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E ho camminato tanto pe’ la macchia
per raggiunge’ la lepre malandrina
e ho camminato tanto e non l’ho chiappa
e mi ha condotto alla spiaggia marina.

(Canto di segatura del grano diffuso in val di Sieve)

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Contents

Introduction

The inspiring subject of this thesis is the following well studied and rather natural question in group theory: *what is the probability that two elements of a group commute?* This probability is called the *degree of commutativity* and is expressed by the following formula

$$\mathrm{dc}(G) = \frac{|\{(x, y) \in G \mid xy = yx\}|}{|G|^2}. \tag{0.0.1}$$

The first result in this area is Gustafson's Theorem (Theorem 2.1.1) which states that for a nonabelian finite group the aforementioned probability cannot be greater than 5/8.

Similar results where commutativity is replaced by other group theoretic properties such as nilpotence and solvability were gained for finite groups by Guralnick and Wilson [13] using the classification of finite simple groups. Namely, for what concerns the probability that two elements generate a nilpotent subgroup, the problem is posed in the same paper whether there exists a proof of a Gustafson-like theorem that does not involve the classification of finite simple groups. Whilst looking for such a proof, a rather interesting and elusive set emerged as useful: the *Wielandt's subnormalizer* of a subgroup. Focussing on a probability similar to (0.0.1) and related to subnormalizers, which we called $\mathrm{sp}(G)$ (see Definition 2.2.6), we found that this can tell us something about the nilpotence and the solvability of G.

Moreover the study of $\mathrm{sp}(G)$, it transpired, was also related to another quite natural problem, which is discussed in the final chapter: given a finite group G and a prime p dividing the order of G, what can be said about the number of p-elements in G with respect to the cardinality of a Sylow p-subgroup?

In the first chapter we recall some well known facts about subnormality and we give the definition of subnormalizer. Then we write the proof of two results about the order of the subnormalizer which were proved in [4] and [3] (Theorem 1.4.1 and Theorem 1.3.6) and which will both be of great importance throughout the thesis.

We start the second chapter introducing the degree of commutativity 0.0.1 and some similar probabilities. We then show how the results on subnormalizers reported in the first chapter can be used to prove Guralnick and Wilson's theorem on the degree of nilpotence without the use of the classification of finite simple groups (Theorem 2.4.1). The main tool used here is Proposition 2.3.4, a probabilistic version of Wielandt's subnormality criterion. Also, we define the probability $\mathrm{sp}(G)$ and we show how it can be strictly related to nilpotence (Theorem 2.4.2).

The third chapter is entirely devoted to the proof of the following result (Theorem 3.1.1): if $\mathrm{sp}(G) > 1/6$ then G is solvable (the bound $1/6$ is tight). The proof involves the classification of finite simple groups. Using results from Chapter 2 we reduce the problem to two main steps. The first one requires the analysis of the order of a subnormalizer of a 3-element in finite almost simple groups. The second one consists in finding a bound on the ratio between the number of 2-elements and the order of a Sylow 2-subgroup in nonsolvable monolithic groups.

The fourth chapter takes inspiration from the last section of the third. We define the *p-Frobenius ratio* of a finite simple group to be the ratio between the number of p-elements and the cardinality of a Sylow p-subgroup. The focus of this chapter is to find lower bounds for the Frobenius ratio depending on the number of Sylow subgroups. For a p-solvable group G we prove in Theorem 4.0.1 that

$$\frac{|\mathfrak{U}_p(G)|}{|G|_p} \geq n_p(G)^{(p-1)/p}, \tag{0.0.2}$$

where $\mathfrak{U}_p(G)$ is the union of Sylow p-subgroups, i.e., the set of all p-elements in G, $|G|_p$ is the cardinality of a Sylow p-subgroup, and $n_p(G)$ is the number of Sylow p-subgroups of G. We also show that the bound (0.0.2) is asymptotically tight.

As for the non-p-solvable case, conjecturing that (0.0.2) is still true, we find a sufficient condition and give a reduction of the proof of this condition to finite almost simple groups.

Chapter 1

Wielandt's subnormalizer

1.1 Some facts about subnormality

We recall the definition together with some well known facts about subnormality. These can be found for example in [20].

Since in the next chapters we will only be interested in finite groups we set G to be a finite group, even if some of the following general statements hold for infinite groups as well.

Definition 1.1.1. Let H a subgroup of G. H is *subnormal in G*, in symbols $H \trianglelefteq\trianglelefteq G$, if there exists $n \in \mathbb{N}$ and a series of subgroups

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G. \tag{1.1.1}$$

The minimum n such that a series like (1.1.1) exists is called the *defect of subnormality of H in G*.

Some facts are obvious from the definition.

Proposition 1.1.2. *Let H, K be subgroups of a group G.*

(i) If $H \trianglelefteq\trianglelefteq G$ of defect k and $H \leq K$, then $H \trianglelefteq\trianglelefteq K$ of defect at most k.

(ii) If $H, K \trianglelefteq\trianglelefteq G$ then $H \cap K \trianglelefteq\trianglelefteq G$.

(iii) If $H \trianglelefteq\trianglelefteq K$ and $K \trianglelefteq\trianglelefteq G$ then $H \trianglelefteq\trianglelefteq G$, i.e., subnormality is a transitive relation. (In fact, subnormality is the transitive closure of the relation of normality.)

Among all the series witnessing the subnormality of a subgroup, one in particular is worth noting. Given a subgroup H of G, the *normal closure of H in G*, denoted by H^G, is the smallest normal subgroup of G containing H, i.e.,

$$H^G = \langle H^g \mid g \in G \rangle.$$

We define a series as follows. Set $K_1 = H^G$ and, for $i \geq 2$, $K_i = H^{K_{i-1}}$. It is easy to see that given a series like 1.1.1, we have $K_i \leq H_{n-i}$ and so H is subnormal in G of defect n if and only if $K_n = H$.

As noted in Proposition 1.1.1 (ii), the intersection of two subnormal subgroups is subnormal. The following theorem due to Wielandt holds.

Theorem 1.1.3. *Let $H, K \leq G$. If both H and K are subnormal in G then their join $\langle H, K \rangle$ is subnormal in G.*

We recall that the Fitting subgroup $\mathbf{F}(G)$ of G is the largest normal nilpotent subgroup of G. This is well defined since the join of two normal nilpotent subgroups of G is normal and nilpotent. What we want to observe is that $\mathbf{F}(G)$ is the largest *subnormal* nilpotent subgroup of G. This is because if $N \trianglelefteq G$ then $\mathbf{F}(N)$ is characteristic in N, whence normal in G, that is $\mathbf{F}(N) \leq \mathbf{F}(G)$. A trivial induction shows then that if $H \trianglelefteq \trianglelefteq G$ then $\mathbf{F}(H) \leq \mathbf{F}(G)$ and so if H is nilpotent $H = \mathbf{F}(H) \leq \mathbf{F}(G)$.

Let p be a prime dividing the order of $\mathbf{F}(G)$. Then $\mathbf{F}(G)$ contains a unique Sylow p-subgroup which is called $O_p(G)$. This is the largest (sub)normal p-subgroup in G and is a characteristic subgroup of G. Moreover calling $Syl_p(G)$ the set of Sylow p-subgroups of G one has

$$O_p(G) = \bigcap_{P \in Syl_p(G)} P.$$

1.2 Wielandt's criteria for subnormality

In this section we recall one of the criteria for subnormality due to Wielandt and some of its consequences. We follow chapter 7 of [20] and chapter 2 of [15].

Given a group G and a subgroup H it is clear that the following condition is necessary and sufficient for H to be normal in G:

$$H \trianglelefteq \langle H, g \rangle, \ \forall g \in G.$$

The analogue condition

$$H \trianglelefteq \trianglelefteq \langle H, g \rangle, \ \forall g \in G \tag{1.2.1}$$

is clearly necessary for the subnormality of H in G. The following theorem, one of the celebrated Wielandt's criteria for subnormality, states that this condition is also sufficient.

Theorem 1.2.1 (Theorem 7.3.3 in [20])**.** *Let H be a subgroup of G. Then each of the following conditions is equivalent to $H \trianglelefteq \trianglelefteq G$.*

(i) $H \trianglelefteq \trianglelefteq \langle H, g \rangle, \ \forall g \in G$;

(ii) $H \trianglelefteq \trianglelefteq \langle H, H^g \rangle, \ \forall g \in G$.

The proof of this theorem relies on a powerful lemma, the so called *zipper lemma*, which is of independent interest and which is due to Wielandt as well.

Theorem 1.2.2 (Lemma 7.3.1 in [20])**.** *Let H be a subgroup of G and suppose that H is not subnormal in G, but H is subnormal in all proper subgroups of G containing H. Then:*

(i) H is contained in a unique maximal subgroup M of G;

(ii) if $g \in G$, then $H^g \leq M$ if and only if $g \in M$.

Proof. We work by induction on $[G : H]$. Since H is not subnormal in G we have $H < G$ and so the induction basis is vacuously satisfied.

As H is not normal, we have $N_G(H) < G$ and thus $N_G(H) \leq M$, for some maximal subgroup M of G. Suppose that $H \subseteq K$, where K is maximal in G. We want to show that $K = M$.

By hypothesis H is subnormal in K. If $H \trianglelefteq K$ then $K \leq N_G(H) \leq M$ and so $K = M$, since K is maximal. We can thus suppose that H is not normal in K. Let

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_r = K \tag{1.2.2}$$

be a series between H and K of minimal length. Then $r \geq 2$ and H is not normal in H_2, since r is minimal. Let $x \in H_2$ with $H^x \neq H$ and write $L = \langle H, H^x \rangle$. We have $L \leq K$ and $H^x \leq H_1^x = H_1 \leq N_G(H)$ so that $L \leq N_G(H)$ which implies that $L \leq M$ and that $H \trianglelefteq L \neq G$.

We claim that L satisfies the hypotheses of the theorem. L is not subnormal in G since $H \trianglelefteq L$. Let S be a subgroup of G containing L. By hypothesis $H \trianglelefteq\trianglelefteq S$ and the same is true for H^x (which shares with H the hypothesis' property). Then L, being the join of two subnormal subgroups of S, is subnormal in S by Theorem 1.1.3.

We can thus apply the inductive hypothesis to L since it strictly contains H. We then have that L is contained in a unique maximal subgroup of G and so $K = M$, which proves (i).

Suppose that $H^g \leq M$. Because H is subnormal in M but not in G, M is not normal in G and since M is maximal we have $M = N_G(M)$. If $H^g \leq M$ then $H^g \leq M \cap M^g$ and so $M = M^g$ which implies $g \in N_G(M) = M$ and so (ii) is also proved. $\square$

We can now give the proof of Wielandt's criterion.

Proof of Theorem 1.2.1. It is enough to prove that (ii) implies $H \trianglelefteq\trianglelefteq G$. Suppose that this is not the case and take G to be a counterexample of least order. By the zipper lemma, H is contained in a unique maximal subgroup M of G. For every $g \in G$, we have that $H \trianglelefteq\trianglelefteq \langle H, H^g \rangle$ and since H is not subnormal in G, $\langle H, H^g \rangle$ must be contained in M. From (ii) of the zipper lemma, it follows that $g \in M$. This implies that $H^G \leq M$ and so $H \trianglelefteq\trianglelefteq H^G \trianglelefteq G$, a contradiction. $\square$

An immediate consequence of Wielandt's criterion is the generalized Baer-Suzuki theorem.

Theorem 1.2.3 (Theorem 2.12 in [15])**.** *Let H be a subgroup of a finite group G. Then $H \leq \mathbf{F}(G)$ if and only if $\langle H, H^g \rangle$ is nilpotent for all $g \in G$.*

Proof. Suppose that H is contained in the Fitting subgroup of G. For all $g \in G$, $H^g \leq \mathbf{F}(G)^g = \mathbf{F}(G)$ and so $\langle H, H^g \rangle$ is nilpotent, since it is contained in the nilpotent subgroup $\mathbf{F}(G)$.

Conversely assume that $\langle H, H^g \rangle$ is nilpotent for all $g \in G$. Taking $g = 1$ we see that H is nilpotent. Since every subgroup of a nilpotent group is subnormal, H is subnormal in $\langle H, H^g \rangle$ for all $g \in G$. Part (ii) of Wielandt's criterion ensures that H is subnormal in G. $\square$

1.3 The subnormalizer of a p-subgroup

Suppose that H is a subgroup of G. Then the normalizer of H is defined by

$$N_G(H) = \{g \in G \mid H \trianglelefteq \langle H, g\rangle\}$$

and H is a normal subgroup of G if and only if $N_G(H) = G$. In [20], pag. 238, the following definition inspired by Wielandt's criterion is given.

Definition 1.3.1. Let H be a subgroup of G. The *subnormalizer of H in G* is the set

$$S_G(H) = \{g \in G \mid H \trianglelefteq\trianglelefteq \langle H, g\rangle\}.$$

In the same fashion, with part (ii) of Theorem 1.2.1 in mind, we define another candidate for the role of subnormalizer, namely the set

$$S^1_G(H) = \{g \in G \mid H \trianglelefteq\trianglelefteq \langle H, H^g\rangle\}.$$

When the subgroup H is cyclic we will write $S_G(x)$ in place of $S_G(\langle x\rangle)$, and $S^1_G(x)$ in place of $S^1_G(\langle x\rangle)$.

It follows from part (i) of Wielandt's criterion that H is subnormal in G if and only if $S_G(H) = G$ if and only if $G = S^1_G(H)$. While $N_G(H)$ is a subgroup of G for any H, this is not true neither for the subnormalizer $S_G(H)$, nor for $S^1_G(H)$, as shown by the following examples.

Example 1.3.2. Let $G = S_5$ be the symmetric group on 5 elements. Take $H = \langle(1, 2)\rangle$, a subgroup of order 2. H is contained in (thus subnormal in) three Sylow 2-subgroups of G. These three Sylow subgroups generate the whole group G and so if $S_G(H)$ were a subgroup, it would have to coincide with G. But this would imply that $H \leq \mathbf{F}(G) = 1$, a contradiction. Since $S_G(H) \subseteq S^1_G(H)$, this counterexample shows that even $S^1_G(H)$ is not a subgroup in general.

From now on p will be a prime dividing the order of G.

We gather some basic results concerning subnormalizers of p-subgroups in the following proposition (whose proof we omit).

Lemma 1.3.3 (Proposition 2.1 and Lemma 2.3 in [4]). *Let H be a p-subgroup of G, I be the intersection of all Sylow p-subgroups of G containing H, N a normal subgroup of G. Then*

(i) $S_G(H) = \bigcup\{N_G(R) \mid R = O_p(N_G(R))$ *and* $H \leq R\}$;

(ii) $S_G(H) = S_G(I)$;

(iii) $S_{G/N}(HN/N) = S_G(H)N/N = \{gN \mid g \in S_G(H)\}$ *and the same holds, mutatis mutandis, for* $S^1_G(H)$;

(iv) $g \in S^1_G(H)$ *if and only if there is* $P \in Syl_p(G)$ *such that* $H \leq P \cap P^{g^{-1}}$.

The main reason why we are interested in subnormalizers is a nice formula to count the elements contained in the subnormalizer in G of a p-subgroup. This formula was obtained in [4] for p-solvable groups and in [3] for any group.

To state this formula we introduce two symbols that will be fundamental in what follows.

Definition 1.3.4. Let H be a p-subgroup of G and P a Sylow p-subgroup of G. We write $\lambda_G(H)$ for the number of Sylow p-subgroups containing H and $\alpha_G(H)$ for the number of conjugates of H contained in P.

Again if H is cyclic we write $\lambda_G(x)$ in place of $\lambda_G(\langle x\rangle)$, while with $\alpha_G(x)$ we mean the number of conjugates of x contained in P. To be explicit, the following relation holds

$$\alpha_G(\langle x\rangle) = \alpha_G(x)\frac{|C_G(x)|}{|N_G(\langle x\rangle)|}.$$

Note that both λ_G and α_G are constant on the conjugacy classes of p-subgroups and that α_G is independent of the particular Sylow p-subgroup chosen. The following lemma is just an easy observation.

Lemma 1.3.5. *Let H be a p-subgroup of G and P be a Sylow p-subgroup of G. Then*

$$\lambda_G(H)\,[G : N_G(H)] = \alpha_G(H)\,[G : N_G(P)]\,.$$

Proof. Let $\mathcal{H}$ be the set of conjugates of H and

$$S = \{(K, Q) \in \mathcal{H} \times Syl_p(G) \mid K \leq Q\}.$$

Then summing on $K \in \mathcal{H}$

$$|S| = \sum_{K\in\mathcal{H}} \lambda_G(K) = \lambda_G(H)\,|\mathcal{H}| = \lambda_G(H)\,[G : N_G(H)]\,,$$

whereas summing on $Q \in Syl_p(G)$

$$|S| = \sum_{Q\in Syl_p(G)} \alpha_G(H) = \alpha_G(H)\,|Syl_p(G)| = \alpha_G(H)\,[G : N_G(P)]\,.$$

□

We can now state the theorem about the order of the subnormalizer of a p-subgroup.

Theorem 1.3.6 (Theorem 2.8 in [4], main theorem in [3])**.** *Let H be a p-subgroup of G and $P \in Syl_p(G)$. Then*

$$|S_G(H)| = \lambda_G(H)|N_G(P)| = \alpha_G(H)|N_G(H)|. \tag{1.3.1}$$

The second and third terms of 1.3.1 are equal by Lemma 1.3.5. If $H = \langle x\rangle$ is cyclic then the thesis of the theorem can be written in a third way, that is $|S_G(x)| = \alpha_G(x)|C_G(x)|$.

Remark 1.3.7. An analogue to 1.3.1 for $S^1_G(H)$ can be easily proved. For a p-subgroup H of G, set $\delta_G(H)$ to be the number of conjugates of H that generate a p-group together with H, i.e.,

$$\delta_G(H) = \{H^g \mid H \trianglelefteq\trianglelefteq \langle H, H^g\rangle\}.$$

Given one of these conjugates H^g and an element $x \in S^1_G(H)$ such that $H^x = H^g$, we have that g and x lie in the same coset of $N_G(H)$. Then $|S^1_G(H)| = \delta_G(H)|N_G(H)|$.

1.4 Proof of Theorem 1.3.6 in the p-solvable case

The proof of Theorem 1.3.6 when G is a p-solvable group is group-theoretical and is based on considerations about the behaviour of subnormalizing elements with respect to normal sections.

In fact the following theorem is proved.

Theorem 1.4.1 (Theorem 2.6 in [4]). *Let G be a p-solvable group and H a p-subgroup of G. Let $\mathcal{M}$ be the set of all p'-factors in a given normal $\{p, p'\}$-series of G. Then*

$$|S_G(H)| = |G|_p \prod_{U/V \in \mathcal{M}} \left|C_{U/V}\left(HV/V\right)\right|$$

The reason why we report this proof of Theorem 1.3.6 is both its different nature with respect to the general one and the fact that we will use Theorem 1.4.1 in the following chapters.

Two lemmas are required to get Theorem 1.4.1. Of the first one we omit the proof.

Lemma 1.4.2 (Lemma 2.5 in [4]). *Let $H \leq G$, $x \in G$ and $A \trianglelefteq G$. Moreover set $X = \langle H, x\rangle$, $C = C_A\left(H^X\right)$ and*

$$\Delta = \bigcup_{b \in C_A(H)} (xC)^b.$$

Then $|\Delta| = |C_A(H)|$.

Lemma 1.4.3 (Lemma 2.4 in [4]). *Let A be a normal p'-subgroup of G, H be a p-subgroup of G and $x \in S_G(H)$. Then*

$$|xA \cap S_G(H)| = |C_A(H)|.$$

Proof. Let $X = \langle H, x\rangle$ and $B = H^X$. Then $H \trianglelefteq\trianglelefteq X$ implies $H \leq O_p(X)$ so that B is a p-group. Let $P \in Syl_p(X)$. Since A is a normal p'-subgroup of G, P is a Sylow p-subgroup of $G_0 = AX$. Furthermore $H \leq P$ and since X/B is cyclic P is normal in X.

Let $a \in A$ such that $xa \in S_G(H)$. Then xa normalizes a p-subgroup Y of G_0 with $H \leq Y$. Let Q be a Sylow p-subgroup of G_0 containing Y. Since P is normal in X and $G_0 = AX$ we have $Q = P^b$ with $b \in A$. Then

$$H^{b^{-1}} \leq P \cap HA = H(P \cap A) = H$$

and $b \in N_A(H) = C_A(H)$.

