Prawitz, 1965], p. 33.
[29] The term 'harmony' is introduced in [Dummett, 1991], but the idea that Inversion Principle is not enough to justify E-rules is already established in [Prawitz, 1965], as we will see later.

- $C_i = C_{i+1}$, for every $i < n$
- C_i for $1 \leqslant i < n$ is the premise of an inference used in $\mathfrak{D}$, the conclusion of which is the next element on the list C_{i+1};
- The last element of the list C_n is the major premise of an E-rule.

Of course maximal formulae are just special cases of maximals sequence such that $n = 1$.

This generalization requires the adoption of another class of reduction steps, which reduces the length of the maximal sequence. From [Prawitz, 1971] they are called *permutative reductions.* Now, we can define normal derivation as:

Definition 1.2.4 (Normal derivation (Prawitz))**.** A derivation of B from A is normal if there are no maximal sequences in it.

With this instrument, we can define harmony:

Definition 1.2.5 (Harmony (Prawitz))**.** The rules for the logical constants of a system are in harmony iff for every derivation in this system there is a sequence of reductions that brings it to normal form.

Someone could object that harmony is not defined in general, since it relates to reduction steps, and these are defined only for **NJ**. To solve this problem, Prawitz proposes a generalization of reduction steps in his definition of validity exposed in [Prawitz, 1973]. We do not consider this issue here (although it is a very serious one), because in chapter 2 we will argue that the existence of normal form is enough for harmony, and while normalization refers to reduction steps and so needs a framework, existence of normal form is already a general property.[30] This change will also make a lot easier our generalization of the notion of maximal formula for intuitionistic and classical systems in chapter 3.

It is important to remark that, although reduction steps and Inversion Principle are purely local, harmony is defined as a global property. That is the rules for a logical constant can be in harmony in a logical system and can be not in harmony in another one. According to Prawitz, this global character of harmony is not something wrong, but an antidote against some misconceptions about the Inversion Principle and the acceptability of rules. Indeed, let us consider the following rules for set operations:

$$\lambda\mathrm{I}\ \frac{A[x/t]}{t \in \lambda x A} \qquad \lambda\mathrm{E}\ \frac{t \in \lambda x A}{A[x/t]}$$

They seem to be in harmony, but they only satisfy Inversion Principle; that is for every maximal formula there is a reduction step that "eliminates" it. Nonetheless, we can not find a normal version of every derivation constructed using these rules. Indeed, let us define $t = \lambda x.\neg(x \in x)$ and consider the following derivation:[31]

$$\supset\!\mathrm{E}\ \dfrac{\lambda\mathrm{I}\ \dfrac{\supset\!\mathrm{I}_1\ \dfrac{\supset\!\mathrm{E}\ \dfrac{[t \in t]^1 \qquad \lambda\mathrm{E}\ \dfrac{[t \in t]^1}{\neg(t \in t)}}{\bot}}{\neg(t \in t)}}{t \in t} \qquad \supset\!\mathrm{I}_2\ \dfrac{\supset\!\mathrm{E}\ \dfrac{[t \in t]^2 \qquad \lambda\mathrm{E}\ \dfrac{[t \in t]^2}{\neg(t \in t)}}{\bot}}{\neg(t \in t)}}{\bot}$$

If we try to normalize it, we discover that the only available reduction sequence is circular, that is in some steps of reduction, we return to its original shape. So there is no normal proof equivalent to this, and λI and λE are not in harmony:[32]

> "We have thus an example of a system for which the inversion principle holds [. . .] and where we hence can remove any given maximum formula, but where it is impossible to remove all maximum formulas from certain deductions."

This example speaks in favour of a full normalizability requirement since it is obvious that we do not want contradictory logical systems to pass our *criterion.* So Prawitz's choice is comprehensible and we will follow him on this path. Moreover, this definition of harmony is enough permissible to justify at least **NJ** since:[33]

[30] On this topic, see also note 50 The defence of 'existence of normal form' property is in section 2.4.2.
[31] [Prawitz, 1965], p. 95.
[32] [Prawitz, 1965], p. 95.
[33] [Prawitz, 1965], p. 50, [Prawitz, 1971] p. 256.

Theorem 1.2.1 (Normalization theorem for **NJ**). *If* $\Gamma \vdash_{NJ} B$*, then there is a normal deduction in* ***NJ*** *of* B *from* Γ*, and this normal form can be found applying the reduction steps for the maximal sequences in* $\Gamma \vdash_{NJ} B$*.*

As a matter of fact, we could reject Prawit's counterexample also using Dummett's version of the complexity principle that we explained in paragraph 1.1.2. Indeed, the inference

$$\lambda\text{I}\ \frac{\neg((\lambda x.\neg(x \in x)) \in (\lambda x.\neg(x \in x)))}{(\lambda x.\neg(x \in x)) \in (\lambda x.\neg(x \in x))}$$

used in the proof of $\perp$ just seen has a premise that is more complex than the conclusion, so the rule λI violates Dummett's *criterion*. Of course, this is just an indication that we could drop normalizability if we accept Dummett's proposal, but it is far from being a complete proof of this result since there could be other unacceptable pairs of rules rejected by normalizability but not by this other restriction. Anyway, we will not stress further this topic since we have important reasons to reject Dummett's criterion that we partially already exposed. Indeed this principle – in addition to raising some philosophical worries about circularity in meaning-conferring rules – would reject also some introduction rules that seem completely fine from a meaning-theoretical point of view and that we will use in chapters 2 and 3 as an essential component for our project.

Harmony and normalizability

The previous reconstruction of the meaning of 'harmony' in proof-theoretic semantics is not uncontroversial. As we have already noticed, in our formulation this notion becomes global. For this and other reasons, some authors have preferred to identify harmony with Inversion Principle.

Stephen Read identifies these two concepts and proposes the acceptability of the logical constant $\bullet$ (bullet) that behaves like the lair's sentence. Its rules are:[34]

$$\bullet\text{I}\ \frac{\neg\bullet}{\bullet} \qquad\qquad \bullet\text{E}\ \frac{\bullet \quad \begin{matrix}[\neg\bullet] \\ \vdots \\ C\end{matrix}}{C}$$

This example is equivalent to Prawitz's $\lambda x.\neg(x \in x)$, but the important difference between the two authors is that while Prawitz rejects $\lambda x.\neg(x \in x)$ asking for normalization, Read accepts $\bullet$ as a harmonious logical constant. He defends his position in [Read, 2010], where he distinguishes between inconsistency and incoherence: $\bullet$ is inconsistent since it allows the proof of $\perp$, but it is not incoherent since its E-rule is faithful to its I-rule.[35]

For this reason, I think I should say a few words of justification for my decision to follow Prawitz's original intuition.

1. First of all, we have to consider our framework. We are not interested in formal systems that do not satisfy weak separability (definition 1.1.10), since we want analyticity of logic (observation 1.1.1), and it is obvious that both $\bullet$ and $\lambda x.\neg(x \in x)$ violate it.[36] So, also accepting Read's proposal of a local reformulation of harmony, we should additionally impose weak separability as a separate requirement.

2. We will use normalization as a(n *a posteriori*)[37] proof of harmony. Although someone rejects the identification between these two concepts, I do not think anyone wants to object this side of the entailment, at least not in the context in which we apply it. Indeed the only controversial applications of the entailment from normalizability to harmony are caused by rules that do not suit clearly the distinction between introduction and elimination rules, like Prawitz's classical *reductio*, while in our case we will follow rigorously this distinction. So Read's rejection seems to be irrelevant in our case.

3. Our main application of normalization will be in the definition of validity that we will give in the following section (definition 1.2.8). I think that a definition of harmony should not be evaluated

[34] [Read, 2000], p. 141.

[35] [Tranchini, 2015] offers an interesting analysis of these two examples and an alternative definition of harmony.

[36] Moreover, even though the principle of ineffectiveness of logic is controversial (as we argued at the end of section 1.1.3), in this case its violation is clearly unacceptable.

[37] [Read, ming]

in isolation, but only in connection with this other notion.[38] Furthermore, since proof-theoretic validity is a global property and the main purpose of harmony is its application in this definition, a local redefinition of it seems pointless.[39]

4. Read's counterexample $\bullet$ can also be rejected for reasons independent from normalization. Indeed:

   (a) $\bullet$I violates both Dummett's complexity requirement on inferences (exposed in 1.1.2) and our non-circularity requirement (definition 1.1.9). These restrictions are so natural that it seems very hard to reject them, and since $\bullet$I is unacceptable, its harmony with $\bullet$E is at most irrelevant.

   (b) Although we call it an introduction rule, $\bullet$I is also an elimination rule for $\neg$.[40] So $\bullet$ only apparently suits Inversion Principle, since there is no reduction step for $\bullet\mathrm{I}\,\dfrac{\neg\mathrm{I}\,\dfrac{\begin{matrix}[\bullet]\\ \vdots\\ \bot\end{matrix}}{\neg\bullet}}{\bullet}$. This is obvious if we look at the derivation:[41]

$$\bullet\mathrm{I}\,\dfrac{\neg\mathrm{I}_2\,\dfrac{\neg\mathrm{E}\,\dfrac{\bullet\mathrm{E}_1\,\dfrac{[\bullet]^2 \qquad [\neg\bullet]^1}{\neg\bullet} \qquad [\bullet]^2}{\bot}}{\neg\bullet}}{\bullet}$$

   Peter Milne stressed that $\bullet$ does not suit Inversion Principle already in his [Milne, 2015]. His reason to believe this is that an equivalent formulation of the rules for this constant is[42]

$$\bullet\mathrm{I}^{new}\,\dfrac{\begin{matrix}[\bullet]\\ \vdots\\ \bot\end{matrix}}{\bullet} \qquad \bullet\mathrm{E}^{new}\,\dfrac{\bullet}{\bot}$$

   and in every transitive logic in which we have *Efq*, $\bullet\mathrm{E}^{new}$ is equivalent to

$$\bullet\mathrm{E}^{new-new}\,\dfrac{\bullet}{C}$$

   Now, $\bullet\mathrm{E}^{new-new}$ is essentially *Efq*, so it should suit Inversion Principle when paired with the introduction rule for $\bot$. As we will see in section 1.2.3, the topic of Inversion Principle applied to $\bot$ is quite controversial.[43] Nonetheless, it is obvious that $\bullet\mathrm{I}^{new}$ is not an acceptable candidate, so something must be wrong about this supposed adequacy to Inversion Principle.[44] Despite this clever intuition, Milne, as opposed to Gabbay, did not furnish any explanation of what is wrong with Read's argument for harmony of $\bullet$-rules. He only argues indirectly that there must be something wrong.

1.2.2 Validity

Of course, the main reason to have a logical system is to generate valid derivations, and the main reason to have semantics is to distinguish between valid and invalid derivations. Proof-theoretic semantics allows us to do this without reference to models (Tarskian or of other kinds).

[38]Indeed, [Schroeder-Heister, 2006] proposes an objection to the identification of normalization with validity that is very convincing. Nonetheless, the precise boundary of harmony seems irrelevant for his argument since he just employs 'harmony' and 'normalizability' as interchangeable terms.

[39]To tell the truth, although it can dismantle some scepticisms about its philosophical pedigree, this connection with the explicit definition of validity may lead also to further criticisms of normalizability, since this definition relies on a controversial fundamental assumption (assumption 1.2.1).

[40]For this clever observation, [Gabbay, 2017], p. S113.

[41]Since this objection holds about Prawitz's conception of Inversion Principle, it also holds about Read's reformulation of it (in [Read, 2010]), that is more demanding.

[42]This is essentially the reformulation of the rules given by Read himself in his [Read, 2010].

[43]To be precise, in this paper Milne endorses the thesis that there is no introduction rule for $\bot$, although this seems to be in contradiction with his former position in [Milne, 1994] that we will consider in section 1.2.3.

[44][Milne, 2015], pp. 215-216; the author can also strengthen this argument, because of the classical natural deduction calculus that he develops. Nonetheless, I believe that this partial formulation is already enough to point out the problematic nature of $\bullet$.

We could think that a good definition of valid derivations just derives from our characterization of harmony. It seems that we can justify I-rules because they are meaning-conferring, that we can justify E-rules by harmony. As a consequence derivations built up from these rules seems justified by construction.

The only problem with this plan is that some meaning-conferring rules already use valid derivations in them, so we risk circularity in definition: some valid applications of I-rules are defined using valid derivations, and valid derivations are defined compositionally from valid applications of rules. In **NJ** we have this problem for I$\supset$ since one of its applications $\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B}$ is valid iff its immediate subderivation $\begin{matrix}A\\ \vdots \\ B\end{matrix}$ is valid. Moreover, both $\supset$ and applications of E-rules can already occur in the derivation of B from A.[45]

This circularity is solved, at least in logic, with an inductive definition of valid derivations. So, in order to avoid circularity, we skip from a local definition of valid derivation to a global one, which has as a conceptual consequence a definition of valid inference rule. This unfolding of the notion of valid derivation is made precise by a definition of canonical proof, that is the starting point to define validity.[46]

There is also another reason why we should consider a definition of valid derivation more primitive than that of valid inference rule. Our justification of inference rules by harmony is based on the meaning of logical constants, but we already stated that sentences have priority over terms in proof-theoretic semantics. An inference rule alone can only partially explain the meaning of a sentence, and its application in real inferences has to be investigated. So a definition of validity that uses sentences and meaning-conferring derivations for sentences is warmly welcomed.

The term 'canonical proof' is coined in [Dummett, 1978a], but the idea that valid closed derivations that end with an application of an I-rule have a key role in the definition of validity is already established in [Prawitz, 1973]. Prawitz adopts this terminology in [Prawitz, 1974].

[Dummett, 1991] proposes a composite definition of canonical proof, since it distinguishes between a canonical proof for a system in which I-rules do not discharge assumptions and canonical proofs in general. In the first case, a derivation is canonical if and only if it has only atomic assumptions and it is built up using only I-rules.[47] In the second case, we have to generalize this definition:[48]

> "Hence, when the canonical argument involves an appeal to introduction rules that discharge one of the hypotheses of their premiss or premisses, we cannot place any restriction on the forms of the rules of inference appealed to in subordinate deductions."

Prawitz instead proposes directly this generalization, and we behave in the same way. So our formal definition is:

Definition 1.2.6 (Canonical derivation (Prawitz-Dummett))**.** A canonical derivation for a non-atomic sentence is a closed derivation that ends with an application of an introduction rule and such that it has only valid immediate subderivations.

Since I-rules are meaning-conferring, a canonical derivation is valid by definition. In order to obtain a general definition of validity, now we need to:

- Define the validity of open derivations using the validity of closed derivations;
- Define the validity of non-canonical closed derivations using canonical derivations.

The two main inductive steps of the definition of validity take care of these issues. But before seeing this, we need a further definition:

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[45] chapter 11 of [Gentzen, 1969a] 11 and [Prawitz, 1971] p. 285.

[46] The problem of circularity is much harder to solve for intuitionism *tout court* (as opposed to the purely logical part of this doctrine), from which (via its **BHK** interpretation) I$\supset$ is taken. In this case, the notion of canonical proof is not so friendly since there is no guarantee for a well-ordering of canonical proofs for mathematical (intuitionist) results. So the purely formal nature of logic is heavily applied here. This observation is related to one of Shapiro's criticisms of Tennant's proof-theoretical logicism ([Shapiro, 1998b], p. 613), but it is essentially a consequence of the old problem of *purity of methods*. See also [Dummett, 2000], p. 269-274.

[47] [Dummett, 1991] p. 254

[48] [Dummett, 1991], p. 260.

Definition 1.2.7 (Atomic Base)**.** An atomic base $\mathscr{B}$ is a set of rules (called 'atomic rules') that apply to atomic sentences and have atomic sentences as conclusions.[49]

An atomic rule can discharge an assumption, and there is no assumption of consistency for atomic bases, that is an atomic base $\mathscr{B}$ can authorise a closed derivation of $\bot$. We have a definition of validity with respect to an atomic base, and then a generalization that gives validity *tout court*. This detour is necessary because we need atomic bases in order to close open derivations:[50]

Definition 1.2.8 (Validity in $\mathscr{B}$ (Prawitz))**.** A derivation $\mathfrak{D}$ is valid in $\mathscr{B}$ iff either:

1. $\mathfrak{D}$ is a closed derivation of an atomic conclusion C and it can be reduced by normalization to a closed proof of the same conclusion C carried on in $\mathscr{B}$; or
2. $\mathfrak{D}$ is a closed derivation of a non-atomic conclusion C and it can be reduced by normalization to a canonical proof of the same conclusion C; or[51]
3. $\mathfrak{D}$ is an open derivation and every closure of $\mathfrak{D}$, obtained by replacing open assumptions by closed derivations for the same sentences that are valid in $\mathscr{B}$, is valid in $\mathscr{B}$.

Definition 1.2.9 (Validity (Prawitz))**.** A derivation is valid iff it is valid with respect to each atomic base $\mathscr{B}$.

So the validity of an open derivation is defined by the validity of its closure. But the availability of closures is not enough: we need closures made with canonical proofs. Indeed, let us consider an application of $E\wedge$, and let us try to prove that it is a valid open derivation. In order to do this we have to prove that all its closures made with valid proofs are valid, but if we just pick

$$\dfrac{\begin{matrix}\mathfrak{D}\\ \vdots\\ A\wedge B\end{matrix}}{A}\,\wedge\text{E}$$

we do not know how to rewrite this proof in a canonical form, also accepting validity of the derivation $\mathfrak{D}$ of $A \wedge B$ and the fact that it is closed. That is we have to assume:

Assumption 1.2.1 (Fundamental assumption (Prawitz))**.** *For every valid closed derivation of B non-atomic, there is a canonical derivation with the same conclusion that can be found by reduction.*

This assumption is a theorem for valid closed derivations in **NJ**, that is without atomic bases. Indeed a closed derivation in normal form always ends with an introduction rule, if there is no application of atomic rules.[52] Nonetheless, we need to assume this property also for closed derivations that use atomic rules, because these are needed to close open derivations.[53] This assumption is the reason why we can accept the second clause of the definition of validity in $\mathscr{B}$. Without this, there is no warrant that to every closed derivation corresponds a canonical one.

Now, since we have a closed derivation of $A \wedge B$, we have a canonical derivation for it that can be found by reduction (2), that is a closed derivation (valid in $\mathscr{B}$) that ends with an introduction rule:

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}'\\ \vdots\\ A\end{matrix}\qquad\begin{matrix}\mathfrak{D}''\\ \vdots\\ B\end{matrix}}{A\wedge B}\,\wedge\text{I}}{A}\,\wedge\text{E}$$

[49] We do not need to speak of non-logical constants since we do not consider quantification.

[50] Prawitz proposes two generalizations of this notion: validity with respect to an atomic base and a set of reduction steps (so that reduction steps are not decided before the definition); strong validity that asks for strong normalization property instead of just normalization. The first generalization is the main improvement of [Prawitz, 1973] compared to [Prawitz, 1971], and poses the problem of completeness of a logical system: a proof system is complete if it is the strongest system that derives only inference rules valid with respect to some set of reduction steps. Prawitz conjectured that minimal logic is complete ([Prawitz, 1973], p. 246), but this is still an open problem. Even though this is a very important generalization, we will not consider it, since it is irrelevant for the investigation we want to carry out. Whereas, about the second proposal of generalization (strong validity), we will reject it, following a criticism already stressed by [Schroeder-Heister, 2006]. Moreover, in our modification of Proof-theoretic semantics contained in chapter 2, we will even propose a weakening of the definition of validity that goes in the opposite direction.

[51] In this case a canonical derivation asks for subderivations valid in $\mathscr{B}$.

[52] This is the reason why Schroeder-Heister speaks of it as a corollary of normalization in [Schroeder-Heister, 2006] (p. 531), receiving some severe criticisms from Stephen Read: [Read, 2015], p. 146, note 17.

[53] Since in pure **NJ**, that is without atomic rules, we have closed derivations only for logical theorems.

To prove that this derivation is valid we just have to apply a step of normalization and obtain:

$$\begin{array}{c}\mathfrak{D}' \\ \vdots \\ A\end{array}$$

This is a valid derivation in $\mathscr{B}$ by assumption, and since it can be found by reduction from every closure of the application of $\wedge$E, also these applications are valid in $\mathscr{B}$. Nonetheless, we did not use any assumption about $\mathscr{B}$, so every example of $\wedge$E is a valid open derivation *simpliciter*.

In a similar way we can justify all the elimination rules, so we have that:

Theorem 1.2.2 (Soundness of $\mathbf{NJ}^{\not\perp}$)**.** *Every derivation $\mathfrak{D}$ in* ***NJ*** *in which there is no application of ex falso quodlibet is valid.*

Ex falso quodlibet is in some way problematic, and we will deal with it in section 1.2.3.

Autonomy, ineffectiveness and separability of logic

Since proof-theoretic semantics gives an explicit justification of E-rules by both harmony and definition of validity, we may ask whether this justification automatically warrants autonomy as well, ineffectiveness and separability of logic. In the first part of this chapter, when we were speaking of inferentialism in general, we decided to accept these properties as extra requirements for a good inferentialist theory of meaning.[54] If these properties follow from harmony and validity, then we can drop them as assumptions, since they become redundant. Unfortunately, this entailment does not hold in general, or at least we do not have it as an established result.[55] On the contrary, if we only ask for Inversion Principle, we can find acceptable rules that do not respect separability, as we will see. Regarding the rules that suit Inversion Principle and violate ineffectiveness, the topic is more complex, also because we saw that this principle is not uncontroversial. As we have already seen Read's constant $\bullet$ can be rejected since $\bullet$I violates non-circularity requirement, but we still have some examples of rules of this kind. They are not logical rules, but together with logical rules they cause lost of ineffectiveness: we have already seen Prawitz's λ-rules and in this section we will see truth predicate and Peano's function '?'. In order to check whether the full requirement of harmony (normalization) solves also these problems, we would probably need a generalization of the notion of maximal sequence. For now, we are happy with a specific approach for every single counterexample.

As a first step, we will analyse autonomy, ineffectiveness and separability for the logical system we are interested in, and we will also reject some violations of the ineffectiveness of logic using harmony. In the absence of a general result, we will follow this strategy for the rest of the thesis. Later, we will deal with some alleged counterexample to these properties.

Autonomy of logic For **NJ** we have a property that is also stronger than that required from definition 1.1.11, since it holds:[56]

Theorem 1.2.3 (Subformula property for **NJ**)**.** *Every formula occurring in a normal derivation $\mathfrak{D}$ in* ***NJ*** *of C from Γ is a subformula of C of or some sentence in Γ.*

Together with theorem 1.2.1, we have that the only (logical and non-logical) vocabulary needed to prove a logical consequence of C from Γ is the vocabulary already used in C or Γ. Of course, when we extend **NJ** with non-logical rules we have to check whether autonomy is preserved or not.

Ineffectiveness of logic From theorem 1.2.3 it follows that we can not prove $A \vdash B$ in **NJ** if A and B are atomic and different each other (and $A \neq \bot$), since it would be provable in **NJ** using only subsentences of A and B, and we do not have rules that go from atoms to atoms. This almost amount to ineffectiveness, indeed we will see that the only exceptions to this principle are very tricky.

A concrete example of violation of the ineffectiveness of logic that is rejected by proof-theoretic semantics is *tonk*. We already saw why we want to reject it, now we see how harmony can make this rejection non *ad hoc*, just by showing the non-normalizable derivation:

$$\cfrac{\cfrac{A}{A tonk B}\,tonk\text{I}}{B}\,tonk\text{E} \quad \rightsquigarrow \quad ???$$

[54]The only requirement that we concluded was controversial, and so acceptable only *prima facie*, was ineffectiveness; indeed, in this section, we will see other reasons to doubts about its full validity.

[55]Cozzo identifies the same problem: [Cozzo, 2008b], p. 312.

[56][Prawitz, 1965], p. 53.

It is quite obvious that we can not normalize this derivation. Indeed, to normalize this derivation there has to be a reduction step for this kind of maximal formulae, and so when A and B are atomic we need a derivation $A \vdash B$ in **NJ**. So in order to extend it with rules that violate ineffectiveness of logic and that are harmonious, **NJ** must already violate ineffectiveness, and we know from theorem 1.2.3 that it does not. So *tonk*-rules are not harmonious and they must be rejected.

Separability Still from theorem 1.2.3 and by the form of the rules, we have weak separability for **NJ** (definition 1.1.10). We also have a stronger property for this logical system: to prove a logical consequence $A \vdash B$ we only need to use the rules for the logical constants that occur in A and B. That is we also have strong separability.

The best way to prove separability is to use conservative extension:

Definition 1.2.10 (Conservative extension)**.** Let $\mathcal{S}$ and $\mathcal{S}'$ be two logics expressed in the languages $\mathcal{L}$ and $\mathcal{L}'$, respectively, where $\otimes \notin \mathcal{L}$ and $\mathcal{L}' = \mathcal{L} \cup \otimes$. $\mathcal{S}'$ is a conservative extension of $\mathcal{S}$ iff

- $\mathcal{S}'$ extends $\mathcal{S}$, that is if $\Gamma \vdash_{\mathcal{S}} C$ then $\Gamma \vdash_{\mathcal{S}'} C$;
- if $\Gamma \vdash_{\mathcal{S}'} C$, $C \in \mathcal{L}$ and for every $\gamma \in \Gamma$ $\gamma \in \mathcal{L}$, then $\Gamma \vdash_{\mathcal{S}} C$.

That is $\mathcal{S}'$ is a conservative extension of $\mathcal{S}$ iff it extends the language of $\mathcal{S}$ and it extends its valid derivations but only with derivations in which the new elements of the language occur in the assumptions or in the conclusion.

Conservative extension is a very powerful method for proving separability, and we will make extensive use of it. With **NJ** this application is quite obvious since its rules are atomistic. Let us abbreviate with $\mathcal{S}^{\otimes\cdots\oslash}$ the logic that we obtain from $\mathcal{S}^{\oplus\cdots\ominus\otimes\cdots\oslash}$ by removing the rules for the connectives $\oplus \ldots \ominus$. That is $\mathcal{S}^{\otimes\cdots\oslash}$ is the sublogic of $\mathcal{S}$ formulated using only the rules for $\otimes, \ldots, \oslash$. Then we have:

Theorem 1.2.4. *For every logical constants $\oplus, \ldots, \ominus, \otimes, \ldots, \oslash$ in the language of **NJ**, every extension of $\mathbf{NJ}^{\oplus\cdots\ominus}$ with the rules for $\otimes, \ldots, \oslash$ is conservative.*

Proof. The fact that $\mathbf{NJ}^{\oplus\cdots\ominus\otimes\cdots\oslash}$ is an extension of $\mathbf{NJ}^{\oplus\cdots\ominus}$ is trivial, so let us consider just conservativeness. If $\Gamma \vdash_{NJ^{\oplus\cdots\ominus}} C$ and none of $\otimes, \ldots, \oslash$ occurs in Γ or in C, let us consider a normal derivation $\mathfrak{D}$ for this consequence. For this derivation, theorem 1.2.3 holds, so none of $\otimes, \ldots, \oslash$ occurs in any of its sentence, since none of them occurs in Γ or in C. Now, every E-rule for one of these constants needs a sentence as a premise in which one of $\otimes, \ldots, \oslash$ occurs, so if it were applied in $\mathfrak{D}$ we would have one of these sentences in it. Similarly, if an I-rule for one of these constants were applied in $\mathfrak{D}$ then we would have one of $\otimes, \ldots, \oslash$ in the conclusion, and so in one of the sentences of the derivation. So there is no application of the rules for $\otimes, \ldots, \oslash$ in $\mathfrak{D}$ and these rules are useless for establishing this consequence, that is the extension is conservative. □

So we have strong separability for **NJ**.

Why we can not establish separability in general? And ineffectiveness? We could think that separability follows generally from harmony and maybe that also the converse is true: every conservative extension can be formalised using harmonious rules. Unfortunately, the relations between these properties are not so simple.

First of all, conservativeness is not enough to establish harmony of the base system or of the extension. Indeed we have non-harmonious systems that are extended conservatively by both harmonious and non-harmonious sets of rules. As an example, the system obtained by adding to the purely implicational fragment of **NJ** the rules for *tonk* is conservatively extended by every set of rules (both harmonious and non-harmonious), since it proves every well-formed formula. Another example of rules that probably are not harmonious but that nonetheless conservatively extend a base system is given by the classical rules for negation, which indeed are conservative over the $\wedge\vee$-fragment of intuitionistic logic (or classical logic, that is the same). Probably there is no harmonious formulation of classical rules for negation in the standard framework of proof-theoretic semantics (we will see this in the next chapter), but nonetheless $\wedge$-rules and $\vee$-rules are complete for the $\wedge\vee$-fragment of classical logic, and so classical rules for negations are conservative over them.[57] About the other direction of the entailment, that is whether harmonious sets of rules always extend harmonious base systems, the issue is less obvious.

[57] [Steinberger, 2013], p. 82.

First of all, we can acknowledge that also this question has a negative answer if we use Inversion Principle instead of the full requirement of harmony. Indeed, already Dummett found rules that suit this principle but do not warrant conservativeness, like quantum disjunction.[58] This connective – that we will represent using $\sqcup$ – has the same introduction rule of standard disjunction, but an E-rule that can be applied only to sub-derivations that have only the disjuncts as open assumptions. Using sequent notation for clarity, the elimination rule for $\sqcup$ is:

$$\sqcup\mathrm{E}\ \dfrac{\Gamma \vdash A \sqcup B \qquad A \vdash C \qquad B \vdash C}{\Gamma \vdash C}$$

A logic that has only standard $\wedge$-rules and $\sqcup$-rules is harmonious.[59] Nonetheless its extension with the (harmonious) rules for standard disjunction is not conservative. Indeed the following derivation can not be normalized:[60]

$$\vee\mathrm{E}_2\ \dfrac{\sqcup\mathrm{E}_1\ \dfrac{\wedge\mathrm{E}\ \dfrac{(A \sqcup B) \wedge C}{A \sqcup B} \quad \vee\mathrm{I}\ \dfrac{[A]^1}{A \vee B} \quad \vee\mathrm{I}\ \dfrac{[B]^1}{A \vee B}}{A \vee B} \quad \sqcup\mathrm{I}\ \dfrac{\wedge\mathrm{I}\ \dfrac{[A]^2 \quad \wedge\mathrm{E}\ \dfrac{(A \sqcup B) \wedge C}{C}}{A \wedge C}}{(A \wedge C) \sqcup (B \wedge C)} \quad \sqcup\mathrm{I}\ \dfrac{\wedge\mathrm{I}\ \dfrac{[B]^2 \quad \wedge\mathrm{E}\ \dfrac{(A \sqcup B) \wedge C}{C}}{B \wedge C}}{(A \wedge C) \sqcup (B \wedge C)}}{(A \wedge C) \sqcup (B \wedge C)}$$

Moreover, we can reject this entailment also substituting the ask for harmony with the ask for normalization *tout court*. Indeed Prawitz proved that the disjunction-free fragment of **NK** is normalizable,[61] but, nonetheless, it is a non-conservative extension of the disjunction-free positive fragment of **NJ**, since for example Peirce's law is provable only in **NK** but it can be formalized using just implication. Of course, this counterexample can not be used for our definition of harmony, since we imposed that every logical rule is an introduction or an elimination rule, while Prawitz's rule of classical *reductio* does not suit this division.

We decided to consider harmony as a general property of the logic, that gathers both Inversion Principle and normalizability, so none of the counterexample seen applies directly to it. Indeed, the logic containing $\wedge$-rules, $\sqcup$-rules and $\vee$-rules is not harmonious, even though the logic containing only $\wedge$-rules and $\sqcup$-rules is harmonious and the logic containing only $\vee$-rules is harmonious. So we do not have a harmonious extension, and the violation of conservativeness is useless. On the other side, **NK** is not acceptable, as we already stressed, so there is no question of conservativeness. In conclusion, in our case, we just do not have real proof of the entailment, but we can be optimistic about its validity. Nonetheless, since we do not have real proof of it, we will continue to prove both harmony (defined as Inversion Principle plus normalization) and separability of our logical systems.

Let us now consider the ineffectiveness of logic. First of all, we can easily see that Inversion Principle is not enough to guarantee ineffectiveness of logic. Indeed, we have already seen that Prawitz's λ-rules suit Inversion Principle but leads to triviality, if added to **NJ**. In the previous section we used this result to argue that some pairs of rules that suit this restriction manifest an unacceptable behaviour, and so to skip to a more refined notion of harmony. The same example can be used to prove that Inversion Principle does not prevent violations of ineffectiveness of logic. Indeed, in order to prove $\perp$ we need to use both λ and $\supset$-rules. So extending a base set theory composed of λ-rules with standard rules for the conditional we obtain a non-conservative extension. As a result, we have an application of logic that suits Inversion Principle and that is not innocent.

The complete criterion of harmony answers to this counterexample. Indeed, in order to prove $\perp$ we need to pass through a maximal formula that is both conclusion of $I \supset$ and major premise of $E \supset$, and this maximal formula can not be removed. As a result, the system that includes λ-rules and $\supset$-rules is not in harmony, so it is not acceptable and it does not work as a counterexample to the entailment from harmony to conservativeness.

[58][Dummett, 1991] p. 288.

[59]A logic that has also standard $\supset$-rules suits Inversion Principle, but nonetheless it is not harmonious. Indeed permutative conversions for standard disjunction do not hold for quantum disjunction if there is implication ([Francez, 2017b]), and some derivations do not have a normal form. As an example, let us consider this derivation:

$$\supset\mathrm{E}\ \dfrac{(A \sqcup B) \wedge C \quad \sqcup\mathrm{E}_3\ \dfrac{\wedge\mathrm{E}\ \dfrac{(A \sqcup B) \wedge C}{A \sqcup B} \quad \supset\mathrm{I}_1\ \dfrac{\sqcup\mathrm{I}\ \dfrac{\wedge\mathrm{I}\ \dfrac{[A]^3 \quad \wedge\mathrm{E}\ \dfrac{[(A \sqcup B) \wedge C]^1}{C}}{A \wedge C}}{(A \wedge C) \sqcup (B \wedge C)}}{(A \sqcup B) \wedge C \supset (A \wedge C) \sqcup (B \wedge C)} \quad \supset\mathrm{I}_2\ \dfrac{\sqcup\mathrm{I}\ \dfrac{\wedge\mathrm{I}\ \dfrac{[B]^3 \quad \wedge\mathrm{E}\ \dfrac{[(A \sqcup B) \wedge C]^2}{C}}{B \wedge C}}{(A \wedge C) \sqcup (B \wedge C)}}{(A \sqcup B) \wedge C \supset (A \wedge C) \sqcup (B \wedge C)}}{(A \sqcup B) \wedge C \supset (A \wedge C) \sqcup (B \wedge C)}}{(A \wedge C) \sqcup (B \wedge C)}$$

It can not be normalized, since from $(A \sqcup B) \wedge C$ we can not derive $(A \wedge C) \sqcup (B \wedge C)$ without using $\supset$.

[60][Dummett, 1991] p. 288.

[61][Prawitz, 1965], chapter III.

Looking at the history of logic we can find some other good candidates for a rejection of ineffectiveness of logic. As an example, let us consider the liar's sentence:

This sentence is false

As it is well known, the assumption that this sentence is true entails that it is false, while the assumption that it is false entails that it is true. That is this sentence is paradoxical. So if in a standard logical system we are able to formalize this sentence, then the system itself is contradictory.

Tarski uses this example to prove that all the languages that allow self-reference and contain a truth predicate that can be applied to every sentence (*semantic closure*) and the standard rules for negation are incoherent.[62] It is generally agreed that this consequence can be avoided by weakening the logical system that we use, and especially by weakening the rules for negation.[63] The traditional proposal is to use non-bivalent logics, like Strong Kleene logic.[64] Nonetheless, Priest raised some philosophical objections to these solutions, and proposed to accept incoherence as an acceptable property of a language but to reject *ex falso quodlibet* as an invalid law.[65] Apart from the details, what is important for our issue here is that logic does not seem to be an innocuous supplement to a theory of truth.

The standard position in proof-theoretic semantics is that we do not want liar's paradox and so our harmony *criterion* should exclude theories of truth and logics that together enable its derivation. Indeed, in this framework, the possibility of developing these kinds of paradoxes is usually considered as a piece of evidence that something in our everyday language is erroneous, and we will follow this approach.[66] After evaluating this issue we will consider other, more controversial consequences of the combination of logic and theory of truth.

Harmony could exclude liar's paradox in two different ways: by restricting only the range of acceptable logics or by restricting also the range of acceptable theories of truth. Technically speaking the issue of ineffectiveness of logic deals only with the first kind of solutions, that is we need logic to be conservative over every kind of theories in order to have ineffectiveness. Nonetheless, we already expressed our doubts about such a strong requirement, so we will consider both alternatives.

As a matter of fact, we do not know precisely how to extend harmony to non-logical fragments of the language. Nonetheless, there seems to be a straightforward application of Inversion Principle to truth predicate. Indeed let us consider the following pair of rules:[67]

$$\mathscr{T}\text{I}\ \frac{A}{\mathscr{T}(\ulcorner A \urcorner)} \qquad \mathscr{T}\text{E}\ \frac{\mathscr{T}(\ulcorner A \urcorner)}{A}$$

Whether we impose some restrictions on their applicability, or we accept them in their full generality, this pair of rules seems to suit perfectly Inversion Principle. Of course, we just need to pair a restriction imposed to the I-rule with the equivalent restriction for the E-rule. Nonetheless at least the fully unrestricted version of these rules seems to lead to paradoxes. So it seems that inversion is not a good *criterion* for excluding dangerous versions of the truth predicate. Anyway, we are neglecting an essential part of Tarski's receipt for the paradox: a quotation device capable of modelling self-reference.[68] Steinberger correctly pointed out that the possibility of developing such a theory following the restrictions of harmony is not obvious, so we have to investigate these issues in formal frameworks like Peano Arithmetic (**PA**) in which we can settle the question about self-reference.[69]

Anyway, at least for the Inversion Principle, we can give another counterexample that shows its inability to preventing ineffectiveness of logic.[70] Indeed, let us consider the following rules:

[62] To be precise, in order to be coherent it has to be a completely positive logic, that is it can not have implication either, since Curry's paradox can be formulated without negation: given the sentence "$\mathscr{T}(p) \supset \varphi$" named p, we can derive the truth of φ both from truth and falseness of p.

[63] This is an oversimplified presentation since, as we just remembered in note 62, there are paradoxes that do not use negation. A revision of the rules for implication is needed as well.

[64] see chapter 9 of [Horsten, 2011].

[65] [Priest, 2006], pp. 12-16. Some other recent proposals to reject Contraction rule in order to fix the problem go in the same direction too: [Hjortland and Standefer, 2018], p. 127.

[66] "[...] as Tarski observed, we cannot prevent the semantic paradoxes from arising in our language as we have it: our linguistic practice is thus not perfectly coherent. We have, therefore, just as Frege believed for quite different reasons, to tidy up the language somewhat before we can begin to construct a systematic account of the way it functions [...]" [Dummett, 1991], p. 67. The only well-known exception is Stephen Read, that as we already saw proposes • as a proof-theoretic acceptable version of the liar's paradox.

[67] [Shapiro, 1998a], p. 616. The author imposes also some restrictions that we will discuss later. See also [Read, 2000], p. 127.

[68] As a matter of fact, Quine showed that self-reference is not indispensable in order to have the paradox, since we can use 'Yields a falsehood when appended to its own quotation' that gives a paradox when you put it down twice and apply quotation marks on its first occurrence; see [Quine, 1976c]. Nonetheless, quotation is still indispensable.

[69] [Steinberger, 2011a], p. 636.

[70] For an extensive discussion, see [Ceragioli, 2019].

$$?\text{I}\ \frac{(a+c)/(b+d)=e/f}{(a/b)?(c/d)=e/f} \qquad ?\text{E}\ \frac{(a/b)?(c/d)=e/f}{(a+c)/(b+d)=e/f}$$

This strange operation has been pointed out for the first time by Giuseppe Peano, obviously not in relation to Prawitz's work.[71] When introduced in very weak fragments of arithmetic, ?-rules allow the derivation of 'concrete' examples of absurdity, like $1+1=1$. Indeed it is enough to have the rules for identity and for addiction between natural numbers to obtain:

$$\text{Add.}\ \dfrac{?\text{E}\ \dfrac{\text{Sub. of Id.}\ \dfrac{?\text{I}\ \dfrac{(1+1)/(2+3)=2/5}{(1/2)?(1/3)=2/5} \qquad 1/2=2/4}{(2/4)?(1/3)=2/5}}{(2+1)/(4+3)=2/5}}{3/7=2/5}$$

Of course the conclusion of this derivation is blatantly false, since it is equivalent with $3\times 5=2\times 7$, that is $15=14$. From this result it is easy to derive $1=0$, which is the gate of all arithmetical absurdities.

It should be possible to give a reformulation that suits Inversion Principle of all the rules used in this derivation. Indeed the only controversial part can be the application of this principle to subsentential elements of the language, like fractions and sums. Nonetheless, also the truth predicate is a subsentential element, so this application is not more controversial than it. And the one based on the truth predicate is a well-established counterexample in proof-theoretic semantics since, as we saw, it is discussed by Prawitz, Read, Steinberger and Tennant *inter alia*. Answered this criticism, the hardest part is given by rules for identity, but we can luckily rely on a work of Read for this.[72]

Let us now consider a negation-free version of this small fragment of **PA**. It manages to prove false statements like $1+1=1$ but not $\bot$, while its extension with $\neg$-rules enables the derivation of $\bot$ as well. So we have a non-conservative extension. Of course, if we have also some non-arithmetical vocabulary in our basic theory the non-conservativeness is also more important, since we can derive $\bot$ using $1+1=1$ and then use *ex falso quodlibet* to derive every kind of sentences. In conclusion, Inversion Principle is not capable of preventing effectiveness of logic, whenever the previous counterexample based on the truth predicate is sound or not.

It is interesting to notice that ?I suits both our *criterion* of non-circularity, and Dummett's *criterion* of complexity. At least it suits what is arguably a good extension of Dummett's *criterion* to the non-logical fragment of the language. Indeed, its conclusion is neither more complex than its premise, nor less complex than it – in contrast with what happens with the already seen application of Dag Prawitz's λI, that is disqualified by Dummett's condition – and it seems plausible that this is all we can ask for a rule about non-logical terms.

Let us now return to the problem of developing a complete base system that suits Inversion Principle and that is extended non-conservatively by the truth predicate. We argued that it is better to address this issue in **PA**, and this raises also a further question: the possibility of developing a non-conservative but nonetheless coherent extension of **PA** with the truth predicate. Indeed we already stressed that eventual extra requirements to $\mathscr{T}$-rules are not problematic. Of course it is well known that we have a quotation operator built in **PA**, that is arithmetization. Nonetheless, we still have two points to solve:

- We have to prove that there is a formulation of **PA** that suits Inversion Principle, and this is problematic especially for the induction schema;[73]

- In order to have a non-conservative extension we need to extend also the induction schema with occurrences for the truth predicate.

Of course, the second problem greatly depends on the first one.[74] The kind of non-conservativeness that we obtain (whether it is coherent or incoherent, etc) depends both on eventual restrictions imposed on the applicability of the $\mathscr{T}$-rules, and on the logical system that we use.

[71] [Peano, 1921].

[72] [Read, 2004]

[73] [Shapiro, 1998a] can overlook this problem since he is working under the hypothesis *ad absurdum* that arithmetic is part of logic and can be formulated using harmonious rules.

[74] Steinberger on the other side argues that also this extension of the Induction Schema is not sufficient to have non-conservativeness, since we need a full, compositional theory of truth and so a further extension ([Steinberger, 2011a], p. 635). However, I think that his reasons to believe this are not very strong since, while it is surely true that in an axiomatic theory of truth we should explicitly postulate the compositionality of truth predicate (See chapter 6 of [Horsten, 2011]), it is not so obvious that we are forced to do the same in a theory based on natural deduction.

Arguably, the step from a formalization of **PA** that suits inversion to a fully harmonious one is short. Anyway, while we could believe that Inversion Principle is too weak to prevent pathological applications of the truth predicate, the complete *criterion* of harmony could be more effective to exclude them. At least this seems to be Prawitz's hope when he considers other paradoxical cases like that caused by λ-rules. Nonetheless, we already saw that the normalizability requirement alone is not enough to guarantee conservativeness, since Prawitz's version of classical logic without disjunction is normalizable but its rules for negation are necessary to prove Peirce's law, and also ineffectiveness is essentially a matter of conservativeness, so maybe our hope is misguided. Lacking a good argument for one or the other answer, we can only report what is the common belief.

Prawitz firmly believe that the truth predicate should not be conservative over arithmetic, and so that harmony should not require conservativeness.[75] In other words, he rejects both the thesis of ineffectiveness of logic and that of deflationism regarding the truth predicate.[76] Nonetheless, Prawitz accepts only some kinds of non-conservativeness regarding the theories of truth. He does not seem to be in favour of a proof-theoretically acceptable version of liar's paradox, as opposed to Read, although he does not impose ineffectiveness of logic or coherence as extra requirements. As a matter of fact, apart from questions of coherence, conservative extension (that is necessary for ineffectiveness of logic) is never a *desideratum* nor a usual result in Prawitz. We already saw that Inversion Principle is useless to discriminate between acceptable and unacceptable theories of truth, so he can only rely on normalizability for this purpose. Anyway, the existence of such a consequence of this property should be clearly exposed.[77]

So in conclusion maybe harmony entails the coherence of the system, but there are good reasons to believe that it does not entail ineffectiveness of logic. So if we want this property (and it is a controversial choice, since we saw that both Dummett and Prawitz rejected it) we need to assume it as an extra property.

1.2.3 Absurd and *ex falso quodlibet*

Prawitz proposes a clear justification of minimal logic since it is characterised by harmonious rules that produce valid derivations. Nonetheless, the extension with *ex falso quodlibet* in order to obtain intuitionistic logic is problematic. Indeed in [Prawitz, 1965] this rule does not enter in the standard distinction between introduction and elimination rules, although it does not cause problems with normalization. *Efq* causes problems also to the definition of validity, due to the presence of non-consistent atomic basis. Let us consider these two problems separately.

Harmony of *Efq*

Technically speaking someone could think that harmony is not such a big problem for *Efq*, since we defined it using normalization and not directly using Inversion Principle. Indeed it seems that, since *Efq* is neither an I-rule, nor an E-rule in Prawitz's analysis, it can not give rise to maximal formulae by definition. If we accept this position, all we have to show is that it does not disturb normalization regarding the other rules. This seems to be the idea behind Prawitz's treatment of this rule in [Prawitz, 1965], where he proves normalization for the complete system of intuitionistic logic, and so that *Efq* does not cause any loss of harmony. But of course this solution would be improper: when we impose normalization we assume that every rule has to be considered as an I or an E-rule, otherwise the requirement loses its *raison d'être*. Also, Dummett is very clear when he stresses that harmony should be a requirement for the entire language, not only for logic.[78] Under these circumstances, it seems reasonable to consider *Efq* as an E-rule.

If we accept the label $\bot$E for *Efq*, that is we decide to consider it an elimination rule, then we have to justify it. The obvious problem is that there seems to be no introduction rule for $\bot$, so no chance of justification via Inversion Principle. Nonetheless there are three suggested solutions:

Dummett $\bot$I has the form $\bot\mathrm{I}\,\dfrac{b_1 \qquad b_2 \qquad \cdots}{\bot}$ where the b_i run through all the atomic sentences of the language;[79]

[75] [Prawitz, 1994] and [Prawitz, 1985] p. 166.

[76] See chapters 5, 7 and 10 of [Horsten, 2011] for an exposition of the traditional relation between conservativeness and deflationism and some proposals of alternative approaches.

[77] Of course, I am not negating that we can apply normalization theorem to prove coherence in some contexts. Some similar applications are well known and completely valid. Nonetheless, we do not have a general warrant that this is always possible.

[78] [Dummett, 1991], p. 287.

[79] [Dummett, 1991], p. 295.

Read $\bot$E is justified by the absence of an introduction rule for $\bot$;[80]

Milne $\bot$I is a non-logical and non-formal rule that depends on the context, like $\bot\text{I}\,\dfrac{0=1}{\bot}$.[81]

The first two solutions consider $\bot$ as a logical constant, while the last one relegates it to the non-logical vocabulary. If we accept the first formulation of $\bot$I it is clear that Inversion Principle holds. Indeed we have the following kind of maximal formulae and reduction step:

$$\bot\text{I}\,\dfrac{\dfrac{\vdots}{b_1} \quad \dfrac{\vdots}{b_2} \quad \dots}{\bot\text{E}\,\dfrac{\bot}{b_i}} \quad \rightsquigarrow \quad \dfrac{\vdots}{b_i}$$

Nonetheless, this kind of solution has a big disadvantage: the meaning of $\bot$ is not independent on the non-logical fragment of the language. Indeed, since we treat it as a kind of conjunction of all the atoms, if we change the class of atoms then we change its meaning. So it seems that the meaning of $\bot$ is not fixed for every language, and it can change in different contexts. Dummett speaks on this regard of a lack of 'invariance' of $\bot$.[82]

Also, nothing says that $\bot$ has to be false, that is that there can not be a valid derivation for it from correctly asserted assumptions, as observed by Nils Kürbis.[83] Indeed let us assume that $\bot$E and the first version of $\bot$I were enough to define absurdity. Let us now consider the following scenario: in our pre-logical language every atomic sentence is true. If we do not need pre-logical knowledge to understand absurdity, the meaning of $\bot$ should be unaltered in this case. But to accept this, we have to reject the idea that absurdity must be false, since it is tantamount to the conjunction of all atomic sentences, which is a true sentence in our toy-language. Indeed, if it were not for non-formal contradictions like 'this is a completely red and completely green coloured spot' or 'this body is heavy and light', or at least factually false sentences like 'the Moon is made of cheese', what problems there would be in accepting $\bot$ in our beliefs? According to Kürbis, from this observation we have to conclude that the rules for $\bot$ do not offer a complete characterization of the meaning of "absurd". Kürbis seems to say that the loss of invariance is enough to reject the idea that we can catch the meaning of $\bot$ using an inferentialist theory of meaning. Nonetheless I think, with Dummett, that all that we can conclude from this observation is that a part of the meaning of absurdity does not pertain to logic. [Dummett, 1991] indeed makes the same observation of Kürbis and concludes:[84]

> "It is, however, important to observe that no appeal has been made to the principle of consistency, and that the logical laws do not imply it. We may know our language to be such that not every atomic statement can be true; but logic does not know that. As far as it is concerned, they might form a consistent set, as they are assumed to do in Wittgenstein's Tractatus. The principle of consistency is not a logical principle: logic does not require it, and no logical laws could be framed that would entail it."

So, far from being a problem of proof-theoretic semantics, this situation seems to be a well-known aspect of this theory, in line with some positions regarding logic. Of course, philosophers who do not share this position see it as a problem, but this holds for every philosophical implication of proof-theoretic semantics.

The second solution, proposed by Stephen Read solves at least the problem of invariance of $\bot$. The idea naturally arises from a reformulation of Inversion Principle proposed by the author, according to which E-rules are obtained as results of the application of a function to a set of I-rules. Nonetheless, I think that this proposal has nothing to say about falseness of $\bot$, for which we have to accept Dummett's position.

The third alternative accepts in some way both Dummett's observation about the lack of invariance of $\bot$ and the conclusion, derivable from Kürbis's observation, that its meaning comes from the

[80] [Read, 2000], p. 139. To be precise, this justification was already present in [Cozzo, 1994b] (p. 110) and the author admits that he heard Prawitz suggesting this idea during the 1980s.

[81] [Milne, 1994], p. 64.

[82] [Dummett, 1991], p. 296. Of course, the same lack applies to negation since its meaning depends on that of absurdity. We could even observe this *phenomenon* directly for negation, by the rule (see [Milne, 1994], p. 81): $\neg\text{I}\,\dfrac{\begin{matrix}A\\ \vdots\\ b_1\end{matrix} \quad \begin{matrix}A\\ \vdots\\ b_2\end{matrix} \quad \dots}{\neg A}$ Nonetheless I think that using $\bot$ severs more clearly logical and non-logical aspects of the meaning.

[83] [Kürbis, 2015a] and [Kürbis, 2015b].

[84] P. 295.

non-logical territory. We could say that while Dummett's solution isolates only the logical part of the meaning of $\bot$, Milne's solution accepts a completely non-logical meaning-conferring rule for this constant, leaving in some way open the question of the logical status of $\bot$E.

There have been a lot of logical investigations regarding the harmony of rules for absurdity, but I think it would be better to focus mainly on the validity of these rules, since this topic could shed some light also on harmony.

Validity of *Efq*

First of all, let us just remember that there is no assumption of consistency for atomic bases $\mathscr{B}$. That is in some bases $\mathscr{B}$ we have a derivation of $\bot$. This poses some problems for validity of $\bot$E, indeed since it is an open derivation, it is valid in a basis $\mathscr{B}$ iff all of its closures with valid derivations are valid in $\mathscr{B}$ (clause 3 of definition 1.2.8). As Prawitz observes, $\bot$E is vacuously valid with respect to consistent bases $\mathscr{B}$ – that is bases that do not derive $\bot$ –, since there are not closures for this derivation according to them.[85] Nonetheless, we have some problems with inconsistent logical bases. Indeed consider one such basis $\mathscr{B}$; by the definition of inconsistency, we have a valid in $\mathscr{B}$ closure for $\bot$. Now, $\bot$E is a valid in $\mathscr{B}$ open derivation iff, its closures with valid in $\mathscr{B}$ derivations are valid in $\mathscr{B}$, and it is not obvious that this is the case. Given an atomic valid in $\mathscr{B}$ derivation $\mathfrak{D}$ of $\bot$, the closed derivation

$$\bot\text{E}\,\frac{\dfrac{\mathfrak{D}}{\bot}}{p}$$

is valid iff there is a closed derivation of p that can be carried on in $\mathscr{B}$ (clause 1 of 1.2.8). The problem is that $\bot$E is not part of the atomic rules, so this derivation itself is not carried on in $\mathscr{B}$ and is not acceptable for this justificatory purpose. As a consequence, this derivation is justified if and only if there is a reduction of it that does not use $\bot$E or other logical rules. Unfortunately, we do not have any warrants that this reduction is possible in general. Indeed an atomic basis can be inconsistent without having a closed derivation for every atomic sentence, or at least no such condition is imposed on inconsistent bases. In conclusion, since logical validity is defined as validity in every atomic basis $\mathscr{B}$ (1.2.9) and there are inconsistent atomic bases such that $\bot$E is not valid in them, $\bot$E is not logically valid.

So $\bot$E gives problems because it wants to be a logical rule, but it has some kind of introduction rule that is non-logical and that is given in some bases $\mathscr{B}$. I think we can see Dummett's solution and Milne's solution as opposite, since:

- The first proposes a logical rule for the introduction of $\bot$, and so decides to consider it as a logical constant *tout court*;
- The second proposes something like non-logical rules for the introduction of $\bot$, which depend on the atomic basis $\mathscr{B}$. According to this interpretation, Milne is nearer to Prawitz than Dummett. I think that a logical conclusion of this position would be to consider also $\bot$E as a non-logical rule, that is as a rule of $\mathscr{B}$.

Read's solution is in some way ineffective here, since the fact that we do not have logical rules for introducing $\bot$ is not enough to solve the problem of its validity when we accept inconsistent atomic bases (although maybe it is enough just for harmony). Indeed, Read tells us that we can reject I-rules for $\bot$ and still have harmony for this constant. Nonetheless, we can see that inconsistent atomic bases $\mathscr{B}$ and so non-logical I-rules for $\bot$ are needed in order to reject unwanted open derivations.

What are Inconsistent bases needed for? We could think that it is not a good idea to deal with inconsistent atomic bases in general, and that this is the reason why $\bot$E seems to be an invalid open derivation. Unfortunately, this diagnosis can not be correct, since we need this kind of bases to reject some invalid derivations.

Let us consider the following open derivation:

$$\frac{\neg p}{q \vee r}$$

We assume that p, q and r are atomic. It seems obvious that we want to reject this derivation as invalid but, in order to do this, we need to have a closed derivation for $\neg p$ in an atomic basis $\mathscr{B}$. Indeed if there is no atomic basis $\mathscr{B}$ that warrants $\neg p$, then the derivation is vacuously valid (as $\bot$E

[85] [Prawitz, 1973], p. 243.

with respect to consistent bases). So let us assume that we have a closed valid derivation $\mathfrak{D}$ for $\neg p$ in $\mathscr{B}$. Since $\neg p$ is logically complex, if $\mathfrak{D}$ is valid, then there is a canonical derivation $\mathfrak{D}'$ of $\neg p$ that ends with an application of ¬I. Its form has to be:

$$\neg\mathrm{I}\ \dfrac{\begin{matrix}[p]\\ \vdots\\ \bot\end{matrix}}{\neg p}$$

Let us keep in mind that this derivation has to be closed and let us consider the open sub-derivation of $\bot$ from p. We will see that it needs to be constructed using an inconsistent basis.

Of course, there is no closed derivation of $\bot$ that uses only logical rules, and there is no closed derivation of $\bot$ that uses only logical rules and rules of consistent atomic bases. Indeed a consistent basis allows only to close open atomic assumptions from a logical derivation, or to change their forms or number. That is it contains rules of the form

$$\dfrac{[p]}{p} \qquad \dfrac{p \qquad q \qquad \dots}{r}$$

or that can be obtained by composition from these. And none of these operations is sufficient to derive $\bot$, since there is no open derivation of it in **NK** (and **NJ** *a fortiori*) from atomic assumptions.[86] From this, it follows also that there are open derivations of $\bot$ from atomic assumptions in **NJ** plus consistent $\mathscr{B}$ neither. Indeed, from an open derivation of this kind we can find an open derivation in **NJ** of $\bot$ from (possibly different) atomic assumptions, just by dropping the applications of the rules in $\mathscr{B}$. Consequently, we need an inconsistent atomic basis in order to derive $\bot$ from atomic assumptions, and so we need it also to derive canonically $\neg p$. It is obvious that also in this basis there are not reduction procedures that render normal the derivation

$$\dfrac{\neg\mathrm{I}\ \dfrac{\begin{matrix}[p]\\ \vdots\\ \bot\end{matrix}}{\neg p}}{q \vee r}$$

Indeed we have a derivation of $\bot$ in $\mathscr{B}$ and then no applications of E-rules but just an application of an I-rule and a strange (evidently invalid) inference. So there are no maximal formulae, and the derivation is 'in normal form'. Nonetheless, it is blatantly not in canonical form, since the last application is not an I-rule. In conclusion, since clause 2 of definition 1.2.8, this derivation is neither valid in $\mathscr{B}$, nor logically valid.

If we weaken the requirement of harmony asking only for the existence of normal form instead of asking for normalization, this derivation is harder to reject. We have to ask that in some inconsistent atomic base $\mathscr{B}$ such that $p \vdash_{\mathscr{B}} \bot$, there are closed derivations neither for q nor for r. Maybe it would just be better to assume that for every set of atoms, there is a class of bases that justify only the closed derivations for each of them, and such that for every set of sets of atoms, there is a basis in this class that derives $\bot$ only from all the elements taken together of each of these sets of atoms. In this way, given a set of atoms, we have every kind of atomic basis: we have every selection of true atoms and every selection of inconsistent sets of atoms.

This requirement regarding the atomic bases seems to be in contradiction with Dummett's rule. Indeed, we just decided to accept I-rules for $\bot$ that do not have all the atoms as premises. We could still consider Dummett's rule as the only canonical way of deriving $\bot$. In this case, it is important to remember that fundamental assumption 1.2.1 holds only for closed derivations: we can have an open derivation of a sentence, without also having a canonical derivation of it. Indeed let us consider the derivation:

$$\wedge\mathrm{E}\ \dfrac{A \wedge (B \vee C)}{B \vee C}$$

It is obviously valid, since it is constituted by a single application of an E-rule. Nonetheless, there is no chance of deriving it canonically, since $A \wedge (B \vee C) \nvDash B$ and $A \wedge (B \vee C) \nvDash C$.

This remark is relevant here because without it we could think that, under the hypothesis that Dummett's rule is the meaning conferring rule for $\bot$, from

[86]This is obvious, since in a truth table we are free to consider true every selection of atomic sentences.

$$\dfrac{\neg\mathrm{I}\;\dfrac{\begin{array}{c}[p]\\ \vdots\\ \bot\end{array}}{\neg p}}{q \vee r}$$

we could conclude the existence of the derivation:

$$\dfrac{\neg\mathrm{I}\;\dfrac{\bot\mathrm{I}\;\dfrac{[p] \qquad \dfrac{}{b_1} \qquad \dfrac{}{b_2} \qquad \dots}{\bot}}{\neg p}}{q \vee r}$$

So, if we could apply the fundamental assumption, we would be sure that in such a derivation of $\bot$, all b_i apart from p were closed, as in the starting derivation.

This would not be an obstacle to reject the validity of the derivation $\neg p \vdash q \vee r$, if we define this notion using normalization, since there is no applicability of normalization to the tree just seen. Nonetheless, if we weaken the definition, and ask only for the existence of normal form, then we would have troubles, since we can find the normal derivation:

$$\vee\mathrm{I}\;\dfrac{\dfrac{}{q}}{q \vee r}$$

Where q is an atom b_i. And this justification would be available for every derivation that has the negation of an atom as its only premise and such that the negated atom does not occur in the conclusion.[87] This disquisition is not just hypothetical since we will consider in later chapters this kind of weakening of harmony. So this is another good reason to restrict the application of fundamental assumption to closed derivations.

We could wonder how to apply a canonical derivation for $\bot$. That is: what does it mean to have a closed derivation of it? Of course we need to have a derivation that has only closed assumptions, so we need a set of rules in $\mathscr{B}$ that enable a derivation $b_1, \dots, b_n \vdash \bot$ plus a set of rules that discharge the open assumptions $b_1, \dots, b_n$, directly or not. Since we said that $\mathscr{B}$ considers true those atoms for which it gives closed derivations, we can apply the canonical derivation for $\bot$ only in those basis in which there is a set of inconsistent atoms that is a subset of the set of the true (according to $\mathscr{B}$) atoms. This *phenomenon* is quite strange and deserves some attention when we consider Dummett's proposal.[88]

Concluding remarks on Absurd

Let us now reconsider the three analyses of absurd sketched at the beginning of this section.

We have seen that Dummett's proposal is more demanding than it looks since it uses a canonical derivation for $\bot$ that can be applied only in strange kinds of atomic bases. Apart from this it is a coherent proposal and should be considered as a good way of saving the logical status of *ex falso quodlibet*. Of course, we also have to neglect the strange problem of the lack of invariance of $\bot$.

Milne's solution seems to be the most natural, if we look at the way in which consistency is dealt with in the definition of validity. We just consider it as a notion that regards atomic bases and not logic, and so downgrade *ex falso quodlibet* to a non-logical status. Of course, a natural consequence of this position is that we should consider also incoherent bases in which *ex falso quodlibet* does not hold.[89] Whichever proposal we accept, we need to have a fully general notion of atomic bases in order to reject unsound open derivations that start from negated atoms.

On the contrary, Read's solution can not be accepted, since it is based on the absence of I-rules for $\bot$, and this condition can not be preserved when we consider atomic bases in general. Should we conclude that in these atomic bases $\bot$E does not hold? This is unacceptable, since logic should be independent of the context. Also, the hypothesis of dropping inconsistent bases has to be rejected too, as just shown.

[87] In this case, we tacitly assumed that $q \neq p$ or $r \neq p$, nonetheless we can have the same problem also in some cases in which the negated atom occurs in the conclusion. We choose the first situation since it is surely problematic, but we can have good counterexamples also in the other case.

[88] Already the fact that a canonical derivation for $\neg p$ has to derive $\bot$ using only open assumptions of the form p clashes with the idea behind Dummett's rule. Nonetheless, we tried to save this rule by pairing it with non-canonical derivations for $\bot$. I think that this reconstruction is the most favourable that we can furnish to Dummett's rule.

[89] I want to specify for clarity that this conclusion is not explicitly stated by Milne, although it seems to follow from its proposal.

Chapter 2

Changing Proof-Theoretic Semantics

2.1 That's all?

As we saw in the previous chapter, the rules for **NJ** suits the *criteria* imposed by proof-theoretic semantics with the only possible exception of *ex falso quodlibet*. Of course, this system is complete only for intuitionistic logic and this leads people to wonder whether it was possible to give a proof-theoretically acceptable system also for this logic. This possibility was explicitly rejected by Prawitz, who conjectured that minimal logic is the strongest system that is justified according to his definition of validity.[1] This conjecture wants to be fully general, that is for every different choice of I-rules and of reduction procedures, the strongest logic acceptable is minimal. The status of the conjecture is controversial: it seems to be false if stated for the entire language but arguably true for some of its fragments, at least according to some authors.[2]

We will not consider this conjecture directly, but we will propose an overview of some alleged solutions to save classical logic. After rejecting some proposals we will endorse a solution proposed by Peter Milne and we will lead it into unexpected directions. In order to do this, we will evaluate some starting points generally accepted in proof-theoretic semantics about the shape that an I-rule should have.

2.2 Single-conclusion unilateral systems and their problems

Problems with the standard formulation The proposed solution that should save classical logic with a very small departure from the standard approach of proof-theoretic semantics uses a single-conclusion formulation and uses rules in which only one logical term occurs. With both these restrictions, which **NJ** without *ex falso quodlibet* naturally suits, it is not easy to justify classical logic.

Prawitz's original attempt to save classical logic extends **NJ** with an extra rule for $\bot$, but as we already stressed it is not clear how to categorise it. The standard opinion today in proof-theoretic semantics is that *ex falso* is an elimination rule, and we saw in section 1.2.3 that there are some good attempts to justify it. Nonetheless, unfortunately, this categorization can not extend to *classical reductio*, since there seems to be no justification procedure for it.

Prawitz considers both *ex falso* and *classical reduction* as external to the distinction between I and E-rules, but we already acknowledged that this choice is not in line with the conception of meaning developed in proof-theoretic semantics. Since there seems to be no clear justification of this rule but it has to be accepted in the introduction/elimination distinction, Milne proposed that *classical reductio* should be considered as an I-rule for A, instead of an E-rule for $\bot$. In this way, we could justify $\dfrac{A \qquad \neg A}{\bot}$ as the respective E-rule (for A, not for $\neg A$).[3] But to do this, it seems to be necessary to reject compositionality of meaning, because to know a meaning-conferring rule for A, we should have to know a more complex sentence, *id est* $\neg A$. Milne recognises this problem and proposes an interesting answer.[4] Let us consider these two rules:

[1] [Prawitz, 1973], p. 246.
[2] [Wansing, 2015], p. 19-20.
[3] [Milne, 1994], p. 58.
[4] [Milne, 1994], p. 60.

$$
cI\,\dfrac{\begin{array}{c}[A]\\ \vdots\\ \bot\end{array}}{cA}
\qquad
cE\,\dfrac{A \qquad cA}{\bot}
$$

The intuitive idea is that cA is the contrary of A, so if A is the negated formula $\neg F$, cA stands for F as much as for $\neg\neg F$. Obviously, the objection of complexity can not be stated against these rules.

The problem with this proposal is that these rules do not manage to define classical negation without some changes. In particular, we have to introduce in the theory the idea of cA as the classical contrary of A. We can formalize this intuitive characterization using the pair of rules:

$$
c\neg E1\,\dfrac{c(\neg A)}{A}
\qquad
c\neg E2\,\dfrac{c(\neg A)}{\neg\neg A}
$$

But let us consider the following proof of Peirce's law:

$$
c\neg E1\,\dfrac{cI_{3,4}\,\dfrac{\neg E\,\dfrac{[\neg(((A\supset B)\supset A)\supset A)]^4 \qquad \supset I_2\,\dfrac{\supset E\,\dfrac{\supset I_1\,\dfrac{c\neg E1\,\dfrac{cI\,\dfrac{\neg E\,\dfrac{[\neg(((A\supset B)\supset A)\supset A)]^3 \qquad \supset I\,\dfrac{[A]^1}{((A\supset B)\supset A)\supset A}}{\bot}}{c(\neg B)}}{B}}{A\supset B} \qquad [(A\supset B)\supset A]^2}{A}}{((A\supset B)\supset A)\supset A}}{\bot}}{c(\neg(((A\supset B)\supset A)\supset A))}}{((A\supset B)\supset A)\supset A}
$$

We can see that the role of this rule is not dissimilar to that of double-negation elimination, and it does not seem easier to justify.[5] It could also seem more natural to assume a double-contrary elimination to have the same result. Milne writes that c should be a metalinguistic constant, so probably we should not formulate $Ec\neg$ as rules, but in this case the inferential one is only a part of the complete meaning of c. If this is acceptable in a proof-theoretic study of meaning, it shall be more simple to admit that only the intuitionistic part of classical negation has an inferential meaning.[6]

Another, more interesting, answer given by Milne to the refusal of labelling classical *reductio* as an introduction rule is that not every I-rule is meaning-conferring. If this were true, the classical *reductio* violation of complexity constraint would be much less scandalous, because compositionality could still work for meanings even though not for I-rules. To support this idea, he proves that the meaning of negation can not be given only using I-rules, because it is impossible to obtain a proof of a negative formula without assuming another negative formula.[7] So in order to obtain something negative, something else negative must be assumed. This is not so surprising if we consider that $\neg$ is introduced using $\bot$ and this does not have an introduction rule. So, to understand $\neg$ we need to understand $\bot$, and the knowledge of this can not consist in the mastery of an introduction rule.[8]

The suggestion that the meaning of $\bot$ is not completely given by logical rules is hardly resistible; I think the only answer, if any, could be the one stressed by Stephen Read and already exposed in section 1.2.3. Nonetheless, we already pointed out that Read's proposal is not acceptable from the point of view of validity. So it seems that we have to pick one of the following reading of this phenomenon:

- Part of the meaning of absurdity (and as a consequence negation) can not be captured by an inferentialist theory of meaning, as stressed by Nils Kürbis;

---

[5]Indeed another option to obtain classical logic is to assume also $\dfrac{\begin{array}{c}[cA]\\ \vdots\\ \bot\end{array}}{A}$ as a rule, but so we end up where we started.

