Parabolic Minkowski convolutions and concavity properties of viscosity solutions to fully nonlinear equations

PARABOLIC MINKOWSKI CONVOLUTIONS AND CONCAVITY PROPERTIES OF VISCOSITY SOLUTIONS TO FULLY NONLINEAR EQUATIONS

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Abstract. This paper is concerned with the Minkowski convolution of viscosity solutions of fully nonlinear parabolic equations. We adopt this convolution to compare viscosity solutions of initial-boundary value problems in different domains. As a consequence, we can for instance obtain parabolic power concavity of solutions to a general class of parabolic equations. Our results are applicable to the Pucci operator, the normalized q-Laplacians with $1 < q \leq \infty$, the Finsler Laplacian, and more general quasilinear operators.

1. Introduction

1.1. **Background and motivation.** This paper is connected to a general theory devised for the elliptic case in [46] and extended to the parabolic framework by two of the authors. In particular, we extend the results in [31] and [32] to a general class of fully nonlinear parabolic equations in the framework of viscosity solutions. In connection with the general theory of [46] and with the results and techniques of this paper, we also address the reader to the twin paper [23], where we consider spatial concavity properties as well as Brunn-Minkowski type inequalities for parabolic and elliptic problems.

Let us first describe the basic setting of our problem and introduce its background.

Let $m \geq 2$ and $n \geq 1$. For any $i = 1, 2, \dots, m$, let Ω_i be a bounded smooth domain in $\mathbb{R}^n$. Let ν_i denote the inward unit normal vector to $\partial\Omega_i$. For any

$$\lambda \in \Lambda_m = \left\{ (\lambda_1, \dots, \lambda_m) \in (0,1)^m : \sum_{i=1}^m \lambda_i = 1 \right\},$$

let Ω_λ be the Minkowski combination of Ω_i, defined by

$$\Omega_\lambda = \sum_{i=1}^m \lambda_i \Omega_i = \left\{ \sum_{i=1}^m \lambda_i x_i : x_i \in \Omega_i, i = 1, 2, \dots, m \right\}. \tag{1.1}$$

It is easy to see that Ω_λ is bounded in $\mathbb{R}^n$. Notice that when $\Omega_i = \Omega$ for $i = 1, \dots, m$, we have of course $\Omega \subseteq \Omega_\lambda$, but the inclusion is in general strict unless Ω is convex. Hereafter for simplicity we set $Q_i = \Omega_i \times (0, \infty)$ and $\partial Q_i = (\partial\Omega_i \times (0,\infty)) \cup (\overline{\Omega}_i \times \{0\})$ for $i = \lambda, 1, \dots, m$. Our first aim is to connect the solution u_λ of some Cauchy-Dirichlet problem in Ω_λ to the solutions $u_1, \dots, u_m$ of similar (but not necessarily the same) Cauchy-Dirichlet problems in $\Omega_1, \dots, \Omega_m$.

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In particular, for $i = \lambda$ and $i = 1, 2, \ldots, m$, let us consider the following fully nonlinear Cauchy-Dirichlet problems:

$$\begin{cases} \partial_t u + F_i(x, t, u, \nabla u, \nabla^2 u) = 0 & \text{in } Q_i, \quad (1.2) \\ u = 0 & \text{on } \partial Q_i, \quad (1.3) \end{cases}$$

where $F_i : \overline{Q}_i \times [0, \infty) \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n \to \mathbb{R}$ for $i = \lambda, 1, 2, \ldots, m$ are given continuous elliptic operators, with F_λ suitably related to $F_1, \ldots, F_m$. As we said, we are interested in finding some kind of relationships (which we will clarify later) between the solution of problem (1.2)–(1.3) with $i = \lambda$ and the solutions with $i = 1, \ldots, m$.

Let u_i be a positive solution of (1.2)–(1.3) in Q_i for every $i = 1, 2, \ldots, m$. Let $1/2 \le \alpha \le 1$ and $p < 1$ be two given parameters and define the *α-parabolic Minkowski p-convolution* of $\{u_i\}_{i=1}^m$ for any $\lambda \in \Lambda_m$ as follows:

$$U_{p,\lambda}(x, t) := \sup \left\{ M_p\left(u_1(x_1, t_1), \ldots, u_m(x_m, t_m); \lambda\right) : (x_i, t_i) \in \overline{Q_i}, \right.$$
$$\left. x = \sum_i \lambda_i x_i,\ t = \left(\sum_i \lambda_i t_i^\alpha \right)^{\frac{1}{\alpha}} \right\}. \tag{1.4}$$

Here, for given $\lambda \in \Lambda_m$ and $p \in [-\infty, +\infty]$, $M_p(a_1, \ldots, a_m; \lambda)$ denotes the usual weighted p-means (with weight λ) of $a = (a_1, \ldots, a_m) \in [0, \infty)^m$, whose precise definition is given later in (2.1).

As shown in [32], when the equations are semilinear with F_i of the form

$$F_i(x, t, r, \xi, X) = -\operatorname{tr} X - f_i(x, t, r, \xi), \quad i = \lambda, 1, \ldots, m, \tag{1.5}$$

then, under suitable assumptions on the behavior of the u_i's on ∂Q_i's, $U_{p,\lambda}$ is a subsolution of (1.2)–(1.3) with $i = \lambda$, provided that f_λ and $\{f_i\}_{i=1}^m$ satisfy

$$g_\lambda \left(\sum_i \lambda_i x_i, \sum_i \lambda_i t_i, \sum_i \lambda r_i, \xi \right) \ge \sum_{i=1}^m \lambda_i g_i(x_i, t_i, r_i, \xi) \tag{1.6}$$

for any fixed $\xi \in \mathbb{R}^n$ and any $(x_i, t_i, r_i) \in Q_i \times (0, \infty)$, where

$$g_i(x, t, r, \xi) = r^{3-\frac{1}{p}} f_i \left(x, t^{\frac{1}{\alpha}}, r^{\frac{1}{p}}, \frac{1}{p} r^{\frac{1}{p}-1} \xi \right), \quad i = \lambda, 1, \ldots, m. \tag{1.7}$$

This, coupled with a comparison principle for (1.2), results in a comparison between the solution of the problem in Ω_λ with the solutions in the Ω_i's, $i = 1, \ldots, m$, which consists in a sort of concavity principle for the solutions of the involved problems with respect to the Minkowski combination of the underlying domains. When the domains $\Omega_1, \ldots, \Omega_m$ differ from each other, interesting applications are Brunn-Minkowski type inequalities for possibly connected functionals. For this, we refer to [46] and to the bibliography therein for the elliptic case and to [32] for the parabolic case.

Notice that the condition (1.6) can be interpreted as a comparison relation between f_λ and a certain type of concave combination of the f_i's ($i = 1, 2, \ldots, m$) under the transformation (1.7).

When all the Ω_i's coincide with a convex domain Ω and all f_i are the same for $i = \lambda, 1, \ldots, m$, all the problems clearly reduce to a single one. Then the above result, combined with a comparison principle for (1.2)–(1.3), immediately implies that the unique solution u of such an equation is α-parabolically p-concave in the sense that

$$u\left(\sum_i \lambda_i x_i, M_\alpha(t_1,\dots,t_m;\lambda)\right) \geq M_p\left(u_1(x_1,t_1)\dots,u_m(x_m,t_m);\lambda\right). \tag{1.8}$$

This type of concavity results was established in [31] and [32] (see also [29, 30]). Note that (1.6) then turns into a concavity assumption for g_λ.

When the Ω_i's truly differ from each other, then our result can be used to obtain Brunn-Minkowski and Urysohn type inequalities for related functionals, as it will be more explicitly described in [23] and has been already done in [32] in the parabolic framework and similarly, suitably treating different specific cases, in [13, 10, 12, 11, 44, 41, 45, 7] in the elliptic case. Notice that a general theory (for elliptic problems) is developed in [46], where however only classical solutions and convex domains were considered, although all the results therein did not really need convexity of the involved domains. And indeed non convex domains have been explicitly treated in [32].

The purpose of this paper is to extend the results described above to a more general setting. Our generalization lies at the following three aspects. First, we study the problem for a general class of fully nonlinear parabolic equations, which certainly includes the known semilinear case. We even allow the equations to bear mild singularity caused by vanishing gradient. By "mild singularity" we mean that for each $i=\lambda,1,\dots,m$, there exists a continuous function $h_i:\overline{Q}_i\times[0,\infty)\to\mathbb{R}$ such that

$$h_i(x,t,r)=(F_i)_*(x,t,r,0,0)=(F_i)^*(x,t,r,0,0) \quad \text{for } (x,t,r)\in\overline{Q}_i\times[0,\infty), \tag{1.9}$$

where $(F_i)_*$ and $(F_i)^*$ respectively stand for the lower and upper semicontinuous envelopes of F_i. Our results are applicable to several important types of nonlinear operators including the Pucci operator, the normalized q-Laplacians ($1<q\leq\infty$), and more general quasilinear operators.

Second, in accordance with our generalization of the equations, another significant contribution of this paper is that we use the weaker notion of viscosity solutions rather than the classical solutions. We thus manage to reduce the C^2 regularity of the solutions in the main theorems of [31, 32]. Let us emphasize that it is indeed possible to investigate spatial convexity of solutions in the framework of viscosity theory; we refer to [18, 20, 1, 35, 42] for viscosity techniques in different contexts and to [39, 40, 37, 6, 26, 27, 24, 25] etc for related results for classical solutions. Our current work provides new results on parabolic power concavity of viscosity solutions, which are not considered in the aforementioned references (but let us point out that, right after completing this work, we have learnt also about [15], where viscosity solutions have been now considered to study Brunn-Minkowski type inequalities for the eigenvalues of fully nonlinear homogeneous elliptic operators).

Third, we allow more freedom to the parameters α and p, so that, depending on the involved operators, we can consider $\alpha\in(0,1]$ and $p\in(-\infty,1]$. Notice that, although there is no special difficulty, negative power concavity properties have not been explicitly treated before to our knowledge.

Throughout this paper we assume the following fundamental well-posedness results for any $i=\lambda,1,\dots,m$.

- There exists a unique viscosity solution, locally Lipschitz in space, to (1.2)–(1.3).

- The comparison principle holds for (1.2)–(1.3), at least for $i = \lambda$; that is, if u_λ and v_λ are respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution satisfying $u_\lambda \le v_\lambda$ on ∂Q_λ, then $u_\lambda \le v_\lambda$ in $\overline{Q}_\lambda$.

We refer to [14] and [19] for existence and uniqueness of viscosity solutions of (1.2)–(1.3). For the reader's convenience, in Appendix (Section A.1), we list more precise structure assumptions on the F_λ besides (1.9), which guarantee the comparison principle; see more details also in [14, Theorem 8.2] and [19, Theorem 3.6.1]. On the other hand, showing local Lipschitz regularity of the unique solution requires extra work and further assumptions on F_i. We refer to the extensive literature on this subject in the context of viscosity solutions, for example [3, 47, 48, 38, 5, 4, 2] and references therein.

1.2. **Assumptions and main result.** Our main result is based on a condition connecting F_λ and F_i $(i = 1, 2, \dots, m)$, which generalizes (1.6) in the fully nonlinear setting. In order to give a clear view of this condition, we introduce the following transformed operators with a parameter $k \in \mathbb{R}$. Given $p \le 1$ and $\alpha \in (0, 1]$, set

$$\mathcal{R}_p = \begin{cases} (0,\infty) & \text{if } p \le 1, p \ne 0, \\ \mathbb{R} & \text{if } p = 0 \end{cases}$$

and let $G^{p,\alpha}_{i,k} : \overline{Q}_i \times \mathcal{R}_p \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n \to \mathbb{R}$ be defined as follows for every $i = \lambda, 1, \dots, m$:

$$\begin{aligned} G^{p,\alpha}_{i,k}(x,t,r,\xi,X) &= r^k F_i\left(x,\ t^{\frac{1}{\alpha}},\ r^{\frac{1}{p}},\ \tfrac{1}{p} r^{\frac{1}{p}-1}\xi,\ \tfrac{1}{p} r^{\frac{1}{p}-1} X + \tfrac{1-p}{p^2} r^{\frac{1}{p}-2} \xi\otimes\xi\right) && \text{if } p \ne 0\,, \\ G^{0,\alpha}_{i,k}(x,t,r,\xi,X) &= e^{kr} F_i\left(x,\ t^{\frac{1}{\alpha}},\ e^r,\ e^r\xi,\ e^r(X+\xi\otimes\xi)\right) && \text{if } p = 0\,, \end{aligned} \tag{1.10}$$

for all $(x, t, r, \xi, X) \in \overline{Q}_i \times \mathcal{R}_p \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n$.

To apply our method, we need to find $k \in \mathbb{R}$ satisfying the following two key assumptions (H1) and (H2).

