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Nilpotence relations in products of groups

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A tutti coloro che avrebbero desiderato studiare,
ma non ne hanno avuto l'opportunità

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Introduction

This work is an attempt to combine two themes in group theory, in particular in the infinite group theory. The first subject concerns some types of relations between two subgroups of a given group. In particular, given a class of groups $\mathcal{C}$ and a group G, we will say that two subgroups A and B of G are $\mathcal{C}$-connected if and only if for all $a \in A$ and for all $b \in B$, the subgroup $\langle a, b \rangle$ belongs to the class $\mathcal{C}$. This definition, that is central in this work, was formulated for the first time by Carocca in [14].

As many authors have reconstructed, for instance, in [32] or [13], the origin of the definition is to be attributed to the study of totally permutable groups. In fact, it is easy to show that given A and B two totally permutable subgroups, for all elements $a \in A$ and $b \in B$, the group $\langle a, b \rangle$ is supersolvable, or in other words total permutability implies what later has been called $\mathcal{U}$-connection, where $\mathcal{U}$ is the class of supersolvable groups. From this consideration Carocca investigated $\mathcal{C}$-connection in finite groups, giving two results about $\mathcal{N}$-connection and $\mathcal{S}$-connection, where, $\mathcal{N}$ is the class of nilpotent groups and $\mathcal{S}$ that of solvable groups. In this work, however, we deal only with $\mathcal{N}$-connection.

In the literature, the results about $\mathcal{N}$-connection mainly concern products of finite groups. An example where the authors did not assume G to be a product is in [27]. However, we are able to find examples that, in our opinion, discourage the continuation of the research in that direction. Briefly, the examples we will show are two groups, i.e. the first Grigorchuk group and S_8, that are generated by two subgroups respectively $\mathcal{N}_4$-connected and $\mathcal{N}_3$-connected and these groups lose most of the property of the generating subgroups.

Furthermore, the fact that in the finite case we know many properties of factorized $\mathcal{N}$-connected groups, confers both motivations and interest in the attempt to generalize some results to infinite groups.

For these reasons the second subject we studied is the groups product theory. We will say that G is a product of its subgroups $A, B \leq G$, i.e. $G = AB$, if

$$G = \{ \ ab \ \mid \ a \in A, \ b \in B \ \}$$

In this theory there are many problems that have been studied for a long time. To give some examples, we cite the existence of a non-trivial factorization of a group, or, in other words, the fact that given a group G there exist (or not) two subgroups $A \neq 1$ and $B \neq 1$, such that $G = AB$. Another interesting issue, given a factorized group $G = AB$, is the existence of a normal group N in G that is contained in A or in B. To conclude this list we cite also the existence of a factorization for a given subgroup of a factorized group. A collection of these and other problems on groups product theory are available for instance in the *Kourovka Notebook* [39]. Anyway, the main question in the study of factorized groups is what we call structure problem, as follows:

Given $G = AB$ a factorized group where A and B are subgroups; suppose that A and B satisfies a certain property $\mathcal{P}$, what can be said about the whole structure of the group G?

Two theorems that are milestones in this field are the theorem of Itô [33] and the theorem of Kegel [36] and Wielandt [55]. The first is about the product of two abelian subgroups that, with an easy and very clever calculation, is proved to be metabelian. The second is based on the famous Burnside $p^\alpha q^\beta$ theorem, proving that the product of two finite nilpotent subgroups is solvable.

Moreover, it is also interesting to give negative answers to the structure problems discovering, identifying and sometimes conceiving group constructions to show that, with certain given hypothesis on the factors, it is not possible to prove an established property for the whole group. A classical example is the theorem of Suchkov [52] that proved the existence of a countable group $G = AB$ where A and B are locally finite while G contains a free group of infinite rank.

Coming back to the structure problem and assuming other hypotheses such as, for instance, some solvability conditions on G, we have many remarkable results. For some detailed surveys on this kind of topic, we suggest those of Kazarin and Kurdachenko [35] or Amberg [4].

This brief overview makes it clear that products of groups are in general difficult to manage. So the introduction of some nilpotency relations between the factors is somehow reasonable. In particular, the fact that in the finite case such nilpotency relations, the so-called $\mathcal{N}$-connection, work well as the results in [13] or [32] witness, led us to study the products of groups with those additional properties.

What we essentially did is to consider a group G factorized by two $\mathcal{N}$-connected subgroups A and B that satisfy a certain property $\mathcal{P}$. Our purpose had been to establish as much as possible about the structure of G. We succeeded in proving three non-trivial results about, respectively, three classes of solvable groups with some finiteness conditions assigned to the factors A e B. The classes in question are the class of supersolvable groups that is a subclass of solvable groups satisfying the maximal condition on subgroups. Furthermore the class of Černikov groups that is an important class of groups satisfying the minimal condition on subgroups.

The third is the class of hypercentral minimax groups.

Our work is divided into five chapters. The first is an introductive chapter in which we present several topics in group theory, starting from elementary facts and definitions to cover many themes about finiteness conditions; then we introduce the most important classes of generalized nilpotent groups. The scope is mainly to give a quick reference and to fix the notations.

The second chapter has, similarly, an introductive nature; in it we focus on the theory of groups product. We start presenting the main definitions and lemmas following the introductive chapter of [1]. Moreover we show some results about the factorizations of certain normalizers of subgroups of a factorized group. In particular we proved 2.1.6 and 2.1.8, that are two non-trivial lemmas, that we used, in our proofs, to construct certain series of factorized groups. Finally we present some issues on the theory of groups.