[6]Milne's main reason to investigate c seems to be the observation that meaning-theoretic complexity and syntactic complexity are not necessarily the same thing. As a consequence cA can be syntactically but not meaning-theoretically more complex than A. I neglected this aspect of his proposal because, although reasonable from a general point of view, it obviously clashes with our inferentialist framework.

[7]Recursive definition of negative formula is: $\bot$ is negative; no other atomic sentence is negative; for all A, $\neg A$ is negative; $A \wedge B$ is negative iff A or B is; $A \vee B$ is negative iff A and B are; $A \supset B$ is negative iff B is; [Milne, 1994], p. 61.

[8]Milne suggests that it could consist in the mastery of something that is not (technically speaking) logical, like $1 = 0$.

- Part of the meaning of absurdity (and as a consequence negation) can be captured by an inferentialist theory of meaning, but is not part of its logical meaning, as stressed by Dummett.

In both cases, it seems that Milne's observation forces us to admit that not always the entire meaning of a logical term is completely given by its logical introduction rule. So, sometimes something else must be used to characterise completely its meaning, and we are forced to – at least partially – abandon *autonomy principle* or to reject *ex falso* as a logical rule. However, what Milne needs to show in order to save the I-rule reading of classical *reductio* is that sometimes I-rules do not contribute at all to the meaning of the principal term in the conclusion, but he shows only that sometimes it does not manage to give it all. Indeed it is quite obvious that ¬I contributes to the meaning of intuitionistic negation: at least in expressing its relationship with implication and absurdity.[9] So classical *reductio* should at least give a part of the meaning of the less complex conclusion, even if maybe not the complete meaning. But if this is true, we still have the same problem: to understand the meaning of a sentence A is to know all its meaning-conferring rules, and so to understand completely A we need to understand $\neg A$.[10]

Moreover, Milne tries to use his observation to respond to Prawitz's criticism that ¬ should not occur in a rule for absurdity like *classical reductio*. I think that Prawitz is worried by the possible violation of what we called non-circularity requirement (definition 1.1.9), since if we interpret *classical reductio* as an I-rule, the usage of ¬ in a meaning-conferring rule for ⊥ is paired with the usage of ⊥ in a meaning-conferring rule for ¬. If this is the point, Milne's observation that purely logical I-rules alone can not define the meaning of ¬ is ineffective.

However, apart from technical reasons why some particular proposals fail, there are also general problems for this kind of extensions of **NJ**. Indeed, one of the reasons why it is hard to justify classical logic is that while intuitionistic rules for implication seem reasonable and complete, they are not enough strong to derive Peirce's law, which nonetheless is a classical law. For this reason, an attempt to reconstruct classical logic by extending **NJ** with only rules for negation or absurdity is unacceptable, due to our condition of separability. Of course, also Prawitz's proposal is rejected by this simple observation.[11]

General-introduction rules More recently, Milne has prosed another approach that deals considerably well with the issue of separability. In [Milne, 2010] the author consider an intuitionistic system in which both ⊃ and ¬ are primitive. His reason for doing this is probably the lack of invariance of ⊥ and as a consequence the lack of invariance of the negation if this is defined using that.[12] Indeed, already in [Milne, 1994] he dealt with this issue, and rejected

$$\dfrac{\begin{matrix}[A]\\ \vdots\\ B \wedge \neg B\end{matrix}}{\neg A}$$

because it is unable to characterise the meaning of ¬ in an invariant way, since the occurrence of B in it hide an implicit universal generalization, like in *ex falso quodlibet*. On the contrary, he opted for the following, that does not have this problem:[13]

[9]To reject the ability of this rule to give meaning to ¬, we should reject also that minimal negation has a meaning at all.

[10]Milne is right when he says that in classical logic sentences and their respective negations seem to be at the same level, but it is not clear whether this intuition can be used in inferentialist theories of meaning. I do not think that Milne's proposal of operator c can work, as I have already said; we will later evaluate whether a bilateral theory of speech acts like [Rumfitt, 2000] is more trustworthy.

[11] Although his normalizability result for the disjunction-free fragment of this formulation of classical logic can be extended to the entire language, as is shown in [Andou, 1995]. Indeed: first of all we defined harmony using both normalizability and the Inversion Principle, and a positive answer to the first *criterion* alone is not enough; secondly normalizability is not sufficient to state separability, as an example the proof already seen of Peirce's law (formulated using classical *reductio* instead of $c - rules$) is in normal form but does not exhibit separability.

[12]We exposed this problem in section 1.2.3.

[13]Pp. 83-83. Or equivalently (p. 64)

$$\dfrac{\begin{matrix}[A]\\ \vdots\\ \neg A\end{matrix}}{\neg A}$$

that we will consider also later in this section.

$$\neg\mathrm{I}_{Milne}\ \dfrac{\begin{array}{c}[A]\\ \vdots\\ A\wedge\neg A\end{array}}{\neg A}$$

In his more recent article, he remains faithful to this line. Indeed he starts to consider Dummett's formulation of intuitionistic logic, that is composed by the standard rules for $\wedge$, $\vee$ and $\supset$ extended with

$$\neg\mathrm{I}_{Dummett}\ \dfrac{\begin{array}{cc}[A] & [A]\\ \vdots & \vdots\\ B & \neg B\end{array}}{\neg A} \qquad \neg\mathrm{E}_{Dummett}\ \dfrac{A \qquad \neg A}{B}$$

but immediately explains that the first rule can be split in the two rules[14]

$$\text{Weak}\neg\text{I}\ \dfrac{\begin{array}{c}[A]\\ \vdots\\ \neg A\end{array}}{\neg A} \qquad \neg\text{Inversion}\ \dfrac{\begin{array}{cc}[A] & \\ \vdots & \\ \neg B & B\end{array}}{A}$$

and later clarifies that the second one is redundant for intuitionistic logic (but not for minimal logic).[15]

Now, in order to formulate classical logic without losing separability, he changes both the rules for implication and that for negation. First of all, he considers the extension of his system with[16]

$$\text{Dilemma}\ \dfrac{\begin{array}{cc}[\neg A] & [A]\\ \vdots & \vdots\\ B & B\end{array}}{B} \qquad \text{Tarski}\ \dfrac{\begin{array}{cc}[A\supset C] & [A]\\ \vdots & \vdots\\ B & B\end{array}}{B}$$

The problem with this system is that it is redundant since Tarki's rule and Dilemma are derivable one from the other. Indeed just one of them is enough to obtain classical logic from intuitionistic logic, but dropping one of them leads to a lack of separability, whether because we need Tarki's rule – and so $\supset$ – in order to derive the classically valid but intuitionistically invalid $\neg\neg A \vdash A$, or because we need Dilemma – and so $\neg$ – in order to derive the classically valid but intuitionistically invalid $(A \supset B) \supset A \vdash A$. To solve this problem, Milne proposes the following two changes:[17]

- By dropping $\neg\mathrm{I}_{Dummett}$, he blocks the derivation of Dilemma from Tarski;
- By substituting $\supset$I with $\text{Weak}{\supset}\text{I}\ \dfrac{B}{A\supset B}$, he blocks the derivation of Tarski from Dilemma.

At least from the point of view of separability, this system seems to work fine, but are these rules acceptable as meaning-conferring rules? Milne justifies his choice asserting that[18]

> "the orthodox formulation of an introduction rule presents certain grounds as sufficient for inferring a formula with some connective dominant; if, instead of inferring that formula, one has already used it as an assumption, then these grounds suffice to show that the assumption is unnecessary, it can be discharged, for one has to hand all that is needed to make do without it."

This explanation of the strange shape of Dilemma and Tarski seems convincing in my opinion, but we still need to consider harmony.

In [Milne, 2015] the author generalises this framework and deals also with harmony. This time the idea is explicitly to rethink the role of introduction rules, using a new interpretation of Prawitz's Inversion Principle and the symmetrical version proposed by Negri and von Plato[19]. He then concludes that

[14] [Milne, 2010], p. 179.
[15] [Milne, 2010], p. 188 note 12.
[16] Pp. 192,4. These are a generalization of the more common Peirce's rule and of the reformulation of *classical reductio* without $\bot$:

$$\textit{classical reductio}_{Milne}\ \dfrac{\begin{array}{c}[\neg A]\\ \vdots\\ A\end{array}}{A} \qquad \text{Peirce}\ \dfrac{\begin{array}{c}[A\supset C]\\ \vdots\\ A\end{array}}{A}$$

[17] [Milne, 2010], p. 196.
[18] [Milne, 2010], p. 197.
[19] [Negri and von Plato, 2001].

> "What is essential to an introduction rule is that it characterises conditions under which a logically complex assumption is unnecessary and hence may be discharged without loss."

As a consequence he changes all the introduction rules in what he calls their general-introduction version:[20]

$$\wedge\text{I}\ \frac{\begin{matrix}[A \wedge B]\\ \vdots \\ C\end{matrix} \qquad A \qquad B}{C} \qquad \vee\text{I}\ \frac{\begin{matrix}[A \vee B]\\ \vdots \\ C\end{matrix} \qquad A}{C} \qquad \vee\text{I}\ \frac{\begin{matrix}[A \vee B]\\ \vdots \\ C\end{matrix} \qquad B}{C}$$

$$\text{Dilemma}\ \frac{\begin{matrix}[\neg A]\\ \vdots \\ B\end{matrix} \qquad \begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{B} \qquad \text{Tarski}\ \frac{\begin{matrix}[A \supset B]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[A]\\ \vdots \\ C\end{matrix}}{C} \qquad \text{Weak}{\supset}\text{I}\ \frac{\begin{matrix}[A \supset B]\\ \vdots \\ C\end{matrix} \qquad B}{C}$$

In this way, Milne is able to prove harmony (or something very similar to it) for every pair of general-introduction and general-elimination rules.[21]

In contrast with Milne's earlier proposal of using $\neg \text{I}_{Milne}$ or $\neg \text{I}_{Dummett}$, that do not suit our non-circularity requirement,[22] this later proposal of a general-introduction reformulation does not violate any of the our *criteria*. The only lacks that we can find are:

- As acknowledged by the author himself, this solution does not work for first-order logic. To be precise, it is not possible to save the subformula property without adding *ad hoc* restrictions to the rules for $\forall$;[23]
- General-introduction rules do not suit the explicit definition of proof-theoretical validity seen in section 1.2.2.

In this work we did not consider first-order logic, nonetheless the first problem can not be overlooked, at least without an explicit motivation. About the second problem, there seems to be no proposal for an application of this kind for general-introduction rules. As a consequence, the endorsement of this solution asks for a bigger departure from standard proof-theoretic semantics than we are ready to accept, at least *prima facie*.[24]

2.3 Bilateral systems and their problems

Rumfitt's systems and their problems In [Rumfitt, 2000], the author proposes to save classical logic using both assertion and rejection as primitive speech acts. According to the traditional view, to reject a sentence is just to assert its negation, so there is no reason to assume the existence of a primitive speech act of rejection. Rumfitt objects that this tradition neglects the differences that there are between asserting the negation of a sentence (or answering "yes" when this negation is posed as a question) and just rejecting the sentence itself (that is answering "no" when it is posed as a question).[25] Regardless of the intuitive justification of the distinction between these two situations, the distinction itself liberalizes a lot the structure that a system of logic can have. We will follow Rumefitt and call this approach to meaning "bilateral", as opposed to the "unilateral" approach based on assertion alone. Rumfitt's idea is that while Dummett is right in maintaining that only intuitionistic logic suits the unilateral approach to theory of meaning, on the other side classical logic suits the bilateral approach to theory of meaning. So we need to interpret the disagreement about which one of these two logics is justified as a disagreement about whether a theory of meaning should be unilateral or bilateral. Rumfitt suggest that maybe a unilateral approach is justified in some applications (like in mathematics), while a bilateral one is justified in other applications (like in history).[26]

[20] The equivalence of the second rule for implication with Weak⊃I is established in [Milne, 2010], p. 198.

[21] Pp. 207-209. General-elimination rules are E-rules that have the form of standard $\vee$E, they have been studied by Read ([Read, 2010]), von Plato and Negri ([von Plato, 2001] and [Negri and von Plato, 2001]), Schroeder-Heister ([Schroeder-Heister, 1984]) and others. For every standard elimination rule, there is an equivalent general-elimination reformulation.

[22] See section 1.1.2.

[23] [Milne, 2015], p. 217.

[24] Peter Milne told me in a private talk that he acknowledges and accepts this aspect of his general-introduction rules.

[25] [Rumfitt, 2000], section V.

[26] [Rumfitt, 2000], section VIII.

$$\wedge\mathrm{I}^{+}\ \frac{+A \qquad +B}{+(A \wedge B)} \qquad \wedge\mathrm{E}^{+}\ \frac{+(A \wedge B)}{+A} \qquad \wedge\mathrm{E}^{+}\ \frac{+(A \wedge B)}{+B}$$

$$\wedge\mathrm{E}^{-}\ \frac{-(A \wedge B) \qquad \begin{matrix}[-A]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[-B]\\ \vdots \\ C\end{matrix}}{C} \qquad \wedge\mathrm{I}^{-}\ \frac{-A}{-(A \wedge B)} \qquad \wedge\mathrm{I}^{-}\ \frac{-B}{-(A \wedge B)}$$

$$\vee\mathrm{E}^{+}\ \frac{+(A \vee B) \qquad \begin{matrix}[+A]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[+B]\\ \vdots \\ C\end{matrix}}{C} \qquad \vee\mathrm{I}^{+}\ \frac{+A}{+(A \wedge B)} \qquad \vee\mathrm{I}^{+}\ \frac{+B}{+(A \wedge B)}$$

$$\vee\mathrm{I}^{-}\ \frac{-A \qquad -B}{-(A \vee B)} \qquad \vee\mathrm{E}^{-}\ \frac{-(A \vee B)}{-A} \qquad \vee\mathrm{E}^{-}\ \frac{-(A \vee B)}{-B}$$

$$\supset\mathrm{I}^{+}\ \frac{\begin{matrix}[+A]\\ \vdots \\ +B\end{matrix}}{+(A \supset B)} \qquad \supset\mathrm{E}^{+}\ \frac{+(A \supset B) \qquad +A}{+B}$$

$$\supset\mathrm{I}^{-}\ \frac{+A \qquad -B}{-(A \supset B)} \qquad \supset\mathrm{E}^{-}\ \frac{-(A \supset B)}{+A} \qquad \supset\mathrm{E}^{-}\ \frac{-(A \supset B)}{-B}$$

$$\neg\mathrm{I}^{+}\ \frac{-A}{+(\neg A)} \qquad \neg\mathrm{E}^{+}\ \frac{+(\neg A)}{-A}$$

$$\neg\mathrm{I}^{-}\ \frac{+A}{-(\neg A)} \qquad \neg\mathrm{E}^{-}\ \frac{-(\neg A)}{+A}$$

Table 2.1: **Operational Rules**

Let us now see the details of Rumfitt's proposal and whether it works as a justification of classical logic. First of all assertion and rejection are expressed with signed formulae: $+A$ represents the assertion of A, while $-A$ represents its rejection. Assertion and rejection occur only one time in every formula, that is they can not be iterated, and they can occur only at the outermost position. As an example, the following sentences are not well formed: $+-A$, $\neg -A$, $-(+A \wedge -A)$. Just for this section, we will follow Rumfitt in using lowercase Greek letters to indicate signed sentences in general, and α^* to indicate the opposite of α, that is $-A$ if α is $+A$ and $+A$ if α is $-A$. Moreover, Rumfitt accepts Tennant's opinion that $\bot$ is not a sentence, but just a punctuation mark that displays that something wrong has happened with a derivation, so $+$ and $-$ do not apply to it.[27]

Rumfitt formulates two different systems for classical logic, that we will call **RUMFITT1** and **RUMFITT2**. The first system contains all and only the operational rules for asserting and rejecting complex sentences that are exposed in table 2.1.[28] According to Rumfitt the only problem with this system is that it is too big, so he proposes the system **RUMFITT2**, that contains only the rules $\wedge\mathrm{I}^{+}$, $\wedge\mathrm{E}^{+}$, $\vee\mathrm{I}^{-}$, $\vee\mathrm{E}^{-}$, $\supset\mathrm{E}^{+}$, $\supset\mathrm{E}^{-}$, $\neg\mathrm{I}^{+}$ and $\neg\mathrm{E}^{+}$ together with the *coordination principles reduction* and *non-contradiction* of table 2.2. The third *coordination principle*, called Smiley, is just equivalent to the sum of the other two, so that an equivalent formulation of **RUMFITT2** can be obtained by extending the same set of operational rules with it. The only difference between the two formulations is that using Smiley we do not need $\bot$.

Rumfitt argues that these extra principles are only needed to make the other operational rules

[27] [Rumfitt, 2000], section IV.

[28] [Rumfitt, 2000], pp. 800-802.

$$\textit{Reductio}\ \dfrac{\begin{array}{c}[+A]\\ \vdots\\ \bot\end{array}}{-A} \qquad \textit{Reductio}\ \dfrac{\begin{array}{c}[-A]\\ \vdots\\ \bot\end{array}}{+A} \qquad \text{non-contradiction}\ \dfrac{+A \qquad -A}{\bot}$$

$$\text{Smiley}\ \dfrac{\begin{array}{c}[+A]\\ \vdots\\ +B\end{array} \qquad \begin{array}{c}[+A]\\ \vdots\\ -B\end{array}}{-A} \qquad \text{Smiley}\ \dfrac{\begin{array}{c}[-A]\\ \vdots\\ +B\end{array} \qquad \begin{array}{c}[-A]\\ \vdots\\ -B\end{array}}{+A}$$

Table 2.2: **Coordination Principles**

derivable, but while it is true that in **RUMFITT2** we can derive all the operational rules of **RUMFITT1** thanks to these principles, they are nonetheless needed in general to obtain an adequate system for classical logic. Indeed **RUMFITT1** formulated without *coordination principles* is weaker than classical logic, although it contains all the operational rules.[29] So we need to assume both *reduction* and *non-contradiction* (or *Smiley*) regardless of our choice to assume all the operational rules or only these of **RUMFITT2**.[30]

After proposing his system for classical logic, Rumfitt argues that this is the only acceptable system in such a bilateral framework. The argument consists of three steps:[31]

1. The coordination principle of *non-contradiction* should be assumed only for the atomic sentences and the operational rules should preserve it;

2. $\neg I^+$ and $\neg E^+$ define the meaning of $\neg$, and $\neg E^-$ is needed to prove *non-contradiction* for negated sentences (from the assumption that it works for atomic sentences);

3. $\neg E^+$ and $\neg E^-$ together prove double negation elimination.

The first point seems to be shareable, even though it will cause some troubles that we will discuss later, and the third point is just a matter of computation:

$$\neg E^-\ \dfrac{\neg E^+\ \dfrac{+(\neg\neg A)}{-(\neg A)}}{+A}$$

The second point is the most controversial. We will neglect to discuss Rumfitt's idea that $\neg I^+$ and $\neg E^+$ define the meaning of $\neg$, and just take for granted that there are good reasons for this choice. The big problem with this step of the argument is that Rumfitt is able only to show that $\neg E^-$ is *sufficient* to derive *non-contradiction* for negated sentences, while on the contrary, he can not prove that the adoption of this rule is *necessary* because, as observed by Gibbard, we have other rules that work fine. Gibbard proposes the following rule[32]

$$\neg E^-_{Gibbard}\ \dfrac{-(A) \qquad -(\neg A)}{\bot}$$

and it indeed preserves *non-contradiction* but can not be used to derive double negation elimination. So the choice of $\neg E^-$ and the subsequent justification of classical logic seem to be more arbitrary than Rumfitt admits.

Rumfitt answers to Gibbard's objection by denying that $\neg E^-_{Gibbard}$, $\neg I^-$, $\neg E^+$ and $\neg I^+$ together are able to specify the meaning of $\neg$.[33] His argument for this conclusion is that Gibbard's rules do not specify the conditions under which a negated sentence could and should be rejected. Indeed, while $\neg I^-$ gives sufficient conditions for the rejection of $\neg A$, $\neg E^-_{Gibbard}$ can not establish their completeness, since it is not in harmony with the first rule.

Nonetheless, while this answer could work as a rejection of $\neg E^-_{Gibbard}$, it does not establishes that $\neg E^-$ must be adopted in a set of rules that preserves *non-contradiction* and is able to specify the

[29] It is a kind of constructive logic with strong negation, in which de Morgan's laws and double negation elimination hold, but *tertium non datur* does not. See [Gibbard, 2002], p. 297 note 2.

[30] Rumfitt accepts this conclusion and argues that coordination principles were already planned to be assumed in his first system, so we will use the label **RUMFITT1** also for this extended system. See [Rumfitt, 2002].

[31] [Rumfitt, 2000], p. 814-816.

[32] [Gibbard, 2002], p. 299-300.

[33] [Rumfitt, 2002], p. 310.

meaning of ¬. Indeed Kürbis proposes an intuitionistic bilateral system that is harmonious and that preserves *non-contradiction*.[34] In order to obtain his intuitionistic system, he opts for a revision of both the operational rules and the coordination principles of **RUMFITT1**.

About the operational rules, Kürbis proposes to substitute Rumfitt's rules with the following:[35]

$$\wedge\mathrm{I}^- \ \dfrac{\begin{array}{c}[+A]\\ \vdots\\ -B\end{array}}{-(A\wedge B)} \qquad\qquad \wedge\mathrm{E}^- \ \dfrac{-(A\wedge B) \qquad +A}{-B}$$

$$\supset\mathrm{I}^- \ \dfrac{-B \qquad \begin{array}{c}[-A]\\ \vdots\\ \alpha\end{array} \qquad \begin{array}{c}[-A]\\ \vdots\\ \alpha^*\end{array}}{-(A\supset B)} \qquad \supset\mathrm{E}^- \ \dfrac{-(A\supset B) \qquad -A}{\beta} \qquad \supset\mathrm{E}^- \ \dfrac{-(A\supset B)}{-B}$$

$$\neg\mathrm{I}^- \ \dfrac{\begin{array}{c}[-A]\\ \vdots\\ \alpha\end{array} \qquad \begin{array}{c}[-A]\\ \vdots\\ \alpha^*\end{array}}{-(\neg A)} \qquad\qquad \neg\mathrm{E}^- \ \dfrac{-(\neg A) \qquad -A}{\beta}$$

Now, in order to obtain an intuitionistic system, we assume only the following version of Smiley:[36]

$$\text{Intuitionistic Smiley} \ \dfrac{\begin{array}{c}[+A]\\ \vdots\\ +B\end{array} \qquad \begin{array}{c}[+A]\\ \vdots\\ -B\end{array}}{-A}$$

Kürbis proves that his intuitionistic system shares all the good properties of **RUMFITT1**, and so concludes that Rumfitt's argument that a bilateralist approach leads necessarily to the justification of classical logic is wrong.

Let us now return to the problem of step 1. While it is plausible that *non-contradiction* should be assumed only for atomic sentences and the operational rules should preserve it for logically complex ones, it seems reasonable to require the same for *reduction*. Unfortunately, this can not be done, as discovered by Ferreira: we need to assume this coordination principle for the entire language.[37] The same situation holds also for Smiley, so we can not solve this problem by changing the formulation.[38] Ferreira correctly points out that, as a consequence of this *phenomenon*, coordination principles are not irrelevant for the meaning of logical terms. Were it enough to postulate them for atomic sentences, they would be just principles about the relation between speech acts of assertion and rejection; we would have coordination principles that characterise assertion and rejection, and operational rules that define the meaning of logical terms. Unfortunately, the restriction of *reduction* to atomic sentences change the behaviour of logical terms, so this distinction can not be so clear.

The situation is even worse, because, not only we need justification of coordination principles for complex sentences and we lack it, but there are also alternative coordination principles that lead to other logical systems. Indeed as we already saw, both Humberstone and Kürdis propose intuitionistic bilateral systems that use other coordination principles. To tell the truth, Kürbis's system is different from Rumfitt's ones also regarding the operational rules. Nonetheless, Kürbis himself seems to acknowledge that, although his intuitionistic system and Rumfitt's classical one differ both in their operational and coordination rules, the main difference is given by the second group. Indeed just the extension of them makes derivable the classical operational rules, while the purely operational extension is not adequate for classical logic (as shown for the first system of Rumfitt). So the disagreement between classical and intuitionistic logic ends up being a disagreement between two coordination principles that are not justified in our bilateral theory of meaning:[39]

> "The question about which logic is the right one has thus been pushed from the operational rules governing the connectives, as was Dummett's proposal, to the structural rules of the

[34] [Kürbis, 2016], p. 634-637. His system is explicitly inspired by that of [Humberstone, 2000].

[35] All the other operational rules in **RUMFITT1** are adopted without any change.

[36] Of course we could obtain also an intuitionistic system that uses ⊥ and the coordination principles *reductio* and *non-contradiction*.

[37] [Ferreira, 2008].

[38] [Kürbis, 2016], p. 635.

[39] [Kürbis, 2016], p. 637.

system. Dummett gave criteria for singling out justified operational rules. Rumfitt has not provided a similar proposal to single out justified structural rules."

Kürbis is quite sympathetic to this conclusion and seems to foresee the possibility for a kind of logical pluralism. This is in line with his idea that proof-theoretic semantics should not try to characterise all the meaning of logical terms, but just some of its aspects. Indeed we already saw his opinion about the impossibility of characterising completely negation and absurdity using inferential rules,[40] and he poses some similar problems also for modal terms.[41] Nonetheless, we will not consider this as a viable alternative. Indeed in order to obtain a pluralism, we need to have the validity of some logical systems, while here we have two systems that contain rules that are not justified.[42]

This far, we have found two main objections to the bilateral justification of classical logic:

- Coordination principles are not justified (and what is worse they can not be restricted to atomic sentences in general);
- They are responsible for the choice of the right logic.

Let us now consider another, more recent criticism that tries to undermine bilateralism in general.

Gabbay's new objections Michael Gabbay proposes the two following problems for bilateralism:[43]

- Although, as we saw in section 1.2.1, Read's proposal of a logical constant $\bullet$ for liar's paradox can not be accepted in unilateral proof-theoretic semantics, there are trivialising rules for this constant in a bilateral framework that can not be rejected;
- Although the usage of *tonk*-rules in a unilateral framework allows the construction of non-normalizable derivations, in a bilateral one we can normalize those derivations.

The first argument is developed in three steps. First of all, Gabbay proposes the following pair of rules, that leads to triviality if paired with coordination principles:

$$\frac{-\bullet}{+\bullet} \qquad \frac{+\bullet}{-\bullet}$$

The author points out that they are neither I nor E-rules, and that for this reason they can not be accepted. His idea is to find sets of rules that behave similarly but that are acceptable in bilateral proof-theoretic semantics. The first set of rules of this kind that he proposes is:

$$\bullet\mathrm{I}^-\ \frac{+A}{-\bullet} \qquad \bullet\mathrm{I}^+\ \frac{-A}{+\bullet} \qquad \bullet\mathrm{E}^-\ \frac{+\bullet \qquad -A}{-A} \qquad \bullet\mathrm{E}^+\ \frac{-\bullet \qquad +A}{+A}$$

As Gabbay observes, clearly an introduction and a subsequent elimination of $\bullet$ can be easily reduced, so these rules suit Inversion Principle. This observation is enough for Gabbay in order to conclude that these rules are harmonious, but we posed more stringent requirements abut normalizability.[44] Nonetheless, there are no problems even for our definition of harmony, since in this case we do not use pairs of I and E-rules to derive $\perp$, but just pairs of I-rules for assertion and rejection. Indeed, just using $\bullet\mathrm{I}^-$, $\bullet\mathrm{I}^+$ and Smiley we can prove both $+p$ and $-p$ for every sentence p.[45]

The third and more interesting set of rules that Gabbay evaluates is:

$$\bullet\mathrm{I}^-\ \frac{+A \qquad -A}{-\bullet} \qquad \bullet\mathrm{I}^+\ \frac{+A \qquad -A}{+\bullet}$$

$$\bullet\mathrm{E}^+\ \frac{+\bullet}{+A} \qquad \bullet\mathrm{E}^+\ \frac{+\bullet}{-A} \qquad \bullet\mathrm{E}^-\ \frac{-\bullet}{+A} \qquad \bullet\mathrm{E}^-\ \frac{-\bullet}{-A}$$

This set of rules has all the interesting properties of the previous one: the rules suits Inversion Principle, qualify as good introduction and elimination rules, and leads to triviality, since:

$$\bullet\mathrm{E}^-\ \dfrac{\text{Smiley}_1\ \dfrac{\bullet\mathrm{E}^+\ \dfrac{[+\bullet]^1}{+B} \qquad \bullet\mathrm{E}^+\ \dfrac{[+\bullet]^1}{-B}}{-\bullet}}{+A} \qquad \bullet\mathrm{E}^-\ \dfrac{\text{Smiley}_1\ \dfrac{\bullet\mathrm{E}^+\ \dfrac{[+\bullet]^1}{+B} \qquad \bullet\mathrm{E}^+\ \dfrac{[+\bullet]^1}{-B}}{-\bullet}}{-A}$$

[40] See section 1.2.3.
[41] [Kürbis, 2015a] section 5.
[42] We will see in chapter 4 our proposal for an inferentialist pluralism.
[43] [Gabbay, 2017].
[44] See definition 1.2.5 in section 1.2.1.
[45] Gabbay's formulation uses Smiley, but of course you could substitute it with *non-contradiction* and *reduction*.

Moreover, the introduction rules are valid instances of Smiley, with a vacuous discharge of $+\bullet$ in $\bullet\text{I}^-$ and of $-\bullet$ in $\bullet\text{I}^+$. So they are not only acceptable because in harmony, but also acceptable because derivable from Smiley. In passing, the fact that Rumfitt's formalization of bivalence does not exclude incoherence is even more problematic if we consider the fact that the author explicitly rejects Dummett's opinion that logic should not entail coherence.[46]

It is important to remark that this counterexample does not prove too much, that is that these sets of harmonious but trivialising rules can not be adapted for undermining unilateral proof-theoretic semantics using negation instead of rejection. Indeed, as observed by Gabbay himself the rules

$$\frac{A \qquad \neg A}{\neg\bullet} \quad \frac{A \qquad \neg A}{\bullet} \quad \frac{\bullet}{A} \quad \frac{\bullet}{\neg A} \quad \frac{\neg\bullet}{A} \quad \frac{\neg\bullet}{\neg A}$$

qualify both as rules for $\bullet$ and as rules for $\neg$.[47] As a consequence if we adopt them, the rules of negation are no longer harmonious and can not be accepted. As an example, it seems uncontroversial that in general we can not reduce derivations in which an application of $\neg$I used to derive $\neg\bullet$ is followed by an application of $\frac{\neg\bullet}{A}$.[48]

In an answer to Gabbay, Francez points out that none of the sets of rules proposed is really acceptable, because they do now show the right relation between assertion and rejection.[49] This is more obvious in the last such set, since $\bullet\text{I}^+$ and $\bullet\text{I}^-$ have the same premises but derive different (opposite) conclusions. Francez's observation about how assertion and rejection usually work in everyday life is shareable: the conditions that justify the assertion of a sentence usually are not the same that justify its rejection. Nonetheless, there are pathological cases in which this happens, like for liar's sentence. Francez's intention of excluding these cases is comprehensible and in line with the proof-theoretic approach to linguistic practice, but we have to evaluate his means to do so.

Francez proposes that there should be a horizontal balance between rules for $+$ and rules for $-$, and, for this task, he proposes a principle that should work like harmony.[50] With his principle, he can obtain the rules for the rejection of a constant functionally from its rules for the assertion. The formal details of his proposal are not important, but this principle manages to exclude Gabbay's rules, at least formally.[51]

I think that there are nonetheless some problems for Francez's solution from the meaning-theoretical point of view. Indeed he proposes it as a "non-technical reason" to believe that classical logic suits bilateralism, but it is a purely formal principle that needs justification. We saw the same problem for the coordination principles of Rumfitt but, in this case, the situation is even worse, since the entire coherence of bilateralism is in doubt. Surely, Francez can rely on the idea that assertion and rejection should be coherent and claims that also Restall's bilateral approach in [Restall, 2005] starts from this assumption, but proof-theoretic semantics asks for something more than an intuitive justification. Traditional request for harmony comes from the idea that I-rules define the meaning of the connectives and E-rules are in some way justified if they are harmonious. In order to ask for horizontal balance, it seems that we need something similar to happen for $+$ and $-$. But this poses some interesting problems.

Rumfitt claims that both his systems are separable, since every classically valid logical consequence can be proved using only rules for the connectives that explicitly occur in the result.[52] Nonetheless, we have seen that it is necessary to assume some principles about the relationship between $+$ and $-$, and that they also interact in strange ways with the behaviour of the logical constants. We saw that we need coordination principles in order to have a classical system, and we also need horizontal balance in order to have coherence. So we could argue:

- That in order to decide about separability for logical terms we should also consider the application of coordination principles for non-atomic sentences;
- That we should also consider the occurrence of $+$ and $-$ when we evaluate separability.

While I doubt that the first point poses some real threats, since the application of a coordination principle for a non-atomic sentence shall be paired with some operational rules for the outermost constant, I think nonetheless that the second point is heavily problematic, since already the proof of $+(\neg\neg A) \vdash +A$ asks for applications of $-$. So apart from the problem of justifying both coordination

[46][Rumfitt, 2000], section IV.
[47][Gabbay, 2017], p. S112, note 8.
[48]We already stressed this point when we were evaluating Read's proposal of $\bullet$ in standard unilateral proof-theoretic semantics. See section 1.2.1.
[49][Francez, ming].
[50][Francez, ming], section 4.
[51]The formal details are developed in [Francez, 2013] and in section 4.4.1.7 of [Francez, 2015].
[52][Rumfitt, 2000], p. 808.

principles and horizontal balance in a bilateral approach, these principles endanger also separability of the system, since they raise the problem of the meaning of + and −. Maybe a weak separability criterion could save the situation, but I could not find any author working in this field that proposes such a solution.

Let us now consider Gabbay's second objection.[53] He believes that the adoption of Smiley entails an enlargement of the notion of "normal derivation" and that, for this reason, it enables the reduction of the *maximal formulae* generated by *tonk*-rules. If this is the case, then any bilateral version of proof-theoretic semantics that includes this rule is not able to exclude *tonk* and so to warrant consistency. His proposal for such a reduction is:

$$
\text{\textit{tonk}E}\,\dfrac{\text{\textit{tonk}I}\,\dfrac{+A}{+(A tonk B)}}{+B}
\quad\rightsquigarrow\quad
\text{Smiley}_2\,\dfrac{\text{\textit{tonk}I}\,\dfrac{+A}{+(A tonk B)} \qquad \text{Smiley}_1\,\dfrac{\text{\textit{tonk}E}\,\dfrac{[+(A tonk B)]^1}{+B} \qquad [-B]^2}{-(A tonk B)}}{+B}
$$

Of course this "reduction" looks very strange, but it is true that it manages to derive B from A without producing maximal *formulae*, since the major premise of *tonk*E is not the conclusion of *tonk*I. Francez argues that the second derivation does not qualify as a real reduction of the first, because it does not solves the problem caused by *tonk*, but just spreads it out in the derivation. That is there still is a detour, although it is not in plain view. According to Francez, reduction procedures should also be specifications of something already contained in the derivation that we reduce, but this is not the case with this alleged reduction.[54]

While I agree with Francez in his evaluation of this "reduction", I believe that he misinterpreted Gabbay's intentions, or at least that his counterexample could be adjusted so as to answer to Francez's objections. Indeed Gabbay seems to acknowledge that what he proposes is not a valid reduction but, nonetheless, there should still be a formal *criterion* that points out this, while in a system with Smiley it is not clear how we can reject it. I think that another, symmetric way of looking at Gabbay's observation could shed light on what I mean. Since according to Francez the derivation on the right does not qualify as in normal form, it should qualify as containing a detour. Nonetheless, the standard definition of *maximal formula* is useless for this purpose, since in it there is no conclusion of I-rule that is the major premise of an E-rule. The observation that the detour is just "spread out" in the derivation is just an intuitive observation, so we need something more concrete. As a consequence, in the bilateral systems we should take care also of these hidden detours (or fake normal derivations), while there seems to be no possibility to do so.

It seems that the issue of distinguishing acceptable and unacceptable reductions, and the issue of acknowledging the detour present in Gabbay's alleged reduction are essentially the same. So a formal *criterion* that discriminates between valid and non-valid reductions should use a more general notion of *maximal formula*. Nonetheless, while Francez's meaning theoretical reasons are arguably enough to disqualify Gabbay's alleged reduction, we need a precise formal *criterion* to discriminate and a precise reduction procedure to deal with this new kind of non-normality. I am not sure whether this is the original attack plan of Gabbay or a reformulation of it, but there seems to be no significant answer to it in Francez's paper.

Something similar happens when we accept disjunction in a natural deduction system and we are forced to consider *maximal sequences* in addition to *maximal formulae*. Indeed if we do not define *maximal sequences*, then we could eventually use ∨-rules to "reduce" some *maximal formulae*, by moving an application of ∨E between the I and the E-rule in the following way:

$$
\vee\text{E}\,\dfrac{A \vee B \qquad \oplus\text{E}\,\dfrac{\oplus\text{I}\,\dfrac{\begin{matrix}[A]\\ \vdots\\ C\end{matrix}}{D}}{E} \qquad \oplus\text{E}\,\dfrac{\oplus\text{I}\,\dfrac{\begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{D}}{E}}{E}
\quad\rightsquigarrow\quad
\oplus\text{E}\,\dfrac{\vee\text{E}\,\dfrac{A \vee B \qquad \oplus\text{I}\,\dfrac{\begin{matrix}[A]\\ \vdots\\ C\end{matrix}}{D} \qquad \oplus\text{I}\,\dfrac{\begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{D}}{D}}{E}
$$

Admittingly, it is not easy to imagine a very dangerous application of this fake reduction procedure, since it asks for a very specific situation. Indeed obviously this "reduction" procedure can not be applied to every *maximal formula*. Nonetheless, there is at least a case in which the availability of this reduction is relevant. Let us consider what happens if we add to *tonk*I the following condition of applicability: the major premise of the *tonk*-rule must depend on an assumption that is discharged

[53] [Gabbay, 2017], pp. S113-S114.
[54] [Francez, ming], section 5.

by an application of $\vee$E such that the minor premise of $\vee$E that depends on the same assumption is the *tonk*-formula or the conclusion of the rule that has this formula as major premise, and in the sub-derivation of the other minor premise there must be another application of *tonk*I that has the same form. That is to say, the only acceptable application of this new kind of *tonk* is

$$\vee\text{E}\,\dfrac{A \vee B \qquad \oplus\text{E}\,\dfrac{tonk\text{I}\,\dfrac{\begin{matrix}[A]\\ \vdots \\ C\end{matrix}}{C tonk D}}{E} \qquad \oplus\text{E}\,\dfrac{tonk\text{I}\,\dfrac{\begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C tonk D}}{E}}{E}$$

This is a variation of the traditional *tonk*-rules that is only apparently less dangerous. Indeed this "weakened" *tonk*I can be used together with standard *tonk*E to prove $\bot$ in the following way:

$$\vee\text{E}_2\,\dfrac{\vee\text{I}\,\dfrac{\supset\text{I}_1\,\dfrac{[A]^1}{A \supset A}}{(A \supset A) \vee \bot} \qquad tonk\text{E}\,\dfrac{tonk\text{I}\,\dfrac{[A \supset A]^2}{(A \supset A) tonk \bot}}{\bot} \qquad tonk\text{E}\,\dfrac{tonk\text{I}\,\dfrac{Efq\,\dfrac{[\bot]^2}{A \supset A}}{(A \supset A) tonk \bot}}{\bot}}{\bot}$$

Since we pose coherence as a minimal requirement for the acceptability of a set of rules, we want to reject this reformulation of *tonk*. Interestingly, if we do not qualify *maximal sequence* as a kind of detour, we are forced to accept this as a good pair of rules. Indeed the following reduction procedure can be given:

$$\vee\text{E}\,\dfrac{A \vee B \qquad tonk\text{E}\,\dfrac{tonk\text{I}\,\dfrac{\begin{matrix}[A]\\ \vdots \\ C\end{matrix}}{C tonk D}}{D} \qquad tonk\text{E}\,\dfrac{tonk\text{I}\,\dfrac{\begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C tonk D}}{D}}{D} \quad \rightsquigarrow \quad tonk\text{E}\,\dfrac{\vee\text{E}\,\dfrac{A \vee B \qquad tonk\text{I}\,\dfrac{\begin{matrix}[A]\\ \vdots \\ C\end{matrix}}{C tonk D} \qquad tonk\text{I}\,\dfrac{\begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C tonk D}}{C tonk D}}{D}$$

On the contrary, if we define *maximal sequences* in the usual way, we can not accept this reduction because the derivation on the right neither is in normal form, nor can be reduced to normal form. That is the rejection of this alleged reduction step is the same as the extension of the notion of *maximal formula*. In the same way, the rejection of Gabbay's proposed reduction should be paired with an extension of this same notion. Unfortunately, such an explicit generalization is missing in Francez's paper, so we can not see his answer as satisfactory.[55]

In conclusion, while in contrast to the system based on general-introduction rules, it comprises a complete characterization of proof-theoretic validity,[56] bilateral proof-theoretic semantics still has some open problems:

- A justification of the coordination principles and of Francez's horizontal equilibrium is needed in order to consider it as a valid solution. It is not improbable to find such a justification, but I suspect that it requires a great departure from the orthodox Dummettian approach. It is not by accident that the only philosopher Francez refers to when he tries to justify his principle of balance is Greg Restall, that explicitly departs from the standard proof-theoretic approach that we are investigating here and adopts a conception of the theory of meaning inspired by Brandom's work.[57] Moreover, such a justification should also solve the worries about the influence that these rules have on the meaning of the logical terms and about their consequent role in the selection of the right logic.

- A generalization of the notion of *maximal formula* is needed in order to deal with new kinds of non-normality that exploit coordination principles. As we just saw, another side of this problem is the lack of a precise reason to reject unwanted reduction steps for *tonk*-rules.

[55]We will see that we need a similar generalization of the notion of *maximal formula* in our favourite reform of proof-theoretic semantics as well, both for intuitionistic and classical systems. Nonetheless, we will deal with it in what I believe is a satisfactory way. See sections 3.2.2 and 3.3.2.

[56][Francez, 2013] and [Francez, 2015].

[57]See [Restall, 2005] and [Restall, 2008] *inter alia*.

Axiom

$$A \Rightarrow A$$

Structural Rules

$$Weak\Rightarrow\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \Rightarrow Weak\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$Con\Rightarrow\ \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \Rightarrow Con\ \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$Perm\Rightarrow\ \frac{\Gamma, A, B, \Theta \Rightarrow \Delta}{\Gamma, B, A, \Theta \Rightarrow \Delta} \qquad \Rightarrow Perm\ \frac{\Gamma \Rightarrow \Delta, A, B, \Lambda}{\Gamma \Rightarrow \Delta, B, A, \Lambda}$$

$$Cut\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Theta, A \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda}$$

Operational Rules

$$\wedge\Rightarrow\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \wedge\Rightarrow\ \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}$$

$$\Rightarrow\wedge\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$\vee\Rightarrow\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}$$

$$\Rightarrow\vee\ \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} \qquad \Rightarrow\vee\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

$$\supset\Rightarrow\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Theta, B \Rightarrow \Lambda}{\Gamma, \Theta, A \supset B \Rightarrow \Delta, \Lambda} \qquad \Rightarrow\supset\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$$

$$\neg\Rightarrow\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad \Rightarrow\neg\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$$

Table 2.3: **LK**

While maybe there could be a solution to the second problem, I am more pessimistic about the first one. Indeed coordination principles seem to be acceptable in proof-theoretic semantics only if they are dispensable – that is if they are used only to make more contract the system, as in the original plan of Rumfitt –, but this can not be the case here, as we saw. Since this approach seems to be unsatisfying, let us now move on to some other proposals.

2.4 Multiple-conclusion systems and their problems

This section is in some way a point of no return for this research. Indeed the attempt to justify classical logic with multiple-conclusion systems inspires a main criticism to a shared assumption of all the proof-theoretic semantics projects seen until now. Since we will not be able to rebut this criticism and we will see that the common objections posed to it are lacking, we will bite the bullet and impose a major change in proof-theoretic semantics, by abandoning the relevant assumption.

First of all, let us consider the standard arguments for a multiple-conclusion justification, and how this attempt to save classical logic gets developed. Then, we will use the standard criticisms of this framework to reform all proof-theoretic semantics.

2.4.1 Sequent calculus

The *status* of sequent calculus One of the main reasons to believe that a multiple-conclusion system could justify classical logic is the sequent calculus system **LK** (table 2.3) proposed by Gentzen.[58] Indeed this system is adequate for classical logic and suits all the sequent calculus equivalents of the

[58] [Gentzen, 1969b].

proof-theoretic requirements that we are looking for: it is separable, it suits subformula property and it suits Cut elimination (that is essentially the counterpart of harmony).

Technically speaking, this system does not use multiple conclusions. Indeed, both the premisses and the conclusions of this system are sequents, and there are no rules with more than one sequent in the conclusion. Nonetheless, sequent calculus can be seen as a meta-calculus that speaks of a derivation system and, according to this interpretation, **LK** speaks about a multiple-conclusion system. This reading of sequent calculus is not uncontroversial, and indeed Shoesmith and Smiley distinguish two distinct interpretations of sequent calculi:[59]

material interpretation: the arrow is an object-language conditional, so that the meaning of $A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n$ is $(A_1 \wedge \ldots \wedge A_m) \supset (B_1 \vee \ldots \vee B_n)$;[60]

metalinguistic interpretation: the arrow is equivalent to the metalinguistic term "$\vdash$", so that the sequent calculus is a meta-calculus that speaks of an objective-language calculus and the meaning of $A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n$ is that in that calculus the conclusions $B_1, \ldots, B_n$ are derivable form the assumptions $A_1, \ldots, A_m$.

Of course, the relevant reading for proof-theoretic semantics is the second one, and it is a consequence of this reading that we need some kind of justification of multiple-conclusion systems in order to accept the mere existence of **LK** as an antirealistic justification of classical logic. The first interpretation is at odd with proof-theoretic interpretation because it leads to rules that seem to be neither I, nor E-rules. Moreover, if we try to interpret $\Rightarrow\supset$ as the meaning conferring rule for $\supset$ we obtain a violation of non-circularity,[61] since in

$$\Rightarrow\supset \frac{(\Gamma \wedge A) \ \supset \ (B \vee \Delta)}{\Gamma \ \supset \ ((A \supset B) \vee \Delta)}$$

$\supset$ occurs both in the premise and in the conclusion, and so there is a circular dependence of meaning. On the other side, the choice of $\supset\Rightarrow$ as meaning conferring rule poses the same problem, and the choice of $\Rightarrow \oplus$ as meaning conferring rule for $\oplus$ is, in general, justified by the similarity between introduction rules in natural deduction and rules on the right in sequent calculus. In addition, this analogy is supported by all the standard translation of derivations from natural deduction to sequent calculus and *vice versa*.[62]

Of course this problem is a consequence of the well-known circularity that we encounter if we try to define validity using $\supset$I: $\supset$I asks for a valid derivation of B, possibly form an open assumption A, in order to introduce $A \supset B$, so a definition of validity that takes for granted an understanding of $\supset$I is in danger of circularity. Nonetheless, we saw in section 1.2.2 that it is possible to avoid this circularity between $\supset$I and validity, at least for purely logical derivations, by an explicit definition of validity that uses canonical proofs and reduction procedures. On the contrary, by accepting the material interpretation of sequent calculus and the subsequent interpretation of $\Rightarrow\supset$ we obtain a direct circularity, that can not be solved in this way. For this reason, the first reading is disqualified in our case.[63]

Endorsing the metalinguistic reading of sequents means that in order to evaluate the acceptability of **LK** or of other sequent calculi for classical logic, we need to evaluate the acceptability of the object-language systems that they describe. This can be done in two different ways:

- By focusing on the properties of the sequent calculus and so only indirectly on the properties of the object-language system described;
- By reconstructing the object-language system and evaluating directly it.

We will focus mainly on the second approach, since it is obviously easier to evaluate directly the properties of a system, than indirectly, using another system that describes it. Nonetheless, it is worth spending some words also on the proposals of the first kind, and this is what we are going to do in this section.

[59] [Shoesmith and Smiley, 1978], p. 33.

[60] We should be more careful in the application of conjunction and disjunction, but nonetheless the idea is clear, so we follow the authors (and Gentzen) by neglecting this issue here.

[61] See definition 1.1.9.

[62] [Gentzen, 1969b], section V subsection 4 for the translation from natural deduction to sequent calculus; [Prawitz, 1965], appendix A section 2 for the opposite translation (Prawitz proposes a translation of multiple-succedent sequent calculus that does not use multiple-conclusion, but it has the same problems of Prawitz's formulation of natural deduction for classical logic). Moreover, [Negri and von Plato, 2001] is a monograph entirely dedicated to the topic of the relationship between natural deduction and sequent calculus, in which the authors evaluate different sequents calculi and translations. One of the few standing points is that rules for the introduction on the right are the sequent calculus equivalent of the I-rules.

[63] Nonetheless, we will see that some of its aspects are shareable, like the connection of multiple-antecedent with conjunction, and of multiple-succedent with disjunction.

Hacking Apart from a general vulgate about the possibility of justifying classical logic using sequent calculus instead of natural deduction, the first well developed proposal in this direction is [Hacking, 1979].[64] Hacking has essentially two purposes in this work: to find a *criterion* to discern logical terms from non-logical terms and to justify logic. The two issues turn out to be heavily connected. Anyway, the first issue is tied to Hacking's endorsement of a weakened version of logicism, that we do not have space to cover, so we will focus mainly on the justificatory part of his speculations.

Hacking's idea is that the logical laws are essentially by-products of the way in which logical terms are used in general. He takes this position from Wittgenstein's *Tractatus*, where logical laws come out from combinatorial properties expressed by truth tables, but reinterprets this idea in an antirealistic perspective:[65]

> "Then some compound sentences take the value true regardless of their components. This fact is a by-product of rules for the introduction of the logical constants."

Hacking uses Gentzen's distinction between operational and structural rules, arguing that:

- Structural rules correspond to basic properties of logical consequence, that is transitivity (Cut), idempotence (Weakening) and reflexivity (Axiom), and so they are complete for the prelogical language of sequent calculus (that is the version of sequent calculus without operational rules derives all the logical consequences that are necessarily present in a language without connectives).[66]

- Operational rules characterise the meaning of the logical terms but, since neither characterizations nor definitions *tout court* should extend or reduce our knowledge of the world, Hacking asks that their addition to structural rules be conservative. Instead of asking directly for conservativeness, Hacking asks for the subformula property and for the following theorems:

  Cut elimination: The Cut rule must be admissible, that is every sequent provable in the system with the Cut rule must be provable also without it;

  Preservation of Axiom and Weakening: The general version of the rules of Axiom and Weakening must be admissible in the system with only their version restricted to atomic main formulae.

  Of course, Cut elimination entails Cut preservation, so we have this requirement for all the structural rules. Hacking's reasons to ask the preservation of the basic properties of deduction is that, otherwise, $\vdash$ (or $\Rightarrow$) would mean something different in the pre-logical language and in the full language. His main inspiration for this requirement is Prior's connective *tonk*, for which he proposes an analysis that is in line with Belnap's, since both asks for conservativeness.[67] According to Hacking, the problem with *tonk* is that it can not be adopted in a language without losing transitivity of deduction, that is without modifying the meaning of $\vdash$, and this is the reason why it must be rejected.[68] The author asks the full admissibility only of the Cut rule because he sees a connection between this theorem and conservativeness.[69] This connection is intuitively acceptable, but we saw (in section 1.2.2) that a similar connection between normalizability and conservativeness is unlikely to hold in general, and it seems implausible that Cut elimination has more chance of success. This is probably the reason why Hacking shows flexibility in his *criteria*.[70]

So the distinctive characteristic of logical terms is that they can be characterised using rules for sequents that suit some structural properties.[71] Later on in the paper, Hacking argues that (model-theoretic) semantic *criteria* can be devised based on these proof-theoretical ones, that anyway do not lose their primacy.

[64] His metalinguistic interpretation of sequents is expressed on p. 292, but he never considers directly the system described by **LK**.

[65] [Hacking, 1979], p. 288. Section IV is entirely devoted to the endorsement of antirealism.

[66] [Hacking, 1979], p. 293 for the basic properties of deduction (that are sufficient, but arguably not necessary, properties to a be a relation of deduction) and p. 311 for the completeness of structural rules.

[67] Hacking's refers explicitly to [Prior, 1960], but he does not mention [Belnap, 1962]. See [Hacking, 1979], p. 296.

[68] Tonk's issue with transitivity is well-known in general, nonetheless there is at least a proposal of a transitive system in which this constant can be accepted: [Cook, 2005]. It could be interesting to evaluate the acceptability of this logic from the point of view of Hacking's *criteria*.

[69] P. 296.

[70] See section XIV

[71] Already on page 291 Hacking asserts that he "give(s) reasons for saying that anything defined by a rule of inference like Gentzen's is a logical constant", but he takes all the article to explain what it means for a rule to be like Gentzen's.

I think that this attempt of justifying classical logic fells victim of a criticism proposed by Rumfitt for justifications based on sequent calculus in general. Rumfitt argues that in sequent calculus "the sentences [...] are mentioned rather than used", while it is usage that gives the meaning of expressions.[72] To be precise, his observation is directed against [Shoesmith and Smiley, 1978], but these authors do not focus exclusively on sequent calculus, so I believe that it is better to evaluate it in relation with other proposals, like Hacking's one.

Let us look deeper into Rumfitt's objection. He observes that if we accept the metalinguistic reading of sequents, then we have to interpret them as meaning something like: {"If it is raining then it is not snowing", "It is raining"} entails {"It is not snowing"}.[73] Indeed they are sets of sentences connected by a relation of entailment, symbolized by $\Rightarrow$. Since we need usage in order to have meaning, we have to investigate the possibility of using the sentences according to the entailments described by the sequent calculus. But while this can easily be done for single-succedent sequent systems, there seems to be no possibility of having such usage for multiple-succedent systems. We will see that this objection is not completely right, indeed we can devise multiple-conclusion systems also accepting most of the restrictions imposed usually in proof-theoretic semantics.[74] Nonetheless, it is surely true that Hacking neglects to answer this issue, taking for granted the possibility of a multiple-conclusion object language system.[75]

Restall's bilateral approach Before considering viable answers to Rumfitt's objection, let us consider a further approach based on sequent calculus that, contrary to Rumfitt's one, can hardly answer to Rumfitt's objection. Greg Restall proposes a meaning-theoretic interpretation of sequent calculus that is metalinguistic but does not describe deduction directly.[76] He assigns to every sequent $\Gamma \Rightarrow \Delta$ a dialogical state $[\Gamma; \Delta]$ in which a speaker asserts each sentence in Γ and deny each sentence in Δ. We already argued when we were evaluating Rumfitt's version of bilateralism that Restall's justification of logic is very far from proof-theoretic semantics, but the criticism that we are rising here is more deep. According to Restall, some states are incoherent and so self-defeating, like $[A; A]$, in which a speaker asserts and denies the same sentence. While incoherent states can never be adopted correctly, depending on the theory of negation that one endorses, inconsistent cases, that is cases in which he asserts both a sentence and its negation, can be acceptable. By distinguishing coherence and consistency Restall obtains a very flexible system, that can be adapted at least to classical, intuitionistic and dual-intuitionistic logics.[77]

Provable sequents, that we will indicate as $\vdash \Gamma \Rightarrow \Delta$, correspond to incoherent states, and so the rules of sequent calculus deal with conditions of coherence. While operational rules describe what we can or can not coherently assert and reject about complex sentences, based on what it is coherent to assert and reject about less complex sentences, Restall pairs structural rules of **LK** with properties of coherence and incoherence of states in general:

- Axiom $A \Rightarrow A$ corresponds to incoherence of $[A; A]$;
- Weakening rules $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$ and $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$ correspond to the property that if $[\Gamma; \Delta]$ is incoherent, then also $[\Gamma'; \Delta']$, with $\Gamma' \supseteq \Gamma$ and $\Delta' \supseteq \Delta$, is incoherent;
- Cut rule $\dfrac{\Gamma \Rightarrow A, \Delta \quad \Theta, A \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Lambda, \Delta}$ corresponds to the property that if $[\Gamma; \Delta]$ is coherent, then so is one of $[\Gamma, A; \Delta]$ and $[\Gamma; A, \Delta]$.

To sum up, the core ideas of Restall's antirealism is that $\Rightarrow$ is not interpreted as a metalinguistic property of entailment or derivability any more, but as a marker for the distinction between asserted and rejected sentences, and that the proof of a sequent $\vdash \Gamma \Rightarrow \Delta$ is interpreted as establishing that the corresponding state $[\Gamma; \Delta]$ is incoherent.

Of course, the principles about incoherence that correspond to structural rules are all intuitively acceptable, but is this a justification of these rules? Restall explicitly assert that this is the case,

[72][Rumfitt, 2000], p. 795.

[73][Rumfitt, 2000], p. 795.

[74]And a justified rejection of multiple-conclusion leads to an equivalent rejection of multiple-assumption, so to abandon standard proof-theoretic semantics.

[75]He explicitly refuses to discuss the issue of multiple-conclusion, [Hacking, 1979], p. 293.

[76][Restall, 2005] is the main reference, but the author developed further his theory in [Restall, 2008] (where he develops the connection of his theory with Brandom's inferentialism), [Restall, 2009b] (where he proposes a reconstruction of truth values and models based on idealised states of assertion and denial), [Restall, 2013] (where he proposes an application of assertion and rejection to non-logical theories) and [Restall, 2014] (where he proposes a logical pluralism based on this antirealistic framework).

[77]See [Restall, 2014]. In this paper, he also defends a version of logical pluralism by exploiting this framework, as we will see in section 4.4.

but I think that we have good reasons to believe the contrary.[78] As Rumfitt tells us, in order to have a proof-theoretic justification, we need to deal with usage of sentences, since use determines the meaning. Nevertheless, Restall's interpretation of sequent calculus does not explain how sentences are used, but just how to evaluate whether a state is coherent or not. Indeed, which inferential use could a rule like

$$\Rightarrow\neg\ \frac{\Gamma, A \ \Rightarrow\ \Delta}{\Gamma\ \Rightarrow\ \neg A, \Delta}$$

describe? If this rule is applied in a (well-formed) derivation, we know that both the premise and the conclusion are incoherent, and indeed this is what the rule says. So it surely describe a way in which sentences can not "be used": you can not assert all the sentences in Γ and reject all the sentences in Δ plus $\neg A$, as you can not assert all the sentences in Γ plus A and reject all the sentences in Δ. Arguably you can read the rule upside-down as proposed in [Restall, 2009b] to infer possible coherent extensions of the state in which you are.[79] Nonetheless what we can at most obtain in this way is a metalinguistic description of what is considered a coherent position, and so of how we can switch between a state and the other. Of course, this is a description of a linguistic practice in some way, but it is not a deductive practice, since deduction is surely something more than randomly skipping from coherent to coherent cognitive states. As a consequence, we have nothing like an inferential usage of sentences that can determine the meaning of logical expressions. This is the reason why $\Rightarrow$ is not interpreted as $\vdash$, but just as a punctuation mark to sever assertion and rejection. If we want to reconstruct an inferential practice from this interpretation of sequent calculus, we need to go in the direction of explicit bilateral systems like Rumfitt's one,[80] but Restall takes another path.

In [Restall, 2005] there is an example of a reconstruction of the inferential multiple-conclusion practice of proof by cases, according to the lines of proof-theoretic semantics. However it does not use rejection, and indeed the approach of the author in the section that contains this example is more similar to Stephen Read's "justification" of classical logic (that we will evaluate soon), than to the rest of Restall's paper. Indeed just for this section, the author uses a unilateral approach and speaks of multiple conclusions, while in the rest of the paper he speaks at most of multiple succedents (that are not interpreted as conclusions, but as rejections, according to Restall himself).[81] I think that Restall included this section in his paper because he realized that he needs a justification of **LK** external to his pragmatical bilateral interpretation. In this way $\Rightarrow$ has the role of bridging entailment and speech acts, that is it describes which states are incoherent based on which derivations are valid. Nevertheless, according to this interpretation, assertion/denial reading of sequents can not have a justificatory role. It is just something that comes later, on the applicative side. This clashes with what Restall says about the basic properties of coherence of states that justify the structural rules of sequent calculus. Moreover, it is not clear how to interpret his justification of multiple-conclusion (that, let us repeat, does not use bilateral speech acts) without using assertion, that is without a unilateral theory of speech acts, while of course a justification that uses assertion clashes with the bilateral approach adopted on the pragmatical part and so is not available. As a conclusion, it seems that regarding bilateralism and multiple-conclusion Restall has a foot in both camps, but that they can hardly coexist.

To sum up, Restall searches for a justification of classical logic, and he tries to obtain it using a bilateral theory of speech acts. This bilateral theory works fine from a pragmatical point of view, but it can not justify classical logic, because it is not a theory of deductive usage and so falls under Rumfitt's criticism. In order to have such a justification Restall has to rely on assertive multiple-conclusion reasoning, that is indeed what he proposes. In this way, he has a system in which sentences are used, instead of just mentioned, and used deductively, but nevertheless this theory of multiple-conclusion is independent of (if not even in contradiction with) his theory of sequent calculi.[82]

This is not a knockout argument against Restall's version of antirealism, but it is nonetheless a very general result. We already saw that his approach is very different from proof-theoretic semantics, now we can assert that it even clashes with the meaning-as-use *dictum*, since there is no obvious way of reconstructing usage from his interpretation of sequent calculus rules, or at least not a deductive usage. On the other side, if we want to accept his justification of proof by cases in a unilateral approach to proof-theoretic semantics, it is not clear which role his bilateral pragmatic theory could

[78] [Restall, 2005], p. 198.

[79] See as an example page 248 for the classical situation.

[80] Seen in the previous section 2.3

[81] That the example of proof provided by Restall is not interpreted according to bilateralism is obvious from the fact that he uses a disjunctive reading of succedent, while according to his bilateralism it should be rejective and conjunctive: the denial of each sentence in it.

[82] A similar criticism to Restall's version of bilateralism can be found also in [Steinberger, 2011b], pp. 349-353.

$$\dfrac{A,\Gamma \qquad B,\Delta}{A \wedge B,\Gamma,\Delta}\wedge I \qquad\qquad \dfrac{A \wedge B,\Gamma \qquad \begin{matrix}[A][B]\\ \vdots \\ \Delta\end{matrix}}{\Delta,\Gamma}\wedge E$$

$$\dfrac{A,\Gamma}{A \vee B,\Gamma}\vee I \qquad \dfrac{A \vee B,\Gamma \qquad \begin{matrix}[A]\\ \vdots \\ \Delta\end{matrix} \qquad \begin{matrix}[B]\\ \vdots \\ \Delta\end{matrix}}{\Delta,\Gamma}\vee E \qquad \dfrac{B,\Gamma}{A \vee B,\Gamma}\vee I$$

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B,\Gamma\end{matrix}}{A \supset B,\Gamma}\supset I \qquad \dfrac{\bot,\Delta}{A,\Delta}\bot E \qquad \dfrac{A \supset B,\Gamma \qquad A,\Theta \qquad \begin{matrix}[B]\\ \vdots \\ \Delta\end{matrix}}{\Gamma,\Delta,\Theta}\supset E$$

Table 2.4: **MAMCNK**

play. Moreover, his discussion of this example is far from constituting a complete proof-theoretic reconstruction. Let us now move on to the unilateral multiple-conclusion approach *tout court*.

2.4.2 Natural deduction and the objections to multiple conclusions

While neither Hacking nor Restall formulates an explicit system of natural deduction with multiple conclusions, a multiple-conclusion natural deduction system suitable for formalizing classical logic has been proposed in [Boričić, 1985] and later scrutinized in [Read, 2000]. We will call this system **MAMCNK** (table 2.4), since it is a multiple-assumption (**MA**) and multiple-conclusion (**MC**) natural deduction system (**N**) for classical logic (**K**). Boričić proved both normalizability and separability for this system in his paper, but he did not focus on proof-theoretic semantics. His observations were mostly devoted to the formal result. Read defended this system, proposing a revision based on his idea of general-elimination harmony. Although we do not consider this reformulation of harmony (based on the idea that E-rules should be functions of the corresponding I-rules), we consider Read's version of the calculus.[83]

While already Prawitz was able to prove normalizability for a fragment of his classical system and we saw that this result can be extended to the whole system,[84] Read manages to prove harmony as well, that is **MAMCNK** is obtained without abandoning the distinction between I and E-rules, while Prawitz's classical rule of absurdity is neither an I nor an E-rule. Moreover, Prawitz's system lacks separability, contrary to Boričić-Read's system that is even strongly separable. As an example, Peirce's law can be proved using only ⊃-rules (and structural rules, of course), and so its truth depends only on the meaning of this logical term:

$$\supset\!\mathrm{I}_2\,\dfrac{Contr\,\dfrac{\supset\!\mathrm{E}\,\dfrac{[(A \supset B) \supset A]^2 \qquad \supset\!\mathrm{I}\,\dfrac{Weak\,\dfrac{[A]^1}{A, B}}{A, A \supset B}}{A, A}}{A}}{((A \supset B) \supset A) \supset A}$$

In a derivation of **MAMCNK**, sentences are used rather than mentioned, so we can avoid Rumfitt's objection about sequent calculus, and we just saw that it is both (strongly) separable and harmonious. Arguably, our standard definition of validity can be extended to cover also this system.[85] As a consequence, if there are no more objections to the usage of multiple-conclusion, we have found a proof-theoretic justification of classical logic.

Unfortunately, there are a lot of objections to multiple-conclusion logics, or at least to their usage in proof-theoretic semantics. While I believe that the strength of some of these objections has been overstated, there are anyway some of them that are strong enough to call in question the entire multiple-conclusion framework (and maybe even more, as we will see in the next section). We will consider three main objections: lack of usage of multiple-conclusion arguments in everyday reasoning; lack of constructivity due to multiple-conclusion logic; circularity.

[83]However, the distinction between the two systems is irrelevant for our discussion.

[84]See note 11.

[85]We could speculate that canonicity should be redefined as asking a derivation that ends with an introduction rule possibly followed by some structural rules. Indeed we will see something similar in section 3.3.3. Already this request could open the door to some objections; anyway, since there are other, even bigger problems, we will neglect to discuss this issue.

Lack of common usage

There should be common usage of multiple conclusions? According to this first objection, even though it is possible to give a natural deduction system for multiple-conclusion logic, this is nonetheless severed from our everyday argumentative practice, and so these formal systems do not give meaning to logical terms. The base assumption of this argument is that a connection with *our* usage – not only usage in general – is needed if we want a real theory of meaning. While the observation that multiple conclusions are rare in practice and that this fact is problematic for a theory of meaning is common in writings of proof-theoretic semantics, the fine structure of the argument is usually neglected.[86] One of the clearer exposition is given by Steinberger:[87]

> "Only those deductive systems that answer to the use we put our logical vocabulary to fit the bill. After all, it is the practice represented, not the formalism as such, that confers meanings. Therefore, the formalism is of meaning-theoretic significance and hence of interest to the inferentialist only if it succeeds in capturing (in a perhaps idealised form) the relevant meaning-constituting features of our practice."

Steinberger calls 'Principle of answerability' this position, and clarifies that it can lead to different consequences, depending on which kinds of idealization is considered acceptable and on how the descriptive and the normative approaches of proof-theoretic semantics are balanced.

If Steinberger's argument is the reason why proof-theoretic semanticists are in general worried about the (lack of) common usage of multiple conclusions, I think that they should drop their concerns. Indeed, while it is surely true that it is the usage and not the formalism that gives meaning to logical terms, the shift from usage to common usage is a *non sequitur*. Why should an unusual usage of language be unable to give meaning to terms? This looks like an unjustified radicalization of the correct objection raised by Rumfitt against sequent calculus: we need a theory of usage, but do we need a theory of common usage?