(H1) If $p \ne 0$, the parameter $k \in \mathbb{R}$ satisfies

$$\text{either} \quad \frac{1}{p} - 1 + k \le 0 \quad \text{or} \quad \alpha\left(\frac{1}{p} - 1 + k\right) \ge 1. \tag{1.11}$$

(H2) For any $\lambda \in \Lambda$ and any $(x, t) \in Q_\lambda$, $r \in \mathcal{R}_p$, $\xi \in \mathbb{R}^n \setminus \{0\}$, and $Y \in \mathbb{S}^n$,

$$G^{p,\alpha}_{\lambda,k}(x,t,r,\xi,Y) \le \sum_{i=1}^m \lambda_i G^{p,\alpha}_{i,k}(x_i,t_i,r_i,\xi,X_i) \tag{1.12}$$

holds for $(x_i, t_i, r_i, X_i) \in Q_i \times \mathcal{R}_p \times \mathbb{S}^n$ $(i = 1, 2, \dots, m)$ satisfying

$$\sum_i \lambda_i x_i = x, \quad \sum_i \lambda_i t_i = t, \quad \sum_i \lambda_i r_i = r, \tag{1.13}$$

and

$$\operatorname{sgn}^*(p)\begin{pmatrix} \lambda_1 X_1 & & & \\ & \lambda_2 X_2 & & \\ & & \ddots & \\ & & & \lambda_m X_m \end{pmatrix} \le \operatorname{sgn}^*(p)\begin{pmatrix} \lambda_1^2 Y & \lambda_1\lambda_2 Y & \cdots & \lambda_1\lambda_m Y \\ \lambda_2\lambda_1 Y & \lambda_2^2 Y & \cdots & \lambda_2\lambda_m Y \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_m\lambda_1 Y & \lambda_m\lambda_2 Y & \cdots & \lambda_m^2 Y \end{pmatrix}, \tag{1.14}$$

where $\operatorname{sgn}^*(p) = 1$ if $p \geq 0$ and $\operatorname{sgn}^*(p) = -1$ if $p < 0$.

We emphasize that when $p = 0$, condition (1.11) can be removed, i.e. we can take any $k \in \mathbb{R}$. When $p \neq 0$, the condition (H1) is equivalent to requiring the function $g_k(r,t) = r^{\frac{1}{p}-1-k} t^{1-\frac{1}{\alpha}}$ ($g_k(r,t) = e^{(k+1)r} t^{1-\frac{1}{\alpha}}$ when $p = 0$) to be convex in $(0,\infty)^2$.

The reason for us to impose (H2) in the form involving $G^{p,\alpha}_{i,k}$ rather than F_i is that we will later transform our equation (1.2) into another form, which is more compatible with our convexity argument. The operator $G^{p,\alpha}_{i,k}$ appears in the new equation. The term $\operatorname{sgn}^*(p)$ is needed in (1.14), since for the transformed equation we will consider subsolutions when $p \geq 0$ but supersolutions when $p < 0$.

Before stating our main result, we set

$$\tilde{\nu}_i(x) := \begin{cases} \nu_i(x) & \text{if } x \in \partial\Omega_i, \\ 0 & \text{if } x \in \Omega_i, \end{cases} \quad \text{and} \quad \mu(t) := \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{if } t > 0. \end{cases} \tag{1.15}$$

Theorem 1.1 (Subsolution property of Minkowski convolution)**.** *Fix $\lambda \in \Lambda_m$. Assume that Ω_i is a bounded smooth domain in $\mathbb{R}^n$ for any $i = 1, 2, \ldots, m$. Let Ω_λ be the Minkowski combination of $\{\Omega_i\}_{i=1}^m$ as defined in (1.1). Let $0 < \alpha \leq 1$ and $p \leq 1$. Suppose that there exists $k \in \mathbb{R}$ such that (H1) and (H2) hold. Let u_i be the unique solution of (1.2)–(1.3) that is positive and locally Lipschitz in space in Q_i for $i = 1, 2, \ldots, m$. Assume in addition that for any $i = 1, 2, \ldots, m$,*

(i) *u_i is monotone in time, i.e.,*
$$u_i(x,t) \geq u_i(x,s) \quad \textit{for any } x \in \Omega_i \textit{ and } t \geq s \geq 0; \tag{1.16}$$
(ii) *if $0 < p \leq 1$, then*
$$\frac{1}{\rho} u_i^p \left(x + \tilde{\nu}_i(x)\rho, t + \mu(t)\rho^{1/\alpha}\right) \to \infty \quad \textit{as } \rho \to 0+ \tag{1.17}$$
for any $(x,t) \in \partial Q_i$.

Then $U_{p,\lambda}$ as in (1.4) is a subsolution of (1.2)–(1.3) with $i = \lambda$.

We can use our general result to cover [32, Theorem 3.2]. Indeed, if $p \neq 0$, by taking $k = 3 - 1/p$ we get

$$G^{p,\alpha}_{i,k}(x,t,r,\xi,X) = -\frac{1}{p} r^2 \operatorname{tr} X - \frac{1-p}{p^2} r - g_i(x,t,r,\xi) \tag{1.18}$$

for all $(x,t,r,\xi,X) \in \Omega_\lambda$. We can verify the assumption (H2) in Theorem 1.1 holds with the choice $k = 3 - 1/p$ and the condition (1.6). In the case $p = 0$, we can choose $k = 1$ to show that the same result holds under condition (1.6) but with

$$g_i(x,t,r,\xi) = e^r f_i\left(x, t^{\frac{1}{\alpha}}, e^r, e^r \xi\right), \quad i = \lambda, 1, \ldots, m. \tag{1.19}$$

See more details in Section 5.1.

Compared to the key conditions (H1) and (H2), the additional assumptions (i)–(ii) are more technical. Notice however that assumption (1.17) is not needed for $p \leq 0$. Moreover, for $p \in (0,1]$, even if in applications F_i may not fulfill (i)–(ii), we can fix the issue by perturbing F_i with a small $\varepsilon > 0$ as

$$F_{i,\varepsilon} = F_i - \varepsilon \quad (i = 1, 2, \ldots, m); \tag{1.20}$$

in other words, we instead consider the equation

$$\partial_t u + F_{i,\varepsilon}(x,t,u,\nabla u,\nabla^2 u) = 0 \quad \text{in } Q_i. \tag{1.21}$$

It turns out that such perturbation meet our needs in most of our applications. For $p \in (0,1)$, we can prove (i) and (ii) for (1.21) with a larger class of parabolic operators F_i; see Appendix A.2 and Appendix A.3 for clarification. Such a perturbation causes no harm to the applications of our main results, since all of the other assumptions continue to hold in Theorem 1.1 with F_i replaced by $F_{i,\varepsilon}$. We can still obtain the desired results by first considering the approximate problem (1.21) and then passing to the limit as $\varepsilon \to 0$ by standard stability theory. Let us finally notice that, although Theorem 1.1 holds the same also for $p = 1$, in this case it is very hard to get assumption (1.17), which would require u_i to have vertical slope on the boundary (and indeed it is very hard to have concave solutions).

Our proof of Theorem 1.1 is based on the following two steps. We first take

$$v_i(x,t) = \begin{cases} u_i^p\left(x, t^{\frac{1}{\alpha}}\right) & \text{if } p \neq 0, \\ \log u\left(x, t^{\frac{1}{\alpha}}\right) & \text{if } p = 0, \end{cases} \tag{1.22}$$

for all $i = \lambda, 1, \ldots, m$. It is not difficult to verify, at least formally, that v_i solves

$$\frac{\alpha}{p} v_i^{\frac{1}{p}-1} t^{1-\frac{1}{\alpha}} \partial_t v_i + F_i\left(x,\ t^{\frac{1}{\alpha}},\ v_i^{\frac{1}{p}},\ \frac{1}{p} v_i^{\frac{1}{p}-1}\nabla v_i,\ \frac{1}{p} v_i^{\frac{1}{p}-1}\nabla^2 v_i + \frac{1-p}{p^2} v_i^{\frac{1}{p}-2} \nabla v_i \otimes \nabla v_i\right) = 0 \tag{1.23}$$

if $p \neq 0$ and

$$e^{v_i} t^{1-\frac{1}{\alpha}} \partial_t v_i + F_i\left(x,\ t^{\frac{1}{\alpha}},\ e^{v_i},\ e^{v_i}\nabla v_i,\ e^{v_i}\nabla^2 v_i + e^{v_i}\nabla v_i \otimes \nabla v_i\right) = 0 \tag{1.24}$$

if $p = 0$, which are respectively equivalent to

$$\begin{aligned} &v_i^{\frac{1}{p}-1+k} t^{1-\frac{1}{\alpha}} \partial_t v_i(x,t) + \frac{p}{\alpha} G_{i,k}^{p,\alpha}\left(x,t,v_i(x,t),\nabla v_i(x,t),\nabla^2 v_i(x,t)\right) = 0, \\ &e^{(k+1)v_i} t^{1-\frac{1}{\alpha}} \partial_t v_i(x,t) + \frac{1}{\alpha} G_{i,k}^{0,\alpha}\left(x,t,v_i(x,t),\nabla v_i(x,t),\nabla^2 v_i(x,t)\right) = 0, \end{aligned} \tag{1.25}$$

for any given parameter $k \in \mathbb{R}$. Here $G_{i,k}^{p,\alpha}$ is given by (1.10). In Section 3, we rigorously show that u_i is a viscosity subsolution of (1.2) if and only if v_i is a viscosity subsolution (resp., supersolution) of (1.25) when $p \geq 0$ (resp., $p < 0$).

After such a transformation, we next take the Minkowski convolution of v_i's as follows:

$$V_{p,\lambda}(x,\tau) := \begin{cases} \sup\left\{\sum_{i=1}^m \lambda_i v_i(x_i,\tau_i) :\ (x_i,\tau_i) \in \overline{Q_i},\ x = \sum_{i=1}^m \lambda_i x_i,\ \tau = \sum_{i=1}^m \lambda_i \tau_i\right\} & \text{if } p \geq 0, \\ \inf\left\{\sum_{i=1}^m \lambda_i v_i(x_i,\tau_i) :\ (x_i,\tau_i) \in \overline{Q_i},\ x = \sum_{i=1}^m \lambda_i x_i,\ \tau = \sum_{i=1}^m \lambda_i \tau_i\right\} & \text{if } p < 0, \end{cases} \tag{1.26}$$

for every $(x,\tau) \in \overline{Q}_\lambda$. It is clear that

$$V_{p,\lambda}(x,\tau) = \begin{cases} U_{p,\lambda}\left(x, \tau^{\frac{1}{\alpha}}\right)^p & \text{if } p \neq 0, \\ \log U_{p,\lambda}\left(x, \tau^{\frac{1}{\alpha}}\right) & \text{if } p = 0. \end{cases}$$

To prove Theorem 1.1, it thus suffices to prove that $V_{p,\lambda}$ is a subsolution if $p \geq 0$ or a supersolution if $p < 0$ of (1.25) with $i = \lambda$. The rest of the proof is inspired by [1], where the supersolution property is studied for the convex envelope of viscosity solutions to fully nonlinear elliptic equations with state constraint or Dirichlet boundary conditions. The key is to establish a relation between the semijets (weak derivatives) of v_i and $V_{p,\lambda}$, which combined with (H1)–(H2), leads to the desired conclusion.

1.3. **Applications to parabolic power concavity.** We can use Theorem 1.1 to study the parabolic power concavity of viscosity solutions to a general class of fully nonlinear parabolic equations. More precisely, when $F_i = F_\lambda$ and $\Omega_i = \Omega_\lambda$ for all $i = 1, 2, \ldots, m$ with $m = n+2$, assuming the convexity of Ω_λ, we can apply the above result to the unique solution u of (1.2)–(1.3) with $i = \lambda$ to deduce that

$$\begin{aligned} &u^\star(x,t)\\ &:= \sup\left\{\left(\sum_{i=1}^m \lambda_i u^p(x_i,t_i)\right)^{\frac{1}{p}} : (x_i,t_i)\in\overline{Q_i},\ x=\sum_{i=1}^m \lambda_i x_i,\ t=\left(\sum_{i=1}^m \lambda_i t_i^\alpha\right)^{\frac{1}{\alpha}}\right\} \end{aligned} \tag{1.27}$$

is a subsolution of (1.2)–(1.3) with $i = \lambda$. Since $u \le u^\star$ by the definition and the comparison principle implies that $u \ge u^\star$ in $\overline{Q}_\lambda$, we obtain $u = u^\star$, i.e. the parabolic power concavity of u in the sense of (1.8). In this case, the assumption (H2) becomes the following convexity assumption on the operator $G^{p,\alpha}_{\lambda,k}$ defined by (1.10):

(H2a) For any $\lambda \in \Lambda$ and any $(x,t) \in Q$, $r \ge 0$, $\xi \in \mathbb{R}^n \setminus \{0\}$, and $Y \in \mathbb{S}^n$,

$$G^{p,\alpha}_{\lambda,k}(x,t,r,\xi,Y) \le \sum_{i=1}^{n+2} \lambda_i G^{p,\alpha}_{\lambda,k}(x_i,t_i,r_i,\xi,X_i)$$

holds whenever $(x_i,t_i) \in Q_i, r_i \in \mathcal{R}_p$ and $X_i \in \mathbb{S}^n$ fulfilling (1.13) and (1.14) with $m = n+2$.

Theorem 1.2 (Parabolic power concavity)**.** *Assume that $\Omega_\lambda \subset \mathbb{R}^n$ is a smooth bounded convex domain and that F_λ satisfies* (1.9) *with $i = \lambda$. Let u be the unique viscosity solution of* (1.2)–(1.3) *with $i = \lambda$ (that is positive and locally Lipschitz in space in $Q_\lambda = \Omega_\lambda \times (0,\infty)$). Let $k \in \mathbb{R}$, $0 < \alpha \le 1$, and $p \le 1$. Assume that* (H1) *and* (H2a) *hold, and, in addition, that*

(i) *u is monotone in time, i.e.,*

$$u(x,t) \ge u(x,s) \quad \text{for any } x \in \Omega_\lambda \text{ and } t \ge s \ge 0;$$

(ii) *if $p > 0$, then*

$$\frac{1}{\rho} u^p\left(x + \tilde\nu_0(x)\rho, t + \mu(t)\rho^{1/\alpha}\right) \to \infty \quad \text{as } \rho \to 0+$$

for any $(x,t) \in \overline{Q}_i$.