But we also have $xa \in N_G(Y)$, so $z = (xa)^{b^{-1}} \in N_G(Y^{b^{-1}})$ and $T = Y^{b^{-1}} \leq P$. Now $z = xb^x ab^{-1} \in xA$ so that $xA = zA$ normalizes TA/A. In particular x normalizes TA and thus normalizes $TA \cap P = T(A \cap P) = T$. Since $H = H^{b^{-1}} \leq T$, we have

$$B = H^X \leq T^X = T \leq P.$$

Furthermore:

$$u = x^{-1}z = b^x ab^{-1} \in A \cap N_G(T) = C_A(T).$$

In particular $u \in C := C_A(B)$. We therefore have

$$xa = z^b = (xu)^b \in (xC)^b, \text{ dove } b \in C_A(H).$$

Hence

$$xA \cap S_G(H) \subseteq \Delta := \bigcup_{b \in C_A(H)} (xC)^b.$$

Conversely, let $a \in C$, $b \in C_A(H)$ and $z = (xa)^b = x(b^{-1})^x ab \in xA$. Then xa normalizes B and so $H \trianglelefteq\trianglelefteq \langle H, xa \rangle$ which gives:

$$H = H^b \trianglelefteq\trianglelefteq \langle H, xa \rangle^b \trianglelefteq\trianglelefteq \langle H^b, (xa)^b \rangle = \langle H, z \rangle$$

that is $z \in xA \cap S_G(H)$. We therefore have that $xA \cap S_G(H) = \Delta$.

The previous lemma gives the thesis. □

We can now prove theorem 1.4.1.

Proof of theorem 1.4.1. We proceed by induction on n, where

$$1 = M_0 \leq \cdots \leq M_n = G$$

is a $\{p, p'\}$-normal series of G whose p'-factors are the elements of the set $\mathcal{M}$.

If $n = 1$ the result is obvious. Let $n > 1$ and $M := M_1$. By inductive hypothesis we have

$$\left|S_{G/M}\left(HM/M\right)\right| = |G/M|_p \prod_{U/V \in \mathcal{M}_0} C_{U/V}\left(HV/V\right) \tag{1.4.1}$$

where $\mathcal{M}_0 = \mathcal{M} \setminus \{M\}$.

If M is a p-group, then H is subnormal in HM and so $HM \trianglelefteq\trianglelefteq \langle HM, x \rangle$ implies that H is subnormal in $\langle HM, x \rangle$ and so in $\langle H, x \rangle$. It follows that

$$|S_G(H)| = \left|S_{G/M}\left(HM/M\right)\right| |M|$$

which together with (1.4.1) gives the desired conclusion.

If M is a p'-group we apply Lemma 1.4.3 and Proposition 1.3.3 (iii), obtaining

$$|S_G(H)| = |S_G(H)M/M|\,|C_M(H)| = \left|S_{G/M}\left(HM/M\right)\right| |C_M(H)|$$

which together with (1.4.1) gives the thesis. □

Corollary 1.4.4. *Let G be a p-solvable group and H a p-subgroup of G. Then $|S_G(H)|$ divides $|G|$.*

Example 1.4.5. Without the hypothesis of p-solvability Corollary 1.4.4 does not hold. Let $G = PSL(2, 7)$ and H be a subgroup of G of order 2. Then $|S_G(H)| = 40$ while $|G| = 168$.

We can now prove Theorem 1.3.6 for p-solvable groups.

Proof of Theorem 1.3.6 when G is p-solvable. By induction on $|G|$. Let M be a minimal normal subgroup of G. By inductive hypothesis we have

$$|S_{G/M}(HM/M)| = \lambda_{G/M}(HM/M)|N_{G/M}(PM/M)|.$$

Assume first that M is a p-group. Then $Q \geq M$ for every Sylow p-subgroup Q and $|N_G(P)| = |N_{G/M}(PM/M)||M|$. Moreover $\lambda_{G/M}(HM/M) = \lambda_G(H)$. Therefore

$$\begin{aligned}|S_G(H)| &= |S_{G/M}(HM/M)||M| \\ &= \lambda_{G/M}(HM/M)|N_{G/M}(PM/M)||M| = \lambda_G(H)|N_G(P)|.\end{aligned}$$

Suppose now that M is not a p-group. Then, G being p-solvable, we have that M is a p'-group. Suppose $H \leq P$. If $H \leq P^a$ with $a \in M$ then for every $h \in H$

$$[h, a] = h^{-1}h^a = (a^{-1})^h a \in P^a \cap M = 1.$$

Then $a \in C_M(H)$ and

$$\lambda_{MP}(H) = [C_M(H) : C_M(P)] \,.$$

Moreover

$$\lambda_G(H) = \lambda_{G/M}(HM/M)\lambda_{MP}(H)$$

Applying Lemma 1.4.3 and the inductive hypothesis we get

$$\begin{aligned}
|S_G(H)| &= \left|S_{G/M}\left(HM/M\right)\right| |C_M(H)| \\
&= \lambda_{G/M}\left(HM/M\right) \left|N_{G/M}\left(PM/M\right)\right| |C_M(P)| \left[C_M(H) : C_M(P)\right] \\
&= \lambda_{G/M}\left(HM/M\right) \lambda_{MP}(H) \left|N_{G/M}\left(PM/M\right)\right| |C_M(P)| \\
&= \lambda_G(H) \left|N_G(P)\right| .
\end{aligned}$$

□

1.5 Proof of Theorem 1.3.6 in the general case

Following [3], we state the essential lemmas and present the proof of the main theorem.

Given a finite poset $\mathcal{P}$, one can define a simplicial complex where the n-simplexes are the n-chains in $\mathcal{P}$. Then concepts from topology (e.g. contractibility) can be used referring to the poset. An order preserving map between two posets induces a simplicial map between the related complexes.

We now introduce some notation. Let $\mathcal{P}$ be a finite poset and $G \leq Aut(\mathcal{P})$. For any $g \in G$ and $x \in \mathcal{P}$

- $\mathcal{P}_{>x}$ is the poset of elements $y \in \mathcal{P}$ such that $y > x$;
- $\mathcal{P}^g$ is the poset of elements in $\mathcal{P}$ fixed by g;
- $S_G(x)$ is the set of elements in G that fix some $y \geq x$ (later we will observe that this notation is coherent with the definition of subnormalizer);
- μ is the Möbius function of $\mathcal{P}$;
- $\chi(\mathcal{P})$ is the Euler characteristic of the simplicial complex associated to $\mathcal{P}$.

Lemma 1.5.1. *Let $f : \mathcal{P} \mapsto \mathcal{Q}$ be an order preserving map of posets. If for each $y \in \mathcal{Q}$ we have that $f^{-1}(\mathcal{Q}_{>y})$ is contractible then f is an homotopy equivalence.*

Lemma 1.5.2. *Let X be a finite meet-semilattice, $x \in X$ and $G \leq Aut(X)$. Then*

$$\sum_{y>x} \mu(x, y)|S_G(y)| = -\sum_{g \in G} \chi(X^g_{>x}).$$

Let $\mathcal{S}_p(G)$ be the poset of nontrivial p-subgroups of G ordered by inclusion. Let $\bar{\mathcal{S}}_p(G) = \mathcal{S}_p(G) \cup \{1\}$. Then $\bar{\mathcal{S}}$ is a meet semilattice on which G acts by conjugation as a group of automorphisms. With respect to this poset, the subnormalizer of a p-subgroup H is the set of all the elements in G that fix some p-subgroup that contains H and so the definition given above of $S_G(x)$ for an element of a poset coincides with that of the subnormalizer if seen in $\mathcal{S}$.

Theorem 1.5.3. *(i) For each $H \in \bar{\mathcal{S}}$*

$$\sum_{H \leq K \in \bar{\mathcal{S}}} \mu(H,K)|S_G(K)| = \begin{cases} |N_G(H)|, & \text{if } H \in Syl_p(G), \\ 0, & \text{if } H \notin Syl_p(G). \end{cases}$$

(ii) For each $H \in \bar{\mathcal{S}}$, $P \in Syl_p(G)$,

$$|S_G(H)| = \lambda_G(H)|N_G(P)|,$$

(iii) $\sum_{g \in G} \chi(\mathcal{S}^g) = |G|$.

Proof. By induction on $|G|$. If G is a p-group then $G = S_G(H)$ for all $H \leq G$. In this case (ii) is clear. From the definition of Möbius function we have that $\mu(G,G) = 1$ and for each $H \leq G$,

$$\mu(H,G) = -\sum_{H \leq K < G} \mu(H,K)$$

so that

$$\sum_{H \leq K} \mu(H,K) = 0.$$

For (iii) observe that using again the definition of the Möbius function and Lemma 1.5.2

$$\sum_{g \in G} \chi(\mathcal{S}^g) = \sum_{1 < K \in \mathcal{S}} \mu(1,K)|S_G(K)|$$

$$\begin{aligned} \sum_{1 < K \in \mathcal{S}} \mu(1,K)|S_G(K)| &= \sum_{1 < K \in \mathcal{S}} \mu(1,K)|G| \\ &= |G| \sum_{1 < K \in \mathcal{S}} \mu(1,K) = |G|. \end{aligned}$$

Thus assume that G is not a p-group. If $H \in Syl_p(G)$ then $S_G(H) = N_G(H)$ and so (i) and (ii) follows at once. Suppose that H is not a Sylow p-subgroup of G. Then p divides $|N_G(H)/H|$. By Lemma 1.5.2 we have

$$\sum_{K > H} \mu(H,K)|S_G(K)| = -\sum_{g \in G} \chi(\mathcal{S}^g_{>H}).$$

Let $g \in G$. If $g \notin S_G(H)$ then $\mathcal{S}^g_{>H} = \emptyset$ and so $\chi(\mathcal{S}^g_{>H}) = 0$. If $g \in S_G(H) \setminus N_G(H)$ then $\mathcal{S}^g_{>H}$ has a minimum (the normal closure of H in $\langle H, g\rangle$), whence is a cone and in particular $\chi(\mathcal{S}^g_{>H}) = 1$. Therefore

$$\begin{aligned} \sum_{H \leq K \in \bar{\mathcal{S}}} \mu(H,K)|S_G(K)| &= |S_G(H)| + \sum_{H < K} \mu(H,K)|S_G(K)| \\ &= |S_G(H)| - \sum_{g \in S_G(H)} \chi(\mathcal{S}^g_{>H}) \\ &= |N_G(H)| - \sum_{g \in N_G(H)} \chi(\mathcal{S}^g_{>H}). \end{aligned} \tag{1.5.1}$$

Let $H < R \in \mathcal{S}$. If $g \in N_G(H)$ and R is fixed by g then g fixes $R \cap N_G(H) = N_R(H)$ and so $gH \in N_G(H)/H$ fixes $N_R(H)/H$. Setting $\mathcal{V} = \mathcal{S}_p(N_G(H)/H)$, we can thus define an order preserving map

$$\begin{aligned} f :& \mathcal{S}^g_{>H} \to \mathcal{V}^{gH} \\ & R \mapsto N_R(H)/H. \end{aligned} \tag{1.5.2}$$

If $\bar{K} = K/H \in \mathcal{V}^{gH}$ we have that

$$f^{-1}(\bar{K}) = \mathcal{S}^g_{\geq K}$$

is a cone with vertex K; in particular it is contractible. By Lemma 1.5.1 $\mathcal{S}^g_{>H}$ and $\mathcal{V}^{gH}$ are homotopy equivalent and so

$$\chi(\mathcal{S}^g_{>H}) = \chi(\mathcal{V}^{gH}). \tag{1.5.3}$$

Since $|N_G(H)/H| < |G|$ by inductive hypotesis

$$|N_G(H)/H| = \sum_{gH \in N_G(H)/H} \chi(\mathcal{V}^{gH}).$$

By (1.5.3)

$$\begin{aligned} \sum_{g \in N_G(H)} \chi(\mathcal{S}^g_{>H}) &= \sum_{g \in N_G(H)} \chi(\mathcal{V}^{gH}) = |H| \sum_{gH \in N_G(H)/H} \chi(\mathcal{V}^{gH}) \\ &= |H||N_G(H)/H| = |N_G(H)|. \end{aligned}$$

Inserting this in (1.5.1)

$$\sum_{H \leq K \in \bar{\mathcal{S}}} \mu(H,K)|S_G(K)| = 0,$$

that is (i).

To get (ii) we once again apply the definition of the Möbius function together with (i), which we have just proved for each nontrivial p-subgroup of G. We can write

$$\lambda_G(H)|N_G(P)| = \sum_{K>H} \left(\sum_{X \geq K} \mu(K,X)|S_G(X)| \right)$$

since every $K \notin Syl_p(G)$ gives no contribution to the above sum. Then

$$\begin{aligned} \lambda_G(H)|N_G(P)| &= \sum_{X \geq H} \left(\sum_{H \leq K \leq X} \mu(K,X) \right) |S_G(X)| \\ &= \sum_{X \geq H} \delta(H,X)|S_G(X)| = |S_G(H)|. \end{aligned}$$

Finally if $H = 1$ statement (ii) is obvious while (i) follows from applying (ii) and the Möbius inversion formula. For (iii) we observe that

$$\begin{aligned} \sum_{g \in G} \chi(\mathcal{S}^g) &= - \sum_{H>1} \mu(1,H)|S_G(H)| \\ &= - \sum_{H \geq 1} \mu(1,H)|S_G(H)| + |S_G(1)| \\ &= 0 + |G| = |G|. \end{aligned}$$

□

Chapter 2

Subnormalizers: a probabilistic approach

2.1 Introduction

Given a finite group G and a positive integer n we consider the uniform probability distribution on $G^n = G \times \cdots \times G$, i.e., for all $S \subseteq G^n$, the probability of S is $|S|/|G^n| = |S|/|G|^n$.

In this chapter we focus on certain probabilities defined on finite groups. The most famous among these probabilities is for sure the *degree of commutativity*, that is the probability that two elements in G commute

$$\mathrm{dc}(G) = \frac{|\{(x, y) \in G \mid xy = yx\}|}{|G|^2} = \frac{1}{|G|}\sum_{x\in G}\frac{|C_G(x)|}{|G|}. \tag{2.1.1}$$

Another way of looking at this probability is the following. Let k be the number of conjugacy classes of G and $\mathcal{C} = \{x_1, \ldots, x_k\}$ be the a set of representatives for these conjugacy classes. Then using (2.1.1) and gathering the elements in the same class we get

$$\mathrm{dc}(G) = \frac{1}{|G|}\sum_{i=1}^{k}\left(\frac{|C_G(x)|}{|G|}|x_i^G|\right) = \frac{k}{|G|}. \tag{2.1.2}$$

Evaluating the degree of commutativity is then equivalent to counting the conjugacy classes of a finite group. This latter problem has been studied for example in [8], where it is also proved with elementary methods that the degree of commutativity grows on the subgroups (a property that is not shared with other probabilities as we will see).

A well known result which is probably the starting point for the developement of this area of group theory is the following theorem of Gustafson. We report the simple and elegant proof.

Theorem 2.1.1 ([14]). *If* $\mathrm{dc}(G) > \frac{5}{8}$ *then* G *is abelian.*

Proof. Let G be a nonabelian group. We just need two elementary observations. The first is that if x does not lie in the center of G then $|C_G(x)|/|G| \leq 1/2$. The second is that since G is nonabelian,$G/Z(G)$ is not cyclic and so $[G : Z(G)] \geq 4$. Then using the second equality in (2.1.1) and dividing the elements in the center from the others

$$\begin{aligned}\mathrm{dc}(G) &= \frac{1}{|G|}\sum_{x\in G}\frac{|C_G(x)|}{|G|} = \frac{1}{|G|}\left(|Z(G)| + \sum_{x\notin Z(G)}\frac{|C_G(x)|}{|G|}\right)\\ &\leq \frac{|Z(G)|}{|G|} + \frac{|G|-|Z(G)|}{2|G|} = \frac{|G|+|Z(G)|}{2|G|} \leq \frac{1}{2}+\frac{1}{8} = \frac{5}{8}.\end{aligned}$$

□

The bound in Gustafson's theorem is the best possible since $\mathrm{dc}(Q_8) = 5/8$ where Q_8 is the quaternion group with 8 elements.

One way of generalizing the notion of degree of commutativity is by considering a word $w \in G^n$ (replacing the commutator) and finding probabilistic results concerning the number of n-uples $\mathbf{x} \in G^n$ such that $w(\mathbf{x}) = 1$. This approach leads the recent works of Martino et al. ([24]) and Shalev ([28]).

Another way in which one can explore the subject derives from the obvious observation that $xy = yx$ if and only if $\langle x, y\rangle$ is an abelian subgroup. Many authors were then interested in obtaining Gustafson-like results by replacing commutativity with some other group theoretic property. The next section is devoted to some remarks about the probability that two elements generate a nilpotent subgroup.

2.2 The degree of nilpotence

We want to give some properties and information about the *degree of nilpotence* of a group G, i.e., the probability that two elements in G generate a nilpotent subgroup:

$$\mathrm{dn}(G) = \frac{|\{(x,y)\in G \mid \langle x,y\rangle \text{ is nilpotent }\}|}{|G|^2} = \frac{1}{|G|}\sum_{x\in G}\frac{|Nil_G(x)|}{|G|}. \tag{2.2.1}$$

In the above definition, in place of the centralizer we put $Nil_G(x)$, the set of elements that together with x generate a nilpotent subgroup. We will talk about this fairly elusive set in the next section.

As for now, we will just prove the following well known proposition, a generalization of the trivial fact that $C_G(x) = G$ if and only if x lies in the center of G.

Proposition 2.2.1. *Let $x \in G$ be an element such that $Nil_G(x) = G$. Then $x \in \zeta_\omega(G)$, the hypercenter of G.*

Proof. It is enough to prove the proposition for a p-element x, where p is a prime dividing the order of G. In our hypothesis, such an x commutes with every element whose order is not divided by p.

Let P be a Sylow p-subgroup of G containing x. Let K be the subgroup generated by all the elements whose order is not divided by p and $C = C_G(K)$. Then both K and C are normal subgroups of G. Moreover, if $R \in Syl_q(G)$, for some $q \neq p$ then $R \leq K$ so that $G = KP$.

Now $x \in C$ and so $x \in P \cap C = Q$, a Sylow p-subgroup of C. We observe that:

(i) $[Q, K] = 1$ since $Q \leq C$;

(ii) $[Q,P] \leq Q$ since $Q \trianglelefteq P$.

It follows that $Q \trianglelefteq G$.

We prove by induction on n that for all $1 \leq n \in \mathbb{N}$, $\zeta_n(P) \cap Q \leq \zeta_n(G)$. If $a \in Z(P) \cap Q$ we have $[a,P] = 1 = [a,K]$ and so $a \in Z(G)$. If $n \geq 2$ and $a \in \zeta_n(P) \cap Q$, we have that $[a,K] = 1$ since $Q \leq C$, $[a,P] \leq Q$ since $Q \trianglelefteq G$ and from the definition of the ζ_i's $[a,P] \leq \zeta_{n-1}(P)$. By inductive hypotesis

$$[a,G] = [a,P] \leq \zeta_{n-1}(P) \cap Q \leq \zeta_{n-1}(G)$$

and so $a \in \zeta_n(G)$. □

The previous proposition tells in particular that the degree of nilpotence is 1 if and only if G is nilpotent. It seems then reasonable to ask whether a theorem analogue to that of Gustafson holds for the degree of nilpotence. Such a theorem was proved by Guralnick and Wilson in [13]. There the authors study the *degree of solvability* $\mathrm{ds}(G)$, that is the probability that a pair of elements in G generates a solvable subgroup.

Theorem 2.2.2 (Theorem A in [13])**.** *Let G be a finite group.*

(i) If $\mathrm{ds}(G) > 11/30$*, then G is solvable.*

(ii) If $\mathrm{dn}(G) > 1/2$*, then G is nilpotent.*

Remarks 2.2.3.

- It is not trivial to prove that if $\mathrm{ds}(G) = 1$ then the group is solvable. This was proved by Thompson ([31]) and later with a more direct proof by Flavell ([5]).
- The bounds are best-possible, since they are realized by A_5 and S_3, respectively.

The proof of theorem 2.2.2 relies on the classification of finite simple groups. After proving part (i), the authors deduce part (ii) as a corollary, proving it just for solvable groups (a proof for solvable groups was obtained also in [7]). They also hypothesize that there may be a proof that does not involves the classification of finite simple groups for this second part when G is any group, not necessarily solvable. In section 2.4 we will present such a classification-free proof.

Let us firstly observe that the degree of nilpotence grows on the quotients.

Lemma 2.2.4. *Let N be a normal subgroup of G. Then* $\mathrm{dn}(G) \leq \mathrm{dn}(G/N)$.

Proof. Let $\mathcal{T}$ be a left transversal for N in G. For all $g \in G$ let $t_g \in \mathcal{T}$ be the representative of the coset gN.

It is clear that if $x, y \in G$ are such that $y \in Nil_G(x)$, we have $yN \in Nil_{G/N}(xN)$. The map

$$\begin{aligned} Nil_G(x) &\to Nil_{G/N}(xN) \times N \\ y &\mapsto (t_y N, t_y^{-1} y) \end{aligned}$$

is then well defined and injective.