Moreover, while Steinberger himself acknowledges that proof-theoretic semantics has also a normative nature, he stresses that the only way of justifying multiple-conclusion logic is by adopting it as an idealization of common reasoning. I wonder why the adoption of intuitionistic logic could work as a reform of common usage while we underestimate the adoption of multiple-conclusion as a mere idealization. Until we find a justification of classical logic, we propose a revision of our ordinary argumentative practice that consists in the adoption of intuitionistic logic, but when we discover a justification of classical logic that uses multiple-conclusion we propose a further revision by adopting it – of course checking the standard requirements of proof-theoretic semantics. I do not see any reason not to consider multiple-conclusion logic as at least a candidate revision, and I believe that this connection with ordinary deductive practice is more than enough.

There is another way of interpreting the ask for common usage: maybe the problem is not the lack of meaning, but the impossibility of understanding the practice that we are describing. Indeed while a set of rules is enough to establish a practice, and so arguably to give meaning to a logical term, in order to understand what is the purpose of this practice, we need to be able to connect it with the rest of our linguistic behaviour. In order to exemplify this concept, Dummett evaluates the following situation.[88] Let us consider the rule

$$\frac{A \mapsto (B \vee C)}{(A \mapsto B) \vee (A \mapsto C)}$$

It is not intuitively valid if we want to interpret $\mapsto$ as a counterfactual conditional. Nonetheless, a speaker could adopt it for this usage and decide to accord his linguistic behaviour consequently, and we could not blame him for anything. Of course, we are assuming that the speaker is capable of harmonising this change with the rest of his linguistic behaviour, that is that we can not reject the adoption of this rule by using some proof-theoretic requirements.

The only big problem of this situation is that it could be completely unclear to the listener what is the point of this change of practice. Obviously, the meaning of counterfactual conditional is changed, the problem is that it is not clear the way in which it is changed. We know (enough) what is the goal of this linguistic instrument in our everyday practice, but we have no idea of the reasons why this speaker decided to adopt this new rule. If he manages to explain how this new kind of connective is connected with everyday life, we could understand his move.

As Dummett stresses the listener is in the same situation of someone that, learning a new game, has understood everything about legitimate and illegitimate moves, but nothing about the strategy

[86] [Dummett, 1991], p. 41 and [Rumfitt, 2008], p. 79 are two examples.
[87] [Steinberger, 2011b], p. 335.
[88] [Dummett, 1991], p. 206.

or the goal of it.[89] In both cases the lack of understanding is real, it is not just a feeling. The only difference between the two situations is that the goal of a game is internal to the game itself, while logic has (also) purposes that are external both to the logical fragment of the language and to the language as a whole.[90] So we could decide to define arbitrarily the meaning of logical terms using sets of rules, but this could lead to logical systems that we do not know how to use. And even well behaved sets of rules could define terms that are essentially unintelligible for us. In summary, pointless usage gives at most uninteresting meaning.

This reformulation of the objection is much more interesting in general. Indeed what is the point in establishing that a set of rules for a term suits all the criteria of proof-theoretic semantics, if the meaning that it gives to the term is unintelligible anyway? Nonetheless, while this is a really important point in inferentialism, I do not think that it is relevant here. Indeed multiple-conclusion logic is not so departed from our common linguistic practice to confuse us about the meaning of the logical constant that it characterises. On the contrary, we could argue that, while it is not very common, it is a smaller departure from common practice than the adoption of intuitionistic logic. The point of this alleged revision of our practice is neither obscure nor inexplicable.

In conclusion, although multiple conclusions are arguably a detachment from standard argumentative practice, they are not so exotic to be unintelligible. On the contrary, we will see in a later section that it is true the opposite claim: their meaning is too near to actual practice. So the argument that exploits the rarity of multiple conclusions "in nature" is without teeth.

Is there common usage of multiple conclusions? Anyway, let us give a look at the debate about the usage of multiple conclusions in everyday reasoning. Even though we argued that their absence from our practices would not constitute a valid reason to reject them in proof-theoretic semantics, this issue will be useful later. It can not be denied that multiple-conclusion reasoning is not very common in everyday arguments. Indeed arguments usually have more premisses and just one conclusion. Nonetheless, to be completely honest, the reasoning is displayed linearly in our natural language, so the tree structure of arguments is just a reconstruction, although very natural.[91] As a consequence, as there are multiple premises hidden (not very well) in the linear structure of arguments we could argue that there are also multiple conclusions hidden (very well) somewhere in the same structure.

Indeed some authors have proposed that the most natural formulation of proof by cases uses multiple conclusions. As an example, [Shoesmith and Smiley, 1978] proposes this formulation:[92]

$$\begin{array}{cccc} \multicolumn{4}{c}{A_1 \vee A_2 \vee \cdots \vee A_n} \\ \hline A_1 & A_2 & \cdots & A_n \\ \vdots & \vdots & & \vdots \\ B & B & & B \end{array}$$

That is we evaluate different exhaustive alternatives $A_1 \vee A_2 \vee \cdots \vee A_n$ and we find out that the same conclusion can be derived from each of them, so we conclude it.

While this formulation is in general considered very natural, it is frequently objected that it deploys only a weakened form of multiple conclusions. Indeed while we have more than one conclusion, they are all occurrences of the same formula. Rumfitt indeed argues that for this reason proof by cases can not be regarded as a real example of multiple conclusions.[93] Steinberger pushes to the limit this line of thought and argues that the real form of proof by cases is standard $\vee$E.[94] Indeed since we have more conclusions of the same form, we can just conclude the derivation with a single token of this formula.

Of course, the substitution of multiple-conclusion with $\vee$E makes impossible to justify classical logic, so Steinberger's proposal entails that proof by cases is not enough to escape from constructivity. Nevertheless Greg Restall proposes an example of proof by cases that seems to derive a purely classical logical consequence, that is $\forall x(F(x) \vee G(x)) \vdash \forall x F(x) \vee \exists x G(x)$. The proof he proposes is this:[95]

[89] [Dummett, 1991], p. 208.

[90] [Dummett, 1991], pp. 204-205.

[91] I find interesting that even Gentzen, that is the main responsible for the widespread adoption of branching formulations of deduction, acknowledged the essential linearity of reasoning; see [von Plato, 2005], pp. 681-682 for his comment to Jaskowski's linear alternative to natural deduction.

[92] P. 4.

[93] [Rumfitt, 2008], p. 79.

[94] Steinberger [2011b], pp. 341-347.

[95] Restall uses sequent calculus instead of natural deduction but, in this context, he refers to its unilateral multiple-conclusion reading, so we can use natural deduction instead. See [Restall, 2005].

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x(F(x) \lor G(x))}{F(a) \lor G(a)}}{F(a), G(a)}}{F(a), \exists x G(x)}}{\forall x F(x), \exists x G(x)}}{\forall x F(x) \lor \exists x G(x)}$$

It is surely a multiple-conclusion proof, but is it an example of proof by cases? According to Restall, this formalization corresponds to the real structure of arguments like the following:

> "Suppose everyone is either happy or tired. Choose a person. It follows that this person is either happy or tired. There are two cases. Case (i) this person is happy. Case (ii) this person is tired, and as a result someone is tired. As a result, either this person is happy or someone (namely that person) is tired. But the person we chose was arbitrary, so either everyone is happy or someone is tired."

Steinberger objects that the last step of this derivation occurs outside the proof by cases and so that the correct formalization is just:[96]

$$\dfrac{\lor E_1\ \dfrac{\dfrac{\forall x(F(x) \lor G(x))}{F(a) \lor G(a)} \qquad \dfrac{[F(a)]^1}{F(a) \lor \exists x G(x)} \qquad \dfrac{\dfrac{[G(a)]^1}{\exists x G(x)}}{F(a) \lor \exists x G(x)}}{F(a) \lor \exists x G(x)}}{\forall y(F(y) \lor \exists x G(x))}$$

That is the purely classical consequence can not be derived using only standard $\lor$E instead of multiple conclusions, and indeed it is not acceptable according to Steinberger's reconstruction.

To be completely honest, it is not clear whether in the natural language reasoning that we are trying to formalize the last step occurs inside the proof by cases or immediately outside it. So we could be in doubt about which one of the two formalizations is the correct one. Nonetheless, we also have to acknowledge that the classical consequence is usually regarded as correct by competent speakers, and we have to take this into account. That is, while Restall's explanation of proof by cases explains why we regard both $\forall x(F(x) \lor G(x)) \vdash \forall x F(x) \lor \exists x G(x)$ and $\forall x(F(x) \lor G(x)) \vdash \forall y(F(y) \lor \exists x G(x))$ as valid, Steinberger's explanation can not explain why we regards the first as valid, and so proposes a revision. If there are no other objections to the multiple-conclusion interpretation of proof by cases, then the fact that it "saves the *phenomena*", while its $\lor$E interpretation does not, speaks in favour of it.[97]

So there seem to be reasons to formalize proof by cases with multiple conclusions. Nonetheless, if this is the only example of multiple-conclusion arguments that we can find in our everyday inferential practice, there is still Rumfitt's objection: in proof by cases the conclusions have all the same form, so this is "at best a degenerate form of multiple-conclusion argument, for the different conclusions are all the same".[98] Restall correctly answers to Rumfitt that multiplicity of occurrences is enough to have multiple conclusions, like it is enough to have multiple assumptions. Indeed the inference

$$\land I\ \dfrac{A \qquad A}{A \land A}$$

asks for two premises, although of the same shape, in order to derive the conclusion. If we do not recognise the status of multiplicity to these assumptions, then we need a special rule to derive $A \land A$ from a single occurrence of A, but there are no reasons to impose such a revision. In the same way, a derivation that ends with two or more occurrences of a formula has multiple conclusions.

Nonetheless, even acknowledging the multiplicity of the conclusions of proof by cases, it is at least strange that the only recognised application of multiple-conclusion arguments in natural reasoning

[96] [Steinberger, 2011b], pp. 343-344.

[97] Steinberger tries to recast his argument arguing that what Restall really needs in order to prove $\forall x(F(x) \lor G(x)) \vdash \forall x F(x) \lor \exists x G(x)$ is not proof by cases, but the unjustified introduction of $\forall$ inside a disjunction; see [Steinberger, 2011b], p. 344. This objection relies on the author's formulation of proof by cases and is unfair without an independent rejection of multiple conclusions, since it focuses on a property that is a prerogative of multiple conclusions. So it can be rejected together with Steinberger's reconstruction of proof by cases, until we find good reasons to object to multiple conclusions. On the other side, we will see that it regains most of its relevance as soon as we find good arguments against them.

[98] Quoting [Shoesmith and Smiley, 1978], p. 5.

has this strange form. For this reason, Restall proposes a strategy to find derivations with multiple conclusions of different forms: it is sufficient to truncate the derivation at some point before arriving at the conclusions of the same form. Restall explains his idea using the following argument:

> "Suppose $A \wedge (B \vee C)$. Then it follows that A. It follows that $B \vee C$. So, we have two cases: (i) B, and (ii) C. [1] Consider case (i). Here, B, and we already have A, so $A \wedge B$. Consider case (ii). Here, C, and we already have A, so $A \wedge C$. [2]. Back in case (i), it follows that $(A \wedge B) \vee (A \wedge C)$. In (ii), it also follows that $(A \wedge B) \vee (A \wedge C)$ [3]. So, we conclude, $(A \wedge B) \vee (A \wedge C)$."

If, instead of deriving the single conclusion $(A \wedge B) \vee (A \wedge C)$ or stopping the derivation at point [3] where we have two conclusions of the same form, we stop the derivation at point [2], then we have a derivation that ends with two conclusions of different forms: one is $A \wedge B$ and the other is $A \wedge C$. So Restall seems to suggest that already proofs by cases contain multiple conclusions with different forms hidden inside them.[99]

It is interesting to note that Steinberger's proposal of a single-conclusion reformulation works fine for the external multiple conclusions with the same form. What we really need are the intermediate multiple conclusions with different forms that, as Restall has shown, lay inside proof by cases. Indeed Restall's example of the derivation of $(A \wedge B) \vee (A \wedge C)$ from $A \wedge (B \vee C)$ ends with a single conclusion, obtained by what seems to be an application of $\vee$E. So it seems that in common reasoning multiple conclusions are always contracted in just one conclusion, at the end of derivations, and so they can occur only as intermediate conclusions. But this situation, of course, needs an explanation: why we can not have multiple conclusions at the end of the derivation? If they are acceptable elements in an argument, we should have derivations that just end with them.

I think that there is a naturalistic explanation of why this is the case and that this is independent of meaning theoretic considerations. Indeed we have to consider the fact that our language is primarily oral, and there are some bad consequences in allowing multiple conclusions to occur outside an argument. As an example, as observed by Gareth Evans, if multiple-conclusion exposes a class of collectively exhaustive possibilities, the speech act of the assertion of them is not complete until we have stated all of them. So when we use multiple conclusions we should find a way to advise our hearer when the statement is complete. While we could argue that the same can be said for multiple premises, the situation is not symmetric. Indeed multiple premises are naturally closed by the inference step that they precede, while the borders of conclusions are much less clear, especially for non-intermediate ones. Moreover, while all the partial expressions of a multiple assumption are true, if the complete expression does, the partial expressions of a multiple conclusion are not only incomplete, but also possibly false. This follows trivially from the conjunctive nature of multiple assumptions and the disjunctive nature of multiple conclusions.[100]

The previous naturalistic reasons explain why multiple conclusions are less practical than multiple assumptions. I think that in spoken language there is also another stronger reason why it is not convenient to have both multiple assumptions and multiple conclusions. That is it is preferable to have only one common usage of multiple assertions, since it is obviously problematic to have usages of multiple assertions that are ambiguous between a conjunctive and a disjunctive reading. Indeed we could be tempted to "store in our memory" a multiple conclusion and then use it as an assumption for a new argument, but this leads to invalid proof steps, going from a disjunctive reading of $A_1, \ldots, A_n$ to a conjunctive one. Of course, nothing prevents us from using as a new assumption a multiple assertion obtained as the conclusion of a derivation, but we need to remember that it was established disjunctively. If this is the situation, it is far better to have just one standard interpretation of multiple assertions in our practice. And since we saw that conjunctive multiple assumptions are easier to use, the adoption of them is far from mysterious.

All these problems are relevant only for spoken language. Indeed, in a written derivation the borders of a conclusion constituted by multiple sentences are always clear, and we do not have any problem at indexing sets of sentences that are disjunctively or conjunctively derived. This is something practically inconvenient in spoken languages, and it contributes to explaining why multiple-conclusion is not normally used. Arguably, the second problem is relevant in general, so maybe also in formal systems we want multiple conclusions to occur only in intermediate positions. Nonetheless, we fully explained the reasons why multiple conclusions occur only there in everyday language, and they have

[99] Both arguments of Restall against Rumfitt can be found in Restall [2009a].

[100] These observations are reported in [Shoesmith and Smiley, 1978], p. 5-6. Steinberger interprets this argument as a reason to reject multiple conclusions in proof-theoretic semantics but I think that, on the contrary, it works far better as an explanation of why they are so rare in common practice, although they are acceptable in it. Also Shoesmith and Smiley seem to agree on my naturalistic interpretation of Evans' argument.

nothing to do with theory of meaning. So we have no reason to be suspicious about the strange way in which multiple conclusions are used in everyday argumentative practice.

To sum up, the existence of multiple conclusions in everyday reasoning is not indispensable for their acceptance in proof-theoretic semantics, but it is welcomed anyway. Indeed it excludes completely the possibility that the appeal to such a practice gives an unintelligible meaning to logical terms. There are good reasons to believe that we use multiple conclusions in proof by cases, but that for limits imposed by the oral medium we use them only for intermediate conclusions. Nonetheless, those limits are irrelevant for a theory of meaning, so we have no reason to reject multiple conclusions in proof-theoretic semantics, or at least we have no reason based on actual argumentative practice.

Lack of constructivity

Antirealism and constructivism Another common criticism to multiple conclusions is that they lead to non-constructivity of the logical system.[101] Of course, this objection applies to every attempt of saving classical logic, so we should deal with it once and for all: should every system used for proof-theoretic semantics be constructive?

I think that constructivism should not be an assumption for proof-theoretic semantics, but at most a conclusion. Indeed, while a lot of proof-theoretic semanticists and the traditional intuitionists ended up endorsing similar positions about many issues, first of all the preference of intuitionistic over classical logic, their philosophical reasons are very different in general. A key example is given by language: while traditional intuitionists like Brouwer searched a foundation of mathematics and logic independent of language, Dummett's philosophy of logic is rooted in linguistic and meaning-theoretic issues.

Epistemic truth and *tertium non datur* However, the general verificationist approach endorsed in antirealism[102] seems to be at odds with a justification of *tertium non datur*. Indeed if $A \vee \neg A$ is a logical theorem, then the utterance of all its exemplifications is always justified, and so there needs to be always a (possible) verification of a sentence or of its negation. But this result asks for a very idealised notion of verification, which makes essentially pointless the verificationist approach. In this work, I did not discuss thoroughly the general approach of antirealism to meaning, but I just focused on logical terms. Nonetheless, we need to deal at least a little with this issue.

First of all, let us consider how the constructivist reading of disjunction influences the understanding of *tertium non datur*. According to the constructivist interpretation of disjunction, in order to prove $A \vee \neg A$ we need to know which one of A and its negation is true. That is we can not have a proof of a disjunction without a proof of one of the disjuncts. So according to the constructivist reading of disjunction, it would be problematic for the verificationist approach to be able to prove $p \vee \neg p$ for every sentence p, since this would require a proof of p or of $\neg p$ for every sentence, something that we can not do in general. Nonetheless, the proof of *tertium non datur* regards classical disjunction, so that in order to prove $p \vee \neg p$ it is enough to prove that at least one of the two must be verifiable. That is we do not need to be able to prove one of the two disjuncts in order to prove the disjunction. Arguably, this is not enough to justify the acceptance of $A \vee \neg A$ in general, but at least makes possible for this logical law to cohere with a verificationist approach to meaning. Indeed classical disjunction considers a gap between the proof of a disjunction and a proof of one of the disjuncts, and so legitimises our position in which we have a proof of all the exemplifications of $A \vee \neg A$ without having a proof of most of the exemplifications of the disjuncts. Moreover, the gap can be so big that a justification of one of the disjuncts is not required in general, not even in future or purely possible knowledge.

Unfortunately, there are reasons to believe that, while this consideration is enough to solve the apparent disagreement between *tertium non datur* and an epistemic notion of truth, it is not enough to solve the disagreement between *tertium non datur* and the particular kind of epistemic notion of truth used in proof-theoretic semantics. Indeed while the classical conception of disjunction requires nothing about the possibility of proving the disjuncts in order to prove their disjunction, proof-theoretic semantics does this, via its fundamental assumption (assumption 1.2.1). According to this assumption, if a disjunction is justified, then there ought to be a justification of it that ends with an application of $\vee$I. But for this to be the case there must be a derivation of one of the two disjuncts (at least taking for granted the shape of this rule). So it seems that a proof-theoretical justification of *tertium non datur* requires the warrant of at least one disjunct for each of their instances, independently on the

[101]As an example [Tennant, 1997], p. 320 (where the author expresses concern also for the absence of multiple conclusions in everyday reasoning, the issue we just discussed).

[102]Integrated by a pragmatist approach connected via harmony, as explained in [Dummett, 1991].

intuitionistic reading of disjunction. That is we need to endorse strong decidability in order to justify it. [103]

Normalization Let us now consider another issue related to constructivism: the nature of the reduction procedure used to define validity. The reduction steps for natural deduction derivations correspond to those for simply typed λ-calculus, which is a formal system that models computation. The well-known correspondence is called the Curry-Howard isomorphism, and we will not discuss it in details.[104] However, we need to consider some consequences of this relation between one of the main properties used in proof-theoretic semantics and computability. As an example, the ambiguity of this relation leads Prawitz to ask for strong normalization in order to have what he calls "strong validity". As we already stressed in note 50, we reject this strengthened version of validity, following the evaluation of Schroeder-Heister. We also propose a more general criticism: since the connection between harmony and computation has been greatly exaggerated, maybe the appeal to normalization in the definition of validity (definition 1.2.8 is unessential, and the appeal to the existence of normal form is more appropriate. What I propose is the following reformulation:

Definition 2.4.1 (Validity in $\mathscr{B}$)**.** A derivation $\mathfrak{D}$ is valid in $\mathscr{B}$ iff either:

1. $\mathfrak{D}$ is a closed derivation of an atomic conclusion C and *there is* a closed normal proof of the same conclusion C carried on in $\mathscr{B}$; or

2. $\mathfrak{D}$ is a closed derivation of a non-atomic conclusion C and *there is* a closed normal proof of the same conclusion C that is canonical; or

3. $\mathfrak{D}$ is an open derivation and every closure of $\mathfrak{D}$, obtained by replacing open assumptions by closed derivations for the same sentences that are valid in $\mathscr{B}$, is valid in $\mathscr{B}$.

The reason behind this change is essentially that validity should ask for a well-grounded relation of logical consequence. Indeed, let us just remember that this kind of definition was needed essentially to solve the "circularity" in the definitions of $\supset$I and of valid derivation, by exploiting its inductive formulation.[105] Now, it seems that the existence of normal form is completely sufficient to carry on this duty. Since there is a normal proof, the consequence is justified from the point of view of meaning; nothing asks for a decidable method for finding this normal proof. That it can be found in some way should be enough.

I suspect that the only good reason why normalization was used in first place in this definition is because of the constructivist scepticism about pure-existence theorems.[106] Nonetheless, if this is the reason, the ask for normalization lacks impartiality. Although it is understandable the ask for a decidable procedure to find an example of normal derivation when we are searching for a justification of intuitionistic logic, why should we not be satisfied with purely existential results when we are dealing with the justification of classical logic?

Circularity

For logical conversion The consideration that ends the previous paragraph is usually reversed in the literature of proof-theoretic semantics, where multiple-conclusion logic is criticized because applies a *classical* disjunctive reading to multiple conclusions. The request for impartiality comes from Dummett's belief that proof-theoretic semantics should be the main tool to solve logical disagreements. Dummett's idea seems to be that in order for this application to be fruitful we can not use controversial instruments in the framework of the theory, and multiple conclusions interpreted using classical disjunction *are* controversial (at least in the debate between classicists and intuitionists[107]). So, since multiple conclusions are not impartial, they can be used to justify classical logic only in a circular way.

According to Steinberger, the structure of Dummett's argument is the following:[108]

[103] I must confess that I find it much harder to answer objections to classical logic based on verificationism than objections based on the theory of meaning. Indeed, except for bare coherence, it does not seem easy to combine verificationism with classical logic. Nonetheless, what we want to stress here is that proof-theoretic semantics should not exclude classical logic, that is that if there are good arguments against *tertium non datur* in antirealism they lay outside the pure logical framework.

[104] See [Sørensen and Urzyczyn, 2006].

[105] See section 1.2.2.

[106] Of course the connections between harmony and computability, though they are very interesting, are not a good reason.

[107] That according to Dummett correspond to the debate between realists and antirealists, although our project of developing an antirealist theory of meaning for classical logic messes up this interpretation.

[108] [Steinberger, 2011b], p. 346, that refers to [Dummett, 1991], p. 187.

1. The discussion between realists and antirealists must rest on a clear specification of the meaning of logical terms, so that they do not talk past each other;

2. The specification must occur in a proof-theoretic semantics framework that all the parties agree on;

3. In order to specify the meaning of multiple conclusions we have to rely on that of classical disjunction;

4. So we have to rely on a prior understanding of classical disjunction, in order to explain the meaning of this connective using multiple conclusions, and that is useless.

Of course, the accuse of circularity in the justification of a logical system is not something new. On the contrary, it is not clear how a non-circular justification of logic could even be possible: are we not going to use arguments in the justification, after all? Dummett interestingly observes that the only kind of circularity that we can not escape with the justification of logic is[109]

> "not the ordinary gross circularity that consists of including the conclusion to be reached among the initial premisses of the argument. We have some argument that purports to arrive at the conclusion that such-and-such a logical law is valid; and the charge is not that this argument must include among its premisses the statement that that logical law is valid, but only that at least one of the inferential steps in the argument must be taken in accordance with that law. We may call this a 'pragmatic' circularity."

We run into this kind of circularity also when we try to justify intuitionistic logic, of course, but Dummett smartly observes that in the debate between classical and intuitionist logicians – that is what he is mostly interested in –, intuitionistic logical principles are not really controversial, so arguably they are not in need of a justification *in this context*.

Nevertheless, the pragmatic circularity of any justification of intuitionistic logic could be problematic for the intuitionist himself. That is, even if his principles had not been debated by the classical logician, they would still have been in need of a non-circular justification. Indeed justification is not only needed in the debate between logicians that disagree about which logic is the right logic, but it is needed in general.

Nonetheless according to Dummett, pragmatic circularity is problematic only when we try to convert someone that is genuinely sceptic about the validity of a logical principle. When we are only searching an explanation for a logical principle that we consider valid, pragmatic circularity is fine:[110]

> "If the justification is intended as suasive, then the pragmatic circularity will defeat its principal objective. That is to say, if the justification is addressed to someone who genuinely doubts whether the law is valid, and is intended to persuade him that it is, it will fail of its purpose [...] A gross circularity is as damaging to an explanatory argument as to a suasive one; but a pragmatic circularity need do it no harm at all."

The reason why pragmatic circularity is less detrimental than gross circularity is that while we can not fail when we search a grossly circular justification of a logical principle, we can fail when we search for a pragmatically circular justification of it. Indeed assuming the validity of *tertium non datur*, we already have a proof of its validity, while the use of purely classical principles in our arguments about validity does not lead necessarily to a proof of their validity. As an example in order to prove the completeness of first order *intuitionistic* logic with respect to Kripke structures, we need to use classical logic at the metalevel, but this does not make possible to prove the validity of *tertium non datur* in those structures.[111] According to Dummett, the possibility of failure means that our success in justifying a logical principle can not be completely pointless, so pragmatical circularity does not invalidate at least explicative justification.[112]

Arguably, if pragmatic circularity is not a problem for the explicatory justification of intuitionistic logic, then it is not a problem for the justification of classical logic either. So we have two logics that can be justified in an explanatory way, that is a classical logician can explain why he believes in classical principles and an intuitionistic logician can explain why he believes in intuitionistic principles. However, when they speak to each other, pragmatical circularity becomes unacceptable and classical

[109][Dummett, 1991], p. 202.
[110][Dummett, 1991], p. 202.
[111][Dummett, 2000], p. 154.
[112]It could be useful to remind that Wittgenstein's main objection against Private Language is that it does not leave open the possibility of a failure in following a rule. See paragraph 202 of [Wittgenstein, 1958] "[...] it is not possible to obey a rule 'privately': otherwise thinking one was obeying a rule would be the same thing as obeying it".

logician cannot convert the intuitionistic logician. The intuitionistic logician, on the contrary, does not need to convince the other speaker of the validity of his principles, so his pragmatical circularity is not problematic even in his debate with him.

We could pose at least two objections to this argument:

- First of all, while Dummett is interested essentially in the debate between classical and intuitionistic logic, there are a lot of other systems, and some of them are endorsed by philosophers as the one true logic. According to Dummett's analysis, the disagreement between philosophers seems to be always solved downgrading the strength of the system. That is if a speaker endorses the system $\mathcal{S}$ and another speaker endorses the system $\mathcal{S}'$, such that $\mathcal{S}'$ proves all the logical consequences provable in $\mathcal{S}$, and they give pragmatically circular justifications of their systems, then the circular justification of the weaker system $\mathcal{S}$ is shared also by the speaker that endorses $\mathcal{S}'$, so in the disagreement the weaker system seems to be preferable. This is not so surprising, but it hardly counts as a good reason to weaken our logical beliefs, especially if we take into consideration very weak logical systems, and there seem to be no non *ad hoc* reasons not to do that. In brief, the cost of a prudence *criterion* could be greater than Dummett acknowledges.

- We want a logical system that is justified, and while the conversion of a logical heretic is something relevant to justification, it can not be the full story, nor probably it is the core of the issue.[113] We started this section searching for impartiality, but the issue here is not impartiality, it is how to convince someone we disagree with. The choice of the weakest system is not impartial, it is only prudent (and prudence is not impartial in any way), as we saw in the previous point. I think that while disagreement can sometimes be solved by considerations regarding proof-theoretic semantics – for example when a philosopher adopts a system of logic that can not be described by a well-behaved system –, we can not expect this to be the paramount case of logical disagreement, nor the paramount kind of application of proof-theoretic semantics. As an example, this procedure seems to be useless to convert a philosopher that endorses a system that is well behaved according to *his* standards. He could decide to downgrade his logical beliefs to share a common ground with some other logicians, but he is not obliged to do so by purely meaning theoretical considerations. That is, let us take for granted for a moment that there is a harmonious system for classical logic in which it is possible to define validity, but that both harmony and validity asks for multiple conclusions with a classical disjunctive reading and purely existential results. Could we convince a classical proof-theoretic semanticist to step down from his endorsement by the existence of a constructivist justification of an intuitionistic system? Maybe yes, but not using proof-theoretic semantics, that is only by external considerations. *Vice-versa* it would be hard to convince the intuitionist to adopt a classical system, but only if he has external (to proof-theoretic semantics, not to constructivism) reasons to be sceptical of purely existential results and/or of classical disjunction. Maybe in those cases a pluralist approach could be more useful.[114]

Between comma and disjunction So the main problem in the justification of classical logic can be neither the difficulty of changing the mind of a non-classical logician (since this is not a fundamental issue for justification), nor pragmatic circularity (since this is a kind of circularity that we always obtain when we try to justify logical principles). I will argue that Dummett correctly rejects the justification of classical logic via multiple conclusions because of gross circularity, and not because of pragmatical circularity, and so that his argument is mischaracterized by Steinberger's description. Indeed let us consider directly the text that Steinberger is interpreting:[115]

> "Sequents with two or more sentences in the succedent, by contrast, have no straightforwardly intelligible meaning, explicable without recourse to any logical constant. Asserting **A** and asserting **B** is tantamount to asserting ⌜ **A and B** ⌝; so, although the sentences in the antecedent of a sequent are in a sense conjunctively connected, we can understand the significance of a sequent with more than one sentence in the antecedent without having to know the meaning of 'and'. But, in a succedent comprising more than one sentence, the sentences are connected disjunctively; and it is not possible to grasp the sense of such a connection otherwise than by learning the meaning of the constant 'or'. A sequent of the

[113] Indeed also Dummett's remarks at the end of the paragraph are slightly pessimistic from this point of view; [Dummett, 1991], p. 204.

[114] We will say more about this in chapter 4. Crispin Wright wonders whether pragmatical circularity should really be a problem for the conversion of a logical heretic and suggests a negative answer in [Wright, 2018], note 10. Nonetheless, he develops this position in a very different way from ours.

[115] [Dummett, 1991], p. 187.

form **A : B, C** cannot be explained by saying, 'If you have asserted **A**, you may with equal right assert either **B** or **C**', for that would imply that you can assert either one at your choice; and the formulation, 'If you have asserted **A**, then either you may assert **B** or you may assert **C**', does not entitle you to make any further assertion until you learn which of them you may assert. A general explanation of this form of sequent becomes possible only when we can say, 'Having asserted **A**, you are thereby entitled to assert ⌜ **A or B** ⌝"'

Accordingly, this passage can be read as criticizing multiple conclusions on the base that their application to explain disjunction is (pragmatically) circular, but this would lead to problems only for non-classical logicians that evaluate the argument. What Dummett is arguing, in my opinion, is something stronger, that is that the definition of classical disjunction that uses comma is grossly circular, since we are able to prove *tertium non datur* for disjunction just because we assume that it holds for comma. Moreover, given the disjunctive reading of multiple conclusions, we can not fail in proving *tertium non datur* for disjunction if we assume it for comma. This is a blatant case of gross circularity rather than pragmatic circularity, so Steinberger's interpretation based on logical disagreement is misplaced.

We could wonder whether this kind of circularity is connected with circularity in meaning as characterised in definition 1.1.9. Nonetheless while maybe *right-comma* $\prec \vee$ since arguably right comma is used to define the meaning of disjunction, the converse does not hold.[116] Indeed the meaning of the right comma is not introduced in the system using $\vee$, but it is just assumed since it is part of the pre-logical framework of the system. This, of course, does not solve the problem since we have an element of the system the meaning of which is erroneously supposed to be transparent.[117] Moreover, the only solution to this lack could be to use $\vee$ in order to define the meaning of *right-comma*, obtaining precisely what we were trying to escape: meaning circularity. In conclusion, while gross circularity between multiple conclusions and disjunction is not equivalent to meaning circularity, they are nonetheless deeply connected.

Albeit this argument works fine to reject multiple-conclusion justifications of classical logic, I think that there is an easy way out to it. Indeed while with multiple-conclusion logic the properties that enable the derivation of *tertium non datur* are postulated from the outside as part of the framework of proof-theoretic semantics, we could reject multiple-conclusions and work directly with disjunction, searching for a justification of the properties that enable this derivation. This is what Milne did in his [Milne, 2002], where he proposes to substitute the I-rules of **NJ** for $\neg$ and $\supset$ with the following:

$$\supset\mathrm{I}_{Mln}\ \dfrac{\begin{matrix}[A]\\ \vdots\\ B\{\vee D\}\end{matrix}}{(A \supset B)\{\vee D\}} \qquad \neg\mathrm{I}_{Mln}\ \dfrac{\begin{matrix}[A]\\ \vdots\\ D\end{matrix}}{\neg A \vee D}$$

Where the meaning of curly brackets is that the formula they contain may either be or not be present, the rule remaining valid anyway.

It is not hard to prove that this extension leads to a complete classical system, what at most can be controversial is its meaning-theoretical justification. First of all, let us observe that in this case we do not have the same circularity problem observed by Dummett regarding the multiple-conclusion solution. Indeed this time the validity of *tertium non datur* does not come as an obvious consequence of the classical nature of some structural elements of the system the properties of which are just assumed. This time the validity of *tertium non datur* depends on the behaviour of the logical constants and the acceptability of this behaviour is evaluated in accordance with the usual proof-theoretic *criteria*. Indeed while the meaning of comma is a starting point in proof-theoretic semantics, the meaning of $\vee$ is given by its I-rule.

Milne's solution is also capable of explaining why we can find examples of applications of multiple-conclusion logic for intermediate conclusions of arguments but not multiple conclusions *tout court* in natural reasoning.[118] Multiple-conclusion arguments are just arguments in which a connective is introduced inside a disjunction, so when we find it applied as the last step of an argument, we just end with a disjunction as conclusion.

[116] I say 'arguably', because while we could use multiple conclusions in the classical I-rule for disjunction, we can also maintain the standard I-rule, as we did for **MAMCNK** (table 2.4). Indeed as observed by Maehara, multiple conclusions are essential only for $\neg$ and $\supset$ in order to obtain classical (propositional) logic (see [Takeuti, 1987], p. 52). In this case, the situation is even worse for the classical proof-theoretic semanticist, since he should explain how multiple-conclusions enable the derivation of purely classical principles without resorting to a postulated equivalence between $\vee$ and classical disjunction. Shortly we will see a solution to this issue that passes through the rejection of multiple conclusions and the direct application of disjunction in their place.

[117] We will see that this problem generalises to multiple assumptions.

[118] We dealt with this problem in paragraph 2.4.2.

Are there good reasons to reject Milne's rules? Dummett defined the following notions regarding the structure of an I-rule:[119]

purity Only one logical constant figures in each rule;

simplicity Every logical constant which occurs in a rule, occurs as principal operator;

directness Discharged assumptions are completely general, rules do not specify some connectives that must occur in them.

All the I-rules of **NJ** have all these properties, apart from that for $\neg$ that is not pure, since $\bot$ occurs in it. Adopting Dummett's theory of meaning, it is nonetheless possible to give a formulation of intuitionistic logic that suits all these restrictions, since Dummett's rule for negation[120] is pure, simple and direct. Unfortunately, we saw that his proposal is unacceptable from our point of view since we imposed a non-circularity condition on the rules rather than on their applications.[121] On the contrary, some rules used to obtain the classical extension of **NJ** that we rejected violate some of these properties. As an example, Prawitz's rule of classical *reductio* is oblique (non direct) and double negation elimination is pure (it deals only with negation) but not simple, since one of the negations is not principal.

Nevertheless, these few pieces of evidence are not enough to decide about these requirements, and indeed the standard way of dealing with negation in intuitionistic logic still remains passing through $\bot$, using an impure rule. Indeed, despite their standard (usually silent) endorsement in proof-theoretic semantics papers,[122] these restrictions are explicitly rejected by Dummett himself, as pointed out from Milne:[123]

> "Reflection shows that this demand is exorbitant. An impure **c**-introduction rule will make the understanding of **c** depend on the prior understanding of the other logical constants figuring in the rule. Certainly we do not want such a relation of dependence to be cyclic; but there would be nothing in principle objectionable if we could so order the logical constants that the understanding of each depended only on the understanding of those preceding it in the ordering."

Dummett's ideas about rejecting purity and simplicity seem to be different from Milne's proposal, since he only proposes the I$\neg$-rules[124]

$$\frac{\neg A \qquad \neg B}{\neg(A \vee B)} \qquad \frac{\neg A}{\neg(A \wedge B)} \qquad \frac{\neg B}{\neg(A \wedge B)} \qquad \frac{A \qquad \neg B}{\neg(A \supset B)} \qquad \frac{A}{\neg\neg A}$$

in which the outermost connective is the one introduced. So impure and non-simple rules could be accepted but constants should nonetheless be introduced as the outermost connective. Nonetheless, Dummett himself clarifies that, in conclusion, he considers complexity condition as the only *criterion* that we should impose on I-rules, leaving open the possibility of Milne's proposal. We will follow this suggestion apart from the substitution of complexity condition with our non-circularity requirement. Of course the endorsement of the weak version of separability, as opposed to its strong version, is just a consequence of this approach.[125]

Steinberger objects to this kind of solutions that it betrays one of the fundamental tasks of logic: that of "identifying the most basic forms of inference from which all other derived rules of inference can be a consequence".[126] I think that this criticism is misplaced, since the lesson Milne is trying to teach us is that there are not any more direct ways of obtaining classical results; we need to use impure and non-simple I-rules in order to gain them. So we already found "the most basic forms of inference" that are at play here, they are just not as simple as we are accustomed to.

Milne deals also with the ask for compositionality, which could be at odds with his rules. The reason for being scared of this possibility is that as an example the meaning of $(A \supset B) \vee C$ is not defined using their disjuncts, if it is derived using $\supset$I. So we could wonder whether the meaning of a disjunction in Milne's system can be still defined by composition from that of its disjuncts. While it is true that in order to gain compositionality a set of meaning-conferring rules should be sufficient to

119 [Dummett, 1991], p. 257.
120 See section 2.2.
121 See definition 1.1.9 and discussion in section 1.1.2.
122 Although standard antirealistic theories of meaning endorse general molecularism (1.1.5), proof-theoretic semantics endorses special atomism regarding logical language (1.1.8).
123 [Dummett, 1991], pp. 256-258; see also [Milne, 2002], pp. 522-.
124 [Dummett, 1991], p. 258.
125 See section 1.1.3.
126 [Steinberger, 2011b], pp. 345-346. The point is made in connection with both Milne and Restall.

introduce the respective connective in the principal position, nothing prevents us from assuming also rules that introduce it in other positions.[127] Doing so, we have to ways of deriving $(A \supset B) \vee C$ by I-rules: one that ends with an introduction of disjunction and one that ends with an introduction of conditional. Compositionality is respected if there is harmony between these two ways of canonical derivation.

So let us consider how Milne's system deals with the usual requirements of proof-theoretic semantics. First of all, since the meaning of both negation and conditional depends on that of disjunction and apart from that all the other connectives have an atomistic meaning, we do not have any problem of circular dependence of meaning. Let us now consider separability. Of course, since the meaning of conditional depends explicitly on that of disjunction, this connective can be used also to derive purely implicational results, like Peirce law. For this Milne proposes the following proof:[128]

$$\dfrac{\dfrac{\dfrac{\dfrac{[p]^1}{p \vee q}\,{\vee}\mathrm{I}}{p \vee (p \supset q)}\,{\supset}\mathrm{I}_1 \qquad \dfrac{[(p \supset q) \supset p]^4 \qquad [p \supset q]^2}{p}\,{\supset}\mathrm{E} \qquad [p]^3}{p}\,{\vee}\mathrm{E}_{2,3}}{((p \supset q) \supset p) \supset p}\,{\supset}\mathrm{I}_4$$

The availability of this derivation is not by accident, but we will not give a complete proof of separability, since this will be an obvious consequence of a stronger result that we will prove in section 3.3. About harmony, the situation is similar: we can not expect to obtain harmony according to the standard definition, since there is no reduction procedure for

$$\dfrac{\dfrac{\begin{matrix}\Gamma, [A]^1 \\ \vdots \\ B \vee C\end{matrix}}{(A \supset B) \vee C}\,{\supset}\mathrm{I}_{Mln\ 1} \qquad \begin{matrix}\Delta, [A \supset B]^2 \\ \vdots \\ D\end{matrix} \qquad \begin{matrix}\Theta, [C]^3 \\ \vdots \\ D\end{matrix}}{D}\,{\vee}\mathrm{E}_{2,3}$$

This is indeed the core of the objection raised by Steinberger against Milne's proposal.[129] Nonetheless, as Milne himself pointed out, we should not expect this derivation to be reduced, since $(A \supset B) \vee C$ is not maximal in it.[130] Indeed $\supset$ is introduced in it but not used by the corresponding E-rule. What we need to reduce is the occurrence of this formula in the following derivation: [131]

$$\dfrac{\dfrac{\begin{matrix}[A]^1 \\ \vdots \\ B \vee C\end{matrix}}{(A \supset B) \vee C}\,{\supset}\mathrm{I}_{Mln\ 1} \qquad \begin{matrix}\dfrac{A \qquad [A \supset B]^2}{B}\,{\supset}\mathrm{E} \\ \vdots \\ D\end{matrix} \qquad \begin{matrix}[C]^3 \\ \vdots \\ D\end{matrix}}{D}\,{\vee}\mathrm{E}_{2,3}$$

And indeed this can be done, given

$$\dfrac{\begin{matrix}A \\ \vdots \\ B \vee C\end{matrix} \qquad \begin{matrix}[B]^1 \\ \vdots \\ D\end{matrix} \qquad \begin{matrix}[C]^2 \\ \vdots \\ D\end{matrix}}{D}\,{\vee}\mathrm{E}_{1,2}$$

In conclusion, the endorsement of Milne's reform asks for a rethinking of the fundamental notions of proof-theoretic semantics. We will investigate them in the following chapter (chapter 3), together with other original changes. But, before doing that extensively, it is better to discuss some consequences of our choice to interpret multiple conclusions as simple disjunctions. Indeed we could ask why we should not follow the same path also for multiple assumptions and conjunctions. This will be the topic of section 2.6. But before doing this, in the next section we will look at an alternative way of developing a classical non-pure system.

[127] [Milne, 2002], p. 526.
[128] [Milne, 2002], p. 527.
[129] This worry is extensively exposed in [Steinberger, 2008], and re-marked in [Steinberger, 2011b], p. 345.
[130] [Milne, 2002], p. 527: "What we must not do is consider this [his I-rule for negation, but the same holds for his rule for implication] a new, canonical, meaning-specifying rule for the introduction of disjunction as dominant operator".
[131] [Milne, 2002], p. 518-519.

2.5 Prawitz's impure ecumenical system

Prawitz recently proposed another system for classical logic that rejects purity.[132] There are two main differences between Milne's and Prawitz's systems:

- In Milne's formulation, classical and intuitionistic constants cannot coexist together, while Prawitz proposes a restriction on the validity of classical logic that enables the coexistence of both sets of constants;
- In Milne's formulation, classical logic is an extension of intuitionistic logic since every intuitionistic consequence is classically valid as well, while in Prawitz's system the opposite holds, that is the classical system is in some way a restriction of the intuitionistic one.[133]

Prawitz adopts a distinction between intuitionistic ($P_i(\,)$) and classical predicates ($P_c(\,)$), and distinguish between an intuitionistic and a classical version of disjunction, implication and existential quantifier. His system is composed of Gentzen's rules plus the following ones:

$$P_c\mathrm{I}\ \dfrac{\begin{array}{c}[\neg P_i(t)]\\ \vdots\\ \bot\end{array}}{P_c(t)} \qquad P_c\mathrm{E}\ \dfrac{P_c(t) \qquad \neg P_i(t)}{\bot} \qquad \vee_c\mathrm{I}\ \dfrac{\begin{array}{c}[\neg A, \neg B]\\ \vdots\\ \bot\end{array}}{A \vee_c B} \qquad \vee_c\mathrm{E}\ \dfrac{A \vee_c B \qquad \neg A \qquad \neg B}{\bot}$$

$$\supset_c\mathrm{I}\ \dfrac{\begin{array}{c}[A, \neg B]\\ \vdots\\ \bot\end{array}}{A \supset_c B} \qquad \supset_c\mathrm{E}\ \dfrac{A \supset_c B \qquad A \qquad \neg B}{\bot} \qquad \exists_c\mathrm{I}\ \dfrac{\begin{array}{c}[\forall x \neg A(x)]\\ \vdots\\ \bot\end{array}}{\exists_c x A(x)} \qquad \exists_c\mathrm{E}\ \dfrac{\exists_c x A(x) \qquad \forall x \neg A(x)}{\bot}$$

Using these rules, we can indeed define an ecumenical system in which classical and intuitionistic consequences coexist. We just assume all Gentzen's rules and these extra rules, and add a subscript to distinguish between the classical and the intuitionistic version of $\vee$, $\supset$ and $\exists$. So Gentzen's rules for $\wedge$, $\neg$, $\bot$ and $\forall$ are common to both the classical and the intuitionistic system, while Gentzen's rules for $\vee$, $\supset$ and $\exists$ define the meaning only of the intuitionistic variant of the corresponding connectives. Prawitz does not specify this, but in some cases Gentzen's rules cannot be derived for the classical connectives (at least not in the complete ecumenical system), as we will see for *Modus Ponens*.

In this system, the intuitionistic theorems are provable if they are formulated using only the intuitionistic vocabulary and the classical theorems are provable if they are formulated using only the classical vocabulary.[134] In this context, classical consequences are weaker than intuitionistic ones, since they do not hold universally. Moreover, the meaning of classical constants is given by means of the intuitionistic ones in Prawitz's system, so the first system in some way depends on the second.

Prawitz stresses that we need to distinguish between the intuitionistic and the classical version of these connectives because, even though the intuitionistic I-rules are always classically valid as well, they sometimes cannot be regarded as meaning conferring.[135] As an example, according to the author, the standard rule of introduction for intuitionistic disjunction entails what is sometimes called "property of disjunction", that is that "$A \vee B$ may be rightly asserted only if it is possible to prove either A or B".[136] Since this position is obviously at odds with the excluded middle, it follows that this rule cannot be meaning conferring for classical disjunction.

It is not completely clear to me that this property of disjunction follows from the adoption of standard $\vee$I. Indeed Milne's classical system endorses it as meaning conferring but nonetheless does not show this property. Maybe Prawitz is conjecturing that this property follows from $\vee$I together with the fundamental assumption (assumption 1.2.1), or from $\vee$I together with some *criterion* stronger than harmony.[137] Nonetheless, this line of reasoning needs a clear reformulation of these extra principles in the context of a non-pure system; a topic that deserves an entire investigation of its own and

[132] [Prawitz, 2015a].
[133] [Prawitz, 2015a], p. 28.
[134] [Prawitz, 2015a], p. 29.
[135] [Prawitz, 2015a], p. 27.
[136] [Prawitz, 2015a], p. 26.
[137] Maybe this could be true for $\vee$I together with the requirement of stability ([Dummett, 1991], chapter 13; [Steinberger, 2011a] for a good introduction). This *criterion* has recently received a lot of attention in the literature, and even though Dummett's original formulation is irreparably useless ([Dicher, 2016]), there have been good proposals for an up to date reformulation: [Tranchini, 2016], [Jacinto and Read, 2017].

that we cannot discuss here. For this reason, we will just consider harmony and the fundamental assumption,[138] and avoid discussing others extra *criteria*.

In general, both Prawitz and Milne ask for a change in I-rules in order to obtain a sound system for classical logic, and they both argue that their systems are harmonious. We will consider thoroughly Milne's system, and we will take for granted that Prawitz's one is harmonious.[139] While Milne modifies the rules for $\supset$ and $\neg$ in order to obtain classical logic, Prawitz modifies those for $\supset$ and $\vee$, and defines purely classical predicates (P_c). Prawitz argues that the connectives that he modifies in order to obtain classical logic are different from the intuitionistic ones, while Milne does not take a position about this issue. Prawitz's reason for asserting their difference is that these connectives are severed in his mixed (classical and intuitionistic) system. We will discuss the issue of the identity of logical connectives in chapter 4, and we will compare these two approaches to the problem.

Of course, the dependence of meaning is different in Prawitz's system and in Milne's system. According to Milne, both conditional and negation depend on disjunction in the classical system, while according to Prawitz, classical predicate, classical disjunction, classical conditional and classical existential all depend on negation, classical existential quantifier depends on universal quantifier and every classical predicate depends on its intuitionistic counterpart. Both systems exhibit weak separability, so they both are good candidates for our project.[140] As an example, let us consider the following proof of Peirce's law:

$$\supset_c\mathrm{I}_2\,\dfrac{\supset_c\mathrm{E}\,\dfrac{[(A \supset_c B) \supset_c A]^2 \qquad \divideontimes_1\,\dfrac{\mathrm{Efq}\,\dfrac{\neg\mathrm{E}\,\dfrac{[A]^1 \qquad [\neg A]^2}{\bot}}{B}}{A \supset_c B} \qquad [\neg A]^2}{\bot}}{((A \supset_c B) \supset_c A) \supset_c A}$$

Where the inference $\divideontimes$ has the same shape of $\supset_i$I and is derivable for $\supset_c$ in the following way:

$$\supset_c\mathrm{I}_1\,\dfrac{\neg\mathrm{E}\,\dfrac{\begin{matrix}[A]^1\\ \vdots \\ B\end{matrix} \qquad [\neg B]^1}{\bot}}{A \supset_c B}$$

It uses rules for $\supset_c$, $\neg$ and $\bot$, and this is completely fine, since the meaning of $\supset_c$ is defined using $\neg$ and this is defined using $\bot$.

Interestingly enough, the previous derivation of $\divideontimes$ holds even if some elements of the intuitionistic vocabulary occur in A or B. That is $\supset_i$I is always derivable for $\supset_c$. If it were possible to derive $\supset_i$E as well, that is *Modus Ponens*, we would have a problem. Indeed it would become possible to derive $A \supset_i B$ from $A \supset_c B$:

$$\supset_i\mathrm{I}\,\dfrac{\text{Derived MP}\,\dfrac{A \supset_c B \qquad [A]^1}{B}}{A \supset_i B}$$

Of course we do not want this, since $\vdash \neg\neg A \supset_c A$ but $\nvdash \neg\neg A \supset_i A$. What we want is the derivability of *Modus Ponens* for $\supset_c$ when both premisses and conclusion are composed using only the classical vocabulary.[141] Luckily, an even stronger result holds:

$$\mathrm{DNE}\,\dfrac{\neg\mathrm{I}_1\,\dfrac{\supset_c\mathrm{E}\,\dfrac{A \supset_c B \qquad A \qquad [B]^1}{\bot}}{\neg\neg B}}{B}$$

[138] In chapter 3.

[139] A complete proof of normalization (missing in [Prawitz, 2015a]) can be found in [Pereira and Rodriguez, 2017].

[140] Nonetheless, in the next section we will adapt these restrictions for the acceptability of the systems to a new single-antecedent framework. We will develop these restrictions in chapter 3 for a system based on Milne's one. Arguably, the same process could be accomplished for Prawitz's system too.

[141] Indeed, since the canonical bases to derive $A \supset_c B$ are weaker than those to derive $A \supset_i B$, it is understandable that they do not justify *Modus Ponens*, but only the weaker rule: $\dfrac{A \supset_c B \qquad A}{\neg\neg B}$. See [Pereira and Rodriguez, 2017], p. 1157.

Here, DNE is derivable if B is a classical constant or its main operator is classical.[142] As a consequence, in this case *Modus Ponens* holds too.

So classical and intuitionistic conditionals are severed, but the fact that *Modus Ponens* holds in general only for the intuitionistic conditional is at least strange. The discussions between classical and intuitionistic logicians rarely regard *Modus Ponens*. Moreover, this means that entailment in Prawitz's system is intuitionistic, and become classical only in a restricted subsystem.[143] At first glance, this seems to suggest that in this system only the intuitionistic one is a real logic, while the classical fragment does not really speak of entailment.

Let us now consider how Prawitz's system deals with negation. We could wonder whether it is possible to sever classical and intuitionistic negations as we did with classical and intuitionistic conditionals. In Prawitz's system they coincide because it is provable $(A \supset_c \bot) \subset\supset_i (A \supset_i \bot) \subset\supset_i \neg A$, and deduction theorem holds for intuitionistic conditional. So, even though $\supset_i$ and $\supset_c$ are different, if there is a classical derivation of $\bot$ then there is an intuitionistic derivation too. Since this situation is sanctioned by Glivenko's theorem, at least for the propositional fragment there seems to be no way out: classical and intuitionistic negations cannot be distinguished.[144] This situation is strange as well since there seem to be disagreements between the classical and the intuitionistic usage of negation. As an example, double negation elimination holds only classically. This difference of behaviour cannot be explained plainly by reference to two different connectives, and this is unpleasant for this kind of ecumenical system.

2.6 Reasons for single-assumption

Taking for granted that multiple conclusions are just disjunctions in disguise, it is hard to resist the temptation of concluding the same about multiple assumptions and conjunctions. Indeed already Dummett deals with this issue and feels an obligation to find a reason why this should not be the case. As we already stressed, according to him, "asserting **A** and asserting **B** is tantamount to asserting ⌜ **A and B** ⌝".[145] Dummett suggests that, for this reason, we can grasp the meaning of multiple assumptions without using that of conjunction.

Dummett's idea is essentially that multiple *assertion*, that is asserting multiple sentences, is understood as conjunctive in nature so, since conjunctive multiple *assertion* is what we need in order to have multiple *assumptions*, multiple *assertion* is enough to define the meaning of multiple *assumption*. Of course, the next step is the introduction of conjunction using multiple assumptions, that gives no problem at all.

While Dummett's argument is very precise and innovative, I think that it rests on a misconception about multiple *assertions*. Arguably the conjunctive reading of multiple *assertions* is so common to be seen as automatic, or natural, but there are no reasons to believe that this is the case. We are just repeating the same mistake, doing here what we did with multiple-conclusion logic: assuming for the comma the properties we want to prove for conjunction. We already exposed in paragraph 2.4.2 the *naturalistic* reasons why conjunctive reading is preferred to disjunctive one, and I believe that they are enough to explain why accidentally multiple assertions are usually read conjunctively. That is there are no bases to claim that conjunctive multiple *assertions* are graspable naturally, without any notion of conjunction.

Steinberger proposes an interesting interpretation of this argument.[146] While if you assert **A** and you assert **B**, you speaks truly if and only if the sentence ⌜ **A and B** ⌝ is true, if you assert **A** or you assert **B**, you can not be sure of speaking truly only because the sentence ⌜ **A or B** ⌝ is true. Indeed if you assert that aardvarks are indigenous of South America, then you are asserting that aardvarks are indigenous of South America or you are asserting that they are nocturnal. Nonetheless (since aardvarks are not indigenous of South America) you are not speaking truly, although it is true that aardvarks are indigenous of South America or nocturnal (since they are nocturnal). So there is a mismatch between the assertion of a disjunction and a disjunction of assertions that is not present between the assertion of a conjunction and a conjunction of assertions.

Even though this observation is very interesting and could be useful to integrate our naturalistic explanation of why the conjunctive reading of multiple *assertions* is more easily graspable, I do not see its relevance to the discussion at issue. Indeed we are not discussing the eventuality of identifying a multiple conclusion with a disjunction of assertions (or better alternative assertions), we are just

142 [Pimentel et al., 2019a], sec. 3.1, Def. 5.

143 In general $\Gamma \vdash C$ iff $\vdash \Gamma \supset_i C$, while only for a subset of sentences $\Gamma \vdash C$ iff $\vdash \Gamma \supset_c C$: [Pimentel et al., 2019a], pp. 10,13.

144 [Pimentel et al., 2019b].

145 [Dummett, 1991], p. 187.

146 [Steinberger, 2011b], p. 248.

questioning the conjunctive reading of multiple *assertions*. Given the conjunctive reading of multiple *assertions* we have $A, B = A \wedge B$, while given the disjunctive reading we have $A, B = A \vee B$; none of these interpretations entails $A = A \vee B$ and $B = A \vee B$, that is what we need to have the equivalence between assertion of a disjunction and alternative assertion. So alternative assertions are not part of the picture, and rejecting their identification with the assertion of disjunctions does not justify the conjunctive reading of multiple assertions.

Moreover, let us consider again the starting point of this argument. Suppose that someone asserts **A** and asserts **B**. Are we sure that these assertions are true if and only if ⌜ **A and B** ⌝ is true, according to our standard linguistic practice? It seems to depend a lot on the context. First of all, it depends on when these statements are asserted. Conjunctive reading is plausible (almost certain) only if the two sentences are asserted together one shortly after the other, and especially if there are no other utterances of words between these two. Otherwise, this reading is blatantly unjustified. So, it seems to me that conjunctive reading of multiple *assertions* is only accepted as a contraction of conjunctions, and if this is the case, there are no reasons to believe in their logical priority with respect to conjunctions *tout court*.

I think that this argument is far from being the real reason why the unmasking of multiple assumptions is so rare in proof-theoretic semantics. The real reason is a worry explicitly expressed by Steinberger himself. He fears that[147]

> "The result [of this unmasking] would be not so much a disproof of inferentialism as a wholesale disqualification of any proof system with multiple premises (and so, in practice, any proof system whatsoever) from playing the role of a proof-theoretic framework."

I believe that this conclusion is far from been necessary. Indeed we will see in the next chapter how it is possible to reconstruct the usual proof systems in a framework that allows neither multiple assumptions (**S**ingle **A**ssumption), nor multiple conclusions (**S**ingle **C**onclusion). The kind of liberalizations that we will need are not stronger in essence than the ones proposed by Milne, we will just need a more extensive application of his ideas. We will discuss also the possibility of accommodating harmony and the other *criteria* of proof-theoretic acceptability.[148]

The only further modification that we need to impose is the substitution of Dummett's complexity condition with our non-circularity condition (definition 1.1.9), that we already discussed extensively. As an example, we can not impose any restriction on the complexity of C in the following rule, that is needed in order to deal with conjunction in **SASC**-systems:[149]

$$\wedge\text{I}\ \frac{C \quad \begin{matrix}[C]\\ \vdots \\ A\end{matrix} \quad \begin{matrix}[C]\\ \vdots \\ B\end{matrix}}{A \wedge B}$$

[147][Steinberger, 2011b], p. 347.

[148]Someone could be worried about the famous lottery paradox according to which, while it is rational for each single lottery ticket to believe that it is not winning, it is absurd to believe that all of them are loosing. In particular, it could seem rational to believe the sentence "the ticket number n is loosing" for every number, but it is surely absurd to believe in the conjunction of all these statements, since at least one of them must be winning. In some way, an argument of this kind goes in the uncharted direction of rejecting the conjunctive reading of assertions in a radical way. Nonetheless I believe that the problem here raises with considering all the sentences together, regardless of their being connected by a conjunction or not. That is, the endorsement of the complete list of sentences "the ticket number n is loosing" is as absurd as the endorsement of their conjunction, according to the conjunctive reading, so this is not a reason to be sceptical about their identification.

[149]This rule has been already proposed by Milne in his [Milne, 1994] (p. 78), although for completely different reasons.

Chapter 3

Harmony and Validity for SASC systems

In this chapter, we introduce a strongly separable very weak system for a sublogic of intuitionistic logic. Since this system suits both the restrictions for intuitionistic logic, and those for dual-intuitionistic logic, I label it '**JDJ**'. Later on, we start weakening the requirement of separability. We obtain in this way a system for both intuitionistic (**J**) and classical (**K**) logic. While the main purpose of this chapter is to deal with natural deduction systems, we will develop sequent calculus systems as well. These systems will be used in order to prove results about the natural deduction formulations. A summary of the systems developed can be found in appendix A, while here we will report only the natural deduction versions. Harmony, separability, and validity of these systems is discussed, while the formal results are demanded to the appendix B.

3.1 Single-assumption and single-conclusion strong-separable system

The first system is the one in table 3.1. The usage of lists, multisets or sets is irrelevant in this case, since we will have only one formula (at most) in the antecedent and in the succedent. Nonetheless, in order to have a **SASC**-system, the discharge of the assumptions can not be optional. Indeed, without this obligation we could obtain multiple assumptions in this way.

$$\vee\mathrm{E}\,\dfrac{A \vee B \qquad \vee\mathrm{I}\,\dfrac{A}{A \vee B} \qquad \vee\mathrm{I}\,\dfrac{B}{A \vee B}}{A \vee B}$$

Observation 3.1.1. *Every valid derivation in **SASCNJDJ** have at most one open assumption and precisely one conclusion.*

Proof. We can show that in every derivation there is at most one open assumption by induction on the length of the derivation. The base is obvious and the only interesting cases are those which compose more than one sub-derivation. □

3.1.1 Equivalence between SASCLJDJ and SASCNJDJ

Theorem 3.1.1 (Equivalence between **SASCLJDJ** and **SASCNJDJ**). *1. (a) If $\vdash_{SASCLJDJ} A \Rightarrow B$, then:*

- *Or $A \vdash_{SASCNJDJ} B$;*
- *Or $\vdash_{SASCLJDJ} \Rightarrow B$ and $\vdash_{SASCNJDJ} B$.*

(b) If $\vdash_{SASCLJDJ} A \Rightarrow$, then $A \vdash_{SASCNJDJ} \bot$.

2. (a) If $A \vdash_{SASCNJDJ} B$, then $\vdash_{SASCLJDJ} A \Rightarrow B$.

3.1.2 Properties of JDJ systems

Theorem 3.1.2 (Cut elimination for **SASCLJDJ**). *If $\vdash_{SASCLJDJ} A \Rightarrow B$, then we can prove it without using the Cut rule. Also, there is a procedure that changes a valid derivation of a sequent, in a Cut-free derivation of the same result.*

$$\wedge\mathrm{I}\ \dfrac{C \qquad \begin{matrix}[C]\\ \vdots \\ A\end{matrix} \qquad \begin{matrix}[C]\\ \vdots\\ B\end{matrix}}{A \wedge B} \qquad \wedge\mathrm{E}\ \dfrac{A \wedge B \qquad \begin{matrix}[A]\\ \vdots \\ C\end{matrix}}{C} \qquad \wedge\mathrm{E}\ \dfrac{A \wedge B \qquad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}$$

$$\vee\mathrm{E}\ \dfrac{A \vee B \qquad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C} \qquad \vee\mathrm{I}\ \dfrac{A}{A \vee B} \qquad \vee\mathrm{I}\ \dfrac{B}{A \vee B}$$

$$\supset\mathrm{I}\ \dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B} \qquad \supset\mathrm{E}\ \dfrac{A \supset B \qquad \begin{matrix}\varnothing\\ \vdots \\ A\end{matrix} \qquad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}$$

$$\neg\mathrm{I}\ \dfrac{\begin{matrix}[A]\\ \vdots \\ \bot\end{matrix}}{\neg A} \qquad Efq\ \dfrac{\bot}{C} \qquad \neg\mathrm{E}\ \dfrac{\neg A \qquad \begin{matrix}\varnothing\\ \vdots \\ A\end{matrix}}{\bot}$$

Table 3.1: **SASCNJDJ**

Proof. Proof is in B.2.1. □

Theorem 3.1.3 (Admissibility of Contraction for **SASCLJDJ**). *1. If $\vdash_{SASCLJDJ} A \wedge A \Rightarrow B$, then $\vdash_{SASCLJDJ} A \Rightarrow B$;*

2. If $\vdash_{SASCLJDJ} A \Rightarrow B \vee B$, then $\vdash_{SASCLJDJ} A \Rightarrow B$.
The admissibility of Contraction does not affect the elimination of Cut.

Proof. We can derive the contraction rules in the following way, using Cut:

$$Cont\ \dfrac{A \wedge A \ \Rightarrow\ B}{A \ \Rightarrow\ B} \quad \text{as} \quad Cut\ \dfrac{\Rightarrow\wedge\ \dfrac{A \ \Rightarrow\ A \qquad A \ \Rightarrow\ A}{A \ \Rightarrow\ A \wedge A} \qquad A \wedge A \ \Rightarrow\ B}{A \ \Rightarrow\ B}$$

$$Cont\ \dfrac{A \ \Rightarrow\ B \vee B}{A \ \Rightarrow\ B} \quad \text{as} \quad Cut\ \dfrac{A \ \Rightarrow\ B \vee B \qquad \vee\Rightarrow\ \dfrac{B \ \Rightarrow\ B \qquad B \ \Rightarrow\ B}{B \vee B \ \Rightarrow\ B}}{A \ \Rightarrow\ B}$$

Contraction is not assumed as a primitive rule in **SASCLJDJ**, and so it is not used in the proof of Cut elimination. The adoption of this rule is just an abbreviation, so Cut elimination remains valid. From Cut elimination and definability of Contraction using Cut, we have Contraction elimination.[1] □

Definition 3.1.1 (Maximal sequence (**SASCNJDJ**)). Given a derivation $\mathfrak{D}$ in **SASCNJDJ**, a maximal sequence in it is a list of formulae $C_1, \cdots, C_n$ such that:

- C_1 is the conclusion of an I-rule.
- $C_i = C_{i+1}$, for every $i < n$
- C_i for $1 \leqslant i < n$ is the premise of an inference used in $\mathfrak{D}$, the conclusion of which is the next element on the list C_{i+1};
- The last element of the list C_n is the major premise of an E-rule.

A maximal sequence composed by just a formula is called 'maximal formula'.

Definition 3.1.2 (Normal derivation (**SASCNJDJ**)). A derivation of B from A in **SASCNJDJ** is normal if there are no maximal sequences in it.

Theorem 3.1.4 (Normalization for **SASCNJDJ**). *If $A \vdash_{SASCNJDJ} B$, then:*

[1]This phenomenon is not common at all in sequent calculus, since in almost every sequent calculus the admissibility of Cut is proved *using* Contraction.

- *Or there is a normal derivation of B from A;*
- *Or there is a normal closed proof of B.*

Proof. Proof is in B.2.1 □

Theorem 3.1.5 (Strong separability for **SASCLJDJ**). *If $\vdash_{SASCLJDJ^{\not\leq}} A \Rightarrow B$, then there is a derivation of $A \Rightarrow B$ in which only the rules and formulae for the constants that occur in the end-sequent are used. If $\vdash_{SASCLJDJ} A \Rightarrow B$, then there is a derivation of $A \Rightarrow B$ in which only the rules and formulae for the constants that occur in the end-sequent are used, apart from $\bot$ that can be used if it or $\neg$ occurs in the end-sequent.*

Proof. Let us consider the Cut-free derivation of $A \Rightarrow B$ that exists according to theorem 3.1.2. Without Cut, we can not erase any logical constant introduced in the derivation by any rule (operational or structural, is irrelevant). The only exception is $\Rightarrow \bot$ that erase $\bot$ from the succedent, we will consider this case later. If a rule is necessary to prove a consequence, it has to be used also in the Cut-free derivation, and so the constant it introduce has to be found also in the antecedent or in the conclusion of the sequent.

Let us now consider the seemingly problematic case of $\bot$. Technically speaking, strong separability holds only for the $\bot$-free fragment of **SASCLJDJ**, but we will see that this is unproblematic, since $\bot$ occurs in the I-rule for $\neg$. To save weak separability, we have to prove that if $\bot$ occurs in a Cut-free derivation for $A \Rightarrow B$, then it or $\neg$ occur in A or B. If $\bot$ occurs in the derivation, it has to be introduced by Axiom or by Weakening.

- In the first case, it occurs both on the left and on the right, and only the occurrence on the right is erased. To erase the one on the left, we should move it on the right without using it to form a more complex formula (complexity ineliminable in a Cut-free derivation, once introduced). But the only rules which move formulae on the right are $\Rightarrow \neg$ and $\Rightarrow\supset$, which violate the requirement. So we will find $\bot$ in the end-sequent.
- In the second case we need a void succedent, and the only way to obtain it is by $\Rightarrow \neg$ or by $\Rightarrow \bot$. If we have obtained it by $\Rightarrow \neg$ we have the conclusion, otherwise we go on and ask how this new $\bot$ is introduced in the derivation. We can not always answer taking this path, since in this way we would obtain an infinite descending chain in the derivation (that obviously has to be finite). So $\bot$ or $\neg$ is in the end-sequent.

So we can conclude that the $\bot$-free calculus is strongly separable, and the complete calculus is weakly separable, since $\bot$ occurs only if it or $\neg$ occurs in the end-sequent.