Then u is α-parabolically p-concave in Q_λ in the sense of (1.8).

It is worth remarking that (H2a) is actually slightly weaker than the usual convexity of $(x,t,r,X) \mapsto G^{p,\alpha}_{\lambda,k}(x,t,r,\xi,X)$ combined with the ellipticity of F_λ, since (1.14) implies that

$$\operatorname{sgn}^*(p) \sum_i \lambda_i X_i \le \operatorname{sgn}^*(p) Y.$$

We also remark that the quasiconcavity (i.e. convexity of superlevel sets) of $(x,t) \mapsto u\left(x, t^{1/\alpha}\right)$ holds as long as the assumptions in Theorem 1.2 hold for a finite p, since by definition quasiconcavity (corresponding to the case $p = -\infty$) is the weakest notion among all possible concavity properties. On the other hand, we cannot treat directly the case $p = -\infty$

here. Indeed, in this case the auxiliary function $v = u^p$ loses its significance and we probably should work directly with the parabolic quasiconcave envelope of the solution u and with the original equation. We are not aware of any result treating directly the mere quasiconcavity of solutions of general elliptic or parabolic equations in convex domains. It is instead discussed for problems posed in annular domains; see for instance [8, 28] and references therein.

As Theorem 1.1, Theorem 1.2 generalizes some previous results, precisely [31, Theorem 3] and [32, Theorem 4.2], which treat in the special case

$$F_i(x,t,r,\xi,X) = -\operatorname{tr} X - f(x,t,r,\xi) \qquad (i=\lambda,1,2,\ldots,m)$$

with $f \geq 0$ a given continuous function such that

$$(x,t,r) \mapsto \begin{cases} r^{3-\frac{1}{p}} f\left(x, t^{\frac{1}{\alpha}}, r^{\frac{1}{p}}, \frac{1}{p} r^{\frac{1}{p}-1}\xi\right) & \text{if } p \neq 0, \\ e^r f(x, t^{1/\alpha}, e^r, e^r\xi) & \text{if } p = 0, \end{cases} \tag{1.28}$$

is concave in $Q_\lambda \times \mathcal{R}_p$ for any $\xi \in \mathbb{R}^n$.

For most applications of Theorem 1.1 and Theorem 1.2, we can take $k = 3 - 1/p$ for $p \neq 0$. It is clear that (H1) is satisfied in this case. Denoting

$$\overline{G} = G^{p,\alpha}_{\lambda,3-1/p} \qquad (p \neq 0), \tag{1.29}$$

we see that the equation (1.25) with $i = \lambda$ reduces to

$$v^2 t^{1-\frac{1}{\alpha}} \partial_t v + \frac{p}{\alpha} \overline{G}\left(x,t,v,\nabla v,\nabla^2 v\right) = 0. \tag{1.30}$$

To meet the requirement (H2a) in Theorem 1.2, we only need to assume the following.

(H2b) For any $\lambda \in \Lambda$ and any $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\sum_i \lambda_i \overline{G}(x_i,t_i,r_i,\xi,X_i) \geq \overline{G}\left(\sum_i \lambda_i x_i, \sum_i \lambda_i t_i, \sum_i \lambda_i r_i, \xi, Y\right) \tag{1.31}$$

for all $(x_i,t_i) \in Q_\lambda, r_i > 0$, and $X_i, Y \in \mathbb{S}^n$ satisfying (1.14) with $m = n+2$.

Corollary 1.3 (A special case for parabolic power concavity)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth convex domain. Assume that F_λ satisfies* (1.9) *with $i = \lambda$. Let $0 < \alpha \leq 1$ and $0 \neq p \leq 1$. Assume that* (H2b) *holds. Let u be a unique viscosity solution of* (1.2)*–*(1.3) *with $i = \lambda$ (that is positive and locally Lipschitz in space in Q_λ). Assume in addition that u satisfies* (i) *and* (ii) *in Theorem 1.2. Then u is α-parabolically p-concave in Q_λ.*

We can verify that (H2b) holds when the operator F_λ is in the form

$$F_\lambda(x,t,r,\xi,X) = \mathcal{L}(\xi,X) - f(x,t,r,\xi),$$

where $\mathcal{L}$ is a degenerate elliptic operator satisfying proper assumptions (for instance $\mathcal{L}$ is 1-homogeneous with respect to X and 0-homogeneous with respect to ξ) and $f \geq 0$ is a continuous function such that (1.28) is concave for any fixed $\xi \in \mathbb{R}^n$. Examples of $\mathcal{L}$ include the Laplacian, the Pucci operator, the normalized q-Laplacian ($1 < q \leq \infty$), the Finsler Laplacian, etc.; see details in Section 5.

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2. Preliminaries

2.1. **Power means of nonnegative numbers.**

For $a = (a_1, \dots, a_m) \in (0,\infty)^m$, $\lambda \in \Lambda_m$, and $p \in [-\infty, +\infty]$, we set

$$M_p(a;\lambda) := \begin{cases} [\lambda_1 a_1^p + \lambda_2 a_2^p + \cdots + \lambda_m a_m^p]^{1/p} & \text{if} \quad p \neq -\infty,\, 0,\, +\infty, \\ \max\{a_1, \dots, a_m\} & \text{if} \quad p = +\infty, \\ a_1^{\lambda_1} \cdots a_m^{\lambda_m} & \text{if} \quad p = 0, \\ \min\{a_1, a_2, \dots, a_m\} & \text{if} \quad p = -\infty, \end{cases} \tag{2.1}$$

which is the (λ-weighted) p-mean of a.

For $a = (a_1, \dots, a_m) \in [0,\infty)^m$, we define $M_p(a;\lambda)$ as above if $p \geq 0$ and $M_p(a;\lambda) = 0$ if $p < 0$ and $\prod_{i=1}^m a_i = 0$.

Notice that $M_p(a;\lambda)$ is a continuous function of the argument a. Due to the Jensen inequality, we have

$$M_p(a;\lambda) \leq M_q(a;\lambda) \quad \text{if} \quad -\infty \leq p \leq q \leq \infty, \tag{2.2}$$

for any $a \in [0,\infty)^m$ and $\lambda \in \Lambda_m$. Moreover, it easily follows that

$$\lim_{p \to +\infty} M_p(a;\lambda) = M_{+\infty}(a;\lambda), \quad \lim_{p\to 0} M_p(a;\lambda) = M_0(a,\lambda), \quad \lim_{p\to -\infty} M_p(a;\lambda) = M_{-\infty}(a;\lambda).$$

For further details, see e.g. [22].

2.2. **Definition of viscosity solutions.**

We recall the definition of viscosity solutions to (1.2), which can also be found in [14, 19]. In Appendix A, we review more properties of viscosity solutions that are needed in this work.

Let Ω be a bounded smooth domain in $\mathbb{R}^n$. Let $\mathcal{O}$ denote an arbitrary open subset of $Q = \Omega \times (0,\infty)$. Consider a general parabolic equation

$$\partial_t u + F(x,t,u,\nabla u,\nabla^2 u) = 0 \tag{2.3}$$

in Q, where F is a proper elliptic operator.

Here, by elliptic we mean that

$$F(x,t,r,\xi,X_1) \leq F(x,t,r,\xi,X_2) \tag{2.4}$$

for all $(x,t,r,\xi) \in \overline{\Omega}_i \times [0,\infty) \times [0,\infty) \times (\mathbb{R}^n \setminus \{0\})$ and $X_1, X_2 \in \mathbb{S}^n$ satisfying $X_1 \geq X_2$. We also recall that F is proper if there exists $c \in \mathbb{R}$ such that

$$F(x,t,r_1,\xi,X) + cr_1 \leq F(x,t,r_2,\xi,X) + cr_2 \tag{2.5}$$

for all $(x,t,\xi,X) \in \overline{\Omega}_i \times [0,\infty) \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n$ and $r_1, r_2 \in [0,\infty)$ satisfying $r_1 \leq r_2$.

We further assume that F satisfies (1.9) with the subindex i omitted.

Definition 2.1. A locally bounded upper (resp., lower) semicontinuous function $u : \mathcal{O} \to \mathbb{R}$ is said to be a subsolution (resp., supersolution) of (2.3) in $\mathcal{O}$ if whenever there exist $(x_0, t_0) \in \mathcal{O}$ and $\phi \in C^2(\mathcal{O})$ such that $u - \phi$ attains a maximum (resp., minimum) at (x_0, t_0), we have

$$\begin{aligned} &\partial_t \phi(x_0,t_0) + F_*(x_0,t_0,u(x_0,t_0),\nabla\phi(x_0,t_0),\nabla^2\phi(x_0,t_0)) \le 0 \\ &\left(\text{resp., } \partial_t \phi(x_0,t_0) + F^*(x_0,t_0,u(x_0,t_0),\nabla\phi(x_0,t_0),\nabla^2\phi(x_0,t_0)) \ge 0\right). \end{aligned}$$

A continuous function $u : \mathcal{O} \to \mathbb{R}$ is called a solution of (2.3) in $\mathcal{O}$ if it is both a subsolution and a supersolution in $\mathcal{O}$.

It is clear that $F_* = F^* = F$ in $\mathcal{O} \times (0,\infty) \times \mathbb{R}^n \times \mathbb{S}^n$ provided that F is assumed to be continuous in $\mathcal{O} \times (0,\infty) \times \mathbb{R}^n \times \mathbb{S}^n$.

Remark 2.2. It is standard in the theory of viscosity solutions to use the semijets to give an equivalent definition. More precisely, for any $(x_0, t_0) \in \mathcal{O}$, setting $P^{2,+}u(x_0,t_0) \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ as

$$\begin{aligned} P^{2,+}u(x_0,t_0) = \Big\{(\tau,\xi,X) : u(x,t) \le & u(x_0,t_0) + \tau(t-t_0) + \langle \xi, x-x_0\rangle \\ & + \frac{1}{2}\langle X(x-x_0),(x-x_0)\rangle + o(|t-t_0| + |x-x_0|^2)\Big\} \end{aligned}$$

and its "closure" as

$$\begin{aligned} \overline{P}^{2,+}u(x_0,t_0) = \Big\{(\tau,\xi,X) : \ & \text{there exist } (x_j,t_j) \in \mathcal{O} \text{ and } (\tau_j,\xi_j,X_j) \in P^{2,+}u(x_j,t_j) \\ & \text{such that } (x_j,t_j,\tau_j,\xi_j,X_j) \to (x_0,t_0,\tau,\xi,X) \text{ as } j \to \infty\Big\}, \end{aligned}$$

we then say u is a subsolution of (2.3) if

$$\tau + F_*(x_0,t_0,u(x_0,t_0),\xi,X) \le 0$$

for every $(\tau,\xi,X) \in \overline{P}^{2,+}u(x_0,t_0)$. The semijet $P^{2,-}u(x_0,t_0)$, its closure, and supersolutions can be analogously defined in a symmetric way.

If $F(x,t,r,\xi,X)$ is mildly singular at $\xi = 0$, i.e. (1.9) holds, one can use the following equivalent definition, called $\mathcal{F}$-solutions as in [19].

Definition 2.3. Suppose that there exists $h \in C(\overline{\Omega} \times [0,\infty) \times [0,\infty))$ such that

$$h(x,t,r) = F^*(x,t,r,0,0) = F_*(x,t,r,0,0)$$

holds for all $(x,t,r) \in \mathcal{O} \times \mathbb{R}$. A locally bounded upper (resp., lower) semicontinuous function $u : \mathcal{O} \to \mathbb{R}$ is said to be a subsolution (resp., supersolution) of (2.3) in $\mathcal{O}$ if, whenever there exist $(x_0,t_0) \in \mathcal{O}$ and $\phi \in C^2(\mathcal{O})$ such that $u - \phi$ attains a maximum (resp., minimum) at (x_0,t_0), we have

$$\begin{aligned} &\partial_t \phi(x_0,t_0) + F(x_0,t_0,u(x_0,t_0),\nabla\phi(x_0,t_0),\nabla^2\phi(x_0,t_0)) \le 0 \\ &\left(\text{resp., } \partial_t \phi(x_0,t_0) + F(x_0,t_0,u(x_0,t_0),\nabla\phi(x_0,t_0),\nabla^2\phi(x_0,t_0)) \ge 0\right) \end{aligned}$$

when $\nabla\phi(x_0,t_0) \ne 0$ and

$$\begin{aligned} &\partial_t \phi(x_0,t_0) + h(x_0,t_0,u(x_0,t_0)) \le 0 \\ &(\text{resp., } \partial_t \phi(x_0,t_0) + h(x_0,t_0,u(x_0,t_0)) \ge 0) \end{aligned}$$

when $\nabla\phi(x_0,t_0) = 0$ and $\nabla^2\phi(x_0,t_0) = 0$.

Remark 2.4. In the definition of subsolutions by semijets, these conditions are written as follows: for any $(\tau, \xi, X) \in \overline{P}^{2,+}u(x_0, t_0)$, we require that

$$\begin{aligned} &\tau + F(x_0, t_0, u(x_0, t_0), \xi, X) \leq 0 \qquad && \text{if} \quad \xi \neq 0, \\ &\tau + h(x_0, t_0, u(x_0, t_0)) \leq 0 && \text{if} \quad \xi = \lambda \quad \text{and} \quad X = 0. \end{aligned}$$

3. A Useful Transformation of the Unknown Function

A straightforward way to study this problem is to directly turn the unknown function into a form that fits the desired parabolic power concavity.