Thus we have the inequality $|Nil_G(x)| \leq \left|Nil_{G/N}(xN)\right| |N|$. Finally

$$\begin{aligned} \mathrm{dn}(G) = \frac{1}{|G|} \sum_{t \in \mathcal{T}} \sum_{a \in N} \left(\frac{|Nil_G(ta)|}{|G|} \right) &\leq \frac{1}{|G|} \sum_{t \in \mathcal{T}} \sum_{a \in N} \left(\frac{|Nil_G(tN)|}{|G/N|} \right) \\ &= \frac{|N|}{|G|} \sum_{t \in \mathcal{T}} \left(\frac{|Nil_G(tN)|}{|G/N|} \right) = \mathrm{dn}\left(\frac{G}{N}\right). \end{aligned}$$

□

Example 2.2.5. Contrary to what happens with the degree of commutativity, it is not the case that $\mathrm{dn}(H) \geq \mathrm{dn}(G)$ for $H \leq G$, not even for a normal subgroup H. Let $A = \langle x_1, x_2, x_3, x_4 \rangle$ be an elementary abelian group with 16 elements and let $U = \langle y, u \rangle \leq Aut(A)$ where U is isomorphic to S_3 acting on A via

$$x_1^y = x_2, \;\; x_2^y = x_1x_2, \;\; x_3^y = x_3x_4, \;\; x_4^y = x_3,$$
$$x_1^u = x_2, \;\; x_2^u = x_1, \;\; x_3^u = x_4, \;\; x_4^u = x_3, \;\; y^u = y^2.$$

Then if $G = A \rtimes U$ and $H = \langle A, y \rangle \leq G$ we have

$$\mathrm{dn}(G) = \frac{7}{24} > \frac{1}{6} = \mathrm{dn}(H).$$

We can now establish the link between subnormalizers and the degree of nilpotence, and show how the former can be used to get information about the latter. The crucial (and trivial) observation is that for all elements $x \in G$ we have that

$$Nil_G(x) \subseteq S_G(x). \tag{2.2.2}$$

This is because if $\langle x, y \rangle$ is nilpotent then all its subgroups (in particular $\langle x \rangle$) are subnormal. The properties of these subsets will be the topic of the next section.

We thus introduce another probability, the analysis of which takes a big part of this thesis.

Definition 2.2.6. Let G be a finite group. Given $x \in G$ we define

$$\mathrm{sp}_G(x) = \frac{|S_G(x)|}{|G|}$$

and

$$\mathrm{sp}(G) = \frac{1}{|G|} \sum_{x \in G} \mathrm{sp}_G(x). \tag{2.2.3}$$

From the definition of subnormalizer we see that $\mathrm{sp}(G)$ is the probability that a random couple of elements $(x, y) \in G \times G$ satisfies

$$\langle x \rangle \trianglelefteq\trianglelefteq \langle x, y \rangle. \tag{2.2.4}$$

By (2.2.2) we have that

$$\mathrm{dn}(G) \leq \mathrm{sp}(G). \tag{2.2.5}$$

Thanks to Theorem 1.3.6 we can characterize the probability $\mathrm{sp}_G(x)$ in two satisfactory ways when x is a p-element. Namely the theorem implies that

$$\mathrm{sp}_G(x) = \frac{|S_G(x)|}{|G|} = \frac{\lambda_G(x)}{[G : N_G(P)]} = \frac{\alpha_G(x)}{[G : C_G(x)]}$$

and so, if $n_p(G)$ is the number of Sylow p-subgroups of G,

$$\mathrm{sp}_G(x) = \frac{\lambda_G(x)}{n_p(G)} = \frac{\alpha_G(x)}{|x^G|}. \tag{2.2.6}$$

In other words $\mathrm{sp}_G(x)$ is the percentage of Sylow p-subgroups containing x or, equivalently, the percentage of conjugates of x contained in a fixed Sylow p-subgroup of G.

We end this section with a remark following from Theorem 1.3.6.

Lemma 2.2.7. *Let p be a prime dividing the order of G and $P \in Syl_p(G)$. Moreover let $\mathfrak{U}_p(G)$ be the set of the p-elements in G (i.e., the union of all Sylow p-subgroups). Then*

$$\sum_{x \in \mathfrak{U}_p(G)} |S_G(x)| = |P||G|.$$

Proof. Let $\mathcal{K}_1, \dots \mathcal{K}_n$ be the conjugacy classes of p-elements in G and, for all i, choose $x_i \in P \cap \mathcal{K}_i$. Then by Theorem 1.3.6

$$\sum_{x \in \mathfrak{U}_p(G)} |S_G(x)| = \sum_{i=1}^{n} |K_i| \alpha_G(\langle x_i \rangle) |N_G(\langle x_i \rangle)|$$

$$= \sum_{i=1}^{n} [G : C_G(x_i)] \alpha_G(x_i) |C_G(x_i)| = |G| \sum_{i=1}^{n} \alpha_G(x_i) = |G||P|.$$

The last equality holds since every element in P is conjugate to one of the x_i's.

□

2.3 The sets $Nil_G(x)$ and $S_G(x)$

In this section we present some properties of the sets which take the place of the centralizer of an element when we consider the degree of nilpotence or the probability $\mathrm{sp}(G)$ instead of the degree of commutativity.

Neither $Nil_G(x)$ nor $S_G(x)$ is a subgroup of G in general. As for the subnormalizer, this was pointed out in the previous chapter (Example 1.3.2), while the following easy counterexample works for $Nil_G(x)$.

Example 2.3.1. Let $G = S_4$, $x = (1,2)(3,4)$, $y_1 = (1,4,3,2)$ and $y_2 = (1,3,4,2)$. Then one sees that $\langle x, y_1 \rangle$ and $\langle x, y_2 \rangle$ are 2-subgroups since $x \in O_2(G)$ while $y_1 y_2$ and x generate A_4.

Another fact which is trivially true for the centralizer of an element is that if $|C_G(x)| > |G|/2$ then x lies in the center of G. Then, with Proposition 2.2.1 in mind, the question could be asked if there exists a constant $0 < c < 1$ such that if $|Nil_G(x)|/|G| > c$ then $x \in \zeta_\omega(G)$. The following proposition shows that this is not the case.

Proposition 2.3.2. *There exists a sequence of groups $(G_k)_{k \in \mathbb{N}}$ together with $x_k \in G_k$ such that $Z(G_k) = 1$ and*

$$\lim_{k \to \infty} \frac{|Nil_{G_k}(x_k)|}{|G_k|} = 1.$$

Proof. For $k \in \mathbb{N}$ and $k \geq 2$, let $n = 2^k$. Moreover let $\mathbb{K} = \mathbb{F}_{2^n}$ be the field with 2^n elements and V be the additive group of $\mathbb{K}$, so that V is an elementary abelian group of size 2^n.

By Zsigmondy's theorem there exists a prime p which divides $2^n - 1$ and doesn't divide $2^l - 1$ for any $1 \leq l < n$. Since the multiplicative group $\mathbb{K}^\times$ is cyclic, it contains a unique subgroup $P = \langle x \rangle$ of order p. P acts fixed point freely on V by multiplication and the elements of the group $V \rtimes P$ have order either 2 or p.

Let $\mathcal{G} = Gal(\mathbb{K}|\mathbb{F}_2)$, a cyclic group of order $n = 2^k$. Then $\mathcal{G}$ acts both on V and on P. If $\sigma \in \mathcal{G}$ is such that $x^\sigma = x$ then $x \in \mathbb{E} = \mathrm{Fix}_{\mathbb{K}}(\langle \sigma \rangle)$ the field fixed by σ. Since

$x \notin \mathbb{F}_2$ we have $\mathbb{E} > \mathbb{F}_2$. By the choice of p, and since $|x| = p$ has to divide $|\mathbb{E}| - 1$, we have that $\mathbb{E} = \mathbb{K}$ so that $\sigma = 1$, i.e., $\mathcal{G}$ acts fixed point freely on P.

Let $g \in \mathcal{G}$, $u \in V$ and $y \in P$. Then, writing $w \cdot v$ for the multiplication of two elements $v, w \in \mathbb{K}$,

$$(u^y)^g = (u \cdot y)^g = u^g \cdot y^g = (u^g)^{y^g}$$

which means that $\mathcal{G}$ acts on $V \rtimes P$. The group $G = (V \rtimes P) \rtimes \mathcal{G}$, whose order is $2^{n+k}p$, is then well defined.

Suppose that a p-element $1 \neq y$ in G centralizes a 2-element g in G. Up to conjugation we can suppose that $g = v\sigma \in V\mathcal{G}$, which is a Sylow 2-subgroup of G. Moreover $y \in P^{w^{-1}}$ for some $w \in V$ and so, setting $u = v^w$ we have

$$[u\sigma^w, x] = 1.$$

Thus

$$uw^{-1}w^{\sigma^{-1}}\sigma = u\sigma^w = (u\sigma^w)^x = \left(uw^{-1}\right)^x w^{\sigma^{-1}x}\sigma^x$$

which implies, in particular, that $[x, \sigma] = 1$. Then $\sigma = 1$ and so $u \in C_V(x) = 1$. Then we have that $C_G(x) = \langle x \rangle$ and that there are not any elements of composite order in G. In particular $Z(G) = 1$.

We observe that $N_G(S) = S$ for all $S \in Syl_2(G)$. This is because if $N_G(S) > S$ then S being maximal would be normal in G and so $G/V \simeq P \rtimes \mathcal{G}$ would be nilpotent. Also, it is true that $N_{G/V}(S/V) = S/V$ for $S \in Syl_2(G)$.

Moreover, we observe that for all $S_1, S_2 \in Syl_2(G), S_1 \neq S_2$, we have $S_1 \cap S_2 = V$, since if $S_1 \cap S_2 > V$ then $(S_1/V) \cap (S_2/V) > 1$ which means that $S_1/V = S_2/V$ and this is the contradiction with the fact that there are p Sylow 2-subgroups in G/V. Then $V = O_2(G)$ and so for all $1 \neq v \in V$

$$Nil_G(v) = \bigcup_{S \in Syl_2(G)} S = V \cup (G \setminus (VP)).$$

Finally

$$\frac{|Nil_G(v)|}{|G|} = \frac{2^n + 2^{n+k}p - 2^n p}{2^{n+k}p} = 1 - \frac{p-1}{2^k p},$$

which tends to 1 as k tends to infinity. □

Wielandt's subnormality criterion shows that if $\mathrm{sp}_G(x) = 1$ then x lies in the Fitting subgroup of G. Next proposition gives a probabilistic version of this result, which makes the subnormalizer a better set than $Nil_G(x)$ from this point of view. Namely, given an element x there is a threshold that the order of $S_G(x)$ cannot exceed if the element is not contained in the Fitting subgroup of G.

Proposition 2.3.3. *Let $1 \neq x \in G$. If $\mathrm{sp}_G(x) > 1/2$, then $x \in \mathbf{F}(G)$.*

Proof. Let p be a prime dividing the order of x, $|x| = p^a m$ with $(m, p) = 1$ and $z = x^m$, the p-part of x. Of course we have $S_G(x) \subseteq S_G(z)$. By Theorem 1.3.6 we have that z is contained in more than a half of Sylow p-subgroups of G and the same is true for all its conjugates. It follows that for every y conjugated to z there exists $P \in Syl_p(G)$ such that $\langle z, y \rangle \leq P$. By the Baer-Suzuki theorem (Theorem 1.2.3), $z \in O_p(G)$. □

The proof of the previous proposition is very simple, but with a slightly more careful argument the bound obtained can be improved as shown in next result. This tight bound will be the main tool for the classification-free proof of part (ii) of Theorem 2.2.2 (see Theorem 2.4.1).

Proposition 2.3.4. *Let p be a prime dividing the order of G, $x \in G$ be a p-element of order p^r and $1 \leq k \leq r$. If $\mathrm{sp}_G(x) > 1/(p^k+1)$ then $x^{p^{k-1}} \in O_p(G)$.*

Proof. By Theorem 1.3.6 $\mathrm{sp}_G(x) = \lambda_G(x)/n_p(G)$. Let $y_1, \dots, y_{p^k+1}$ be p^k+1 distinct conjugates of x. Then there exists $P \in Syl_p(G)$ such that two of these conjugates both belong to P. For if not,

$$U_i = \{P \in Syl_p(G) \mid y_i \in P\},\ i \in \{1, \dots, p^k+1\},$$

would be disjoint sets, each of cardinality $\lambda_G(x)$ (since λ_G is constant on the conjugacy classes of p-elements) and we would have

$$n_p(G) \geq \left| \bigcup_{i=1}^{p^k+1} U_i \right| = (p^k+1)\lambda_G(x)$$

against the hypothesis.

Let $g \in G$ and set $y_0 = x$, $y_i = x^{gx^{i-1}}$ for $1 \leq i \leq p^k$. We then have two cases: either there exist $0 \leq i < j \leq p^k$ such that $y_i = y_j$ or the set of the y_i's has cardinality p^k+1. In any case we then have that the following statement holds.

$$\text{There exist } 0 \leq i < j \leq p^k \text{ such that } \langle y_i, y_j \rangle \text{ is a } p\text{-group} \qquad (*)$$

We want to prove that if $(*)$ holds for all $g \in G$ then $x^{p^{k-1}} \in O_p(G)$. Arguing by induction on $|G|$ we can suppose that $x^{p^{k-1}} \in O_p(H)$, i.e., $\langle x^{p^{k-1}} \rangle$ is subnormal in H, for all proper subgroups H of G containing $x^{p^{k-1}}$. By the zipper lemma (1.2.2) $x^{p^{k-1}}$ is contained in a unique maximal subgroup M of G.

If $1 \leq s < p^k$ then $x^{p^{k-1}} \in \langle x^s \rangle$ and so M is the unique maximal subgroup containing x^s. Moreover if for some $a \in G$, $(x^s)^a \in M$ then $x^s \in M^{a^{-1}}$ and so $M = M^{a^{-1}}$. Since M is maximal and is not normal in G, we have $a \in M$.

Let then $g \in G$, y_i be defined as above and suppose that $(*)$ holds. We separately consider two cases: one in which $i = 0$ and the other in which $i \geq 1$. If $i = 0$ then $\langle x, y_j \rangle$ is a p-group, which implies that $y_j = x^{gx^{j-1}} \in M$. It follows that $gx^{j-1} \in M$ and so $g \in M$. If instead $i \geq 1$ then $\langle y_i, y_j \rangle$ is a p-group and so is the subgroup

$$\langle x^{(x^{j-i})^{g^{-1}}}, x \rangle = \langle y_i, y_j \rangle^{x^{-i}g^{-1}}.$$

It follows that $x^{(x^{j-i})^{g^{-1}}} \in M$, so $(x^{j-i})^{g^{-1}} \in M$ and finally $g^{-1} \in M$

We proved that if $(*)$ holds then $g \in M$. It follows that $G \leq M$, a contradiction. □

The bound in the previous proposition is the best possible, for if $G = PSL(2, p)$ we have that each Sylow p-subgroup of G has cardinality p, $n_p(G) = p+1$ and $O_p(G) = 1$. Then if $x \in G$ is a p-element we have that $\mathrm{sp}(G) = 1/(p+1)$.

Remark 2.3.5. Let $x \in G$ be an element that does not lie in the Fitting subgroup of G. Then there exists a prime p dividing the order of x such that the p-part x_p of x does not lie in $O_p(G)$. Then we have $\mathrm{sp}_G(x) \leq \mathrm{sp}_G(x_p) \leq 1/(p+1)$. In particular for all $x \in G \setminus \mathbf{F}(G)$, $\mathrm{sp}_G(x) \leq 1/3$.

Proposition 2.3.4 is a probabilistic version of the fact that if $S_G(x) = G$ then $\langle x \rangle \trianglelefteq\trianglelefteq G$, which is Theorem 1.2.1 (i). Part (ii) of the same theorem states that if $S_G^1(x) = G$ then $\langle x \rangle \trianglelefteq\trianglelefteq G$.The next example tells us that this second criterion (just like Proposition 2.2.1) doesn't have a probabilistic analogue.

Example 2.3.6. Let $n = 2k$ for $k \in \mathbb{N}, k \geq 2$ and let $G = S_n, x = (1, 2)$. By Remark 1.3.7 we have that

$$\frac{|S_G^1(x)|}{|G|} = \frac{\delta_G(x)}{|x^G|}$$

where $\delta_G(x)$ is the number of conjugates of x that generate a p-group together with x. We then want to count the number of transpositions that generates a 2-group together with x. If y is such a transposition then y commutes with x, because otherwise xy would be a 3-cycle. Then

$$\delta_G(x) = |\{x\} \cup \{(i,j) \mid 2 < i < j \leq n\}| = 1 + \frac{(n-2)(n-3)}{2}$$

and so

$$\frac{|S_G^1(x)|}{|G|} = \frac{1 + \frac{(n-2)(n-3)}{2}}{\frac{n(n-1)}{2}}$$

which tends to 1 as n goes to infinity. On the other hand, coherently with Proposition 2.3.4,

$$\mathrm{sp}_G(x) = \frac{\frac{n}{2}}{\frac{n(n-1)}{2}} = 1/(n-1),$$

which tends to 0 as n goes to infinity.

We end this section with a description of p-elements in solvable groups that assume the limit value in Proposition 2.3.4.

Proposition 2.3.7. *Let G be a solvable group with $O_p(G) = 1$ and x a p-element in G such that $\mathrm{sp}_G(x) = 1/(p+1)$. Then x has order p and is contained in a subnormal subgroup $H \simeq U\langle x \rangle$, where U is a q-group for some prime $q \neq p$. Moreover either $p = 2$ and $H \simeq S_3$, or U is a 2-group, $\Phi(U) = Z(H)$ and p is a Mersenne prime.*

Proof. The fact that $|x| = p$ is an immediate consequence of Proposition 2.3.4 and the fact that $O_p(G) = 1$. Let H be the minimal subnormal subgroup of G such that $x \in H$. Then $O_p(H) = 1$. By Theorem 1.4.1 it follows easily that $\mathrm{sp}_H(x) \geq \mathrm{sp}_G(x) = 1/(p+1)$ and so $\mathrm{sp}_H(x) = 1/(p+1)$. Moreover $U = \mathbf{F}(G)$ is a p'-group and so again

$$\mathrm{sp}_{U\langle x \rangle}(x) = 1/(p+1). \tag{2.3.1}$$

Let

$$1 < U < U_2 < \cdots < U_{n-1} < U_n = H$$

a $\{p, p'\}$-series in H and let $\mathcal{M}$ be the set of its p'-factors. By Theorem 1.4.1, if x_L is the image of x in the factor L,

$$\mathrm{sp}_H(x) = \prod_{L \in \mathcal{M}} \frac{|C_L(x_L)|}{|L|},$$

and so, by (2.3.1), x centralizes every L but U. It follows that xU is a left Engel element in H/U and so it lies in $\mathbf{F}(H/U)$ (12.3.3 in [27]). This implies that $U\langle x\rangle$ is subnormal in H and so $U\langle x\rangle = H$.

If $\{Q_i \mid i = 1, \ldots, k\}$ are the Sylow subgroups of U one get a $\{p, p'\}$-series

$$1 < Q_1 < Q_1 Q_2 < \cdots < \prod_{i=1}^{k-1} Q_i < U < H$$

and so x centralizes all but one of the Q_i's, say Q_1. It follows that $Q_1\langle x\rangle$ is subnormal in H and so $Q_1 = U$, i.e., U is a q-group for a prime q.

We now consider the $\{p, p'\}$-series

$$1 \le \Phi(U) < U < H.$$

Since the action of x on U is coprime we have from 3.29 in [15] that x acts nontrivially on $\frac{U}{\Phi(U)}$ and so $[\Phi(U), x] = 1$. Since $\langle x\rangle^H = H$ it follows that $\Phi(U) \le Z(H)$. Now the action of x on the elementary abelian group $\bar{U} = \frac{U}{\Phi(U)}$ decomposes as

$$\bar{U} = C_{\bar{U}}(x) \times \left[\bar{U}, x\right].$$

Calling C (respectively V) the subgroup in H whose homomorphic image in $\frac{U}{\Phi(U)}$ is $C_{\bar{U}}(x)$ (respectively $\left[\bar{U}, x\right]$) one has that $V\langle x\rangle \trianglelefteq H$ and so $C_{\bar{U}}(x) = 1$. In particular $Z(H) = \Phi(U)$. Since the Sylow p-subgroups have order p, any two of them intersect trivially and so $\lambda_G(x) = 1$. It follows that $n_p(G) = p + 1$. Since $N_{H/\Phi(U)}(\langle x\rangle) = C_{\bar{U}}(x)\langle x\rangle$ we have that $N_{H/\Phi(U)}(\langle x\rangle) = \langle x\rangle$ and so

$$p + 1 = n_p(G) = \left|\frac{U}{\Phi(U)}\right| = q^l$$

for some $l \in \mathbb{N}$. If $p = 2$ then $q = 3$ and $l = 1$ so that $H \simeq S_3$. If p is odd, then $q = 2$ and $p = q^l - 1$ is a Mersenne prime. □

2.4 Global probabilistic results

In this section we use Proposition 2.3.4 to get a classification-free proof of Theorem 2.2.2, as well as another similar probabilistic result about $\mathrm{sp}(G)$. We want to stress that both the proofs are very similar to that of Theorem 2.1.1, with the added ingredient of Remark 2.3.5.