□

Theorem 3.1.6 (Strong separability for **SASCNJDJ**). *If $A \vdash_{SASCNJDJ^{\not\leq,+}} B$, then there is a derivation of B from A in which only the rules and formulae for the constants that occur in those formulae are used. If $A \vdash_{SASCNJDJ} B$, then there is a derivation of B from A in which only the rules and formulae for the constants that occur in those formulae are used, apart from $\bot$ that can be used if it or $\neg$ occurs in A or B.*

Proof. From the previous theorem, we have a Cut-free derivation of $A \Rightarrow B$ in **SASCLJDJ**. By taking this **SASCLJDJ**-derivation and applying the translation seen in proof B.1.1 we obtain a **SASCNJDJ**-derivation of B from A in which $A \otimes B$ is assumed iff it is the principal formula in an application of Axiom, $\otimes$I is applied iff $\Rightarrow \otimes$ is applied, $\otimes$E is applied iff $\otimes \Rightarrow$ is applied and *Efq* is applied iff $\Rightarrow Weak$ is applied. From this, we obtain weak separability for **SASCNJDJ** since in order to have the occurrence of a rule or a formula in the **SASCNJDJ**-derivation we should have it also in the corresponding **SASCLJDJ**-derivation. The only delicate case is with absurdity and negation, since $\Rightarrow Weak$ and void succedent translate in applications of *Efq* and usage of $\bot$.

We have already shown that in order to use $\Rightarrow Weak$ we need a void succedent, and to obtain it we have to use $\Rightarrow \bot$ or $\neg \Rightarrow$, and in those cases $\bot$ or $\neg$ occur in the end-sequent. In other words, we already connected void succedent to absurdity. So if $\bot$ occurs in the **SASCNJDJ**-derivation obtained from translation, it does not violate separability, and the same holds for applications of *Efq*. In conclusion, weak separability holds also for **SASCNJDJ**.

The only consequence of the translation of Weakening with *Efq* is that we can state strong separability only for the $\bot$-free and $\neg$-free fragment of **SASCNJDJ** (and not in the $\bot$-free fragment), because Weakening is enough to give us *Efq* in the translation. For example the strongly separable **SASCLJDJ**-derivation:

$$\wedge\mathrm{I}\ \frac{C \quad \begin{matrix}[C]\\ \vdots \\ A\end{matrix} \quad \begin{matrix}[C]\\ \vdots \\ B\end{matrix}}{A \wedge B} \qquad \wedge\mathrm{E}\ \frac{A \wedge B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix}}{C} \qquad \wedge\mathrm{E}\ \frac{A \wedge B \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}$$

$$\vee\mathrm{E}\ \frac{(A \vee B)\{\wedge D\} \quad \begin{matrix}[A\{\wedge D\}]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B\{\wedge D\}]\\ \vdots \\ C\end{matrix}}{C} \qquad \vee\mathrm{I}\ \frac{A}{A \vee B} \qquad \vee\mathrm{I}\ \frac{B}{A \vee B}$$

$$\supset\mathrm{I}\ \frac{\{C\} \quad \begin{matrix}[A\{\wedge C\}]\\ \vdots \\ B\end{matrix}}{A \supset B} \qquad \supset\mathrm{E}\ \frac{A \supset B\{\wedge(D \wedge E)\} \quad \begin{matrix}[\{E\}]\\ \vdots \\ A\end{matrix} \quad \begin{matrix}[B\{\wedge D\}]\\ \vdots \\ C\end{matrix}}{C}$$

$$\neg\mathrm{I}\ \frac{\{C\} \quad \begin{matrix}[A\{\wedge C\}]\\ \vdots \\ \bot\end{matrix}}{\neg A} \qquad Efq\ \frac{\bot}{C} \qquad \neg\mathrm{E}\ \frac{\neg A\{\wedge C\} \quad \begin{matrix}[\{\ C\}]\\ \vdots \\ A\end{matrix}}{\bot}$$

Table 3.2: **SASCNJ**

$$\Rightarrow Weak\ \dfrac{\neg \Rightarrow\ \dfrac{\Rightarrow\supset\ \dfrac{A \Rightarrow A}{\Rightarrow A \supset A}}{\neg(A \supset A) \Rightarrow}}{\neg(A \supset A) \Rightarrow C}$$

Is translated using *Efq*:

$$Efq\ \dfrac{\neg\mathrm{E}\ \dfrac{\neg(A \supset A) \qquad \supset\mathrm{I}_1\ \dfrac{[A]^1}{A \supset A}}{\bot}}{C}$$

□

In conclusion, let us note that strong separability could not be established for **SASCNJDJ** without *Efq*, so for minimal **SASCNJDJ**. Indeed, even though *Efq* could not be used in this system, the usage of ⊥ could still be needed in a derivation regarding ¬. Of course, the recourse to ⊥ is still justified, since it occurs in ¬I in minimal **SASCNJDJ** too, and the only result we achieve by using a minimal negation is to keep undefined the meaning of ⊥.

The proof of normalization can be used to obtain the validity of every derivation of **SASCJDJ**, defined as in standard proof-theoretic semantics. We will see that this result is not so obvious for the more powerful systems that we develop in the following sections.

3.2 Intuitionistic logic

The **SASC**-system for intuitionistic logic is resented in table 3.2. Its distinctive character is that it allows the elimination of connectives that occur in a subordinate position with respect to the conjunction.

As for Milne's system, '$\{A\}$' means that in the rule this formula is not necessary.[2] For example ⊃I can be used to transform a derivation of B from A in a closed proof of $A \supset B$, or to transform a derivation of B from $A \wedge C$ in a derivation of $A \supset B$ from C.[3]

This notation has the same meaning in the sequent calculus system as well.[4] That is, ⇒⊃ can be

[2] See paragraph 2.4.2.

[3] Technically speaking we could avoid the use of brackets, assuming two rules for each formula $\{A\}$: one with it and one without.

[4] The sequent calculus systems are developed in appendix A.

used to derive $\Rightarrow A \supset B$ from $A \Rightarrow B$, or $C \Rightarrow A \supset B$ from $A \wedge C \Rightarrow B$. This would be impossible with '$\wedge C$' in $\Rightarrow\supset$ without brackets. Obviously also the formulae in the context are not needed but, in this case, we omit the curly brackets.

About sequent calculus, some observations are needed for the transition from a multiple-antecedent framework to our single-antecedent one. If we formulate **LJ** using sets or multisets we can omit Permutation from the rules. An effect of this choice is that formulae in the antecedent become all 'at the same level', and we can apply the rules to every selection of them. For example, in the rule $\Rightarrow\supset$ we can have several formulae in the antecedent, and they are all 'at the same level' if we use sets or multisets of formulae in the definition of sequents, so that we can apply the rule to any one of these. In this case, for example, all the following applications of the rule are correct:

$$\Rightarrow\supset \frac{A, B, C \Rightarrow E}{B, C \Rightarrow A \supset E} \qquad \Rightarrow\supset \frac{A, B, C \Rightarrow E}{A, C \Rightarrow B \supset E} \qquad \Rightarrow\supset \frac{A, B, C \Rightarrow E}{A, B \Rightarrow C \supset E}$$

Of course, it is not possible to behave in the same way in a single formula sequent calculus like **SASCLJ**, since conjunction is a binary connective, and an expression like $A \wedge B \wedge C$ is not grammatical. In this case, we have that in:

$$\Rightarrow\supset? \frac{A \wedge (B \wedge C) \Rightarrow E}{B \wedge C \Rightarrow A \supset E} \qquad \Rightarrow\supset? \frac{A \wedge (B \wedge C) \Rightarrow E}{A \wedge C \Rightarrow B \supset E} \qquad \Rightarrow\supset? \frac{A \wedge (B \wedge C) \Rightarrow E}{A \wedge B \Rightarrow C \supset E}$$

$$\Rightarrow\supset? \frac{B \wedge (A \wedge C) \Rightarrow E}{B \wedge C \Rightarrow A \supset E} \qquad \Rightarrow\supset? \frac{B \wedge (A \wedge C) \Rightarrow E}{A \wedge C \Rightarrow B \supset E} \qquad \Rightarrow\supset? \frac{B \wedge (A \wedge C) \Rightarrow E}{A \wedge B \Rightarrow C \supset E}$$

$$\Rightarrow\supset? \frac{C \wedge (A \wedge B) \Rightarrow E}{B \wedge C \Rightarrow A \supset E} \qquad \Rightarrow\supset? \frac{C \wedge (A \wedge B) \Rightarrow E}{A \wedge C \Rightarrow B \supset E} \qquad \Rightarrow\supset? \frac{C \wedge (A \wedge B) \Rightarrow E}{A \wedge B \Rightarrow C \supset E}$$

Only the first application of the first raw, the second application of the second raw and the third of the third are correct.

In the same way, if we use sets to construct antecedents and succedents of sequents, we can remove Contraction from the rules of **LJ**. In this way we can accept inferences like:

$$\supset\Rightarrow \frac{\Gamma, A \supset B \Rightarrow A \qquad B, \Delta \Rightarrow C}{\Gamma, \Delta, A \supset B \Rightarrow C}$$

Nonetheless, it is obvious that the corresponding inference in **SASCLJ** is not correct:

$$\supset\Rightarrow \frac{D \wedge (A \supset B) \Rightarrow A \qquad B \wedge E \Rightarrow C}{(D \wedge E) \wedge (A \supset B) \Rightarrow C}$$

Since the conclusion should be $(A \supset B) \wedge (E \wedge (D \wedge (A \supset B))) \Rightarrow C$.

Of course, even though these are not correct single steps of inference, they can be derived using Associativity, Commutativity, and Idempotence of conjunction in **SASCLJ**, together with the rule of **SASCLJ** corresponding to that of **LJ**. Nonetheless, since we use the multiple-assumption sequent calculus **LJ** just as an instrument to study our single-assumption systems, we prefer to use lists of formulae. In this way, Permutation and Contraction correspond to Commutativity and Idempotence of conjunction, and the proof of equivalence between **SASCLJ** and **LJ** become easier and more informative. We will know that when **LJ** uses Permutation we have to use Commutativity of conjunction and when **LJ** uses Contraction we have to use Idempotence of conjunction.

While this solution solves much of our problems, the correspondence is not perfect, since Permutation can be applied without consideration for associativity. Indeed the following application of Permutation is correct:

$$\Rightarrow\supset \frac{Per \Rightarrow \dfrac{A, B, C \Rightarrow E}{B, A, C \Rightarrow E}}{A, C \Rightarrow B \supset E}$$

While the corresponding derivation in **SASCLJ** can not relate just to Commutativity:[5]

$$Com \Rightarrow \frac{\Rightarrow\supset \dfrac{As \Rightarrow \dfrac{Com \Rightarrow \dfrac{A \wedge (B \wedge C) \Rightarrow E}{(B \wedge C) \wedge A \Rightarrow E}}{B \wedge (C \wedge A) \Rightarrow E}}{C \wedge A \Rightarrow B \supset E}}{A \wedge C \Rightarrow B \supset E}$$

[5]It is the first derivation in the second raw of the previous table.

Indeed it is obvious that also Associativity of conjunction is needed, since Commutativity just allows us to derive $A \wedge (C \wedge B) \Rightarrow E$ or $(B \wedge C) \wedge A \Rightarrow E$ or $(C \wedge B) \wedge A \Rightarrow E$ from the given premise. The same problem can be seen also in other examples regarding permutation. This problem can not be solved since it refers to a radical difference between comma and conjunction. We can render explicit the need for Permutation in **LJ**, as we did, by using lists of formulae. In the same way, we render explicit the need for Contraction in **LJ**.[6] Nonetheless, there is no possibility for associativity of conjunction to be made explicit in **LJ**, since antecedent formulae are all at the same level. This will influence the conclusion we can obtain from the translation between our system and **LJ**.

Definition 3.2.1 (Division of lists)**.** Given a list of formulae Γ, the pair (Δ, Θ) is a division of it iff $\Delta \cup \Theta = \Gamma$, $\Delta \cap \Theta = \varnothing$ and $\forall x_{\in\Delta} y_{\in\Theta}(x < y)$. Where $<$ is the relation of ordering of the list. We write $(\Delta, \Theta)_\Gamma$ if (Δ, Θ) is a division of Γ.

Definition 3.2.2 (Conjunction of antecedent)**.** Given a list of formulae Γ, $\Gamma^\wedge$ is the class of all the possible associations of conjunctions of all the formulae $\gamma \in \Gamma$. Formally, $\Gamma^\wedge = \{(x) \wedge (y) \mid x \in \Delta^\wedge \;\; \& \;\; y \in \Theta^\wedge \;\; \& \;\; (\Delta, \Theta)_\Gamma\}$, if Γ has more than one element. $\Gamma^\wedge = \Gamma$, otherwise. $\vdash \Gamma^\wedge \Rightarrow C$ and $\Gamma^\wedge \vdash C$ are used as abbreviations for $\vdash \{A\} \Rightarrow \{C\}$ and $\{A\} \vdash \{C\}$ for every $A \in \Gamma^\wedge$. In brief, the result has to be provable for every element of $\Gamma^\wedge$, regardless of the position of parenthesis.

As an example, let us consider $\Gamma = \{A, B, C, D\}$. It has three divisions, with several divisions themselves:

1. $\Delta_1 = \{A, B, C\}$ and $\Theta_1 = \{D\}$, where Δ_1 has two divisions:
   - $\Lambda_1 = \{A, B\}$ and $\Xi_1 = \{C\}$;
   - $\Lambda_2 = \{A\}$ and $\Xi_2 = \{B, C\}$.

   and Θ_1 only the obvious one;
2. $\Delta_2 = \{A, B\}$ and $\Theta_2 = \{C, D\}$, where both sets have only one obvious division;
3. $\Delta_3 = \{A\}$ and $\Theta_3 = \{B, C, D\}$, where Δ_3 has only one obvious division, while Θ_3 has two divisions:
   - $\Upsilon_1 = \{B, C\}$ and $\Omega_1 = \{D\}$;
   - $\Upsilon_2 = \{B\}$ and $\Omega_2 = \{C, D\}$.

Let us start from the first division of Γ. We have that $\Lambda_1^\wedge = \{A \wedge B\}$, $\Xi_1^\wedge = \{C\}$, $\Lambda_2^\wedge = \{A\}$ and $\Xi_2^\wedge = \{B \wedge C\}$. So $\Delta_1^\wedge = \{(x) \wedge (y) \mid x \in \Lambda_1^\wedge \;\; \& \;\; y \in \Xi_1^\wedge \;\; or \;\; x \in \Lambda_2^\wedge \;\; \& \;\; y \in \Xi_2^\wedge\} = \{(A \wedge B) \wedge C, A \wedge (B \wedge C)\}$, while obviously $\Theta_1^\wedge = \{D\}$

The second division is easier. $\Delta_2^\wedge = \{A \wedge B\}$ and $\Theta_2^\wedge = \{C \wedge D\}$.

The third division is similar to the first. We have that $\Upsilon_1^\wedge = \{B \wedge C\}$, $\Omega_1^\wedge = \{D\}$, $\Upsilon_2^\wedge = \{B\}$ and $\Omega_2^\wedge = \{C \wedge D\}$. So $\Delta_3^\wedge = \{A\}$ and $\Theta_3^\wedge = \{(x) \wedge (y) \mid x \in \Upsilon_1^\wedge \;\; \& \;\; y \in \Omega_1^\wedge \;\; or \;\; x \in \Upsilon_2^\wedge \;\; \& \;\; y \in \Omega_2^\wedge\} = \{(B \wedge C) \wedge D, B \wedge (C \wedge D)\}$.

In conclusion: $\Gamma^\wedge = \{(x) \wedge (y) \mid x \in \Delta_1^\wedge \;\; \& \;\; y \in \Theta_1^\wedge \;\; or \;\; x \in \Delta_2^\wedge \;\; \& \;\; y \in \Theta_2^\wedge \;\; or \;\; x \in \Delta_3^\wedge \;\; \& \;\; y \in \Theta_3^\wedge\} = \{((A \wedge B) \wedge C) \wedge D, (A \wedge (B \wedge C)) \wedge D, (A \wedge B) \wedge (C \wedge D), A \wedge ((B \wedge C) \wedge D), A \wedge (B \wedge (C \wedge D))\}$.

Definition 3.2.3. Given a formula C, C° is defined as the result of the uniform substitution of $\neg(E \supset E)$ for $\perp$ in C. Given a set of formulae Γ, Γ° is the set of all γ° for $\gamma \in \Gamma$.

Theorem 3.2.1 (Equivalence between **LJ** and **SASCLJ**)**.** *Sequent calculi **LJ** and **SASCLJ** are equivalent to each other, that is:*

1. *If* $\vdash_{LJ} \Gamma \Rightarrow C$*, then* $\vdash_{SASCLJ} \Gamma^\wedge \Rightarrow C$*;*
2. *If* $\vdash_{SASCLJ} D \Rightarrow C$*, then* $\vdash_{LJ} D^\circ \Rightarrow C^\circ$*.*

Proof. The proof is in B.1.2. □

Theorem 3.2.2 (Equivalence between **SASCLJ** and **SASCNJ**)**.** *The sequent calculus **SASCLJ** and the natural deduction system **SASCNJ** are equivalent to each other, that is:*

1. *(a) if* $\vdash_{SASCLJ} A \Rightarrow B$*, then:*

[6] In this case multisets would be enough.

- *Or* $A \vdash_{SASCNJ} B$;
- *Or* $\vdash_{SASCLJ} \Rightarrow B$ *and* $\vdash_{SASCNJ} B$.

(b) *If* $\vdash_{SASCLJ} A \Rightarrow$, *then* $A \vdash_{SASCNJ} \bot$.

2. (a) *If* $A \vdash_{SASCNJ} B$ *then* $\vdash_{SASCLJ} A \Rightarrow B$.

Proof. The proof is in B.1.2. □

Theorem 3.2.2 in combination with theorem 3.2.1 shows that **SASCNJ** is adequate for intuitionistic logic.

3.2.1 Separability

To prove weak separability of this system, we use the principle:

Observation 3.2.1. *If* $\ominus_1, \ldots, \ominus_m$ *are all and only the logical constants used in* $\otimes_1 I, \ldots, \otimes_n I$*, then all valid derivations of consequences regarding only* $\otimes_1, \ldots, \otimes_n$ *must be provable using only* $\otimes_1, \ldots, \otimes_n, \ominus_1, \ldots, \ominus_m$ *(so just by using* $\ominus_i$*-formulae,* $\ominus_i$*-rules,* $\otimes_i$*-formulae and* $\otimes_i$*-rules).*

This observation specifies what we have to prove in order to obtain separability. The *desiderata* is that there is no dependence of the meaning of one connective from the meaning of another one if this does not occur in the I-rule of the first. This is just a more practical reformulation of the analyticity of logic.

By looking at the I-rules of **SASCNJ** we obtain the following clauses:

1. To prove a consequence in which only $\wedge$ occurs, we need to use only $\wedge$-rules and $\wedge$-formulae;
2. To prove a consequence in which only $\vee$ occurs, we need to use only $\vee$-rules and $\vee$-formulae;
3. To prove a consequence in which only $\wedge$ and $\vee$ occur, we need to use only $\vee$-rules, $\wedge$-rules, $\vee$-formulae and $\wedge$-formulae;
4. To prove a consequence in which only $\supset$ occurs, we need to use only $\supset$-rules, $\wedge$-rules, $\supset$-formulae and $\wedge$-formulae (the same holds for consequences in which only $\supset$ and $\wedge$ occur);
5. To prove a consequence in which only $\supset$ and $\vee$ (and $\wedge$) occur, we need to use only $\supset$-rules, $\wedge$-rules, $\vee$-rules, $\supset$-formulae, $\wedge$-formulae and $\vee$-formulae;
6. To prove a consequence in which only $\neg$ occurs, we need to use only $\neg$-rules, $\wedge$-rules, *Efq*, $\neg$-formulae, $\wedge$-formulae and $\bot$ (the same holds for consequences in which only $\neg$ and $\wedge$ occur);
7. To prove a consequence in which only $\neg$ and $\vee$ (and $\wedge$) occur, we need to use only $\neg$-rules, $\wedge$-rules, $\vee$-rules, *Efq*, $\neg$-formulae, $\wedge$-formulae, $\vee$-formulae and $\bot$;
8. To prove a consequence in which only $\neg$ and $\supset$ (and $\wedge$) occur, we do not need to use rules of formulae for $\vee$;
9. Of course if the consequence regards the full language, we can fully use it in the derivation.

We should have also some clauses regarding the occurrence of $\bot$ in assumption and conclusion, but we will deal with it in the next section.

Disjunction and conjunction

Let us start from clause *1*.

Conjunction is independent To show this, let us prove that **SASCNJ** conservatively extends **SASCNJ**$^\wedge$. The result will be that the meaning of conjunction is independent of that of the other connectives.[7]

Theorem 3.2.3 (**SASCNJ** conservatively extends **SASCNJ**$^\wedge$)**.** *If* $A \vdash_{SASCNJ} B$ *and the only connective that occurs in* $\{A\} \cup \{B\}$ *is* $\wedge$*, then* $A \vdash_{SASCNJ^\wedge} B$.

[7]Let us keep in mind that, since we use separability to define independence of meaning, this relation is not symmetrical.

Proof. We prove that $\mathbf{SASCNJ}^{\wedge}$ can prove all the consequences of $\mathbf{NJ}^{\wedge}$. **NJ** is strongly separable, that is $\wedge$ rules are sufficient to prove all the **NJ**-consequences regarding only $\wedge$, so by this result we obtain that $\wedge$ rules of **SASCNJ** are sufficient to prove all the **SASCNJ**-consequences regarding only $\wedge$.

Let us assume that $\mathtt{d}$ is a proof of $\Gamma \vdash_{NJ} B$. We define the proof $\mathtt{d}^*$ of $\Gamma^{\wedge} \vdash_{SASCNJ} B$ by induction on the length of $\mathtt{d}$. The base is obvious, so let us see the steps:

$\wedge$I:

$$\wedge\mathrm{I}\ \dfrac{\begin{matrix}\Delta\\ \vdots\mathtt{d}_1\\ A\end{matrix} \qquad \begin{matrix}\Theta\\ \vdots\mathtt{d}_2\\ B\end{matrix}}{A \wedge B} \quad \rightsquigarrow \quad \wedge\mathrm{I}_{1,2}\ \dfrac{\Delta^{\wedge} \wedge \Theta^{\wedge} \qquad \begin{matrix}\wedge\mathrm{E}\ \dfrac{[\Delta^{\wedge} \wedge \Theta^{\wedge}]^1}{\Delta^{\wedge}}\\ \vdots\mathtt{d}_1^*\\ A\end{matrix} \qquad \begin{matrix}\wedge\mathrm{E}\ \dfrac{[\Delta^{\wedge} \wedge \Theta^{\wedge}]^2}{\Theta^{\wedge}}\\ \vdots\mathtt{d}_2^*\\ B\end{matrix}}{A \wedge B}$$

In this way we obtain $\Delta^{\wedge} \wedge \Theta^{\wedge} \vdash_{SASCNJ} A \wedge B$. To have $(\Delta \cup \Theta)^{\wedge} \vdash_{SASCNJ} A \wedge B$ in its full generality, we have to apply associativity of conjunction to the major premise of $\wedge$I. This is not a problem, since we have already seen that it can be derived using only $\wedge$-rules.

$\wedge$E:

$$\wedge\mathrm{E}\ \dfrac{\begin{matrix}\Delta\\ \vdots\mathtt{d}\\ A \wedge B\end{matrix}}{A} \quad \rightsquigarrow \quad \wedge\mathrm{E}_1\ \dfrac{\begin{matrix}\Delta^{\wedge}\\ \vdots\mathtt{d}^*\\ A \wedge B\end{matrix} \qquad [A]^1}{A}$$

So, in order to prove the purely conjunctive consequences of **NJ** we have to use only $\wedge$-rules of **SASCNJ**. If $A \vdash_{SASCNJ} B$, then $A \vdash_{NJ} B$ by theorems 3.2.1, 3.2.2 and equivalence between **LJ** and **NJ** in [Gentzen, 1969b].[8] By strong separability, there is a derivation of $A \vdash_{NJ} B$ in which only $\wedge$-rules are applied, and by this translation there is such a derivation also in **SASCNJ**.[9]

We consider the $\wedge$-fragment as the fragment in which the other logical constants are removed from the vocabulary and their rules are removed from the proof systems. So also the separability restriction on formula occurrence holds.

□

Disjunction is independent (also from conjunction) The rules for $\wedge$ are pure both in **SASCNJDJ** and in **SASCNJ**, while those for $\vee$ are pure only in the first system. The fact that $\wedge$ occurs in the elimination but not in the introduction rule for $\vee$ can be confusing; is conjunction necessary to define the meaning of disjunction? I do not think so, indeed $\vee$I defines the meaning of disjunction, and it is pure. We can make rigorous this intuition by proving that all the valid consequences in **SASCNJ** in which only $\vee$ occurs are already provable in **SASCNJDJ**, and so can be proved using only pure rules for $\vee$ (clause *2*).

Theorem 3.2.4 (**SASCNJ** conservatively extends $\mathbf{SASCNJDJ}^{\vee}$)**.** *If $A \vdash_{SASCNJ} B$ and the only connective that occurs in $\{A\} \cup \{B\}$ is $\vee$, then $A \vdash_{SASCNJDJ} B$.*

Proof. If $A \vdash_{SASCNJ} B$ and $\vee$ is the only connective in A and B, then we know by theorems 3.2.2 and 3.2.1 that $\vdash_{LJ} A \Rightarrow B$.[10] By the equivalence between **LJ** and **NJ** established in [Gentzen, 1969b], we obtain $A \vdash_{NJ} B$. Now, since this system is strongly separable, $A \vdash_{NJ} B$ can be proved using only $\vee$-rules and $\vee$-formulae. Since these rules are the same as those of **SASCNJDJ**, and it is impossible to obtain multiple assumptions just using them, the derivation of **NJ** works also for **SASCNJDJ**. So we have that the $\vee$-fragment of **SASCNJDJ** (that is equivalent to that of **SASCNJ**) proves every purely disjunctive consequence of **SASCNJ**. □

So the conjunction in $\vee$E is needed only to prove consequences in which $\vee$ do occurs alongside with $\wedge$, like distribution of conjunction over disjunction:

$$\dfrac{(B \vee C) \wedge A \qquad \vee\mathrm{I}\ \dfrac{[A \wedge B]^1}{(A \wedge B) \vee (A \wedge C)} \qquad \vee\mathrm{I}\ \dfrac{[A \wedge C]^1}{(A \wedge B) \vee (A \wedge C)}}{(A \wedge B) \vee (A \wedge C)}\ \vee\mathrm{E}_1$$

[8] Since $\perp$ does not occur neither in A nor in B, the conversion indicated by $^{\circ}$ is not used.

[9] Of course we could obtain the same result using **LJ** instead of **NJ**.

[10] Since $\perp$ does not occur neither in A nor in B, the conversion indicated by $^{\circ}$ is not used.

Disjunction and conjunction does not depend on other connectives Let us conclude the section on conjunction and disjunction by proving clause *3*.

Theorem 3.2.5 (**SASCNJ** conservatively extends **SASCNJ**$^{\wedge\vee}$)**.** *If* $A \vdash_{SASCNJ} B$ *and the only connectives that occur in* $\{A\} \cup \{B\}$ *are* $\wedge$ *and* $\vee$*, then* $A \vdash_{SASCNJ^{\wedge\vee}} B$*.*

Proof. If $A \vdash_{SASCNJ} B$ and the only connectives that occur in $\{A\} \cup \{B\}$ are $\wedge$ and $\vee$, then we know by theorems 3.2.2 and 3.2.1 that $\vdash_{LJ} A \Rightarrow B$.[11] Let us consider a Cut-free derivation of $A \Rightarrow B$ in **LJ** in which only $\wedge$ and $\vee$-formulae occur.[12] The translation used in proof B.1.2 gives us a **SASCLJ**-derivation in which only the rules used by **LJ** are applied, apart from those obtained by the translation of $Cont \Rightarrow$, $Weak \Rightarrow$, $Perm \Rightarrow$ and by occurrences of $As \Rightarrow$. Those extra rules request also the application of Cut. Nonetheless, it can be seen that the translation used in proof B.1.2 to obtain a **SASCNJ**-derivation from a **SASCLJ**-derivation allow us to translate all those extra rules by using only $\wedge$-rules. Also, it can be easily seen that only atomic, $\wedge$ and $\vee$-formulae occur in the obtained derivation. So separability holds for the derivation that we obtain at the end of this process. □

But how is it possible that the meanings of $\wedge$ and $\vee$ are the same in **SASCNJDJ** and **SASCNJ**, if the two logics disagree on some derivations like distribution? The answer is that, even though the connectives have the same meanings, $\vee$E of **SASCNJDJ** does not allow to completely use the meaning given to this connective by $\vee$I. In some way, it is too weak for the respective I-rule and, in order to solve this problem, we have to introduce $\wedge$. This answer could seem a little *ad hoc*, but we will see that also another *criterion* for deciding of the acceptability of E-rules – existence of normal derivation – save **SASCNJ** extension of $\vee$E.

Conjunction and Implication

We will prove clause *4* in two steps.

Implication depends on conjunction Since $\wedge$ occurs in $\supset$I, the meaning of intuitionistic implication depends on the meaning of conjunction. To show this, it is enough to find a purely implicational consequence that can not be derived without using conjunction.[13] A good example is transitivity. Let us use the abbreviations: $\phi =_{def} ((B \supset C) \wedge (A \supset B)) \wedge A$ and $\psi =_{def} (B \supset C) \supset (A \wedge B) \vee (A \wedge C)$.

$$
\dfrac{[\phi]^5 \qquad \dfrac{\dfrac{[\phi]^2}{(B \supset C) \wedge (A \supset B)}\,{\scriptstyle \wedge E}}{B \supset C}\,{\scriptstyle \wedge E} \qquad \dfrac{\dfrac{[\phi]^2 \qquad \dfrac{\dfrac{[\phi]^1}{(B \supset C) \wedge (A \supset B)}\,{\scriptstyle \wedge E}}{A \supset B}\,{\scriptstyle \wedge E} \qquad \dfrac{[\phi]^1}{A}\,{\scriptstyle \wedge E}}{(A \wedge B) \wedge A}\,{\scriptstyle \wedge I_1}}{((A \wedge B) \wedge A)}\,{\scriptstyle \wedge I_2}}{\psi}\,{\scriptstyle \wedge I_2}
$$

$$\vdots$$

$$
\dfrac{\dfrac{[A \supset B]^7 \qquad \dfrac{[(B \supset C) \wedge (A \supset B)]^6 \qquad \dfrac{\begin{matrix}\vdots\\ \psi\end{matrix} \qquad \dfrac{[(A \supset B) \wedge A]^4 \qquad [A]^3 \qquad [B]^3}{B}\,{\scriptstyle \supset E_3} \qquad [C]^4}{C}\,{\scriptstyle \supset E_4}}{A \supset C}\,{\scriptstyle \supset I_5}}{(B \supset C) \supset (A \supset C)}\,{\scriptstyle \supset I_6}}{(A \supset B) \supset ((B \supset C) \supset (A \supset C))}\,{\scriptstyle \supset I_7}
$$

To see that the usage of $\wedge$ is necessary to prove transitivity of $\supset$, we show that in **JDJ** (that is where $\wedge$ is not used to define $\supset$) transitivity does not hold.

Theorem 3.2.6. $\nvdash_{JDJ} (A \supset B) \supset ((B \supset C) \supset (A \supset C))$.

[11] Since $\bot$ does not occur neither in A nor in B, the conversion indicated by $^\circ$ is not used.

[12] Such a derivation exists, since Cut is admissible in **LJ** even if Weakening and Axiom can be applied only with atoms as principal formulae.

[13] Of course this is a far less important restriction since it just prevents from useless weakenings of separability.

Proof. We have already established that strong separability holds for **LJDJ**, so if this formula is a theorem, a sequent with void antecedent and it as succedent is provable using only rules for $\supset$ and Weakening. In particular this derivation will not use Cut. *Ad absurdum* let us consider such a derivation of $\Rightarrow (A \supset B) \supset ((B \supset C) \supset (A \supset C))$ in **LJDJ**. The last rule applied to derive this conclusion can only be $\Rightarrow\supset$, since the only other alternative $\Rightarrow Weak$ is excluded by $\nvdash_{LJDJ}\Rightarrow$. So the penultimate sequent of the derivation is $A \supset B \Rightarrow (B \supset C) \supset (A \supset C)$. Since the antecedent is not void, this can not be the conclusion of an application of $\Rightarrow\supset$. It can not be the conclusion of Weakening because $\nvdash_{LK}\Rightarrow A \supset B$ and $\nvdash_{LK}\Rightarrow (B \supset C) \supset (A \supset C)$, so *a fortiori* they can not be provable in **LJDJ**. So it has to be the conclusion of $\supset\Rightarrow$, with premisses: $\Rightarrow A$ and $B \Rightarrow (B \supset C) \supset (A \supset C)$. But none of these is derivable in **LJDJ**, since they are not derivable in **LK**. This is absurd, so $\nvdash_{LJDJ}\Rightarrow (A \supset B) \supset ((B \supset C) \supset (A \supset C))$. ☐

It is obvious that if $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$ were provable in **SASCLJ** using only $\supset$ then it would be provable also in **SASCLJDJ**. So we have that the meaning of intuitionistic implication depends on that of conjunction.

Implication does not depend on other constants apart from conjunction

Theorem 3.2.7 (**SASCNJ** conservatively extends **SASCNJ**$^{\wedge\supset}$)**.** *If $A \vdash_{SASCNJ} B$ and the only connectives that occur in $\{A\} \cup \{B\}$ is $\supset$, then $A \vdash_{SASCNJ^{\wedge\supset}} B$. The same holds if also $\wedge$ occurs in $\{A\} \cup \{B\}$.*

Proof. If $A \vdash_{SASCNJ} B$ and the only connectives that occur in $\{A\} \cup \{B\}$ is $\supset$, then we know by theorems 3.2.2 and 3.2.1 that $\vdash_{LJ} A \Rightarrow B$.[14] Let us consider a Cut-free derivation of $A \Rightarrow B$ in **LJ** in which only $\supset$-formulae occur.[15] The translations used in proof B.1.2 and B.1.2 give us the desired **SASCNJ**-derivation, in which only the rules corresponding to those used by **LJ** are applied, apart from those obtained by the translation of $Cont \Rightarrow$, $Weak \Rightarrow$, $Perm \Rightarrow$, Cut and by occurrences of $As \Rightarrow$, that are all $\wedge$-rules. Also, it can be easily seen that only atomic, $\wedge$ and $\supset$-formulae occur in the obtained derivation. So separability holds for the derivation that we obtain at the end of this process.

☐

Disjunction and Implication

The same proof procedure seen until now can be used to prove clause *5* regarding consequences about $\supset$ and $\vee$ (and $\wedge$).

Theorem 3.2.8 (**SASCNJ** conservatively extends **SASCNJ**$^{\wedge\vee\supset}$)**.** *If $A \vdash_{SASCNJ} B$ and the only connectives that occur in $\{A\} \cup \{B\}$ are $\supset$ and $\vee$, then $A \vdash_{SASCNJ^{\wedge\vee\supset}} B$. The same holds if also $\wedge$ occurs in $\{A\} \cup \{B\}$.*

Conjunction and negation

The same proof procedure seen until now can be used to prove clause *6*.

Theorem 3.2.9 (**SASCNJ** conservatively extends **SASCNJ**$^{\wedge\neg}$)**.** *If $A \vdash_{SASCNJ} B$ and the only connective that occurs in $\{A\} \cup \{B\}$ is $\neg$, then $A \vdash_{SASCNJ^{\wedge\neg}} B$. The same holds if also $\wedge$ occurs in $\{A\} \cup \{B\}$.*

Disjunction and negation

The same proof procedure seen until now can be used to prove clause *7*.

Theorem 3.2.10 (**SASCNJ** conservatively extends **SASCNJ**$^{\wedge\vee\neg}$)**.** *If $A \vdash_{SASCNJ} B$ and the only connectives that occur in $\{A\} \cup \{B\}$ are $\neg$ and $\vee$ then $A \vdash_{SASCNJ^{\wedge\vee\neg}} B$. The same holds if also $\wedge$ occurs in $\{A\} \cup \{B\}$.*

[14] Since $\perp$ does not occur neither in A nor in B, the conversion indicated by $^\circ$ is not used.

[15] Such a derivation exists, since Cut is admissible in **LJ** even if Weakening and Axiom can be applied only with atoms as principal formulae.

Negation and implication

The same proof procedure seen until now can be used to prove clause *8*.

Theorem 3.2.11 (**SASCNJ** conservatively extends **SASCNJ**$^{\wedge\supset\neg}$)**.** *If* $A \vdash_{SASCNJ} B$ *and the only connectives that occur in* $\{A\} \cup \{B\}$ *are* $\neg$ *and* $\supset$ *then* $A \vdash_{SASCNJ^{\wedge\supset\neg}} B$*. The same holds if also* $\wedge$ *occurs in* $\{A\} \cup \{B\}$*.*

Absurdity

We discuss separately the occurrence of $\bot$ in the derivation, since in this case we can not prove separability in the usual way, due to $^\circ$-translation of theorem 3.2.1. Let us start by observing the following theorem:

Theorem 3.2.12 (**SASCNJ** conservatively extends **SASCNJ**$^{\bot}$)**.** *If* $A \vdash_{SASCNJ} B$ *and the only logical constant that occurs in* $\{A\} \cup \{B\}$ *is* $\bot$ *then* $A \vdash_{SASCNJ^{\bot}} B$*.*

Proof. Since $\bot$ is a 0-ary connective and we are developing a single-assumption, single-conclusion system, the only possibilities are:

- $\bot \vdash \bot$, that is just an assumption;
- $\bot \vdash B$, that is an application of *Efq*;
- $A \vdash \bot$, that is impossible in **SASCNJ**.[16]

So in every case, if the derivation holds in **SASCNJ**, it holds just applying *Efq* or Axiom, so without using any other logical constant apart from $\bot$. ☐

We can not extend the result to consequences in which $\bot$ occurs together with other logical constants in the same way done previously for $\neg$, $\supset$, etc. Indeed by relating to translation in **LJ** and Cut elimination, we could be forced to use $\supset$ and $\neg$ all the time $\bot$ is in the assumption or in the conclusion. Indeed $\bot^\circ = \neg(E \supset E)$. But we can not accept this conclusion since the meaning of $\bot$ does not depends on that of $\neg$ and $\supset$.[17] We propose a different solution.

Theorem 3.2.13. *If* $A \vdash_{SASCNJ} B$*, the only connectives necessary to derive* B *from* A *are those in* $\{A\} \cup \{B\}$ *together with at most* $\wedge$*.*

Proof. Let us consider the translation from **LJ** to **SASCLJ** in proof B.1.2. We can observe that if we translate only Cut-free derivations of **LJ**, we obtain a derivation of **SASCLJ** in which we use Cut only to translate applications of $Con \Rightarrow$, $Per \Rightarrow$ or to apply Associativity of conjunction. Let us call these kind of Cut respectively: $(Con \Rightarrow)^*$, $(Per \Rightarrow)^*$ and $(As \Rightarrow)^*$. Let us also call semi-Cut-free a derivation of **SASCLJ** in which only these Cuts occur. Since Cut-free **LJ** is equivalent to full **LJ**, we conclude that semi-Cut-free **SASCLJ** is equivalent to full **SASCLJ**.[18]

By this result and theorem 3.2.2 (from **SASCNJ** to **SASCLJ**) we can assume that there is a semi-Cut-free derivation of $A \Rightarrow B$ in **SASCLJ**. All the logical constants that occur in a semi-Cut-free derivation occurs in the end-sequent apart from $\wedge$. This is the only constant that can vanish in the derivation, since $(Con \Rightarrow)^*$, $(Per \Rightarrow)^*$ and $(As \Rightarrow)^*$ are the only Cuts in the derivation and they can erase only $\wedge$.[19]

Let us now consider the translation from **SASCLJ** to **SASCNJ** in proof B.1.2. In the **SASCNJ**-derivation obtained from a Cut-free **SASCLJ**-derivation we use all and only the rules and formulae used in the **SASCLJ**-derivation, apart from *Efq* used to translate $\Rightarrow Weak$ and $\bot$ used to translate the void succedent. Nonetheless, these two exceptions are not relevant, since in order to apply $\Rightarrow Weak$ we need a void succedent, and in order to have it, we have to use $\Rightarrow \bot$ or $\neg \Rightarrow$. So if $\bot$ occurs in the **SASCNJ**-derivation, it or $\neg$ occur in the open assumption or in the conclusion.[20] In our case the **SASCLJ**-derivation is just semi-Cut-free, so we could have extra applications of $\wedge$-rules and formulae. So in order to prove $A \vdash B$ in **SASCNJ** we need to use only the rules for the connectives in $\{A\} \cup \{B\}$ together with $\wedge$. This conclusion holds also for $\bot$, since there is no application of the translation in **LJ**, and the extra occurrences of *Efq* are unproblematic. ☐

[16]At least without further specification about atomic language.

[17]It is not so clear how to define the meaning of this term, but we will see that a solution can be found. Nonetheless, it is blatantly obvious that it does not depend on that of $\supset$.

[18]This result has the strength of a theorem of existence of Cut-free derivation, not of a theorem of Cut elimination.

[19]To be honest, also this eventuality is very rare. It only happens with $(Con \Rightarrow)^*$, and only when we contract a conjunction that does not have any conjunctions between its subformulae.

[20]The proof is identical to that for **SASCLJDJ**.

This theorem solve all the problems for separability apart from $A \vdash_{SASCNJ} B$ when the only connectives in $\{A\} \cup \{B\}$ are $\vee$ and $\bot$. In this case, since $\wedge$ does not occur neither in $\vee$I, nor in $\bot$I (vacuously), it can not be necessary to derive $A \vdash B$.

Theorem 3.2.14. *If $A \vdash_{SASCNJ} B$, and the only logical constants in $\{A\} \cup \{B\}$ are $\vee$ and $\bot$, then $A \vdash_{SASCNJ^{\vee\bot}} B$.*

Proof. If $A \vdash_{SASCNJ} B$, then $A \vdash_{NJ} B$. Since **NJ** is strongly separable, there is a derivation of $A \vdash B$ in **NJ** in which only $\vee$ and $\bot$ occur. Since the rules for these constants are the same in **NJ** and in **SASCNJ**, this derivation holds also for this second system. □

With this result, we conclude our proof of weak separability for **SASCNJ**.

3.2.2 Harmony

Definition of normal form

The idea of normalization is that if only I-rules define the meaning of logical constants, E-rules have to be justified according to them. From the point of view of validity, this means that E-rules are non-canonical derivations and so they are not self-justified. We can not derive E-rules from I-rules in the standard meaning of the term 'derive'. Otherwise, we could have a proof-system with only I-rules for intuitionist and classical logic, and this is not the case.

The idea of harmony is that an application of E-rule can be erased when its major premise is immediately derived using an I-rule. We call this process normalization, and two sets of rules (I and E-rules) are in harmony if they allow for such a process. This characterization of harmony (that we presented in section 1.2.1) is fine both for the traditional formulation of **NJ** and for **JDJ**, as we have already seen, but it must be adapted to fit our other systems. The problem is that in our systems a rule can manipulate more logical constants at the same time, so we need to consider lists of applications of I and E-rules.

As an example, let us consider a derivation in which $\supset$E occurs.

$$\supset\mathrm{E}_1\ \dfrac{\begin{array}{c}\mathtt{d}\\ (A \supset B) \wedge (D \wedge E)\end{array} \qquad \begin{array}{c}[E]^1\\ \vdots\\ A\end{array} \qquad \begin{array}{c}[B \wedge D]^1\\ \vdots\\ C\end{array}}{C}$$

The major premise $(A \supset B) \wedge (D \wedge E)$ can not be derived directly using $\supset$I, since $\supset$ is not its principal connective. Does this mean that all derivations of this kind are normal? I do not think so. Indeed let us consider these two rules:

$$tonk\mathrm{I}\ \dfrac{A}{A tonk B} \qquad tonk\mathrm{E}_1\ \dfrac{(A tonk B)\{\wedge C\} \qquad \begin{array}{c}[B\{\wedge C\}]^1\\ \vdots\\ D\end{array}}{D}$$

We can not criticize this pair of rules using our separability constraint, at least not *prima facie*. Indeed there is not evident circularity in the way in which connectives are used in these rules since the situation seems to be the same of $\vee$, so we respect meaning molecularity (definition 1.1.8). Nonetheless, we find out that this pair of rules are unacceptable, for two connected reasons. Consider the following derivation:

$$\dfrac{\wedge\mathrm{I}_1\ \dfrac{A \qquad tonk\mathrm{I}\ \dfrac{[A]^1}{A tonk B} \qquad [A]^1}{(A tonk B) \wedge A} \qquad [B \wedge A]^2}{B \wedge A}\ tonk\mathrm{E}_2$$

It proves that *tonk* violates separability since $A \nvdash_{SASCNJ} B \wedge A$, and that it violates harmony since the conclusion can not be derived from the direct ground for $(A tonk B) \wedge A$. So we can violate harmony even though E-rules use more than one logical constant. We can also show that this pair of rules rejects the thesis that the meaning of non-logical terms is independent of that of logical-terms, since we can continue the derivation applying $\wedge$E and derive B from A. Indeed, as we already observed, separability and innocence of logic do not follow from meaning molecularity and absence of meaning conferring logical-rules for atomic sentences, since they are global properties of the language.[21] What we want to develop in this section is a notion of harmony apt to reject the rules that violate these restrictions, in the same way as we did in the section 1.2.2 of the first chapter.

The key idea of harmony is that if all the meaning of the major premise is given by I-rules, the E-rule will just exploit this ground to derive its conclusion. Doing so, it gives us a non-canonical ground for the conclusion. Harmony requires that, in this case, we can derive the same conclusion canonically, using the grounds for the major premise of the E-rule. The full development of this idea is demanded to the explicit formulation of validity in proof-theoretic semantics, as we saw.[22]

This picture is adequate also for our single-assumption system, but we have to acknowledge that some E-rules exploit the meaning of more than one connective, and so we will be able to normalize the derivation only if we have more than one I-rule in the direct derivation of the major premise. The connectives used by an E-rules are those that occur in its schema. In case an E-rule uses the meaning of both $\wedge$ and $\supset$ to derive the conclusion, it is natural to ask that all these connectives are introduced by their respective canonical grounds, in order to have normalization. We could be tempted to say that $\supset$E is an elimination rule both for $\supset$ and $\wedge$, but we must remember our separability results: the meaning of $\supset$E depends on that of $\wedge$, but the opposite does not hold. So it is not strange to ask an application of a $\wedge$-rule in order to use $\supset$. In other words, the applications of $\supset$E that we have to normalize has the structure:

$$
\dfrac{\dfrac{F \quad \dfrac{[F]^3 \quad \begin{matrix}[A\wedge F]^1\\ \vdots \\ B\end{matrix}}{A\supset B}{\supset}\mathrm{I}_1 \quad \dfrac{[F]^3 \quad \begin{matrix}[F]^2\\ \vdots\\ D\end{matrix} \quad \begin{matrix}[F]^2\\ \vdots\\ E\end{matrix}}{D\wedge E}{\wedge}\mathrm{I}_2}{(A\supset B)\wedge(D\wedge E)}{\wedge}\mathrm{I}_3 \quad \begin{matrix}[E]^4\\ \vdots\\ A\end{matrix} \quad \begin{matrix}[B\wedge D]^4\\ \vdots\\ C\end{matrix}}{C}{\supset}\mathrm{E}_4
$$

The other occurrences of E-rules that we have to normalize are:

$$
\dfrac{\dfrac{D \quad \dfrac{[D]^2 \quad \begin{matrix}[A\wedge D]^1\\ \vdots\\ \bot\end{matrix}}{\neg A}\neg\mathrm{I}_1 \quad \begin{matrix}[D]^2\\ \vdots\\ C\end{matrix}}{\neg A\wedge C}{\wedge}\mathrm{I}_2 \quad \begin{matrix}[C]^3\\ \vdots\\ A\end{matrix}}{\bot}\neg\mathrm{E}_3
$$

$$
\dfrac{\dfrac{E \quad \dfrac{\begin{matrix}[E]^1\\ \vdots\\ A\end{matrix}}{A\vee B}{\vee}\mathrm{I} \quad \begin{matrix}[E]^1\\ \vdots\\ C\end{matrix}}{(A\vee B)\wedge C}{\wedge}\mathrm{I}_1 \quad \begin{matrix}[A\wedge C]^2\\ \vdots\\ D\end{matrix} \quad \begin{matrix}[B\wedge C]^2\\ \vdots\\ D\end{matrix}}{D}{\vee}\mathrm{E}_2
$$

Of course, apart from these derivations, we still have those of **SASCNJDJ** to normalize. A derivation in which none of these subderivations occur is in normal form. We can restate the same definition using maximal formulae:

Definition 3.2.4 (Maximal formulae (**SASCNJ**)). Given a derivation $\mathfrak{D}$, a formula that occurs in it is a maximal formula iff:

- it is conclusion of $\oplus$I and major premise of $\oplus$E, for a logical constant $\oplus$; or
- it has the form $(A \vee B) \wedge D$, it is the conclusion of $\wedge$I and major premise of an application of $\vee$E that discharges two assumptions that are major premises of two $\wedge$E; or
- it has the form $A \wedge B$, it is the conclusion of an application of $\wedge$I that has a premise derived using $\oplus$I, and it is the major premise of $\oplus$E.

[21] See section 1.1.3.

[22] See section 1.2.2.

Existence of normal form

Theorem 3.2.15 (Existence of normal form for **SASCNJ**). *If $A \vdash_{SASCNJ} B$, then there is a normal derivation of B from A in* ***SASCNJ***.

Proof. If $A \vdash_{SASCNJ} B$, then $\vdash_{SASCLJ} A \Rightarrow B$ (theorem 3.2.2). We have already seen that semi-Cut-free derivations are complete for **SASCLJ**, so we know that there is such a derivation `d` for $A \Rightarrow B$.

The translation from **SASCLJ** to **SASCNJ** in proof B.1.2 uses only non-derived major premises to translate Cut-free **SASCLJ**-derivations. So the only non-assumed major premises in the derivation `d`*, obtained by `d` by the translation, are due to $(Con \Rightarrow)^*$, $(Per \Rightarrow)^*$ and $(As \Rightarrow)^*$, that are the only Cut in the **SASCLJ**-derivation. We call the **SASCNJ**-sub-derivations obtained by translating these **SASCLJ**-sub-derivations $(Con \Rightarrow)^{**}$, $(Per \Rightarrow)^{**}$ and $(As \Rightarrow)^{**}$. It can be shown that the only non-normality that we can obtain by these translations is when the end-formula of the chain of $(Con \Rightarrow)^{**}$, $(Per \Rightarrow)^{**}$ and $(As \Rightarrow)^{**}$ is a major premise of $\wedge$E. We show this and that in this case we can normalize the derivation in proof B.2.2. At the end of the process, we have a derivation in normal form.

□

It is just a theorem of existence because, despite the fact that it provides instructions to obtain a normal **SASCNJ**-derivation starting from a semi-Cut-free **SASCLJ**-derivation, it does not provide any instructions to obtain this semi-Cut-free **SASCLJ**-derivation from a standard **SASCNJ**-derivation. We just know there has to be one because of the completeness of semi-Cut-free **SASCLJ**.

It is usually assumed that this kind of theorem is not enough for proof-theoretic semantics, and so that normalization is necessary. This requirement is justified only by an intuitionist scepticism regarding purely existential results. Since we will prove that also classical logic can be accepted in our framework, we are not forced to prove normalization theorem in the strict sense.

3.2.3 Validity

Definitions 1.2.8 of validity in an atomic basis $\mathscr{B}$ and 1.2.9 of validity *tout court* can be used also for our single-assumption and single-conclusion version of the calculi. What is in need of clarification is the definition 1.2.6 of the standard notion of canonical derivation. Indeed, while in multiple-assumption systems the normalization of a closed derivation always ends with the application of an I-rule, in a single-assumption system this is not the case, so normalization can not be seen as a reduction to the traditional canonical form. This observation follows from our characterization of the maximal *formulae* for **SASCNJ**, since we decided to consider as maximal only *some* combinations of I and E-rules. Indeed, our definition 3.2.4 of maximal formulae does not characterise formulae that are conclusions of $\wedge$I and premises of $\vee$E, $\supset$E or $\neg$E as maximal, if other contextual conditions do not occur. As a consequence, it seems that we can have closed derivations that end with an application of an E-rule and that are not normalizable.

The main effect of this *phenomenon* is that while in standard proof-theoretic semantics the fundamental assumption (assumption 1.2.1) is controversial in general but at least provable for purely logical closed derivations as a corollary to normalization,[23] in this generalization this is not the case. We will see that this is a big problem, especially for the classical system **SASCNK**. Nonetheless, at least for the intuitionistic system **SASCNJ** that we are considering here, we can find a solution. Indeed, while it does not follow as a corollary of normalization, we can nonetheless prove that for every intuitionistic theorem, there is a closed derivation of it in this system that ends with an application of an I-rule, that is we can prove that fundamental assumption holds at least in this case.[24] Let us look at this result:

Theorem 3.2.16 (Existence of canonical form for theorems of **SASCNJ**). *If $\vdash_{SASCNJ} A$, then there is a proof of A in* ***SASCNJ*** *that ends with an application of an I-rule.*

Proof. Let us consider a Cut-free proof of $\Rightarrow A$ in **LJ**. First of all, let us notice that this proof must end with an application of a rule on the right. Indeed, in order to have a void antecedent (without using Cut), we need to use $\Rightarrow \neg$ and/or $\Rightarrow\supset$ multiple times. This means that, tracing back the void antecedent from the endsequent, we find a number of applications of these rules such that every sequent depending on them has a void antecedent, and so such that every rule applied after them is a rule on the right.

[23]See note 52.
[24]That is all we can do also in standard proof-theoretic semantics regarding **NJ**.

$$\wedge\mathrm{I}\ \dfrac{C \qquad \begin{matrix}[C]\\ \vdots\\ A\end{matrix} \qquad \begin{matrix}[C]\\ \vdots\\ B\end{matrix}}{A \wedge B} \qquad \wedge\mathrm{E}\ \dfrac{A \wedge B \qquad \begin{matrix}[A]\\ \vdots\\ C\end{matrix}}{C} \qquad \wedge\mathrm{E}\ \dfrac{A \wedge B \qquad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}$$

$$\vee\mathrm{E}\ \dfrac{(A \vee B)\{\wedge D\} \qquad \begin{matrix}[A\{\wedge D\}]\\ \vdots\\ C\end{matrix} \qquad \begin{matrix}[B\{\wedge D\}]\\ \vdots\\ C\end{matrix}}{C} \qquad \vee\mathrm{I}\ \dfrac{A}{A \vee B} \qquad \vee\mathrm{I}\ \dfrac{B}{A \vee B}$$

$$\supset\mathrm{I}\ \dfrac{\{C\} \qquad \begin{matrix}[A\{\wedge C\}]\\ \vdots\\ B\{\vee D\}\end{matrix}}{(A \supset B)\{\vee D\}} \qquad \supset\mathrm{E}\ \dfrac{A \supset B\{\wedge(D \wedge E)\} \qquad \begin{matrix}[\{E\}]\\ \vdots\\ A\{\vee F\}\end{matrix} \qquad \begin{matrix}[B\{\wedge D\}]\\ \vdots\\ C\end{matrix}}{C\{\vee F\}}$$

$$\neg\mathrm{I}\ \dfrac{\{C\} \qquad \begin{matrix}[A\{\wedge C\}]\\ \vdots\\ \bot\{B\}\end{matrix}}{\neg A\{\vee B\}} \qquad Efq\ \dfrac{\bot}{C} \qquad \neg\mathrm{E}\ \dfrac{\neg A\{\wedge C\} \qquad \begin{matrix}[\{\ C\}]\\ \vdots\\ A\{\vee B\}\end{matrix}}{\bot\{\vee B\}}$$

Table 3.3: **SASCNK**

According to the translation from **LJ** to **SASCLJ** in B.1.2, the order of the rules in the two derivation is the same, apart from extra applications of $(Con \Rightarrow)^*$, $(Per \Rightarrow)^*$ and $(As \Rightarrow)^*$. So we can obtain a derivation of $\Rightarrow A$ in **SASCLJ** that ends with rules on the right followed at most by $(Con \Rightarrow)^*$, $(Per \Rightarrow)^*$ and $(As \Rightarrow)^*$.

According to the translation from **SASCLJ** to **SASCNJ** in B.1.2, the rules of introduction on the right of **SASCLJ** are translated using I-rules of **SASCNJ** and the order of the application of the rules is preserved if there are no applications of Cut. Moreover, the translation tells us also that $(Con \Rightarrow)^{**}$, $(Per \Rightarrow)^{**}$ and $(As \Rightarrow)^{**}$ do not deal with the conclusion of the derivation in which they are applied, but with the assumptions instead.

As a consequence, the translation that we obtain by composing the one from **LJ** to **SASCLJ** and the other from **SASCLJ** to **SASCNJ** takes Cut-free derivations of $\Rightarrow A$ in **LJ** in derivations that end with an I-rule in **SASCNJ**. So for every intuitionistic theorem, we have a derivation that ends with an introduction rule. □

This result poses us in the same situation in which we are in standard proof-theoretic semantics. We have no more and no fewer reasons to believe in the fundamental assumption, since it is provable for purely logical theorems but controversial in the other cases. Adopting this assumption, we can endorse the standard definition of validity and also the standard definition of canonical proof, although normalization is not enough to give canonicity in this framework.

3.3 Classical logic

The **SASC**-system for classical logic is presented in table 3.3. Its peculiarity is that it allows both the introduction of connectives in a subordinate position with respect to disjunction, and the elimination of connectives that occurs in a subordinate position with respect to conjunction.

The meaning of the curly brackets is the same seen for **SASCNJ**. Nonetheless, its application to the rules for negation is a little peculiar, so we should briefly address this point. For the $\neg$-rules, the occurrence of $\bot$ is an alternative to that of its disjunct. That is we take the abbreviation $\bot\{B\}$ to mean $\bot$ if there is no B, and B otherwise. So we have the intuitionist rules plus:

$$\neg\mathrm{I}\ \dfrac{\{C\} \qquad \begin{matrix}[A\{\wedge C\}]\\ \vdots\\ B\end{matrix}}{\neg A \vee B} \qquad \neg\mathrm{E}\ \dfrac{\neg A\{\wedge C\} \qquad \begin{matrix}[\{\ C\}]\\ \vdots\\ A \vee B\end{matrix}}{B}$$

One could think that ¬I has to work also as a canonical way of deriving some kind of disjunction. For example, consider the difference between this rule and ⊃I in which ∨ already occurs in a premise. Nonetheless, we will see that this strangeness does not pose any problem for separability, and the meaning of ∨ in general does not depends on ¬I.[25] So ¬I defines the meaning of ¬ and refers to ∨I for the definition of ∨. Since ∨ occurs in ¬I but is not canonically introduced, the meaning of ¬ depends on that of ∨ and not *vice-versa*.

We could instead consider ¬E as both an elimination rule for ¬, and an elimination rule for ∨. Indeed, differently from ⊃E, ∨ disappear in the conclusion, and there is no restriction of canonicity for being an elimination rule.[26] We will see that this is not a problem for harmony.

As for intuitionistic logic, the structural rules of **LK** corresponds to provable properties of conjunction and disjunction: idempotence of ∧ and ∨ to Contraction on the left and on the right respectively, and commutativity of ∧ and ∨ to Permutation on the left and on the right respectively. Associativity of ∧ and ∨ corresponds instead to the fact that formulae in antecedent and succedent are all at the same level.

Definition 3.3.1 (Disjunction of succedent)**.** Given a list of formulae Δ, $\Delta^{\vee}$ is the class of all the possible associations of disjunction of all the formulae $\delta \in \Delta$. Formally, $\Delta^{\vee} = \{(x) \vee (y) \mid x \in \Lambda^{\vee} \quad \& \quad y \in \Theta^{\vee} \quad \& \quad (\Lambda, \Theta)_{\Delta}\}$, if Δ has more than one element. $\Delta^{\vee} = \Delta$, otherwise. $\vdash \Gamma^{\wedge} \Rightarrow \Delta^{\vee}$ and $\Gamma^{\wedge} \vdash \Delta^{\vee}$ are used as abbreviations for $\vdash \{A\} \Rightarrow \{C\}$ and $\{A\} \vdash \{C\}$ for every combination of $A \in \Gamma^{\wedge}$ and $C \in \Delta^{\vee}$. In brief, the result has to be provable for every element of $\Gamma^{\wedge}$ and $\Delta^{\vee}$, regardless of the position of parenthesis.

Theorem 3.3.1 (Equivalence between **LK** and **SASCLK**)**.** *Sequent calculi **LK** and **SASCLK** are equivalent to each other, that is:*

1. *If $\vdash_{LK} \Gamma \Rightarrow \Delta$, then $\vdash_{SASCLK} \Gamma^{\wedge} \Rightarrow \Delta^{\vee}$;*
2. *If $\vdash_{SASCLK} D \Rightarrow C$, then $\vdash_{LK} \Gamma^{\circ} \Rightarrow C^{\circ}$.*

Proof. The proof is in B.1.3. □

Theorem 3.3.2 (Equivalence between **SASCLK** and **SASCNK**)**.** *The sequent calculus **SASCLK** and the natural deduction system **SASCNK** are equivalent to each other, that is:*

1. (a) *if $\vdash_{SASCLK} A \Rightarrow B$, then:*
    - *Or $A \vdash_{SASCNK} B$;*
    - *Or $\vdash_{SASCLK} \Rightarrow B$ and $\vdash_{SASCNK} B$.*

   (b) *If $\vdash_{SASCLK} A \Rightarrow$, then $A \vdash_{SASCNK} \bot$.*
2. (a) *If $A \vdash_{SASCNK} B$ then $\vdash_{SASCLK} A \Rightarrow B$.*

Proof. The proof is in B.1.3. □

Theorem 3.3.2 in combination with theorem 3.3.1 shows that **SASCNK** is adequate for classical logic.

3.3.1 Separability

By looking at the I-rules of **SASCNK** we obtain the following clauses:

1. To prove a consequence in which only ∧ occurs, we need to use only ∧-rules and ∧-formulae;
2. To prove a consequence in which only ∨ occurs, we need to use only ∨-rules and ∨-formulae;
3. To prove a consequence in which only ∧ and ∨ occur, we need to use only ∨-rules, ∧-rules, ∨-formulae and ∧-formulae;
4. To prove a consequence in which only ⊃ occurs, we need to use only ⊃-rules, ∧-rules, ∨-rules, ⊃-formulae, ∧-formulae and ∨-formulae (the same holds for consequences in which only ⊃ and ∧ and/or ∨ occur);
5. To prove a consequence in which only ¬ occurs, we need to use only ¬-rules, ∧-rules, ∨-rules, *Efq*, ¬-formulae, ∧-formulae, ∨-formulae and ⊥ (the same holds for consequences in which only ¬ and ∧ and/or ∨ occur);
6. In every other cases, we can use every constant to derive the consequence.

[25] Indeed the purely disjunctive fragment of classical logic is equivalent to the same fragment of intuitionistic logic.

[26] For the same reason we proposed to consider ∨E, ⊃E and ¬E of **SASCNJ** as E-rules also for ∧, even though this is almost irrelevant for harmony due to the particular structure of non-normalities in intuitionistic logic.

Conjunction and disjunction

Since the $\wedge\vee$-fragment of classical logic is equivalent to that of intuitionistic logic and we have shown the adequacy of our systems for those logics, the proof already used for **SASCNJ** still holds for **SASCNK**. So there is no need to restate clauses *1*, *2* and *3*.

Implication

Disjunction is necessary to derive purely implicational consequences. The standard example is Peirce law. To complete the proof of clause *4*, we have:

Theorem 3.3.3 (**SASCNK** conservatively extends $\mathbf{SASCNK}^{\wedge\vee\supset}$)**.** *If $A \vdash_{SASCNK} B$ and the only connectives that occur in $\{A\} \cup \{B\}$ is $\supset$, then $A \vdash_{SASCNK^{\wedge\vee\supset}} B$. The same holds if also $\wedge$ and/or $\vee$ occurs in $\{A\} \cup \{B\}$.*

Proof. If $A \vdash_{SASCNK} B$ and the only connectives that occur in $\{A\} \cup \{B\}$ is $\supset$, then we know by theorems 3.3.2 and 3.3.1 that $\vdash_{LK} A \Rightarrow B$.[27] Let us consider a Cut-free derivation of $A \Rightarrow B$ in **LK** in which only $\supset$-formulae occur.[28] The translation used in proof B.1.3 gives us a **SASCLK**-derivation in which the only Cuts are obtained by translations of $Cont \Rightarrow$, $Weak \Rightarrow$, $Perm \Rightarrow$, $\Rightarrow Cont$, $\Rightarrow Weak$, $\Rightarrow Perm$ and $\Rightarrow \wedge$ or from occurrences of $As \Rightarrow$ and $\Rightarrow As$. These Cuts do not erase any logical constants, apart from those obtained from Contractions that can erase $\wedge$ or $\vee$. So in the derivation there are only the logical constants in A and B plus $\wedge$ and $\vee$, and only the rules for these constants are applied. The translation used in proof B.1.3 translates $\Rightarrow \otimes$ with $\otimes$I, $\otimes \Rightarrow$ with $\otimes$E and *Cut* using $\wedge$ and $\vee$-rules, so we have a derivation of $A \vdash B$ in **SASCNK** in which only the rules for $\wedge$, $\vee$ and $\supset$ are applied, and in which no other connective occurs.

□

Negation

The same proof procedure can be used for clause *5*:

Theorem 3.3.4 (**SASCNK** conservatively extends $\mathbf{SASCNK}^{\wedge\vee\neg}$)**.** *If $A \vdash_{SASCNJ} B$ and the only connective that occurs in $\{A\} \cup \{B\}$ is $\neg$, then $A \vdash_{SASCNJ^{\wedge\vee\neg}} B$. The same holds if also $\wedge$ and/or $\vee$ occurs in $\{A\} \cup \{B\}$.*

Absurdity

Theorem 3.3.5 (**SASCNJ** conservatively extends $\mathbf{SASCNJ}^{\perp}$)**.** *If $A \vdash_{SASCNK} B$ and the only logical constant that occurs in $\{A\} \cup \{B\}$ is $\perp$ then $A \vdash_{SASCNK^{\perp}} B$.*

Proof. The same proof of theorem 3.2.12. □

Theorem 3.3.6. *If $A \vdash_{SASCNK} B$, the only connectives necessary to derive B from A are those in $\{A\} \cup \{B\}$ together with at most $\wedge$ and $\vee$.*

Proof. Let us consider the translation from **LK** to **SASCLK** in proof B.1.3. We can observe that if we translate only Cut-free derivations of **LK**, we obtain a derivation of **SASCLK** in which we use Cut only to translate applications of $Con \Rightarrow$, $Per \Rightarrow$, $\Rightarrow Con$, $\Rightarrow Per$, $\Rightarrow \wedge$ or to apply Associativity of conjunction or disjunction. Let us call these kind of Cut respectively: $(Con \Rightarrow)^*$, $(Per \Rightarrow)^*$, $(As \Rightarrow)^*$, $(\Rightarrow Con)^*$, $(\Rightarrow Per)^*$, $(\Rightarrow As)^*$ and $Distri^{\vee\wedge}$. Let us also call semi-Cut-free a derivation of **SASCLK** in which only these Cuts occur. Since Cut-free **LK** is equivalent to full **LK**, we conclude that semi-Cut-free **SASCLK** is equivalent to full **SASCLK**.[29]

By this result and theorem 3.3.2 (from **SASCNK** to **SASCLK**) we can assume that there is a semi-Cut-free derivation of $A \Rightarrow B$ in **SASCLK**. All the logical constants that occur in a semi-Cut-free derivation occurs in the end-sequent apart from $\wedge$ and $\vee$. This is the only constant that can vanish in the derivation, since $(Con \Rightarrow)^*$, $(Per \Rightarrow)^*$, $(As \Rightarrow)^*$, $(\Rightarrow Con)^*$, $(\Rightarrow Per)^*$, $(\Rightarrow As)^*$ and $Distri^{\vee\wedge}$ are the only Cuts in the derivation and they can erase only $\wedge$ and $\vee$.[30]

[27] Since $\perp$ does not occur neither in A nor in B, the conversion indicated by $^\circ$ is not used.

[28] Such a derivation exists, since Cut is admissible in **LK** even if Weakening and Axiom can be applied only with atoms as principal formulae.

[29] This result has the strength of a theorem of existence of Cut-free derivation, not of a theorem of Cut elimination.

[30] To be honest, also this eventuality is very rare. It only happens with $(Con \Rightarrow)^*$ for $\wedge$ and $(\Rightarrow Con)^*$ for $\vee$, and only when we contract a conjunction (disjunction) that does not have any conjunctions (disjunctions) between its subformulae.