If u_i is a smooth positive subsolution of (1.2) and F is not mildly singular, then by direct calculations we see that v_i defined in (1.22) is a smooth subsolution of (1.23) for all $i = \lambda, 1, \ldots, m$. In fact, we have

$$u_i(x,t) = v_i^{\frac{1}{p}}(x, t^\alpha), \quad \partial_t u_i(x,t) = \frac{\alpha}{p} v_i^{\frac{1}{p}-1} t^{\alpha-1} \partial_t v_i(x, t^\alpha), \quad \nabla u_i(x,t) = \frac{1}{p} v_i^{\frac{1}{p}-1} \nabla v_i(x, t^\alpha),$$
$$\nabla^2 u_i(x,t) = \frac{1}{p} v_i^{\frac{1}{p}-1} \nabla^2 v_i(x, t^\alpha) + \frac{1-p}{p^2} v_i^{\frac{1}{p}-2} \nabla v_i(x, t^\alpha) \otimes \nabla v_i(x, t^\alpha).$$

Plugging these into (1.2), we easily obtain (1.23). It is clear that positive smooth solutions of (1.23) are equivalent to positive smooth solutions of (1.25), where $G_{i,k}^{p,\alpha}$ is given by (1.10).

When u_i is not necessarily smooth, we can interpret such a result in the viscosity sense.

Lemma 3.1 (Sub/supersolution properties under transformation)**.** *Fix $i = \lambda, 1, \ldots, m$ arbitrarily. Assume that* (1.9) *holds. Let u_i be positive and upper semicontinuous in Q_i. Let v_i be given by* (1.22)*. Then*

- *if $p \geq 0$, u_i is a viscosity subsolution of* (1.2) *if and only if v_i is a viscosity subsolution of* (1.25) *in Q_i;*
- *for $p < 0$, u_i is a viscosity subsolution of* (1.2) *if and only if v_i is a viscosity supersolution of* (1.25) *in Q_i.*

Moreover, a symmetric result holds also for supersolutions.

Proof. Let us give the proof in details for the case $p > 0$ and u_i is a subsolution of (1.2), then let us prove that this implies that v_i is a subsolution of (1.23). The converse implication can be similarly shown.

Assume that there exist $(x_0, t_0) \in Q_i$ and $\phi \in C^2(\overline{Q}_i)$ such that

$$\max_{\overline{Q}_i}(v_i - \phi) = (v_i - \phi)(x_0, t_0) = 0.$$

In other words, we have

$$v_i(x_0, t_0) = \phi(x_0, t_0), \quad v_i(x,t) \leq \phi(x,t) \quad \text{for all } (x,t) \in Q_i.$$

Since $v_i > 0$ in Q_i, it follows that

$$u_i(x_0, t_0^{1/\alpha}) = \phi^{1/p}(x_0, t_0), \quad u_i(x, t^{1/\alpha}) \leq \phi^{1/p}(x,t) \quad \text{for all } (x,t) \in Q_i.$$

This implies that $u_i(x,t) - \psi(x,t)$ attains a maximum over Q_i at $(x_0, t_0^{1/\alpha})$, where $\psi(x,t) = \phi^{\frac{1}{p}}(x, t^\alpha)$.

Suppose that $\nabla\phi(x_0,t_0)\neq 0$. Then $\nabla\psi(x_0,t_0)\neq 0$. Since u_i is a subsolution of (1.2), we see that

$$\partial_t\psi + F_i\left(x_0, t_0^{1/\alpha}, u_i, \nabla\psi, \nabla^2\psi\right)\leq 0 \quad \text{at } (x_0, t_0^{1/\alpha}).$$

By direct calculations it follows that at (x_0,t_0)

$$\frac{\alpha}{p}v_i^{\frac{1}{p}-1}t^{1-\frac{1}{\alpha}}\partial_t\phi + F_i\left(x_0,\ t_0^{\frac{1}{\alpha}},\ v_i^{\frac{1}{p}},\ \frac{1}{p}v_i^{\frac{1}{p}-1}\nabla\phi,\ \frac{1}{p}v_i^{\frac{1}{p}-1}\nabla^2\phi + \frac{1-p}{p^2}v_i^{\frac{1}{p}-2}\nabla\phi\otimes\nabla\phi\right)\leq 0. \tag{3.1}$$

Multiplying (3.1) by $pv_i(x_0,t_0)^k$, we obtain

$$v_i(x_0,t_0)^{\frac{1}{p}-1+k}t^{1-\frac{1}{\alpha}}\partial_t\phi_i(x_0,t_0) + \frac{p}{\alpha}G_{i,k}^{p,\alpha}(x_0,t_0,v_i(x_0,t_0),\nabla\phi_i(x_0,t_0),\nabla^2 v_i(x_0,t_0))\leq 0.$$

If $\nabla\phi(x_0,t_0)=0$, we have $\nabla\psi(x_0,t_0^{1/\alpha})=0$. Using Definition 2.3, we assume $\nabla^2\phi(x_0,t_0)=0$, which is equivalent to $\nabla^2\psi\left(x_0,t_0^{1/\alpha}\right)=0$. We thus can apply the definition of subsolution on u_i to obtain

$$\partial_t\psi(x_0,t_0^{1/\alpha}) + h_i\left(x_0, t_0^{1/\alpha}, u_i(x_0,t_0^{1/\alpha})\right)\leq 0,$$

which yields

$$v_i(x_0,t_0)^{\frac{1}{p}-1}t_0^{1-\frac{1}{\alpha}}\partial_t\phi(x_0,t_0) + \frac{p}{\alpha}\tilde{h}_i\left(x_0, t_0^{1/\alpha}, v_i(x_0,t_0)\right)\leq 0.$$

The proof of the case $p>0$ and u_i is a subsolution is thus complete. The cases $p=0$ and $p<0$ can be treated similarly, and the same for the symmetric case when u_i is a supersolution. □

If F is mildly singular, it is not difficult to see that

$$G_{i,k}^{p,\alpha}(x,t,r,\xi,X)\to r^k h_i\left(x, t^{\frac{1}{\alpha}}, r^{\frac{1}{p}}\right) \quad \text{as } \xi\to 0,\ X\to 0 \tag{3.2}$$

locally uniformly for all $(x,t,r)\in\overline{Q}_i\times\mathcal{R}_p$ and all $i=\lambda,1,\ldots,m$. In other words, the operator $G_{i,k}^{p,\alpha}$ satisfies the same properties as in (1.9). We are thus able to apply Definition 2.3 to define the sub- and supersolutions of (1.25). Let us denote

$$\tilde{h}_i(x,t,r) = r^k h_i\left(x, t^{\frac{1}{\alpha}}, r^{\frac{1}{p}}\right) \tag{3.3}$$

for all $(x,t,r)\in\overline{Q}_i\times\mathcal{R}_p$ and $i=\lambda,1,\ldots,m$.

Proposition 3.2. *Assume that (1.9) holds for each $i=\lambda,1,\ldots,m$. Suppose that there exists $k\in\mathbb{R}$ such that (H2) holds. Then $\tilde{h}_i$ given by (3.3) satisfies*

$$\sum_i\lambda_i\tilde{h}_i(x_i,t_i,r_i)\geq\tilde{h}_0\left(\sum_i\lambda_i x_i, \sum_i\lambda_i t_i, \sum_i\lambda_i r_i\right) \tag{3.4}$$

for any $\lambda\in\Lambda$, $(x_i,t_i)\in Q_i$, and $r_i\in\mathcal{R}_p$.

Proof. Since

$$G_{\lambda,k}^{p,\alpha}(x,t,r,\xi,0)\to\tilde{h}_0(x,t,r) \quad \text{locally uniformly as } \xi\to 0$$

for any $\varepsilon>0$, there exists $\xi_\varepsilon\in\mathbb{R}^n\setminus\{0\}$ such that

$$G_{i,k}^{p,\alpha}\left(\sum_i\lambda_i x_i, \sum_i\lambda_i t_i, \sum_i\lambda_i r_i, \xi_\varepsilon, 0\right)\geq\tilde{h}_0\left(\sum_i\lambda_i x_i, \sum_i\lambda_i t_i, \sum_i\lambda_i r_i\right)-\varepsilon. \tag{3.5}$$

Since (1.14) clearly holds with $Y=X_i=0$ for all $i=1,2,\ldots,,n+2$, by (H2) we get

$$\sum_i\lambda_i G_{i,k}^{p,\alpha}(x_i,t_i,r_i,\xi_\varepsilon,0)\geq G_{\lambda,k}^{p,\alpha}\left(\sum_i\lambda_i x_i, \sum_i\lambda_i t_i, \sum_i\lambda_i r_i, \xi_\varepsilon, 0\right).$$

which by (3.5) yields

$$\sum_i \lambda_i G^{p,\alpha}_{i,k}(x_i,t_i,r_i,\xi_\varepsilon,0) \geq \tilde{h}_0\left(\sum_i \lambda_i x_i, \sum_i \lambda_i t_i, \sum_i \lambda_i r_i\right) - \varepsilon$$

Sending $\varepsilon \to 0$, we obtain (3.4) by (3.2) and (3.3). ☐

When $p > 0$, we easily see that v_i satisfies the same initial and boundary conditions as u_i. Therefore we can write the Cauchy-Dirichlet problem for v_i $(i = \lambda, 1, \dots, m)$ as

$$\begin{cases} v^{\frac{1}{p}-1+k} t^{1-\frac{1}{\alpha}} \partial_t v + \dfrac{p}{\alpha} G^{p,\alpha}_{i,k}\left(x,t,v,\nabla v,\nabla^2 v\right) = 0 & \text{in } Q_i, \quad (3.6)\\ v = 0 & \text{on } \partial Q_i. \quad (3.7)\end{cases}$$

Since we assume that a comparison principle holds for sub- and supersolutions of (1.2)–(1.3) that are positive in $\Omega \times (0,\infty)$, Lemma 3.1 implies that positive sub- and supersolutions of (3.6)–(3.7) also enjoy a comparison principle (which is what we truly need).

When $p \leq 0$, in place of (3.7), v_i satisfies a blow-up boundary and initial condition (precisely $v_i \to -\infty$ for $p = 0$, while $v_i \to +\infty$ when $p < 0$ on ∂Q_i), which enter into the case of state constraints boundary conditions. Then we have to go back to u_i and use the comparison principle for the problem satisfied by u_λ.

We conclude this section by pointing out the equivalence between (1.16) and the condition

$$v_i(x,t) \geq v_i(x,s) \quad \text{for any } x \in \Omega_i,\ t \geq s \geq 0, \text{ and } i = 1, \dots, m. \tag{3.8}$$

The monotonicity with respect to time will be used in the proof of Theorem 1.1.

4. The Minkowski Convolution

4.1. **Achievability in the interior.** For any given $\lambda \in \Lambda$ and $(x,t) = (\hat{x},\hat{t})$, we show that the supremum in (1.4) can be attained at some $(x_i,t_i) \in Q_i$ for $i = 1, 2, \dots, m$. Our proof is essentially the same of [32, Lemma 3.1]. We give the details for the sake of completeness.

Lemma 4.1 (Interior maximizers for the envelope)**.** *Suppose that the assumptions of Theorem 1.1 hold. Then for any $(\hat{x},\hat{t}) \in Q_\lambda$, there exist $(x_1,t_1) \in Q_1$, $(x_2,t_2) \in Q_2$, $\dots$, $(x_m,t_m) \in Q_m$ such that*

$$\hat{x} = \sum_{i=1}^m \lambda_i x_i,\ \ (\hat{t})^\alpha = \sum_{i=1}^m \lambda_i t_i^\alpha, \tag{4.1}$$

and

$$U_{p,\lambda}(\hat{x},\hat{t}) = \left(\sum_{i=1}^m \lambda_i u_i^p(x_i,t_i)\right)^{\frac{1}{p}}. \tag{4.2}$$

Proof. Let us only discuss the case $p > 0$, since the results with $p < 0$ or $p = 0$ clearly hold. In view of the compactness of the set

$$\left\{(y_1,s_1,y_2,s_2,\dots,y_m,s_m) \in \prod_{i=1}^m \overline{Q}_i : \hat{x} = \sum_{i=1}^m \lambda_i y_i,\ (\hat{t})^\alpha = \sum_{i=1}^m \lambda_i s_i^\alpha\right\}$$

and the continuity of

$$(y_1,s_1,y_2,s_2,\dots,y_m,s_m) \mapsto \left(\sum_{i=1}^m \lambda_i u_i^p(x_i,t_i)\right)^{\frac{1}{p}},$$

we can find $(x_i, t_i) \in \overline{Q}_i$ for $i = 1, 2, \ldots, m$ such that (4.1) and (4.2) hold.

Let $\hat{\tau} = (\hat{t})^\alpha$ and $\tau_i = t_i^\alpha$ and recall that v_i is given by (1.22). We have

$$\sum_{i=1}^m \lambda_i x_i = \hat{x}, \quad \sum_{i=1}^m \lambda_i \tau_i = \hat{\tau}, \tag{4.3}$$

$$U_{p,\lambda}(\hat{x}, \hat{t})^p = V_{p,\lambda}(\hat{x}, \hat{\tau}) = \sum_{i=1}^m \lambda_i v(x_i, \tau_i). \tag{4.4}$$

It thus suffices to show that $(x_i, \tau_i) \in Q_i$ for all $i = 1, 2, \ldots, m$.

Assume by contradiction that $(x_i, \tau_i) \in \partial Q_i$ for some $i = 1, 2, \ldots, m$. We derive a contradiction in the following two cases.