Theorem 2.4.1. *Let G be a finite group. If $\mathrm{dn}(G) > 1/2$ then G is nilpotent.*

Proof. First of all we observe that $[G : \mathbf{F}(G)] \le 3$. For if $[G : \mathbf{F}(G)] \ge 4$ then by Remark 2.3.5

$$\begin{aligned}
\mathrm{dn}(G) \le \mathrm{sp}(G) &= \frac{1}{|G|}\sum_{x\in G}\frac{|S_G(x)|}{|G|} \\
&= \frac{1}{|G|}\left(|\mathbf{F}(G)| + \sum_{x\notin \mathbf{F}(G)}\frac{|S_G(x)|}{|G|}\right) \\
&\le \frac{1}{|G|}\left(|\mathbf{F}(G)| + \frac{1}{3}\frac{|G| - |\mathbf{F}(G)|}{|G|}\right) \\
&= \frac{|G| + 2|\mathbf{F}(G)|}{3|G|} \le \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.
\end{aligned} \tag{2.4.1}$$

Thus G is solvable and $[G : \mathbf{F}(G)] \in \{2,3\}$. Let G be a counterexample of minimal order. By Proposition 2.2.4 every quotient of G is nilpotent. Setting $N := \mathbf{F}(G)$, we have $G = N\langle x \rangle$ with $|x| = q \in \{2,3\}$ and N is an elementary abelian group of order p^k for some prime $p \neq q$ and some $k \in \mathbb{N}$. Moreover $C_N(x) = 1$. G is then a Frobenius group with kernel N. For every $1 \neq a \in N$ we have

$$Nil_G(a) = N,$$

while for every $y \notin N$ we have

$$Nil_G(y) = \langle y \rangle.$$

Therefore

$$\begin{aligned}\mathrm{dn}(G) &= \frac{1}{|G|}\sum_{g\in G}\frac{|Nil_G(g)|}{|G|} = \frac{1}{|G|}\left(\sum_{g\in N}\frac{|Nil_G(g)|}{|G|} + \sum_{g\notin N}\frac{|Nil_G(g)|}{|G|}\right)\\ &= \frac{1}{p^k q}\left(1 + (p^k-1)\frac{1}{q} + (q-1)p^k\frac{1}{p^k}\right) = \frac{1}{p^k q} + \frac{p^k-1}{p^k}\frac{1}{q^2} + \frac{q-1}{p^k q}\end{aligned}$$

which is greater then $1/2$ if and only if $q = 2$ and $p^k = 3$, that is if and only if $G \simeq S_3$. By direct calculation one see that $\mathrm{dn}(S_3) = 1/2$ and so we have the thesis. $\square$

Theorem 2.4.2. *If $\mathrm{sp}(G) > 2/3$ then G is nilpotent, and the bound is the best possible.*

Proof. This is just a calculation which follows easily from Remark 2.3.5. Let G be a nonnilpotent group: then $|\mathbf{F}(G)| \leq |G|/2$. Thus

$$\begin{aligned}\mathrm{sp}(G) &= \frac{1}{|G|}\sum_{g\in G}\mathrm{sp}_G(x) = \frac{1}{|G|}\sum_{g\in\mathbf{F}(G)}\mathrm{sp}_G(x) + \frac{1}{|G|}\sum_{g\notin\mathbf{F}(G)}\mathrm{sp}_G(x)\\ &\leq \frac{|\mathbf{F}(G)|}{|G|} + \frac{1}{3}\frac{|G\setminus\mathbf{F}(G)|}{|G|} = \frac{1}{3} + \frac{2}{3}\frac{|\mathbf{F}(G)|}{|G|} \leq \frac{2}{3}.\end{aligned}$$

The fact that the bound is tight follows from an easy calculation that gives $\mathrm{sp}(S_3) = 2/3$. $\square$

We want to point out that the existence of a threshold like the one provided by the Theorem 2.4.2 is not trivially deductible from the results of Guralnick and Wilson. What follows trivially from those results is that there exists $c \in (0,1)$ such that if $\mathrm{sp}(G) > c$ then G is solvable. The next chapter will be focused on the search for the best possible c.

In [32] it is proved that the degree of nilpotence of a finite group is linked with the index of the Fitting subgroup. More precisely, there exists a function $f : [0,1] \to \mathbb{N}$ such that if G is a finite group with $\mathrm{dn}(G) = \varepsilon$ then $[G : \mathbf{F}(G)] \leq f(\varepsilon)$.

Looking at Proposition 2.3.4 and Theorem 2.4.2 the question could be asked if a theorem like Wilson's could hold replacing $\mathrm{dn}(G)$ with $\mathrm{sp}(G)$. This is still an open question for us.

It seems that one of the main problems in considering this question is to treat elements of composite order, for which we do not have a formula analogous to the one for p-elements given by Theorem 1.3.6.

Chapter 3

Subnormalizers and solvability

3.1 Main statement and sketch of proof

In the previous chapter we studied the relation between the ratio $\mathrm{sp}(G)$ and nilpotence.

Now we want to see what $\mathrm{sp}(G)$ can say about the solvability of G. In the proof of Theorem 2.4.1 we observed that if $\mathrm{sp}(G) > 1/2$ then the group is solvable (in fact, the Fitting subgroup has index less than 4). A better threshold is given by Theorem 2.2.2. This is because if $\langle x \rangle \trianglelefteq\trianglelefteq \langle x, y \rangle$ then $\langle x, y \rangle$ is solvable (nilpotent-by-cyclic) and so $\mathrm{ds}(G) \geq \mathrm{sp}(G)$. Thus if $\mathrm{sp}(G) > 11/30$ then G is solvable.

Still, one can expect that the threshold on $\mathrm{sp}(G)$ above which G is forced to be solvable, is smaller than $11/30$. Indeed the main result of this chapter is the following theorem.

Theorem 3.1.1. *If $\mathrm{sp}(G) > 1/6$ then G is solvable.*

Let us observe at first that the bound $1/6$ is tight, since $\mathrm{sp}(A_5) = 1/6$. The following easy lemma will be very useful in the rest of the chapter.

Lemma 3.1.2. *If $N \trianglelefteq G$ then $\mathrm{sp}(G/N) \geq \mathrm{sp}(G)$.*

Proof. The proof is the same as that of Lemma 2.2.4, replacing $Nil_G(x)$ with $S_G(x)$. □

We are now able to sketch the proof of Theorem 3.1.1.

Let G be a nonsolvable group. We want to prove that $\mathrm{sp}(G) \leq 1/6$. Thanks to the previous lemma and arguing by induction we can assume that G is nonsolvable while all its proper quotients are solvable.

Suppose that N and M are minimal normal subgroups of G. Since G is nonsolvable, if N (or M) is solvable then G/N (or G/M) is nonsolvable, which contradicts our assumption. N and M are then nonsolvable. If $M \neq N$ then $M \cap N = 1$ and

$$\frac{MN}{N} \simeq \frac{N}{M \cap N} = N$$

so that G/N would be nonsolvable (since it would contain a nonsolvable subgroup). Thus we may assume that G has a unique minimal normal subgroup N, N is nonsolvable and is then the direct product of k copies of a nonabelian simple group L:

$$N = L_1 \times \cdots \times L_k,\ L_i \simeq L,\ \forall i \in \{1, \dots, k\}.$$

Moreover $C_G(N) = 1$, since otherwise $N \leq C_G(N)$, $C_G(N)$ being normal in G, i.e., N would be abelian.

A group having a unique minimal normal subgroup is called monolithic. We will call *minimal nonsolvable monolithic group* a monolithic group whose proper quotients are solvable. We call the class of minimal nonsolvable monolithic groups $\mathfrak{M}_{ns}$.

From Lemma 2.3.4 we know that given $x \in G$ and p a prime dividing the order of x, if $\mathrm{sp}_G(x) > 1/(p+1)$, $x \in O_p(G)$. Since in our assumptions G has trivial Fitting subgroup we can assume that

$$\mathrm{sp}_G(x) \leq \frac{1}{q+1},$$

where q is the maximum prime power dividing $|x|$.

Therefore if the order of an element x is divided by a prime power $q \geq 5$, its contribution in the sum (2.2.3) is at most $1/6$.

The proof of Theorem 3.1.1 goes then as follows. Let P be a 2-Sylow subgroup of G and $\mathfrak{U}_2(G)$ be the set of 2-elements in G, i.e., the union of all 2-Sylow subgroups of G. Moreover let S be the set of $\{2,3\}$-elements x in G such that $|x| = 2^n 3$ for some $n \in \mathbb{N}$. For such an element set $x_3 := x^{2^n}$. If $x \in G$ is not in $S \cup \mathfrak{U}_2(G)$ then $\mathrm{sp}_G(x) \leq 1/6$.

Thus, using Lemma 2.2.7 we have

$$\begin{aligned}
\mathrm{sp}(G) &= \frac{1}{|G|}\sum_{g\in G}\mathrm{sp}_G(g)\\
&\leq \frac{1}{|G|}\left(\sum_{g\in\mathfrak{U}_2(G)}\mathrm{sp}_G(g)\right) + \frac{1}{|G|}\left(\sum_{g\in S}\mathrm{sp}_G(g)\right) + \frac{|G\setminus(\mathfrak{U}_2(G)\cup S)\,|}{6|G|} \qquad (3.1.1)\\
&\leq \frac{|P|}{|G|} + \frac{1}{|G|}\left(\sum_{g\in S}\mathrm{sp}_G(g_3)\right) + \frac{|G\setminus(\mathfrak{U}_2(G)\cup S)\,|}{6|G|}.
\end{aligned}$$

We divide the rest of the argument in two steps.

Step 1 We want to bound the term

$$\frac{1}{|G|}\left(\sum_{g\in S}\mathrm{sp}_G(g_3)\right).$$

More precisely, we prove that if G is a group in $\mathfrak{M}_{ns}$ and $x \in G$ is an element of order 3 then

$$\mathrm{sp}_G(x) \leq \frac{1}{6}.$$

Step 2 Given *Step 1* we can rewrite (3.1.1) as follows:

$$\begin{aligned}
\mathrm{sp}(G) &\leq \frac{|P|}{|G|} + \frac{|G\setminus(\mathfrak{U}_2(G))\,|}{6|G|}\\
&= \frac{|P|}{|\mathfrak{U}_2(G)|}\frac{|\mathfrak{U}_2(G)|}{|G|} + \frac{|G\setminus(\mathfrak{U}_2(G))\,|}{6|G|}.
\end{aligned} \qquad (3.1.2)$$

As a consequence it is enough to prove that

$$\frac{|P|}{|\mathfrak{U}_2(G)|} \leq \frac{1}{6} \qquad (3.1.3)$$

for a group G in $\mathfrak{M}_{ns}$. We will prove this, except for a finite number of cases which we will treat separately (see Lemma 3.3.4).

3.2 Elements of order 3

In this section we prove *Step 1* of the proof. Namely we want to prove

Proposition 3.2.1. *Let G be a finite nonsolvable group with no normal solvable subgroups and let x be an element of G of order 3. Then*

$$\mathrm{sp}_G(x) \leq \frac{1}{6}.$$

The proof depends on the classification of finite simple groups and the reduction is given by the following two lemmas.

Lemma 3.2.2. *Let p be a prime dividing $|G|$, H a subgroup of G and $x \in H$ be a p-element.*

i) $\mathrm{sp}_G(x) \leq \mathrm{sp}_H(x)$.

ii) If $H \trianglelefteq G$ then $\mathrm{sp}_G(x) = \mathrm{sp}_H(x)$.

Proof. Let $\mathcal{S} := Syl_p(G)$ and let

$$\Lambda = \{(y, P) \in x^H \times \mathcal{S} \mid y \in P\}.$$

We count the elements in Λ first by summing on the first component, and then on the second one. We then get, using Theorem 1.3.6 and the fact that λ_G is invariant on the conjugacy classes

$$|\Lambda| = \sum_{y \in x^H} \lambda_G(y) = \lambda_G(x)|x^H|$$

and

$$|\Lambda| = \sum_{P \in \mathcal{S}} |x^H \cap P| = \sum_{P \in \mathcal{S}} |x^H \cap (P \cap H)| \leq \sum_{P \in \mathcal{S}} \alpha_H(x) = \alpha_H(x)|\mathcal{S}| \qquad (3.2.1)$$

since $|x^H \cap (P \cap H)| \leq |x^H \cap Q|$ where Q is a Sylow p-subgroup containing $P \cap H$. Finally we get

$$\frac{\lambda_G(x)}{|\mathcal{S}|} \leq \frac{\alpha_H(x)}{|x^H|}$$

which is the (i), again by Theorem 1.3.6. (ii) follows by noticing that the inequality in 3.2.1 is an equality when $H \trianglelefteq G$. $\square$

Lemma 3.2.3. *It is enough to prove Proposition 3.2.1 in the following three cases:*

1. *G is a finite simple group.*
2. *$G = L\langle x \rangle$, with L a nonabelian finite simple group and $x \in Aut(L) \setminus L$.*
3. *$G = (L \times L \times L)\langle x \rangle$, L a nonabelian finite simple group and x permutes the three factors.*

Proof. Let G and $x \in G$, $|x| = 3$ be a counterexample to Proposition 3.2.1, with G of minimal order. Then $\mathrm{sp}_G(x) > 1/6$. Let N be a minimal normal subgroup of G. Being nonsolvable, N is a direct product of k copies of a simple group L:

$$N = L_1 \times \cdots \times L_k,\ L_i \simeq L,\ \forall i \in \{1, \ldots, k\}.$$

First suppose that $x \in N$. Then $x = (x_1, \ldots, x_k)$ with $|x_i| \in \{1, 3\}$. Without loss of generality we can suppose that $x_1 \neq 1$ and so

$$x \in H = L_1 \times \langle x_2 \rangle \times \cdots \times \langle x_k \rangle.$$

By Lemma 3.2.2 $\mathrm{sp}_G(x)$ is smaller than $\mathrm{sp}_H(x)$ which is equal to $\mathrm{sp}_{L_1}(x_1)$.

Suppose now that $x \notin N$. We can assume $L_1 \not\leq C_N(x)$. If $x \in N_G(L_1)$ then we can take $H = L_1\langle x \rangle$ and we are in case 2. Otherwise for every $a \in N$

$$((L_1)^x)^a = (L_1)^{a^{x^{-1}}x} = (L_1)^x$$

so that L_1^x is a normal subgroup of N. We know that the only normal subgroups of a direct product of simple groups are the products of the factors and so L_1^x must be one of the L_i, say L_2. If $x \in N_G(L_2)$ then $L_1^{x^2} = L_1^x$ contradicting $x \notin N_G(L_1)$. Also $L_2^x \neq L_1$ for otherwise

$$L_1 = (L_1)^{x^3} = (L_2)^{x^2} = (L_1)^x.$$

We conclude that L_2^x is a third factor different from L_1, L_2, say L_3. Of course $(L_3)^x = (L_1)^{x^3} = L_1$ and so x cyclically permutes L_1, L_2 and L_3. $\square$

We treat the third case first.

Proposition 3.2.4. *Let p be a prime, H be a finite group and $G = (H_1 \times \cdots \times H_p)\langle x \rangle$, with $H_i \simeq H$ for $i = 1, \ldots, p$ and x an element of order p such that $H_i^x = H_{i+1}$ for $i = 1, \ldots, p-1$ and $H_p^x = H_1$. Then*

$$\mathrm{sp}_G(x) \leq \frac{1}{n_p(H)^{p-1}}.$$

Proof. Let $K = H_1 \times \cdots \times H_p$. The Sylow p-subgroups of G containing x are exactly those of the form $Q\langle x \rangle$ where Q is a Sylow p-subgroup of K normalized by x. Moreover, given $P \in Syl_p(G)$ such that $x \in P$, Q is uniquely determined as $P \cap K$. Thus

$$\lambda_G(x) = |\{Q \in Syl_p(K) \mid Q^x = Q\}|.$$

Now a Sylow p-subgroup Q of K is the direct product of Sylow p-subgroups Q_i of H_i. If P is x-invariant then $Q_i^x = Q_{i+1}$ for $i = 1, \ldots, p-1$ and $Q_p^x = Q_1$, so that the number of x-invariant Sylow p-subgroups of K is less or equal to the number of Sylow p-subgroups of H.

Finally

$$\begin{aligned} \mathrm{sp}_G(x) =& \frac{\lambda_G(x)}{n_p(G)} = \frac{|\{Q \in Syl_p(K) \mid Q^x = Q\}|}{n_p(G)} \\ \leq& \frac{n_p(H)}{n_p(K)} = \frac{n_p(H)}{n_p(H)^p} = \frac{1}{n_p(H)^{p-1}}. \end{aligned} \tag{3.2.2}$$

$\square$

We can now apply this fact to case 3 in Lemma 3.2.3. Since L is a finite simple group, $n_3(L) \geq 4$ and so

$$\mathrm{sp}_G(x) \leq \frac{1}{n_3(L)^2} \leq \frac{1}{16}.$$

We now turn our attention to finite simple groups.

3.2.1 Alternating groups

Here we consider $G = A_n$, where $5 \leq n \in \mathbb{N}$. This is the easiest case but it is representative for the strategy used in all the others. In order to use Lemma 3.2.2 we want to find a subgroup H of G such that $x \in H$ and $\mathrm{sp}_H(x) \leq 1/6$.

A direct calculation shows that in $G = A_5$ or $G = A_5 \times C_3$ every noncentral element y of order 3 is such that $\mathrm{sp}_G(y) = 1/10$.

Let $n \geq 6$. Each element of order 3 is product of k 3-cycles with pairwise disjoint supports. Without loss of generality we can assume that

$$x = (1,2,3)(4,5,6)\dots(3k-2, 3k-1, 3k),$$

with $3k \leq n$. Put $x_1 = (1,2,3)(4,5,6)$, $x_2 = xx_1^2$ and $y = (3,5)(4,6)$. Clearly $x_2 \in C_G(x_1) \cap C_G(y)$. It follows that

$$H = \langle x_1, x_2, y\rangle \simeq \langle x_1, y\rangle \times \langle x_2\rangle \simeq A_5 \times C_3.$$

By Lemma 3.2.2 we have that

$$\mathrm{sp}_{A_n}(x) \leq \mathrm{sp}_H(x) = 1/10.$$

The almost simple case here is trivial, since 3 does not divide $Out(A_n)$ for any n.

3.2.2 Finite groups of Lie type: the *fixed point ratio*

Thanks to (2.2.6) we know that for a p-element x of a finite group G,

$$\mathrm{sp}_G(x) = \frac{\lambda_G(x)}{n_p(G)},$$

i.e., the ratio between the number of Sylow p-subgroups containing x and the total number of Sylow p-subgroups of G. If we consider the transitive action of G on its Sylow p-subgroups, we see that $\lambda_G(x)$ is the number of Sylow p-subgroups fixed (normalized) by x. Therefore $\mathrm{sp}_G(x)$ is what is commonly called the *fixed point ratio* (fpr) of x with respect to this action.

A first remark to be made is that the equality (2.2.6) is still true in this more general setting. Namely, if the action of G on a set Λ is transitive and H is a point stabilizer then

$$\mathrm{fpr}_\Lambda(g) = \frac{|x^G \cap H|}{|x^G|}.$$

The fixed point ratio has been much studied (see for example [1], [22], [10] and the survey by Burness [2]), especially for what concerns primitive actions of finite groups of Lie type.

The most general result in this area, which is very useful for our very particular and circumscribed case, is the main theorem in [21].

Theorem 3.2.5 (Theorem 1 in [21]). *Let L be a finite simple group of Lie type on $\mathbb{F}_q$, $q = p^e$, and let G be an almost simple group with socle L acting faithfully and primitively on a set Ω. Then for all $1 \neq g \in G$*

$$\operatorname{fpr}_G(x) = \frac{|\operatorname{Fix}_\Omega(g)|}{|\Omega|} \leq \frac{4}{3q}, \tag{3.2.3}$$

apart from a short list of known exceptions.

Let us observe that the result is true for transitive actions as well, since if an action of G on Λ is transitive then

$$\operatorname{fpr}_\Lambda(x) = \frac{|x^G \cap H|}{|x^G|} \leq \frac{|x^G \cap K|}{|x^G|} = \operatorname{fpr}_{G/K}(x)$$

where K is any maximal subgroup of G containing H and G/K is the set of cosets of K in G, which G acts primitively on.

Consequently, if G is an almost simple group of Lie type on $\mathbb{F}_q$ with $q \geq 8$ we get the result. Moreover, there are even better bounds for the exceptional groups of Lie type, studied in [19], which give fixed point ratios smaller than $1/6$ even for small q.