Let us now consider the translation from **SASCLK** to **SASCNK** in proof B.1.3. In the **SASCNK**-derivation obtained from a Cut-free **SASCLK**-derivation we use all and only the rules and formulae used in the **SASCLK**-derivation, apart from *Efq* used to translate $\Rightarrow Weak$ and $\bot$ used to translate the void succedent. Nonetheless, these two exceptions are not relevant, since in order to apply $\Rightarrow Weak$ we need a void succedent, and in order to have it, we have to use $\Rightarrow \bot$ or $\neg \Rightarrow$. So if $\bot$ occurs in the **SASCNK**-derivation, it or $\neg$ occur in the open assumption or in the conclusion.[31] In our case the **SASCLK**-derivation is just semi-Cut-free, so we could have extra applications of $\wedge\vee$-rules and formulae. So in order to prove $A \vdash B$ in **SASCNK** we need to use only the rules for the connectives in $\{A\} \cup \{B\}$ together with $\wedge$ and $\vee$. This conclusion holds also for $\bot$, since there is no application of the translation in **LJ**, and the extra occurrences of *Efq* are unproblematic. ☐

To complete the proof of separability, we still have to consider two cases: $A \vdash_{SASCNK} B$ when the only connectives in $\{A\} \cup \{B\}$ are $\vee$ and $\bot$, and when the only connectives in $\{A\} \cup \{B\}$ are $\wedge$ and $\bot$. In the first case, since $\wedge$ occurs neither in $\vee$I, nor in $\bot$I (vacuously), it should not be necessary to derive $A \vdash B$. In the second case, since $\vee$ occurs neither in $\wedge$I, nor in $\bot$I (vacuously), it should not be necessary to derive $A \vdash B$. Let us prove it.

Theorem 3.3.7. *If $A \vdash_{SASCNK^{\vee\wedge\bot}} B$, and the only logical constants in $\{A\} \cup \{B\}$ are $\vee$ and $\bot$, then $A \vdash_{SASCNK^{\vee\bot}} B$.*

Proof. $\mathbf{SASCNK}^{\vee\wedge\bot}$ is formally equivalent to $\mathbf{SASCNJ}^{\vee\wedge\bot}$.[32] So, since we know by theorem 3.2.14 that $\mathbf{SASCNJ}^{\vee\wedge\bot}$ conservatively extends $\mathbf{SASCNJ}^{\vee\bot}$, we know that $\mathbf{SASCNK}^{\vee\wedge\bot}$ conservatively extends $\mathbf{SASCNK}^{\vee\bot}$. ☐

Theorem 3.3.8. *If $A \vdash_{SASCNK^{\vee\wedge\bot}} B$, and the only logical constants in $\{A\} \cup \{B\}$ are $\wedge$ and $\bot$, then $A \vdash_{SASCNK^{\wedge\bot}} B$.*

Proof. $\mathbf{SASCNK}^{\vee\wedge\bot}$ is formally equivalent to $\mathbf{SASCNJ}^{\vee\wedge\bot}$. We know by theorem 3.2.13 that $\mathbf{SASCNJ}^{\vee\wedge\bot}$ conservatively extends $\mathbf{SASCNJ}^{\wedge\bot}$, since the only logical constant that can be necessary to prove a consequence and does not occur in it is $\wedge$. So $\mathbf{SASCNK}^{\vee\wedge\bot}$ conservatively extends $\mathbf{SASCNK}^{\wedge\bot}$.[33] ☐

3.3.2 Harmony

Definition of normal form

Let us see the effect of allowing the introduction of connectives in a subordinate position for the definition of normal form. Steinberger[34] observes that in the derivation

$$\vee\mathrm{E}_2\,\dfrac{\supset\!\mathrm{I}_1\,\dfrac{\begin{matrix}[A\wedge C]^1\\ \vdots\\ C \qquad B\vee D\end{matrix}}{(A\supset B)\vee D} \qquad \begin{matrix}[A\supset B]^2\\ \vdots\\ E\end{matrix} \qquad \begin{matrix}[D]^2\\ \vdots\\ E\end{matrix}}{E}$$

the major premise $(A \supset B) \vee D$ of $\vee$E is derived by an I-rule, so its derivation is the direct ground for it. But if this is right, it should be possible to normalize this derivation, while there is no reason to believe that this is the case.

Let us argue that this reduction is not necessary to have normalizability, since it does not deal with a real violation of normality. $\supset$I does not define the meaning of $\vee$, since $\mathbf{SASCNK}^{\vee\supset}$ is a conservative extension of $\mathbf{SASCNK}^{\vee}$. So, we do not have a real maximal form in this case: Steinberger's counterexample is just obtained applying Prawitz's definition of maximal formula to Milne's system without questioning its suitability. In order to obtain a maximal formula, we have to use completely the meaning of $\supset$, that is we have to apply its E-rule. Since the meaning of $\supset$ depends on that of $\vee$ (and $\wedge$, of course) it is not absurd that we need an application of $\vee$E as a precondition for the applicability of $\supset$E. A reduction should remove the application of $\supset$-rules, possibly without touching $\vee$-rules. Indeed, following a suggestion of Milne himself, we consider maximal a conclusion of $\supset$I only of the following kind:[35]

[31] The proof is identical to that for **SASCLJDJ**.

[32] They are composed by the same rules.

[33] Let us notice that this proof procedure works only for $\wedge$, $\vee$ and $\bot$. Luckily we already have the other cases for theorem 3.3.6.

[34] This worry is extensively exposed in [Steinberger, 2008], and re-marked in [Steinberger, 2011b], p. 345. It is directly used against Milne's system, but it can be used also against my reformulation.

[35] [Milne, 2002], p. 518-519.

$$
\dfrac{\dfrac{C \qquad \begin{array}{c}[A \wedge C]^1\\ \vdots \\ B \vee D\end{array}}{(A \supset B) \vee D}\,{\supset}\mathrm{I}_1 \qquad \begin{array}{c}\dfrac{[A \supset B]^3 \qquad \begin{array}{c}\varnothing\\ \vdots\\ A \vee F\end{array} \qquad \begin{array}{c}[B]^2\\ \vdots\\ G\end{array}}{G \vee F}\,{\supset}\mathrm{E}_2\\ \vdots\\ E\end{array} \qquad \begin{array}{c}[D]^3\\ \vdots\\ E\end{array}}{E}\,{\vee}\mathrm{E}_3
$$

According to this position, the only derivations that we have to make normal are this and:

$$
\dfrac{\dfrac{C \qquad \begin{array}{c}[A \wedge C]^1\\ \vdots \\ B\end{array}}{\neg A \vee B}\,{\neg}\mathrm{I}_1 \qquad \begin{array}{c}\dfrac{[\neg A]^2 \qquad \begin{array}{c}\varnothing\\ \vdots\\ A \vee \{D\}\end{array}}{\bot\{D\}}\,{\neg}\mathrm{I}_1\\ \vdots\\ E\end{array} \qquad \begin{array}{c}[B]^2\\ \vdots\\ E\end{array}}{E}\,{\vee}\mathrm{E}_2
$$

In order to show why we want to be able to eliminate this kind of maximal formulae, let us consider the following generalization of *tonk*:

$$
\dfrac{A\{\vee C\}}{(A tonk B)\{\vee C\}}\,tonk\mathrm{I} \qquad \dfrac{(A tonk B)\{\wedge C\} \qquad \begin{array}{c}[B\{\wedge C\}]^1\\ \vdots\\ D\end{array}}{D}\,tonk\mathrm{E}_1
$$

This pair of rules is unacceptable because they generate a maximal formula that can not be reduced:

$$
\dfrac{\dfrac{A \vee B}{(A tonk B) \vee B}\,tonk\mathrm{I} \qquad \dfrac{[A tonk B]^2 \qquad [B]^1}{B}\,tonk\mathrm{E}_1 \qquad [B]^2}{B}\,{\vee}\mathrm{E}_2
$$

Indeed, since $A \vee B \nvdash_{SASCNK} B$, there is no possibility of finding a normal derivation with the same conclusion and assumption. In order for the reduction to be possible, from *tonk*I, $\vee$E and *tonk*E we should derive only consequences already derivable without *tonk*-rules.[36] So our request regarding maximal formulae is well-posed.

Instead, we should not ask for normalization of derivations in which an application of $\vee$I and an application of $\supset$I are used to derive the major premise of an application of $\vee$E. Indeed the meaning of $\vee$ does not depend on that of $\supset$, but the opposite does hold. So it is completely acceptable that in order to apply $\supset$-rules we need to introduce disjunction and later remove it.

We also have to consider the generalizations of the other non-normality already seen for **SASCNJDJ** and **SASCNJ**, obviously. In these cases we have:

$$
\dfrac{\dfrac{F \qquad \dfrac{[F]^3 \qquad \begin{array}{c}[A \wedge F]^1\\ \vdots\\ B\end{array}}{A \supset B}\,{\supset}\mathrm{I}_1 \qquad \dfrac{[F]^3 \qquad \begin{array}{c}[F]^2\\ \vdots\\ D\end{array} \qquad \begin{array}{c}[F]^2\\ \vdots\\ E\end{array}}{D \wedge E}\,{\wedge}\mathrm{I}_2}{(A \supset B) \wedge (D \wedge E)}\,{\wedge}\mathrm{I}_3 \qquad \begin{array}{c}[E]^4\\ \vdots\\ A \vee F\end{array} \qquad \begin{array}{c}[B \wedge D]^4\\ \vdots\\ C\end{array}}{C \vee F}\,{\supset}\mathrm{E}_4
$$

and

[36] In the same way as, regarding SASCNJ version of *tonk*, from *tonk*I, $\wedge$I and *tonk*E we should derive only consequences already derivable without *tonk*-rules. See section 3.2.2.

$$\neg\mathrm{E}_3\,\dfrac{\wedge\mathrm{I}_2\,\dfrac{D \qquad \neg\mathrm{I}_1\,\dfrac{[D]^2 \qquad \begin{matrix}[A \wedge D]^1\\ \vdots\\ \bot\end{matrix}}{\neg A} \qquad \begin{matrix}[D]^2\\ \vdots\\ C\end{matrix}}{\neg A \wedge C} \qquad \begin{matrix}[C]^3\\ \vdots\\ A \vee B\end{matrix}}{\bot}$$

Let us now consider the problem if the usage of $\vee$ made in the minor premises of $\supset$E and $\neg$E is justified. This is not a real problem at all, since we already proved that $\supset$-rules and $\neg$-rules do not modify the meaning of $\vee$ already given by its proper rules. Indeed we already applied this reasoning to intuitionistic logic, in order to establish that $\supset$E does not count also as an elimination rule for conjunction. This is the reason why we discuss separability before harmony. So there are no maximal formulae of this kind.

To sum up, we have the following definition of maximal formulae:

Definition 3.3.2 (Maximal formulae (**SASCNK**))**.** Given a derivation $\mathfrak{D}$, a formula that occurs in it is a maximal formula iff:

- it is conclusion of $\oplus$I and major premise of $\oplus$E, for a logical constant $\oplus$; or
- it has the form $A \vee B$, it is the conclusion of $\oplus$I for a logical constant $\oplus \neq \vee$ and $\oplus \neq \wedge$ and major premise of an application of $\vee$E that discharges an assumption that is major premise of $\oplus$E; or
- it has the form $(A \vee B) \wedge D$, it is the conclusion of $\wedge$I and major premise of an application of $\vee$E that discharges two assumptions that are major premises of two $\wedge$E; or
- it has the form $A \wedge B$, it is the conclusion of an application of $\wedge$I that has a premise derived using $\oplus$I, and it is the major premise of $\oplus$E.

Existence of normal form

Theorem 3.3.9 (Existence of normal form for **SASCNK**)**.** *If $A \vdash_{SASCNK} B$, then there is a normal derivation of B from A in **SASCNK**.*

Proof. If $A \vdash_{SASCNK} B$, then $\vdash_{SASCLK} A \Rightarrow B$ (theorem 3.3.2). We have already seen that semi-Cut-free derivations are complete for **SASCLK**, so we know that there is such a derivation `d` for $A \Rightarrow B$.

The translation from **SASCLK** to **SASCNK** in proof B.1.3 uses only non-derived major premises to translate Cut-free **SASCLK**-derivations. So the only non-assumed major premises in the derivation `d`*, obtained by `d` by the translation, are due to $(Con \Rightarrow)^*$, $(Per \Rightarrow)^*$, $(As \Rightarrow)^*$, $(\Rightarrow Con)^*$, $(\Rightarrow Per)^*$, $(\Rightarrow As)^*$ and $Distri^{\vee\wedge}$ that are the only Cut in the **SASCLK**-derivation. We call the **SASCNK**-sub-derivations obtained by translating these **SASCLK**-sub-derivations $(Con \Rightarrow)^{**}$, $(Per \Rightarrow)^{**}$, $(As \Rightarrow)^{**}$, $(\Rightarrow Con)^{**}$, $(\Rightarrow Per)^{**}$, $(\Rightarrow As)^{**}$ and $(Distri^{\vee\wedge})^{**}$. It can be shown that the only non-normality that we can obtain by these translations is when the end-formula of the chain of $(Con \Rightarrow)^{**}$, $(Per \Rightarrow)^{**}$ and $(As \Rightarrow)^{**}$ is a major premise of $\wedge$E or when the first-formula of the chain of $(\Rightarrow Con)^{**}$, $(\Rightarrow Per)^{**}$ and $(\Rightarrow As)^{**}$ is a conclusion of $\vee$I. So in particular, $(Distri^{\vee\wedge})^{**}$ does not cause any non-normality. We show this and that, in this case, we can normalize the derivation in proof B.2.3. At the end of the process, we have a derivation in normal form.

□

3.3.3 Validity

For the validity of **SASCNK** we have the same problems that we saw about **SASCNJ**: there are normal closed derivations that are purely logical but that do not end with an application of I-rules. Indeed let us consider the following derivation:

$$\vee\mathrm{E}_2\,\dfrac{\supset\mathrm{I}_1\,\dfrac{\vee\mathrm{I}\,\dfrac{[A]^1}{\bot \vee A}}{(A \supset \bot) \vee A} \qquad \vee\mathrm{I}\,\dfrac{[A \supset \bot]^2}{A \vee (A \supset \bot)} \qquad \vee\mathrm{I}\,\dfrac{[A]^2}{A \vee (A \supset \bot)}}{A \vee (A \supset \bot)}$$

We need the last application of $\vee$E in order to permute the disjunction, since $\supset$I can introduce the conditional only on the left disjunct.

Of course, we could think to extend the system with extra rules to deal with this problem (as an example we could assume another I-rule for the introduction of $\supset$ on the right disjunct). Nonetheless, this is a very easy case in which we need a concluding application of E-rules, there could be other more complicated. Moreover, the main problem remains: we have pairs of I and E-rules that are not maximal, so we can not expect that the normalization of closed derivation gives us always derivations that end with an I-rule.

We saw in section 3.2.3 that we can circumvent this problem for **SASCNJ**, at least in order to prove the existence of canonical proofs for intuitionistic theorems (that is the fundamental assumption for intuitionistic theorems). Unfortunately, we can not have the same result for **SASCNK**. Indeed in proving theorem 3.2.16, we used the fact that the composed translation from **LJ** to **SASCNJ** requires only well-behaved rules that do not add extra E-rules on the end of the derivation. Nonetheless, a similar translation from **LK** to **SASCNK** needs applications of $(\Rightarrow Con)^{**}$, $(\Rightarrow Per)^{**}$, $(\Rightarrow As)^{**}$ and $(Distri^{\vee\wedge})^{**}$ *inter alia*, and these cause extra applications or E-rules in the end. So we can not extend this result to the classical system, and we have to consider some revisions of the notions of canonicity and/or of the role of fundamental assumption in the definition of validity.

Since both the notion of canonicity and the fundamental assumption are at the bottom of the notion of validity developed in proof-theoretic semantics, the only way of saving this framework is to extend the notion of canonicity in the following way:

Definition 3.3.3 (Canonical derivation (**SASCNK**))**.** A canonical derivation for a non-atomic sentence is a normal closed derivation that has only valid immediate subderivations.

With this redefinition, we can justify the fundamental assumption at least for logical derivations. The only other option is a complete reorganization of the plan of proof-theoretic semantics.

We have good reasons both to believe and to disbelieve that this redefinition of concepts is acceptable. Indeed on one side we argued that an eventual normalization when we have a pair of I and E-rules that does not constitute a maximal formula is not only impossible, but also pointless from the point of view of our theory of meaning, so it seems reasonable to conclude that those normal closed derivations are canonical as well.[37] On the other side this could be problematic for our direct characterization of validity because, while derivations that end with an application of an I-rule and that have only valid immediate sub-derivations are valid by definition, this is not the case for derivations that end with an application of an E-rule. Indeed E-rules are not meaning-conferring rules, and so they need a justification. This problem displays a situation in which harmony and validity seem to point to opposite solutions, although probably we could solve the disagreement by letting to harmony and normalizability an even stronger role in the definition of validity. In this work, we will accept provisionally the redefinition of canonical derivation, in such a way to warrant validity for all well-formed derivations of **SASCNK**. Nonetheless, this point has to be recognised as a loose end.[38]

[37]In this way the availability of canonical proofs for intuitionistic theorems according to the old notion of canonicity is just a frivolous extra property, devoid of real meaning-theoretic significance.

[38]I want to specify that, given our reinterpretation of Dummett's argument against multiple-conclusion logic discussed in chapter 2, a single-assumption and single-conclusion approach is mandatory for proof-theoretic semantics. So a negative answer to this proposal of redefinition of canonicity would mark a great problem for proof-theoretic semantics in general.

Chapter 4

Pluralism

4.1 Which relation between the logics?

4.1.1 Deviance or extension?

As a matter of fact, we developed different logical systems and we proved that all of them are valid according to our criterion. This situation forces us to deal with two topics that are nowadays very developed: logical disagreement and logical pluralism. We will try to find a position that is both coherent with the inferentialist framework that we developed in the rest of the thesis and acceptable from a general philosophical perspective.

As a starting point, I want to specify which kind of logical disagreement we are dealing with. Following a classification proposed by Susan Haack, we distinguish between deviation and extension:[1]

Extension: A logic is an extension of another logic iff it is a conservative extension (definition 1.2.10) of it.

Deviation: A logic $\mathcal{S}$ is a deviance of another logic $\mathcal{S}'$ iff they are expressed in the same language $\mathcal{L}$ and prove different sets of consequences.

The category of 'deviation' is quite various: it collects both cases in which one of the two logics is a sub-logic of the other one (like classical logic and intuitionist logic, at least accepting the homophonic translation), and cases in which this is not the case (like relevance logic and intuitionist logic, at least accepting the homophonic translation). We also have another possibility, that is:

Quasi-deviation: A logic is a quasi-deviation of another logic iff they are not expressed in the same language, but none of them is a conservative extension of the other one.

For example, classical logic is a quasi-deviance in relation to the negation-free fragment of intuitionistic logic. Indeed we have a syntactical extension, since we add $\neg$ to the vocabulary, but we do not have a conservative extension, since $((A \supset B) \supset A) \supset A$ is a classical but not intuitionist theorem that can be expressed in the old, negation-free vocabulary. Susan Haack points out that every time in which we have a quasi-deviation, we can obtain a deviation between the logic formulated using the narrower vocabulary and a subsystem of the other logic, obtained by removing the additional logical terms. Indeed, in the example just seen, we have a deviation between classical and intuitionistic negation-free fragments. Haack's observation is unobjectionable from a formal point of view, but it is not clear at all that we could find a good formulation of the required subsystem.[2] In general, it could even be an open question the existence of an axiomatization for this subsystem, and in the inferentialist perspective that we outlined in the first chapter we posed quite strong restrictions to be considered apart from pure axiomatizability.[3] For this reason, we will see that in our inferentialist approach quasi-deviation represent a core problem, tied to the desideratum of separability and analyticity.

Apart from the problematic point of quasi-deviation, the main difference between deviance and extension is based on conservative extension. We applied the notion of conservative extension already at the beginning of this thesis. Nonetheless, we saw its application only inside a language, when we

[1] [Haack, 1974], p. 4.

[2] Indeed we already saw the problem of formulating the classical negation-free fragment in a single conclusion system in section 1.1.3 and later in more details in section 2.2.

[3] A case in which the axiomatizability of a fragment of logic was disputed is relevance logic. For many years it was dubious whether it were possible to axiomatize completely the implicational fragment of **R**, even though it is nonetheless true that exists a set of purely implicational theorems of this logic. [Dunn and Restall, 2002], p. 7.

were dealing with the problem of separability of the logical constants. Here instead the problem is the relation between terms of different languages, so here we have a much more complex philosophical problem because we have to deal with translations.[4] That is, taken for granted Haack's distinction, its applicability is not so clear in cases where the homophonic translation is not indisputable.[5]

Let us briefly consider two examples of logics that apparently deviate from classical logic, but that could be considered extensions of it if we decide to abandon homophonic translation: intuitionistic logic and relevance logic.

We could see intuitionistic logic as deviant with respect to classical logic, since they are usually formulated using the same vocabulary and validate different sets of logical consequence. Nonetheless, classical logic can be translated into intuitionistic logic in such a way that a formula is a classical theorem iff its translation is an intuitionistic theorem. Indeed, let us call this translation $*$:

- $A^* = \neg_i \neg_i A$ for atomic A;
- $(A \wedge_c B)^* = (A^* \wedge_i B^*)$;
- $(A \vee_c B)^* = \neg_i(\neg_i A^* \wedge_i \neg_i B^*)$;
- $(\neg_c A)^* = \neg_i A^*$;
- $(A \supset_c B)^* = (A^* \supset_i B^*)$.

It can be proved that $\Gamma \vdash_c C$ iff $\Gamma^* \vdash_i C^*$.[6]

There is also a well-known translation between intuitionistic logic and modal logic **S4**, so we could argue that the real face of intuitionistic logic is a modal extension of classical logic. The translation * is defined as:

- $p^\star = \Box A$ for atomic A;
- $(A \wedge_i B)^\star = \Box(A^\star \wedge_c B^\star)$;
- $(A \vee_i B)^\star = \Box(A^\star \vee_c B^\star)$;
- $(\neg_i A)^\star = \Box \neg_c A^\star$;
- $(A \supset_i B)^\star = \Box(A^\star \supset_c B^\star)$.

In this way, A is an intuitionistic theorem iff $A^\star$ is a theorem of **S4**.[7] This translation could seem to be related with a very common and intuitive reading of intuitionistic connectives; for example the reading of $\neg_i p$ as "it is impossible that p". This interpretation is not available for $*$, since it translates negation homophonically, while for * it is arguably acceptable. Susan Haack rejects this explanation because she argues that * does not translate negation in the expected way.[8] Nonetheless, her analysis is deeply related to the traditional conception of intuitionistic logic originated from the works of Brouwer and Heyting. Maybe another non-classical reading of intuitionistic logical constants could suit well to the translation *.

In the same way, relevant logic rejects classical material conditional, proposing a deviance at least regarding this constant. Nonetheless, classical logic and **FDE** (that is a subsystem common both to **R** and **E**, the major relevant systems, in which conditional does not occur) share the same theorems in their $\wedge \vee \neg$-fragments. So, since classical conditional can be defined using disjunction and negation, there is a clear sense in which relevance logic is an extension of classical logic instead of a deviation.[9]

It seems clear that it is impossible to find a good answer to these doubts between homophonic deviances or non-homophonic extensions without a precise theory of meaning. So already the application of Haack's categorisation is a controversial departure point. Anyway, we can apply it at least in a negative way: we will not consider logics that are explicitly and unquestionably extensions one of the other. We can have doubts about the existence of genuine rivalry between logics, but it is unquestionable that, for example, modal logic is a (non-deviant) extension of classical logic. Here we will consider only logics that are at least candidate for deviance, although without taking for granted their philosophical status.

[4] A hardcore Wittgensteinian could object that there is the same problem also inside one language. This at least is the interpretation given by Dummett of the philosophical works of Wittgenstein about foundations of mathematics.

[5] And I will argue that there is no such thing as an indisputable homophonic translation.

[6] This result was discovered independently by Gentzen ([Gentzen, 1969c]) and Gödel ([Gödel, 1986b]), and is grounded on Glivenko's theorem. For a recent exposition: [Mints, 2000], pp. 23-24.

[7] Also this result has been pointed out for the first time by Gödel (in his [Gödel, 1986a]), and has since then become common knowledge in the literature about modal logic: [Hughes and Cresswell, 1996], p. 225.

[8] [Haack, 1974], p. 97.

[9] [Dunn and Restall, 2002], pp. 30-31.

4.1.2 Disagreement and pluralism

We selected apparent deviance as our topic, we now have to deal with its philosophical nature. Fist of all, let us define precisely what are the problems that we want to solve. I think that we can summarize them in two questions:

1. Is there disagreement between two different (and non-equivalent) logical systems? That is:
   (a) Can there be disagreement between two different (and non-equivalent) logical systems?
   (b) Does there has to be disagreement between two different (and non-equivalent) logical systems?
2. Can there be more than one (correct) logic?

Using Haack's expressions, essentially the first question is about the existence of real deviance between logics, while the second asks about consequences for the monism-pluralism debate about logic. An answer to the first question is a major ingredient in an answer to the second, of course, so we will try to follow this order. The answers to these questions determine whether for example **SASCNK** has to be considered as a revision of **SASCNJ** or as an alternative logical system that can coexist with the second one. It is an intuitive belief that if two logics disagree, then at most one of them can be right, while if there is no real disagreement they can be both true "at the same time". In the first case, **SASCNK** can only be seen as a revision of **SASCNJ**, while in the second case they are in some way independent.

It is interesting to notice that a negative answer to the first question does not directly speak for a case of extension, even though it speaks against the existence of real deviance. Indeed, Haack's distinction between deviance and extension is complete (apart from the already mentioned intermediate case of quasi-deviance) only if we accept some kind of translation. Nonetheless, this seems to be an unquestioned starting point of Haack's analysis.

Given a logic, an apparently deviant logic can speak about

- The same logical terms; or
- some other logical terms.[10]

It seems that according to Susan Haack in the second case there has to be a suitable translation between one of the two logic into (an extension of) the other one. Of course, this is always achievable from a formal point of view, but it is not clear whether this formal result can be paired with a philosophical result about the intended meaning of the logics. Indeed the interpretation of an apparently deviant logic as an extension does not seem to suit well to some reading of deviance as "change of topic" that we will encounter. The search for an interpretation of a logic into another one is only one of the arguments against real deviance. So if in some cases we will find reasons to reject real deviance without the possibility of reinterpreting one of the logics as extending the other, we will reject the completeness of Haack's categorization: we can have two logics that are neither in real disagreement, nor one the extension of the other.

The division of the first question in two different sub-questions allows us to give different answers about different pairs of logics, at least if we find good reasons to answer positively to the first and negatively to the second sub-question. In the midst of all these problems, we can at least relate to some principles that I will not question.

We will accept the principle:

Observation 4.1.1 (Disagreement and subject). *There is disagreement between two logics iff they speak about the same logical terms and the do not prove the same logical consequences.*

So the main problem with disagreement will become a problem of identity for logical terms.

A second principle that we consider acceptable is:

Observation 4.1.2 (Disagreement and right logic). *If there is real disagreement between two logics, then at most one of them can be right.*

This principle has been criticized by Beall and Restall in a series of papers. These authors argue for a kind of logical pluralism in which two rival logics can be both true. We will see that their proposal is unacceptable in our inferentialist perspective (and maybe also non-conclusive in their realist perspective).

[10]The reference of Susan Haack to classical logic is inessential in this case: we are interested in the possibility of deviance in general.

Since the discrimination between real disagreement and purely apparent rivalry is given by the identity of logical terms, and since it seems obvious to propose an identity criterion based on meaning, the development of a theory of meaning for logical terms seems to be a *conditio sine qua non* for answering our worries.[11] Anyway, first of all we will try to address the first question in general, without focusing on any theory of meaning in general. We will try to answer this problem about the identity of logical terms *quidquid ea sunt*. We will nonetheless see that this approach reaches its limits very soon, and change approach, grounding our research on a theory of meaning.

4.2 *Quidquid ea sunt* approach

4.2.1 Quine and the Principle of Charity

The problem of disagreement between different logical systems, that is the problem of the possibility of real deviance in logic dates back at least to the debate between Carnap and Quine.

Carnap has been the first to propose a kind of logical pluralism with his Principle of Tolerance.[12] Nonetheless, he never focused on how to translate a language into another one. Indeed most of Quine's criticisms on his philosophy of logic are based on this problem. The reason for this lack is that in his opinion every logic corresponds to a linguistic framework, and linguistic frameworks are essentially theoretically incomparable, even though they can be compared pragmatically.[13] There can not be real disagreement between logics like there is no real disagreement between Euclidean and non-Euclidean geometries. They are in contradiction only if they are applied to the same pragmatical purpose – for example, only one geometry will describe correctly the physical universe – and the existence of translations between them tells nothing about the meaning of their terms. So, in the same way in which the only validity criterion for a geometrical sentence is relative to a geometry, the only validity criterion for an argument is relative to a logical system.[14] That is Carnap rejects the notion of external validity, the only kind of validity is internal to a system: the question if a rule of inference is valid in general (and not in a precise system) makes no sense.[15] Since this attitude regarding validity and his focus on pragmatical criteria for selecting the 'right' logic, it is not surprising that he did not consider central the issue of real disagreement between logics. We can so conclude that Carnap answers negatively to our question 1: since there is nothing like external validity, there is nothing like two different logical systems speaking about the same logical terms. In some sense, the question whether two logics speak about the same logical terms seems to be meaningless according to Carnap, and we need a strong, positive answer in order to have a disagreement. The conclusion about pluralism is that there is nothing that speaks against it, even though without a clear notion of validity it is not clear whether we obtain what we were looking for.[16]

Quine gives a somewhat opposite solution both to the problem of identity of logical terms and to the problem of logical pluralism. His main reasons for challenging Carnap's position are grounded in observations about ideal conditions of translation.[17]

In his [Quine, 1976a] (and then in more detail in [Quine, 1986]) the author argues that when we have to translate a logical system into another one, we can choose essentially between two principles:

Principle of homophony: when in two languages $\mathcal{L}$ and $\mathcal{L}'$ we have two logical terms $\oplus$ and $\oplus'$ that sound the same, we should translate one with the other;

Principle of charity: when we translate a sentence from one language $\mathcal{L}$ into another language $\mathcal{L}'$, we should assume that this sentence is both true and rational.[18]

[11] Of course, although it seems obvious that a good identity criterion for logical terms should be grounded on a theory of meaning, neither it is obvious which kind of criterion is adequate, nor which kind of theory of meaning should be applied. We could also doubt about the existence of a good criterion of identity or a good theory of meaning.

[12] [Carnap, 1937], first part, section 17.

[13] [Steinberger, 2016], for a defence of the coherence of this position.

[14] [Coffa, 1991], p. 309 for the analogy with geometry. [Restall, 2002] gives a clear picture of the kind of pluralism that we can achieve following Carnap's approach.

[15] For the distinction between internal and external validity, [Haack, 1978], pp. 14-15; for Carnap's rejection of an external notion of validity, just consider the Foreword of [Carnap, 1937] in which he disapproves the "striving after 'correctness"' of traditional logicians.

[16] Technically speaking, Carnap has a conventional theory of meaning, so its inclusion in this chapter is in some way objectionable; nonetheless, his theory is so permissive that does not pose any restriction for formal systems and does not give any identity criterion for logical terms.

[17] Indeed his famous rejection of analytic sentences in [Quine, 1951] does not apply to propositional logic (even though he consider this topic when he presents his famous picture of the language as a web), and also his rejection of truth by convention in [Quine, 1976b] is essentially devoted to refuting Carnap's version of logicism.

[18] Of course, this is not a full characterization of this principle, but it is nonetheless enough detailed for our investigation about translation between logical languages.

Of course, both principles are fallible in general that is there are cases in which they lead to a bad translation. A counterexample to the first is, for example, the world "gift", which means present in English but poison in German. And about the second, just imagine what would happen if we apply it when we are translating a sentence of an old Babylonian book about the origin of the universe. However both principles are very useful when we are trying a radical translation, that is a translation of a new language we know nothing about. For this reason, these principles are very useful for investigating the purely rational limits of translation, and in particular translations between logics.

According to Quine, even though the principle of homophony is commonly applied by philosophers who deal with deviance in logic, we should follow the second principle. In this way, the existence of deviant logics and deviant logicians would be just the outcome of a bad habit of translation, since denying a logical law they fail to speak of the standard logical terms. His reason for this conclusion is that logic is something so fundamental that every time we encounter someone that seems to reject a logical law, we should search for another explanation, like a problem of translation.

It is interesting to notice that this consequence of the principle of charity does not need a theory of meaning in order to work. So we still have a *quidquid ea sunt* approach to connectives (at least to non-classical ones, as we will see), but nonetheless we can exclude the acceptability of homophonic translation.

This reading of the phenomenon of deviance in logic is very controversial and seems at odds also with other papers of the same philosopher. Indeed in his [Quine, 1951], he famously sustained that logic is (at least in principle) revisable, and it is not so clear how this could be possible if logical disagreement is impossible. This mismatch between the two views is particularly apparent in [Quine, 1986], where Quine argues that there can not be something like a disagreement about logic, and then evaluates non-classical logical systems as if they were real rivals to classical logic.

I think that a good solution to this apparently irresolvable contradiction can be found in Quine's attempt to reconstruct truth tables for logical terms via radical interpretation.[19] His idea is that since classical logical terms are truth-functional, it should be possible to identify logical terms in an unknown language if we manage to find a word for assertion and one for rejection. In this way, it should be possible to identify a truth table for a logical term and the identification of a logical term via radical translation would be identical to the recognition of the validity of the classical logical laws for that term. For this reason, it would be impossible to reject a logical law without changing the subject.[20]

Despite this theory of classical logical terms, I nonetheless believe that Quine should be included in this chapter, since he does not give any theory of meaning for non-classical logical terms. His position is probably the most idiosyncratic that we will encounter, since it considers a theory of meaning but for a single logic: classical logic. For all the other systems the *quidquid ea sunt* approach is applied.[21]

Even though we will defend the "change of logic, change of subject" position, I believe that many aspects of Quine's approach are disputable. Indeed, also accepting his plan of reconstructing truth-tables via radical translations, the gap between these and classical logic is bigger than it would seem, since we still need a justification of bivalence. Without bivalence truth-tables can suit well intuitionistic logic, as observed by McDowell[22] and Read[23], so we are still far from a good reason to select classical logic as the one true logic. Also, his position regarding the possibility of reconstructing truth-tables could entail monism regarding classical logic only if we accept his general pessimistic position about meaning and translation, and our general inferentialist approach is already in contrast with it.

4.2.2 Reasons for identity

Even though the idea that apparently deviant logics speak about different connectives is the more acceptable aspect of Quine's picture, there are nonetheless reasons to believe that different logical systems can speak about the same logical terms and that, for this reason, they can be genuinely deviant. There is an argument for this conclusion, that starts with an old paper of Williamson and arrives at recent articles written by Beall and Restall.[24]

The argument is based on the observation that some connectives, like classical and intuitionistic negation, can not occur in the same language and be distinct. As far as I know, the first observation

[19] The clearest exposition of this idea is in [Quine, 1992], p. 44-47.

[20] Of course, this position has to be considered as a change of mind with respect to Quine's earlier position about the changeability of logic.

[21] We will see that a realist approach to logical disagreement differs from that of Quine because it attributes meaning also to deviant systems, regardless of their validity!

[22] [McDowell, 1976]

[23] [Read, 1988], p. 23.

[24] Nonetheless, they apply this argument to reach essentially different conclusions. It seems that their position regarding deviance is their only *trait d'union*.

about this property of the two negations is in [Popper, 1948], where Popper uses it to exclude the interpretation of "$\neg_i \ldots$" as "it is impossible that…". Popper's reasoning is essentially this:

- "it is impossible that…" and "it is not the case that…" can coexist in natural language;
- the meaning of classical negation is "it is not the case that…";
- if the meaning of intuitionistic negation were "it is impossible that…", then we could have a language (natural language, indeed) in which both it and classical negation occur as distinct operators;
- but this is not possible, since if $\neg_i$ and $\neg_c$ occur in the same language, it is always possible to derive $\neg_i\phi \dashv\vdash \neg_c\phi$ for every sentence ϕ;
- so the meaning of intuitionistic negation is not "it is impossible that…".

The penultimate point is a purely formal one, and it is nowadays widely known. It can be easily shown by the following natural deduction derivations

$$\neg_c\text{I}\,\dfrac{Efq_i\,\dfrac{\neg_i\text{E}\,\dfrac{\neg_i A \qquad [A]}{\bot_i}}{\bot_c}}{\neg_c A} \qquad\qquad \neg_i\text{I}\,\dfrac{Efq_c\,\dfrac{\neg_c\text{E}\,\dfrac{\neg_c A \qquad [A]}{\bot_c}}{\bot_i}}{\neg_i A}$$

and it trivially follows by uniqueness of intuitionistic negation.[25]

We have to distinguish between $\bot_i$ and $\bot_c$ because otherwise the identification between the two negations would be caused by the unjustified identification between the two absurdities. Indeed we know that minimal negation is not unique, so the two derivations could not work for two logical constants, $\bot_{m_1}$ and $\bot_{m_2}$, characterised by the rules for minimal negation. The reason for this is essentially that *ex falso quodlibet* does not work for it.[26]

This observation is used to show the uniqueness of logical constants for the first time in [Williamson, 1988], while Restall uses it explicitly only in [Restall, 2014],[27] but it is in the background of its and Beall's arguments against an identification between his pluralism and the kind of Carnappian pluralism we already saw.[28]

Williamson considers the possibility that the disagreement between intuitionists and classicists is purely verbal, that is that the classical logician is speaking about something (classical negation) and the intuitionist logician is speaking about something completely different (intuitionistic negation). If this were the case, it should be possible to have both terms in the same language, but as we just saw, Popper told us that this is not possible.[29] So the disagreement can not be purely verbal, at least for classical and intuitionistic negations.[30]

Restall uses the same argument in one of his papers in order to defend his conception of logical pluralism against Carnap's one. The main difference between these two kinds of pluralism is that according to Restall classical logicians and deviant logicians speak about the same logical terms, so there is a real deviance and a real disagreement; while as we already saw, according to Carnap classical logicians and deviant logicians speak different languages and as a consequence speak about different logical terms, so there is no real disagreement (nor deviance, we could argue), at least as long as we do not need to select a logic for an application. Since we can not have a language in which there are both classical and intuitionistic negations, we should conclude that there is just one negation, differently characterised by the different logics.

There are just two differences between Restall's argument and Williamson's one: Williamson speaks about the impossibility to have classical and intuitionistic negations in the same language, while Restall speaks about the impossibility to have them in the same (model-theoretic) frame; Williamson speaks

25 [Belnap, 1962]

26 [Milne, 1994], p. 66. Although we did not rely on it, it is nonetheless interesting to note that the version of Inversion Principle formulated by Negri and von Plato ([Negri and von Plato, 2001]) and frequently identified with uniqueness holds for minimal negation (this identification is for example supported in [Milne, 2015], p. 196). Indeed, even though the rules for minimal negation are too weak to give uniqueness, they are enough strong to permit a derivation of direct grounds for negation from a negated formula: $\neg_m\text{E}\,\dfrac{\neg_m A \qquad A}{\bot_m}$ So we can conclude in passing that we have to reject the identification between Negri-vonPlato Inversion Principle and uniqueness.

27 In the sections "three negations in one logic?" and "or one negation in three logics?"

28 His version of Williamson's argument is essentially developed in [Restall, 2002], but it also occurs in other papers; we will give precise references in the next section, in which we will deal with a realist approach to logical disagreement.

29 Interestingly, Williamson ascribes this observation to [Belnap, 1962], and not to [Popper, 1948].

30 Williamson then applies this result to other questions regarding predicate ascription and existence.

only about classical and intuitionistic logic, while Restall speaks also about dual-intuitionistic logic.[31] The first distinction is inessential, while the second is very interesting in my opinion.

I think that Restall's version of this argument gives good grounds to doubt about its strength. Indeed it contains the seeds of its rejection, since it argues for both identity of intuitionistic and classical negation, and identity of dual-intuitionistic and classical negation. Now we have two options: we can consider intuitionistic negation identical with dual-intuitionistic negation or we can consider them different. Both options lead to unacceptable conclusions. Indeed:

- If they are different, then we lose transitivity of identity, since $\neg_c = \neg_i$ and $\neg_c = \neg_{di}$ but $\neg_i \neq \neg_{di}$;
- If they are the same, then it should be impossible to have both of them in the same language. But we have a logical system that contains both $\neg_i$ and $\neg_{di}$, that is BI-intuitionistic logic, so according to the same criterion proposed by Williamson and Restall they should be distinct connectives.[32]

  Also, we can prove that if $\neg_i$ and $\neg_{di}$ were the same negation, we could not have BI-intuitionistic logic. Indeed if $\Gamma \vdash_i \Delta$ or $\Gamma \vdash_{di} \Delta$, then $\Gamma \vdash_{bi} \Delta$, so, since we can prove that $\Gamma \vdash_c \Delta$ iff there is a classical formula A such that $\Gamma \vdash_i A[\neg_i/\neg_c]$ and $A[\neg_{di}/\neg_c] \vdash_{di} \Delta$, if $\neg_i$ and $\neg_{di}$ were identical we should have $\Gamma \vdash_{bi} \Delta$ by transitivity, and so classical logic and BI-intuitionistic logic should be identical.[33] In conclusion the identification between $\neg_i$ and $\neg_{di}$ not only is not dictated by BI-intuitionistic logic, the same existence of this logic is in danger if we assume this identity.

What is worse is that Restall's extension of Williamson's argument seems completely justified. Indeed, it is true that we can not have classical and dual-intuitionistic negations in the same language, so this problem regarding Restall's position affects also Williamson's one.

Someone could argue that the last point regarding BI-intuitionistic logic misses the target, since both Williamson and Restall focus on the entailment "if they can not coexist in the same language then they are the same" and not on the opposite entailment "if they can coexist in the same language then they are not the same".[34] Nonetheless here we are not evaluating Williamson's and Restall's positions *per se*, we are interested in the problem of identity of logical constants, and if they assert that their criterion is applicable only in one direction, then they refuse to give an answer about identity when their criterion is not satisfied, and so we can not help but consider this criterion as (at least) incomplete.

In conclusion, the *quidquid ea sunt* approach can lead us to a generalized adoption of the principle of charity, to a generalized adoption of the principle of homophony or to just mentioned coexistence criterion. None of this choice seems to be fully satisfactory and, for this reason, we need to investigate the inner structure of logical constants in order to find a more fruitful approach. That is we have to deal with the characterization of logical terms.[35]

4.3 Realist theories of meaning

In a series of papers Beall and Restall argue for logical pluralism in a realist approach.[36] Their arguments for logical pluralism are quite complex and would deserve a deeper consideration. Nonetheless, we are mostly interested in antirealism in this thesis, so we will deal with their project only briefly. First of all, we will consider their argument for the existence of genuine rivalry between logics, then we will focus on pluralism.

[31] "There is no frame in which all three negations coexist as propositional operators on the same class of propositions, giving the distinct classical, intuitionist and dual-intuitionist properties." [Restall, 2014], p. 287. A classical reference for this logic is [Urbas, 1996].

[32] [Rauszer, 1974].

[33] The proof of this result is given in Appendix B.3.

[34] Williamson, smartly enough, asserts only that if they can not coexist distinct in the same language, there can not be just a purely verbal disagreement about them. Nonetheless, I believe that this conclusion rests on the identity problem we focused on.

[35] There is another argument for the identity of logical terms that could be collected in this section: proponents of deviant logics usually assert that they are not postulating new meanings for connectives, but new theories for the old connectives ([Williamson, 2014], p. 225). Nonetheless, I do not think that the intents of people are so influential in this kind of debate.

[36] [Beall and Restall, 2000], [Beall and Restall, 2001], [Beall and Restall, 2006].

4.3.1 Logical disagreement

Beall and Restall give essentially a modified version of Williamson's argument. I think that in this way the argument works far better, although at the cost of relying on a realist conception of the meaning.

First of all, the authors define logical validity following essentially Tarskian tradition:

Definition 4.3.1 (Generalised Tarski Thesis (GTT))**.** An argument is valid$_x$ if and only if, in every case$_x$ in which the premises are true, so is the conclusion.

So validity is defined as truth preservation over a set of models, but different sets of models give different notions of logical consequences. The authors prefer the term "case" to "model" in order to specify that they do not want just formal structures, but objects with a philosophical dignity. It is natural to interpret their reading of (GTT) as tied to some kind of correspondence theory of truth, following the same interpretation given to Tarskian semantics by Davidson.[37]

What is new about (GTT) is the idea of proposing this conception of logical consequence without a selection of just one class of cases over which truth should be preserved. In this way, we can define different relations of logical consequences, and if we find good reasons to assert that there are at least two sets of cases that correctly define a notion of logical consequence, then we can conclude that there are at least two correct logics.

We need to be very clear about this point: even if we manage to find two (philosophically and formally) acceptable sets of cases that define two relations of logical consequence, this does not entail that we should be pluralist. We just have a realist philosophical interpretation of two logics. We still have to argue that they can coexist and that there are no reasons to impose a selection between them.

About the possibility of coexistence, we will see that according to Beall and Restall this is grounded on the fact that they speak of the same logical terms. So in some way, Beall and Restall obtain the real disagreement between deviant logics and, at the same time, the availability of pluralism. This is a peculiar aspect of their kind of pluralism.

Let us consider classical, intuitionistic and relevance (**FDE**) logics. The authors detect these systems from the following sets of cases:

- Worlds or Tarskian models for classical logic: that is cases that are both consistent and complete;
- Constructions for intuitionistic logic: cases that are consistent but possibly incomplete;
- Situations for relevance logic: cases that are possibly inconsistent and incomplete.

Essentially, starting from possible worlds or Tarskian models, we obtain constructions from the phases of a process of discovery carried out in that world or from the phases of a process of proof regarding that model. For this reason, we can have incomplete constructions. This is essentially the idea behind Kripke's models for intuitionistic logic, and gives us our first result important for the relation between logics:

Observation 4.3.1 (Classical and intuitionistic cases)**.** *Every classical case is also a case for intuitionistic logic, that is every Tarskian model and possible world is a special kind of construction, but the converse does not hold.*

The authors are enough smart not to explain in detail what they mean with incompleteness. Indeed an intuitive way in which a case could be incomplete is by accepting gaps in truth values, especially if we consider the realist flavour of this approach. Nonetheless, it is well known that intuitionistic logic can not be characterised using gaps in truth values. Indeed even though $A \vee \neg A$ is not an intuitionistic law, $\neg(A \vee \neg A)$ leads to contradiction and $\neg\neg(A \vee \neg A)$ is an intuitionistic logical law. So it is intuitionistically contradictory to assert that there are gaps in truth values; the kind of incompleteness that we need in this case is subtler.

In order to obtain the cases for relevance logic, we need two steps. First of all, we consider situations as parts of possible worlds, that is, incomplete reports of what is going on in a possible world: this gives us consistent situations. We could think that consistent situations and constructions are essentially the same thing, but this is not the case. Indeed constructions are built according to precise rules, while the definition of consistent situations is freer. Also, the way in which the logical constants are interpreted in the two sets of cases is very different. Our second step is the inclusion of inconsistent situations, that are situations in which some incompatible events happen. For example,

[37] [Davidson, 1969]

a situation which the same box is absolutely empty and contain a figurine in it.[38] So inconsistent situations can be used to describe "ways that things could not be".[39]

Now we can assert our second result:

Observation 4.3.2 (Classical and relevance cases)**.** *Every classical case is also a case for relevance logic, that is every Tarskian model and possible world is a special kind of situation, but the converse does not hold.*

Although in this case gaps and gluts in truth values are enough to obtain our logic[40] and so there is a characterization of both incompleteness and inconsistency of situations that is unquestionably in agreement with our realist disposition, Beall and Restall are strangely silent about this interpretation.

Using our two observations, we can argue that the three logics speak about the same logical terms. First of all, let us observe that the set of classical cases is the intersection of the three sets of cases that detect the logics, that is, classical cases are also (degenerate) situations and (degenerate) constructions. If we look at the properties of the cases that define the logics this property is almost obvious. So we can ask how the supposed different negations behave in a classical case.

Let us consider the negation, as an example. Its behaviour is characterised in the three logics by the following clauses:

classical logic given w possible world $w \Vdash \neg A$ iff $w \nVdash A$;[41]

intuitionistic logic given c construction $c \Vdash \neg A$ iff for every successive step of construction c', $c' \nVdash A$;

relevance logic given s situation $s \Vdash \neg A$ iff for every situation s' compatible with s, $s' \nVdash A$.

Without considering the details, it should not be too controversial that a possible world is a situation that is compatible only with himself, and a construction that does not have any further steps. If we grant this, when we evaluate a negation in a classical case we find out that all three clauses are equivalent, that is in a classical case an intuitionistic or relevant negation behave as a classical negation. The same conclusion can be found about all the other connectives.

Beall and Restall argue that this is evidence that all three logics speak about the same connectives, and that the differences in behaviour are due to differences in the properties of the cases. Indeed when these external differences are absent, the same clause works fine for all three logics. So classical, intuitionistic and relevance logics do not speak of different connectives, but just of different sets of cases.

A consequence of this observation is that we can not have a severed classical negation in a framework for intuitionistic logic or in a framework for relevant logic. A moment of reflection gives also the opposite entailment. For this reason, I consider this argument a more refined version of Williamson's early argument for the same conclusion.

Criticisms

The general criticism is that the model-theoretic approach is not apt to define some logics: as an example, it is not possible to define logics that are not transitive or reflexive. Of course, also for some logics that are technically definable in this way there is nonetheless the problem of the philosophical acceptability of this definition. [Read, 2006], as an example, criticizes Beall and Restall's definition of relevance logic using model-theoretic semantics because it essentially relies on classical metatheory. Of course, the criticism of this habit of using different logics on the metalanguage and on the object language is shared also by Timothy Williamson.[42]

Shapiro agrees on these objections to the model-theoretic approach to pluralism and argues that for intuitionistic logic the problem is also deeper. Indeed we can not model the intuitive idea of the gradual construction of natural numbers using Kripke semantics.[43] This problem does not hold only for intuitionism, but for every kind of constructivist account of arithmetic.

[38] [Priest, 1997]

[39] This description is not in contradiction with dialetheism. Indeed this doctrine accepts a distinction between possible and impossible worlds, but suspends the judgment about whether the actual world is possible or impossible; see the conclusion of [Priest, 1997].

[40] [Priest, 2008], p. 146.

[41] The clause for Tarskian model is similar.

[42] [Williamson, 2014].

[43] [Shapiro, 2014] chapter 2 section 3.

4.3.2 Logical pluralism

Beall and Restall obtain logical pluralism from (GTT) arguing that we are not committed to select just one set of cases for the definition of a valid logic. Also, the fact that different logics speak about the same logical terms but in different cases makes them disagree without the possibility to have only one right logic. Indeed Steinberger has proposed the label "structural meaning-variance" for this kind of pluralism, where 'valid' has different meanings but the logical conncetives have the same meaning in all the acceptable systems.[44]

Criticisms

Is it a real pluralism? The fact that intuitionistic and relevant structures have as special cases classical structures is not a new discovery. Intuitionists for example usually assert that there is nothing bad with the application of classical logic in some special contexts (for example when we deal with finite domains or decidable predicates). For this reason, Priest is sceptical about the advantages of using a pluralist approach instead of a monist one in which extra pieces of information about specific contexts are used.[45]

Priest poses two other problems for pluralism, related to this. Let us consider the two logics $\mathcal{S}_\infty$ and $\mathcal{S}_\in$, such that $\mathcal{S}_\infty$ is a sub-logic of $\mathcal{S}_\in$. If and $\mathcal{S}_\in$ can be justified using (GTT), then it preserves truth, and so we are always entitled to use it to justify a conclusion. So logics are not equal: the true logic is the strongest logic that preserves truth. From another perspective, it seems that is we take (GTT) seriously enough, we should consider valid only the logic that preserves validity over all the cases. In this way, the true logic is the weakest logic that preserves truth.[46]

Stephen Read points out that for logics that are not sub-logics of classical logic we have also more dangerous consequences. As an example, while for classical logic $\neg A, B \vDash_c \neg(((A \supset B) \supset B) \supset A)$, for Abelian logic $\neg A, B \vDash_a ((A \supset B) \supset B) \supset A$. This situation raises the worry: we should be entitled to assert $((A \supset B) \supset B) \supset A$, its negation or both of them? [47] Beall and Restall could object that it is not obvious how to define Abelian logic following (GTT), but Read rejects this hypothesized objection, arguing that it is always possible to find a "possible world semantics" for this kind of logic. I suspect that Read's formal observation does not satisfy the philosophical requirements imposed for cases by Beall and Restall. Indeed they are very clear at least about their intention to distinguish between cases and formal models.

Gillian Russell points out that the "collapse arguments" for logical pluralism could also give ground for endorsing logical nihilism, the thesis that there is no valid logic. To be precise, she started arguing for nihilism without taking into account any precise definition of validity. Essentially she assumes that truth-preservation is at least necessary for validity and proposes a way to find a counterexample for every alleged logical consequence, but without a clear theory of meaning for logical terms. So she argues for this bad consequence of the "collapse arguments" in a *Quidquid ea sunt* approach to logical constants.[48]

Someone could think that this is another bad consequence of (GTT), but I do not think so. Indeed the kinds of counterexamples given by Russell for some seemingly obvious logical consequences seem to be in friction with a realist approach to logical consequence. Let us consider her most significant examples:

- In order to reject $A, B \vDash A \wedge B$ she proposes the atomic sentence 'SOLO', that is always true when it does not occur as a subsentence and always false otherwise. In this way we can have two true premises for $A, B \vDash A \wedge B$, if A is substituted with SOLO and B with any true sentence, but a false conclusion.

- In order to reject $A \vDash A$ she proposes the atomic sentence 'PREM', that is always true when it occurs as a premise and always false otherwise.

It seems evident that these counterexamples are at odd with a realist interpretation of logical consequence in which sentences should be about worlds, situations, or related entities. Indeed the author recognises this way out in a later paper, even though she points out that this is not a zero-cost solution, metaphysically speaking.[49]

[44] [Steinberger, 2019], p. 8. Although Hjortland raised some doubts about the possibility that a variation in the meaning of 'valid' does not entail a variation in the meaning of the logical constant in Beall and Restall's approach: see section 5 of [Hjortland, 2012].

[45] [Priest, 2001]

[46] These two arguments are standardly called "collapse arguments", for obvious reasons, in the literature.

[47] [Read, 2006], p. 197.

[48] [Russell, 2018].

[49] [Russell, 2019].

4.4 Antirealist theories of meaning

In this thesis, we developed an antirealistic theory of meaning for logical terms and we reconstructed different logical systems inside this framework. We started imposing some restrictions on the acceptable systems and considering some *desiderata*, like autonomy and innocence of logic, and (weak) separability of meaning. We then saw a special kind of inferentialist theory of meaning, that is proof-theoretic semantics, that imposes extra requirements for logical systems to be acceptable. In the second chapter, we proposed liberalization of traditional proof-theoretic semantics, by pushing to its extreme limits a reasoning proposed by Dummett to reject multiple-conclusion logic and integrating this conclusion with an observation made by Milne. With this machinery, we were able to propose a single-assumption and single-conclusion formulation of three different logical systems:

SASCJDJ: a system that suits both the restrictions for intuitionistic logic and dual-intuitionistic logic;

SASCJ: an intuitionistic system;

SASCK: a classical system.

Now we want to evaluate the possibility to explain logical disagreement and to adopt pluralism in inferentialism, in general, and in our theory of meaning, in particular.

First of all, let us just observe the special status of quasi-deviance in this framework. We defined quasi-deviance as a situation in which two logics are not expressed in the same language, but none of them is a conservative extension of the other one. This situation is problematic for our kind of inferentialism because we want an analytic justification of logical consequences and an inferentialist notion of meaning, and quasi-deviance can be at odds with the combination of these two principles. Indeed when a theory is extended with some rules for a new connective that have repercussions on the meaning of the terms used in the starting system (a kind of quasi-deviance) there seems to be a chaotic relation between the meaning of logical terms, and separability is in particular danger.

Sometimes it is possible to save the situation, by finding other sets of rules that suit separability of meaning. Indeed this is what we did with full classical logic, that in traditional proof-theoretic semantics is a quasi-deviance of negation-free intuitionistic logic, due to differences in the purely implicational fragment of the two logics that are nonetheless formulated using the same pair of rules for implication. We saved the acceptability of classical logic by showing a complete system for negation-free classical logic that suits weak separability. In this way we have deviance between classical and intuitionistic negation-free fragments and the complete systems are conservative extensions of their sub-theories. In order to save this good result, we have to reject identity between connectives of different logical systems, at least when they do not agree on valid consequences. That is classical implication and intuitionistic implication can neither be the same, nor have the same meaning, if we want both the analytic status of logical consequences and separability of meaning. In brief, in addition to the standard reasons to accept meaning variance in inferentialism, we also have reasons base on separability issues.

There is a proposal to circumvent this and related arguments by adopting a distinction between two kinds of meaning: an *operational* one, given by the rules that govern the constant in isolation, and a *global* one, given by those rules together with structural rules.[50] As far as this thesis is concerned, we can not endorse this answer, since we reject a strong distinction between multiple assumption and assumption of a conjunction and between multiple conclusion and conclusion of a disjunction. So, according to our analysis, a structural difference between two logics is just a logical difference in disguise. Indeed the differences between our **SASC**-systems are essentially grounded on differences in the way in which logical terms interact with conjunction and disjunction. Nonetheless, this does not lead to a distinction between two different kinds of meanings in this perspective.

Another similar proposal uses restrictions on the number of formulas in the assumption and in the conclusion in order to reconstruct different notions of logical consequence inside the same system.[51] Of course, the observation that this kind of restriction can be used to detect interesting sub-logics is not new; what is new is the proposal to use this fact to have multiple notions of validity inside one system. The argument is essentially the following: since there is no variation in the rules there is no variation in the meaning of the logical terms. In this way, we can have a plurality of notions of consequence that does not entail a variance in the meaning of logical terms.[52] Hjortland also proposes a generalization

[50] [Paoli, 2003].

[51] This is the main proposal of Restall in his [Restall, 2014].

[52] Even Hjortland, which is usually sceptical about the possibility to have a variance in the meaning of "validity" that does not entail a variance in the meaning of the logical constants (see note 44), is sympathetic with this conclusion, in this case: [Hjortland, 2012], pp. 11-12.

of this argument, based on n-sided sequent calculus.[53] Unfortunately, also this option relies on a notion of multiplicity that does not depend on conjunction or disjunction, so it is not acceptable in our perspective. Moreover, it also has some independent flaws, since it relies on sequent calculus, which has a much more controversial philosophical status than natural deduction. The individuation of different notions of "validity" should be justified philosophically (or at least meaning-theoretically) as well. From this point of view, Restall proposes a bilateral reading of sequents as preventing the assertion of the antecedents and the rejection of the succedent.[54] I am not completely sure of how to use this interpretation of sequents to accept the presence of more logical consequences inside one system, but at least he proposes an antirealistic interpretation of what logical consequence is. Unfortunately, this reading has to be abandoned if we want to accept Hjortland's proposed generalization, since Restall's bilateral approach to meaning can not justify an n-sided version of sequent calculus.[55]

In this work we essentially developed a version of Restall's proposal in which multiplicity of formulae is interpreted as conjunction or disjunction of formulae, exploiting Milne's interpretation of the restriction and liberalisation on the number of formulae in antecedent and succedent of sequents.[56] What we found is a pluralistic justification of three different logics that can rely on a well-established meaning-theoretical interpretation of natural deduction. The main difference between Restall's proposal based on sequent calculus and our proposal based on natural deduction is that in our perspective the logical terms of different logical systems can not be the same or have the same meaning.

It is usually believed that in this kind of framework in which differences in the set of theorems entail differences in meaning it is not possible to account for logical disagreement. As a consequence, it seems that the only philosophical position that can spring out from this framework is a kind of irenic pluralism devoid of real philosophical significance. We will reject both conclusions.

4.4.1 Disagreement

There are two kinds of disagreement about logic that we can find in an antirealistic perspective:

- A disagreement between two different theories of meaning: as an example a disagreement between Prawitz's original theory of meaning that justifies intuitionistic logic but rejects classical logic, adopting a multiple-assumption, multiple-conclusion approach[57] and our revised theory that justifies classical logic, adopting neither multiple assumptions nor multiple conclusions;
- A disagreement between two logics inside a single theory of meaning: as an example, we justify both classical and intuitionistic logic within our theory of meaning, but these two logics seem to disagree about the validity of some logical consequence.

The first kind of disagreement is just a theoretical disagreement about what is meaning, what is a good theory of multiple assertions, etc. We do not have any reason to deny that there can be a genuine disagreement between proponents of different theories of meaning, like we do not have any reason to deny that there can be a genuine disagreement between proponents of different theories of electrons, planets, etc.

Nonetheless, the second kind of disagreement is more problematic because, as we already saw, different logical systems speak about different logical terms when they justify different consequences. That is, when there seems to be disagreement, there is just "change of subject". Indeed every objection to meaning change that we considered in the previous sections has to be abandoned in our new approach, as we just discussed. Then our next question can only be: can we explain the apparent disagreement between logics, by using disagreement between theories of meaning? I think that there are good reasons to believe that we can, at least in most cases. Indeed a lot of traditional rivalry between logics can be understood as rivalry between which form a true theory of meaning should have: the disagreement between classical logic and intuitionistic logic corresponds to disagreements about multiple conclusions, identification of denial with negation, weak or strong separability, etc; the disagreement between classical and substructural logic corresponds to the disagreement about the substructural properties of logic.[58] Of course, this can not be a good solution for realistic-borne logics, like trivalent logics. Nonetheless, this is not surprising for an antirealistic reconstruction of the debate, and maybe we should wait until we have a good antirealistic reading of these logics to evaluate the possibility of explaining logical disagreement in their respect.

[53] [Hjortland, 2012], p. 14.
[54] [Restall, 2005].
[55] Indeed Hjortland obtains his system by a reflection on multiple-valued logics [Hjortland, 2012], p. 13.
[56] See chapter 2.
[57] At most, since the problems with *ex falso quodlibet* that we discussed in section 1.2.3 of chapter 1.
[58] [Restall, 2000]

A consequence of this reformulation of logical disagreement is that when a theory of meaning justifies more than one logic (as happens in our case), there can not be a real disagreement between them, but just a kind of pragmatic rivalry. Two logics can both be justified and a consequence can hold according to the first and do not hold according to the second. In this case, if the theory of meaning is right, the consequence is valid if formulated with the connectives of the first logic and invalid if formulated with the homonymous connectives of the second logic.

Apart from disagreements about the theory of meaning, the only disagreements possible are about which one of the various logics should be used for a given task. As an example, given our theory of meaning, *tertium non datur* is valid if formulated using the connectives of **SASCK** but invalid if formulated using the connectives of **SASCJ**. Nonetheless, the question of which connectives are more useful as tools for the construction of some kind of theory remains open. Indeed, while **SASCK** is stronger, so it seems to be preferable, some mathematical theories that are coherent if formulated using **SASCJ** are provably incoherent if formulated using classical logic.[59] Let me stress for clarity that the most useful logic for a given purpose is not in any way the most rigorous or the most valid one: we take into account only systems that are justified by our theory of meaning. In summary, we can have a theoretical disagreement about the acceptability of a theory of meaning that has as a consequence a disagreement about the justification of a logic, and we can have a pragmatical disagreement about the convenience of the usage of a (justified) logical system in place of another (justified) logical system.

This approach to the issue of logical disagreement is not entirely new. Dummett distinguishes between two different kinds of apparent logical disagreement: conceptually deep and conceptually trivial.[60] Quine's doctrine of change of subject is endorsed in general, so two logics that apparently disagree with each other, are really just speaking of different connectives. Nonetheless, an apparent disagreement about the properties of a logical constant can hide a real disagreement about which theory of meaning we should endorse. According to Dummett, this disagreement is conceptually trivial when:

GoodMeaning both parties in the dispute accept that both meanings are well characterised and acceptable, that is they just verbally disagree about the respective label that they use;

Coexistence using different labels we can have two logical constants in the same language that correspond to the two meanings attached by the speakers to the same constant.

Dummett seems to treat these two conditions as equivalent, and so characterises a disagreement as conceptually deep when:

~~GoodMeaning~~ at least one of the parties in the dispute rejects that both meanings are well characterised and acceptable.

I will argue that **GoodMeaning** and **Coexistence** should not be considered equivalent in general. Then I will consider a recent interpretation of Dummett's idea offered by Prawitz's and argue that it is in some way misleading. In conclusion, I will explain what we can say from our perspective about Dummett's position.

Let us start with **GoodMeaning** and **Coexistence** by considering intuitionistic and dual-intuitionistic negations ($\neg_i$ and $\neg_{di}$). We already saw that the issue of the identity of these *prima facie* distinct negations is a very controversial one. Of course, they can coexist as distinct connectives in the same formal system, as shown by BI-intuitionistic logic, so they suit **Coexistence** and the disagreement between an intuitionist and a dual-intuitionist should be trivial. Nonetheless, it is surely possible to characterise it as a deep disagreement between two different approaches to meaning: verificationism and falsificationism.[61] As a consequence, two speakers can be sceptical about whether the other's point of view about meaning is acceptable or not. The mere existence of a formal system that contains both connectives is not enough to settle the issue of whether meaning should be defined via verification or via falsification. Nor it is enough to establish that these two approaches to meaning can go together, since they could be incompatible for purely philosophical and non-technical reasons. So the two speakers can still fail to satisfy **GoodMeaning** and be in conceptually deep disagreement.

We can have the opposite situation as well, that is to say, we can have an apparent disagreement that is obviously conceptually trivial but such that it cannot be solved by showing a system that contains two connectives, one for each meaning. Let us consider the following situation: Alice asserts $p \vee q$ and Bob rejects this assertion.[62] Let us assume also that p and q are sentences regarding *momentum*

[59][Shapiro, 2014] explains very well how pragmatical reasons could lead one to chose one or another logic, and not necessarily the strongest one, as commonly believed. His examples regarding classical and intuitionistic logic can be adapted to this case since **SASCK** is classical while **SASCJ** is intuitionistic.

[60][Dummett, 1991], p. 193.

[61]We already saw the connection between intuitionism and verification. For the connection between dual-intuitionism and falsification, see [Shramko, 2005].

[62]That is to say, he refuses to assert it. I leave open the issue of his acceptance of $\neg(p \vee q)$.

and position of a subatomic particle, and that Bob is interpreting $\vee$ as a quantum disjunction while Alice is interpreting it as a plain and simple classical disjunction. Both Alice and Bob could agree that quantum logic is just a formal system of derivation, practically useful but conceptually pointless, and that it should not be considered as a good proposal for a revision of our logical practice. In this case, if they find out that they are applying different connectives, they could agree that their disagreement is purely trivial: there is no real disagreement about the correct theory of meaning, since Bob is just using $\vee$ in a technical way, without suggesting any revision of Alice's linguistic practice, which he endorses too.[63] Nevertheless, since classical and quantum disjunctions cannot coexist in the same system, we should consider this as a case of real disagreement, according to Dummett. So it seems that the possibility of coexistence in the same system is not a good *criterion* of real disagreement, after all.

In his [Prawitz, 2015a] the author applies Dummett's distinction to conclude that it is possible to correct the apparent rivalry between classical and intuitionistic logics.[64] Indeed, as we saw in section 2.5, he proposes an ecumenical system in which both classical and intuitionistic logics hold. In particular, it is possible to have both classical and intuitionistic disjunction and implication in the same language. So **Coexistence** holds for both these connectives and there seems to be only a trivial disagreement between a logician that endorses classical logic and a logician that endorses intuitionistic logic, at least about the validity of sentences like $((A \supset B) \supset B) \supset B$ and $(A \supset B) \vee (B \supset A)$. If they adopt an ecumenical language, then their disagreement disappear: they agree that both sentences are valid laws if they are constructed with classical connectives and they are not if they are constructed with intuitionistic connectives.[65]

The situation with negation is a little different, but similar. The validity of a single inference of double negation elimination does not depend on the kind of negation that is applied, since Prawitz's ecumenical system contains only one negation, but depends on which logical terms are used in the sentences that occur in this particular instance: if only classical terms are used, then it holds, otherwise it does not. So, in the end, the classical logician has to withdraw his assertion that double negation elimination holds in general, and what he has to admit about Peirce's and Dummett's laws is only barely less detrimental.

There seems to be one main difference between Dummett's original idea and Prawitz's adaptation: Dummett speaks of logical terms that coexist in the same language with universal applicability, while Prawitz accepts logical terms with a restricted range of applicability. As we saw, in Prawitz's ecumenical system *Modus Ponens* does not hold in general for classical implication. So we have only two options:

- We can recognise that this rule is not valid for classical conditional;
- We can restrict the field of applicability of classical conditional, by changing the definition of well-formed formula, so that the counterexamples to *Modus Ponens* become ill-formed.

I think that none of these alternatives can be accepted without pain.