Case 1. Suppose that $(x_i, \tau_i) \in \partial Q_i$ for all $i = 1, 2, \ldots, m$, then by (1.3) and (4.2) we have $U_{p,\lambda}(\hat{x}, \hat{t}) = 0$, which is a contradiction, since $U_{p,\lambda}(\hat{x}, \hat{t}) > 0$ for every $(\hat{x}, \hat{t}) \in Q_\lambda$.

Case 2. Assume, without loss of generality, that $(x_1, \tau_1) \in \partial Q_1$ and $(x_2, \tau_2) \in Q_2$. Take $\rho \in (0, 1)$ and put

$$\tilde{x}_1 = x_1 + \frac{\rho}{\lambda_1}\tilde{\nu}_1, \quad \tilde{x}_2 = x_2 - \frac{\rho}{\lambda_2}\tilde{\nu}_1, \quad \tilde{x}_i = x_i \quad (i = 3, 4, \ldots, m),$$

$$\tilde{\tau}_1 = \tau_1 + \mu(t_1)\frac{\rho}{\lambda_1}, \quad \tilde{\tau}_2 = \tau_2 - \mu(t_1)\frac{\rho}{\lambda_2}, \quad \tilde{\tau}_i = \tau_i \quad (i = 3, 4, \ldots, m).$$

Then it is clear that $\sum_i \lambda_i \tilde{x}_i = \sum_i \lambda_i x_i = \hat{x}$, $\sum_i \lambda_i \tilde{\tau}_i = \sum_i \lambda_i \tau_i = \hat{\tau}$. By taking $\rho > 0$ small enough we also have $(\tilde{x}_1, \tilde{\tau}_1) \in Q_1$, $(\tilde{x}_2, \tilde{\tau}_2) \in Q_2$.

Adopting the local Lipschitz regularity of u_2, we get $M > 0$ and $\delta_1 > 0$ such that

$$|\nabla v_2| + |\partial_t v_2| \leq M \quad \text{a.e. in } B_{\delta_1}(x_2) \times (\tau_2 - \delta_1, \tau_2 + \delta_1) \subset Q_2.$$

It follows that

$$\lambda_2 v_2(\tilde{x}_2, \tilde{\tau}_2) - \lambda_2 v_2(x_2, \tau_2) \geq -\lambda_2 M \left(|\tilde{x}_2 - x_2| + |\tilde{\tau}_2 - \tau_2|\right) \geq -2M\rho. \tag{4.5}$$

On the other hand, the condition (1.17) implies that

$$v_1(\tilde{x}_1, \tilde{\tau}_1) - v_1(x_1, \tau_1) = u_1\left(x_1 + \tilde{\nu}_1(x_1)\frac{\rho}{\lambda_1}, t_1 + \mu(t_1)\left(\frac{\rho}{\lambda_1}\right)^{\frac{1}{\alpha}}\right)^p \geq (2M+1)\frac{\rho}{\lambda_1},$$

which yields that

$$\lambda_1 v(\tilde{x}_1, \tilde{t}_1) - \lambda_1 v(x_1, t_1) \geq (2M+1)\rho \tag{4.6}$$

when ρ is sufficiently small. By (4.5) and (4.6), we have

$$\begin{aligned}\sum_i \lambda_i v_i(\tilde{x}_i, \tilde{\tau}_i) &\geq \lambda_1 \left(v_1(\tilde{x}_1, \tilde{\tau}_1) - v_1(x_1, \tau_1)\right) + \lambda_2 \left(v_2(\tilde{x}_2, \tilde{\tau}_2) - v_2(x_2, \tau_2)\right) + \sum_i \lambda_i v_i(x_i, t_i) \\ &> \sum_{i=1}^m \lambda_i v_i(x_i, \tau_i) = U_{p,\lambda}(\hat{x}, \hat{t})^p,\end{aligned}$$

which contradicts (4.4). □

4.2. **A key lemma.** To show our main result, instead of using $U_{p,\lambda}$ defined in (1.4), we consider the Minkowski convolution $V_{p,\lambda}$ for v_i as given in (1.26).

It turns out that the following lemma plays a central role in the proof of Theorem 1.1.

Lemma 4.2 (Minkowski convolution preserves subsolutions)**.** *Fix $\lambda \in \Lambda_m$. Assume that Ω_i is a bounded smooth domain in $\mathbb{R}^n$ for any $i = 1, 2, \dots, m$. Let Ω_λ be the Minkowski combination as defined in* (1.1)*. Assume that F_i satisfies* (1.9) *for all $i = \lambda, 1, \dots, m$. Let $0 < \alpha \leq 1$ and $p \leq 1$. Suppose that there exists $k \in \mathbb{R}$ such that* (H1) *and* (H2) *hold, where $G^{p,\alpha}_{i,k}$ is given by* (1.10)*. Then:*

- *Case $0 \leq p \leq 1$.*

  Let v_i be a nondecreasing in time upper semicontinuous subsolution of (3.6) *for every $i = 1, 2, \dots, m$. Suppose that for any fixed $(\hat{x}, \hat{t}) \in Q$, the supremum in the definition of $V_{p,\lambda}$ in* (1.26) *at $(\hat{x}, \hat{\tau})$ is attained at some $(x_i, \tau_i) \in Q_i$ for $i = 1, 2, \dots, m$, in other words,* (4.3) *and* (4.4) *hold. Then $V_{p,\lambda}$ satisfies the subsolution property for* (3.6) *at $(\hat{x}, \hat{t})$.*

- *Case $p < 0$.*

  Let v_i be a nonincreasing in time lower semicontinuous supersolution of (3.6) *for every $i = 1, 2, \dots, m$. Suppose that for any fixed $(\hat{x}, \hat{t}) \in Q$, the infimum in the definition of $V_{p,\lambda}$ in* (1.26) *at $(\hat{x}, \hat{\tau})$ is attained at some $(x_i, \tau_i) \in Q_i$ for $i = 1, 2, \dots, m$, in other words,* (4.3) *and* (4.4) *hold. Then $V_{p,\lambda}$ satisfies the supersolution property for* (3.6) *at $(\hat{x}, \hat{t})$.*

Using Lemma 4.2, we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Fix $\lambda \in \Lambda$ arbitrarily. Let $U_{p,\lambda}$, v_i, and $V_{p,\lambda}$ be given respectively by (1.4), (1.22), and (1.26) ($i = 1, 2, \dots, m$). Adopting Lemma 3.1, we can show that if $p \geq 0$ then v_i is a subsolution of (3.6), while for $p < 0$ it is a supersolution of (3.6) for any $i = 1, 2, \dots, m$.

For any $(\hat{x}, \hat{t}) \in Q$, by Lemma 4.1 we see that the maximizers in the definition of $U_{p,\lambda}(\hat{x}, \hat{t})$ appear in Q_i for all $i = 1, 2, \dots, m$. This enables us to apply Lemma 4.2 with $\hat{\tau} = \hat{t}^{1/\alpha}$ to deduce that $V_{p,\lambda}$ is a subsolution or a supersolution, according to the value of p, of (3.6) with $i = \lambda$. Adopting Lemma 3.1 again yields that $U_{p,\lambda}$ is a subsolution of (1.2) with $i = \lambda$. □

We next present a proof of of Lemma 4.2.

Proof of Lemma 4.2. Let us present the proof in details in the case $p > 0$. The cases $p = 0$ and $p < 0$ can be treated similarly.

Suppose that $\phi \in C^2(\overline{Q})$ is a test function of $V_{p,\lambda}$ at $(\hat{x}, \hat{\tau}) \in Q_\lambda$, that is,

$$(V_{p,\lambda} - \phi)(x, \tau) \leq (V_{p,\lambda} - \phi)(\hat{x}, \hat{\tau}) = 0$$

for all $(x, \tau) \in \overline{Q}_0$. Due to the maximality of

$$(y_1, s_1, \dots, y_m, s_m) \mapsto \sum_{i=1}^m \lambda_i v_i(y_i, s_i) - V_{p,\lambda}\left(\sum_{i=1}^m \lambda_i y_i, \sum_{i=1}^m \lambda_i s_i\right) \tag{4.7}$$

over $\prod_{i=1}^m \overline{Q}_i$ at $(x_1, \tau_1, \ldots, x_m, \tau_m) \in \prod_{i=1}^m Q_i$, we see that

$$(y_1, s_1, \ldots, y_m, s_m) \mapsto \sum_{i=1}^m \lambda_i v_i(x_i, s_i) - \phi\left(\sum_{i=1}^m \lambda_i y_i, \sum_{i=1}^m \lambda_i s_i\right) \tag{4.8}$$

also attains a maximum over $\prod_{i=1}^m \overline{Q}_i$ at $(x_1, \tau_1, \ldots, x_m, \tau_m)$.

We next apply the Crandall-Ishii lemma [14]: For any $\varepsilon > 0$, there exist $(\eta_i, \xi_i, A_i) \in \overline{P}^{2,+} v_i(x_i, \tau_i)$ $(i = 1, 2, \ldots, m)$ such that

$$\eta_i = \partial_t \phi(\hat{x}, \hat{\tau}), \tag{4.9}$$

$$\xi_i = \nabla \phi(\hat{x}, \hat{\tau}), \tag{4.10}$$

$$\begin{pmatrix} \lambda_1 A_1 & & & \\ & \lambda_2 A_2 & & \\ & & \ddots & \\ & & & \lambda_m A_m \end{pmatrix} \leq Z + \varepsilon Z^2, \tag{4.11}$$

where Z is given by

$$Z = \begin{pmatrix} \lambda_1^2 B & \lambda_1 \lambda_2 B & \cdots & \lambda_1 \lambda_m B \\ \lambda_2 \lambda_1 B & \lambda_2^2 B & \cdots & \lambda_2 \lambda_m B \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_m \lambda_1 B & \lambda_{m-1} \lambda_2 B & \cdots & \lambda_m^2 B \end{pmatrix} \tag{4.12}$$

and $B = \nabla^2 \phi(\hat{x}, \hat{t})$. It follows that there exists $C > 0$ depending on $\|B\|$ and λ such that

$$\begin{pmatrix} \lambda_1 \tilde{A}_1 & & & \\ & \lambda_2 \tilde{A}_2 & & \\ & & \ddots & \\ & & & \lambda_m \tilde{A}_m \end{pmatrix} \leq Z, \tag{4.13}$$

where $\tilde{A}_i = A_i - C\varepsilon I$ for $i = 1, 2, \ldots, n+2$.

Adopting the time monotonicity together with (4.9), we have

$$\eta_i = \partial_t \phi(\hat{x}, \hat{\tau}) \geq 0 \quad \text{for all } i = 1, 2, \ldots, m. \tag{4.14}$$

Let us consider two different cases.

Case 1. Suppose that $\nabla \phi(\hat{x}, \hat{\tau}) \neq 0$. Then, applying the definition of subsolutions of (3.6), we have

$$v_i(x_i, \tau_i)^{\frac{1}{p}-1+k} \tau_i^{1-1/\alpha} \eta_i + \frac{p}{\alpha} G_{i,k}^{p,\alpha}(x_i, \tau_i, v_i(x_i, \tau_i), \xi_i, A_i) \leq 0. \tag{4.15}$$

Multiplying (4.15) by λ_i and summing up the inequalities, we are led to

$$\sum_i \lambda_i v_i(x_i, \tau_i)^{\frac{1}{p}-1+k} \tau_i^{1-\frac{1}{\alpha}} \eta_i + \frac{p}{\alpha} \sum_i \lambda_i G_{i,k}^{p,\alpha}(x_i, \tau_i, v_i(x_i, \tau_i), \xi_i, A_i) \leq 0,$$

which by (4.9) yields that

$$\left(\sum_i \lambda_i v_i(x_i, \tau_i)^{\frac{1}{p}-1+k} \tau_i^{1-\frac{1}{\alpha}}\right) \partial_t \phi(\hat{x}, \hat{\tau}) + \frac{p}{\alpha} \sum_i \lambda_i G_{i,k}^{p,\alpha}(x_i, \tau_i, v_i(x_i, \tau_i), \xi_i, A_i) \leq 0. \tag{4.16}$$

By (H1) we can easily verify that the function $(r,t) \mapsto r^{\frac{1}{p}-1-k} t^{1-\frac{1}{\alpha}}$ is convex in $(0,\infty)^2$, which implies that

$$\begin{aligned}\sum_i \lambda_i v_i(x_i,\tau_i)^{\frac{1}{p}-1+k} \tau_i^{1-\frac{1}{\alpha}} &\geq \left(\sum_i \lambda_i v(x_i,\tau_i)\right)^{\frac{1}{p}-1+k} \left(\sum_i \lambda_i \tau_i\right)^{1-\frac{1}{\alpha}} \\ &= V_{p,\lambda}^{\frac{1}{p}-1+k}(\hat{x},\hat{\tau})\hat{\tau}^{1-\frac{1}{\alpha}}.\end{aligned} \tag{4.17}$$

The last equality is due to (4.3) and (4.4). Using (4.14), (4.16), and (4.17), we thus obtain that

$$V_{p,\lambda}(\hat{x},\hat{\tau})^{\frac{1}{p}-1+k} \hat{\tau}^{1-\frac{1}{\alpha}} \partial_t \phi(\hat{x},\hat{\tau}) + \frac{p}{\alpha} \sum_i \lambda_i G_{i,k}^{p,\alpha}(x_i,\tau_i,v_i(x_i,\tau_i),\xi_i,A_i) \leq 0. \tag{4.18}$$

We next apply (H2) with $X_i = \tilde{A}_i = A_i - C\varepsilon I$ and $Y = B$ to deduce that

$$\sum_i \lambda_i G_{i,k}^{p,\alpha}(x_i,\tau_i,v_i(x_i,\tau_i),\xi_i,A_i - C\varepsilon I) \geq G_{\lambda,k}^{p,\alpha}(\hat{x},\hat{\tau},V_{p,\lambda}(\hat{x},\hat{\tau}),\xi_i,B).$$

It follows from the continuity of F_i (and therefore of $G_{i,k}^{p,\alpha}$) in $\overline{Q}_i \times (0,\infty) \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n$ that

$$\sum_i \lambda_i G_{i,k}^{p,\alpha}(x_i,\tau_i,v(x_i,\tau_i),\xi_i,A_i) \geq G_{\lambda,k}^{p,\alpha}(\hat{x},\hat{\tau},V_{p,\lambda}(\hat{x},\hat{\tau}),\xi_i,B) - \omega_F(\varepsilon), \tag{4.19}$$

where ω_F denotes a modulus of continuity describing the locally uniform continuity of F_i.