Thus we only have to deal with classical groups on $\mathbb{F}_q$ with $q < 8$, besides the exceptions mentioned in Theorem 3.2.5, which can be worked out with GAP ([9]).

Also, the following easy observation allows us to work with nonprojective classical groups.

Lemma 3.2.6. *Let G be a group, p a prime, $x \in G$ a p-element and $Z = Z(G)$. Then $\operatorname{sp}_G(x) = \operatorname{sp}_{G/Z}(xZ)$.*

Proof. Let P be a Sylow p-subgroup of G. For $g \in G$, $H \leq G$ we write $\bar{g}$ and $\bar{H}$ for gZ and HZ/Z.

Since

$$N_{\bar{G}}(\bar{P}) = \{\bar{g} \in \bar{G} \mid \bar{P}^{\bar{g}} = \bar{P}\} = \{\bar{g} \in \bar{G} \mid P^g Z = PZ\}$$

and PZ is nilpotent, we have that $N_{\bar{G}}(\bar{P}) = \overline{N_G(P)}$, and since $Z \leq N_G(P)$

$$[G : N_G(P)] = \left[\bar{G} : \overline{N_G(P)}\right].$$

If $\bar{x} \in \bar{P}$ then $x \in P$ and so $\lambda_G(x) = \lambda_{\bar{G}}(\bar{x})$. □

We distinguish between two cases, whether or not $q = 3$.

3.2.3 Finite groups of Lie type: $q \neq 3$

We begin our analysis from the linear case.

Proposition 3.2.7. *Let $q = p^r$, with $p \neq 3$ a prime. Let $x \in GL_n(q)$, be a noncentral element such that $x^3 \in Z(GL_n(q))$. Then there is a noncentral 3-element $y \in GL_m(q)$, with $m \in \{2, 3\}$, such that*

$$\operatorname{sp}_{SL_n(q)\langle x\rangle}(x) \leq \operatorname{sp}_{SL_m(q)\langle y\rangle}(y).$$

Proof. We can assume that x has order a power of 3. Since $x^3 \in Z(GL_n(q))$, $x^3 = \omega I$, with ω a root of unity.

Let then $V = \mathbb{F}_q^n$. Since $x \notin Z(G)$ we can take $v_1 \in V$ which is not an eigenvector for x.

We then take $v_2 = v_1^x$ and $v_3 = v_2^x$. Let W be the span of $\{v_1, v_2, v_3\}$. Since

$$v_3^x = v_1^{x^3} = \omega v_1$$

it is clear that W is an x-invariant subspace of dimension 2 or 3. Since $p \neq 3$, by Maschke's theorem there exists an x-invariant complement U of W.

Then $x \in A = GL(W) \times GL(U)$, say $x = x_1 x_2$ in this decomposition. Clearly, for our choice of W, $x_1 \notin C_{GL(V)}(SL(W))$.

Let $N = A \cap SL(V)$. We have

$$\begin{aligned} \mathrm{sp}_{SL(V)\langle x\rangle}(x) &\leq \mathrm{sp}_{N\langle x\rangle}(x) \\ \leq \mathrm{sp}_{(SL(W)\times SL(U))\langle x\rangle}(x) &\leq \mathrm{sp}_{SL(W)\langle x\rangle}(x) \\ &= \mathrm{sp}_{SL(W)\langle x_1\rangle}(x_1). \end{aligned}$$

All the inequalities follow from Lemma 3.2.2, while last equality follows from the fact that x_2 centralizes $SL(W)\langle x_1\rangle$. ☐

Now we look at classical groups with forms, i.e., symplectic, unitary and orthogonal groups. The next lemma is our main tool in inspecting these groups.

Lemma 3.2.8. *Let V be a vector space over $\mathbb{F}_q$ of dimension n and f a nondegenerate alternating, sesquilinear or symmetric bilinear form on V. Let x be an endomorphism of V such that for all $v, w \in V$, $f(v, w) = 0$ if and only if $f(v^x, w^x) = 0$. Moreover let W be an x-invariant subspace of dimension d. Then there is a nonsingular x-invariant subspace of dimension $d \leq k \leq 2d$.*

Proof. The subspace $\tilde{W} = W + W^{\perp}$ is x-invariant.

Let U be an x-invariant complement of $\tilde{W}$ in V, whose existence is guaranteed by Maschke's theorem. We observe that

$$\begin{aligned} \dim(U) &= \dim(V) - \dim(\tilde{W}) = \\ &= \dim(V) - (\dim(W) + \dim(W^{\perp}) - \dim(W \cap W^{\perp})) = \\ &= \dim(W \cap W^{\perp}) \end{aligned}$$

We now consider $Y := W \oplus U$. Moreover, let $W = W_0 \oplus W_1$ with $W_0 = W \cap W^{\perp}$.

We have $Y^{\perp} = W^{\perp} \cap U^{\perp}$ and $V = W^{\perp} \oplus (W_1 \oplus U)$.

Let $v \in Y \cap Y^{\perp}$. Then $v = w_0 + w_1 + u$ with $w_0 \in W_0$, $w_1 \in W_1$ and $u \in U$. Since v and w_0 belong to $W^{\perp}$, we have $w_1 + u \in W^{\perp}$ and so $w_1 + u = 0$. Then $v = w_0 \in W \cap W^{\perp} \cap U^{\perp}$ and so

$$\langle v\rangle^{\perp} \geq W + W^{\perp} + U = V$$

which implies $v = 0$. ☐

We can now write the conclusion of our argument for $q \neq 3$.

- $[\mathbf{PSL_n(q)}]$ As for linear groups, using Proposition 3.2.7 we immediately see that the only groups to be checked are $PSL_m(q)\langle y\rangle$ with $y \in PGL_m(q)$ of order 3 ($m \in \{2,3\}$). If $q \in \{4,5,7\}$ this is enough, as we can see with GAP ([9]) that for these groups every 3-element g is such that $\mathrm{sp}_G(g) \leq 1/6$.

  When $q = 2$ we have to consider a subspace of dimension at least 4. To get this we repeat the argument in Proposition 3.2.7 on the complement U of W and then we take the direct sum. We have to check 3-elements in groups of the form $PSL_d(2)\langle y\rangle$ with $4 \leq d \leq 6$ and y a 3-element. Again using GAP ([9]) we see that these elements are such that $\mathrm{sp}_G(g) \leq 1/6$.

  This gives the result in cases 1 and 2 of Lemma 3.2.3 when L is a $PSL_n(q)$, since any automorphism of order 3 of $PSL_n(q), q < 8$, come from conjugation by elements of $GL_n(q)$.

We now look at the other classical groups. If L is a finite simple symplectic, unitary or orthogonal group on $\mathbb{F}_q$, $q < 8$, an element x of order 3 in $Aut(L)$ is inside L unless $L = PSU_n(q)$ and x comes from an element in $GU_n(q)$, or $L = P\Omega_8^+(q)$. Suppose we are not in the latter case, which will be treated later.

Let Δ be one of the following groups:

$$GU_n(q),\ Sp_n(q),\ O_n^{\pm}(q).$$

As described in [18] the stabilizer in Δ of a decomposition

$$V = W \perp U \tag{3.2.4}$$

is the direct product $\Delta(W) \times \Delta(U)$ (the sign of the orthogonal group can change). Moreover, let x be a 3-element in Δ, and $G = S\Delta\langle x\rangle$ where $S\Delta$ is the quasi-simple group contained in Δ. If x stabilizes a decomposition (3.2.4) without centralizing W we have that $x = x_1x_2 \in \Delta(W)\times\Delta(U)$, $S\Delta(W)$ is stabilized by x, x_1 and x_1 doesn't centralize it. Then

$$\mathrm{sp}_G(x) \leq \mathrm{sp}_{S\Delta(W)\langle x_1\rangle}(x_1)$$

Our strategy will then be to get a nonsingular subspace of small dimension W such that $PS\Delta(W)$ is not solvable and then to check the finitely many resulting groups with GAP ([9]).

- $[\mathbf{PSU_n}]$ We consider the $q \neq 2$ case first. As in the proof of Proposition 3.2.7, we find an invariant subspace W of dimension 2 or 3 on which the action of x is not trivial. By Lemma 3.2.8 we find an x-invariant nonsingular subspace Y of dimension $2 \leq d \leq 6$. Then we need only to check the groups $PSU_d(q), 2 \leq d \leq 6$ and their extensions by a 3-element. Since some of these groups are too big for a direct calculation with GAP we use some bounds on the size of conjugacy classes of semisimple elements contained in [1]. Namely, it is enough to use the inequality

  $$\mathrm{sp}_G(x) \leq \frac{|P|}{|x^G|} \tag{3.2.5}$$

  in order to get ratios smaller than $1/6$.

  When $q = 2$ we start with an x-invariant subspace W of dimension 2 or 3 which is not centralized by x. Suppose at first that the nonsingular subspace Y of Lemma 3.2.8 has dimension $4 \leq d \leq 6$.

Suppose $\dim Y = 2$ or 3. If there is a vector $w \in W^{\perp}$ such that $w^x \notin \langle w \rangle$, we can find an x-invariant subspace W' of dimension 2 or 3 and again apply Lemma 3.2.8 to this subspace to obtain a nonsingular subspace Y' of dimension $2 \leq d' \leq 6$. If $d' \leq 3$ we consider $Y \oplus Y'$ which is nonsingular. If $d' \geq 4$ we just consider Y'.

The last case is when $\dim Y \in \{2, 3\}$ and $Y^{\perp}$ is an autospace for x. In that case we can take a subspace W' of dimension 2, apply Lemma 3.2.8 and finally obtain a nonsingular subspace of dimension $4 \leq d \leq 6$.

- $[\mathbf{PSp_n}]$ Here the argument is the same used for PSU.
- $[\mathbf{P\Omega_n}]$ If $q \in \{5, 7\}$ the argument used for PSU work in this case as well, since the quadratic form is equivalent to the related bilinear form.

  In characteristic 2 we are only interested with $P\Omega^{\pm}_{2n}$, since

  $$P\Omega_{2n+1}(2^r) \simeq PSp_{2n}(2^r).$$

  When the dimension is even, calling Q the quadratic form and f the related bilinear symmetric form we have

  $$Rad(f) = 0 \text{ if and only if } Rad(Q) = 0.$$

  Thus we have that $Rad(f) = 0$ and Lemma 3.2.8 still gives a subspace Y, nonsingular with respect to the bilinear form and so also to the quadratic form (since $Rad(Q) \subseteq Rad(f)$).

  If $q = 2$ with arguments similar to those of the SU case we get a nonsingular subspace of dimension $5 \leq d \leq 8$. If $q = 4$ the dimension will be between 3 and 6.

The only case to be worked out concerns the outer automorphisms of $L = P\Omega^+_8(q)$ with $q \leq 7$ (this is the only case having graph automorphisms involved). In [1], II, Lemma 3.48 pag. 107, the following bound is given for a 3-element $x \in Aut(L) \setminus L$:

$$|x^L| \geq \frac{1}{8} q^{14}.$$

Set $G = L\langle x \rangle$. For $q \in 2, 4, 5, 7$ the 3-Sylow subgroup P in $P\Omega^+_8(q)$ has order 243 and so

$$\mathrm{sp}_G(x) = \frac{|x^G \cap P\langle x \rangle|}{|x^G|} \leq \frac{3|P|}{|x^L|} \leq \frac{8 \cdot 729}{q^{14}}$$

which is less than $1/6$ for $q \geq 4$.
As for $P\Omega^+_8(2)$ we check with GAP ([9]).

3.2.4 Finite groups of Lie type: $q = 3$

Again the case of an outer automorphism of $P\Omega^+_8(3)$ can be checked with GAP ([9]). We thus only have to deal with elements inside L. In [11] a description of unipotent classes in classical groups is given in terms of Jordan blocks. Since the element x has order 3, its Jordan blocks have dimensions between 1 and 3.

In the linear case it is enough to take the subspace related to a Jordan block of dimension 3, two Jordan blocks of order 2 or one of order 1 and the other of order 2. We can use GAP to check $PSL_3(3)$ and $PSL_4(3)$.

In the other cases call J_i a Jordan block of dimension $i \in \{1, 2, 3\}$, and if r_i is the multiplicity of that block we write the Jordan form of x as

$$r_1 J_1 + r_2 J_2 + r_3 J_3. \tag{3.2.6}$$

Following the description of [11], we describe what happens when $L = PSU_n(3)$. Proposition 2.2 in [11] tells that for all $i \in \{1, 2, 3\}$ there exist r_i subspaces of V of dimension i which are pairwise orthogonal, x-invariant and such that the sesquilinear form is nonsingular on all of them. Choosing a sum of these subspaces on which x acts nontrivially we get a nonsingular subspace W of dimension $d \in \{3, 4\}$ such that $x|_W \neq id_W$. $W^\perp$ is x-invariant too and so

$$\mathrm{sp}_L(x) \leq \mathrm{sp}_{SU_d(3)}(y)$$

for some 3-element y.

In the symplectic and orthogonal case things go likewise, but $\dim(W) \in \{4, 5, 6\}$.

3.2.5 Sporadic groups

Looking at the ATLAS of finite simple groups it comes out that if L is each one of the twentysix sporadic simple groups, $P \in Syl_3(L)$, it is true the loose inequality

$$\mathrm{sp}_L(x) \leq \frac{|P|}{|x^L|} \leq \frac{1}{6}.$$

The almost simple case is trivial with sporadic groups, since for such an L we have $Out(L) \in \{1, 2\}$ (see for example the survey from Richard Lyons [23]).

In the last paragraphs we proved Proposition 3.2.1 for G almost simple. This, together with Lemma 3.2.3 and Proposition 3.2.4 completes the proof of Proposition 3.2.1.

3.3 The number of 2-elements in nonsolvable groups

In this section we focus on *Step 2* of the proof of Theorem 3.1.1, described at the end of section 3.1. We thus want to deal with the ratio $|G|_2/|\mho_2(G)|$ for nonsolvable groups.

A celebrated theorem of Frobenius gives the following result.

Theorem 3.3.1 ([6]). *Let G be a finite group and p a prime dividing the order of G. Then the order of a Sylow p-subgroup of G divides the number of p-elements in G.*

We will call the integer $|\mho_p(G)|/|G|_p$ the *p-Frobenius ratio* of G (or just Frobenius ratio, in case it is clear which prime we are referring to).

The Frobenius ratio will be the main topic of the next chapter. As for now, we just focus on what will be useful for *Step 2* of our proof.

We thus collect some information about $\mho_p(G)$ and the Frobenius ratio in the following lemma.

Lemma 3.3.2. *Let P be a Sylow p-subgroup of G.*

i) If $P \leq H \leq G$ then the p-Frobenius ratio of G is greater than that of H.

ii) If N is a normal subgroup of G then

$$|\mathfrak{U}_p\left(G/N\right)|/|(G/N)|_p \leq |\mathfrak{U}_p(G)|/|G|_p.$$

If $N \leq Z(G)$ then an equality occurs.

iii) If H is a subgroup normalized by a p-element g and Q is a Sylow p-subgroup of H then

$$|\mathfrak{U}_p(Hg)| \geq |Q|.$$

Proof.

i) This follows trivially from the definition since $|H_p| = |G_p| = |P|$.

ii) We observe that a $\mathfrak{U}_p(G/N) = \{Nx \mid x \in \mathfrak{U}_p(G)\}$. For if $Nx \in \mathfrak{U}_p(G/N)$ then $x \in NP/N$ a Sylow p-subgroup of G/N and so $Nx = Ny$ with $y \in P$. Moreover if $y \in P$ and $g \in P \cap N$, then gy is a p-element in Ny and so

$$|\mathfrak{U}_p(Ny)| \geq |P \cap N|.$$

So we get

$$\begin{aligned}|\mathfrak{U}_p(G)| = \sum_{Nx \in G/N} |\mathfrak{U}_p(Nx)| &\geq |\mathfrak{U}_p\left(G/N\right)|\,|P \cap N| \\ &= |\mathfrak{U}_p\left(G/N\right)|\,|N|\,|P|\,/\,|NP|\,.\end{aligned}$$

If $N \leq Z(G)$ then the inequality above is an equality, since if $z \in Z(G)$ and $x \in \mathfrak{U}_p(G)$ then $|zx| = \mathrm{lcm}(|z|, |x|)$.

iii) Let R be a Sylow p-subgroup of $H\langle g\rangle$ containing g. Then R contains a Sylow p-subgroup Q of H. The coset Qg is then entirely contained in R. ☐

The next lemma will be useful to deal with groups in $\mathfrak{M}_{ns}$.

Lemma 3.3.3. *If $N = N_1 \times \cdots \times N_t \trianglelefteq G$ and $g \in \mathfrak{U}_p(G)$ is an element such that*

$$g \in N_G(N_i), \forall i \in \{1, \ldots, t\}$$

we have

$$|\mathfrak{U}_p(Ng)| = \prod_{i=1}^{t} |\mathfrak{U}_p(N_i g)|.$$

In particular

$$|\mathfrak{U}_p(N)| = \prod_{i=1}^{t} |\mathfrak{U}_p(N_i)|.$$

Proof. It is enough to prove the lemma for $t = 2$ as the general case follows from it with an easy induction. Arguing by induction on $s \in \mathbb{N}$ we see that

$$(xy)^s = x x^{y^{-1}} x^{y^{-2}} \cdots x^{y^{-(s-1)}} y^s$$

for all $x, y \in G$.
Let $|g| = p^r$ and $a \in N_1, b \in N_2$ such that $|(a,b)g| = p^l$. If $l \geq r$

$$\begin{aligned}1 = ((a,b)g)^{p^l} &= (a,b)(a,b)^{g^{-1}}(a,b)^{g^{-2}} \cdots (a,b)^{g^{-(p^l-1)}} g^{p^l} = \\ &= (a a^{g^{-1}} \cdots a^{g^{-(p^l-1)}}, b b^{g^{-1}} \cdots b^{g^{-(p^l-1)}}).\end{aligned}$$

and so

$$\begin{aligned} aa^{g^{-1}} \cdots a^{g^{-(p^l-1)}} &= 1 \\ bb^{g^{-1}} \cdots b^{g^{-(p^l-1)}} &= 1, \end{aligned}$$

i.e., $|ag|$ and $|bg|$ divide p^l.

If on the contrary $l < r$ we have

$$1 = \left(((a,b)g)^{p^l}\right)^{p^{r-l}} = \left(aa^{g^{-1}} \cdots a^{g^{-(p^r-1)}}, bb^{g^{-1}} \cdots b^{g^{-(p^r-1)}}\right)$$

and so again $|ag|$ and $|bg|$ divide p^l.

The following map is then well defined and injective

$$\begin{gathered} \Phi : \mathfrak{U}_p((N_1 \times N_2)g) \to \mathfrak{U}_p(N_1 g) \times \mathfrak{U}_p(N_2 g), \\ (a,b)g \mapsto (ag, bg). \end{gathered}$$

As for the surjectivity we observe that if $|ag| = p^{l_1}$ and $|bg| = p^{l_2}$ then setting $l = \max\{r, l_1, l_2\}$ we have

$$((a,b)g)^{p^l} = (aa^{g^{-1}} \cdots a^{g^{-(p^l-1)}}, bb^{g^{-1}} \cdots b^{g^{-(p^l-1)}}) = 1.$$

□

Now let G be a group in $\mathfrak{M}_{ns}$, i.e., G contains a unique minimal normal subgroup N which is the direct product

$$L_1 \times \cdots \times L_k$$

of k copies of the same nonabelian simple group L ($L_i \simeq L,\ \forall i \in \{1, \ldots, k\}$).

We shall prove that the 2-Frobenius ratio of G is greater than 6. The proof relies on the classification of finite simple groups.

As we said before, our arguments fail for a finite number of groups which we treat separately in the following lemma.

Lemma 3.3.4. *Let $G \in \mathfrak{M}_{ns}$ and N be its minimal normal subgroup. If N is isomorphic to one of the following groups*

$$A_5, A_5 \times A_5, PSL_2(7), PSL_2(16)$$

then $\mathrm{sp}(G) \leq 1/6$.

Proof. Since $C_G(N) = 1$ we have that G is isomorphic to a subgroup of $Aut(N)$. If N is simple then G is an almost simple group with socle N and this cases can be checked by direct calculation with GAP ([9]).

If $N \simeq A_5 \times A_5$ then G is an extension of N of index at most 8. Again by calculation one can see that the 2-Frobenius ratio of any such group is greater than 6. □

To be clearer we can now give the exact statement that is proved in the rest of the chapter.

Proposition 3.3.5. *Let $G \in \mathfrak{M}_{ns}$ and N be its minimal normal subgroup. If N is not isomorphic to one of the groups in Lemma 3.3.4 then the 2-Frobenius ratio of G is greater or equal than 6.*

Using Lemma 3.3.2 i) it is enough to prove Proposition 3.3.5 in the case $G = NP$ with P a Sylow 2-subgroup of G. The following easy lemma gives some more information on G.