There is a similar, related issue: if we accept logical laws that do not hold for the entire language, we should accept logical laws that hold in a subsystem of a system as well. That is, if we can find a translation of a stronger logic inside a weaker one that we regard as correct, then we should acknowledge that both logics are correct. Indeed what we can at most lose is the universal applicability of the logical rules. In this case, Prawitz's proposal is just a reappraisal of the standard observation that there is a translation of classical logic inside intuitionistic logic, since in his ecumenical system the purely classical connectives can be defined in the following way:[66]

- $\vdash (A \vee_c B) \subset\supset_i \neg(\neg A \wedge \neg B)$
- $\vdash (A \supset_c B) \subset\supset_i \neg(A \wedge \neg B)$
- $\vdash (\exists_c x A) \subset\supset_i \neg(\forall x \neg A)$[67]

This does not speak directly against Prawitz's ecumenical system. On the contrary, it could help to give a meaning-theoretical interpretation of these translations analysed at the beginning of this chapter. Anyway, this issue makes controversial Prawitz's entire project of an ecumenical system.

[63] A similar situation could arise when Alice is using a classical connective, Bob is using a substructural connective and they are speaking of pieces of information or resources.

[64] [Prawitz, 2015a], p. 17.

[65] [Prawitz, 2015a], p. 30.

[66] Remember that in Prawitz's system, deduction theorem holds for intuitionistic conditional.

[67] [Pimentel et al., 2019a], p. 9.

About Dummett's position in general, we already argued that the impossibility of coexistence is not a good *criterion* to detect a real disagreement between logicians. We still need to investigate what is the difference between our kind of pluralism – that is a pluralist adoption of different systems with distinct connectives – and Prawitz's proposal of an ecumenical system – that contain more than one logic in one single system. We will speak of this and of pluralism in general in the next section.

4.4.2 Pluralism

The pluralism that arises from our considerations can not be a purely Carnapian one, in which every useful formal system is justified. In the first two chapters of this writing, we imposed some restrictions on the shape that a good theory of meaning should have, in order for logical consequences to be analytically valid. Even though we can give a good theory of meaning for more than one logic, we impose some *criteria* on the acceptability of logical systems. Indeed we argued that both multiple-valued and substructural logics are at least problematic to justify in our perspective. Moreover, logics are not all on the same level: if one of the main heredity of Dummett is the observation that[68]

> "Inconsistency [...] though the worst, is not the only possible defect of a linguistic practice."

what we propose here is that different logics can be more or less defective, depending on how strongly they are separable. Indeed, while harmony and weak separability are required for a logic to be valid, we can not ask for strong separability in general. As a conclusion, we can order acceptable systems depending on their degree of separability. That is, even though there can be good reasons to prefer a system that is weakly separable over another one that is strongly separable, separability is surely a virtue and its lack is a flaw.

At this stage of our investigation, we are considering only acceptable systems and valid consequences. What is at issue is only pragmatics. Remembering this, we can explain what is the difference between the kind of systems we constructed and the ecumenical systems proposed by Prawitz.[69] What at most is controversial in Prawitz's system is the logical status of its classical component. Indeed we saw that classical laws are not generally valid and some rules that seem to hold for classical connectives are not even admissible in his system. Moreover, logical consequence in Prawitz's system has an innate intuitionitic character, since deduction theorem holds only for $\supset_i$. Nonetheless the acceptability of Prawitz's system from the meaning-theoretical point of view is beyond doubt: $\supset_c$ maybe is not classical implication, but its meaning is defined without circularity and it gives ground to analytic consequences. So the only relevant differences between our systems and Prawitz's one can be pragmatical. Far from being grounded on a deep difference between two theories of meaning or between two different approaches to logical disagreement, they are just different systems that can be justified in our inferentialist framework. The pragmatical reasons that can lead someone to endorse an ecumenical system in which some rules lose their universal applicability are beyond the scope of this work. Surely this flaws is counterbalanced by other good properties of the ecumenical systems. Anyway, what is central to our philosophical point is that both systems have the same attitude towards the identity of logical terms: different rules give different connectives.

Cesare Cozzo proposed a fallibilist and pluralist version of proof-theoretic semantics, which shares some issues with my view. He considers pragmatical and holistic *criteria* relevant for the choice of the logical system[70] and, since the relevance of these pragmatical *criteria* varies according to the context, he consider a pluralistic approach to logic too.[71] All these issues are common to both our proposals, so there might seem to be a big overlap between our positions. Nonetheless, there are some fundamental differences. According to Cozzo, pragmatical considerations already have a role in the justification of logic, while in our version of proof-theoretic semantics there are two distinct moments of evaluation: the first, in which purely meaning-theoretical considerations are applied in order to justify logical truths as analytic; the second, in which valid logical systems are evaluated according to their usefulness and pragmatical utility.

In Cozzo's opinion, in order to construct a fallibilist theory of meaning, we need to reject analyticity of logic (and analytic truths in general).[72] As a consequence, he rejects both the thesis that meaning-conferring rules are self-justified, and the thesis that there is a structural, meaning-theoretical

[68] [Dummett, 1991], p. 215.
[69] [Prawitz, 2015a].
[70] And in general for the choice of a well-behaved language: [Cozzo, 2008b] pp. 313-4, [Cozzo, 2008a], p. 271, [Cozzo, 2019], section 6.
[71] [Cozzo, 1994a], pp. 259-262.
[72] [Cozzo, 1994a], p. 260, [Cozzo, 2008b], pp. 313-314 and [Cozzo, 2002], pp. 42-43.

justification of non-meaning-conferring rules, that is harmony.[73] The distinction between meaning-conferring rules and non-meaning-conferring rules is not rejected, but it is considered relevant only for a theory of understanding, not for a theory of justification of the language: naively, I understand the meaning of a term if I know – but possibly I do not endorse – its meaning-conferring rules. Justification is dealt with from a holistic standpoint, meaning-conferring rules staying on the same level of the other rules. Justification comes after understanding.[74]

I think that Cozzo's position regarding analyticity rests on a subtle mistake. He assumes that analytic truths cannot be rationally revised.[75] However, this depends greatly on how revision is defined. In our reformulation of proof-theoretic semantics, while it is certainly true that logical laws are analytically valid, and so cannot be rejected *tout court remaining in the same language*, nothing prevents us from changing language. Indeed the choice of language is based on purely pragmatical reasons. As an example, there is no possibility of rejecting $A \vee_c \neg_c A$ in its own language, but we can 'reject' the phonemically identical sentence $A \vee_i \neg_i A$, and refuse to apply classical logic for purely pragmatical reasons. Regardless of whether this situation really stands for a revision of analytic truths, I believe that it suffices for explaining why we sometimes do not want to adopt a well-behaved language (and as a corollary its analytic sentences). In other words, it explains perfectly well how we can use language to "impose an order on reality as it is presented to us".[76]

To make a parallelism with theory change, let us consider the redefinition of *momentum* adopted in special relativity. The old, Newtonian definition is:

$$\vec{q} =_{def} m\vec{v} \tag{4.1}$$

Einstein's proposed revision is:

$$\vec{q} =_{def} m\vec{v}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{4.2}$$

Technically speaking, since these laws are definitional, they should be considered analytically valid. Indeed the meaning of $\vec{q}$ (that is, the *momentum* of a body) is defined by the formula on the right.

Nonetheless, there are good reasons to reject the first definition and choose the second. It is a result of Newtonian physics that the *momentum* of a closed system never changes, in other words it is conserved.[77] Unfortunately, this conservation is in contradiction with some postulates of special relativity, specifically with its assumption that the speed of light is independent of the motion of the source and of the observer. In order for *momentum* to be conserved, we need to adopt Einstein's redefinition of this notion.

Since conservation of *momentum* is so useful in physics to be a key property of this entity, it is fully rational to choose to change the definition according to Einstein's proposal. Putnam suggests that in this kind of cases what happens is that a definitional property, that should hold analytically, is rejected. As a consequence, this definition is only apparently analytical. To describe the status of this kind of sentences he says that they are "as analytic as any nonanalytic statements ever get".[78] Indeed they cannot be refuted only by experiments: we need a new theory that proposes an alternative definition or characterization.[79] He shares the opinion of Cozzo that as a consequence we cannot speak of a real analytic sentence for the Newtonian definition of *momentum*.[80] I propose a different analysis of the phenomenon. We have essentially two entities:

- The Newtonian *momentum* $\vec{q_N}$, that is not always conserved;
- The Einsteinian *momentum* $\vec{q_E}$, that is always conserved.

The formula 4.1 is analytically true of $\vec{q_N}$ and the formula 4.2 is analytically true of $\vec{q_E}$. Since we want conservation property to hold of our notion of *momentum*, it is pragmatically more profitable to adopt the $\vec{q_E}$. This is all we need in order to explain the transition from the old definition to the

[73][Cozzo, 2002], pp. 40-43.
[74][Cozzo, 2008b], p. 315.
[75][Cozzo, 2008a], p. 269, [Cozzo, 2002], p. 43.
[76][Dummett, 1978b], p. 308. Cozzo argues that neither a conventionalist approach to logic, nor an approach that identifies understanding and justifying can carry out this task, since the choice of language is arbitrary in the first case and unrelated to the scientific enterprise in the second case. I hope I have explained why our approach (that is a mix between these two) can instead solve this problem.
[77]This property can be derived from Newton's law: $\vec{F} = m\vec{a}$.
[78][Putnam, 1962], p. 374.
[79]Putnam characterises the principles of Euclidean geometry in the same way: [Putnam, 1962], pp. 372-374.
[80]To be precise, he speaks of velocity and kinetic energy, but the situation is completely alike; [Putnam, 1962], pp. 368-381.

new definition of *momentum*, we need neither to reject analyticity, nor to consider conservation of *momentum* as an analytic truth.

I argue that this is the same situation in which we are when we assert that excluded middle holds for $\neg_c$ but not for $\neg_i$. We do not need to reject the thesis that this law is analytic for $\neg_c$, we just need to integrate this observation with some pragmatical considerations about the choice of the negation. So, in conclusion, analytic truths can be rationally "revised".[81]

Since analytic sentences are revisable in our theory, someone could wonder what is the difference between analytic and synthetic sentences. It seems to me that while synthetic sentences can be revised without changing the meaning of the terms that occur in it, analytic sentences can be revised only changing the meaning of these terms. I argue that, since logical laws are always analytically valid, they can never be revised without changing language.[82] Dummett proposes this kind of approach to take seriously the dynamic aspects of language – every sentence can be rejected – without rejecting its static properties – the distinction between analytic and synthetic sentences –, and to pacify Quine's rejection of analyticity with his "change of logic, change of subject" thesis.[83]

> "Quine allows, as he must, that any particular sentence identified only phonemically, could be rejected; but he maintains that no system of sentential operators of a foreign language could be translated into our own unless they were subject to the laws of classical logic. It plainly follows that these laws are constitutive of the meanings of these logical constants in our language."

Nonetheless, the fact that every well-behaved logical system renders its laws analytic tells nothing about the epistemic usefulness of that system. As a consequence, in logic it is possible to understand a term by knowing its meaning-conferring rule without endorsing it, but only for pragmatical reasons, not because we do not consider the rule valid *tout court*. In conclusion, contrary to what is claimed by Cozzo, understanding is sufficient for the justification of the validity of logical sentences, even though it is not sufficient for warranting the epistemic usefulness of the logical notion under investigation.

In order to reject this vision of logic, Putnam considers analytic sentences revisable only for "unintended and unexplained historical changes in the use of language".[84] That is to say, if we change completely the meaning of a term without any rational reason. The reason to ask for a complete change of meaning is Putnam's adoption of cluster concepts: a concept is usually not individuated by a single analytic sentence, but by more sentences that are accepted as true by the community of the speakers, so that by changing idea about one of these sentences you do not change the subject.[85] So, in order to specify that we are not looking for a change of one single property of the same subject but for a complete change of the language, we ask for a complete redefinition of the meaning. Needless to say, as far as we are concerned we cannot adopt the idea of cluster concepts for logical terms, since it is blatantly in conflict with our belief that set of sentences are to be interpreted as conjunctions or as disjunctions. Its applicability for other non-logical concepts should be evaluated carefully, but this topic is beyond the scope of this work. The reason to ask for the absence of rational reasons for the change is however obscure to me. Moreover, it seems to have a big role in Putnam's conclusion that there are no interesting analytic sentences.[86] If, as we propose, it is possible to have a rational revision of analytic sentences and a rational change of language, Putnam's motive to restrict analyticity to uninteresting sentences can be avoided.

The difference between our and Putnam's approaches to analyticity is made clear by the fact that he describes analytic sentences in formal languages as sentences for which there are extra clauses that prevent to give them up.[87] Of course he does not believe that this is what happens with natural language, since there are very few stipulations in strict sense in natural language, and "all bachelors are unmarried" – the prototypical example of analytic sentence – is not based on stipulations of any kind. Nonetheless this is a good picture of what we intend with "true by stipulation" neither, at least according to our theory of meaning. Introduction rules are intended to be meaning-conferring rules, and so true by stipulation (at least for formal languages), nonetheless they can be rejected. Moreover no extra clauses are imposed to determine that their are true-by-convention. What it happens is that when we give them up, we are not speaking of the same logical terms any more. And we do not have any choice about this issue: the identity of the logical terms is determined by the theory of

[81]Cozzo deals with different but equivalent examples, the impact of the discovery of platypus on the classification of mammals and the rejection of phlogiston in contemporary chemistry: [Cozzo, 2002], p. 43.

[82]To be precise, I-rules are directly analytic since meaning-conferring, while E-rules are analytic since justified by I-rules.

[83][Dummett, 1978b], p. 416.

[84][Putnam, 1962], p. 375.

[85][Putnam, 1962], pp. 378-379.

[86][Putnam, 1962], p. 362 and pp. 380-381.

[87][Putnam, 1962], p. 382.

meaning, it cannot be determined independently of that. Putnam's conception of analytic sentences in formal languages explains why he shares Quine's opinion that there should be no interesting analytic sentences in science. That is, to assume the existence of analytic sentences is a bad scientific habit.[88] Of course, if what he means is that it is a bad scientific habit to assume, by extra clauses, the existence of sentences in the theory that cannot be revised rationally, but that can be revised only by random historical changes in the language, we share his negative opinion. In conclusion, the fate of Putnam's conception of analytic sentences is indissolubly tied to his cluster concept theory of meaning. Since it clearly clashes with what we established in this thesis,[89] we can reject Putnam's vision, at least for logical terms.

A much more controversial topic is the existence of analytic truths in the non-logical fragment of the language. Indeed it is not clear how it could be possible to identify analytic sentences in a natural language.[90] Nonetheless, the idea that in order to revise the truth value of some sentences in general we do not need to change the meaning of the terms involved, while in some special cases the revision of the truth value entails such a change is at least *prima facie* appealing. This work is devoted almost exclusively to the logical language, but Dummett gives good reasons to be optimistic about the possibility of investigating such a distinction in natural language too.[91]

4.4.3 Criticisms

Answers to old criticisms

Let us consider the criticisms already seen for the realist version of pluralism.

First of all, in this case, we can not object that we are not proposing a real pluralism. Indeed, the difference between logics can not be explained by a difference in contexts of application this time. It is true that intuitionistic logic accepts the validity of classical logic in some special contexts (finite contexts, decidable context, etc.), but that there seems to be no pluralism in this *phenomenon*. Nonetheless, our justification of logics is purely linguistic and does not depends on the context of application. In addition, and more importantly, different logics speak about different logical connectives, so the fact that in some particular contexts $A \vee_i \neg_i A$ can be proved has nothing to do with the provability of $A \vee_c \neg_c A$ in general.

Priest's two collapse arguments can not be used against our pluralism, indeed:

- The first argument stresses that if we are entitled to use different logics, we know that the strongest is valid (preserves truth, in the old, realist formulation) so it is useless to argue for the validity of its sublogics. Indeed it seems to be in some way useless and in some way obvious, since we will never use them and they are obviously valid, as sublogics of a valid logic.

  This argument can not be used against our pluralism since we strongly distinguish between the connectives of different logics. Indeed the validity of a subset of all the logical consequences provable in a valid logic is obvious, but the validity of a reformulation of these consequences that uses other logical terms is not obvious at all. This is not a purely abstract specification: we impose some formal requirements for the acceptability of a logical system, like harmony and separability, and the fact that a logic satisfies them does not entail that all its sublogics do the same. Indeed we already proved that an intuitionistic system in which we assume quantum-disjunction in place of standard disjunction is not harmonious but, nonetheless, it is a sublogic of both **SASCJ** and **SASCK**, which we accept as valid logics.[92] Also the argument that weaker logics are useless can be refuted for Carnapian kinds of disagreements, as we remarked at the end of the previous section.

- The second argument asserts that we should consider valid only the weakest logic that suits (GTT), since the realist notion of logical consequence asks for necessary truth preservation, interpreted as truth preservation in all cases. So the largest set of cases, that detects the weakest logic, is the only one apt to give valid logical consequence according to realism.

  In an antirealistic conception of logic we can define validity:

  - As derivability in an acceptable system;
  - Explicitly, in an inductive way starting from canonical derivations and then generalising, as we did in section 1.2.2.

[88] [Putnam, 1962], p. 389.
[89] Particularly with our analysis of sets of sentences in section 2.6.
[90] Surely this part of Quine's thesis in [Quine, 1951] is fully shareable and uncontroversial.
[91] [Dummett, 1978b], especially section 7.
[92] We explained the problems with this connective in section 1.2.2.

Some authors prefer the first alternative,[93] grounding all their theory on harmony and other properties of the system, while others prefer an explicit definition of validity.[94] It seems at first glance that none of these alternatives can be used to raise Priest's objection, but the second one is some times reformulated as asking for propagation of *grounds for assertion* from assumptions to conclusions.[95] We can try to use this reformulation to adapt Priest's objection to our antirealist pluralism: we should consider valid only the logic that preserves *grounds for assertion* in every case, that is the weakest of the justified logics.

This would be a smart move for an opponent to our pluralism but, nonetheless, we can easily block it. Indeed, as an example let us consider double negation elimination. For the realist pluralism devised by Beall and Restall, that were the target of Priest's objection, what is at issue is $\neg\neg\phi \vDash_x \phi$, that is truth preservation in a set x of cases. In this case, we can argue that, since logical consequence asks for truth preservation in every case, this consequence is valid if and only if x contains all the cases. Indeed classical logic and intuitionistic logic evaluate the same consequence and the first manages to prove it only because it takes into account a subset of constructions (that are adequate to work as classical models). In our antirealist pluralism, what is at issue is $\neg_l\neg_l\phi \vDash \phi$, that is, preservation of grounds for assertions in general for sentences formulated using the logical terms of a specific logic l. The variety of logics is given by different sets of logical constants, so weaker logics are not more general than stronger ones, they just speak of different terms. To argue that $\neg_c\neg_c\phi \vDash \phi$ is not valid because $\neg_i\neg_i\phi \nvDash \phi$ and so grounds for assertions are not preserved in general makes no sense at all: we are making a comparison between two completely different consequences.

Stephen Read's observation that Priest's argument is even more dangerous when we justify two logics that strongly disagree – that is one proves a result and the other one proves its negation – goes away with Priest's argument, of course. Indeed there is nothing problematic in accepting both $\neg_c A, B \vDash \neg_c(((A \supset_c B) \supset_c B) \supset_c A)$ for classical connectives, and $\neg_a A, B \vDash ((A \supset_a B) \supset_a B) \supset_a A$ for Abelian connectives.[96] We could be in doubt which of the two consequences we should apply for a precise purpose, but both of them preserve grounds for assertion, and so are valid. In the same way, we can deal with Gillian Russell's argument for logical nihilism based on Priest's second collapse argument.

In conclusion, we have to deal with Williamson's arguments against change of subject, based on the observation that we can not have both $\neg_i$ and $\neg_c$ in the same language. We rejected an identity *criterion* based on this property in section 4.2.2. Nonetheless, there is still the problem of explaining why it is not possible to have both connectives in the same language. Even though we managed to weaken this objection by pointing out some of its undesired consequences, the issue of explaining the impossibility of having more logical constants in the same language remains an open problem.

New criticisms

External validity The standard criticism against this view of logic is that it rejects any external notion of logical validity.[97] Indeed, even though our antirealistic perspective manages to distinguish between formal derivability and validity in a well-defined system, we still do not have something like validity *simpliciter*. To be honest, it is not completely clear to me what this extra-systematic notion of validity should be. Priest is very clear about his ideas on this topic: validity is logical validity in natural language, or vernacular reasoning.[98] If this is the standard interpretation of external validity, I will argue that it is not such an important notion and that our requirements grounded on the theory of meaning give an external *criterion* that is much more philosophically pregnant.[99]

Another recent criticism of proof-theoretic validity proposed by Stephen Read can be interpreted in this direction too.[100] I will explain why this is the case, and propose an objection to Read's criticism

[93] Read is surely one of them (see [Read, 2000] and [Read, 2010], *inter alia*), like Tennant ([Tennant, 1997]). One of the main reasons to abandon an explicit characterization of validity is that it requires the infamous *fundamental assumption* (assumption 1.2.1), that is not easy to justify.

[94] Like Prawitz and Dummett: [Prawitz, 1971] and [Dummett, 1991].

[95] [Francez, 2017a] explains very clearly this issue, although it is present in some earlier papers about proof-theoretic semantics too.

[96] As there is nothing problematic in accepting both $\neg_c\phi, \psi \vDash \neg_c(((\phi \supset_c \psi) \supset_c \psi) \supset_c \phi)$ and $\neg_a\phi, \psi \vDash ((\phi \supset_a \psi) \supset_a \psi) \supset_a \phi$ for specific sentences ϕ and ψ.

[97] [Haack, 1978], p. 14-15.

[98] [Priest, 2016], section. 2.5.

[99] Williamson proposes a similar criticism about logical pluralism in general. He argues that although there is variance in how terms are used in different logics, there is no difference in how they should be used, and only this could lead to a change of meaning for logical terms: [Williamson, 2014], p. 224-225. Since this objection is a particular reformulation of the one based on external validity, we will not deal with it directly.

[100] [Read, 2015].

in particular and to all the objections based on external validity in general. He proposes to use both harmony and truth preservation as requirements for justifying a logical system; harmony gives us analyticity, that is it warrants that logical truths follow from the meaning of the logical constants, while truth preservation assures us that the meaning given to logical terms is correct, that is it is not flawed *per se*. Read's examples of terms with a flawed meaning include: 'Boche', 'phlogiston', his logical constant • that we already encountered in section 1.2.1, and both intuitionistic and classical conditional.[101]

The observation that we should reject some terms of common usage in natural language has a long tradition that dates back at least to logical positivism. The most famous example in the community of logicians is Tarski's observation that we have to restrict the applicability of the truth predicate in our natural language, if we want to avoid paradoxes. In addition, criticisms to established linguistic practices are not unusual in proof-theoretic semantics, since most of the logicians that share this perspective are deeply convinced that classical logic is unjustified. What is new in Read's proposal is that he consider proof-theoretical properties as apt only to prove the faithfulness of a set of rules to the meaning given to terms by the introduction rules, but not as apt to prove validity. His main reasons to believe this are his scepticism for non-relevant conditionals and his conviction that his connective bullet is well-defined. These are in his opinion harmonious connectives that lead to unacceptable conclusions.

We already saw that it is not hard to reject •, and indeed logicians seem to have rejected it *en bloc*, and even though some strange terms like Peano's operator '?' can raise doubts about harmony as a complete *criterion* of validity, there are no reasons to look at external validity in order to find a solution. Read's rejection of non-relevant conditionals too can be explained without referring to a strong notion of external validity. Indeed, when we have a proof of a logical law in a harmonious system and we find an alleged counterexample of it in natural language, we are just proving that natural language does not follow the inference rules of that system. But this tells nothing about the validity of that law in that system and tells also nothing about the validity of that law in general. It only tells us something about the validity of this law in that natural language.[102]

Maybe every language gives us the same report about valid inferences, but this is relevant only for cognitive reasons. As an example, it could just indicate that our common brain structure determines some formal aspects of our language; something completely distinct from logical validity. According to this reasoning, the famous problem regarding logical alien that puzzled Frege is just a fake problem: we could have logical aliens (maybe we already have some), but nothing could force us to translate their sentences homophonically.[103] That is if they are logical aliens then they speak different languages.

About this part of logic (but only about it), I agree with philosophers that argue that logic is not exceptional but uses the same methods of the other sciences.[104] This does not mean that the problem of the relationship between formal logic and natural reasoning is uninteresting. I think that the reconstruction of the uprising of the distinction between *de dicto* and *de re* modalities in [Read, ming] greatly exemplifies the attractiveness of this research. Neither it is clear whether what is happening there is the explication of the meaning of modal terms that already the ancient Greeks possessed or a transformation of their meaning. What is sure is that natural languages evolve and that, although we can be competent speakers of a natural language, this competence leads to explicit knowledge only in an imperfect way. This is the reason why we need to study the natural language from the outside, scientifically. However, the issue of the justification of logical consequence is completely independent of these problems.

Moreover, I think that we should really be concerned with the fact that if there is a place in which Gillian Russell's argument for logical nihilism works well, it is in natural language. And defining logical validity as truth preservation without a specification of any theory of meaning prevents us from finding a good objection to this conclusion (for example one based on the change of meaning). As a consequence, probably the combination of Williamson's argument against meaning variance and Russell's argument for logical nihilism entail that natural language does not have any logic at all.

In conclusion as far as logic is the study of the behaviour of logical terms in natural language, logic is a science, but as I already stressed this is a very deep conclusion only if you ascribe a special logical status to natural language, something I do not think you should do (especially if you want to

[101] [Read, 2015] and [Read, ming].

[102] One of the reasons why Read is unsatisfied with proof-theoretic validity is that he believes that Prawitz's definition of validity rests on an erroneous assumption, that is, the fundamental assumption. Indeed, he gives back to harmony and truth preservation *simpliciter* after discarding justified assertion preservation.

His criticism of Prawitz's account of validity could also be right, but, in this case, Read still needs to answer the "change of logic, change of subject" argument. That is natural language counterexample and formal laws still speak about different objects.

[103] [Frege, 2016], p. XVI.

[104] [Priest, 2016], [Russell, 2015], [Williamson, 2017] and [Hjortland, 2017] *inter alia*.

avoid logical nihilism). So I think that it is preferable to keep the investigation about the validity of logical consequences completely severed from issues about natural language.[105]

[105]Someone argues that we need an external notion of validity in order for our knowledge to be grounded. I think that Steinberger's defence of Carnap's philosophy of logic against this kind of criticisms works fine also for our position.

Appendices

Appendix A

Systems

In this appendix we list the formal systems developed during the thesis.

A.1 JDJ systems

A.1.1 LJDJ

Axioms

$$A \Rightarrow A$$

Structural rules

$$Weak \Rightarrow \frac{\Rightarrow \; C}{A \; \Rightarrow \; C} \qquad \Rightarrow Weak \frac{C \; \Rightarrow}{C \; \Rightarrow \; A}$$

$$Cut \frac{C \; \Rightarrow \; A \qquad A \; \Rightarrow \; D}{C \; \Rightarrow \; D}$$

Operational rules

$$\wedge \Rightarrow \frac{A \; \Rightarrow \; C}{A \wedge B \; \Rightarrow \; C} \qquad \wedge \Rightarrow \frac{B \; \Rightarrow \; C}{A \wedge B \; \Rightarrow \; C}$$

$$\Rightarrow \wedge \frac{C \; \Rightarrow \; A \qquad C \; \Rightarrow \; B}{C \; \Rightarrow \; A \wedge B}$$

$$\vee \Rightarrow \frac{A \; \Rightarrow \; C \qquad B \; \Rightarrow \; C}{A \vee B \; \Rightarrow \; C}$$

$$\Rightarrow \vee \frac{C \; \Rightarrow \; B}{C \; \Rightarrow \; A \vee B} \qquad \Rightarrow \vee \frac{C \; \Rightarrow \; A}{C \; \Rightarrow \; A \vee B}$$

$$\supset\Rightarrow \frac{\Rightarrow \; A \qquad B \; \Rightarrow \; C}{A \supset B \; \Rightarrow \; C} \qquad \Rightarrow\supset \frac{A \; \Rightarrow \; B}{\Rightarrow \; A \supset B}$$

$$\neg \Rightarrow \frac{\Rightarrow \; A}{\neg A \; \Rightarrow} \qquad \Rightarrow \neg \frac{A \; \Rightarrow}{\Rightarrow \; \neg A}$$

$$\Rightarrow \bot \frac{A \; \Rightarrow \; \bot}{A \; \Rightarrow}$$

A.1.2 NJDJ

$$\wedge\mathrm{I}\ \dfrac{C \qquad \begin{matrix}[C]\\ \vdots\\ A\end{matrix} \qquad \begin{matrix}[C]\\ \vdots\\ B\end{matrix}}{A \wedge B} \qquad \wedge\mathrm{E}\ \dfrac{A \wedge B \qquad \begin{matrix}[A]\\ \vdots\\ C\end{matrix}}{C} \qquad \wedge\mathrm{E}\ \dfrac{A \wedge B \qquad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}$$

$$\vee\mathrm{E}\ \dfrac{A \vee B \qquad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C} \qquad \vee\mathrm{I}\ \dfrac{A}{A \vee B} \qquad \vee\mathrm{I}\ \dfrac{B}{A \vee B}$$

$$\supset\mathrm{I}\ \dfrac{\begin{matrix}[A]\\ \vdots\\ B\end{matrix}}{A \supset B} \qquad \supset\mathrm{E}\ \dfrac{A \supset B \qquad \begin{matrix}\varnothing\\ \vdots\\ A\end{matrix} \qquad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}$$

$$\neg\mathrm{I}\ \dfrac{\begin{matrix}[A]\\ \vdots\\ \bot\end{matrix}}{\neg A} \qquad Efq\ \dfrac{\bot}{C} \qquad \neg\mathrm{E}\ \dfrac{\neg A \qquad \begin{matrix}\varnothing\\ \vdots\\ A\end{matrix}}{\bot}$$

A.2 Intuitionistic systems

A.2.1 LJ

Axiom

$$A \Rightarrow A$$

Structural Rules

$$Weak \Rightarrow\ \dfrac{\Gamma \ \Rightarrow\ C}{\Gamma, A \ \Rightarrow\ C} \qquad \Rightarrow Weak\ \dfrac{\Gamma \ \Rightarrow}{\Gamma \ \Rightarrow\ A}$$

$$Con \Rightarrow\ \dfrac{\Gamma, A, A \ \Rightarrow\ C}{\Gamma, A \ \Rightarrow\ C} \qquad Perm \Rightarrow\ \dfrac{\Gamma, A, B, \Theta \ \Rightarrow\ C}{\Gamma, B, A, \Theta \ \Rightarrow\ C}$$

$$Cut\ \dfrac{\Gamma \ \Rightarrow\ A \qquad \Delta, A \ \Rightarrow\ C}{\Gamma, \Delta \ \Rightarrow\ C}$$

Operational Rules

$$\wedge \Rightarrow\ \dfrac{\Gamma, A \ \Rightarrow\ C}{\Gamma, A \wedge B \ \Rightarrow\ C} \qquad \wedge \Rightarrow\ \dfrac{\Gamma, B \ \Rightarrow\ C}{\Gamma, A \wedge B \ \Rightarrow\ C}$$

$$\Rightarrow \wedge\ \dfrac{\Gamma \ \Rightarrow\ A \qquad \Gamma \ \Rightarrow\ B}{\Gamma \ \Rightarrow\ A \wedge B}$$

$$\vee \Rightarrow\ \dfrac{\Gamma, A \ \Rightarrow\ C \qquad \Gamma, B \ \Rightarrow\ C}{\Gamma, A \vee B \ \Rightarrow\ C}$$

$$\Rightarrow \vee\ \dfrac{\Gamma \ \Rightarrow\ B}{\Gamma \ \Rightarrow\ A \vee B} \qquad \Rightarrow \vee\ \dfrac{\Gamma \ \Rightarrow\ A}{\Gamma \ \Rightarrow\ A \vee B}$$

$$\supset\Rightarrow\ \dfrac{\Gamma \ \Rightarrow\ A \qquad \Delta, B \ \Rightarrow\ C}{\Gamma, \Delta, A \supset B \ \Rightarrow\ C} \qquad \Rightarrow\supset\ \dfrac{\Gamma, A \ \Rightarrow\ B}{\Gamma \ \Rightarrow\ A \supset B}$$

$$\neg \Rightarrow\ \dfrac{\Gamma \ \Rightarrow\ A}{\Gamma, \neg A \ \Rightarrow} \qquad \Rightarrow \neg\ \dfrac{\Gamma, A \ \Rightarrow}{\Gamma \ \Rightarrow\ \neg A}$$

A.2.2 SASCLJ

Axioms

$$A \Rightarrow A$$

Structural rules

$$Weak \Rightarrow \frac{\Rightarrow\ C}{A\ \Rightarrow\ C} \qquad \Rightarrow Weak \frac{C\ \Rightarrow}{C\ \Rightarrow\ A}$$

$$Cut\ \frac{C\ \Rightarrow\ A \qquad \{G\wedge\}A\{\wedge H\}\ \Rightarrow\ D}{\{G\wedge\}C\{\wedge H\}\ \Rightarrow\ D}$$

Operational rules

$$\wedge\Rightarrow \frac{A\ \Rightarrow\ C}{A\wedge B\ \Rightarrow\ C} \qquad \wedge\Rightarrow \frac{B\ \Rightarrow\ C}{A\wedge B\ \Rightarrow\ C}$$

$$\Rightarrow\wedge \frac{C\ \Rightarrow\ A \qquad C\ \Rightarrow\ B}{C\ \Rightarrow\ A\wedge B}$$

$$\vee\Rightarrow \frac{A\{\wedge D\}\ \Rightarrow\ C \qquad B\{\wedge D\}\ \Rightarrow\ C}{(A\vee B)\{\wedge D\}\ \Rightarrow\ C}$$

$$\Rightarrow\vee \frac{C\ \Rightarrow\ B}{C\ \Rightarrow\ A\vee B} \qquad \Rightarrow\vee \frac{C\ \Rightarrow\ A}{C\ \Rightarrow\ A\vee B}$$

$$\supset\Rightarrow \frac{\{E\}\ \Rightarrow\ A \qquad B\{\wedge D\}\ \Rightarrow\ C}{(A\supset B)\{\wedge(D\wedge E)\}\ \Rightarrow\ C} \qquad \Rightarrow\supset \frac{A\{\wedge C\}\ \Rightarrow\ B}{\{C\}\ \Rightarrow\ A\supset B}$$

$$\neg\Rightarrow \frac{\{C\}\ \Rightarrow\ A}{\neg A\{\wedge C\}\ \Rightarrow} \qquad \Rightarrow\neg \frac{A\{\wedge C\}\ \Rightarrow}{\{C\}\ \Rightarrow\ \neg A}$$

$$\Rightarrow\bot \frac{A\ \Rightarrow\ \bot}{A\ \Rightarrow}$$

A.2.3 SASCNJ

$$\wedge\mathrm{I}\ \frac{C \qquad \begin{array}{c}[C]\\ \vdots\\ A\end{array} \qquad \begin{array}{c}[C]\\ \vdots\\ B\end{array}}{A\wedge B} \qquad \wedge\mathrm{E}\ \frac{A\wedge B \qquad \begin{array}{c}[A]\\ \vdots\\ C\end{array}}{C} \qquad \wedge\mathrm{E}\ \frac{A\wedge B \qquad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}$$

$$\vee\mathrm{E}\ \frac{(A\vee B)\{\wedge D\} \qquad \begin{array}{c}[A\{\wedge D\}]\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[B\{\wedge D\}]\\ \vdots\\ C\end{array}}{C} \qquad \vee\mathrm{I}\ \frac{A}{A\vee B} \qquad \vee\mathrm{I}\ \frac{B}{A\vee B}$$

$$\supset\mathrm{I}\ \frac{\{C\} \qquad \begin{array}{c}[A\{\wedge C\}]\\ \vdots\\ B\end{array}}{A\supset B} \qquad \supset\mathrm{E}\ \frac{A\supset B\{\wedge(D\wedge E)\} \qquad \begin{array}{c}[\{E\}]\\ \vdots\\ A\end{array} \qquad \begin{array}{c}[B\{\wedge D\}]\\ \vdots\\ C\end{array}}{C}$$

$$\neg\mathrm{I}\ \frac{\{C\} \qquad \begin{array}{c}[A\{\wedge C\}]\\ \vdots\\ \bot\end{array}}{\neg A} \qquad Efq\ \frac{\bot}{C} \qquad \neg\mathrm{E}\ \frac{\neg A\{\wedge C\} \qquad \begin{array}{c}[\{\ C\}]\\ \vdots\\ A\end{array}}{\bot}$$

A.3 Classical systems

A.3.1 LK

Axiom

$$A \Rightarrow A$$

Structural Rules

$$Weak \Rightarrow \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \Rightarrow Weak \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$Con \Rightarrow \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \Rightarrow Con \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$Perm \Rightarrow \frac{\Gamma, A, B, \Theta \Rightarrow \Delta}{\Gamma, B, A, \Theta \Rightarrow \Delta} \qquad \Rightarrow Perm \frac{\Gamma \Rightarrow \Delta, A, B, \Lambda}{\Gamma \Rightarrow \Delta, B, A, \Lambda}$$

$$Cut \frac{\Gamma \Rightarrow A, \Delta \qquad \Theta, A \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda}$$

Operational Rules

$$\wedge \Rightarrow \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \wedge \Rightarrow \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}$$

$$\Rightarrow \wedge \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$\vee \Rightarrow \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}$$

$$\Rightarrow \vee \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} \qquad \Rightarrow \vee \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

$$\supset\Rightarrow \frac{\Gamma \Rightarrow A, \Delta \qquad \Theta, B \Rightarrow \Lambda}{\Gamma, \Theta, A \supset B \Rightarrow \Delta, \Lambda} \qquad \Rightarrow\supset \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$$

$$\neg \Rightarrow \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad \Rightarrow \neg \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$$

A.3.2 SASCLK

Axioms

$$A \Rightarrow A$$

Structural rules

$$Weak \Rightarrow \; \frac{\Rightarrow \; C}{A \; \Rightarrow \; C} \qquad\qquad \Rightarrow Weak \; \frac{C \; \Rightarrow}{C \; \Rightarrow \; A}$$

$$Cut \; \frac{C \; \Rightarrow \; \{E\vee\}A\{\vee F\} \qquad \{G\wedge\}A\{\wedge H\} \; \Rightarrow \; D}{\{G\wedge\}C\{\wedge H\} \; \Rightarrow \; \{E\vee\}D\{\vee F\}}$$

Operational rules

$$\wedge \Rightarrow \; \frac{A \; \Rightarrow \; C}{A \wedge B \; \Rightarrow \; C} \qquad\qquad \wedge \Rightarrow \; \frac{B \; \Rightarrow \; C}{A \wedge B \; \Rightarrow \; C}$$

$$\Rightarrow \wedge \; \frac{C \; \Rightarrow \; A \qquad C \; \Rightarrow \; B}{C \; \Rightarrow \; A \wedge B}$$

$$\vee \Rightarrow \; \frac{A\{\wedge D\} \; \Rightarrow \; C \qquad B\{\wedge D\} \; \Rightarrow \; C}{(A \vee B)\{\wedge D\} \; \Rightarrow \; C}$$

$$\Rightarrow \vee \; \frac{C \; \Rightarrow \; B}{C \; \Rightarrow \; A \vee B} \qquad\qquad \Rightarrow \vee \; \frac{C \; \Rightarrow \; A}{C \; \Rightarrow \; A \vee B}$$

$$\supset\Rightarrow \; \frac{\{E\} \; \Rightarrow \; A\{\vee F\} \qquad B\{\wedge D\} \; \Rightarrow \; C}{(A \supset B)\{\wedge(D \wedge E)\} \; \Rightarrow \; C\{\vee F\}} \qquad \Rightarrow\supset \; \frac{A\{\wedge C\} \; \Rightarrow \; B\{\vee F\}}{\{C\} \; \Rightarrow \; (A \supset B)\{\vee F\}}$$

$$\neg \Rightarrow \; \frac{\{C\} \; \Rightarrow \; A\{\vee F\}}{\neg A\{\wedge C\} \; \Rightarrow \; \{F\}} \qquad\qquad \Rightarrow \neg \; \frac{A\{\wedge C\} \; \Rightarrow \; \{F\}}{\{C\} \; \Rightarrow \; \neg A\{\vee F\}}$$

$$\Rightarrow \bot \; \frac{A \; \Rightarrow \; \bot}{A \; \Rightarrow}$$

A.3.3 SASCNK

$$\wedge\text{I} \; \frac{C \qquad \begin{matrix}[C]\\ \vdots \\ A\end{matrix} \qquad \begin{matrix}[C]\\ \vdots \\ B\end{matrix}}{A \wedge B} \qquad \wedge\text{E} \; \frac{A \wedge B \qquad \begin{matrix}[A]\\ \vdots \\ C\end{matrix}}{C} \qquad \wedge\text{E} \; \frac{A \wedge B \qquad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}$$

$$\vee\text{E} \; \frac{(A \vee B)\{\wedge D\} \qquad \begin{matrix}[A\{\wedge D\}]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B\{\wedge D\}]\\ \vdots \\ C\end{matrix}}{C} \qquad \vee\text{I} \; \frac{A}{A \vee B} \qquad \vee\text{I} \; \frac{B}{A \vee B}$$

$$\supset\text{I} \; \frac{\{C\} \qquad \begin{matrix}[A\{\wedge C\}]\\ \vdots \\ B\{\vee D\}\end{matrix}}{(A \supset B)\{\vee D\}} \qquad \supset\text{E} \; \frac{A \supset B\{\wedge(D \wedge E)\} \qquad \begin{matrix}[\{E\}]\\ \vdots \\ A\{\vee F\}\end{matrix} \qquad \begin{matrix}[B\{\wedge D\}]\\ \vdots \\ C\end{matrix}}{C\{\vee F\}}$$

$$\neg\text{I} \; \frac{\{C\} \qquad \begin{matrix}[A\{\wedge C\}]\\ \vdots \\ \bot\{B\}\end{matrix}}{\neg A\{\vee B\}} \qquad Efq \; \frac{\bot}{C} \qquad \neg\text{E} \; \frac{\neg A\{\wedge C\} \qquad \begin{matrix}[\{\;C\}]\\ \vdots \\ A\{\vee B\}\end{matrix}}{\bot\{B\}}$$

Appendix B

Formal Results

B.1 Equivalence between logical systems

B.1.1 JDJ systems

Equivalence between SASCLJDJ and SASCNJDJ

Theorem B.1.1 (Equivalence between **SASCLJDJ** and **SASCNJDJ**).

1. (a) *If* $\vdash_{SASCLJDJ} A \Rightarrow B$*, then:*
   - *Or* $A \vdash_{SASCNJDJ} B$*;*
   - *Or* $\vdash_{SASCLJDJ} \Rightarrow B$ *and* $\vdash_{SASCNJDJ} B$*.*

   (b) *If* $\vdash_{SASCLJDJ} A \Rightarrow$*, then* $A \vdash_{SASCNJDJ} \bot$*.*
2. (a) *If* $A \vdash_{SASCNJDJ} B$*, then* $\vdash_{SASCLJDJ} A \Rightarrow B$*.*

Let us first define a translation from **SASCLJDJ** to **SASCNJDJ**.

Proof. By induction on the length of the derivation $\mathtt{d}$ of **SASCLJDJ**, we obtain the derivation $\mathtt{d}^*$ of **SASCNJDJ**:

Base: The only derivation of length 1 in **SASCLJDJ** is an application of the axiom rule $A \Rightarrow A$ $(\bot \Rightarrow \bot)$, which we translate to the assumption A $(\bot)$ in **SASCNJDJ**;

Steps: By case on the last rule applied in the **SASCLJDJ** derivation of $A \Rightarrow B$ (or $A \Rightarrow$).

$\Rightarrow\wedge$ [1]

$$\Rightarrow\wedge\ \frac{\overset{\mathtt{d}_1}{C \Rightarrow A} \qquad \overset{\mathtt{d}_2}{C \Rightarrow B}}{C \Rightarrow A \wedge B} \quad \rightsquigarrow \quad \wedge\mathrm{I}_{1,2}\ \frac{C \qquad \overset{[C]^1}{\underset{A}{\vdots \mathtt{d}_1^*}} \qquad \overset{[C]^2}{\underset{B}{\vdots \mathtt{d}_2^*}}}{A \wedge B}$$

$\wedge\Rightarrow$

$$\wedge\Rightarrow\ \frac{\overset{\mathtt{d}}{A \Rightarrow C}}{A \wedge B \Rightarrow C} \quad \rightsquigarrow \quad \wedge\mathrm{E}_1\ \frac{A \wedge B \qquad \overset{[A]^1}{\underset{C}{\vdots \mathtt{d}^*}}}{A \wedge B} \qquad C$$ [2]

$\Rightarrow\vee$

$$\Rightarrow\vee\ \frac{\overset{\mathtt{d}}{C \Rightarrow A}}{C \Rightarrow A \vee B} \quad \rightsquigarrow \quad \overset{C}{\vdots \mathtt{d}^*} \quad \vee\mathrm{I}\ \frac{A}{A \vee B}$$ [3]

$\vee\Rightarrow$

$$\vee\Rightarrow\ \frac{\overset{\mathtt{d}_1}{A \Rightarrow C} \qquad \overset{\mathtt{d}_2}{B \Rightarrow C}}{A \vee B \Rightarrow C} \quad \rightsquigarrow \quad \vee\mathrm{E}\ \frac{A \vee B \qquad \overset{[A]}{\underset{C}{\vdots \mathtt{d}_1^*}} \qquad \overset{[B]}{\underset{C}{\vdots \mathtt{d}_2^*}}}{C}$$

[1] Technically speaking we could also have $\vdash_{SASCNJDJ} A$ and $\vdash_{SASCNJDJ} B$, but this is not a problem at all.

[2] Other case is symmetrical.

[3] Other case is symmetrical.

$\Rightarrow\supset$ $\quad \dfrac{\overset{\mathtt{d}}{A \Rightarrow B}}{\Rightarrow A \supset B}\Rightarrow\supset \quad \rightsquigarrow \quad \dfrac{\begin{matrix}[A]^1 \\ \vdots \mathtt{d}^* \\ B\end{matrix}}{A \supset B}\supset\mathrm{I}_1$ [4]

$\supset\Rightarrow$ $\quad \dfrac{\overset{\mathtt{d}_1}{\Rightarrow A} \qquad \overset{\mathtt{d}_2}{B \Rightarrow C}}{A \supset B \Rightarrow C}\supset\Rightarrow \quad \rightsquigarrow \quad \dfrac{A \supset B \qquad \begin{matrix}\varnothing \\ \vdots \mathtt{d}_1^* \\ A\end{matrix} \qquad \begin{matrix}[B]^1 \\ \vdots \mathtt{d}_2^* \\ C\end{matrix}}{C}\supset\mathrm{E}_1$

$\Rightarrow\neg$ $\quad \dfrac{\overset{\mathtt{d}}{A \Rightarrow}}{\Rightarrow \neg A}\Rightarrow\neg \quad \rightsquigarrow \quad \dfrac{\begin{matrix}[A]^1 \\ \vdots \mathtt{d}^* \\ \bot\end{matrix}}{\neg A}\neg\mathrm{I}_1$

$\neg\Rightarrow$ $\quad \dfrac{\overset{\mathtt{d}}{\Rightarrow A}}{\neg A \Rightarrow}\neg\Rightarrow \quad \rightsquigarrow \quad \dfrac{\dfrac{\neg A \qquad \begin{matrix}\varnothing \\ \vdots \mathtt{d}^* \\ A\end{matrix}}{\bot}\neg\mathrm{E}}{C}$

$\Rightarrow Weak$ $\quad \dfrac{\overset{\mathtt{d}}{C \Rightarrow}}{C \Rightarrow A}\Rightarrow Weak \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\vdots \mathtt{d}^* \\ \bot\end{matrix}}{C}Efq$

$Weak\Rightarrow$ $\quad \dfrac{\overset{\mathtt{d}}{\Rightarrow A}}{C \Rightarrow A}Weak\Rightarrow \quad \rightsquigarrow \quad \begin{matrix}\varnothing \\ \vdots \mathtt{d}^* \\ A\end{matrix}$

$\Rightarrow\bot$ $\quad \dfrac{\overset{\mathtt{d}}{A \Rightarrow \bot}}{A \Rightarrow}\Rightarrow\bot \quad \rightsquigarrow \quad \begin{matrix}A \\ \vdots \mathtt{d}^* \\ \bot\end{matrix}$ Note that this conclusion is all we need, according to the statement of the theorem.

Cut Since it is an admissible rule, it is not necessary to translate it (theorem 3.1.2). Nonetheless, for completeness we expose its translation.

$$\dfrac{\overset{\mathtt{d}_1}{C \Rightarrow A} \qquad \overset{\mathtt{d}_2}{A \Rightarrow D}}{C \Rightarrow D}Cut \quad \rightsquigarrow \quad \begin{matrix}C \\ \vdots \mathtt{d}_1^* \\ A \\ \vdots \mathtt{d}_2^* \\ D\end{matrix}$$

It is important to note that in order to translate cut-free derivations of **SASCLJDJ** in derivations of **SASCNJDJ** we used only E-rules with assumed major premises. *A fortiori* there are no major premises of E-rules derived by I-rules in our **SASCNJDJ** derivations. This observation will be used in the proof of theorem 3.1.4.

□

Let us now define a translation from **SASCNJDJ** to **SASCLJDJ**.

Proof. By induction on the length of the derivation $\mathtt{d}$ of **SASCNJDJ**, we define the derivation $\mathtt{d}^*$ of **SASCLJDJ**:

Base: To the assumption A ($\bot$) in **SASCNJDJ**, it corresponds the axiom $A \Rightarrow A$ ($\bot \Rightarrow \bot$) of **SASCLJDJ**;

Step: By case on the last rule applied in the derivation $\mathtt{d}$ of $A \vdash_{SASCNJDJ} B$:

$\wedge\mathbf{I}$ $\quad \dfrac{\begin{matrix}D \\ \vdots \mathtt{d}_1 \\ C\end{matrix} \qquad \begin{matrix}[C]^1 \\ \vdots \mathtt{d}_2 \\ A\end{matrix} \qquad \begin{matrix}[C]^2 \\ \vdots \mathtt{d}_3 \\ B\end{matrix}}{A \wedge B}\wedge\mathrm{I}_{1,2} \quad \rightsquigarrow \quad \dfrac{\overset{\mathtt{d}_1^*}{D \Rightarrow C} \qquad \dfrac{\overset{\mathtt{d}_2^*}{C \Rightarrow A} \qquad \overset{\mathtt{d}_3^*}{C \Rightarrow B}}{C \Rightarrow A \wedge B}\Rightarrow\wedge}{C \Rightarrow A \wedge B}Cut$

[4]If we have $\vdash_{SASCLJDJ} \Rightarrow B$ and $\vdash_{SASCNJDJ} B$, there are no problems, since **SASCNJDJ** allows vacuous dischargement.

∧E

$$\begin{array}{c} C \\ \vdots\mathtt{d} \\ \dfrac{A \wedge B}{A}\,{\scriptstyle \wedge\mathrm{E}} \end{array} \quad\rightsquigarrow\quad \dfrac{\overset{\mathtt{d}^*}{C \Rightarrow A \wedge B} \qquad \dfrac{A \Rightarrow A}{A \wedge B \Rightarrow A}\,{\scriptstyle \wedge\Rightarrow}}{C \Rightarrow A}\,{\scriptstyle Cut}$$ [5]

∨I

$$\begin{array}{c} C \\ \vdots\mathtt{d} \\ \dfrac{A}{A \vee B}\,{\scriptstyle \vee\mathrm{I}} \end{array} \quad\rightsquigarrow\quad \dfrac{\overset{\mathtt{d}^*}{C \Rightarrow A}}{C \Rightarrow A \vee B}\,{\scriptstyle \Rightarrow\vee}$$ [6]

∨E

$$\begin{array}{c} \begin{array}{ccc} D & [A] & [B] \\ \vdots\mathtt{d}_1 & \vdots\mathtt{d}_2 & \vdots\mathtt{d}_3 \end{array} \\ \dfrac{A \vee B \qquad C \qquad C}{C}\,{\scriptstyle \vee\mathrm{E}} \end{array} \quad\rightsquigarrow\quad \dfrac{\overset{\mathtt{d}_1^*}{D \Rightarrow A \vee B} \qquad \dfrac{\overset{\mathtt{d}_2^*}{A \Rightarrow C} \qquad \overset{\mathtt{d}_3^*}{B \Rightarrow C}}{A \vee B \Rightarrow C}\,{\scriptstyle \vee\Rightarrow}}{D \Rightarrow C}\,{\scriptstyle Cut}$$

⊃I

$$\begin{array}{c} [A]^1 \\ \vdots\mathtt{d} \\ \dfrac{B}{A \supset B}\,{\scriptstyle \supset\mathrm{I}_1} \end{array} \quad\rightsquigarrow\quad \dfrac{\overset{\mathtt{d}^*}{A \Rightarrow B}}{\Rightarrow A \supset B}\,{\scriptstyle \Rightarrow\supset}$$

⊃E

$$\begin{array}{c} \begin{array}{ccc} D & \varnothing & [B]^1 \\ \vdots\mathtt{d}_1 & \vdots\mathtt{d}_2 & \vdots\mathtt{d}_3 \end{array} \\ \dfrac{A \supset B \qquad A \qquad C}{C}\,{\scriptstyle \supset\mathrm{E}_1} \end{array} \quad\rightsquigarrow\quad \dfrac{\overset{\mathtt{d}_1^*}{D \Rightarrow A \supset B} \qquad \dfrac{\overset{\mathtt{d}_2^*}{\Rightarrow A} \qquad \overset{\mathtt{d}_3^*}{B \Rightarrow C}}{A \supset B \Rightarrow C}\,{\scriptstyle \supset\Rightarrow}}{D \Rightarrow C}\,{\scriptstyle Cut}$$

¬I

$$\begin{array}{c} [A]^1 \\ \vdots\mathtt{d} \\ \dfrac{\bot}{\neg A}\,{\scriptstyle \supset\mathrm{I}_1} \end{array} \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\mathtt{d}^*}{A \Rightarrow \bot}}{A \Rightarrow}\,{\scriptstyle \Rightarrow\bot}}{\Rightarrow \neg A}\,{\scriptstyle \Rightarrow\neg}$$

¬E

$$\begin{array}{c} \begin{array}{cc} C & \varnothing \\ \vdots\mathtt{d}_1 & \vdots\mathtt{d}_2 \end{array} \\ \dfrac{\neg A \qquad A}{\bot}\,{\scriptstyle \neg\mathrm{E}} \end{array} \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\mathtt{d}_1^*}{C \Rightarrow \neg A} \qquad \dfrac{\overset{\mathtt{d}_2^*}{\Rightarrow A}}{\neg A \Rightarrow}\,{\scriptstyle \neg\Rightarrow}}{C \Rightarrow}\,{\scriptstyle Cut}}{C \Rightarrow \bot}\,{\scriptstyle \Rightarrow Weak}$$

Efq

$$\begin{array}{c} A \\ \vdots \\ \dfrac{\bot}{C}\,{\scriptstyle Efq} \end{array} \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\mathtt{d}^*}{A \Rightarrow \bot}}{A \Rightarrow}\,{\scriptstyle \Rightarrow\bot}}{A \Rightarrow C}\,{\scriptstyle \Rightarrow Weak}$$

□

B.1.2 Intuitionist systems

Equivalence between SASCLJ and LJ

First of all, let us consider the proof of Associativity, Commutativity and Idempotence of conjunction in **SASCLJ**.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \wedge B \Rightarrow A}\,{\scriptstyle \wedge\Rightarrow}}{(A \wedge B) \wedge C \Rightarrow A}\,{\scriptstyle \wedge\Rightarrow} \qquad \dfrac{\dfrac{\dfrac{B \Rightarrow B}{A \wedge B \Rightarrow B}\,{\scriptstyle \wedge\Rightarrow}}{(A \wedge B) \wedge C \Rightarrow B}\,{\scriptstyle \wedge\Rightarrow} \qquad \dfrac{C \Rightarrow C}{(A \wedge B) \wedge C \Rightarrow C}\,{\scriptstyle \wedge\Rightarrow}}{(A \wedge B) \wedge C \Rightarrow B \wedge C}\,{\scriptstyle \Rightarrow\wedge}}{(A \wedge B) \wedge C \Rightarrow A \wedge (B \wedge C)}\,{\scriptstyle \Rightarrow\wedge} \qquad (E \wedge (A \wedge (B \wedge C))) \wedge F \Rightarrow D}{(E \wedge (A \wedge B) \wedge C) \wedge F \Rightarrow D}\,{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \wedge (B \wedge C) \Rightarrow A}\,{\scriptstyle \wedge\Rightarrow} \qquad \dfrac{\dfrac{B \Rightarrow B}{B \wedge C \Rightarrow B}\,{\scriptstyle \wedge\Rightarrow}}{A \wedge (B \wedge C) \Rightarrow B}\,{\scriptstyle \wedge\Rightarrow}}{A \wedge (B \wedge C) \Rightarrow A \wedge B}\,{\scriptstyle \Rightarrow\wedge} \qquad \dfrac{\dfrac{C \Rightarrow C}{B \wedge C \Rightarrow C}\,{\scriptstyle \wedge\Rightarrow}}{A \wedge (B \wedge C) \Rightarrow C}\,{\scriptstyle \wedge\Rightarrow}}{A \wedge (B \wedge C) \Rightarrow (A \wedge B) \wedge C}\,{\scriptstyle \Rightarrow\wedge} \qquad (E \wedge ((A \wedge B) \wedge C)) \wedge F \Rightarrow D}{(E \wedge (A \wedge (B \wedge C))) \wedge F \Rightarrow D}\,{\scriptstyle Cut}$$

[5] Other case is symmetrical.

[6] Other case is symmetrical.

$$\dfrac{\dfrac{A \Rightarrow A \qquad A \Rightarrow A}{A \Rightarrow A \wedge A}{\scriptstyle \Rightarrow\wedge} \qquad C \wedge (A \wedge A) \Rightarrow D}{C \wedge A \Rightarrow D}{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\dfrac{D \Rightarrow D}{D \wedge ((B \wedge A) \wedge C) \Rightarrow (B \wedge A) \wedge C}{\scriptstyle \wedge\Rightarrow} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{B \wedge A \Rightarrow A}{\scriptstyle \wedge\Rightarrow} \quad \dfrac{B \Rightarrow B}{B \wedge A \Rightarrow A}{\scriptstyle \wedge\Rightarrow}}{B \wedge A \Rightarrow A \wedge B}{\scriptstyle \Rightarrow\wedge} \quad \dfrac{C \Rightarrow C}{(B \wedge A) \wedge C \Rightarrow C}{\scriptstyle \wedge\Rightarrow}}{(B \wedge A) \wedge C \Rightarrow (A \wedge B) \wedge C}{\scriptstyle \Rightarrow\wedge}}{D \wedge ((B \wedge A) \wedge C) \Rightarrow (A \wedge B) \wedge C}{\scriptstyle \wedge\Rightarrow}}{D \wedge ((B \wedge A) \wedge C) \Rightarrow D \wedge ((A \wedge B) \wedge C)}{\scriptstyle \Rightarrow\wedge} \qquad D \wedge ((A \wedge B) \wedge C) \Rightarrow E}{D \wedge ((B \wedge A) \wedge C) \Rightarrow E}{\scriptstyle Cut}$$

We will abbreviate them with:

$$(As_1 \Rightarrow)^* \dfrac{(E \wedge (A \wedge (B \wedge C))) \wedge F \Rightarrow D}{(E \wedge (A \wedge B) \wedge C) \wedge F \Rightarrow D} \qquad (As_2 \Rightarrow)^* \dfrac{(E \wedge ((A \wedge B) \wedge C)) \wedge F \Rightarrow D}{(E \wedge (A \wedge (B \wedge C))) \wedge F \Rightarrow D}$$

$$(Idem \Rightarrow)^* \dfrac{C \wedge (A \wedge A) \Rightarrow D}{C \wedge A \Rightarrow D} \qquad (Comm \Rightarrow)^* \dfrac{D \wedge ((A \wedge B) \wedge C) \Rightarrow E}{D \wedge ((B \wedge A) \wedge C) \Rightarrow E}$$

Definition B.1.1. A **SASCLJ**-derivation is semi-Cut-free iff all its applications of Cut are in an occurrence of $(As \Rightarrow)^*$, $(Idem \Rightarrow)^*$ or $(Comm \Rightarrow)^*$.

Antecedent and succedent in **LJ** are lists of formulae, while antecedent and succedent of **SASCLJ** are composed by just one formula. Nonetheless in the translation we will frequently associate a derivation in **LJ** with more derivations in **SASCLJ**, since this system is much more precise in the representation of the logical structure of the antecedent and succedent. For this reason we will frequently use **SASCLJ** as if it were a system of derivation for sets of formulae, with a little notational abuse.

Theorem B.1.2 (Equivalence between **LJ** and **SASCLJ**). *Sequent calculi **LJ** and **SASCLJ** are equivalent to each other, that is:*

1. *If* $\vdash_{LJ} \Gamma \Rightarrow C$, *then* $\vdash_{SASCLJ} \Gamma^\wedge \Rightarrow C$;
2. *If* $\vdash_{SASCLJ} D \Rightarrow C$, *then* $\vdash_{LJ} D^\circ \Rightarrow C^\circ$.

Let us start from the translation *1* from **LJ** to **SASCLJ**.

Proof. By induction on the length of the derivation $\mathtt{d}$ of **LJ**, we define the equivalent derivation $\mathtt{d}^*$ of **SASCLJ**. Let us remember that, by definition of $\Gamma^\wedge$ we want to derive the end-sequent no matter how the conjunctions in its antecedent are associated. Of course this means that by inductive hypothesis we will have a derivation of a sequent no matter how the conjunctions in its antecedent are associated, so the translation is not a function, but a relation. With a little notational abuse, we will write the derivation in **SASCLJ** using $\Gamma^\wedge$ instead of its elements. In this way we can deal with the translation as if it were a function.

Base: If the proof of **LJ** is just an application of the Axiom $\vdash_{LJ} C \Rightarrow C$, then the same conclusion can be proved using the Axiom of **SASCLJ**, since $\{C\}^\wedge = C$.

Step: By cases on the last rule applied:

Step $\Rightarrow \wedge$:

$$\Rightarrow\wedge \dfrac{\begin{matrix}\mathtt{d}_1 \\ \Gamma \Rightarrow_{LJ} A\end{matrix} \qquad \begin{matrix}\mathtt{d}_2 \\ \Gamma \Rightarrow_{LJ} B\end{matrix}}{\Gamma \Rightarrow_{LJ} A \wedge B} \quad \rightsquigarrow \quad \Rightarrow\wedge \dfrac{\begin{matrix}\mathtt{d}_1^* \\ \Gamma^\wedge \Rightarrow_{SASCLJ} A\end{matrix} \qquad \begin{matrix}\mathtt{d}_2^* \\ \Gamma^\wedge \Rightarrow_{SASCLJ} B\end{matrix}}{\Gamma^\wedge \Rightarrow_{SASCLJ} A \wedge B}$$

In this case there is no need to apply $(As \Rightarrow)^*$ to the end, since we do not modify the antecedent. We already have the conclusion in all its generality by the generality of the inductive hypothesis.

Step $\wedge \Rightarrow$:

$$\wedge\Rightarrow \dfrac{\begin{matrix}\mathtt{d} \\ \Gamma, A \Rightarrow_{LJ} C\end{matrix}}{\Gamma, A \wedge B \Rightarrow_{LJ} C} \quad \rightsquigarrow \quad \text{Eventually several } (As \Rightarrow)^* \dfrac{\wedge\Rightarrow \dfrac{\begin{matrix}\mathtt{d}^* \\ \Gamma^\wedge \wedge A \Rightarrow_{SASCLJ} C\end{matrix}}{\Gamma^\wedge \wedge (A \wedge B) \Rightarrow_{SASCLJ} C}}{(\Gamma \cup \{A \wedge B\})^\wedge \Rightarrow_{SASCLJ} C}$$

I use $\Gamma^\wedge \wedge A$ to indicate the set of all the conjunctions $\delta \wedge A$ where $\delta \in \Gamma^\wedge$. In this way, $\Gamma^\wedge \wedge A \Rightarrow C$ is used to indicate that every sequent $\delta \wedge A \Rightarrow C$ is **SASCLJ**-derivable. Inductive hypothesis allows the derivation of $\gamma \Rightarrow C$ for every $\gamma \in (\Gamma \cup \{A\})^\wedge$. So our top-sequent is justified, since $\Gamma^{\wedge} \wedge A \subsetneq (\Gamma \cup \{A\})^{\wedge}$. The several applications of $(As \Rightarrow)^*$ are eventually used to derive all the elements of $(\Gamma \cup \{A \wedge B\})^{\wedge}$. Indeed $\Gamma^{\wedge} \wedge (A \wedge B) \subsetneq (\Gamma \cup \{A \wedge B\})^{\wedge}$.[7]

Step $\Rightarrow \vee$:

$$\Rightarrow\vee\ \dfrac{\overset{\mathtt{d}}{\Gamma \Rightarrow_{LJ} A}}{\Gamma \Rightarrow_{LJ} A \vee B} \quad \rightsquigarrow \quad \Rightarrow\vee\ \dfrac{\overset{\mathtt{d}^*}{\Gamma^{\wedge} \Rightarrow_{SASCLJ} A}}{\Gamma^{\wedge} \Rightarrow_{SASCLJ} A \vee B}$$ [8]

Step $\vee \Rightarrow$:

$$\vee\Rightarrow\ \dfrac{\overset{\mathtt{d}_1}{\Gamma, A \Rightarrow_{LJ} C} \qquad \overset{\mathtt{d}_2}{\Gamma, B \Rightarrow_{LJ} C}}{\Gamma, A \vee B \Rightarrow_{LJ} C}$$

$\rightsquigarrow$

$$\text{Eventually several } (As \Rightarrow)^*\ \dfrac{\vee\Rightarrow\ \dfrac{\overset{\mathtt{d}_1^*}{\Gamma^{\wedge} \wedge A \Rightarrow_{SASCLJ} C} \qquad \overset{\mathtt{d}_2^*}{\Gamma^{\wedge} \wedge B \Rightarrow_{SASCLJ} C}}{\Gamma^{\wedge} \wedge (A \vee B) \Rightarrow_{SASCLJ} C}}{(\Gamma \cup \{A \vee B\})^{\wedge} \Rightarrow_{SASCLJ} C}$$

As for $\wedge \Rightarrow$, we can derive $\Gamma^{\wedge} \wedge A \Rightarrow C$ from inductive hypothesis, even though to be precise it would be enough to derive a more general result.

Step $\Rightarrow\supset$:

$$\Rightarrow\supset\ \dfrac{\overset{\mathtt{d}}{\Gamma, A \Rightarrow_{LJ} B}}{\Gamma \Rightarrow_{LJ} A \supset B} \quad \rightsquigarrow \quad \Rightarrow\supset\ \dfrac{\overset{\mathtt{d}^*}{\Gamma^{\wedge} \wedge A \Rightarrow_{SASCLJ} B}}{\Gamma^{\wedge} \Rightarrow_{SASCLJ} A \supset B}$$

As for $\wedge \Rightarrow$, we can derive $(\Gamma \cup \{A\})^{\wedge} \Rightarrow B$ by inductive hypothesis and $\Gamma^{\wedge} \wedge A \Rightarrow B$ is just a special case of it.[9]

Step $\supset\Rightarrow$:

$$\supset\Rightarrow\ \dfrac{\overset{\mathtt{d}_1}{\Gamma \Rightarrow_{LJ} A} \qquad \overset{\mathtt{d}_2}{\Delta, B \Rightarrow_{LJ} C}}{\Gamma, \Delta, A \supset B \Rightarrow_{LJ} C}$$

$\rightsquigarrow$

$$\text{Ev. sev. } (As \Rightarrow)^*\ \dfrac{\supset\Rightarrow\ \dfrac{\overset{\mathtt{d}_1^*}{\Gamma^{\wedge} \Rightarrow_{SASCLJ} A} \qquad \overset{\mathtt{d}_2^*}{\Delta^{\wedge} \wedge B \Rightarrow_{SASCLJ} C}}{(\Gamma^{\wedge} \wedge \Delta^{\wedge}) \wedge (A \supset B) \Rightarrow_{SASCLJ} C}}{(\Gamma \cup \Delta \cup \{A \supset B\})^{\wedge} \Rightarrow_{SASCLJ} C}$$

Step $\Rightarrow \neg$:

$$\Rightarrow\neg\ \dfrac{\overset{\mathtt{d}}{\Gamma, A \Rightarrow_{LJ}}}{\Gamma \Rightarrow_{LJ} \neg A} \quad \rightsquigarrow \quad \Rightarrow\neg\ \dfrac{\overset{\mathtt{d}^*}{\Gamma^{\wedge} \wedge A \Rightarrow_{SASCLJ}}}{\Gamma^{\wedge} \Rightarrow_{SASCLJ} \neg A}$$

Step $\neg \Rightarrow$:

$$\neg\Rightarrow\ \dfrac{\overset{\mathtt{d}}{\Gamma \Rightarrow_{LJ} A}}{\Gamma, \neg A \Rightarrow_{LJ}} \quad \rightsquigarrow \quad \text{Ev. sev. } (As \Rightarrow)^*\ \dfrac{\neg\Rightarrow\ \dfrac{\overset{\mathtt{d}^*}{\Gamma^{\wedge} \Rightarrow_{SASCLJ} A}}{\Gamma^{\wedge} \wedge \neg A \Rightarrow_{SASCLJ}}}{(\Gamma \cup \{\neg A\})^{\wedge} \Rightarrow_{SASCLJ} C}$$

Step $\Rightarrow$ *Weak*: If the succedent of the premise is empty, we use the homologous rule of **SASCLJ**. Otherwise we use $\Rightarrow \vee$ and eventually several applications of $(As \Rightarrow)^*$.