Plugging (4.19) into (4.18), we get

$$V_{p,\lambda}(\hat{x},\hat{\tau})^{\frac{1}{p}-1+k} \hat{\tau}^{1-\frac{1}{\alpha}} \partial_t \phi(\hat{x},\hat{\tau}) + \frac{p}{\alpha} G_{\lambda,k}^{p,\alpha}(\hat{x},\hat{\tau},V_{p,\lambda}(\hat{x},\hat{\tau}),\xi_i,B) \leq \frac{p}{\alpha}\omega_F(\varepsilon),$$

which yields, by letting $\varepsilon \to 0$, that

$$V_{p,\lambda}(\hat{x},\hat{\tau})^{\frac{1}{p}-1+k} \hat{\tau}^{1-\frac{1}{\alpha}} \partial_t \phi(\hat{x},\hat{\tau}) + \frac{p}{\alpha} G_{\lambda,k}^{p,\alpha}\left(\hat{x},\hat{\tau},V_{p,\lambda}(\hat{x},\hat{\tau}),\nabla\phi(\hat{x},\hat{\tau}),\nabla^2\phi(\hat{x},\hat{\tau})\right) \leq 0.$$

Case 2. Suppose that $\nabla\phi(\hat{x},\hat{t}) = 0$. We are able to apply Definition 2.3 for $V_{p,\lambda}$ by assuming $\nabla^2\phi(\hat{x},\hat{t}) = 0$, which by (4.11) further yields that $A_i \leq 0$ for all $i = 1,2,\ldots,m$. Using Definition 2.3 for v_i and the ellipticity of $G_{i,k}^{p,\alpha}$ with $i = 1,2,\ldots,m$ along with (3.2) and (3.3), we then have

$$v_i(x_i,\tau_i)^{\frac{1}{p}-1+k} \tau_i^{1-1/\alpha} \eta_i + \frac{p}{\alpha} \tilde{h}_i(x_i,\tau_i,v_i(x_i,\tau_i)) \leq 0.$$

Multiplying this inequality by λ_i and summing up over $i = 1,2,\ldots,n+2$, we deduce that

$$\left(\sum_i \lambda_i v_i(x_i,\tau_i)^{\frac{1}{p}-1+k} \tau_i^{1-\frac{1}{\alpha}}\right) \partial_t \phi(\hat{x},\hat{\tau}) + \frac{p}{\alpha} \sum_i \lambda_i \tilde{h}_i(x_i,\tau_i,v(x_i,\tau_i)) \leq 0.$$

Thanks to (4.17) again and (3.4), we may use (4.3) and (4.4) to conclude that

$$V_{p,\lambda}(\hat{x},\hat{\tau})^{\frac{1}{p}-1+k} \hat{\tau}^{1-\frac{1}{\alpha}} \partial_t \phi(\hat{x},\hat{\tau}) + \frac{p}{\alpha} \tilde{h}_0(\hat{x},\hat{\tau},V_{p,\lambda}(\hat{x},\hat{\tau})) \leq 0.$$

The proof of the case $p > 0$ is now complete. As we mentioned at the beginning, the proof for the cases $p = 0$ and $p < 0$ can be done similarly and we leave the details to the reader. In the latter case, several inequalities need to be changed; for example, (4.14) should be reverted and (4.13) will become

$$\begin{pmatrix} \lambda_1 \tilde{A}_1 & & & \\ & \lambda_2 \tilde{A}_2 & & \\ & & \ddots & \\ & & & \lambda_m \tilde{A}_m \end{pmatrix} \geq Z,$$

where $\tilde{A}_i = A_i + C\varepsilon I$ for $i = 1, 2, \ldots, n+2$ this time. □

5. Applications

Let us discuss applications of Theorem 1.1, Theorem 1.2, and Corollary 1.3 in this section. We will mainly verify (H1) and (H2) in Theorem 1.1 for various concrete examples of F_i. Most of our examples below satisfy the assumptions along with the conditions $1/2 \leq \alpha \leq 1$ and $p < 1$.

5.1. The Laplacian.

We are able to use Theorem 1.1 and Theorem 1.2 to recover the main results in [31, 32]. Let us first consider Theorem 1.1 when $p \neq 0$ and

$$F_i(x, t, r, \xi, X) = -\operatorname{tr} X - f_i(x, t, r, \xi),$$

where we assume that $f_i \geq 0$ and (1.6) holds for g_i given in (1.7).

Taking $k = 3 - 1/p$, we see that (H1) holds for any $1/2 \leq \alpha \leq 1$ and $0 \neq p < 1$. We can verify (H2) in this case with $k = 3 - 1/p$. Since $G_{i,k}^{p,\alpha}$ is given by (1.18) and (1.6) holds, it suffices to show that

$$\sum_i \lambda_i H(r_i, X_i) \geq H\left(\sum_i \lambda_i r_i, Y\right) \tag{5.1}$$

for $(x_i, t_i, r_i, X_i) \in Q_i \times (0, \infty) \times \mathbb{S}^n$ $(i = 1, 2, \ldots, m)$ satisfying (1.13) and (1.14), where

$$H(r, X) = -\frac{1}{p} r^2 \operatorname{tr} X, \quad (r, X) \in (0, \infty) \times \mathbb{S}^n.$$

In fact, multiplying both sides of (1.14) by $(r_1\eta, \ldots, r_m\eta) \in \mathbb{R}^{mn}$ for an arbitrary $\eta \in \mathbb{R}^n$ from left and right, we have

$$\operatorname{sgn}(p) \sum_i \lambda r_i^2 \langle X_i \eta, \eta \rangle \leq \operatorname{sgn}(p) \left(\sum_i \lambda_i r_i\right)^2 \langle Y\eta, \eta \rangle;$$

in other words,

$$\operatorname{sgn}(p) \sum_i \lambda r_i^2 X_i \leq \operatorname{sgn}(p) \left(\sum_i \lambda_i r_i\right)^2 Y.$$

Here $\operatorname{sgn}(p)$ denotes the sign of $p \in \mathbb{R}$. This immediately implies that

$$-\frac{1}{p} \sum_i \lambda r_i^2 \operatorname{tr} X_i \geq -\frac{1}{p} \left(\sum_i \lambda_i r_i\right)^2 \operatorname{tr} Y,$$

which is equivalent to (5.1).

We can further use Corollary 1.3 to obtain the parabolic power concavity of the solution. Since the operator $\overline{G}$ defined by (1.29) in this case is

$$\overline{G}(x,t,r,\xi,X) = -\frac{r^2}{p}\operatorname{tr} X - \frac{(1-p)}{p^2} r|\xi|^2 - r^{3-\frac{1}{p}} f\left(x, t^{\frac{1}{\alpha}}, r^{\frac{1}{p}}, \frac{1}{p} r^{\frac{1}{p}-1}\xi\right),$$

the assumption (H2b) in Corollary 1.3 requires concavity of (1.28) in $Q_\lambda \times (0,\infty)$.

We remark that, although the case of the Laplacian has been of course largely and deeply investigated, negative power concavity has never been considered before, to our knowledge.

We can treat the case $p = 0$ in an analogous way. When we apply Theorem 1.1 in this case, since (1.11) in (H1) is not required, we can choose $k \in \mathbb{R}$ according to the given nonlinear terms f_i in order to guarantee (H2). We may take $k = 1$ provided that (1.6) holds with g_i given by (1.19). With such a choice, we can follow the argument in the case $p > 0$ to verify (5.1) under (1.13) and (1.14), where this time we take

$$H(r, X) = -e^{2r}\operatorname{tr} X \quad \text{for} \quad (r, X) \in (0,\infty)\times\mathbb{S}^n.$$

5.2. **The normalized q-Laplacian.** We can apply our results to the normalized q-Laplacian operator with $1 < q < \infty$. Suppose that F_i is given by

$$F_i(x,t,r,\xi,X) = -\operatorname{tr}\left[\left(I + (q-2)\frac{\xi\otimes\xi}{|\xi|^2}\right)X\right] - f_i(x,t,r,\xi), \tag{5.2}$$

where $1 < q < \infty$ and $f_i \geq 0$ $(i = \lambda, 1, \ldots, m)$. We take $k = 3 - 1/p$ and assume that (1.6) holds for g_i in (1.7). Suppose that $1/2 \leq \alpha \leq 1$ and $0 \neq p < 1$. Let us verify the assumption (H2) with $p \neq 0$ again in this case.

Similar to the case $q = 2$ in Section 5.1, the key is to prove that for any fixed $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\sum_i \lambda_i H_q(r_i, \xi, X_i) \geq H_q\left(\sum_i \lambda_i r_i, \xi, Y\right) \tag{5.3}$$

holds for any $(x_i, t_i, r_i, X_i) \in Q_i \times (0,\infty)\times\mathbb{S}^n$ $(i = 1, 2, \ldots, m)$ satisfying (1.13) and (1.14), where

$$H_q(r,\xi,X) = -\frac{1}{p} r^2 \operatorname{tr}\left[\left(I + (q-2)\frac{\xi\otimes\xi}{|\xi|^2}\right)X\right]$$

for $(r,\xi,X) \in (0,\infty)\times(\mathbb{R}^n\setminus\{0\})\times\mathbb{S}^n$. To see this, we first notice that

$$M(\xi) := I + (q-2)\frac{\xi\otimes\xi}{|\xi|^2}$$

is a positive semi-definite matrix in $\mathbb{S}^n$. We thus can write

$$H_q(r,\xi,X) = -\frac{1}{p} r^2 \operatorname{tr}\left(M^{\frac{1}{2}}(\xi) X M^{\frac{1}{2}}(\xi)\right),$$

where $M^{1/2}$ is the (nonnegative) square root of M. If (1.14) holds, then by multiplying (1.14) by $\left(r_1 M^{1/2}(\xi)\eta, r_2 M^{1/2}(\xi)\eta, \ldots, r_m M^{1/2}(\xi)\eta\right) \in \mathbb{R}^{mn}$ from both sides for any $\eta \in \mathbb{R}^n$ we can obtain

$$\operatorname{sgn}(p)\sum_i \lambda_i r_i^2 \operatorname{tr}(M(\xi)X_i) \leq \operatorname{sgn}(p)\left(\sum_i \lambda_i r_i\right)^2 \operatorname{tr}(M(\xi)Y),$$

which immediately yields the desired property (5.3) for H_q.

In this case we also have the parabolic power concavity result in Corollary 1.3 provided that (1.28) is concave in $Q_\lambda \times (0,\infty)$ for any $\xi \in \mathbb{R}^n$. The operator $\overline{G}$ in (1.29) is now given by

$$\begin{aligned}\overline{G}(x,t,r,\xi,X) = -\frac{r^2}{p}\operatorname{tr}\left[\left(I + (q-2)\frac{\xi\otimes\xi}{|\xi|^2}\right)X\right] + \frac{r}{p^2}(1-p)(1-q)|\xi|^2 \\ - r^{3-\frac{1}{p}} f\left(x, t^{\frac{1}{\alpha}}, r^{\frac{1}{p}}, \frac{1}{p} r^{\frac{1}{p}-1}\xi\right).\end{aligned} \tag{5.4}$$

To show that $\overline{G}$ verifies (H2b), we again need to assume the concavity of (1.28) in $Q_\lambda \times (0,\infty)$ for any $\xi \in \mathbb{R}^n$.

We omit the discussion for the case $p = 0$, since it can be handled analogously under appropriate assumptions on f_i.

5.3. **General quasilinear operators.** We can further extend the situation in Section 5.2 to more general quasilinear operators in the form of

$$F_i(x,t,r,\xi,X) = -\operatorname{tr}(A_i(x,\xi)X) - f_i(x,t,r,\xi),$$

where $f_i \geq 0$ and $A_i(x,\xi)$ a given nonnegative matrix for any $x \in \overline{\Omega}_i$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ for all $i = \lambda, 1, \ldots, m$. Let $1/2 \leq \alpha \leq 1$ and $p < 1$. We assume that $A_i(x,\xi)$ is uniformly continuous and bounded in $\overline{\Omega}_i \times (\mathbb{R}^n \setminus \{0\})$ for all $i = 1, 2, \ldots, m$.