Lemma 3.3.6. *i) $C_G(N) = 1$ and so $G \lesssim Aut(N)\,(\simeq Aut(L) \wr S_k)$.*

ii) P acts transitively by conjugation on the set $\{L_1, \dots, L_k\}$.

Proof. i) This is obvious since $C_G(N) \trianglelefteq G$ and $C_G(N) \cap N \leq Z(N) = 1$

ii) If $\mathcal{O}$ is an orbit with respect to this action, then

$$\prod_{L_i \in \mathcal{O}} L_i$$

is a normal subgroup of G.

□

We now give a sketch of the proof of Proposition 3.3.5.

A group $G \in \mathfrak{M}_{ns}$ can be embedded in the wreath product $Aut(L) \wr S_k$. Let B be the base of this wreath product,

$$B = Aut(L_1) \times \cdots \times Aut(L_k).$$

Set $K = B \cap G \trianglelefteq G$, $\mathcal{T}_K$ a right transversal of $P \cap N$ in $P \cap K$ and $\mathcal{T}_G$ a right transversal of $P \cap N$ in P such that

$$1 \in \mathcal{T}_K \subseteq \mathcal{T}_G.$$

These are right transversals of N in K and G, respectively, and so

$$G = K \,\dot{\cup} \dot{\bigcup_{g \in \mathcal{T}_G \setminus \mathcal{T}_K}} (Ng)\,.$$

Observing that

$$|P| = \frac{|P|}{|P \cap N|}\,|P \cap N| = \frac{|PN|}{|N|}\,|P \cap N| = \left|\frac{G}{N}\right|\,|P \cap N|$$

and $|P| = |G/K|\,|P \cap K|$ in the same way, we get

$$\frac{|\mathfrak{U}_2(G)|}{|P|} = \frac{|\mathfrak{U}_2(K)|}{|P|} + \frac{1}{|P|}\sum_{g \in \mathcal{T}_G \setminus \mathcal{T}_K} |\mathfrak{U}_2(Ng)| \geq \mathcal{A} + \mathcal{B}, \tag{3.3.1}$$

where we set

$$\begin{aligned} \mathcal{A} =& \frac{|\mathfrak{U}_2(K)|}{|\frac{G}{K}|\,|P \cap K|} \\ \mathcal{B} =& \frac{1}{|\frac{G}{N}|\,|P \cap N|}\left(\sum_{g \in \mathcal{T}_G \setminus \mathcal{T}_K} |\mathfrak{U}_2(Ng)|\right). \end{aligned} \tag{3.3.2}$$

We now treat the two terms $\mathcal{A}$ and $\mathcal{B}$ separately.

3.3.1 A bound for $\mathcal{B}$

First we deal with $\mathcal{B}$. First of all we observe that up to conjugation of G by an element of $Aut(N)$, we can take P inside $Q \wr R$, the wreath product of a $Q \in Syl_2(Aut(L))$ and $R \in Syl_2(S_k)$. Let $P_0 = Q \cap L$, a Sylow 2-subgroup of L.

Next proposition (and the following corollary) give a bound for any summand of $\mathcal{B}$, i.e., the number of 2-elements in a coset of N not contained in K.

Proposition 3.3.7. *Suppose that $k = 2^t$ and let $\sigma = (1, 2, \dots, k) \in S_k$. Let, moreover, $v = (v_1, \dots, v_k) \in B$ such that $v\sigma \in P$. Then*

$$|\mathfrak{U}_2(Nv\sigma)| \geq \frac{|L|^k}{|C_L(au)|},$$

where $u = v_1 v_2 \dots v_k \in Aut(L)$ and a is any element of P_0.

Proof. Let $a \in P_0$ and $\tilde{a} = (a, 1, \dots, 1) \in N$. Since $P_0^k \leq P$ we have $g := \tilde{a}v\sigma \in P$. Moreover, for all $x \in N$ we have

$$g^x = \tilde{a}^x (v\sigma)^x = a^x x^{-1} x^{(v\sigma)^{-1}} v\sigma,$$

and so $g^N \subseteq \mathfrak{U}_2(Nv\sigma)$.

The size of the N-orbit of g is given by

$$|g^N| = \frac{|N|}{|C_N(g)|}.$$

Let $x = (x_1, \dots, x_k) \in N$. Then $x \in C_N(g)$ if and only if

$$(x_1^{av_1}, x_2^{v_2} \dots, x_k^{v_k})^\sigma = x$$

that is

$$(x_k^{v_k}, x_1^{av_1}, \dots, x_{k-1}^{v_{k-1}}) = (x_1, \dots, x_k).$$

Finally we have that $x \in C_N(g)$ if and only if

$$x_1 = x_k^{v_k} = (x_{k-1}^{v_{k-1}})^{v_k} = \dots = x_1^{av_1 \dots v_k} = x_1^{au}.$$

For every choice of $x_1 \in C_L(au)$ the other components of x are uniquely determined and so

$$|C_N(g)| = |C_L(au)|.$$

$\square$

Corollary 3.3.8. *Let $1 \neq \sigma \in R$ and $v = (v_1, \dots, v_k) \in Q^k$ so that $v\sigma \in Q \wr R$. Let s be the length of a maximal orbit of σ. Then*

$$|\mathfrak{U}_2(Nv\sigma)| \geq \frac{|L|^s}{|C_L(g)|} |P_0|^{k-s},$$

where $g \in Aut(L)$.

Proof. Let $\sigma = \sigma_1 \dots \sigma_m$ be the expression of σ as a product of disjoint cycles (including those of length 1) with decreasing lengths (so that σ_1 has length s) and set

$$\mathcal{O}_i = Supp(\sigma_i), \;\; N_i = \prod_{j \in \mathcal{O}_i} L_j, \;\; \tau = \prod_{i>1} \sigma_i.$$

Moreover, through the identification

$$Aut(L_j) \simeq 1 \times \cdots \times Aut(L_j) \times \cdots \times 1$$

we write v as $v_1 \dots v_k$ and set

$$w_i = \prod_{j \in \mathcal{O}_i} v_j,$$
$$w = \prod_{i>1} w_i.$$

Then N is the direct product of the N_is and $v\sigma$ normalizes each of the N_i. We can then apply Lemma 3.3.3 and obtain

$$|\mho_2(Nv\sigma)| = \prod_{i=1}^{m} |\mho_2(N_i v\sigma)|. \tag{3.3.3}$$

We observe that $\langle N_1, w_1, \sigma_1 \rangle$ and $\langle w, \tau \rangle$ commute, we have

$$v\sigma = w_1 w \sigma_1 \tau = (w_1 \sigma_1)(w\tau)$$

and an element $aw_1\sigma_1 w\tau \in N_1 w_1 \sigma_1 w \tau$ is a 2-element if and only if $a(w_1\sigma_1)$ is such. We can then apply Lemma 3.3.7 and get

$$|\mho_2(N_1 v\sigma)| \geq \frac{|L|^s}{|C_L(g)|},$$

for some $g \in Aut(L)$. For the other terms of the product (3.3.3) we just use (iii) in Lemma 3.3.2 and finally we get

$$|\mho_2(Nv\sigma)| = \frac{|L|^s}{|C_L(g)|} \prod_{i=2}^{m} |P_0|^{|\mathcal{O}_i|} = \frac{|L|^s}{|C_L(g)|} \, |P_0|^{k-s}.$$

□

We can now give a bound for $\mathcal{B}$. First of all $P \cap N$ is a Sylow 2-subgroup of N and so its cardinality is $|P_0|^k$. Every element $g \in \mathcal{T}_G \setminus \mathcal{T}_K$ is the product of an element $v \in Q^k$ by an element $1 \neq \sigma \in S_k$, so we can use Corollary 3.3.8. For such a g set s_g the maximal length of an orbit of σ.

Setting

$$c = \max_{x \in Aut(L)} |C_L(x)|$$

we get

$$
\begin{aligned}
\mathcal{B} =& \frac{1}{|\frac{G}{N}|\,|P\cap N|}\left(\sum_{g\in\mathcal{T}_G\setminus\mathcal{T}_K}|\mho_2(Ng)|\right)\\
\geq& \frac{1}{|\frac{G}{N}|\,|P_0|^k}\left(\sum_{g\in\mathcal{T}_G\setminus\mathcal{T}_K}\frac{|L|^{s_g}}{c}\,|P_0|^{k-s_g}\right)\\
\geq& \frac{1}{|\frac{G}{N}|\,|P_0|^k}\,\frac{1}{c}\left(\sum_{g\in\mathcal{T}_G\setminus\mathcal{T}_K}|L|^{s_g}\,|P_0|^{k-s_g}\right)\\
\geq& \frac{|\mathcal{T}_G\setminus\mathcal{T}_K|}{|\frac{G}{N}|}\,\frac{1}{c}\,\frac{|L|^2}{|P_0|^k}|P_0|^{k-2}\\
=& \frac{|G|-|K|}{|G|}\,\frac{|L|}{c}\,\frac{|L|}{|P_0|^2}
\end{aligned}
$$

The following theorem, whose proof relies on CFSG, ensures that the third term of the last product is greater than 1.

Theorem 3.3.9. *[17] Let L be a nonabelian finite simple group, p a prime dividing $|L|$ and $P\in Syl_p(G)$. Then $|P|^2<|L|$.*

Finally we observe that a finite simple group has a proper subgroup of index smaller then 6 if and only if it is isomorphic to A_5. For A_5 we can calculate c directly to see that $c=10$. Finally we obtain the bound

$$
\mathcal{B}\geq 6\frac{|G|-|K|}{|G|}. \tag{3.3.4}
$$

3.3.2 A bound for $\mathcal{A}$

In this section we deal with the term $\mathcal{A}$ of (3.3.2).

Since $N\leq K$ we have that $|\mho_2(K)|\geq|\mho_2(N)|=|\mho_2(L)|^k$, while if $\hat{P}$ is a Sylow 2-subgroup of $Aut(L)$ we get $|P\cap K|\leq|\hat{P}|^k$ and so

$$
\mathcal{A}\geq\frac{|\mho_2(L)|^k}{|\frac{G}{K}|\,|\hat{P}|^k}=\frac{|K|}{|G|}\left(\frac{|\mho_2(L)|}{|\hat{P}|}\right)^k. \tag{3.3.5}
$$

We want to show that this loose bound is enough to see that $\mathcal{A}\geq 6|K|/|G|$, that, together with (3.3.4) will complete the proof of Proposition 3.3.5. We thus want to deal with the ratio

$$
\phi(L)=\frac{|\mho_2(L)|}{|\hat{P}|},
$$

for any L nonabelian finite simple group, except for the cases treated in Lemma 3.3.4. Namely we will show that for such groups this ratio is strictly greater than 5.

In most cases, this is proved by taking a 2-element $s\in L$ such that

$$
|s^L|\geq 6|Aut(L)|_2.
$$

For finite groups of Lie type the existence of these elements is proved in [12] (Theorem 7.5 and following).

Alternating groups

Consider first the case $L \simeq A_n$. Let $n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_l}$, with $m_1 > m_2 > \cdots > m_l \geq 0$ and set $n_i = \sum_{j=1}^{i} 2^{m_j}$ for $i \in \{1, \ldots, l\}$. Take

$$s = (1, 2, \ldots, n_1) \ldots \big(n_{l-1} + 1, \ldots, n\big),$$

if this is an element of A_n, or

$$s = (1, 2, \ldots, n_1) \ldots \big(n_{l-1} + 1, \ldots, n_{l-1} + 2^{m_l - 1}\big)(n_{l-1} + 2^{m_l - 1} + 1, \ldots n),$$

otherwise.

Now the conjugacy class of s in S_n has more than

$$|s^L| \geq \frac{n!}{2 \cdot 2^{m_1} \ldots 2^{2m_l}} = \frac{n!}{2^{(\sum_{i=1}^{l} m_i) + m_l + 1}}.$$

elements. Since $m_l \leq \lfloor \log_2(n) \rfloor$ we get that

$$\sum_{i=1}^{l} m_j \leq \sum_{i=1}^{\lfloor \log_2(n) \rfloor} i \leq \log_2^2(n) + \log_2(n)$$

and so

$$|s^L| \geq \frac{n!}{2^{\frac{1}{2}(\log_2^2(n) + 3\log_2(n)) + 1}}.$$

The order of a Sylow 2-subgroup of L is

$$|P_0| = \frac{1}{2} 2^{2^{m_1} - 1} \cdots 2^{2^{m_l} - 1} = \frac{2^n}{2^{l+1}} \leq 2^{n-1}.$$

Finally for $n > 6$ we have

$$\phi(L) > \frac{|s^L|}{2|P_0|} \geq \frac{n!}{2^{\frac{1}{2}(\log_2^2(n) + 3\log_2(n)) + n + 1}}.$$

The last term of this inequality is increasing in n. Since $\phi(A_{12}) > 6$, we have that $\phi(A_n) \geq 6$ for $n \geq 12$. For $6 \leq n < 12$ we can calculate $\phi(A_n)$ directly to get the desired bound.

We observe that when $L \simeq A_5$, we have $\phi(L) = 2$. The bound (3.3.5) fails when $k = 1, 2$, and this is the reason why these cases were treated separately in Lemma 3.3.4.

Classical groups of Lie type in odd characteristic

In this paragraph, $q = p^f$ with p an odd prime and $1 \leq f \in \mathbb{N}$ ($q^2 = p^f$ for PSU and $P\Omega^-$). For this family of groups we use the results on 2-regular elements contained in [12]: all the bounds on the size of the conjugacy classes of elements can be found in that paper (Theorem 7.5 and following).

We start with $L \simeq PSL_n(q)$. Writing $d := (n, q-1)$ we have

$$|L| = \frac{q^{\frac{1}{2}n(n-1)}}{d} \prod_{i=2}^{n} (q^i - 1).$$

and so

$$|P_0| \leq \prod_{i=2}^{n}(q^i-1)_2$$

while

$$|Out(L)| = \begin{cases} 2f, \text{ if } n=2, \\ 2df, \text{ if } n>2. \end{cases} \tag{3.3.6}$$

There exists a 2-element $s \in L$ such that

$$|C_L(s)| \leq \frac{q^n-1}{q-1}.$$

We then get

$$\phi(L) \geq \frac{|s^L|}{2df|P_0|} \geq \frac{q^{\frac{1}{2}n(n-1)}}{q^n-1}\left(\prod_{i=2}^{n}\frac{q^i-1}{(q^i-1)_2}\right)\frac{1}{2df}.$$

For $n \geq 5$ the inequality

$$\phi(L) \geq q^{\frac{1}{2}(n^2-3n-4)}$$

is enough.
If $n=4$ we have

$$\begin{aligned}\phi(L) &\geq \frac{q^6}{8fq^4-1}\frac{(q^4-1)(q^3-1)(q^2-1)}{(q^3-1)_2(q^3-1)_2(q^2-1)_2} \geq \\ &\geq \frac{q(q^2-1)(q^2+q+1)(q-1)}{32},\end{aligned}$$

which is enough for $q \geq 5$.
If $n=3$ we get

$$\begin{aligned}\phi(L) &\geq \frac{q^3}{3f(q^3-1)}\frac{(q^3-1)(q^2-1)}{(q^3-1)_2(q^2-1)_2} \geq \\ &\geq \frac{(q^2+q+1)(q-1)}{6f},\end{aligned}$$

which is enough for $q \geq 7$.
If $n=2$ we get

$$\phi(L) \geq \frac{q(q-1)(q+1)}{4f(q-1)_2(q+1)_2} \geq \frac{q(q-1)}{8f(q+1)}.$$

If $f=1$ this bound is enough for $p \geq 43$, if $f=2$ for $p \geq 11$, if $f=3$ for $p \geq 5$, if $f \geq 4, 5$ for $p \geq 3$. The groups for which the bounds do not suffice can be checked directly.

If $L \simeq PSU_n(q)$ we set $d=(n,q+1)$. In [12] it is shown that there exists $s \in \mathfrak{U}_2(L)$ such that

$$|C_L(s)| \leq (q+1)^{n-1}.$$

Since

$$|L| = \frac{q^{\frac{1}{2}n(n-1)}}{d}\prod_{i=2}^{n}(q^i-(-1)^i),$$

$$|P_0| \leq \prod_{i=2}^{n}(q^i-(-1)^i)_2,$$

$$|Out(L)| = df,$$

we get

$$\phi(L) \geq \frac{q^{\frac{1}{2}n(n-1)}}{(q+1)^n} \prod_{i=2}^{n} \frac{q^i - (-1)^i}{(q^i - (-1)^i)_2} \frac{1}{df} \geq \frac{q^{\frac{1}{2}n(n-1)}}{f(q+1)^{n+1}},$$

which is enough for $n \geq 5$.
If $n = 4$ we have

$$\phi(L) \geq \frac{q^6}{(q+1)^4} \frac{(q^2-q+1)(q-1)^2}{4f}$$

which is enough for $q \geq 5$. The case $(n,q) = (3,3)$ can be checked directly.
If $n = 3$ we have

$$\phi(L) \geq \left(\frac{q}{q+1}\right)^3 \frac{(q^2-q+1)(q-1)}{6f}$$

which is enough for $q \geq 5$.

If $L \simeq PSp_{2n}(q)$ or $P\Omega_{2n+1}(q)$, with $n \geq 2$, we have

$$|L| = \frac{q^{n^2}}{2} \prod_{i=1}^{n} (q^{2i} - 1)$$
$$|C_L(s)| \leq (q+1)^n$$
$$|P_0| = \frac{1}{2} \prod_{i=1}^{n} (q^{2i} - 1)_2$$
$$|Out(L)| = 2f$$

and so

$$\phi(L) \geq \frac{q^{n^2}}{(q+1)^n} \prod \frac{(q^{2i}-1)}{(q^{2i}-1)_2} \frac{1}{2f} \geq \frac{q^{n^2}}{2f(q+1)^n}$$

which give the desired bound in all cases but $(n,q) = (2,3)$, which can be checked directly.

If $L \simeq P\Omega^{\pm}_{2n}(q)$, for $n \geq 4$, we get

$$|L| = \frac{q^{n^2-n}(q^n \mp 1)}{(4, q^n \mp 1)} \prod_{i=1}^{n-1} (q^{2i} - 1)$$
$$|C_L(s)| \leq (q+1)^n$$
$$|P_0| \leq \frac{1}{2}(q^n \mp 1)_2 \prod_{i=1}^{-} 1(q^{2i} - 1)_2$$
$$|Out(L)|_2 \leq 8f$$

and so

$$\phi(L) \geq \frac{q^{n^2-n}}{8f(q+1)^n}$$

which is enough.

Exceptional groups of Lie type in odd characteristic

We collect information about this family of groups in some tables, organized as follows. In the first row we put the order of L. In the second row we put an upper bound C (taken from [12], Lemma 7.16) for the centralizer of a particular 2-element in L which gives a lower bound for the size of its conjugacy class. In the third row we write the 2-part of the order of $Out(L)$. All the above information is summed up in the last row, where we write D such that $\phi(L) \geq D > 5$.

$L \simeq E_6(q)$	
$\lvert L\rvert$	$q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)$ $(q^5-1)(q^2-1)/(3,q-1)$
C	q^8
$\lvert Out(L)\rvert_2$	$2f_2$
D	$q^{28}/(6f_2)$

$L \simeq {}^2E_6(q)$	
$\lvert L\rvert$	$q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)$ $(q^5+1)(q^2-1)/(3,q+1)$
C	$(q+1)q^7$
$\lvert Out(L)\rvert_2$	f_2
D	$q^{29}/(18f_2)$

$L \simeq E_7(q)$	
$\lvert L\rvert$	$q^{63}(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)$ $(q^8-1)(q^6-1)(q^2-1)/(2,q-1)$
C	$(q+1)^2q^7$
$\lvert Out(L)\rvert_2$	$2f_2$
D	$q^{52}/(4f_2)$

$L \simeq E_8(q)$	
$\lvert L\rvert$	$q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)$ $(q^{12}-1)(q^8-1)(q^2-1)/(2,q-1)$
C	q^8-1
$\lvert Out(L)\rvert_2$	f_2
D	$q^{112}/(2f_2)$

$L \simeq F_4(q)$	
$\lvert L\rvert$	$q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$
C	$(q+1)^3q^3$
$\lvert Out(L)\rvert_2$	f_2
D	$q^{12}/(f_2)$

$L \simeq G_2(q)$	
$\lvert L\rvert$	$q^6(q^6-1)(q^2-1)$
C	q^2-1
$\lvert Out(L)\rvert_2$	$2f_2$
D	$q^6/(f_2)$

$L \simeq {}^2G_2(q),\ q>3$	
$\lvert L\rvert$	$q^3(q^3+1)(q-1)$
C	$q+1$
$\lvert Out(L)\rvert_2$	f_2
D	q^3

$L \simeq {}^3D_4(q),\ q>3$	
$\lvert L\rvert$	$q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$
C	$(q^3-1)(q+1)$
$\lvert Out(L)\rvert_2$	f_2
D	q^3

The remaining groups $G_2(3), {}^2G_2(3)'$ and ${}^3D_4(3)$ can be checked directly on the ATLAS.