Step *Weak* $\Rightarrow$: If the antecedent of the premise is empty, we use the homologous rule of **SASCLJ**.

Otherwise:

$$Weak\Rightarrow\ \dfrac{\overset{\mathtt{d}}{\Gamma \Rightarrow_{LJ} C}}{\Gamma, A \Rightarrow_{LJ} C} \quad \rightsquigarrow \quad \text{Ev. sev. } (As \Rightarrow)^*\ \dfrac{\wedge\Rightarrow\ \dfrac{\overset{\mathtt{d}^*}{\Gamma^{\wedge} \Rightarrow_{SASCLJ} C}}{\Gamma^{\wedge} \wedge A \Rightarrow_{SASCLJ} C}}{(\Gamma \cup \{A\})^{\wedge} \Rightarrow_{SASCLJ} C}$$

Step *Con* $\Rightarrow$:

$$Con\Rightarrow\ \dfrac{\overset{\mathtt{d}}{\Gamma, A, A \Rightarrow_{LJ} C}}{\Gamma, A \Rightarrow_{LJ} C} \quad \rightsquigarrow \quad \text{Ev. sev. } (As \Rightarrow)^*\ \dfrac{(Idem \Rightarrow)^*\ \dfrac{\overset{\mathtt{d}^*}{\Gamma^{\wedge} \wedge (A \wedge A) \Rightarrow_{SASCLJ} C}}{\Gamma^{\wedge} \wedge A \Rightarrow_{SASCLJ} C}}{(\Gamma \cup \{A\})^{\wedge} \Rightarrow_{SASCLJ} C}$$

Step *Per* $\Rightarrow$:

$$Per\Rightarrow\ \dfrac{\overset{\mathtt{d}}{\Gamma, A, B, \Delta \Rightarrow_{LJ} C}}{\Gamma, B, A, \Delta \Rightarrow_{LJ} C} \quad \rightsquigarrow \quad \text{Ev. sev. } (As \Rightarrow)^*\ \dfrac{(Comm \Rightarrow)^*\ \dfrac{\overset{\mathtt{d}^*}{\Gamma^{\wedge} \wedge ((A \wedge B) \wedge \Delta^{\wedge}) \Rightarrow_{SASCLJ} C}}{\Gamma^{\wedge} \wedge ((B \wedge A) \wedge \Delta^{\wedge}) \Rightarrow_{SASCLJ} C}}{(\Gamma \cup \{B \wedge A\} \cup \Delta)^{\wedge} \Rightarrow_{SASCLJ} C}$$

[7] Other case is symmetrical.
[8] Other case is symmetrical.
[9] From now on, we will no more specify this point.

Step ***Cut*****:** Cut is admissible in **LJ**,[10] so we do not need to translate it.

In the proof that all sequents provable in **LJ** are provable in **SASCLJ** we do not need any application of Cut apart from these in ($As \Rightarrow$)*, ($Idem \Rightarrow$)* and ($Comm \Rightarrow$)*. So the semi-Cut-free fragment of **SASCLJ** is enough strong to derive all intuitionistically valid sequents.

□

Let us now consider the part *2* of the theorem.

Proof. The proof is by induction on the length of the derivation in **SASCLJ**, and by cases on the last rule applied.

Base: If the proof of **SASCLJ** is just an application of the Axiom $\vdash_{SASCLJ} C \Rightarrow C$ (or $\vdash_{SASCLJ} \bot \Rightarrow \bot$), then the same conclusion can be proved using the Axiom of **LJ**: $\vdash_{LJ} C \Rightarrow C$ (or $\vdash_{LJ} \neg(E \supset E) \Rightarrow \neg(E \supset E)$).

Steps $\Rightarrow\wedge$, $\wedge\Rightarrow$, $\Rightarrow\vee$ and *Weak*: The rule of **SASCLJ** is a particular case of the homologous rule of **LJ**.

Step $\vee\Rightarrow$: The derivation of **SASCLJ** ends with: $\vee\Rightarrow \dfrac{A \wedge C \ \Rightarrow_{SASCLJ}\ D \qquad B \wedge C \ \Rightarrow_{SASCLJ}\ D}{(A \vee B) \wedge C \ \Rightarrow_{SASCLJ}\ D}$

By inductive hypothesis we have that in **LJ** $A \wedge C \Rightarrow D$ and $B \wedge C \Rightarrow D$ are provable, so we obtain the conclusion:

$$Cut\ \dfrac{\Rightarrow\wedge\ \dfrac{Weak\Rightarrow \dfrac{A \ \Rightarrow_{LJ}\ A}{A, C \ \Rightarrow_{LJ}\ A} \qquad Weak\Rightarrow \dfrac{C \ \Rightarrow_{LJ}\ C}{A, C \ \Rightarrow_{LJ}\ C}}{A, C \ \Rightarrow_{LJ}\ A \wedge C} \qquad A \wedge C \ \Rightarrow_{LJ}\ D}{A, C \ \Rightarrow_{LJ}\ D}$$

$$\vdots$$

$$Cont\Rightarrow \dfrac{\wedge\Rightarrow \dfrac{\wedge\Rightarrow \dfrac{\vee\Rightarrow \dfrac{\vdots \qquad Cut\ \dfrac{\Rightarrow\wedge\ \dfrac{Weak\Rightarrow \dfrac{B \ \Rightarrow_{LJ}\ B}{B, C \ \Rightarrow_{LJ}\ B} \qquad Weak\Rightarrow \dfrac{C \ \Rightarrow_{LJ}\ C}{B, C \ \Rightarrow_{LJ}\ C}}{B, C \ \Rightarrow_{LJ}\ B \wedge C} \qquad B \wedge C \ \Rightarrow_{LJ}\ D}{B, C \ \Rightarrow_{LJ}\ D}}{(A \vee B), C \ \Rightarrow_{LJ}\ D}}{(A \vee B) \wedge C, C \ \Rightarrow_{LJ}\ D}}{(A \vee B) \wedge C, (A \vee B) \wedge C \ \Rightarrow_{LJ}\ D}}{(A \vee B) \wedge C \ \Rightarrow_{LJ}\ D}$$

[11]

Step $\Rightarrow\supset$: In the non-trivial case, the derivation of **SASCLJ** ends with: $\Rightarrow\supset \dfrac{A \wedge C \ \Rightarrow_{SASCLJ}\ B}{C \ \Rightarrow_{SASCLJ}\ A \supset B}$

By inductive hypothesis we have that in **LJ** $A \wedge C \Rightarrow B$ is provable, so we obtain the conclusion:[12]

$$\Rightarrow\supset \dfrac{Cut\ \dfrac{\overset{\vdots}{A, C \ \Rightarrow_{LJ}\ A \wedge C} \qquad A \wedge C \ \Rightarrow_{LJ}\ B}{A, C \ \Rightarrow_{LJ}\ B}}{C \ \Rightarrow_{LJ}\ A \supset B}$$

Step $\supset\Rightarrow$: In the non-trivial case, the derivation of **SASCLJ** ends with:
$\supset\Rightarrow \dfrac{E \ \Rightarrow_{SASCLJ}\ A \qquad B \wedge D \ \Rightarrow_{SASCLJ}\ C}{(A \supset B) \wedge (D \wedge E) \ \Rightarrow_{SASCLJ}\ C}$ By inductive hypothesis we have that in **LJ** $E \Rightarrow A$ and $B \wedge D \Rightarrow C$ are provable, so we obtain the conclusion:[13]

$$\wedge\Rightarrow \text{ and } Cont\Rightarrow \dfrac{\supset\Rightarrow \dfrac{\overset{B \wedge D \ \Rightarrow_{LJ}\ C}{\overset{\vdots}{B, D \ \Rightarrow_{LJ}\ C}} \qquad E \ \Rightarrow_{LJ}\ A}{A \supset B, D, E \ \Rightarrow_{LJ}\ C}}{(A \supset B) \wedge (D \wedge E) \ \Rightarrow_{LJ}\ C}$$

[10] As established in [Gentzen, 1969b].
[11] If the curly brackets are empty, we have a trivial modification.
[12] We use the already established result $\vdash_{LJ} A, B \Rightarrow A \wedge B$.
[13] We use the already established derivation of $A, B \Rightarrow C$ from $A \wedge B \Rightarrow C$.

Step $\Rightarrow \neg$: In the non-trivial case, the derivation of **SASCLJ** ends with: $\Rightarrow \neg \dfrac{A \wedge C \ \Rightarrow_{SASCLJ}}{C \ \Rightarrow_{SASCLJ} \ \neg A}$
By inductive hypothesis we have that in **LJ** $A \wedge C \Rightarrow$ is provable, so we obtain the conclusion:

$$\begin{array}{c} A \wedge C \ \Rightarrow_{LJ} \\ \vdots \\ \Rightarrow \neg \ \dfrac{A, C \ \Rightarrow_{LJ}}{C \ \Rightarrow_{LJ} \ \neg A} \end{array}$$

Step $\neg \Rightarrow$: In the non-trivial case, the derivation of **SASCLJ** ends with: $\neg \Rightarrow \dfrac{C \ \Rightarrow_{SASCLJ} \ A}{\neg A \wedge C \ \Rightarrow_{SASCLJ}}$
By inductive hypothesis we have that in **LJ** $C \Rightarrow A$ is provable, so we obtain the conclusion:

$$\wedge \Rightarrow \text{ and } Cont \Rightarrow \dfrac{\neg \Rightarrow \dfrac{C \ \Rightarrow_{LJ} \ A}{\neg A, C \ \Rightarrow_{LJ}}}{\neg A \wedge C \ \Rightarrow_{LJ}}$$

Step $\Rightarrow \bot$: The derivation of **SASCLJ** ends with: $\Rightarrow \bot \dfrac{A \ \Rightarrow_{SASCLJ} \ \bot}{A \ \Rightarrow_{SASCLJ}}$ By inductive hypothesis we have that in **LJ** $A \Rightarrow \neg(E \supset E)$ is provable, so we obtain the conclusion:

$$Cut \ \dfrac{A \ \Rightarrow_{LJ} \ \neg(E \supset E) \qquad \neg \Rightarrow \dfrac{\Rightarrow\supset \dfrac{E \ \Rightarrow_{LJ} \ E}{\Rightarrow_{LJ} \ E \supset E}}{\neg(E \supset E) \ \Rightarrow_{LJ}}}{A \ \Rightarrow_{LJ}}$$

Step Cut: The derivation of **SASCLJ** ends with: $Cut \ \dfrac{C \ \Rightarrow_{SASCLJ} \ A \qquad (G \wedge A) \wedge H \ \Rightarrow_{SASCLJ} \ D}{(G \wedge C) \wedge H \ \Rightarrow_{SASCLJ} \ D}$
By inductive hypothesis we have that in **LJ** $C \Rightarrow A$ and $(G \wedge A) \wedge H \Rightarrow D$ are provable, so we obtain the conclusion:[14]

$$\begin{array}{c} (G \wedge A) \wedge H \ \Rightarrow_{LJ} \ D \\ \vdots \\ (G \wedge C), H \ \Rightarrow_{LJ} \ D \\ \vdots \end{array}$$

$$\wedge \Rightarrow \text{ and } Cont \Rightarrow \dfrac{Cut \ \dfrac{C \ \Rightarrow_{LJ} \ A \qquad G, C, H \ \Rightarrow_{LJ} \ D}{G, A, H \ \Rightarrow_{LJ} \ D}}{(G \wedge A) \wedge H \ \Rightarrow_{LJ} \ D}$$

□

Equivalence between SASCLJ and SASCNJ

Theorem B.1.3 (Equivalence between **SASCLJ** and **SASCNJ**). *The sequent calculus **SASCLJ** and the natural deduction system **SASCNJ** are equivalent to each other, that is:*

1. *(a) if $\vdash_{SASCLJ} A \Rightarrow B$, then:*
   - *Or $A \vdash_{SASCNJ} B$;*
   - *Or $\vdash_{SASCLJ} \Rightarrow B$ and $\vdash_{SASCNJ} B$.*

   (b) If $\vdash_{SASCLJ} A \Rightarrow$, then $A \vdash_{SASCNJ} \bot$.
2. *(a) If $A \vdash_{SASCNJ} B$ then $\vdash_{SASCLJ} A \Rightarrow B$.*

Let us start from the first point.

Proof. The proof is by induction on the length of the derivation in **SASCLJ**, and by cases on the last rule applied.

Base: The only derivation of just 1 step in **SASCLJ** is an application of the Axiom $A \Rightarrow A$ ($\bot \Rightarrow \bot$), to which we associate the assumption A ($\bot$) in **SASCNJ**;

Steps $\Rightarrow \wedge$, $\wedge \Rightarrow$, $\Rightarrow \vee$, $\Rightarrow \bot$, $\Rightarrow$ *Weak* and *Weak* $\Rightarrow$: We treat them as in the proof B.1.1.

[14] If there is no G or H, the proof remain valid; if there is none of them the rule is a special case of the corresponding rule of **LJ**.

Step $\vee\Rightarrow$: From $\vee\Rightarrow \dfrac{A \wedge C \Rightarrow D \qquad B \wedge C \Rightarrow D}{(A \vee B) \wedge C \Rightarrow D}$, we have $A \wedge C \vdash_{SASCNJ} D$ e $B \wedge C \vdash_{SASCNJ} D$ by induction. We then conclude:

$$\vee\text{E}\ \dfrac{(A \vee B) \wedge C \qquad \begin{matrix}[A \wedge C]\\ \vdots \\ D\end{matrix} \qquad \begin{matrix}[B \wedge C]\\ \vdots \\ D\end{matrix}}{D}$$

[15]

Step $\Rightarrow\supset$: From $\Rightarrow\supset \dfrac{A \wedge C \Rightarrow B}{C \Rightarrow A \supset B}$, we have $A \wedge C \vdash_{SASCNJ} B$ by induction. So, we can conclude:

$$\supset\text{I}_1\ \dfrac{C \qquad \begin{matrix}[A \wedge C]^1\\ \vdots \\ B\end{matrix}}{A \supset B}$$

Step $\supset\Rightarrow$: From $\supset\Rightarrow \dfrac{E \Rightarrow A \qquad B \wedge D \Rightarrow C}{(A \supset B) \wedge (D \wedge E) \Rightarrow C}$, we have $E \vdash_{SASCNJ} A$ and $B \wedge D \vdash_{SASCNJ} C$ by induction. So, we can conclude:

$$\supset\text{E}_{1,2}\ \dfrac{(A \supset B) \wedge (D \wedge E) \qquad \begin{matrix}[E]^1\\ \vdots \\ A\end{matrix} \qquad \begin{matrix}[B \wedge D]^2\\ \vdots \\ C\end{matrix}}{C}$$

Step $\Rightarrow\neg$: From $\Rightarrow\neg \dfrac{A \wedge C \Rightarrow}{C \Rightarrow \neg A}$, we have $A \wedge C \vdash_{SASCNJ} \bot$ by induction. So, we can conclude:

$$\neg\text{I}_1\ \dfrac{C \qquad \begin{matrix}[A \wedge C]^1\\ \vdots \\ \bot\end{matrix}}{\neg A}$$

Step $\neg\Rightarrow$: From $\neg\Rightarrow \dfrac{C \Rightarrow A}{\neg A \wedge C \Rightarrow}$, we have $\vdash_{SASCNJ} A$ by induction. So, we can conclude:

$$\neg\text{E}_1\ \dfrac{\neg A \wedge C \qquad \begin{matrix}[C]^1\\ \vdots \\ A\end{matrix}}{\bot}$$

Step *Cut*: From

$$Cut\ \dfrac{\begin{matrix}\vdots \mathtt{d}_1\\ C \Rightarrow A\end{matrix} \qquad \begin{matrix}\vdots \mathtt{d}_1\\ (F \wedge A) \wedge G \Rightarrow H\end{matrix}}{(F \wedge C) \wedge G \Rightarrow H}$$

we obtain:

$$\begin{matrix}\wedge I_2\ \dfrac{(F \wedge C) \wedge G \qquad \wedge I_1\ \dfrac{[(F \wedge C) \wedge G]^2 \qquad \wedge E\ \dfrac{[(F \wedge C) \wedge G]^1}{F} \qquad \begin{matrix}\wedge E\ \dfrac{[(F \wedge C) \wedge G]^1}{C}\\ \vdots \mathtt{d}_1^*\\ A\end{matrix}}{F \wedge A} \qquad \wedge E\ \dfrac{[(F \wedge C) \wedge G]^2}{G}}{(F \wedge A) \wedge G}\\ \vdots \mathtt{d}_2^*\\ H\end{matrix}$$

The other cases of Cut are easy variations of this or identical with that of **SASCLJDJ**.

□

Let us now turn to the second part of the theorem.

Proof. The proof is by induction on the length of the derivation in **SASCNJ**, and by cases on the last rule applied.

Base: To the assumption of A ($\bot$) in **SASCNJ**, we associate the Axiom $A \Rightarrow A$ ($\bot \Rightarrow \bot$) of **SASCLJ**;

Steps $\wedge$I, $\wedge$E, $\vee$I and Efq: We treat them as in the proof B.1.1.

[15] If there is not C, we behave as in proof B.1.1; in general we always deal in this way with derivations in which the curly brackets in the rules are empty.

Step ∨E: The derivation ends with: $$\vee\text{E}\,\dfrac{\begin{array}{c}E\\ \vdots\\ (A\vee B)\wedge C\end{array}\qquad\begin{array}{c}[A\wedge C]\\ \vdots\\ D\end{array}\qquad\begin{array}{c}[B\wedge C]\\ \vdots\\ D\end{array}}{D}$$, so by inductive hypothesis we have that in **SASCLJ** $A\wedge C\Rightarrow D$, $B\wedge C\Rightarrow D$ and $E\Rightarrow(A\vee B)\wedge C$ are provable. So, we conclude:

$$Cut\,\dfrac{E\ \Rightarrow\ (A\vee B)\wedge C\qquad \vee\Rightarrow\,\dfrac{A\wedge C\ \Rightarrow\ D\qquad B\wedge C\ \Rightarrow\ D}{(A\vee B)\wedge C\ \Rightarrow\ D}}{E\ \Rightarrow\ D}$$

Step ⊃I: The derivation ends with: $$\supset\text{I}_1\,\dfrac{\begin{array}{c}D\\ \vdots\\ C\end{array}\qquad\begin{array}{c}[A\wedge C]^1\\ \vdots\\ B\end{array}}{A\supset B}$$, so by inductive hypothesis we have that in **SASCLJ** $A\wedge C\Rightarrow B$ and $D\Rightarrow C$ are provable. So, we conclude:

$$Cut\,\dfrac{D\ \Rightarrow\ C\qquad \Rightarrow\supset\,\dfrac{A\wedge C\ \Rightarrow\ B}{C\ \Rightarrow\ A\supset B}}{D\ \Rightarrow\ A\supset B}$$

Step ⊃E: The derivation ends with: $$\supset\text{E}_{1,2}\,\dfrac{\begin{array}{c}D\\ \vdots\\ (A\supset B)\wedge(E\wedge F)\end{array}\qquad\begin{array}{c}[E]^1\\ \vdots\\ A\end{array}\qquad\begin{array}{c}[B\wedge F]^2\\ \vdots\\ C\end{array}}{C}$$, so by inductive hypothesis we have that in **SASCLJ** $D\Rightarrow(A\supset B)\wedge(E\wedge F)$, $E\Rightarrow A$ and $B\wedge F\Rightarrow C$ are provable. So, we conclude:

$$Cut\,\dfrac{D\ \Rightarrow\ (A\supset B)\wedge(E\wedge F)\qquad \supset\Rightarrow\,\dfrac{E\ \Rightarrow\ A\qquad B\wedge F\ \Rightarrow\ C}{(A\supset B)\wedge(E\wedge F)\ \Rightarrow\ C}}{D\ \Rightarrow\ C}$$

Step ¬I: The derivation ends with: $$\supset\text{I}_1\,\dfrac{\begin{array}{c}D\\ \vdots\\ C\end{array}\qquad\begin{array}{c}[A\wedge C]^1\\ \vdots\\ \bot\end{array}}{\neg A}$$, so by inductive hypothesis we have that in **SASCLJ** $A\wedge C\Rightarrow\bot$ and $D\Rightarrow C$ are provable. So, we conclude:

$$Cut\,\dfrac{D\ \Rightarrow\ C\qquad \Rightarrow\neg\,\dfrac{\Rightarrow\bot\,\dfrac{A\wedge C\ \Rightarrow\ \bot}{A\wedge C\ \Rightarrow}}{C\ \Rightarrow\ \neg A}}{D\ \Rightarrow\ \neg A}$$

Step ¬E: The derivation ends with: $$\neg\text{E}_1\,\dfrac{\begin{array}{c}D\\ \vdots\\ \neg A\wedge C\end{array}\qquad\begin{array}{c}[C]^1\\ \vdots\\ A\end{array}}{\bot}$$, so by inductive hypothesis we have that in **SASCLJ** $D\Rightarrow\neg A\wedge C$ and $C\Rightarrow A$ are provable. So, we conclude:

$$\Rightarrow Weak\,\dfrac{Cut\,\dfrac{D\ \Rightarrow\ \neg A\wedge C\qquad \neg\Rightarrow\,\dfrac{C\ \Rightarrow\ A}{\neg A\wedge C\ \Rightarrow}}{D\ \Rightarrow}}{D\ \Rightarrow\ \bot}$$

□

Let us now consider the proof of Associativity, Commutativity and Idempotence of conjunction in **SASCNJ**. Even though these do not correspond to primitive rules of **SASCLJ**, they will be very useful in the future.

$$\alpha\quad \dfrac{\dfrac{[\alpha]^4\qquad \wedge\text{E}\,\dfrac{[\alpha]^3}{E}\qquad \dfrac{[\alpha]^3\qquad \wedge\text{E}\,\dfrac{[\alpha]^2}{A}\qquad \wedge\text{I}_1\,\dfrac{[\alpha]^2\qquad \wedge\text{E}\,\dfrac{[\alpha]^1}{B}\qquad \wedge\text{E}\,\dfrac{[\alpha]^1}{C}}{B\wedge C}\,\wedge\text{I}_2}{A\wedge(B\wedge C)}\,\wedge\text{I}_3}{E\wedge(A\wedge(B\wedge C))}\qquad \wedge\text{E}\,\dfrac{[\alpha]^4}{F}}{\begin{array}{c}\beta\\ \vdots\\ D\end{array}}\,\wedge\text{I}_4$$

$$\dfrac{\beta \qquad \dfrac{[\beta]^4 \qquad \dfrac{[\beta]^3}{E}{\scriptstyle \wedge E} \qquad \dfrac{[\beta]^3 \qquad \dfrac{[\beta]^2 \qquad \dfrac{[\beta]^1}{A}{\scriptstyle \wedge E} \qquad \dfrac{[\beta]^1}{B}{\scriptstyle \wedge E}}{A \wedge B}{\scriptstyle \wedge I_1} \qquad \dfrac{[\beta]^2}{C}{\scriptstyle \wedge E}}{(A \wedge B) \wedge C}{\scriptstyle \wedge I_2}}{E \wedge ((A \wedge B) \wedge C)}{\scriptstyle \wedge I_3} \qquad \dfrac{[\beta]^4}{F}{\scriptstyle \wedge E}}{\alpha}{\scriptstyle \wedge I_4}$$

$$\vdots$$

$$D$$

Where $\alpha = (E \wedge ((A \wedge B) \wedge C)) \wedge F$ and $\beta = (E \wedge (A \wedge (B \wedge C))) \wedge F$. We will call the first derivation $(As_1 \Rightarrow)$** and the second $(As_2 \Rightarrow)$**.

$$\dfrac{\gamma \qquad \dfrac{[\gamma]^3}{C}{\scriptstyle \wedge E} \qquad \dfrac{[\gamma]^3 \qquad \dfrac{[\gamma]^2 \qquad \dfrac{[\gamma]^1}{A}{\scriptstyle \wedge E} \qquad \dfrac{[\gamma]^1}{B}{\scriptstyle \wedge E}}{A \wedge B}{\scriptstyle \wedge I_1} \qquad \dfrac{[\gamma]^2}{D}{\scriptstyle \wedge E}}{(A \wedge B) \wedge D}{\scriptstyle \wedge I_2}}{\delta}{\scriptstyle \wedge I_3}$$

$$\vdots$$

$$D$$

Where $\gamma = C \wedge ((B \wedge A) \wedge D)$ and $\delta = C \wedge ((A \wedge B) \wedge D)$. We call this derivation $(Comm \Rightarrow)$**.

$$\dfrac{A \wedge B \qquad \dfrac{[A \wedge B]^2}{A}{\scriptstyle \wedge E} \qquad \dfrac{\dfrac{[A \wedge B]^2}{B}{\scriptstyle \wedge E} \qquad [A \wedge B]^1 \qquad [A \wedge B]^1}{B \wedge B}{\scriptstyle \wedge I_1}}{A \wedge (B \wedge B)}{\scriptstyle \wedge I_2}$$

$$\vdots$$

$$D$$

We call this derivation $(Idem \Rightarrow)$**.

B.1.3 Classical systems

Equivalence between SASCLK and LK

First of all, let us consider the proof of Associativity, Commutativity and Idempotence of disjunction in **SASCLK**.

$$\dfrac{A \Rightarrow (E \vee ((B \vee C) \vee D)) \vee F \qquad \dfrac{\dfrac{\dfrac{B \Rightarrow B}{B \Rightarrow B \vee (C \vee D)}{\scriptstyle \Rightarrow \vee} \qquad \dfrac{C \Rightarrow C}{B \Rightarrow B \vee (C \vee D)}{\scriptstyle \Rightarrow \vee}}{B \vee C \Rightarrow B \vee (C \vee D)}{\scriptstyle \vee \Rightarrow} \qquad \dfrac{D \Rightarrow D}{D \Rightarrow B \vee (C \vee D)}{\scriptstyle \Rightarrow \vee}}{(B \vee C) \vee D \Rightarrow B \vee (C \vee D)}{\scriptstyle \vee \Rightarrow}}{A \Rightarrow (E \vee (B \vee (C \vee D))) \vee F}{\scriptstyle Cut}$$

$$\dfrac{A \Rightarrow (E \vee (B \vee (C \vee D))) \vee F \qquad \dfrac{\dfrac{B \Rightarrow B}{B \Rightarrow (B \vee C) \vee D}{\scriptstyle \Rightarrow \vee} \qquad \dfrac{\dfrac{C \Rightarrow C}{B \Rightarrow (B \vee C) \vee D}{\scriptstyle \Rightarrow \vee} \qquad \dfrac{D \Rightarrow D}{D \Rightarrow (B \vee C) \vee D}{\scriptstyle \Rightarrow \vee}}{C \vee D \Rightarrow (B \vee C) \vee D}{\scriptstyle \vee \Rightarrow}}{B \vee (C \vee D) \Rightarrow (B \vee C) \vee D}{\scriptstyle \vee \Rightarrow}}{A \Rightarrow (E \vee ((B \vee C) \vee D)) \vee F}{\scriptstyle Cut}$$

$$\dfrac{C \Rightarrow (A \vee A) \vee D \qquad \dfrac{A \Rightarrow A \qquad A \Rightarrow A}{A \vee A \Rightarrow A}{\scriptstyle \vee \Rightarrow}}{C \Rightarrow A \vee D}{\scriptstyle Cut}$$

$$
{\scriptstyle Cut}\,\dfrac{E \Rightarrow D \vee ((A \vee B) \vee C) \qquad {\scriptstyle \vee\Rightarrow}\,\dfrac{{\scriptstyle \Rightarrow\vee}\,\dfrac{D \Rightarrow D}{D \Rightarrow D \vee ((B \vee A) \vee C)} \qquad {\scriptstyle \Rightarrow\vee}\,\dfrac{{\scriptstyle \vee\Rightarrow}\,\dfrac{{\scriptstyle \Rightarrow\vee}\,\dfrac{{\scriptstyle \vee\Rightarrow}\,\dfrac{{\scriptstyle \Rightarrow\vee}\,\dfrac{A \Rightarrow A}{A \Rightarrow B \vee A} \qquad {\scriptstyle \Rightarrow\vee}\,\dfrac{B \Rightarrow B}{B \Rightarrow B \vee A}}{A \vee B \Rightarrow B \vee A}}{A \vee B \Rightarrow (B \vee A) \vee C} \qquad {\scriptstyle \Rightarrow\vee}\,\dfrac{C \Rightarrow C}{C \Rightarrow (B \vee A) \vee C}}{(A \vee B) \vee C \Rightarrow (B \vee A) \vee C}}{(A \vee B) \vee C \Rightarrow D \vee ((B \vee A) \vee C)}}{D \vee ((A \vee B) \vee C) \Rightarrow D \vee ((B \vee A) \vee C)}}{E \Rightarrow D \vee ((B \vee A) \vee C)}
$$

We will abbreviate them with:

$$
(\Rightarrow As_1)^*\,\dfrac{A \Rightarrow (E \vee ((B \vee C) \vee D)) \vee F}{A \Rightarrow (E \vee (B \vee (C \vee D))) \vee F} \qquad (\Rightarrow As_2)^*\,\dfrac{A \Rightarrow (E \vee (B \vee (C \vee D))) \vee F}{A \Rightarrow (E \vee ((B \vee C) \vee D)) \vee F}
$$

$$
(\Rightarrow Idem)^*\,\dfrac{C \Rightarrow (A \vee A) \vee D}{C \Rightarrow A \vee D} \qquad (\Rightarrow Comm)^*\,\dfrac{E \Rightarrow D \vee ((A \vee B) \vee C)}{E \Rightarrow D \vee ((B \vee A) \vee C)}
$$

Let us also consider the following proof of distributivity of disjunction over conjunction:

$$
{\scriptstyle (Comm\Rightarrow)^*}\,\dfrac{{\scriptstyle \vee\Rightarrow}\,\dfrac{{\scriptstyle (Comm\Rightarrow)^*}\,\dfrac{{\scriptstyle \vee\Rightarrow}\,\dfrac{A \wedge B \Rightarrow A \wedge B \qquad {\scriptstyle \wedge\Rightarrow}\,\dfrac{{\scriptstyle \Rightarrow\vee}\,\dfrac{C \Rightarrow C}{C \Rightarrow (A \wedge B) \vee C}}{C \wedge B \Rightarrow (A \wedge B) \vee C}}{(A \vee C) \wedge B \Rightarrow (A \wedge B) \vee C}}{B \wedge (A \vee C) \Rightarrow (A \wedge B) \vee C} \qquad {\scriptstyle (Comm\Rightarrow)^*}\,\dfrac{{\scriptstyle \vee\Rightarrow}\,\dfrac{{\scriptstyle \wedge\Rightarrow}\,\dfrac{{\scriptstyle \Rightarrow\vee}\,\dfrac{C \Rightarrow C}{C \Rightarrow (A \wedge B) \vee C}}{A \wedge C \Rightarrow (A \wedge B) \vee C} \qquad {\scriptstyle \wedge\Rightarrow}\,\dfrac{{\scriptstyle \Rightarrow\vee}\,\dfrac{C \Rightarrow C}{C \Rightarrow (A \wedge B) \vee C}}{C \wedge C \Rightarrow (A \wedge B) \vee C}}{(A \vee C) \wedge C \Rightarrow (A \wedge B) \vee C}}{C \wedge (A \vee C) \Rightarrow (A \wedge B) \vee C}}{(B \vee C) \wedge (A \vee C) \Rightarrow (A \wedge B) \vee C}}{(A \vee C) \wedge (B \vee C) \Rightarrow (A \wedge B) \vee C}
$$

We will then call $Distri^{\vee\wedge}$ the derivation:

$$
{\scriptstyle Cut}\,\dfrac{D \Rightarrow (A \vee C) \wedge (B \vee C) \qquad \begin{matrix}\vdots\\ (A \vee C) \wedge (B \vee C) \Rightarrow (A \wedge B) \vee C\end{matrix}}{D \Rightarrow (A \wedge B) \vee C}
$$

Theorem B.1.4 (Equivalence between **LK** and **SASCLK**)**.** *Sequent calculi **LK** and **SASCLK** are equivalent to each other, that is:*

1. *If* $\vdash_{LK} \Gamma \Rightarrow \Delta$*, then* $\vdash_{SASCLK} \Gamma^{\wedge} \Rightarrow \Delta^{\vee}$*;*
2. *If* $\vdash_{SASCLK} D \Rightarrow C$*, then* $\vdash_{LK} \Gamma^{\circ} \Rightarrow C^{\circ}$*.*

Let us start from translation 1 from **LK** to **SASCLK**.

Proof. By induction on the length of the derivation $\mathtt{d}$ of **LK**, we define the equivalent derivation $\mathtt{d}^*$ of **SASCLK**. Let us remember that, by definition of $\Gamma^{\wedge}$ and $\Delta^{\vee}$ we want to derive the end-sequent no matter how the conjunctions in its antecedent and the disjunctions in its succedent are associated. Of course this means that by inductive hypothesis we will have a derivation of a sequent no matter how the conjunctions in its antecedent and the disjunctions in its succedent are associated, so the translation is not a function, but a relation. With a little notational abuse, we will write the derivation in **SASCLJ** using $\Gamma^{\wedge}$ and $\Delta^{\vee}$ instead of their elements. In this way we can deal with the translation as if it were a function.

Base: If the proof of **LK** is just an application of the Axiom $\vdash_{LK} C \Rightarrow C$, then the same conclusion can be proved using the Axiom of **SASCLK**, since $\{C\}^{\wedge} = C$.

Step: By cases on the last rule applied:

Step $\Rightarrow\wedge$**:**

$$
{\scriptstyle \Rightarrow\wedge}\,\dfrac{\begin{matrix}\mathtt{d}_1\\ \Gamma \Rightarrow_{LK} A, \Delta\end{matrix} \qquad \begin{matrix}\mathtt{d}_2\\ \Gamma \Rightarrow_{LK} B, \Delta\end{matrix}}{\Gamma \Rightarrow_{LK} A \wedge B, \Delta}
$$

$\rightsquigarrow$

$$
{\scriptstyle (\Rightarrow As)^*}\,\dfrac{{\scriptstyle Distri^{\vee\wedge}}\,\dfrac{{\scriptstyle \Rightarrow\wedge}\,\dfrac{\begin{matrix}\mathtt{d}_1^*\\ \Gamma^{\wedge} \Rightarrow_{SASCLK} A \vee \Delta^{\vee}\end{matrix} \qquad \begin{matrix}\mathtt{d}_2^*\\ \Gamma^{\wedge} \Rightarrow_{SASCLK} B \vee \Delta^{\vee}\end{matrix}}{\Gamma^{\wedge} \Rightarrow_{SASCLK} (A \vee \Delta^{\vee}) \wedge (B \vee \Delta^{\vee})}}{\Gamma^{\wedge} \Rightarrow_{SASCLK} (A \wedge B) \vee \Delta^{\vee}}}{\Gamma^{\wedge} \Rightarrow_{SASCLK} (\{A \wedge B\} \cup \Delta)^{\vee}}
$$

Step $\wedge\Rightarrow$**:** Like in proof B.1.2.

Step $\Rightarrow \vee$:

$$\Rightarrow\vee\ \dfrac{\overset{\mathtt{d}}{\Gamma \Rightarrow_{LK} A, \Delta}}{\Gamma \Rightarrow_{LK} A \vee B, \Delta} \quad \rightsquigarrow \quad (\Rightarrow As)^*\ \dfrac{(\Rightarrow Comm)^*(\Rightarrow As)^*\ \dfrac{\Rightarrow\vee\ \dfrac{\overset{\mathtt{d}^*}{\Gamma^\wedge \Rightarrow_{SASCLK} A \vee \Delta^\vee}}{\Gamma^\wedge \Rightarrow_{SASCLK} (A \vee \Delta^\vee) \vee B}}{\Gamma^\wedge \Rightarrow_{SASCLK} (A \vee B) \vee \Delta^\vee}}{\Gamma^\wedge \Rightarrow_{SASCLK} (\{A \vee B\} \cup \Delta)^\vee}$$

[16]

I use $A \vee \Delta^\vee$ to indicate the set of all the disjunctions $A \vee \delta$ where $\delta \in \Delta^\vee$. In this way, $\Gamma^\wedge \Rightarrow A \vee \Delta^\vee$ is used to indicate that every sequent $\gamma \Rightarrow A \vee \delta$, where $\gamma \in Gamma^\wedge$, is **SASCLJ**-derivable. Inductive hypothesis allows the derivation of $\gamma \Rightarrow \eta$ for every $\gamma \in \Gamma^\wedge$ and $\eta \in (\{A\} \cup \Delta)^\vee$. So our top-sequent is justified, since $A \vee Delta^\vee \subsetneq (\{A\} \cup \Delta)^\vee$. The several applications of $(As \Rightarrow)^*$ are eventually used to derive all the elements of $(\{A \vee B\} \cup \Delta)^\vee$. Indeed $\Gamma^\wedge \wedge (A \wedge B) \subsetneq (\{A \vee B\} \cup \Delta)^\vee$.[17]

Step $\vee \Rightarrow$:

$$\vee\Rightarrow\ \dfrac{\overset{\mathtt{d}_1}{\Gamma, A \Rightarrow_{LK} \Delta} \qquad \overset{\mathtt{d}_2}{\Gamma, B \Rightarrow_{LK} \Delta}}{\Gamma, A \vee B \Rightarrow_{LK} \Delta}$$

$\rightsquigarrow$

$$\text{Eventually several } (As \Rightarrow)^*\ \dfrac{\vee\Rightarrow\ \dfrac{\overset{\mathtt{d}_1^*}{\Gamma^\wedge \wedge A \Rightarrow_{SASCLK} \Delta^\vee} \qquad \overset{\mathtt{d}_2^*}{\Gamma^\wedge \wedge B \Rightarrow_{SASCLK} \Delta^\vee}}{\Gamma^\wedge \wedge (A \vee B) \Rightarrow_{SASCLK} \Delta^\vee}}{(\Gamma \cup \{A \vee B\})^\wedge \Rightarrow_{SASCLK} \Delta^\vee}$$

We can derive $\Gamma^\wedge \wedge A \Rightarrow \Delta^\vee$ from inductive hypothesis, even though technically it would give a more general result.

Step $\Rightarrow\supset$:

$$\Rightarrow\supset\ \dfrac{\overset{\mathtt{d}}{\Gamma, A \Rightarrow_{LK} B, \Delta}}{\Gamma \Rightarrow_{LK} A \supset B, \Delta} \quad \rightsquigarrow \quad \Rightarrow\supset\ \dfrac{\overset{\mathtt{d}^*}{\Gamma^\wedge \wedge A \Rightarrow_{SASCLK} B \vee \Delta^\vee}}{\Gamma^\wedge \Rightarrow_{SASCLK} (A \supset B) \vee \Delta^\vee}$$

We can derive $(\Gamma \cup \{A\})^\wedge \Rightarrow (\{B\} \cup \Delta)^\vee$ by inductive hypothesis and $\Gamma^\wedge \wedge A \Rightarrow B \vee \Delta^\vee$ is just a special case of it.[18]

Step $\supset\Rightarrow$:

$$\supset\Rightarrow\ \dfrac{\overset{\mathtt{d}_1}{\Gamma \Rightarrow_{LK} A, \Delta} \qquad \overset{\mathtt{d}_2}{\Theta, B \Rightarrow_{LK} \Lambda}}{\Gamma, \Theta, A \supset B \Rightarrow_{LK} \Delta, \Lambda}$$

$\rightsquigarrow$

$$\text{Ev. sev. } (\Rightarrow As)^*\ \dfrac{\text{Ev. sev. } (As \Rightarrow)^*\ \dfrac{\supset\Rightarrow\ \dfrac{\overset{\mathtt{d}_1^*}{\Gamma^\wedge \Rightarrow_{SASCLK} A \vee \Delta^\vee} \qquad \overset{\mathtt{d}_2^*}{\Theta^\wedge \wedge B \Rightarrow_{SASCLK} \Lambda^\vee}}{(\Gamma^\wedge \wedge \Theta^\wedge) \wedge (A \supset B) \Rightarrow_{SASCLK} \Delta^\vee \vee \Lambda^\vee}}{(\Gamma \cup \Theta \cup \{A \supset B\})^\wedge \Rightarrow_{SASCLK} \Delta^\vee \vee \Lambda^\vee}}{(\Gamma \cup \Theta \cup \{A \supset B\})^\wedge \Rightarrow_{SASCLK} (\Delta \cup \Lambda)^\vee}$$

Step $\Rightarrow \neg$:

$$\Rightarrow\neg\ \dfrac{\overset{\mathtt{d}}{\Gamma, A \Rightarrow_{LK} \Delta}}{\Gamma \Rightarrow_{LK} \neg A, \Delta} \quad \rightsquigarrow \quad \text{Ev. sev. } (\Rightarrow As)^*\ \dfrac{\Rightarrow\neg\ \dfrac{\overset{\mathtt{d}^*}{\Gamma^\wedge \wedge A \Rightarrow_{SASCLK} \Delta^\vee}}{\Gamma^\wedge \Rightarrow_{SASCLK} \neg A \vee \Delta^\vee}}{\Gamma^\wedge \Rightarrow_{SASCLK} (\{\neg A\} \cup \Delta)^\vee}$$

Step $\neg \Rightarrow$:

$$\neg\Rightarrow\ \dfrac{\overset{\mathtt{d}}{\Gamma \Rightarrow_{LK} A, \Delta}}{\Gamma, \neg A \Rightarrow_{LK} \Delta} \quad \rightsquigarrow \quad \text{Ev. sev. } (As \Rightarrow)^*\ \dfrac{\neg\Rightarrow\ \dfrac{\overset{\mathtt{d}^*}{\Gamma^\wedge \Rightarrow_{SASCLK} A \vee \Delta^\vee}}{\Gamma^\wedge \wedge \neg A \Rightarrow_{SASCLK} \Delta^\vee}}{(\Gamma \cup \{\neg A\})^\wedge \Rightarrow_{SASCLK} \Delta^\vee}$$

Step $\Rightarrow Weak$: If the succedent of the premise is empty, we use the homologous rule of **SASCLK**.

Otherwise:

$$\Rightarrow Weak\ \dfrac{\overset{\mathtt{d}}{\Gamma \Rightarrow_{LK} \Delta}}{\Gamma \Rightarrow_{LK} A, \Delta} \quad \rightsquigarrow \quad \text{Ev. sev. } (\Rightarrow As)^*\ \dfrac{\wedge\Rightarrow\ \dfrac{\overset{\mathtt{d}^*}{\Gamma^\wedge \Rightarrow_{SASCLK} \Delta^\vee}}{\Gamma^\wedge \Rightarrow_{SASCLK} A \vee \Delta^\vee}}{\Gamma^\wedge \Rightarrow_{SASCLK} (\{A\} \cup \Delta)^\vee}$$

[16] Other case is symmetrical.
[17] Other case is symmetrical.
[18] From now on, we will no more specify this point.

Step $Weak \Rightarrow$**:** If the antecedent of the premise is empty, we use the homologous rule of **SASCLJ**.

Otherwise:

$$Weak \Rightarrow \frac{\overset{\mathtt{d}}{\Gamma \Rightarrow_{LK} \Delta}}{\Gamma, A \Rightarrow_{LK} \Delta} \quad \rightsquigarrow \quad \text{Ev. sev. } (As \Rightarrow)^* \frac{\wedge \Rightarrow \frac{\overset{\mathtt{d}^*}{\Gamma^\wedge \Rightarrow_{SASCLK} \Delta^\vee}}{\Gamma^\wedge \wedge A \Rightarrow_{SASCLK} \Delta^\vee}}{(\Gamma \cup \{A\})^\wedge \Rightarrow_{SASCLK} \Delta^\vee}$$

Step $Con \Rightarrow$**:**

$$Con \Rightarrow \frac{\overset{\mathtt{d}}{\Gamma, A, A \Rightarrow_{LK} \Delta}}{\Gamma, A \Rightarrow_{LK} \Delta} \quad \rightsquigarrow \quad \text{Ev. sev. } (As \Rightarrow)^* \frac{(Idem \Rightarrow)^* \frac{\overset{\mathtt{d}^*}{\Gamma^\wedge \wedge (A \wedge A) \Rightarrow_{SASCLK} \Delta^\vee}}{\Gamma^\wedge \wedge A \Rightarrow_{SASCLK} \Delta^\vee}}{(\Gamma \cup \{A\})^\wedge \Rightarrow_{SASCLK} \Delta^\vee}$$

Step $\Rightarrow Con$**:**

$$\Rightarrow Con \frac{\overset{\mathtt{d}}{\Gamma \Rightarrow_{LK} A, A, \Delta}}{\Gamma \Rightarrow_{LK} A, \Delta} \quad \rightsquigarrow \quad \text{Ev. sev. } (\Rightarrow As)^* \frac{(\Rightarrow Idem)^* \frac{\overset{\mathtt{d}^*}{\Gamma^\wedge \Rightarrow_{SASCLK} (A \vee A) \vee \Delta^\vee}}{\Gamma^\wedge \Rightarrow_{SASCLK} A \vee \Delta^\vee}}{(\Gamma \Rightarrow_{SASCLK} (\{A\} \cup \Delta)^\vee}$$

Step $Per \Rightarrow$**:**

$$Per \Rightarrow \frac{\overset{\mathtt{d}}{\Gamma, A, B, \Delta \Rightarrow_{LK} \Theta}}{\Gamma, B, A, \Delta \Rightarrow_{LK} \Theta} \quad \rightsquigarrow \quad \text{Ev. sev. } (As \Rightarrow)^* \frac{(Comm \Rightarrow)^* \frac{\overset{\mathtt{d}^*}{\Gamma^\wedge \wedge ((A \wedge B) \wedge \Delta^\wedge) \Rightarrow_{SASCLK} \Theta^\vee}}{\Gamma^\wedge \wedge ((B \wedge A) \wedge \Delta^\wedge) \Rightarrow_{SASCLK} \Theta^\vee}}{(\Gamma \cup \{B \wedge A\} \cup \Delta)^\wedge \Rightarrow_{SASCLK} \Theta^\vee}$$

Step $\Rightarrow Per$**:**

$$\Rightarrow Per \frac{\overset{\mathtt{d}}{\Gamma \Rightarrow_{LK} \Delta, A, B, \Theta}}{\Gamma \Rightarrow_{LK} \Delta, B, A, \Theta} \quad \rightsquigarrow \quad \text{Ev. sev. } (\Rightarrow As)^* \frac{(\Rightarrow Comm)^* \frac{\overset{\mathtt{d}^*}{\Gamma^\wedge \Rightarrow_{SASCLK} \Delta^\vee \vee ((A \vee B) \vee \Theta^\vee)}}{\Gamma^\wedge \Rightarrow_{SASCLK} \Delta^\vee \vee ((B \vee A) \vee \Theta^\vee)}}{\Gamma^\wedge \Rightarrow_{SASCLK} (\Delta \cup \{B \wedge A\} \cup \Theta)^\vee}$$

Step Cut**:** Since Cut is admissible in **LK**,[19] we do not need to translate it.

In the proof that all sequents provable in **LK** are provable in **SASCLK** we do not need any application of Cut apart from these in $(As \Rightarrow)^*$, $(Idem \Rightarrow)^*$, $(Comm \Rightarrow)^*$, $(\Rightarrow As)^*$, $(\Rightarrow Idem)^*$ and $(\Rightarrow Comm)^*$. So the semi-Cut-free fragment of **SASCLK** is enough strong to derive all classically valid sequents.

□

Let us now consider the part *2* of the theorem.

Proof. The proof is by induction on the length of the derivation in **SASCLK**, and by cases on the last rule applied.

Base: If the proof of **SASCLK** is just an application of the Axiom $\vdash_{SASCLK} C \Rightarrow C$ (or $\vdash_{SASCLK} \bot \Rightarrow \bot$), then the same conclusion can be proved using the Axiom of **LK**: $\vdash_{LK} C \Rightarrow C$ (or $\vdash_{LK} \neg(E \supset E) \Rightarrow \neg(E \supset E)$).

Steps $\Rightarrow \wedge$**,** $\wedge \Rightarrow$ **,** $\Rightarrow \vee$**and** $Weak$**:** The rule of **SASCLK** is a particular case of the homologous rule of **LK**.

Steps $\vee \Rightarrow$ **and** $\Rightarrow \bot$**:** We proceed as in proof B.1.2.

Step $\Rightarrow\supset$**:** In the non-trivial case, the derivation of **SASCLK** ends with: $\Rightarrow\supset \frac{A \wedge C \Rightarrow_{SASCLK} B \vee D}{C \Rightarrow_{SASCLK} (A \supset B) \vee D}$

By inductive hypothesis we have that in **LK** $A \wedge C \Rightarrow B \vee D$ is provable, and the already seen proof of $\vdash_{LJ} A, C \Rightarrow A \wedge C$ still holds for **LK**. We can prove that $\vdash_{LK} B \vee D \Rightarrow B, D$:

$$\vee \Rightarrow \frac{B \Rightarrow B, D \qquad D \Rightarrow B, D}{B \vee D \Rightarrow B, D}$$

So we obtain the conclusion:

$$\Rightarrow Cont \frac{\Rightarrow \vee \frac{\Rightarrow\supset \frac{Cut \frac{Cut \frac{\overset{\vdots}{A, C \Rightarrow_{LK} A \wedge C} \qquad A \wedge C \Rightarrow_{LK} B \vee D}{A, C \Rightarrow_{LK} B \vee D} \qquad \vee \Rightarrow \frac{B \Rightarrow_{LK} B, D \qquad D \Rightarrow_{LK} B, D}{B \vee D \Rightarrow_{LK} B, D}}{A, C \Rightarrow_{LK} B, D}}{C \Rightarrow_{LK} A \supset B, D}}{C \Rightarrow_{LK} (A \supset B) \vee D, (A \supset B) \vee D}}{C \Rightarrow_{LK} (A \supset B) \vee D}$$

[19] As established in [Gentzen, 1969b].

Step $\supset\Rightarrow$: In the more interesting case, the derivation of **SASCLK** ends with: $\supset\Rightarrow \dfrac{E \Rightarrow_{SASCLK} A \vee F \qquad B \wedge D \Rightarrow_{SASCLK} C}{(A \supset B) \wedge (D \wedge E) \Rightarrow_{SASCLK} C \vee F}$ By inductive hypothesis we have that in **LK** $E \Rightarrow A \vee F$ and $B \wedge D \Rightarrow C$ are provable, so we obtain the conclusion:[20]

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}B \wedge D \Rightarrow_{LK} C \\ \vdots \\ B, D \Rightarrow_{LK} C\end{array} \qquad \begin{array}{c}E \Rightarrow_{LK} A \vee F \\ \vdots \\ E \Rightarrow_{LK} A, F\end{array}}{A \supset B, D, E \Rightarrow_{LK} C, F}\supset\Rightarrow}{(A \supset B) \wedge (D \wedge E) \Rightarrow_{LK} C, F}\wedge\Rightarrow \text{ and } Cont\Rightarrow}{(A \supset B) \wedge (D \wedge E) \Rightarrow_{LK} C \vee F}\Rightarrow\vee \text{ and } \Rightarrow Cont$$

Step $\Rightarrow\neg$: In the non-trivial case, the derivation of **SASCLK** ends with: $\Rightarrow\neg \dfrac{A \wedge C \Rightarrow_{SASCLK} D}{C \Rightarrow_{SASCLK} \neg A \vee D}$
By inductive hypothesis we have that in **LK** $A \wedge C \Rightarrow D$ is provable, so we obtain the conclusion:

$$\dfrac{\dfrac{\begin{array}{c}A \wedge C \Rightarrow_{LK} D \\ \vdots \\ A, C \Rightarrow_{LK} D\end{array}}{C \Rightarrow_{LK} \neg A, D}\Rightarrow\neg}{C \Rightarrow_{LK} \neg A \vee D}\Rightarrow\vee \text{ and } \Rightarrow Cont$$

Step $\neg\Rightarrow$: In the non-trivial case, the derivation of **SASCLK** ends with: $\neg\Rightarrow \dfrac{C \Rightarrow_{SASCLK} A \vee D}{\neg A \wedge C \Rightarrow_{SASCLK} D}$
By inductive hypothesis we have that in **LK** $C \Rightarrow A \vee D$ is provable, so we obtain the conclusion:

$$\dfrac{\dfrac{\begin{array}{c}C \Rightarrow_{LK} A \vee D \\ \vdots \\ C \Rightarrow_{LK} A, D\end{array}}{\neg A, C \Rightarrow_{LK} D}\neg\Rightarrow}{\neg A \wedge C \Rightarrow_{LK} D}\wedge\Rightarrow \text{ and } Cont\Rightarrow$$

Step *Cut*: The derivation of **SASCLK** ends with: $Cut \dfrac{C \Rightarrow_{SASCLK} (F \vee A) \vee G \qquad (H \wedge A) \wedge I \Rightarrow_{SASCLK} D}{(G \wedge C) \wedge H \Rightarrow_{SASCLK} D}$ By inductive hypothesis we have that in **LK** $C \Rightarrow (F \vee A) \vee G$ and $(G \wedge A) \wedge H \Rightarrow D$ are provable, so we obtain the conclusion:[21]

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}C \Rightarrow_{LK} (F \vee A) \vee G \\ \vdots \\ C \Rightarrow_{LK} F \vee A, G \\ \vdots \\ C \Rightarrow_{LK} F, A, G\end{array} \qquad \begin{array}{c}(G \wedge A) \wedge H \Rightarrow_{LK} D \\ \vdots \\ (G \wedge C), H \Rightarrow_{LK} D \\ \vdots \\ G, C, H \Rightarrow_{LK} D\end{array}}{G, A, H \Rightarrow_{LK} F, D, G} Cut}{(G \wedge A) \wedge H \Rightarrow_{LK} F, D, G}\wedge\Rightarrow \text{ and } Cont\Rightarrow}{(G \wedge A) \wedge H \Rightarrow_{LK} (F \vee D) \vee G}\Rightarrow\vee \text{ and } \Rightarrow Cont$$

□

Equivalence between SASCLK and SASCNK

Theorem B.1.5 (Equivalence between **SASCLK** and **SASCNK**). *The sequent calculus **SASCLK** and the natural deduction system **SASCNK** are equivalent to each other, that is:*

1. (a) *if* $\vdash_{SASCLK} A \Rightarrow B$, *then:*
   - *Or* $A \vdash_{SASCNK} B$;
   - *Or* $\vdash_{SASCLK} \Rightarrow B$ *and* $\vdash_{SASCNK} B$.

   (b) *If* $\vdash_{SASCLK} A \Rightarrow$, *then* $A \vdash_{SASCNK} \bot$.
2. (a) *If* $A \vdash_{SASCNK} B$ *then* $\vdash_{SASCLK} A \Rightarrow B$.

[20] We use the already established derivations of $A, B \Rightarrow C$ from $A \wedge B \Rightarrow C$ and $A \Rightarrow B, C$ from $A \Rightarrow B \vee C$.
[21] If there is no F or G, the proof remain valid.

Let us start from point *1*.

Proof. The proof is by induction on the length of the derivation in **SASCLK**, and by cases on the last rule applied.

Base: Obvious;

Steps $\Rightarrow \wedge$, $\wedge \Rightarrow$, $\Rightarrow \vee$, $\vee \Rightarrow$, $\Rightarrow \bot$, $\Rightarrow Weak$ **and** $Weak \Rightarrow$: We treat them as in the proof B.1.2.

Step $\Rightarrow\supset$: From $\Rightarrow\supset \dfrac{A \wedge C \;\Rightarrow\; B \vee D}{C \;\Rightarrow\; (A \supset B) \vee D}$, we have $A \wedge C \vdash_{SASCNK} B \vee D$ by induction.

So, we can conclude:

$$\supset\mathrm{I}_1 \dfrac{C \qquad \begin{matrix}[A \wedge C]^1 \\ \vdots \\ B \vee D\end{matrix}}{(A \supset B) \vee D}$$

Step $\supset\Rightarrow$: From $\supset\Rightarrow \dfrac{E \;\Rightarrow\; A \vee F \qquad B \wedge D \;\Rightarrow\; C}{(A \supset B) \wedge (D \wedge E) \;\Rightarrow\; C \vee F}$, we have $E \vdash_{SASCNK} A \vee F$ and $B \wedge D \vdash_{SASCNK} C$ by induction. So, we can conclude:

$$\supset\mathrm{E}_{1,2} \dfrac{(A \supset B) \wedge (D \wedge E) \qquad \begin{matrix}[E]^1 \\ \vdots \\ A \vee F\end{matrix} \qquad \begin{matrix}[B \wedge D]^2 \\ \vdots \\ C\end{matrix}}{C \vee F}$$

Step $\Rightarrow \neg$: From $\Rightarrow\neg \dfrac{A \wedge C \;\Rightarrow\; D}{C \;\Rightarrow\; \neg A \vee D}$, we have $A \wedge C \vdash_{SASCNK} D$ by induction. So, we can conclude:

$$\neg\mathrm{I}_1 \dfrac{C \qquad \begin{matrix}[A \wedge C]^1 \\ \vdots \\ D\end{matrix}}{\neg A \vee D}$$

Step $\neg \Rightarrow$: From $\neg\Rightarrow \dfrac{C \;\Rightarrow\; A \vee D}{\neg A \wedge C \;\Rightarrow\; D}$, we have $\vdash_{SASCNJ} A \vee D$ by induction. So, we can conclude:

$$\neg\mathrm{E}_1 \dfrac{\neg A \wedge C \qquad \begin{matrix}[C]^1 \\ \vdots \\ A \vee D\end{matrix}}{D}$$

Step *Cut*: From

$$Cut \dfrac{\begin{matrix}\vdots\mathtt{d}_1 \\ C \;\Rightarrow\; (D \vee A) \vee E\end{matrix} \qquad \begin{matrix}\vdots\mathtt{d}_1 \\ (F \wedge A) \wedge G \;\Rightarrow\; H\end{matrix}}{(F \wedge C) \wedge G \;\Rightarrow\; (D \vee H) \vee E}$$

we obtain:

$$\wedge I_2 \dfrac{[A \wedge ((F \wedge C) \wedge G)]^4 \qquad \wedge I_1 \dfrac{[A \wedge ((F \wedge C) \wedge G)]^2 \qquad \wedge E \dfrac{[A \wedge ((F \wedge C) \wedge G)]^1}{F} \qquad \wedge E \dfrac{[A \wedge ((F \wedge C) \wedge G)]^1}{A}}{F \wedge A} \qquad \wedge E \dfrac{[A \wedge ((F \wedge C) \wedge G)]^2}{G}}{(F \wedge A) \wedge G}$$

$$\vdots\mathtt{d}_2^*$$

$$\vee E_4 \dfrac{[(D \vee A) \wedge ((F \wedge C) \wedge G)]^5 \qquad \vee I \dfrac{\wedge E \dfrac{[D \wedge ((F \wedge C) \wedge G)]^4}{D}}{(D \vee H) \vee E} \qquad \vee I \dfrac{\begin{matrix}\vdots\mathtt{d}_2^* \\ H\end{matrix}}{(D \vee H) \vee E}}{(D \vee H) \vee E}$$

$$\vdots$$

$$\wedge E \dfrac{[(F \wedge C) \wedge G]^3}{C}$$

$$\vee E_5 \dfrac{\wedge I_3 \dfrac{(F \wedge C) \wedge G \qquad \begin{matrix}\vdots\mathtt{d}_1^* \\ (D \vee A) \vee E\end{matrix} \qquad [(F \wedge C) \wedge G]^3}{((D \vee A) \vee E) \wedge ((F \wedge C) \wedge G)} \qquad \begin{matrix}\vdots \\ (D \vee H) \vee E\end{matrix} \qquad \vee I \dfrac{\wedge E \dfrac{[E \wedge ((F \wedge C) \wedge G)]^5}{E}}{(D \vee H) \vee E}}{(D \vee H) \vee E}$$

The other cases of Cut are easy variations of this or identical with that of **SASCLJ**.

□

Let us now turn to part *2* of the theorem.

Proof. The proof is by induction on the length of the derivation in **SASCNK**, and by cases on the last rule applied.

Base: To the assumption of A ($\bot$) in **SASCNK**, we associate the Axiom $A \Rightarrow A$ ($\bot \Rightarrow \bot$) of **SASCLK**;

Steps $\wedge$I, $\wedge$E, $\vee$I, $\vee$E and Efq: We treat them as in the proof B.1.2.

Step $\supset$I: The derivation ends with:

$$\supset\mathrm{I}_1\,\dfrac{\begin{array}{cc} D & [A \wedge C]^1 \\ \vdots & \vdots \\ C & B \vee E \end{array}}{(A \supset B) \vee E}$$

, so by inductive hypothesis we have that in **SASCLK** $A \wedge C \Rightarrow B \vee E$ and $D \Rightarrow C$ are provable. So, we conclude:

$$Cut\,\dfrac{D \Rightarrow C \qquad \Rightarrow\supset\dfrac{A \wedge C \Rightarrow B \vee E}{C \Rightarrow (A \supset B) \vee E}}{D \Rightarrow (A \supset B) \vee E}$$

Step $\supset$E: The derivation ends with:

$$\supset\mathrm{E}_{1,2}\,\dfrac{\begin{array}{ccc} D & [E]^1 & [B \wedge F]^2 \\ \vdots & \vdots & \vdots \\ (A \supset B) \wedge (E \wedge F) & A \vee G & C \end{array}}{C \vee G}$$

, so by inductive hypothesis we have that in **SASCLK** $D \Rightarrow (A \supset B) \wedge (E \wedge F)$, $E \Rightarrow A \vee G$ and $B \wedge F \Rightarrow C$ are provable. So, we conclude:

$$Cut\,\dfrac{D \Rightarrow (A \supset B) \wedge (E \wedge F) \qquad \supset\Rightarrow\dfrac{E \Rightarrow A \vee G \qquad B \wedge F \Rightarrow C}{(A \supset B) \wedge (E \wedge F) \Rightarrow C \vee G}}{D \Rightarrow C \vee G}$$

Step $\neg$I: The derivation ends with:

$$\supset\mathrm{I}_1\,\dfrac{\begin{array}{cc} D & [A \wedge C]^1 \\ \vdots & \vdots \\ C & B \end{array}}{\neg A \vee B}$$

, so by inductive hypothesis we have that in **SASCLK** $A \wedge C \Rightarrow B$ and $D \Rightarrow C$ are provable. So, we conclude:

$$Cut\,\dfrac{D \Rightarrow C \qquad \Rightarrow\neg\dfrac{A \wedge C \Rightarrow B}{C \Rightarrow \neg A \vee B}}{D \Rightarrow \neg A \vee B}$$

The other case, in which B is absent, is already treated in the intuitionistic version of this theorem. Remember that $\bot\{B\}$ is B or $\bot$.

Step $\neg$E: The derivation ends with:

$$\neg\mathrm{E}_1\,\dfrac{\begin{array}{cc} D & [C]^1 \\ \vdots & \vdots \\ \neg A \wedge C & A \vee B \end{array}}{B}$$

, so by inductive hypothesis we have that in **SASCLK** $D \Rightarrow \neg A \wedge C$ and $C \Rightarrow A \vee B$ are provable. So, we conclude:

$$Cut\,\dfrac{D \Rightarrow \neg A \wedge C \qquad \neg\Rightarrow\dfrac{C \Rightarrow A \vee B}{\neg A \wedge C \Rightarrow B}}{D \Rightarrow B}$$

□

Let us now consider the proof of Associativity, Commutativity and Idempotence of disjunction in **SASCNK**. Even though these do not correspond to primitive rules of **SASCLK**, they will be very useful in the future.

$$\dfrac{\begin{matrix}A\\ \vdots\, d_1^*\\ \beta\end{matrix} \quad \dfrac{[E\vee((B\vee C)\vee D)]^4 \quad \dfrac{[E]^3}{\alpha}\vee\mathrm{I} \quad \dfrac{[(B\vee C)\vee D]^3 \quad \dfrac{[B\vee C]^2 \quad \dfrac{[B]^1}{\alpha}\vee\mathrm{I} \quad \dfrac{[C]^1}{\alpha}\vee\mathrm{I}}{\alpha}\vee\mathrm{E}_1 \quad \dfrac{[D]^2}{\alpha}\vee\mathrm{I}}{\alpha}\vee\mathrm{E}_2}{\alpha}\vee\mathrm{E}_3 \quad \dfrac{[F]^4}{\alpha}\vee\mathrm{I}}{\alpha}\vee\mathrm{E}_4$$

$$\vdots$$

$$\dfrac{\begin{matrix}A\\ \vdots\, d_1^*\\ \alpha\end{matrix} \quad \dfrac{[E\vee(B\vee(C\vee D))]^4 \quad \dfrac{[E]^3}{\beta}\vee\mathrm{I} \quad \dfrac{[B\vee(C\vee D)]^3 \quad \dfrac{[B]^2}{\beta}\vee\mathrm{I} \quad \dfrac{[B\vee C]^2 \quad \dfrac{[C]^1}{\beta}\vee\mathrm{I} \quad \dfrac{[D]^1}{\beta}\vee\mathrm{I}}{\beta}\vee\mathrm{E}_1}{\beta}\vee\mathrm{E}_2}{\beta}\vee\mathrm{E}_3 \quad \dfrac{[F]^4}{\beta}\vee\mathrm{I}}{\beta}\vee\mathrm{E}_4$$

$$\vdots$$

Where $\alpha = (E \vee (B \vee (C \vee D))) \vee F$ and $\beta = (E \vee ((B \vee C) \vee D)) \vee F$. We will call the first derivation $(\Rightarrow As_1)^{**}$ and the second $(\Rightarrow As_2)^{**}$.

$$\dfrac{\begin{matrix}E\\ \vdots\, d_1^*\\ \gamma\end{matrix} \quad \dfrac{[C]^3}{\delta}\vee\mathrm{I} \quad \dfrac{[(A\vee B)\vee D]^3 \quad \dfrac{[A\vee B]^2 \quad \dfrac{[A]^1}{\delta}\vee\mathrm{I} \quad \dfrac{[B]^1}{\delta}\vee\mathrm{I}}{\delta}\vee\mathrm{E}_1 \quad \dfrac{[D]^2}{\delta}\vee\mathrm{I}}{\delta}\vee\mathrm{E}_2}{\delta}\vee\mathrm{E}_3$$

$$\vdots$$

Where $\gamma = C \vee ((A \vee B) \vee D)$ and $\delta = C \vee ((B \vee A) \vee D)$. We will call this derivation $(\Rightarrow Comm)^{**}$.

$$\dfrac{\begin{matrix}E\\ \vdots\, d_1^*\\ (A\vee A)\vee B\end{matrix} \quad \dfrac{\dfrac{[A\vee A]^2 \quad [A]^1 \quad [A]^1}{A}\vee\mathrm{E}_1}{A\vee B}\vee\mathrm{I} \quad \dfrac{[B]^2}{A\vee B}\vee\mathrm{I}}{A\vee B}\vee\mathrm{E}_2$$

$$\vdots$$

We will call this derivation $(\Rightarrow Idem)^{**}$.

In conclusion, let us see the translation of $Distri^{\vee\wedge}$:

$$\dfrac{\dfrac{\begin{matrix}D\\ \vdots\, d_1^*\\ (A\vee C)\wedge(B\vee C)\end{matrix}}{(B\vee C)\wedge(A\vee C)}(\Rightarrow Comm)^{**} \quad \dfrac{\dfrac{[B\wedge(A\vee C)]^4}{\epsilon}(\Rightarrow Comm)^{**} \quad \dfrac{[A\wedge B]^2}{\epsilon}\vee\mathrm{I} \quad \dfrac{[C\wedge B]^2 \quad \dfrac{[C]^1}{\epsilon}\vee\mathrm{I}}{\epsilon}\wedge\mathrm{E}_1}{\epsilon}\vee\mathrm{E}_2 \quad \dfrac{[C\wedge(A\vee C)]^4 \quad \dfrac{[C]^3}{\epsilon}\vee\mathrm{I}}{\epsilon}\wedge\mathrm{E}_3}{\epsilon}\vee\mathrm{E}_4$$

Where $\epsilon = (A \wedge B) \vee C$ and the applications of $(\Rightarrow Comm)^{**}$ have form:

$$\dfrac{A\wedge B \quad \dfrac{[A\wedge B]^3 \quad [B]^1}{B}\wedge\mathrm{E}_1 \quad \dfrac{[A\wedge B]^3 \quad [A]^2}{A}\wedge\mathrm{E}_2}{B\wedge A}\wedge\mathrm{I}_3$$

We will call this derivation $(Distri^{\vee\wedge})^{**}$.