Let us again only consider the case $p \neq 0$. Besides the condition (1.6) with g_i in (1.7), the assumption (H2) with $k = 3 - 1/p$ requires that

$$\sum_{i=1}^m \lambda_i H_{A_i}(x_i, r_i, \xi, X_i) \geq H_{A_\lambda}\left(\sum_{i=1}^m \lambda_i x_i, \sum_{i=1}^m \lambda_i r_i, \xi, Y\right) \tag{5.5}$$

for $(x_i, t_i, r_i, X_i) \in Q_i \times (0,\infty) \times \mathbb{S}^n$ $(i = 1, 2, \ldots, m)$ satisfying (1.13) and (1.14), where

$$H_{A_i}(x,r,\xi,X) = -\frac{1}{p} r^2 \operatorname{tr}(A_i(x,\xi)X) - \frac{1-p}{p^2} r\langle A_i(x,\xi)\xi,\xi\rangle \tag{5.6}$$

for $i = \lambda, 1, \ldots, m$. This can be verified easily as in Section 5.2 when all A_i coincide and do not depend on the variable x.

As for the application of Corollary 1.3, we see that $\overline{G}$ in this case is given by

$$\overline{G}(x,t,r,\xi,X) = -r^2 \operatorname{tr}(A(x,\xi)X) - \frac{1-p}{p^2} r\langle A(x,\xi)\xi,\xi\rangle - g(x,t,r,\xi),$$

where g is as in (1.28).

Since the first term on the right hand side can be handled analogously as in Section 5.2, we omit the details. Hence, a sufficient condition to guarantee the assumption (H2b) is the concavity of

$$(x,t,r) \mapsto \frac{1-p}{p^2} r\langle A(x,\xi)\xi,\xi\rangle + r^{3-\frac{1}{p}} f\left(x, t^{\frac{1}{\alpha}}, r^{\frac{1}{p}}, \frac{1}{p} r^{\frac{1}{p}-1}\xi\right)$$

in $Q_\lambda \times (0,\infty)$ for any fixed $\xi \neq 0$. In particular, if the coefficient matrix A does not depend on x, i.e., $A = A(\xi)$, then we require

$$(x,t,r) \mapsto r^{3-\frac{1}{p}} f\left(x, t^{\frac{1}{\alpha}}, r^{\frac{1}{p}}, \frac{1}{p} r^{\frac{1}{p}-1}\xi\right)$$

is concave for any $\xi \neq 0$, as needed in the previous examples.

We remark that in addition to the normalized q-Laplacian discussed in Section 5.2, applicable quasilinear operators also include the so-called Finsler Laplacian as a special case. Recall that the Finsler-Laplace operator is defined by

$$\mathcal{F}u = -\operatorname{div}(J(\nabla u)\nabla J(\nabla u)),$$

where $J : \mathbb{R}^n \to \mathbb{R}$ is a given nonnegative convex function of class $C^2(\mathbb{R}^n \setminus \{0\})$ which is positively homogeneous of degree 1, i.e., $J(k\xi) = |k|J(\xi)$ for all $k \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. We can write

$$\mathcal{F}u = -\operatorname{tr}\left(A_J(\nabla u)\nabla^2 u\right), \quad \text{where} \quad A_J(\xi) = \frac{1}{2}\nabla^2 J^2(\xi).$$

The homogeneity and regularity of the function J imply that the coefficient matrix A_J is bounded and continuous in $\mathbb{R}^n \setminus \{0\}$.

It is now easily seen that Theorem 1.1 does apply to the equations with

$$F_i(x, t, r, \xi, X) = -\operatorname{tr}\left(A_J(\xi)X\right) - f_i(x, t, \xi), \qquad i = \lambda, 1, \ldots, m.$$

Note that the boundedness and continuity of A_J in $\mathbb{R}^n \setminus \{0\}$ enable us to apply the standard viscosity theory to equations involving $\mathcal{F}$; see basic structure assumptions (F1)–(F5) in Appendix A.1 for well-posedness.

Moreover, since in this case H_{A_i} in (5.6) is given by

$$H_{A_i}(x, r, \xi, X) = -\frac{1}{p}r^2 \operatorname{tr}(A_J(\xi)X) - \frac{1-p}{p^2}r\langle A_J(\xi)\xi, \xi\rangle,$$

for $i = \lambda, 1, \ldots, m$, we can show that (5.5) holds for $(x_i, t_i, r_i, X_i) \in Q_i \times (0, \infty) \times \mathbb{S}^n$ ($i = 1, 2, \ldots, m$) satisfying (1.13) and (1.14), due to the convexity and nonnegativity of J.

One can use a similar argument to justify the application of Corollary 1.3 to the Finsler Laplacian.

5.4. **The Pucci operator.** A typical example of fully nonlinear operators is the Pucci operator

$$\mathcal{M}^-_{a,b}(X) = \inf_{aI \le A \le bI} \operatorname{tr}(AX) = a\sum_{e_i \ge 0} ae_i + b\sum_{e_i < 0} be_i,$$

where $0 < a \le b$ are given and $e_i = e_i(X)$ denotes the eigenvalues of any $X \in \mathbb{S}^n$.

Consider

$$F_i(x, t, r, \xi, X) = -\mathcal{M}^-_{a,b}(X) - f_i(x, t, r, \xi) \tag{5.7}$$

for $(x, t) \in \overline{Q}$, $r \in (0, \infty)$, $\xi \in \mathbb{R}^n$, and $X \in \mathbb{S}^n$. As in the examples in Section 5.1 and Section 5.2, we again assume that f_i is nonnegative and satisfies the relation (1.6) with g_i defined in (1.7).

Assume that $1/2 \le \alpha \le 1$, $p < 1$, and $p \ne 0$ so that (H1) holds with $k = 3 - 1/p$. With such a choice of k, we can also verify (H2). In fact, the operator $G_{i,3-1/p}$ in this case reads

$$G_{i,3-1/p}(x, t, r, \xi, X) = \sup_{aI \le A \le bI} H_A(r, \xi, X) - g_i(x, t, r, \xi),$$

where

$$H_A(r, \xi, X) = -\frac{r^2}{p}\operatorname{tr}(AX) - \frac{(1-p)r}{p^2}\langle A\xi, \xi\rangle.$$

As shown in Section 5.3, for any fixed $A \in \mathbb{S}^n$ such that $aI \le A \le bI$ and $\lambda \in \Lambda_m$, by (1.6) we have

$$\sum_i \{\lambda_i H_A(r_i, \xi, X_i) - \lambda_i g_i(x_i, t_i, r_i, \xi)\}$$
$$\ge H_A\left(\sum_i \lambda_i r_i, \xi, Y\right) - g_0\left(\sum_i \lambda_i x_i, \sum_i \lambda_i t_i, \sum_i \lambda_i r_i, \xi\right)$$

for any $(x_i, t_i, r_i, X_i) \in Q_i \times (0,\infty) \times \mathbb{S}^n$ $(i = 1, 2, \dots, m)$ satisfying (1.13)–(1.14). Maximizing both sides over $aI \le A \le bI$, we are led to

$$\sum_i \left\{\lambda_i \sup_{aI \le A \le bI} H_A(r_i, \xi, X_i) - \lambda_i g_i(x_i, t_i, r_i, \xi)\right\}$$
$$\ge \sup_{aI \le A \le bI} H_A\left(\sum_i \lambda_i r_i, \xi, Y\right) - g_0\left(\sum_i \lambda_i x_i, \sum_i \lambda_i t_i, \sum_i \lambda_i r_i, \xi\right),$$

which completes the verification of (H2). Similar applications can be obtained in the case $p = 0$. One needs to fix $k \in \mathbb{R}$ in accordance with assumptions on f_i $(i = \lambda, 1, 2, \dots, m)$.

We can therefore use Corollary 1.3 to give a corresponding parabolic power concavity result. Suppose that f is a given nonnegative continuous function and (1.28) is concave with respect to (x, t, r). Noticing that $\overline{G}$ in (1.29) in this case is

$$\overline{G}(x, t, r, \xi, X) = \sup_{aI \le A \le bI} H_A(r, \xi, X) - g(x, t, r, \xi), \tag{5.8}$$

we can show that it satisfies (H2b).

We remark that although the result of Theorem 1.1 holds for the operator in (5.7), in general, it may not apply to the other type of Pucci operator, which reads

$$\mathcal{M}^+_{a,b}(X) = \sup_{aI \le A \le bI} \operatorname{tr}(AX) = a\sum_{e_i \le 0} ae_i + b\sum_{e_i > 0} be_i, \quad X \in \mathbb{S}^n.$$

Note that $-\mathcal{M}^-_{a,b}(X)$ is convex in X but $-\mathcal{M}^+_{a,b}(X)$ is concave.

5.5. **Porous medium equation.** We also show an application of our concavity result to the porous medium equation. Suppose that the equation (1.2) reduces to

$$\partial_t u - \Delta(u^\sigma) = f_i(x, t, u, \nabla u) \quad \text{in } \Omega \times (0, \infty)$$

for a given $\sigma > 1$ and $f_i \ge 0$ satisfying assumptions to be specified later. In this case, the elliptic operator F_i becomes

$$F_i(x, t, r, \xi, X) = -\sigma r^{\sigma-1} \operatorname{tr} X - \sigma(\sigma - 1) r^{\sigma-2} |\xi|^2 - f_i(x, t, r, \xi).$$

The situation for this operator is different from the previous applications. In the case $p \ne 0$, we are actually not able to apply Theorem 1.1 with $k = 3 - 1/p$ or obtain a corresponding concavity result through Corollary 1.3, since our operators hardly satisfy (H2) no matter what assumption is imposed on f_i. Instead, we use Theorem 1.1 with

$$k = \frac{\sigma}{p} - 3 \tag{5.9}$$

so as to meet the requirement (H2). Note that due to the choice of k as in (5.9), we have

$$G^{p,\alpha}_{i,k}(x, t, r, \xi, X) = -\frac{\sigma}{p} r^2 \operatorname{tr} X - \frac{\sigma(\sigma - p)}{p^2} r|\xi|^2 - r^{3-\frac{\sigma}{p}} f_i\left(x, t^{\frac{1}{\alpha}}, r^{\frac{1}{p}}, \frac{1}{p} r^{\frac{1}{p}-1} \xi\right),$$

where the major part (the first and second terms) is convex in the sense of (H2). (They are in fact the same as those major terms in the previous examples.) This is precisely the reason why we chose k as in (5.9).

Moreover, the assumption on f_i is still as in (1.6) but with g_i given by

$$g_i(x,t,r) = r^{3-\frac{\sigma}{p}} f_i\left(x, t^{\frac{1}{\alpha}}, r^{\frac{1}{p}}, \frac{1}{p} r^{\frac{1}{p}-1}\xi\right).$$

In order to meet the condition (H1), we need to additionally assume that

$$\text{either} \quad 2p-\sigma+1 \leq 0 \quad \text{or} \quad \frac{p}{2p-\sigma+1} \leq \alpha \leq 1. \tag{5.10}$$

We again omit the details for the case $p = 0$.

For the concavity result in Theorem 1.2, we can make the same choice of parameters k, α, and p as in (5.9) and (5.10), and assume that

$$(x,t,r) \mapsto r^{3-\frac{\sigma}{p}} f\left(x, t^{\frac{1}{\alpha}}, r^{\frac{1}{p}}, \frac{1}{p} r^{\frac{1}{p}-1}\xi\right) \tag{5.11}$$

is concave for any ξ.

Remark 5.1. The condition (5.10) means that, for any given $\sigma > 1$, in order to obtain parabolic power concavity with $p \in (-\infty, 1)$ satisfying $\sigma \geq 2p+1$, we basically have no restrictions on $\alpha \in (0,1]$ except for (5.11) when $p \neq 0$ (or a variant of (5.11) when $p = 0$). On the other hand, for $p \in (-\infty, 1)$ fulfilling $\sigma < 2p+1$, (5.10) requires that $p \geq \sigma - 1$ so as to allow room for α; there is no result for the range $(\sigma-1)/2 < p < \sigma - 1$ no matter what α is taken.

Note that when $\sigma = 1$ the conditions (5.11) and (5.10) reduce to the assumptions for the Laplacian case as discussed in Section 5.1.

Appendix A. Related issues on viscosity solutions

Let us discuss several important properties of viscosity solutions, which are used in the previous sections. We first review well-posedness for general fully nonlinear parabolic equations and then give sufficient conditions for the time monotonicity of solutions and a Hopf-type property.

A.1. **Well-posedness.** Let Ω be a bounded smooth domain. We below consider the equation (2.3) in $Q = \Omega \times (0,\infty)$ with a homogeneous initial boundary condition, i.e.,

$$\begin{cases} \partial_t u + F(x,t,u,\nabla u,\nabla^2 u) = 0 & \text{in } Q, \quad \text{(A.1)} \\ u = 0 & \text{on } \partial Q. \quad \text{(A.2)} \end{cases}$$

We denote by ν the inward unit normal vector to $\partial\Omega$. Set $\tilde{\nu}(x) = \nu(x)$ if $x \in \partial\Omega$ and $\tilde{\nu}(x) = 0$ if $x \in \Omega$.

Let us impose the following basic structure assumptions on F:

- (F1) $F : \overline{\Omega} \times [0,\infty) \times \mathbb{R} \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n \to \mathbb{R}$ is continuous.
- (F2) F is degenerate elliptic, i.e., (2.4) holds for all $x \in \overline{\Omega}_i$, $t \in [0,\infty)$, $r \in [0,\infty)$, $\xi \in \mathbb{R}^n \setminus \{0\}$, and $X_1, X_2 \in \mathbb{S}^n$ satisfying $X_1 \geq X_2$.
- (F3) F is monotone in the sense that there exists $c \in \mathbb{R}$ such that (2.5) holds for all $(x,t,\xi,X) \in \overline{\Omega}_i \times [0,\infty) \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n$ and $r_1, r_2 \in [0,\infty)$ satisfying $r_1 \leq r_2$.