Groups of Lie type in characteristic 2

Here $q = 2^f$ for nontwisted groups, $q^2 = 2^f$ for $PSU, P\Omega^-, {}^2E_6$ and $q^3 = 2^f$ for ${}^3D_4$. A theorem of Steinberg gives an exact result about the Frobenius ratio.

Theorem 3.3.10. *(15.2 in [30]). Let G be a connected reductive algebraic group over a field of characteristic p and let F be a Frobenius map. Then*

$$|\mathfrak{U}_p(G^F)| = (|G^F|_p)^2.$$

This means that the p-Frobenius ratio in this case coincides with the order of a Sylow p-subgroup.

Each finite quasisimple group of Lie type can be seen as the group of fixed points of a certain Frobenius map on a connected reductive group. Thus we can use Steinberg's theorem together with Lemma 3.3.2 (ii) to get

$$\phi(L) = \frac{|\mathfrak{U}_2(L)|}{|Out(L)|_2\,|P_0|} = \frac{\left|\mathfrak{U}_2(G^F)\right|}{|Out(L)|_2\,|G^F|_2} = \frac{|G^F|_2}{|Out(L)|_2}.$$

So we examine the above ratio in every instance.

- $PSL_n(q)$. If $n=2$ then $f \geq 3$ (since $PSL_2(2)$ is not simple, while $PSL_2(4) \simeq A_5$). We have $\phi(L) = 2^f/f_2$ which is greater than 6 except for $f=4$ (one of the cases of Lemma 3.3.4).
  If $n=3$ we have $\phi(L) = 8^f/(2f_2)$, which is enough except for $f=1$, when we have $PSL_3(2) \simeq PSL_2(7)$ (Lemma 3.3.4). If $n \geq 4$, $\phi(L) = 64^f/(2f_2)$ which gives the desired bound for all f.
- $PSU_n(q), n \geq 3, q^2 = p^f$. We have $\phi(L) = q^{\frac{1}{2}n(n-1)}/f_2$.
- $PSp_{2n}(q) \simeq P\Omega_{2n+1}(q), n \geq 2$. $\phi(L) = q^{n^2}/f_2$.
- $P\Omega^+_{2n}(q), n \geq 4$. $\phi(L) = q^{n(n-1)}/(2f_2)$.
- $P\Omega^-_{2n}(q), n \geq 4, q^2 = p^f$. $\phi(L) = q^{n(n-1)}/f_2$.
- $E_6(q)$. $\phi(L) = q^{36}/(2f_2)$.
- ${}^2E_6(q), q^2 = p^f$. $\phi(L) = q^{36}/f_2$.
- $E_7(q)$. $\phi(L) = q^{63}/f_2$.
- $E_8(q)$. $\phi(L) = q^{120}/(f_2)$.
- $F_4(q)$. $\phi(L) = q^{24}/(2f_2)$.
- $G_2(q), q \geq 4$. $\phi(L) = q^6/(f_2)$.
- ${}^3D_4(q)$. $\phi(L) = q^{12}/(f_2)$.
- ${}^2B_2(2^{2n+1}), n \geq 1$. $\phi(L) = 4^{2n+1}$.
- ${}^2F_4(2^{2n+1}), n \geq 1$. $\phi(L) = 2^{12(2n+1)}$.
- ${}^2F_4(2)'$, $\phi(L) = 2^{10}$.

Sporadic groups

The CTblLib library of GAP ([9]) contains the sizes of elements' centralizers for the 26 sporadic groups. One can thus find the exact value of $\phi(L)$ for L sporadic. We collect this information in the following table: in the first row we write the group's name, in the second the value $\phi(L)$.

M_{11}	M_{12}	J_1	M_{22}	J_2	M_{23}	HS	J_3	M_{24}	McL
196	239	183	383.5	331	12482	8999	26674	19966	475171

He	Ru	Suz	$O'N$	$Co3$	$Co2$
146529.5	1653289	1547764	142395974.5	20511724	14474629

Fi_{22}	HN	Ly	Th
26457274	238505507	2537631842146	36426974732

Fi_{23}	Co_1	$J4$	Fi'_{24}
1097523664363	68934773888	1461467295316	10067474874319006

B	M
79013591454691398863	183158957839149723553047352346054126359

If N is not one of the groups in Lemma 3.3.4 then we have

$$\frac{|\mathfrak{U}_2(G)|}{|G|_2} \geq \mathcal{A} + \mathcal{B} \geq 6\frac{|K|}{|G|} + 6\frac{|G| - |K|}{|G|} = 6$$

and so this completes the proof of Proposition 3.3.5 and so that of Theorem 3.1.1 too.

Chapter 4

The Frobenius ratio in solvable groups

In this chapter we will say something more in general about the number of p-elements and the p-Frobenius ratio of a finite group introduced in last chapter to prove Proposition 3.1.1. Along the chapter, G will be a finite group and p a prime dividing the order of G.

Several different proofs of Frobenius' theorem (Theorem 3.3.1) have been given. Just to cite two of them, an elegant proof using elementary methods is the one by Isaacs and Robinson ([16]), while Speyer found a proof that uses combinatorial methods ([29]). Nevertheless it is still unknown if the Frobenius ratio has a combinatorial meaning.

The focus of this chapter is on bounding the number of p-elements of a finite group G in terms of the number of Sylow p-subgroups. We prove the following.

Theorem 4.0.1. *Let G be a finite p-solvable group. If $n_p(G)$ is the number of Sylow p-subgroups in G then*

$$\frac{|\mathfrak{U}_p(G)|}{|G|_p} \geq n_p(G)^{\frac{p-1}{p}}.$$

We conjecture that Theorem 4.0.1 is true without the hypothesis of p-solvability of G. In the last section we state a sufficient condition for the conjecture and prove a reduction to finite almost simple groups.

We start our investigation with some generical results about $\mathfrak{U}_p(G)$.

4.1 The number of p-elements in a finite group: some examples

First of all we report a well known result from Miller, which gives a general bound for the p-Frobenius ratio. For the sake of readability we start with an easy lemma.

Lemma 4.1.1. *Let H be a p-subgroup of G. Then $\lambda_G(H) \equiv 1 \pmod{p}$.*

Proof. H acts by conjugation on the set $\Lambda = Syl_p(G)$. Since each H-orbit has order a power of p we have

$$|\Lambda| \equiv |\operatorname{Fix}_\Lambda(H)| \pmod{p}.$$

But the fixed points of H are exactly the Sylow p-subgroups containing H and thus we have the thesis. $\square$

Proposition 4.1.2 ([25]). *Let P be a Sylow p-subgroup of G. If P is not normal in G then*

$$|\mho_p(G)| > p|P|.$$

Proof. Consider a pair of different Sylow p-subgroups, P_1 and P_2 whose intersection M has order as large as possible. If R is another Sylow p-subgroup containing M then $R \cap P_1 = R \cap P_2 = M$ by the maximality of M. Moreover as stated in the previous Lemma 4.1.1 $\lambda_G(M)$ is congruent to 1 modulo p, so that it is at least $p+1$. It follows that

$$\begin{aligned} |\mho_p(G)| \geq \Big| \bigcup_{M \leq Q \in Syl_p(G)} Q \Big| \geq \\ = (p+1)(|P_1| - |M|) + |M| = p|P| + |P| - p|M| \geq p|P|. \end{aligned} \tag{4.1.1}$$

$\square$

Next we prove an easy formula for the number of p-elements in a p-nilpotent group, which will be useful for some calculations.

Lemma 4.1.3. *Let G have a normal p-complement N. Then if P is a Sylow p-subgroup of G one has*

$$|\mho_p(G)| = \sum_{x \in P} \frac{|N|}{|C_N(x)|}.$$

Proof. Let g be a p-element of G, then there exist $x \in P$ and $a \in N$ such that $g = x^a$. To show this, choose a Sylow p-subgroup Q containing g and an element h such that $P^h = Q$. Since $G = PN$, we have $h = ya$ with $y \in P$ and $a \in N$. Then $P^a = Q$ and so $x^a = g$ for some $x \in P$.

Moreover if $x \in P$ and $a \in N$ then $x^a \in P$ if and only if $x^a = x$. This is because in this case

$$x^a x^{-1} = [a, x^{-1}] = a^{-1} a^{x^{-1}} \in P \cap N = 1.$$

Therefore

$$|\mho_p(G)| = \sum_{x \in P} |x^N| = \sum_{x \in P} \frac{|N|}{|C_N(x)|}.$$

$\square$

In the previous chapter we pointed out (Lemma 3.3.2) that the Frobenius ratio decreases on quotients. This is not the case for subgroups as one can see by the following example.

Example 4.1.4. Let $N = \langle a \rangle \times \langle b \rangle$ be an elementary abelian group of order 9. Then the diedral group of order 8 acts on N faithfully as follows. Let $P = \langle x, y \rangle$, with $|y| = 4, |x| = 2, y^x = y^{-1}$. We define an action of P on N by

$$a^y = b^{-1},\ b^y = a,\ a^x = a^{-1},\ b^x = b. \tag{4.1.2}$$

Set $z = y^2$. Then we have $a^z = a^{-1}$ and $b^z = b^{-1}$ and so $C_N(z) = 1$. Since $z = y^2$ this implies that $C_N(y) = 1$. Obiously $C_N(x) = \langle b \rangle$ so that $|C_N(x)| = 3$ and the same holds for every noncentral involution of P. Let G be the semidirect product $N \rtimes P$ and H its normal subgroup (having index 2) generated by N and y.

Using Lemma 4.1.3 we have

$$|\mho_2(G)| = \sum_{g \in P} \frac{|N|}{|C_N(g)|} = 1 + 3 \cdot 9 + 4 \cdot 3 = 40$$

while

$$|\mho_2(H)| = \sum_{g \in \langle y \rangle} \frac{|N|}{|C_N(g)|} = 1 + 3 \cdot 9 = 28.$$

It follows that $|\mho_2(G)|/|P| = 5 < 7 = |\mho_2(H)|/|P \cap H|$.

The previous example shows a group with a subgroup H of index 2 such that the 2-elements contained inside H are more than those contained in $G \setminus H$. By taking an elementary abelian group of order p^2 instead of the one of order 9 considered above, we can show that the ratio between these two quantities can be arbitrarily big.

Proposition 4.1.5. *There exists a sequence G_n of groups each having a subgroup H_n of index 2 such that*

$$\lim_{n \to \infty} \frac{|\mho_2(G_n \setminus H_n)|}{|\mho_2(H_n)|} = 0$$

Proof. For every p prime let $N = \langle a \rangle \times \langle b \rangle$ be an elementary abelian group of order p^2 and $P = \langle x, y \mid x^2, y^4, (xy)^2 \rangle$ be a dihedral group of order 8 acting on N by (4.1.2). Take G_p to be the semidirect product $N \rtimes P$. Let H_p be $N \rtimes \langle y \rangle$. All the calculations made above work here as well, replacing 3 with p. So we have

$$|\mho_2(G)| = \sum_{g \in P} \frac{|N|}{|C_N(g)|} = 1 + 3p^2 + 4p$$

while

$$|\mho_2(H)| = \sum_{g \in \langle y \rangle} \frac{|N|}{|C_N(g)|} = 1 + 3p^2.$$

Finally

$$\lim_{p \to \infty} \frac{|\mho_2(G_p \setminus H_p)|}{|\mho_2(H_p)|} = \lim_{p \to \infty} \frac{4p}{1 + 3p^2} \to 0.$$

$\square$

We end this section with another fact that might look odd at a first glance. It seems reasonable to think that the number of p-elements of a finite group G depends on the number of Sylow p-subgroups of G as well as on $|G|_p$. For example one could think that given a sequence of groups the number of p-elements grows at least as fast as the number of Sylow p-subgroups. The following proposition gives an example that contradicts this naive idea.

Proposition 4.1.6. *There exists a sequence G_n of groups such that*

$$\lim_{n \to \infty} \frac{|\mho_2(G_n)|}{n_2(G_n)} = 0.$$

Proof. Let p be an odd prime and $N = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ with $|a| = p, |b| = |c| = p$. Let $P = \langle x \rangle \times \langle y \rangle$ be an elementary abelian group of order 4 acting on N as follows:

$$a^x = a^{-1},\ b^x = b,\ c^x = c^{-1}$$
$$a^y = a^{-1},\ b^y = b^{-1},\ c^y = c.$$

Let G be the semidirect product $N \rtimes P$. Clearly $N_G(P) = C_G(P) = P$ so that $n_2(G) = [G : P] = p^3$.

Moreover, we have

$$C_N(x) = \langle b \rangle,\ C_N(y) = \langle c \rangle,\ C_N(xy) = \langle a \rangle,$$

so that using Lemma 4.1.3

$$|\mathfrak{U}_2(G)| = 1 + 3p^2$$

and the result follows. □

4.2 A bound for the number of p-elements in p-solvable groups

In this section we focus on the search for good bounds for the p-Frobenius ratio in p-solvable groups depending on the number of Sylow p-subgroups. Namely we give a proof of Theorem 4.0.1.

An upper bound for the Frobenius ratio in a finite group G can be easily found with the following trivial counting argument. Since all the Sylow p-subgroups intersect in the identity, the maximal number of p-elements is achieved when each pair of Sylow p-subgroups intersects trivially. Therefore

$$|\mathfrak{U}_p(G)| \leq n_p(G)\,(|G|_p - 1) + 1$$

and so

$$\frac{|\mathfrak{U}_p(G)|}{|G|_p} \leq n_p(G) - \frac{n_p(G) - 1}{|G|_p} \leq n_p(G).$$

Proposition 4.1.6 shows that a lower bound for the Frobenius ratio can not be linear in $n_p(G)$.

We now turn to the proof of Theorem 4.0.1, starting with a lemma which holds for every finite group.

Lemma 4.2.1. *Let p be a prime dividing the order of $|G|$ and P be a Sylow p-subgroup of G. Then*

$$|\mathfrak{U}_p(G)| = \sum_{x \in P} \frac{|G|}{|S_G(x)|}.$$

Proof. This is a straightforward application of Theorem 1.3.6. In the sum

$$\sum_{x \in P} |x^G|$$

every class of p-elements is involved and it is repeated as many times as the cardinality $|x^G \cap P| = \alpha_G(x)$. Hence

$$|\mathfrak{U}_p(G)| = \sum_{x \in P} \frac{|x^G|}{\alpha_G(x)} = \sum_{x \in P} \frac{|G|}{\alpha_G(x)|C_G(x)|} = \sum_{x \in P} \frac{|G|}{|S_G(x)|}.$$

□

The fundamental tools that we are going to use in the proof of Theorem 4.0.1 are Theorem 1.4.1 and the following formula proved by Navarro and Rizo.

Theorem 4.2.2 (Theorem A in [26])**.** *Suppose that P is a p-group acting on a p'-group G. Then*

$$|C_G(P)| = \left(\prod_{x\in P} \frac{|C_G(x)|}{|C_G(x^p)|^{1/p}}\right)^{\frac{p}{(p-1)|P|}}.$$

We can now prove Theorem 4.0.1.

Proof of Proposition 4.0.1. Let P be a Sylow p-subgroup of G. By applying Lemma 4.2.1 we can see that the p-Frobenius ratio is the arithmetic mean of the ratios

$$\frac{|G|}{|S_G(x)|}$$

as x runs across P. We can then apply the inequality with the geometric mean, getting

$$\frac{|\mho_p(G)|}{|G|_p} = \frac{1}{|P|}\left(\sum_{x\in P}\frac{|G|}{|S_G(x)|}\right) \geq \left(\prod_{x\in P}\frac{|G|}{|S_G(x)|}\right)^{1/|P|}. \tag{4.2.1}$$

Since G is p-solvable we can take a normal $\{p,p'\}$-series, whose set of p'-factors we call $\mathcal{M}$. Then, by Theorem 1.4.1 we have for all $x \in P$

$$\frac{|G|}{|S_G(x)|} = \frac{|G|_p|G|_{p'}}{|G|_p\prod_{U/V\in\mathcal{M}}|C_{U/V}(xV)|} = \prod_{U/V\in\mathcal{M}}\frac{|U/V|}{|C_{U/V}(xV)|}.$$

We insert this last term in (4.2.1) and swap the products to get

$$\frac{|\mho_p(G)|}{|G|_p} \geq \left(\prod_{U/V\in\mathcal{M}}\left(\prod_{x\in P}\frac{|U/V|}{|C_{U/V}(xV)|}\right)\right)^{1/|P|} =$$
$$= \left(\prod_{U/V\in\mathcal{M}}\left|\frac{U}{V}\right|^{|P|}\left(\prod_{x\in P}\frac{1}{|C_{U/V}(xV)|}\right)\right)^{1/|P|}.$$

Now for all $U/V \in \mathcal{M}$, P is a p-group that acts on the p'-group U/V. We can then apply Theorem 4.2.2 and use the trivial inequality $|C_{U/V}((xV)^p)| \leq |U/V|$, so that we have

$$\prod_{x\in P}\frac{1}{|C_{U/V}(xV)|} = \left(\prod_{x\in P}\frac{1}{|C_{U/V}((xV)^p)|^{1/p}}\right)\frac{1}{|C_{U/V}(P)|^{|P|(p-1)/p}} \geq$$
$$\geq \frac{|U/V|^{|P|/p}}{|C_{U/V}(P)|^{|P|(p-1)/p}}$$

and so

$$\frac{|\mho_p(G)|}{|G|_p} \geq \left(\prod_{U/V\in\mathcal{M}}\frac{|U/V|}{|C_{U/V}(P)|}\right)^{\frac{p-1}{p}}.$$

Finally we apply Theorem 1.4.1 once more and we observe that for a Sylow p-subgroup $S_G(P) = N_G(P)$, so that

$$\frac{|\mho_p(G)|}{|G|_p} \geq \left(\frac{|G|}{|S_G(P)|}\right)^{\frac{p-1}{p}} = \left(\frac{|G|}{|N_G(P)|}\right)^{\frac{p-1}{p}} = n_p(G)^{\frac{p-1}{p}}.$$

□

It is worth mentioning that this bound is asymptotically tight in the sense specified by the following example.

Example 4.2.3. Let p be a prime and, for $n \in \mathbb{N}$, let P be an elementary abelian group of order p^n. Moreover set $\mathcal{M}$ to be the set of the maximal subgroups of P. Choose a prime q such that p divides $q-1$. Then for any $M \in \mathcal{M}$ we have that $P/M \simeq C_p$ acts faithfully and fixed point freely as a group of automorphisms on a cyclic group $\langle a_M \rangle \simeq C_q$.

It follows that $P = P/\left(\bigcap_{M \in \mathcal{M}} M\right)$ acts faithfully on the direct product of the groups $\langle a_M \rangle$. To be more explicit, calling N this direct product, the following injective homomorphism is defined

$$\begin{aligned} P &\to Aut(N) \\ x &\mapsto \phi_x, \end{aligned}$$

where $\phi_x(a_M) = a_M^{xM}$ for all $M \in \mathcal{M}$.

We consider the semidirect product $G_n = N \rtimes P$. The normalizer of P in G_n is $C_N(P)P = P$ so that the number of Sylow p-subgroups of G_n is

$$n_p(G_n) = |N| = q^{|\mathcal{M}|} = q^{\frac{p^n-1}{p-1}}. \tag{4.2.2}$$

In order to count the number of p-elements in G_n we want to use the equality

$$|\mho_p(G_n)| = \sum_{x \in P} \frac{n_p(G_n)}{\lambda_{G_n}(x)},$$

descending directly from Lemma 4.2.1 and remark (2.2.6). We thus have to evaluate $\lambda_{G_n}(x)$ for $x \in P \setminus \{1\}$. Using again (2.2.6) we have

$$\lambda_{G_n}(x) = \frac{\alpha_{G_n}(x) n_p(G_n)}{|x^{G_n}|}.$$

Now since P is abelian we have $\alpha_{G_n}(x) = 1$ and $|x^{G_n}| = |x^N| = |N|/|C_N(x)|$ so that $\lambda_{G_n}(x) = |C_N(x)|$. Given $x \in P \setminus \{1\}$ the centralizer of x in N is generated by those a_M such that

$$a_M^{xM} = a_M. \tag{4.2.3}$$

Since P/M acts fixed point freely on $\langle a_M \rangle$, (4.2.3) holds if and only if $x \in M$ and so

$$C_N(x) = \langle a_M \mid x \in M \rangle.$$

The number of maximal subgroups in P containing a fixed nontrivial element is $\frac{p^{n-1}-1}{p-1}$ and so

$$|C_N(x)| = q^{\frac{p^{n-1}-1}{p-1}}.$$

We can then calculate the p-Frobenius ratio of G_n

$$\begin{aligned}\frac{|\mho_p(G_n)|}{|G_n|_p} &= \frac{1}{|P|}\sum_{x\in P}\frac{n_p(G_n)}{\lambda_{G_n}(x)}\\ &= \frac{1}{p^n}\left(1+(p^n-1)\frac{q^{\frac{p^n-1}{p-1}}}{q^{\frac{p^{n-1}-1}{p-1}}}\right)\\ &= \frac{1}{p^n}+\frac{p^n-1}{p^n}q^{p^{n-1}}.\end{aligned} \tag{4.2.4}$$

By (4.2.2) we have

$$n_p(G_n)^{\frac{p-1}{p}} = q^{\frac{p^n-1}{p}} = q^{p^{n-1}-\frac{1}{p}}.$$

We can now compare the two members of the inequality stated by Theorem 4.0.1

$$\lim_{n\to\infty}\frac{|\mho_p(G_n)|/|G_n|_p}{n_p(G_n)^{\frac{p-1}{p}}} = \lim_{n\to\infty}\left(\frac{1}{p^n q^{p^{n-1}-\frac{1}{p}}}+\frac{p^n-1}{p^n}\frac{q^{p^{n-1}}}{q^{p^{n-1}-\frac{1}{p}}}\right) = q^{1/p},$$

i.e., the p-Frobenius ratio of G_n and the $\frac{p-1}{p}$th power of the number of Sylow p-subgroups have the same asymptotic behaviour.