B.2 Cut eliminations and normalizations

B.2.1 JDJ systems

Cut elimination for SASCNJDJ

Theorem B.2.1 (Cut elimination for **SASCLJDJ**). *If $\vdash_{SASCLJDJ} A \Rightarrow B$, then we can prove it without using the Cut rule. Also there is a procedure that change a valid derivation of a sequent, in a Cut-free derivation of the same result.*

Our proof is a slightly modified version of the one from [Gentzen, 1969b]. We need some definitions before we can start.

Definition B.2.1 (Cut degree)**.** The degree of an application of Cut is the degree of the principal formula in the application of the rule. The degree of a formula is inductively defined: the degree of an atomic formula is 1; the degree of $\neg A$ is 1 plus the degree of A; the degree of $A \otimes B$ is the sum of the degree of A and B.

Definition B.2.2 (Cut rank)**.** Given a derivation in **SASCLJDJ** and an application of the Cut rule in it, we define:

Left rank: the largest number of consecutive sequents in the sub-derivation of the left-hand premise of the Cut such that every sequents contains the Cut formula in the succedent.

Right rank: the largest number of consecutive sequents in the sub-derivation of the right-hand premise of the Cut such that every sequents contains the Cut formula in the antecedent.

We define the rank of an application of Cut as the sum of its left and right ranks.

Proof. Let us consider an application of Cut such that there are no other application of this rule above it. If we manage to eliminate it, we can eliminate every Cut. So we have to deal only with derivations with just one occurrence of the Cut rule, as last rule.[22]

We prove the theorem by a primary induction on the degree and a secondary induction on the rank of this single Cut. When the rank of the Cut is 2, we will reduce the degree of it; when the rank is greater than 2, we will reduce the rank of it.

Base: $Cut\ \dfrac{C \Rightarrow A \qquad A \Rightarrow D}{C \Rightarrow D}$ We have degree 1 and rank 2. Since the Cut-formula is atomic (degree 1) and it have been introduced in the premises of the Cut for the first time (rank 2), it can be the principal formula of a Weakening or of an Axiom. If one of the two formulae is the principal formula of a Weakening, then we pick its sub-derivation and use Weakening to introduce the context of the other premise of the Cut. As an example, if we obtain $A \Rightarrow D$ by Weakening from $\Rightarrow D$, we use Weakening to obtain directly $C \Rightarrow D$. Otherwise, if both formulae are principal formulae of Axiom, all the formulae in the Cut are identical: $A = B = C$. So the conclusion of the Cut is identical to the premises and it can be eliminated.

Step, rank 2: Let us assume that $Cut\ \dfrac{C \Rightarrow A \qquad A \Rightarrow D}{C \Rightarrow D}$ has degree n and rank 2. If A is introduced by Weakening or atomic, we can behave as in the base. We prove the other steps by cases on the outermost connective in A:

- $\wedge$ Since rank is 2, the derivation should be:
$$Cut\ \dfrac{\Rightarrow\wedge\ \dfrac{A \Rightarrow D \qquad A \Rightarrow E}{A \Rightarrow D \wedge E} \qquad \wedge\Rightarrow\ \dfrac{D \Rightarrow B}{D \wedge E \Rightarrow B}}{A \Rightarrow B}$$
We can reduce the degree of the Cut in this way: $Cut\ \dfrac{A \Rightarrow D \qquad D \Rightarrow B}{A \Rightarrow B}$ [23]

- $\vee$ Since rank is 2, the derivation should be:
$$Cut\ \dfrac{\Rightarrow\vee\ \dfrac{A \Rightarrow D}{A \Rightarrow D \vee E} \qquad \vee\Rightarrow\ \dfrac{D \Rightarrow B \qquad E \Rightarrow B}{D \vee E \Rightarrow B}}{A \Rightarrow B}$$
We can reduce the degree of the Cut in this way: $Cut\ \dfrac{A \Rightarrow D \qquad D \Rightarrow B}{A \Rightarrow B}$ [24]

- $\supset$ Since rank is 2, the derivation should be:
$$Cut\ \dfrac{\Rightarrow\supset\ \dfrac{A \Rightarrow B}{\Rightarrow A \supset B} \qquad \supset\Rightarrow\ \dfrac{B \Rightarrow C \qquad \Rightarrow A}{A \supset B \Rightarrow C}}{\Rightarrow C}$$
We can reduce the degree of the Cut in this way:
$$Cut\ \dfrac{\Rightarrow A \qquad Cut\ \dfrac{A \Rightarrow B \qquad B \Rightarrow C}{A \Rightarrow C}}{\Rightarrow C}$$
We obtain two applications of Cut, but this is not a problem, since both of them are eliminable by inductive hypothesis, being of lesser degree.

- $\neg$ Since rank is 2, the derivation should be:
$$Cut\ \dfrac{\Rightarrow\neg\ \dfrac{A \Rightarrow}{\Rightarrow \neg A} \qquad \neg\Rightarrow\ \dfrac{\Rightarrow A}{\neg A \Rightarrow}}{\Rightarrow}$$
We can reduce the degree of the Cut in this way: $Cut\ \dfrac{\Rightarrow A \qquad A \Rightarrow}{\Rightarrow}$

[22]We consider, given a derivation, the sub-derivation that ends with the first application of Cut.
[23]Other case is symmetrical.
[24]Other case is symmetrical.

We have so proved the inductive step for Cut applications of rank 2 and arbitrary degree.

Step, rank > 2: Let us assume that $Cut\,\dfrac{C \Rightarrow A \qquad A \Rightarrow D}{C \Rightarrow D}$ has degree n and rank $m_{>2}$. If $A = B = C$, the Cut is useless and it can be erased. Since Cut rank is strictly greater than 2, at least one of left and right rank has to be strictly greater than 1. Let us consider the two complete but not mutually exclusive cases:

Right rank > 1: The last rule applied in the sub-derivation of the right-hand premise of the Cut must have a premise with the same antecedent of the conclusion. The only rules of **SASCLJDJ** that fulfil this requirement are $\Rightarrow \wedge$, $\Rightarrow \vee$, $\Rightarrow Weak$ and Cut. We can dismiss the last alternative, since by hypothesis we are dealing with the upper occurrence of such a rule. Let us see the other cases:[25]

$\Rightarrow \wedge$ The derivation is:

$$Cut\,\dfrac{A \Rightarrow C \qquad \Rightarrow\wedge\,\dfrac{C \Rightarrow D \qquad C \Rightarrow E}{C \Rightarrow D \wedge E}}{A \Rightarrow D \wedge E}$$

We can obtain the same conclusion using only Cut applications with lesser rank:

$$\Rightarrow\wedge\,\dfrac{Cut\,\dfrac{A \Rightarrow C \qquad C \Rightarrow D}{A \Rightarrow D} \qquad Cut\,\dfrac{A \Rightarrow C \qquad C \Rightarrow E}{A \Rightarrow E}}{A \Rightarrow D \wedge E}$$

The two Cuts obtained have the same left-rank of the one deleted, but lesser right-rank, so the rank of the two Cuts is $m-1$, and they can be eliminated by inductive hypothesis.

$\Rightarrow \vee$ The derivation is:

$$Cut\,\dfrac{A \Rightarrow C \qquad \Rightarrow\vee\,\dfrac{C \Rightarrow D}{C \Rightarrow D \vee E}}{A \Rightarrow D \vee E}$$

We can substitute this Cut with one of rank $m-1$:

$$\Rightarrow\vee\,\dfrac{Cut\,\dfrac{A \Rightarrow C \qquad C \Rightarrow D}{A \Rightarrow D}}{A \Rightarrow D \vee E}$$ [26]

$\Rightarrow Weak$ The derivation is:

$$Cut\,\dfrac{A \Rightarrow C \qquad \Rightarrow Weak\,\dfrac{C \Rightarrow}{C \Rightarrow B}}{A \Rightarrow B}$$

We can substitute this Cut with one of rank $m-1$:

$$\Rightarrow Weak\,\dfrac{Cut\,\dfrac{A \Rightarrow C \qquad C \Rightarrow}{A \Rightarrow}}{A \Rightarrow B}$$

$\Rightarrow \bot$ The derivation is:

$$Cut\,\dfrac{A \Rightarrow C \qquad \Rightarrow\bot\,\dfrac{C \Rightarrow \bot}{C \Rightarrow}}{A \Rightarrow}$$

We can substitute this Cut with one of rank $m-1$:

$$\Rightarrow\bot\,\dfrac{Cut\,\dfrac{A \Rightarrow C \qquad C \Rightarrow \bot}{A \Rightarrow \bot}}{A \Rightarrow}$$

This end the proof of the inductive step for right-rank greater than 1. Let us consider now the other possibility:

Left rank > 1: The last rule applied in the sub-derivation of the left-hand premise of the Cut must have a premise with the same succedent of the conclusion. The only rules of **SASCLJDJ** that fulfil this requirement are $\wedge \Rightarrow$, $\vee \Rightarrow$, $\supset\Rightarrow$, $Weak \Rightarrow$ and Cut. As before, we can dismiss the last alternative. Let us see the other cases:

$\wedge \Rightarrow$ The derivation is:

$$Cut\,\dfrac{\wedge\Rightarrow\,\dfrac{E \Rightarrow C}{D \wedge E \Rightarrow C} \qquad C \Rightarrow B}{D \wedge E \Rightarrow B}$$

We can reduce by 1 the rank of this Cut, in this way:

$$\wedge\Rightarrow\,\dfrac{Cut\,\dfrac{E \Rightarrow C \qquad C \Rightarrow B}{E \Rightarrow B}}{D \wedge E \Rightarrow B}$$

$\vee \Rightarrow$ The derivation is:

$$Cut\,\dfrac{\vee\Rightarrow\,\dfrac{D \Rightarrow C \qquad E \Rightarrow C}{D \vee E \Rightarrow C} \qquad C \Rightarrow B}{D \vee E \Rightarrow B}$$

We can substitute this Cut with two of rank $m-1$, in this way:

$$\vee\Rightarrow\,\dfrac{Cut\,\dfrac{D \Rightarrow C \qquad C \Rightarrow B}{D \Rightarrow B} \qquad Cut\,\dfrac{E \Rightarrow C \qquad C \Rightarrow B}{E \Rightarrow B}}{D \vee E \Rightarrow B}$$

$\supset\Rightarrow$ The derivation is:

$$Cut\,\dfrac{\supset\Rightarrow\,\dfrac{E \Rightarrow C \qquad \Rightarrow D}{D \supset E \Rightarrow C} \qquad C \Rightarrow B}{D \supset E \Rightarrow B}$$

We can substitute this Cut with one of rank $m-1$, in this way:

$$\supset\Rightarrow\,\dfrac{Cut\,\dfrac{E \Rightarrow C \qquad C \Rightarrow B}{E \Rightarrow B} \qquad \Rightarrow D}{D \supset E \Rightarrow B}$$

[25] This time the division in cases is not based on the rule that derive the Cut formula, but just on the last rule applied in the derivation of the right premise of the Cut.

[26] Other case is symmetric.

$Weak \Rightarrow$ The derivation is: $$\dfrac{\dfrac{\Rightarrow C}{A \Rightarrow C}\,{\scriptstyle Weak \Rightarrow} \qquad C \Rightarrow B}{A \Rightarrow B}\,{\scriptstyle Cut}$$ We can substitute this Cut with one of rank $m-1$, in this way: $$\dfrac{\dfrac{\Rightarrow C \qquad C \Rightarrow B}{\Rightarrow B}\,{\scriptstyle Cut}}{A \Rightarrow B}\,{\scriptstyle Weak \Rightarrow}$$

This ends our proof. We have shown by cases that: if the rank of the Cut is greater than 2, we can reduce it to some Cuts of rank 2; if the rank of the Cut is 2, we can reduce it to some Cuts of lesser degree. Cuts of degree 1 and rank 2 are dispensable.

□

Normalization for SASCLJDJ

Theorem B.2.2 (Normalization for **SASCNJDJ**). *If $A \vdash_{SASCNJDJ} B$, then:*

- *Or there is a normal derivation of B from A;*
- *Or there is a normal closed proof of B.*

Proof. Given a derivation $\mathfrak{D}$ of $A \vdash_{SASCNJDJ} B$, since theorem 3.1.1 (clause 2), we have that $\vdash_{SASCLJDJ} A \Rightarrow B$. We apply the Cut elimination procedure (theorem 3.1.2) and find the Cut-free derivation $\mathfrak{D}^{st}$ of the same sequent. Applying the (clause 1a of) theorem 3.1.1, we obtain the **SASCNJDJ** derivation $\mathfrak{D}^{*}$ of B

- from A;
- or without open assumptions.[27]

Since in the translation used we only apply E-rules with non-derived major premises (and composition is not used, since the derivation $\mathfrak{D}^{st}$ is Cut-free), $\mathfrak{D}^{*}$ is normal.

□

B.2.2 Existence of normal form in SASCNJ

All we have to show to complete the proof of theorem 3.2.15 is that:

1. The only anormalities in a SASCNJ-derivation obtained by translating a semi-Cut-free SASCLJ-derivation can arise from $(As \Rightarrow)^{**}$, $(Comm \Rightarrow)^{**}$ or $(Idem \Rightarrow)^{**}$;
2. We can deal with these anormality.

Let us start from the first point.

Proof. The translation in proof B.1.2 uses only non-derived major premises, so *a fortiori* only non-derived major premises of E-rules. It follows that the only way to obtain anormality is by composition, and the only rule of **SASCLJ** that ask for composition is Cut. We have already argued that the semi-Cut-free fragment of **SASCLJ** is complete, so we have to deal only with $(As \Rightarrow)^{**}$, $(Comm \Rightarrow)^{**}$ or $(Idem \Rightarrow)^{**}$.

These steps of derivation allow to compose deductions on the bottom, that is the bottom-formula of $(X \Rightarrow)^{**}$ (that is obviously derived) can be the major premise of an E-rule.[28] Of course the open assumption of $(X \Rightarrow)^{**}$ can itself be derived by $(X \Rightarrow)^{**}$, thanks to composition. So we have to deal with chains of $(X \Rightarrow)^{**}$ that derive major premises of E-rules.

The only situation in which we obtain an anormality is when the end-formula of the chain of $(X \Rightarrow)^{**}$ is the major premise of $\wedge E$. Indeed, each $(X \Rightarrow)^{**}$ derive its conclusion with $\wedge I$ as last rule and applies only $\wedge$-rules. It is important also to notice that there are no anormalities in $(X \Rightarrow)^{**}$, and that their open assumption is always a major premise of $\wedge I$. So there are no anormalities in the chains of $(X \Rightarrow)^{**}$ (neither in a single $(X \Rightarrow)^{**}$, nor in the assumption of an $(X \Rightarrow)^{**}$ derived by the preceding $(X \Rightarrow)^{**}$), only its end-formula can give anormality with the next rule.

□

Now let us see that we can deal with these anormalities.

[27] To be honest, it is quite obvious that in the translation from $\mathfrak{D}$ and Cut-elimination we do not use Weakening on the left, and so A is present as an open assumption in $\mathfrak{D}^{*}$.

[28] I will use $(X \Rightarrow)^{**}$ to indicate each of $(As \Rightarrow)^{**}$, $(Comm \Rightarrow)^{**}$ and $(Idem \Rightarrow)^{**}$.

Proof. Let us consider the chain that derives an anormality, and in particular the last $(X \Rightarrow)^{**}$, that directly causes the anormality. We will remove completely the anormality or we will remove the last $(X \Rightarrow)^{**}$, so by induction on the length of the chain we obtain the normalization. By cases on the last $(X \Rightarrow)^{**}$.

$(Idem \Rightarrow)^{**}$

$$\wedge E_3\,\dfrac{\wedge I_2\,\dfrac{\begin{matrix}\vdots_{(X\Rightarrow)^{**}}\\ A\wedge B\end{matrix} \qquad \wedge E\,\dfrac{[A\wedge B]^2}{A} \qquad \wedge I_1\,\dfrac{\wedge E\,\dfrac{[A\wedge B]^2}{B} \qquad [A\wedge B]^1 \qquad [A\wedge B]^1}{B\wedge B}}{A\wedge(B\wedge B)} \qquad \begin{matrix}[\alpha]^3\\ \vdots\\ G\end{matrix}}{G}$$

If $\alpha = A$, we remove the last $(Idem \Rightarrow)^{**}$ and reduce to:

$$\wedge E_1\,\dfrac{\begin{matrix}\vdots_{(X\Rightarrow)^{**}}\\ A\wedge B\end{matrix} \qquad \begin{matrix}[A]^1\\ \vdots\\ G\end{matrix}}{G}$$

If $\alpha = B \wedge B$, we reduce to

$$\begin{matrix}\wedge I_1\,\dfrac{\wedge E\,\dfrac{\begin{matrix}\vdots_{(X\Rightarrow)^{**}}\\ A\wedge B\end{matrix}}{B} \qquad [B]^1 \qquad [B]^1}{B\wedge B}\\ \vdots\\ G\end{matrix}$$

In this case we have an anormality for $A \wedge B$ if $\vdots_{(X\Rightarrow)^{**}} \neq \emptyset$, and an anormality for $B \wedge B$ if $B \wedge B \vdash G$ begins with $\wedge E$. This last anormality is easily treated, since the only discharged assumption of such an E-rule could be B, and so we could reduce to

$$\begin{matrix}\wedge E\,\dfrac{\begin{matrix}\vdots_{(X\Rightarrow)^{**}}\\ A\wedge B\end{matrix}}{B}\\ \vdots\\ G\end{matrix}$$

So we only have the anormality generated by $A \wedge B$, if $\vdots_{(X\Rightarrow)^{**}}$ is not empty, and we have normalized or at least reduced the length of the chain.

$(As_1 \Rightarrow)^{**}$

$$\wedge E_5\,\dfrac{\begin{matrix}\wedge I_4\,\dfrac{\begin{matrix}\vdots_{(X\Rightarrow)^{**}}\\ \alpha\end{matrix} \qquad \wedge I_3\,\dfrac{[\alpha]^4 \qquad \wedge E\,\dfrac{[\alpha]^3}{E} \qquad \wedge I_2\,\dfrac{[\alpha]^3 \qquad \wedge E\,\dfrac{[\alpha]^2}{A} \qquad \wedge I_1\,\dfrac{[\alpha]^2 \qquad \wedge E\,\dfrac{[\alpha]^1}{B} \qquad \wedge E\,\dfrac{[\alpha]^1}{C}}{B\wedge C}}{A\wedge(B\wedge C)}}{E\wedge(A\wedge(B\wedge C))} \qquad \wedge E\,\dfrac{[\alpha]^4}{F}}{\beta}\\ \vdots\\ D\end{matrix} \qquad \begin{matrix}[\gamma]^5\\ \vdots\\ G\end{matrix}}{G}$$

Where $\alpha = (E \wedge ((A \wedge B) \wedge C)) \wedge F$ and $\beta = (E \wedge (A \wedge (B \wedge C))) \wedge F$.

If $\gamma = F$, then we reduce the length of the chain by:

$$\begin{matrix}\wedge E\,\dfrac{\begin{matrix}\vdots_{(X\Rightarrow)^{**}}\\ (E\wedge((A\wedge B)\wedge C))\wedge F\end{matrix}}{F}\\ \vdots\\ G\end{matrix}$$

If $\gamma = E \wedge ((A \wedge B) \wedge C)$, then we reduce the length of the chain by:

$$\begin{array}{c}\dfrac{\overset{\vdots\,(X\Rightarrow)^{**}}{\alpha} \qquad \wedge\mathrm{E}\,\dfrac{[\alpha]^3}{E} \qquad \dfrac{[\alpha]^3 \qquad \wedge\mathrm{E}\,\dfrac{[\alpha]^2}{A} \qquad \wedge\mathrm{I}_1\,\dfrac{[\alpha]^2 \qquad \wedge\mathrm{E}\,\dfrac{[\alpha]^1}{B} \qquad \wedge\mathrm{E}\,\dfrac{[\alpha]^1}{C}}{B \wedge C}}{A \wedge (B \wedge C)}\,\wedge\mathrm{I}_2}{E \wedge (A \wedge (B \wedge C))}\,\wedge\mathrm{I}_3 \\ \vdots \\ G\end{array}$$

If it create anormality, then $E \wedge (A \wedge (B \wedge C))$ is the major premise of an $\wedge E$, let us call δ its discharged assumption.

If $\delta = E$, then we reduce to $$\begin{array}{c}\wedge\mathrm{E}\,\dfrac{\overset{\vdots\,(X\Rightarrow)^{**}}{(E \wedge ((A \wedge B) \wedge C)) \wedge F}}{E} \\ \vdots \\ G\end{array}$$

If $\delta = A \wedge (B \wedge C)$, then we reduce to

$$\begin{array}{c}\dfrac{\overset{\vdots\,(X\Rightarrow)^{**}}{\alpha} \qquad \wedge\mathrm{E}\,\dfrac{[\alpha]^2}{A} \qquad \wedge\mathrm{I}_1\,\dfrac{[\alpha]^2 \qquad \wedge\mathrm{E}\,\dfrac{[\alpha]^1}{B} \qquad \wedge\mathrm{E}\,\dfrac{[\alpha]^1}{C}}{B \wedge C}}{A \wedge (B \wedge C)}\,\wedge\mathrm{I}_2 \\ \vdots \\ G\end{array}$$

If $A \wedge (B \wedge C)$ creates anormality, then it is major premise of $\wedge E$, let us call ε its discharged assumption.

If $\varepsilon = A$, then we reduce to $$\begin{array}{c}\wedge\mathrm{E}\,\dfrac{\overset{\vdots\,(X\Rightarrow)^{**}}{(E \wedge ((A \wedge B) \wedge C)) \wedge F}}{A} \\ \vdots \\ G\end{array}$$

If $\varepsilon = B \wedge C$, then we reduce to

$$\begin{array}{c}\wedge\mathrm{I}_1\,\dfrac{\overset{\vdots\,(X\Rightarrow)^{**}}{\alpha} \qquad \wedge\mathrm{E}\,\dfrac{[\alpha]^1}{B} \qquad \wedge\mathrm{E}\,\dfrac{[\alpha]^1}{C}}{B \wedge C} \\ \vdots \\ G\end{array}$$

If $B \wedge C$ creates anormality, then it is major premise of $\wedge E$, and its discharged premise can only be B or C. We then reduce to

$$\begin{array}{c}\wedge\mathrm{E}\,\dfrac{\overset{\vdots\,(X\Rightarrow)^{**}}{(E \wedge ((A \wedge B) \wedge C)) \wedge F}}{B} \\ \vdots \\ G\end{array} \quad \text{or} \quad \begin{array}{c}\wedge\mathrm{E}\,\dfrac{\overset{\vdots\,(X\Rightarrow)^{**}}{(E \wedge ((A \wedge B) \wedge C)) \wedge F}}{C} \\ \vdots \\ G\end{array}$$

In both cases we have reduced the length of the chain of $(X \Rightarrow)^{**}$ or eliminated the anormality if there is no chain.

$(As_2 \Rightarrow)^{**}$

$$\dfrac{\dfrac{\begin{matrix}\vdots_{(X\Rightarrow)^{**}}\\ \beta\end{matrix} \quad \dfrac{[\beta]^4 \quad \wedge\mathrm{E}\,\dfrac{[\beta]^3}{E} \quad \dfrac{[\beta]^3 \quad \wedge\mathrm{I}_1\,\dfrac{[\beta]^2 \quad \wedge\mathrm{E}\,\dfrac{[\beta]^1}{A} \quad \wedge\mathrm{E}\,\dfrac{[\beta]^1}{B}}{A \wedge B} \quad \wedge\mathrm{E}\,\dfrac{[\beta]^2}{C}}{(A \wedge B) \wedge C}\,\wedge\mathrm{I}_2}{E \wedge ((A \wedge B) \wedge C)}\,\wedge\mathrm{I}_3 \quad \wedge\mathrm{E}\,\dfrac{[\beta]^4}{F}}{\alpha}\,\wedge\mathrm{I}_4 \quad \begin{matrix}[\gamma]^5\\ \vdots\\ G\end{matrix}}{G}\,\wedge\mathrm{E}_5$$

Where $\alpha = (E \wedge ((A \wedge B) \wedge C)) \wedge F$ and $\beta = (E \wedge (A \wedge (B \wedge C))) \wedge F$.

If $\gamma = F$, then we reduce to

$$\begin{matrix}\vdots_{(X\Rightarrow)^{**}}\\ \wedge\mathrm{E}\,\dfrac{(E \wedge (A \wedge (B \wedge C))) \wedge F}{A}\\ \vdots\\ G\end{matrix}$$

If $\gamma = E \wedge ((A \wedge B) \wedge C)$, then we reduce to

$$\begin{matrix}\dfrac{\begin{matrix}\vdots_{(X\Rightarrow)^{**}}\\ \beta\end{matrix} \quad \wedge\mathrm{E}\,\dfrac{[\beta]^3}{E} \quad \dfrac{[\beta]^3 \quad \wedge\mathrm{I}_1\,\dfrac{[\beta]^2 \quad \wedge\mathrm{E}\,\dfrac{[\beta]^1}{A} \quad \wedge\mathrm{E}\,\dfrac{[\beta]^1}{B}}{A \wedge B} \quad \wedge\mathrm{E}\,\dfrac{[\beta]^2}{C}}{(A \wedge B) \wedge C}\,\wedge\mathrm{I}_2}{E \wedge ((A \wedge B) \wedge C)}\,\wedge\mathrm{I}_3\\ \vdots\\ G\end{matrix}$$

If $E \wedge ((A \wedge B) \wedge C)$ creates anormality, then it is the major premise of $E\wedge$. Let us call δ the assumption it discharges.

If $\delta = E$, then we reduce to

$$\begin{matrix}\vdots_{(X\Rightarrow)^{**}}\\ \wedge\mathrm{E}\,\dfrac{(E \wedge (A \wedge (B \wedge C))) \wedge F}{E}\\ \vdots\\ G\end{matrix}$$

If $\delta = (A \wedge B) \wedge C$, then we reduce to

$$\begin{matrix}\dfrac{\begin{matrix}\vdots_{(X\Rightarrow)^{**}}\\ \beta\end{matrix} \quad \wedge\mathrm{I}_1\,\dfrac{[\beta]^2 \quad \wedge\mathrm{E}\,\dfrac{[\beta]^1}{A} \quad \wedge\mathrm{E}\,\dfrac{[\beta]^1}{B}}{A \wedge B} \quad \wedge\mathrm{E}\,\dfrac{[\beta]^2}{C}}{(A \wedge B) \wedge C}\,\wedge\mathrm{I}_2\\ \vdots\\ G\end{matrix}$$

If $(A \wedge B) \wedge C$ creates anormality, then it is the major premise of $\wedge$E. Let us call ε the assumption it discharges.

If $\varepsilon = C$, then we reduce to

$$\begin{matrix}\vdots_{(X\Rightarrow)^{**}}\\ \wedge\mathrm{E}\,\dfrac{(E \wedge (A \wedge (B \wedge C))) \wedge F}{C}\\ \vdots\\ G\end{matrix}$$

If $\delta = A \wedge B$, then we reduce to

$$\wedge\mathrm{I}_1\ \dfrac{\overset{\vdots_{(X\Rightarrow)^{**}}}{\beta} \qquad \wedge\mathrm{E}\ \dfrac{[\beta]^1}{A} \qquad \wedge\mathrm{E}\ \dfrac{[\beta]^1}{B}}{\begin{array}{c} A \wedge B \\ \vdots \\ G \end{array}}$$

If $A \wedge B$ creates anormality, then it is the major premise of $\wedge$E, and the assumption it discharges can only be A or B. We then reduce to

$$\wedge\mathrm{E}\ \dfrac{\overset{\vdots_{(X\Rightarrow)^{**}}}{(E \wedge (A \wedge (B \wedge C))) \wedge F}}{\begin{array}{c} A \\ \vdots \\ G \end{array}} \quad \text{or} \quad \wedge\mathrm{E}\ \dfrac{\overset{\vdots_{(X\Rightarrow)^{**}}}{(E \wedge (A \wedge (B \wedge C))) \wedge F}}{\begin{array}{c} B \\ \vdots \\ G \end{array}}$$

In both cases we have reduced the length of the chain of $(X \Rightarrow)^{**}$ or eliminated the anormality if there is no chain.

$(Comm \Rightarrow)^{**}$

$$\wedge\mathrm{E}_4\ \dfrac{\begin{array}{c} \wedge\mathrm{I}_3\ \dfrac{\overset{\vdots_{(X\Rightarrow)^{**}}}{\alpha} \qquad \wedge\mathrm{E}\ \dfrac{[\alpha]^3}{C} \qquad \wedge\mathrm{I}_2\ \dfrac{[\alpha]^3 \qquad \wedge\mathrm{I}_1\ \dfrac{[\alpha]^2 \qquad \wedge\mathrm{E}\ \dfrac{[\alpha]^1}{A} \qquad \wedge\mathrm{E}\ \dfrac{[\alpha]^1}{B}}{A \wedge B} \qquad \wedge\mathrm{E}\ \dfrac{[\alpha]^2}{D}}{(A \wedge B) \wedge D}}{\beta} \\ \vdots \\ D \end{array} \qquad \begin{array}{c} [\gamma]^4 \\ \vdots \\ G \end{array}}{G}$$

Where $\alpha = C \wedge ((B \wedge A) \wedge D)$ and $\beta = C \wedge ((A \wedge B) \wedge D)$.

If $\gamma = C$, then we reduce to $\wedge\mathrm{E}\ \dfrac{\overset{\vdots_{(X\Rightarrow)^{**}}}{C \wedge ((B \wedge A) \wedge D)}}{\begin{array}{c} C \\ \vdots \\ G \end{array}}$

If $\gamma = (A \wedge B) \wedge D$, then we reduce to

$$\wedge\mathrm{I}_2\ \dfrac{\overset{\vdots_{(X\Rightarrow)^{**}}}{\alpha} \qquad \wedge\mathrm{I}_1\ \dfrac{[\alpha]^2 \qquad \wedge\mathrm{E}\ \dfrac{[\alpha]^1}{A} \qquad \wedge\mathrm{E}\ \dfrac{[\alpha]^1}{B}}{A \wedge B} \qquad \wedge\mathrm{E}\ \dfrac{[\alpha]^2}{D}}{\begin{array}{c} (A \wedge B) \wedge D \\ \vdots \\ D \\ \vdots \\ G \end{array}}$$

If $(A \wedge B) \wedge D$ creates anormality, then it is the major premise of $\wedge$E. Let us call δ the assumption that it discharges.

If $\delta = D$, then we reduce to

$$\dfrac{\vdots_{(X\Rightarrow)^{**}} \\ C \wedge ((B \wedge A) \wedge D)}{D}\ {\scriptstyle \wedge E} \quad \vdots \quad G$$

If $\delta = A \wedge B$, then we reduce to

$$\wedge I_1 \dfrac{\dfrac{\vdots_{(X\Rightarrow)^{**}}}{\alpha} \quad \wedge E\dfrac{[\alpha]^1}{A} \quad \wedge E\dfrac{[\alpha]^1}{B}}{A \wedge B} \quad \vdots \quad D \quad \vdots \quad G$$

If $A \wedge B$ creates anormality, then it is the major premise of $\wedge$E. The discharged assumption can only be A or B. We then reduce to

$$\wedge E \dfrac{\vdots_{(X\Rightarrow)^{**}} \quad C \wedge ((B \wedge A) \wedge D)}{A} \quad \vdots \quad G \qquad \text{or} \qquad \wedge E \dfrac{\vdots_{(X\Rightarrow)^{**}} \quad C \wedge ((B \wedge A) \wedge D)}{B} \quad \vdots \quad G$$

In both cases we have reduced the length of the chain of $(X \Rightarrow)^{**}$ or eliminated the anormality if there is no chain.

This last case ends our proof.

□

B.2.3 Existence of normal form in SASCNK

All we have to show to complete the proof of theorem 3.3.9 is that:

1. The only anormalities in a SASCNK-derivation obtained by translating a semi-Cut-free SASCLK-derivation can arise from $(As \Rightarrow)^{**}$, $(Comm \Rightarrow)^{**}$, $(Idem \Rightarrow)^{**}$, $(\Rightarrow As)^{**}$, $(\Rightarrow Comm)^{**}$ or $(\Rightarrow Idem)^{**}$;
2. We can deal with these anormality.

Let us start from the first point. First of all let us see that:

Lemma B.2.1. $(Distri^{\vee\wedge})^{**}$ *does not create maximal formulae.*

Proof. First of all, let us check that there are no maximal formulae inside $(Distri^{\vee\wedge})^{**}$. In $(Distri^{\vee\wedge})^{**}$ we have two major premises of E-rules that are derived, that correspond to the two applications of $(Comm \Rightarrow)^{**}$. The first on the right could create a maximal formula of the kind $\wedge$I-$\vee$E-$\wedge$E. However, only one of the assumptions discharged by $\vee$E is major premise of $\wedge$E, while in order to have a maximal formula, both should be so (definition 3.3.2). The second has the same structure, and can not create a maximal formula, because the discharged assumption on the left is major premise of $\wedge$I and not of $\wedge$E (in the example of translation given at the end of section B.1.3, $[C \wedge (A \vee C)]^4$ is major premise of $\wedge$E, while $[B \wedge (A \vee C)]^4$ is major premise of $\wedge$I). So there is no maximal formula inside $(Distri^{\vee\wedge})^{**}$.

Let us now consider the possibility that $(Distri^{\vee\wedge})^{**}$ creates maximal formulae by composition, that is by the rule used to derive $(A \vee C) \wedge (B \vee C)$ as last step of $\mathtt{d}_1^*$, together with the rules of $(Distri^{\vee\wedge})^{**}$. The only rule that could create a maximal formula in this way is $\vee$I (in this way we could have a maximal formula of the kind $\vee$I-$\wedge$I-$\vee$E). Nonetheless this possibility is excluded by the form of $(A \vee C) \wedge (B \vee C)$ itself, since the only I-rule that could derive it is $\wedge$I. So $(Distri^{\vee\wedge})^{**}$ does not create maximal formulae neither inside itself, nor by composition. □

With this lemma, we can now prove clause *1*:

Proof. The translation in proof B.1.3 uses only non-derived major premises, so *a fortiori* only non-derived major premises of E-rules. It follows that the only way to obtain anormality is by composition, and the only rule of **SASCLK** that ask for composition is Cut. We have already argued that the semi-Cut-free fragment of **SASCLK** is complete, so we have to deal only with $(As \Rightarrow)^{**}$, $(Comm \Rightarrow)^{**}$, $(Idem \Rightarrow)^{**}$, $(\Rightarrow As)^{**}$, $(\Rightarrow Comm)^{**}$ and $(\Rightarrow Idem)^{**}$. By lemma B.2.1, we know that $(\Rightarrow Idem)^{**}$ does not create maximal formulae, so let us consider the other cases.

Since $(X \Rightarrow)^{**}$ give rise to chains, we have to deal with chains of $(X \Rightarrow)^{**}$ that derive major premises of $\wedge E$ as in the intuitionistic case. We will see that also the solution for this cases is the same as in the intuitionistic one.

Let us now consider $(\Rightarrow X)^{**}$.[29] These steps of derivation allow to compose deductions on the top, that is the top-formula of $(\Rightarrow X)^{**}$ (that is a major premise of $\vee E$) can be derived by an I-rule. Of course the conclusion of $(\Rightarrow X)^{**}$ can itself be a top-formula of $(\Rightarrow X)^{**}$, thanks to composition. So we have to deal with chains of $(\Rightarrow X)^{**}$ that eliminate conclusions of I-rules.

The only situation in which we obtain an anormality is when the top-formula of the chain of $(\Rightarrow X)^{**}$ is the conclusion of $\vee I$. Indeed, each $(\Rightarrow X)^{**}$ derives its conclusion by $\vee E$ with its open assumption as major premise, and applies only $\vee$-rules.

It is important also to notice that there are no anormalities in $(\Rightarrow X)^{**}$, and that their conclusion is always a conclusion of $\vee E$. So there are no anormalities in the chains of $(\Rightarrow X)^{**}$ (neither in a single $(\Rightarrow X)^{**}$, nor in the conclusion of an $(\Rightarrow X)^{**}$ being the major premise of the next $(\Rightarrow X)^{**}$), only its first formula can give anormality with the rule that derives it. So we have chains of $(\Rightarrow X)^{**}$ that can create a maximal formula only at the point of application at their top.

□

Now let us see that we can deal with these anormalities.

Proof. We can deal with maximal formulae caused by chains of $(X \Rightarrow)^{**}$ in the same way as we did regarding **SASCNJ** in preivous section B.2.2.

Let us consider a chain of $(\Rightarrow X)^{**}$ that derives an anormality, and in particular the first $(\Rightarrow X)^{**}$, that directly causes the anormality. We will remove completely the anormality or we will remove the first $(\Rightarrow X)^{**}$, so by induction on the length of the chain we obtain the normalization. By cases on the last $(\Rightarrow X)^{**}$.

$(\Rightarrow Idem)^{**}$

$$\vee\mathrm{E}_2\,\dfrac{\vee\mathrm{I}\,\dfrac{\overset{\vdots}{\alpha}}{(A \vee A) \vee B} \qquad \vee\mathrm{I}\,\dfrac{\vee\mathrm{E}_1\,\dfrac{[A \vee A]^2 \qquad [A]^1 \qquad [A]^1}{A}}{A \vee B} \qquad \vee\mathrm{I}\,\dfrac{[B]^2}{A \vee B}}{\underset{\vdots_{(\Rightarrow X)^{**}}}{A \vee B}}$$

If $\alpha = B$, we remove the last $(\Rightarrow Idem)^{**}$ and reduce to:

$$\vee\mathrm{I}\,\dfrac{\overset{\vdots}{B}}{\underset{\vdots_{(\Rightarrow X)^{**}}}{A \vee B}}$$

If $\alpha = A \vee A$, we reduce to

$$\wedge\mathrm{E}_1\,\dfrac{\overset{\vdots}{A \vee A} \qquad [A]^1 \qquad [A]^1}{\vee\mathrm{I}\,\dfrac{A}{\underset{\vdots_{(\Rightarrow X)^{**}}}{A \vee B}}}$$

In this case we have an anormality for $A \vee B$ if $\vdots_{(\Rightarrow X)^{**}} \neq \varnothing$, and an anormality for $A \vee A$ if it is derived by $\vee$I. This last anormality is easily treated, since the only premise of such an introduction discharged could be A, and so we could reduce to

[29] I will use $(\Rightarrow X)^{**}$ to indicate each of $(\Rightarrow As)^{**}$, $(\Rightarrow Comm)^{**}$ and $(\Rightarrow Idem)^{**}$.

$$\vee\mathrm{I}\,\dfrac{\begin{matrix}\vdots\\ A\end{matrix}}{A \vee B}$$
$$\vdots_{(\Rightarrow X)^{**}}$$

So we only have the anormality generated by $A \vee B$, if $\vdots_{(\Rightarrow X)^{**}}$ is not empty, and we have normalized or at least reduced the length of the chain.

$(\Rightarrow As_1)^{**}$.

$$\vee\mathrm{E}_4\,\dfrac{\vee\mathrm{I}\,\dfrac{\begin{matrix}\vdots\\ \alpha\end{matrix}}{\gamma} \qquad \vee\mathrm{E}_3\,\dfrac{[E \vee ((B \vee C) \vee D)]^4 \qquad \vee\mathrm{I}\,\dfrac{[E]^3}{\delta} \qquad \vee\mathrm{E}_2\,\dfrac{[(B \vee C) \vee D]^3 \qquad \vee\mathrm{E}_1\,\dfrac{[B \vee C]^2 \qquad \vee\mathrm{I}\,\dfrac{[B]^1}{\delta} \qquad \vee\mathrm{I}\,\dfrac{[C]^1}{\delta}}{\delta} \qquad \vee\mathrm{I}\,\dfrac{[D]^2}{\delta}}{\delta}}{\delta} \qquad \vee\mathrm{I}\,\dfrac{[F]^4}{\delta}}{\delta}$$
$$\vdots_{(\Rightarrow X)^{**}}$$

Where $\gamma = (E \vee ((B \vee C) \vee D)) \vee F$ and $\delta = (E \vee (B \vee (C \vee D))) \vee F$.

If $\alpha = F$, then we reduce the length of the chain by:
$$\vee\mathrm{I}\,\dfrac{\begin{matrix}\vdots\\ F\end{matrix}}{(E \vee (B \vee (C \vee D))) \vee F}$$
$$\vdots_{(\Rightarrow X)^{**}}$$

If $\alpha = E \vee ((B \vee C) \vee D)$, then we reduce the length of the chain by:

$$\vee\mathrm{E}_3\,\dfrac{\begin{matrix}\vdots\\ E \vee ((B \vee C) \vee D)\end{matrix} \qquad \vee\mathrm{I}\,\dfrac{[E]^3}{\delta} \qquad \vee\mathrm{E}_2\,\dfrac{[(B \vee C) \vee D]^3 \qquad \vee\mathrm{E}_1\,\dfrac{[B \vee C]^2 \qquad \vee\mathrm{I}\,\dfrac{[B]^1}{\delta} \qquad \vee\mathrm{I}\,\dfrac{[C]^1}{\delta}}{\delta} \qquad \vee\mathrm{I}\,\dfrac{[D]^2}{\delta}}{\delta}}{\delta}$$
$$\vdots_{(\Rightarrow X)^{**}}$$

If it creates anormality, then $E \vee ((B \vee C) \vee D)$ is the conclusion of an $\vee I$, let us call β its premise.

If $\beta = E$, then we reduce to

$$\vee\mathrm{I}\,\dfrac{\begin{matrix}\vdots\\ E\end{matrix}}{(E \vee (B \vee (C \vee D))) \vee F}$$
$$\vdots_{(\Rightarrow X)^{**}}$$

If $\beta = (B \vee C) \vee D$, then we reduce to

$$\vee\mathrm{E}_2\,\dfrac{\begin{matrix}\vdots\\ (B \vee C) \vee D\end{matrix} \qquad \vee\mathrm{E}_1\,\dfrac{[B \vee C]^2 \qquad \vee\mathrm{I}\,\dfrac{[B]^1}{\delta} \qquad \vee\mathrm{I}\,\dfrac{[C]^1}{\delta}}{\delta} \qquad \vee\mathrm{I}\,\dfrac{[D]^2}{\delta}}{\delta}$$
$$\vdots_{(\Rightarrow X)^{**}}$$

If $(B \vee C) \vee D$ creates anormality, then it is conclusion of $\vee I$, let us call ε its premise.

If $\varepsilon = D$, then we reduce to

$$\vee\mathrm{I}\ \dfrac{\begin{array}{c}\vdots\\ D\end{array}}{(E \vee (B \vee (C \vee D))) \vee F}$$
$$\vdots_{(\Rightarrow X)^{**}}$$

If $\varepsilon = B \vee C$, then we reduce to

$$\vee\mathrm{E}_1\ \dfrac{\begin{array}{c}\vdots\\ B \vee C\end{array} \qquad \vee\mathrm{I}\ \dfrac{[B]^1}{\delta} \qquad \vee\mathrm{I}\ \dfrac{[C]^1}{\delta}}{\delta}$$
$$\vdots_{(\Rightarrow X)^{**}}$$

If $B \vee C$ creates anormality, then it is conclusion of $\vee I$, and its premise can only be B or C. We then reduce to

$$\vee\mathrm{I}\ \dfrac{\begin{array}{c}\vdots\\ B\end{array}}{(E \vee (B \vee (C \vee D))) \vee F} \quad \text{or} \quad \vee\mathrm{I}\ \dfrac{\begin{array}{c}\vdots\\ C\end{array}}{(E \vee (B \vee (C \vee D))) \vee F}$$
$$\vdots_{(\Rightarrow X)^{**}} \qquad\qquad \vdots_{(\Rightarrow X)^{**}}$$

In both cases we have reduced the length of the chain of $(\Rightarrow X)^{**}$ or eliminated the anormality if there is no such chain.

$(\Rightarrow As_2)^{**}$.

$$\vee\mathrm{E}_4\ \dfrac{\vee\mathrm{I}\ \dfrac{\begin{array}{c}\vdots\\ \alpha\end{array}}{\gamma} \qquad \vee\mathrm{E}_3\ \dfrac{[E \vee (B \vee (C \vee D))]^4 \qquad \vee\mathrm{I}\ \dfrac{[E]^3}{\delta} \qquad \vee\mathrm{E}_2\ \dfrac{[B \vee (C \vee D)]^3 \qquad \vee\mathrm{I}\ \dfrac{[B]^2}{\delta} \qquad \vee\mathrm{E}_1\ \dfrac{[C \vee D]^2 \qquad \vee\mathrm{I}\ \dfrac{[C]^1}{\delta} \qquad \vee\mathrm{I}\ \dfrac{[D]^1}{\delta}}{\delta}}{\delta}}{\delta} \qquad \vee\mathrm{I}\ \dfrac{[F]^4}{\delta}}{\delta}$$
$$\vdots_{(\Rightarrow X)^{**}}$$

Where $\gamma = (E \vee (B \vee (C \vee D))) \vee F$ and $\delta = (E \vee ((B \vee C) \vee D)) \vee F$.

If $\alpha = F$, then we reduce the length of the chain by: $\vee\mathrm{I}\ \dfrac{\begin{array}{c}\vdots\\ F\end{array}}{(E \vee ((B \vee C) \vee D)) \vee F}$
$$\vdots_{(\Rightarrow X)^{**}}$$

If $\alpha = E \vee (B \vee (C \vee D))$, then we reduce the length of the chain by:

$$\vee\mathrm{E}_3\ \dfrac{\begin{array}{c}\vdots\\ E \vee (B \vee (C \vee D))\end{array} \qquad \vee\mathrm{I}\ \dfrac{[E]^3}{\delta} \qquad \vee\mathrm{E}_2\ \dfrac{[B \vee (C \vee D)]^3 \qquad \vee\mathrm{I}\ \dfrac{[B]^2}{\delta} \qquad \vee\mathrm{E}_1\ \dfrac{[C \vee D]^2 \qquad \vee\mathrm{I}\ \dfrac{[C]^1}{\delta} \qquad \vee\mathrm{I}\ \dfrac{[D]^1}{\delta}}{\delta}}{\delta}}{\delta}$$
$$\vdots_{(\Rightarrow X)^{**}}$$

If it creates anormality, then $E \vee (B \vee (C \vee D))$ is the conclusion of an $\vee I$, let us call β its premise.

If $\beta = E$, then we reduce to

$$\vee\mathrm{I}\ \dfrac{\begin{array}{c}\vdots\\ E\end{array}}{(E \vee ((B \vee C) \vee D)) \vee F}$$
$$\vdots_{(\Rightarrow X)^{**}}$$

If $\beta = B \vee (C \vee D)$, then we reduce to

$$\vee\mathrm{E}_2\ \dfrac{\begin{matrix}\vdots\\ B \vee (C \vee D)\end{matrix} \qquad \vee\mathrm{I}\,\dfrac{[B]^2}{\delta} \qquad \vee\mathrm{E}_1\,\dfrac{[C \vee D]^2 \qquad \vee\mathrm{I}\,\dfrac{[C]^1}{\delta} \qquad \vee\mathrm{I}\,\dfrac{[D]^1}{\delta}}{\delta}}{\delta}$$

$$\vdots_{(\Rightarrow X)^{**}}$$

If $B \vee (C \vee D)$ creates anormality, then it is conclusion of $\vee I$, let us call ε its premise.
If $\varepsilon = B$, then we reduce to

$$\vee\mathrm{I}\,\dfrac{\begin{matrix}\vdots\\ B\end{matrix}}{(E \vee ((B \vee C) \vee D)) \vee F}$$

$$\vdots_{(\Rightarrow X)^{**}}$$

If $\varepsilon = C \vee D$, then we reduce to

$$\vee\mathrm{E}_1\ \dfrac{\begin{matrix}\vdots\\ C \vee D\end{matrix} \qquad \vee\mathrm{I}\,\dfrac{[C]^1}{\delta} \qquad \vee\mathrm{I}\,\dfrac{[D]^1}{\delta}}{\delta}$$

$$\vdots_{(\Rightarrow X)^{**}}$$

If $C \vee D$ creates anormality, then it is conclusion of $\vee I$, and its premise can only be C or D. We then reduce to

$$\vee\mathrm{I}\,\dfrac{\begin{matrix}\vdots\\ C\end{matrix}}{(E \vee ((B \vee C) \vee D)) \vee F} \qquad \text{or} \qquad \vee\mathrm{I}\,\dfrac{\begin{matrix}\vdots\\ D\end{matrix}}{(E \vee ((B \vee C) \vee D)) \vee F}$$

$$\vdots_{(\Rightarrow X)^{**}} \qquad\qquad\qquad \vdots_{(\Rightarrow X)^{**}}$$

In both cases we have reduced the length of the chain of $(\Rightarrow X)^{**}$ or eliminated the anormality if there is no such chain.

$$(\Rightarrow Comm)^{**}$$

$$\dfrac{\vee\mathrm{I}\,\dfrac{\begin{matrix}\vdots\\ \alpha\end{matrix}}{\gamma} \qquad \vee\mathrm{I}\,\dfrac{[C]^3}{\delta} \qquad \dfrac{[(A \vee B) \vee D]^3 \qquad \vee\mathrm{E}_1\,\dfrac{[A \vee B]^2 \qquad \vee\mathrm{I}\,\dfrac{[A]^1}{\delta} \qquad \vee\mathrm{I}\,\dfrac{[B]^1}{\delta}}{\delta} \qquad \vee\mathrm{I}\,\dfrac{[D]^2}{\delta}}{\delta}\,\vee\mathrm{E}_2}{\delta}\,\vee\mathrm{E}_3$$

$$\vdots_{(\Rightarrow X)^{**}}$$

Where $\gamma = C \vee ((A \vee B) \vee D)$ and $\delta = C \vee ((B \vee A) \vee D)$.

If $\alpha = C$, then we reduce to $\vee\mathrm{I}\,\dfrac{\begin{matrix}\vdots\\ C\end{matrix}}{C \vee ((B \vee A) \vee D)}$

$$\vdots_{(\Rightarrow X)^{**}}$$

If $\alpha = (A \vee B) \vee D$, then we reduce to

$$\cfrac{\begin{array}{c}\vdots\\(A\lor B)\lor D\end{array}\qquad {\lor}\mathrm{E}_1\,\cfrac{[A\lor B]^2\qquad {\lor}\mathrm{I}\,\cfrac{[A]^1}{\delta}\qquad {\lor}\mathrm{I}\,\cfrac{[B]^1}{\delta}}{\delta}\qquad {\lor}\mathrm{I}\,\cfrac{[D]^2}{\delta}}{\delta}\,{\lor}\mathrm{E}_2$$

$$\vdots_{(\Rightarrow X)^{**}}$$

If $(A \lor B) \lor D$ creates anormality, then it is the conclusion of $\lor$I. Let us call β its premise.

If $\beta = D$, then we reduce to $${\lor}\mathrm{I}\,\cfrac{\begin{array}{c}\vdots\\D\end{array}}{C\lor((B\lor A)\lor D)}$$

$$\vdots_{(\Rightarrow X)^{**}}$$

If $\beta = A \lor B$, then we reduce to

$${\lor}\mathrm{E}_1\,\cfrac{\begin{array}{c}\vdots\\A\lor B\end{array}\qquad {\lor}\mathrm{I}\,\cfrac{[A]^1}{\delta}\qquad {\lor}\mathrm{I}\,\cfrac{[B]^1}{\delta}}{\delta}$$

$$\vdots_{(\Rightarrow X)^{**}}$$

If $A \lor B$ creates anormality, then it is the conclusion of $\lor$I. Its premise can only be A or B. We then reduce to

$${\lor}\mathrm{I}\,\cfrac{\begin{array}{c}\vdots\\A\end{array}}{C\lor((B\lor A)\lor D)}\quad\text{or}\quad {\lor}\mathrm{I}\,\cfrac{\begin{array}{c}\vdots\\B\end{array}}{C\lor((B\lor A)\lor D)}$$

$$\vdots_{(\Rightarrow X)^{**}}\qquad\qquad\vdots_{(\Rightarrow X)^{**}}$$

In both cases we have reduced the length of the chain of $(\Rightarrow X)^{**}$ or eliminated the anormality if there is no such chain.

This last case ends our proof.

$\square$

B.3 Intuitionistic and dual-intuitionistic negations

[30]

Craig-Lyndon interpolation theorem for classical logic says that:[31]

Theorem B.3.1 (Craig-Lyndon Interpolation)**.** *If* $\Gamma \vdash_K \Delta$ *then:*

- $\Gamma \vdash_K$ *or;*
- $\vdash_K \Delta$ *or;*
- *There is a formula* χ *(interpolant) such that all its non-logical terms belong both to* Γ *and to* Δ*, and such that* $\Gamma \vdash_K \chi$ *and* $\chi \vdash_K \Delta$*.*

[Milne, 2017] refined this theorem showing that:

Theorem B.3.2 (Milne Interpolation)**.** *If* $\Gamma \vdash_K \Delta$ *then:*

- $\Gamma \vdash_{K3}$ *or;*

[30] For brevity, in this section we will use interchangeably $\Rightarrow$ and $\vdash$ since the adoption of the metalinguistic interpretation of sequents; see section 2.4.1.

[31] [Takeuti, 1987], lemma 6.5 and theorem 6.6, with minimal changes.

- $\vdash_{LP} \Delta$ *or;*
- *There is a formula* χ *(the interpolant) such that all its non-logical terms belong both to* Γ *and to* Δ*, and such that* $\Gamma \vdash_{K3} \chi$ *and* $\chi \vdash_{LP} \Delta$.

Where **K3** is Kleene's three-valued strong logic and **LP** is Priest's logic of paradox. In this section, we will prove that a similar result holds about intuitionistic and dual-intuitionistic logics, that (as we argued in section 4.2.2) can be used to show some bad consequences of the thesis of the identity of intuitionistic and dual-intuitionistic negations.

Milne's proof is based on a very precise analysis of tableaux derivation for $\Gamma \vdash_K \Delta$, while we will prove our result metatheoretically, using Glivenko theorem.[32] This theorem only holds for the propositional fragment of **K** and **I**, and it can be adapted for **DI** only as long as we confine ourselves to the propositional fragment, so we will refine only the propositional part of the interpolation theorem. Since we will use different versions of Glivenko theorem, we will prove their equivalence.

Theorem B.3.3 (Glivenko)**.** *1.* $A \vdash_K$ *iff* $A \vdash_I$*;*

2. $\vdash_K \neg A$ *iff* $\vdash_I \neg A$*;*

3. $\vdash_K A$ *iff* $\vdash_I \neg\neg A$.

Proof. We will show that *1* entails *2*, *2* entails *3*, and *3* entails *1*. In this way, we obtain equivalence between the three formulations.[33]

1 entails 2: Assuming $\vdash_K \neg A$, we derive $\neg\neg A \vdash_K$, so for double negation introduction and transitivity we obtain $A \vdash_K$. Applying *1*, we obtain $A \vdash_I$, and so $\vdash_I \neg A$;

2 entails 3: Assuming $\vdash_K A$, we derive $\vdash_K \neg\neg A$. Now we apply *2* and conclude $\vdash_I \neg\neg A$.

3 entails 1: Assuming $A \vdash_K$, we derive $\vdash_K \neg A$ and so, by *3*, $\vdash_I \neg\neg\neg A$. Since $\neg\neg\neg A \vdash_I \neg A$, we obtain $\vdash_I \neg A$ and so $\neg\neg A \vdash_I$. Introduction of double negation and transitivity give us the result $A \vdash_I$.

□

We also need a stronger version of this theorem:[34]

Theorem B.3.4 (Strong Glivenko)**.** $\Gamma \vdash_K \neg A$ *iff* $\Gamma \vdash_I \neg A$

Proof. Let us assume $\Gamma \vdash_K \neg A$ and use $\Gamma^\wedge$ to refer to the conjunction of all formulae in Γ. From this we obtain $\vdash_K \Gamma^\wedge \supset \neg A$ and so, applying the formulation *3* of standard Glivenko theorem B.3.3, we derive $\vdash_I \neg\neg(\Gamma^\wedge \supset \neg A)$. We can derive $\neg\neg(B \supset \neg C) \vdash_I B \supset \neg C$ by:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \Rightarrow C}{C, \neg C \Rightarrow}\neg\Rightarrow \qquad B \Rightarrow B}{B, C, B \supset \neg C \Rightarrow}\supset\Rightarrow}{B, C \Rightarrow \neg(B \supset \neg C)}\Rightarrow\neg}{B, C, \neg\neg(B \supset \neg C) \Rightarrow}\neg\Rightarrow}{B, \neg\neg(B \supset \neg C) \Rightarrow \neg C}\Rightarrow\neg}{\neg\neg(B \supset \neg C) \Rightarrow B \supset \neg C}\Rightarrow\supset$$

So we obtain $\vdash_I \Gamma^\wedge \supset \neg A$ and from this, since *Modus Ponens* and transitivity hold in **I**, we obtain $\Gamma \vdash_I \neg A$. □

We still need to prove equivalent results for dual-intuitionistic logic (**DI**), in order to be able to prove our main theorem. Dual-intuitionistic logic can be obtained by restricting sequent calculus system **LK** for classical logic to having at most one formula in the antecedent. In this way we obtain the system displayed in table B.1. The connective $\dot{-}$, that have to be taken as primitive since it is not definable in **DI**, is introduced in order to have theorem B.3.5, and it is sometimes called 'subtraction'. On the other hand, we do not need to take $A \supset B$ as primitive in **DI**, since it is definable as $\neg A \vee B$, and we do not need to take $A \dot{-} B$ as primitive in **I**, since it is definable as $A \wedge \neg B$.[35]

[32] [Glivenko, 1929].

[33] We overlook the obvious direction of each formulation of the theorem.

[34] Chapter 3 of [Mints, 2000] and [Humberstone, 2011], p. 306. The proof presented here is based on Mint's one.

[35] This is a very concise definition of this system, I refer to [Urbas, 1996] for a deeper introduction.

Axiom

$$\frac{}{A \Rightarrow A}$$

Structural rules

$$Weakening \Rightarrow \frac{\Rightarrow \ \Delta}{A \ \Rightarrow \ \Delta} \qquad \Rightarrow Weakening \frac{C \ \Rightarrow \ \Delta}{C \ \Rightarrow \ A, \Delta}$$

$$Cut \frac{C \ \Rightarrow \ A, \Delta \qquad A \ \Rightarrow \ \Lambda}{C \ \Rightarrow \ \Delta, \Lambda}$$

Operational rules

$$\wedge \Rightarrow \frac{A \ \Rightarrow \ \Delta}{A \wedge B \ \Rightarrow \ \Delta} \qquad \wedge \Rightarrow \frac{B \ \Rightarrow \ \Delta}{A \wedge B \ \Rightarrow \ \Delta}$$

$$\Rightarrow \wedge \frac{C \ \Rightarrow \ A, \Delta \qquad C \ \Rightarrow \ B, \Delta}{C \ \Rightarrow \ A \wedge B, \Delta}$$

$$\vee \Rightarrow \frac{A \ \Rightarrow \ \Delta \qquad B \ \Rightarrow \ \Delta}{A \vee B \ \Rightarrow \ \Delta}$$

$$\Rightarrow \vee \frac{C \ \Rightarrow \ B, \Delta}{C \ \Rightarrow \ A \vee B, \Delta} \qquad \Rightarrow \vee \frac{C \ \Rightarrow \ A, \Delta}{C \ \Rightarrow \ A \vee B, \Delta}$$

$$\Rightarrow \dot{-} \frac{C \ \Rightarrow \ A, \Delta \qquad B \ \Rightarrow \ \Lambda}{C \ \Rightarrow \ A \dot{-} B, \Delta, \Lambda} \qquad \dot{-} \Rightarrow \frac{A \ \Rightarrow \ B, \Delta}{A \dot{-} B \ \Rightarrow \ \Delta}$$

$$\neg \Rightarrow \frac{\Rightarrow \ A, \Delta}{\neg A \ \Rightarrow \ \Delta} \qquad \Rightarrow \neg \frac{A \ \Rightarrow \ \Delta}{\Rightarrow \ \neg A, \Delta}$$

$$\forall \Rightarrow \frac{F(a) \ \Rightarrow \ \Delta}{\forall x F(x) \ \Rightarrow \ \Delta} \qquad \Rightarrow \forall \frac{C \ \Rightarrow \ F(a), \Delta}{C \ \Rightarrow \ \forall x F(x), \Delta}$$

$$\exists \Rightarrow \frac{F(a) \ \Rightarrow \ \Delta}{\exists x F(x) \ \Rightarrow \ \Delta} \qquad \Rightarrow \exists \frac{C \ \Rightarrow \ F(a), \Delta}{C \ \Rightarrow \ \exists x F(x), \Delta}$$

Restriction on the variables: a must not occur in C or Δ for $R\forall$ and $L\exists$ to be applicable (that is it must not occur in the lower sequent of the rule).

Definition B.3.1 (Duality)**.** Given a formula A, we define by induction its dual A^d: $A^d = A$, for A atomic; $(\neg A)^d = \neg A^d$; $(A \wedge B)^d = A^d \vee B^d$; $(A \vee B)^d = A^d \wedge B^d$; $(A \supset B)^d = B^d \dot{-} A^d$; $(A \dot{-} B)^d = B^d \supset A^d$.

Theorem B.3.5 (Duality)**.** *Given the previous definition B.3.1, we have:*

1. $\Gamma \vdash_I C$ *iff* $C^d \vdash_{DI} \Gamma^d$;[36]
2. $\Gamma \vdash_K \Delta$ *iff* $\Delta^d \vdash_K \Gamma^d$.

Proof. Technically speaking, result *2* is absent in [Urbas, 1996], but it can be easily derived by the first one. From $\Gamma \vdash_K \Delta$, we derive $\Gamma \vdash_K \neg\neg\Delta^\vee$ (defined in the obvious way), and so for strong Glivenko theorem we have $\Gamma \vdash_I \neg\neg\Delta^\vee$. By *1*, we obtain $(\neg\neg\Delta^\vee)^d \vdash_{DI} \Gamma^d$. From this, by definition of duality, we first obtain $\neg\neg(\Delta^\vee)^d \vdash_{DI} \Gamma^d$, and then $\neg\neg(\Delta^d)^\wedge \vdash_{DI} \Gamma^d$. Now, since **DI** is a sublogic of **K**, we have $\neg\neg(\Delta^d)^\wedge \vdash_K \Gamma^d$, and since elimination of double negation holds in **K** and multiple antecedent is equivalent to conjunction, we have $\Delta^d \vdash_K \Gamma^d$. □

As we already stated, we need connective $\dot{-}$ only to have theorem B.3.5. Indeed, this connective will not occur in the **DI** derivations that we will consider for our strengthening of Craig-Lyndon theorem. In order to show this, we will use:

Theorem B.3.6 (Conservative extension)**.** ***DI*** *is a conservative extension of its* $\dot{-}$*-free fragment.*[37]

Theorem B.3.7 (Dual-Glivenko)**.** $\vdash_K A$ *iff* $\vdash_{DI} A$.[38]

While there is no residuation property in **DI**, there is something similar:

Theorem B.3.8 (Dual-residuation)**.** $A \vdash_{DI} B, \Delta$ *iff* $A \dot{-} B \vdash_{DI} \Delta$.[39]

In order to prove our refinement of Craig-Lyndon interpolation theorem, we still need the strong version of dual-Glivenko theorem:

Theorem B.3.9 (Strong dual-Glivenko)**.** $\neg A \vdash_K \Delta$ *iff* $\neg A \vdash_{DI} \Delta$.

Proof. From $\neg A \vdash_K \Delta$ we obtain $\Delta^d \vdash_K (\neg A)^d$ by clause *2* of theorem B.3.5. Now, by definition B.3.1 of duality we obtain $\Delta^d \vdash_K \neg(A^d)$, so that we can apply theorem B.3.4 and derive $\Delta^d \vdash_I \neg(A^d)$. By definition of duality, we return to the formulation $\Delta^d \vdash_I (\neg A)^d$ and then apply clause *1* of theorem B.3.5. In this way, we obtain $(\neg A)^{dd} \vdash_{DI} \Delta^{dd}$. We can prove by induction on the complexity of the dualized formula that duality is involutive (*i.e.* $A^{dd} = A$), so we conclude that $\neg A \vdash_{DI} \Delta$.[40] □

Now we can prove our main result:

Theorem B.3.10 (I and DI Interpolation)**.** *If* $\Gamma \vdash_K \Delta$ *then:*

- $\Gamma \vdash_I$ *or;*
- $\vdash_{DI} \Delta$ *or;*
- *There is a formula* χ *(interpolant) such that all its non-logical terms belong both to* Γ *and to* Δ*, and such that* $\Gamma \vdash_I \chi$ *and* $\chi \vdash_{DI} \Delta$.

Proof. Given $\Gamma \vdash_K \Delta$, we know by theorem B.3.1 that one of the following must hold:

- $\Gamma \vdash_K$, and so by formulation *1* of theorem B.3.3 we have $\Gamma \vdash_I$;
- $\vdash_K \Delta$, and so by theorem B.3.7 we have $\vdash_{DI} \Delta$;
- There is a classical interpolant ϕ for $\Gamma \vdash_K \Delta$, such that all its non-logical terms belong both to Γ and to Δ, and such that $\Gamma \vdash_K \phi$ and $\phi \vdash_K \Delta$. In this case, let us notice that if ϕ is a classical interpolant, so is $\neg\neg\phi$, since the condition about non-logical terms still applies, and both double negation introduction and double negation elimination hold in **K**. So we obtain $\Gamma \vdash_K \neg\neg\phi$ and $\neg\neg\phi \vdash_K \Delta$, with $\neg\neg\phi$ intrepolant. From $\Gamma \vdash_K \neg\neg\phi$ we obtain $\Gamma \vdash_I \neg\neg\phi$ by strong Glivenko theorem B.3.4, and from $\neg\neg\phi \vdash_K \Delta$ we obtain $\neg\neg\phi \vdash_{DI} \Delta$ by strong dual-Glivenko theorem B.3.9. So we just impose $\chi = \neg\neg\phi$ and we can obtain our interpolant from a standard classical one.

[36] Theorem 3.1 of [Urbas, 1996].
[37] Corollary 4.5 of [Urbas, 1996].
[38] Theorem 2.1 of [Urbas, 1996].
[39] Theorem 5.5 of [Urbas, 1996].
[40] For atoms and negated formulae it is obvious, for the other cases, an example is sufficient: $(A \wedge B)^{dd} = (A^d \vee B^d)^d = A^{dd} \wedge B^{dd}$, and from this we obtain $A^{dd} \wedge B^{dd} = A \wedge B$ by inductive hypothesis.

□

Let us notice that, since we did not assume $\dot{-}$ in **K**, it occurs neither in Δ, nor in χ (if present). It follows from this observation and from theorem B.3.6 that there is a $\dot{-}$-free version of the derivation given by previous theorem B.3.10 of $\vdash_{DI} \Delta$ or of $\chi \vdash_{DI} \Delta$.[41]

It is possible to generalize the last theorem by joining it with Milne's one:

Theorem B.3.11 (I, DI, K3 and LP Interpolation)**.** *If* $\Gamma \vdash_K \Delta$ *then:*

- $\Gamma \vdash_I$ *and* $\Gamma \vdash_{K3}$*, or;*
- $\vdash_{DI} \Delta$ *and* $\vdash_{LP} \Delta$*, or;*
- *There are some formulae* χ*,* ϕ*,* ψ *and* ζ *(interpolants) such that all their non-logical terms belong both to* Γ *and to* Δ*, and such that:*
  - $\Gamma \vdash_{K3} \chi$ *and* $\chi \vdash_{LP} \Delta$*;*
  - $\Gamma \vdash_I \phi$ *and* $\phi \vdash_{DI} \Delta$*;*
  - $\Gamma \vdash_{K3} \psi$ *and* $\psi \vdash_{DI} \Delta$*;*
  - $\Gamma \vdash_I \zeta$ *and* $\zeta \vdash_{LP} \Delta$*.*

Proof. All we have to prove are the two last clauses. To do this, we assume $\phi = \psi = \zeta = \neg\neg\chi$, so that we already have $\psi \vdash_{DI} \Delta$ and $\Gamma \vdash_I \zeta$. We can easily prove $\Gamma \vdash_{K3} \psi$ from $\Gamma \vdash_{K3} \chi$ and $\chi \vdash_{K3} \neg\neg\chi$. Double negation introduction holds in **K3**, since if a formula has value 1, then its double negation has value 1. In the same way, we prove $\zeta \vdash_{LP} \Delta$ from $\chi \vdash_{LP} \Delta$ and $\neg\neg\chi \vdash_{LP} \chi$. Double negation elimination holds in **LP**, since if $\neg\neg A$ has value 1 then A has value 1, and if $\neg\neg A$ has value 1/2 then A has value 1/2. □

[41]We could obviously find a symmetric result, if we decide to assume $\dot{-}$ and not $\supset$ in **K**.

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