(F4) For any $R > 0$, there exists a modulus of continuity ω_R such that

$$|F(x,t,r,\xi,X) - F(y,t,r,\xi,X)| \leq \omega_R(|x-y|(|\xi|+1))$$

for all $x, y \in \overline{\Omega}$, $t \in [0,\infty)$, $|r| \leq R$, $\xi \in \mathbb{R}^n \setminus \{0\}$, and $X \in \mathbb{S}^n$.

(F5) There exists a continuous function $h : \overline{Q} \times [0,\infty) \to \mathbb{R}$ such that

$$h(x,t,r) = (F)_*(x,t,r,0,0) = (F)^*(x,t,r,0,0) \quad \text{for } (x,t,r) \in Q \times (0,\infty). \tag{A.3}$$

Under these assumptions, viscosity solutions (sub- and supersolutions) of (A.1) are defined as in Section 2.2. It is known that the following comparison theorem holds.

Theorem A.1 (Theorem 3.6.1 in [19])**.** *Assume that Ω is bounded and* (F1)–(F5) *hold. Let u and v be respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution of* (A.1)*. If $u \leq v$ on ∂Q, then $u \leq v$ in $\overline{Q}$.*

Uniqueness of viscosity solutions to (A.1)–(A.2) is an immediate consequence of Theorem A.1. One can obtain existence of a positive viscosity solution by adopting Perron's method for viscosity solutions if a positive subsolution exists; see precise arguments in [33] and [14, Section 4] for nonsingular equations and [19, Section 2.4] for singular one.

Moreover, a standard argument ([19, Theorem 2.2.1] for instance) yields that the unique solution u is stable in the sup norm under uniform perturbation of the operator and initial boundary data.

As for the spatial Lipschitz regularity, which is needed in Theorem 1.1, we refer to relevant results in the literature. Lipschitz or Hölder regularity of viscosity solutions to fully nonlinear nonsingular parabolic equations is given in [3, 47, 48, 38, 5, 4] etc. We also consult local Lipschitz estimates for singular parabolic equations such as the normalized q-Laplace equations in [17, 43, 34] ($1 < q < \infty$) and in [36] ($q = \infty$).

A.2. **Monotonicity in time.** The next two subsections are devoted to discussion on the assumptions (i) and (ii) in Theorem 1.1 (and in Theorem 1.2) for $p \in (0,1)$. Since it is in general quite restrictive to assume (i) and (ii) on F_i, we consider the approximate equation (1.21). We will actually provide sufficient conditions to guarantee (i) and (ii) for (1.21) instead of (1.2).

Let us first study the time monotonicity in (i). Suppose that

$$h(x,0,0) = F_*(x,0,0,0,0) = F^*(x,0,0,0,0) \leq 0, \tag{A.4}$$

$$\begin{aligned} &F(x,t,r,p,X) \leq F(x,s,r,p,X) \\ &\text{for any } t \geq s \geq 0 \text{ and } (x,r,p,X) \in \overline{\Omega} \times [0,\infty) \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n. \end{aligned} \tag{A.5}$$

Lemma A.2 (Monotonicity in time)**.** *Assume that Ω is bounded and F satisfies* (F1)–(F5)*. If* (A.4) *and* (A.5) *hold, then the unique solution u of* (1.2)–(1.3) *satisfies* (1.16)*.*

Proof. The assumptions (A.4) and (A.5) imply that the constant zero is a subsolution of (1.2)–(1.3). If follows that $u \geq 0$ in $\overline{Q}$ by the comparison principle. Fix $\tau > 0$ arbitrarily and set $w_\tau(x,t) = u(x,t+\tau)$ for all $(x,t) \in \overline{Q}$. Then by (A.5) we can easily show that w_τ is a supersolution of (1.2). Since $w_\tau(\cdot,0) = u(\cdot,\tau) \geq u(\cdot,0)$ in $\overline{\Omega}$, we can use the comparison principle again to prove that $w_\tau \geq u$ in $\overline{Q}$, which immediately yields (1.16) due to the arbitrariness of τ. □

A more specific situation fulfilling (A.4), (A.5), and other assumptions needed in our main results is the case when

$$F(x,t,r,\xi,X)=\mathcal{L}(\xi,X)-f(x,t,r)$$

for all $(x,t,r,\xi,X)\in\overline{Q}\times[0,\infty)\times(\mathbb{R}^n\setminus\{0\})\times\mathbb{S}^n$, where f is nonnegative in $\overline{Q}\times[0,\infty)$ and nondecreasing in t, and $\mathcal{L}_*(0,0)=\mathcal{L}^*(0,0)=0$. Concrete examples of $\mathcal{L}$ include the Pucci operator, normalized q-Laplacian ($1<q\leq\infty$), and more general quasilinear operators as discussed in Section 5.

We next discuss the assumption (ii) in Theorem 1.1 and Theorem 1.2. Assume that $0<p<1$ for the rest of this section. Note that the condition (1.17) can be divided into two parts. One part is the following growth behavior near the initial moment:

$$\frac{1}{\rho}u^p\left(x+\tilde{\nu}(x)\rho,\rho^{1/\alpha}\right)\to\infty\quad\text{as }\rho\to 0\text{ for every }x\in\overline{\Omega}. \tag{A.6}$$

The other part can be expressed by

$$\frac{1}{\rho}u^p\left(x+\nu(x)\rho,t\right)\to\infty\quad\text{as }\rho\to 0\text{ for every }x\in\partial\Omega\text{ and }t>0. \tag{A.7}$$

We will later see that for $p\in(0,1)$ (A.7) is a consequence of the Hopf lemma; consult Section A.3.

In order to obtain (A.6), we need to strengthen the condition (A.4) in Lemma A.2.

Lemma A.3 (Rapid initial growth)**.** *Let $0<p<1$. Assume that Ω is bounded and F satisfies* (F1)–(F5)*. Assume that* (A.5) *holds. Assume that*

$$\begin{cases}\text{there exist }\beta,\beta',t_0>0\text{ and }\psi_0\in C^2(\overline{\Omega})\text{ with }\psi_0>0\text{ in }\Omega\text{ and }\psi_0=0\text{ on }\partial\Omega\\ \text{such that}\quad p\beta'+\dfrac{p\beta}{\alpha}<1,\quad \sup_{\Omega}\dfrac{\operatorname{dist}(\cdot,\partial\Omega)^{\beta'}}{\psi_0}<\infty,\quad\text{and}\\ \beta\psi_0(x)t^{\beta-1}+F_*\left(x,t,\psi_0(x)t^\beta,\nabla\psi_0(x)t^\beta,\nabla^2\psi_0(x)t^\beta\right)\leq 0\quad\text{in }\Omega\times(0,t_0).\end{cases} \tag{A.8}$$

Then the unique solution u of (1.2)–(1.3) *satisfies* (A.6)*.*

Proof. By the last inequality in (A.8) we observe that the function $(x,t)\mapsto\psi_0(x)t^\beta$ is a subsolution of (1.2) restricted in $\Omega\times(0,t_0)$. Noticing that $\psi_0=0$ on $\partial\Omega$, we can use the comparison principle to obtain that $u(x,t)\geq\psi_0(x)t^\beta$ for $(x,t)\in\Omega\times(0,t_0)$. When $x\in\Omega$, we easily deduce (A.6), since $\psi_0>0$ in Ω and $\beta<\alpha/p$.

If $x\in\partial\Omega$, noticing that $\psi_0(x+\rho\nu(x))\geq c\rho^{\beta'}$ in Ω for some $c>0$, we have

$$u^p\left(x+\rho\nu(x),\rho^{1/\alpha}\right)\geq c^p\rho^{p\beta'+p\beta/\alpha},$$

which implies (A.6), due to the condition that $p\beta'+p\beta/\alpha<1$. □

Let us discuss how to apply Lemma A.3 in our applications under the assumption $\alpha\geq p$. Suppose that F_i ($i=\lambda,1,\ldots,m$) satisfies (A.4) and (A.5), i.e.,

$$\begin{aligned}&h_i(x,0,0)=(F_i)*(x,0,0,0,0)=(F_i)^*(x,0,0,0,0)\leq 0,\\ &F_i(x,t,r,p,X)\leq F_i(x,s,r,p,X)\\ &\text{for any }t\geq s\geq 0\text{ and }(x,r,p,X)\in\overline{\Omega}_i\times[0,\infty)\times(\mathbb{R}^n\setminus\{0\})\times\mathbb{S}^n.\end{aligned}$$

As mentioned in the beginning of this section, these assumptions in general may not guarantee (A.8) for $F=F_i$. However, we can first turn to study the perturbed equation (1.21) first and then let $\varepsilon\to 0$. In addition to the perturbation for the operators, we put $p_\varepsilon=p-\varepsilon$

with $\varepsilon > 0$ small so that $\alpha > p_\varepsilon$. We can show (A.8) holds for $p = p_\varepsilon$ and $F = F_{i,\varepsilon}$ for any $i = \lambda, 1, \ldots, m$. Indeed, we can choose

$$1 < \beta < \frac{\alpha}{p_\varepsilon}, \quad 0 < \beta' < 1 - \frac{\beta p_\varepsilon}{\alpha},$$

and $\psi_0 \in C^2(\overline{\Omega})$ such that $\psi_0 = \operatorname{dist}(\cdot, \partial\Omega)^{\beta'}$ near $\partial\Omega$. Then we can verify the last inequality in (A.8) with $F = F_{i,\varepsilon}$ and $p = p_\varepsilon$ provided that t_0 is sufficiently small.

A.3. **The Hopf-type property.** We finally discuss the property (A.7), which is used to derive the condition (ii) of Theorem 1.1. It is in fact related to the so-called Hopf-type property:

(HP) Fix any $x_0 \in \partial\Omega$ and $t_0 > 0$. Assume that there exist $0 < \delta < t_0$ and $y_0 \in \Omega$ such that

- $B_\delta(y_0) \subset \Omega$ and $\overline{B_\delta(y_0)} \cap \partial\Omega = \{x_0\}$;
- u is a supersolution of (A.1);
- u satisfies $u(x,t) > u(x_0, t_0) = 0$ for any $(x,t) \in B_\delta(y_0) \times [t_0 - \delta, t_0]$.

Then

$$\liminf_{\rho \to 0+} \frac{1}{\rho} u\left(x_0 + \rho \frac{y_0 - x_0}{|y_0 - x_0|}, t_0\right) > 0. \tag{A.9}$$

It is obvious that (A.7) is an immediate consequence of (A.9) when $0 < p < 1$. See [16, 21, 9] for sufficient conditions on F in order to obtain (HP).

For our own purpose in this work, following the same method described in Section A.2, we use (A.9) for the approximate problem (1.21), where $F_{i,\varepsilon}$ is the perturbed operator given in (1.20). It turns out that we still only need (A.4) and (A.5) on $F = F_i$ to show (A.9) for a supersolution of (1.21).

Note that (A.4) and (A.5) imply that

$$h_i(x,t,0) = (F_i)_*(x,t,0,0,0) = (F_i)^*(x,t,0,0,0) \leq 0 \quad \text{for all } (x,t) \in \overline{Q}_i. \tag{A.10}$$

Let $u_{i,\varepsilon}$ be a supersolution of (1.21). Since the constant zero is a subsolution, we have $u_{i,\varepsilon} \geq 0$ in $\overline{\Omega}_i \times [0,\infty)$. Denote $\zeta_0 = (y_0, t_0)$ and $z = (x,t)$. We take

$$v_\gamma(z) = e^{-\gamma|z - \zeta_0|^2} - e^{-\gamma\delta^2}$$

with $\gamma > 0$ large. Since

$$\sup_{\overline{B_\delta(y_0)} \times [t_0 - \delta, t_0 + \delta]} \left(|v_\gamma| + |\partial_t v_\gamma| + |\nabla v_\gamma| + |\nabla^2 v_\gamma|\right) \to 0 \qquad \text{as } \gamma \to \infty,$$

it follows from (A.10) and (F5) that, when $\gamma > 0$ is large,

$$\begin{aligned} &\partial_t v_\gamma(z) + (F_{i,\varepsilon})_*(z, v_\gamma(z), \nabla v_\gamma(z), \nabla^2 v_\gamma(z)) \\ &\leq \partial_t v_\gamma(z) + (F_i)_*(z, v_\gamma(z), \nabla v_\gamma(z), \nabla^2 v_\gamma(z)) - \varepsilon \leq -\frac{\varepsilon}{2} \end{aligned}$$

for any $z \in B_\delta(y_0) \times (t_0 - \delta, t_0 + \delta)$.

We have shown that v_γ is a subsolution of (1.21) in $B_\delta(y_0) \times (t_0 - \delta, t_0)$. Noticing that

$$u_{i,\varepsilon} \geq 0 \geq v_\gamma \quad \text{on } \left(\overline{B_\delta(y_0)} \times \{t_0 - \delta\}\right) \cup (\partial B_\delta(y_0) \times (t_0 - \delta, t_0 + \delta)),$$

by comparison principle we have

$$u_{i,\varepsilon} \geq v_\gamma \quad \text{in } \overline{B_\delta(y_0)} \times [t_0 - \delta, t_0 + \delta],$$

which implies that

$$u_{i,\varepsilon}\left(x_0+\rho\frac{y_0-x_0}{|y_0-x_0|},t_0\right)\geq v_\gamma\left(x_0+\rho\frac{y_0-x_0}{|y_0-x_0|},t_0\right)\geq \rho\gamma|x_0-y_0|e^{-\gamma|x_0-y_0|^2}+o(\rho).$$

We thus complete the proof of (A.9) for any supersolution $u_{i,\varepsilon}$ of (1.21).

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