4.3 The general case

In this last section we want to explore the non-p-solvable case. We conjecture that Theorem 4.0.1 is still true without the hypothesis of p-solvability. We now derive a sufficient condition for this conjecture for which we will prove a reduction to finite almost simple groups.

The first step in the proof of Theorem 4.0.1 is applying the inequality between arithmetic and geometric mean. Since Lemma 4.2.1 is true for any finite group, we can still apply that inequality to get

$$\frac{|\mho_p(G)|}{|G|_p} \geq \left(\prod_{x\in P}\frac{|G|}{|S_G(x)|}\right)^{1/|P|}$$

and recalling Theorem 1.3.6 we can write

$$\frac{|\mho_p(G)|}{|G|_p} \geq \left(\prod_{x\in P}\frac{n_p(G)}{\lambda_G(x)}\right)^{1/|P|}.$$

A sufficient condition for our conjecture is then

$$\left(\prod_{x\in P}\frac{n_p(G)}{\lambda_G(x)}\right)^{1/|P|} \geq n_p(G)^{\frac{p-1}{p}}$$

that is true if and only if the following condition, which we state as our conjecture, holds.

Conjecture 4.3.1. *Let G be a finite group, p a prime dividing $|G|$ and $P\in Syl_p(G)$. Then*

$$\left(\prod_{x\in P}\lambda_G(x)\right)^{1/|P|} \leq n_p(G)^{\frac{1}{p}}. \tag{4.3.1}$$

For the sake of clarity, we explicitly observe that Conjecture 4.3.1 states that the geometric mean of the number of Sylow p-subgroups containing an element of a Sylow p-subgroup is less or equal then the pth root of the total number of Sylow p-subgroups (cfr. Proposition 2.3.4).

Inequality (4.3.1) can be also written as

$$\left(\prod_{x\in P}\lambda_G(x)\right) \leq (n_p(G)^{|P|})^{\frac{1}{p}}. \tag{4.3.2}$$

This gives another statement of our conjecture: the number of maps $f : P \to Syl_p(G)$ such that $x \in f(x)\ \forall x \in P$, is less or equal then the pth root of the total number of maps from P in $Syl_p(G)$.

Remark 4.3.2. The bound in Conjecture 4.3.1, if true, is best possible in a strict sense. If we calculate the terms of inequality (4.3.1) in the groups defined in Example 4.2.3, an equality occurs.

We now want to obtain a reduction of Conjecture 4.3.1 to finite almost simple groups.

Remark 4.3.3. We can assume $O_p(G) = 1$. This is because if N is a normal p-subgroup of G then for all $x \in \mathfrak{U}_p(G)$ we have $\lambda_G(x) = \lambda_{G/N}(xN)$ and so Conjecture 4.3.1 holds for G if and only if it holds for G/N.

First of all we reduce the conjecture to groups in $\mathfrak{M}_{ns}$.

Lemma 4.3.4. *Let N be a normal subgroup of G, P be a Sylow p-subgroup of G, $x \in P$ and suppose that $G = NP$. Then*

$$|S_G(x)| = |S_G(x) \cap N||NP/N|$$

Proof. If $x \in N$ the thesis follows from part (ii) of Lemma 3.2.2.

We work by induction on m such that $|NP/N| = p^m$. If $m = 0$ then $G = N$ and there is nothing to prove. Suppose that $m > 0$ and $G \neq N\langle x\rangle$. Let M be a maximal subgroup of G containing $N\langle x\rangle$. Then by inductive hypothesis

$$|S_G(x)| = |S_M(x)|p = |S_N(x)|p^{m-1}p.$$

Finally, if $G = N\langle x\rangle$ then we observe that $ax^t \in S_G(x)$ with $a \in N$ if and only if $\langle x\rangle$ is subnormal in $\langle x, ax^t\rangle = \langle x, a\rangle$, that is if and only if $a \in S_G(x)$. It follows that

$$|S_G(x)| = |S_G(x) \cap N||NP/N|.$$

□

Proposition 4.3.5. *Let N be a normal subgroup of G, P be a Sylow p-subgroup of G and $x \in P$. Then*

$$\lambda_G(x) = \lambda_{\frac{G}{N}}(xN)\lambda_{NP}(x).$$

Proof. First of all we show that given a p-element x, $\lambda_{NP}(x)$ is independent of the particular Sylow p-subgroup P containing x. By Theorem 1.3.6 and the previous lemma

$$\lambda_{NP}(x) = \frac{|S_{NP}(x)|}{|N_{NP}(P)|} = \frac{|S_{NP}(x) \cap N|}{|N_{NP}(P)|}\left|\frac{NP}{N}\right| = \frac{|S_{NP}(x) \cap N|}{|N|}|n_p(NP)|.$$

If Q is another Sylow p-subgroup such that $x \in Q$ then of course $n_p(NP) = n_p(NQ)$ since NP and NQ are conjugated in G. Moreover

$$|S_{NP}(x) \cap N| = \{a \in N \mid \langle x \rangle \trianglelefteq \trianglelefteq \langle a, x \rangle\}$$

depends on N and x only.

Now let Δ_G^x be the set of Sylow p-subgroups of G containing x. We define the map

$$\begin{aligned} \Delta_G^x &\to \Delta_{\frac{G}{N}}^{xN} \\ Q &\mapsto NQ/N. \end{aligned} \tag{4.3.3}$$

The fiber of $N\tilde{Q}/N \in \Delta_{\frac{G}{N}}^{xN}$ is the set of Sylow p-subgroups Q of G containing x and such that $NQ = N\tilde{Q}$, i.e., $\Delta_{N\tilde{Q}}^x$. Since we proved that $\lambda_{N\tilde{Q}}(x)$ is independent of $\tilde{Q}$ we have

$$\lambda_G(x) = |\Delta_G^x| = \left|\Delta_{\frac{G}{N}}^{xN}\right| |\Delta_{NP}^x| = \lambda_{\frac{G}{N}}(xN)\lambda_{NP}(x).$$

□

Proposition 4.3.6. *A counterexample of minimal order of Conjecture 4.3.1 is a group in $\mathfrak{M}_{ns}$.*

Proof. Let G be a counterexample of minimal order to Conjecture 4.3.1 and P be a Sylow p-subgroup of G. By Remark 4.3.3 we have that $O_p(G) = 1$. Since conjecture holds for p-solvable groups, G is nonsolvable. We want to prove that $G \in \mathfrak{M}_{ns}$. Let N be a proper nontrivial normal subgroup of G. Then by Proposition 4.3.5 we have

$$\prod_{x \in P} \lambda_G(x) = \prod_{x \in P} \lambda_{\frac{G}{N}}(xN)\lambda_{NP}(x) = \left(\prod_{xN \in \frac{PN}{N}} \lambda_{\frac{G}{N}}(xN)\right)^{|P \cap N|} \left(\prod_{x \in P} \lambda_{NP}(x)\right) \tag{4.3.4}$$

By the minimality of G we have

$$\prod_{xN \in \frac{PN}{N}} \lambda_{\frac{G}{N}}(xN) \leq n_p\left(\frac{G}{N}\right)^{\frac{|PN/P|}{p}}.$$

If $NP < G$ we can again assume conjecture for NP and so

$$\prod_{x \in P} \lambda_G(x) \leq n_p\left(\frac{G}{N}\right)^{\frac{|PN/P|}{p}|P \cap N|} n_p(NP)^{\frac{|P|}{p}} = n_p(G)^{\frac{|P|}{p}}.$$

Then $G = NP$ for every nontrivial normal subgroup N of G. Since G is not solvable and $O_p(G) = 1$ we have that G has a unique minimal normal subgroup N which is nonsolvable and $G = NP$. □

We can then assume that G is the extension of a direct product N of k copies of a finite simple group L by a p-group P which acts transitively on the factors of N. We now prove an easy lemma which loosely bounds the number of Sylow p-subgroups of N normalized by an element $x \in P$ in terms of the action of x on the direct factors of N.

Lemma 4.3.7. *Let $N \trianglelefteq G$ be a direct product of k copies of a group L, $N = L_1 \times \cdots \times L_k$ and let $x \in G$ be a p-element such that $L_i^x \in \{L_1, \dots, L_k\} = \Delta$, that is, x permutes the factors of N. Then the number of Sylow p-subgroups of N normalized by x is less or equal than $n_p(L)^s$ where s is the number of orbits of $\langle x \rangle$ on Δ.*

Proof. A Sylow p-subgroup P of N is the direct product of k Sylow p-subgroups of L, $P = P_1 \times \cdots \times P_k$. Suppose that P is normalized by x. If $L_i = L_j^{x^r}$ then $P_i = P_j^{x^r}$ and so one has at most $n_p(L)$ choices for each $\langle x \rangle$-orbit in Δ. $\square$

We can now prove the reduction of Conjecture 4.3.1 to almost simple groups.

Theorem 4.3.8. *Conjecture 4.3.1 holds if and only if it holds for every finite almost simple groups.*

Proof. Suppose the conjecture true for all finite almost simple groups and let G be a counterexemple of minimal order. By Proposition 4.3.6, G is a group in $\mathfrak{M}_{ns}$. Then $G = NP$ with P a Sylow p-subgroup and N a nonsolvable minimal normal subgroup

$$N = L_1 \times \cdots \times L_k,\ L_i \simeq L,\ \forall i \in \{1, \dots, k\}.$$

for some nonabelian finite simple group L. Moreover P acts transitively on $\Delta = \{L_1, \dots, L_k\}$. Since we are assuming the result for almost simple groups, we have $k > 1$.

We now fix some notation and write down some elementary observations. We set $Q = P \cap N$ and for any subgroup $Q \leq X \leq P$ we set m_X to be the ratio

$$m_X = \frac{|N_N(X)|}{|N_N(P)|} = \frac{n_p(G)}{n_p(NX)}.$$

The last equality holds since $n_p(NX) = [NX : N_{NX}(X)] = [N : N_N(X)]$. Moreover observe that if $g \in N_N(P)$ and $x \in X$ then

$$x^g = x[x, g] \in X(N \cap P) = XQ = X,$$

and so $N_N(P) \leq N_N(X)$.

With a slight abuse of notation we denote with $\lambda_N(x)$ the number of Sylow p-subgroups in N normalized by x even for $x \notin N$. It is easy to convince oneself that $\lambda_N(x) = \lambda_{N\langle x \rangle}(x)$.

Let $H_0 = N_P(L_1)$ be the stabilizer of S_1 in the action of P on Δ. Since P is transitive, $H_0 \neq P$. Choose then a subgroup H of P of index p containing H_0. Since H is normal and the stabilizers of the L_i's are all conjugated in P we have that H contains all of them. It follows that every element $x \in P \setminus H$ has at most k/p orbits on Δ and so by Lemma 4.3.7

$$\lambda_N(x) \leq n_p(L)^{\frac{k}{p}}. \tag{4.3.5}$$

We now consider separately the elements inside and outside H. As for the elements inside H we get, using part (ii) of Lemma 3.2.2 that

$$\begin{aligned}\prod_{x \in H} \lambda_G(x) &= \prod_{x \in H} \frac{n_p(G)}{n_p(NH)} \lambda_{NH}(x) \\ &= \prod_{x \in H} m_H \lambda_{NH}(x) = (m_H)^{|H|} \prod_{x \in H} \lambda_{NH}(x).\end{aligned}$$

Since H is a Sylow p-subgroup of NH and the conjecture holds for $NH < G$ we get

$$\begin{aligned}\prod_{x\in H}\lambda_G(x) &= (m_H)^{|H|}\prod_{x\in H}\lambda_{NH}(x)\\ &\le (m_H)^{|H|}n_p(NH)^{\frac{|H|}{p}} = m_H^{\frac{p-1}{p}|H|}n_p(G)^{\frac{|H|}{p}}.\end{aligned} \tag{4.3.6}$$

Now we turn our attention to elements in $P \setminus H$. Let $\mathcal{T}$ be a set of representatives for the right cosets of Q in P that are not contained in H. The cardinality of $\mathcal{T}$ is then

$$|\mathcal{T}| = [P : Q] - [H : Q] = \frac{|P| - |H|}{|Q|} = (p-1)\frac{|H|}{|Q|}.$$

We have

$$\begin{aligned}\prod_{x\in P\setminus H}\lambda_G(x) = \prod_{x\in\mathcal{T}}\prod_{g\in Q}\lambda_G(gx) &= \prod_{x\in\mathcal{T}}\left(\prod_{g\in Q}m_{Q\langle gx\rangle}\lambda_{N\langle gx\rangle}(gx)\right)\\ &= \prod_{x\in\mathcal{T}}\left(m_{Q\langle x\rangle}^{|Q|}\prod_{g\in Q}\lambda_N(gx)\right).\end{aligned}$$

Now the elements gx in the previous product are not in H and so by (4.3.5)

$$\begin{aligned}\prod_{x\in P\setminus H}\lambda_G(x) &\le \left(\prod_{x\in\mathcal{T}}m_{Q\langle x\rangle}\right)^{|Q|}n_p(L)^{\frac{k}{p}|Q||\mathcal{T}|}\\ &= \left(\prod_{x\in\mathcal{T}}m_{Q\langle x\rangle}\right)^{|Q|}n_p(N)^{\frac{p-1}{p}|H|}\end{aligned}$$

We now want to evaluate the product $\prod_{x\in\mathcal{T}}m_{Q\langle x\rangle}$. In the following we use the bar notation for the quotients modulo Q (so that for example $\bar{P} = P/Q$). If $R = N_N(Q)$ we have a coprime action of $\bar{P}$ on the p'-group $\bar{R}$. We apply Theorem 4.2.2 to this action and get

$$\begin{aligned}|C_{\bar{R}}(\bar{P})|^{|\bar{P}|\frac{p-1}{p}} &= \prod_{x\in\bar{P}}\frac{|C_{\bar{R}}(x)|}{|C_{\bar{R}}(x^p)|^{1/p}}\\ &= \left(\prod_{x\in\bar{H}}\frac{|C_{\bar{R}}(x)|}{|C_{\bar{R}}(x^p)|^{1/p}}\right)\left(\prod_{x\in\bar{P}\setminus\bar{H}}\frac{|C_{\bar{R}}(x)|}{|C_{\bar{R}}(x^p)|^{1/p}}\right) =\\ &\quad |C_{\bar{R}}(\bar{H})|^{|\bar{H}|\frac{p-1}{p}}\left(\prod_{x\in\bar{P}\setminus\bar{H}}\frac{|C_{\bar{R}}(x)|}{|C_{\bar{R}}(x^p)|^{1/p}}\right)\end{aligned}$$

and so, using the bound $|C_{\bar{R}}(x^p)| \le |\bar{R}|$ and the fact that $\bar{P} \setminus \bar{H} = (p-1)\bar{H}$,

$$\begin{aligned}\prod_{x\in\bar{P}\setminus\bar{H}}|C_{\bar{R}}(x)| &= |C_{\bar{R}}(\bar{P})|^{|\bar{P}|\frac{p-1}{p}}|C_{\bar{R}}(\bar{H})|^{-|\bar{H}|\frac{p-1}{p}}\left(\prod_{x\in\bar{P}\setminus\bar{H}}|C_{\bar{R}}(x^p)|^{1/p}\right)\\ &\le |C_{\bar{R}}(\bar{P})|^{|\bar{P}|\frac{p-1}{p}}|C_{\bar{R}}(\bar{H})|^{|\bar{H}|\frac{p-1}{p}}|\bar{R}|^{|\bar{H}|\frac{p-1}{p}}.\end{aligned}$$

What allows us to use this bound is that if X is a subgroup of P containing Q, then $\overline{N_N(X)} = C_{\bar{R}}(\bar{X})$. Going back to our product we then get

$$\begin{aligned} \prod_{x \in \mathcal{T}} m_{Q\langle x\rangle} = \prod_{x \in \mathcal{T}} \frac{|C_{\bar{R}}(xQ)|}{|C_{\bar{R}}(\bar{P})|} = \prod_{x \in \bar{P}\setminus\bar{H}} \frac{|C_{\bar{R}}(x)|}{|C_{\bar{R}}(\bar{P})|} \\ \leq |C_{\bar{R}}(\bar{P})|^{-|\bar{P}|\frac{p-1}{p}} |C_{\bar{R}}(\bar{P})|^{|\bar{P}|\frac{p-1}{p}} |C_{\bar{R}}(\bar{H})|^{|\bar{H}|\frac{p-1}{p}} |\bar{R}|^{|\bar{H}|\frac{p-1}{p}} \\ = |C_{\bar{R}}(\bar{H})|^{|\bar{H}|\frac{p-1}{p}} |\bar{R}|^{|\bar{H}|\frac{p-1}{p}}. \end{aligned}$$

Finally we get

$$\begin{aligned} \prod_{x \in P\setminus H} \lambda_G(x) \leq \left(\prod_{x \in \mathcal{T}} m_{Q\langle x\rangle}\right)^{|Q|} n_p(N)^{\frac{p-1}{p}|H|} \\ \leq |C_{\bar{R}}(\bar{H})|^{\frac{p-1}{p}|H|} |\bar{R}|^{\frac{p-1}{p}|H|} n_p(N)^{\frac{p-1}{p}|H|} \end{aligned} \tag{4.3.7}$$

Now we put (4.3.6) and (4.3.7) together to obtain

$$\begin{aligned} \prod_{x \in P} \lambda_G(x) = \left(\prod_{x \in H} \lambda_G(x)\right)\left(\prod_{x \in P\setminus H} \lambda_G(x)\right) \\ \leq m_H^{\frac{p-1}{p}|H|} n_p(G)^{\frac{|H|}{p}} |C_{\bar{R}}(\bar{H})|^{|\bar{H}|\frac{p-1}{p}} |\bar{R}|^{|\bar{H}|\frac{p-1}{p}} n_p(N)^{\frac{p-1}{p}|H|} \\ = n_p(G)^{\frac{|H|}{p}} \left(\frac{|N_N(H)|}{|N_N(P)|}\frac{|N|}{|N_N(Q)|}\frac{|N_N(Q)|}{|Q|}\frac{|Q|}{|N_N(H)|}\right)^{\frac{p-1}{p}|H|} \\ = n_p(G)^{\frac{|H|}{p}} \left(\frac{|N|}{|N_N(P)|}\right)^{\frac{p-1}{p}|H|} = n_p(G)^{\frac{|P|}{p}} \end{aligned}$$

which is against the fact that G is a counterexample. □

We end this final chapter with a remark. If Theorem 4.0.1 held in general, without the hypothesis of p-solvability, then we would get the following bound for the average of the ratios $\mathrm{sp}_G(x)$ for $x \in \mathfrak{U}_p(G)$

$$\frac{1}{|\mathfrak{U}_p(G)|}\sum_{x \in \mathfrak{U}_p(G)} \mathrm{sp}_G(x) = \frac{|P|}{|\mathfrak{U}_p(G)|} \leq n_p(G)^{-\frac{p-1}{p}}$$

where we used Lemma 2.2.7.

Now, if H is the normalizer of a Sylow p-subgroup P of G, considering the action of G on the right cosets of H we have

$$\frac{G}{H_G} \hookrightarrow Sym(Lat(G,H)) \simeq S_{n_p(G)}.$$

Since $O_p(G) \leq P \cap H_G$ we would get

$$\frac{|P|}{|O_p(G)|} = \frac{|P|}{|P \cap H_G|} \leq \frac{|G|}{|H_G|} \leq n_p(G)!\ .$$

Putting these two facts together we get that a lower bound on the average on the p-elements of $\mathrm{sp}_G(x)$ would give an upper bound on the index in P of $O_p(G)$. This goes partially in the direction of the unsolved problem we described at the end of Chapter 2.

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