The third chapter is completely devoted to $\mathcal{C}$-connection. We list the principal classes of groups that are mainly involved in the connection, we try to give a more precise idea of the origin of the definition and we try to do a survey and illustrate the state of the art.

In the fourth chapter, we show some properties that are particular to the $\mathcal{N}$-connection and that we investigated. In particular, we present properties about groups that somehow belong to the infinite groups theory with a special regard to local nilpotency. First of all, we show that in a factorized group $G = AB$, under $\mathcal{N}$-connection and few other hypotheses, both the torsion group and the isolator group of subgroup containing $A \cap B$ factorize. Then we point out how in a product G of $\mathcal{N}$-connected locally nilpotent subgroups, this $FC(G) \leq Z_\infty(G)$ holds. Furthermore, we are able to generalize one of his theorems involving FC-hypercentre and hypercentre in locally nilpotent groups to the product of $\mathcal{N}$-connected locally nilpotent subgroups. The last section is dedicated to some examples of $\mathcal{N}$-connected subgroups.

The last chapter is the main part of this thesis. Essentially here we present the statements and the detailed proofs we demonstrated. We start with two solvable products of groups. The first proposition is to continue with the pattern laid out about abelian factorization. It turns out that the theorem of Itô together with $\mathcal{N}$-connection simplify the description of these groups. The second result on solvable factorization is however interesting for the generality in which it is done and it turns out to be convenient for many proofs we did in the rest of the chapter, because it essentially gives local nilpotence that is not, in general, an easy property to verify. In the following section we deal with a product of supersolvable $\mathcal{N}$-connected subgroups. We prove that this product comes out to be supersolvable. The proof is done essentially by induction on the Hirsch length and we use several properties characterizing this class. After that, we continue our study on products of groups satisfying chain condition approaching Černikov groups. We demonstrate that a product of two $\mathcal{N}$-connected Černikov subgroups is Černikov. The proof is divided in the case of product of Černikov p-subgroups and then the general case. The third result is about the product of $\mathcal{N}$-connected hypercentral minimax subgroup. This proof is also divided in two parts, the first assuming that the set of torsion elements is trivial, the second assuming that the isolator of the intersection of the factor is the whole group. We would like to stress the fact that in all the three proofs the properties of the factors are transmitted to the group, so this means that we are able to generalize all these results to a finite number of pairwise permutable $\mathcal{N}$-connected subgroups. At the end of the chapter we present some examples and remarks to conclude.

Chapter 1

Basic concepts

This first chapter of the thesis is conceived to introduce and fix the notations, recall the principal definitions and the main standard results we will need throughout the rest of the work. The topics we introduce are basic facts of group theory and they are available in many text books. The main objective is, actually, to give a quick reference for the readers that need some explanations and to attempt to offer a self-contained structure of the work.

1.1 Commutators

Let G be a group and let x, y be elements of G. The conjugate of x by y is denoted by $x^y = y^{-1}xy$, while the commutator of x and y is defined as

$$[x, y] = x^{-1}y^{-1}xy$$

We can also define commutators involving more than two elements. In fact, given $n \in \mathbb{N}$, $n \geq 2$ and $x_1, \ldots, x_n \in G$, we define single commutator of weight n recursively as follows

$$[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$$

Another symbol it is possible to find very often is the iterated commutator $[x,_n y]$ that is recursively defined as

$$[x,_1 y] = [x, y]$$

and

$$[x,_n y] = [[x,_{n-1} y], y]$$

The next lemma summarizes the main properties of commutators and manipulations in commutator calculus.

Lemma 1.1.1. *Let G be a group and x, y, z elements of G. Then:*

1) $[x, y]^{-1} = [y, x]$;

2) $[xy, z] = [x, z]^y[y, z]$ *and* $[x, yz] = [x, z][x, y]^z$;

3) $[x, y^{-1}] = \left([x, y]^{y^{-1}}\right)^{-1}$ *and* $[x^{-1}, y] = \left([x, y]^{x^{-1}}\right)^{-1}$;

4) $[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1$ *(the Hall-Witt identity).*

Then, one proves the following simple formulas.

Lemma 1.1.2. *Let G be a group and x, y, z elements of G and $n \in \mathbb{N}$, here $x^{y+z} = x^y x^z$. Then:*

$$[x^n, y] = [x, y]^{x^{n-1}+x^{n-2}+\ldots+x+1}$$

Moreover if $[x, y, x] = 1$ and $[x, y, y] = 1$, we get

$$(xy)^n = x^n y^n [y, x]^{\binom{n}{2}}.$$

If X and Y are non empty subset of the group G, we set

$$X^Y = \langle x^y \mid x \in X,\ y \in Y\rangle$$

and

$$[X, Y] = \langle [x, y] \mid x \in X,\ y \in Y\rangle$$

From this last definition we can define recursively for all $X_1, \ldots, X_n \subseteq G$

$$[X_1, \ldots, X_n] = [[X_1, \ldots, X_{n-1}], X_n]$$

and

$$[X,_n Y] = [[X,_{n-1} Y], Y]$$

Given a group G we define the derived subgroup G' such as $[G, G]$. We define inductively $G^{(1)} = G'$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$, $n \in \mathbb{N}$. These subgroups are all characteristic and they form the derived series. The group G is said to be solvable if there exists $t \in \mathbb{N}$ such that $G^{(t)} = 1$. The smallest t for which this fact holds is called derived length.

Using commutators it is possible to define another fundamental series of characteristic subgroups of a group G. Inductively we call $\gamma_1(G) = G$, and for $1 \leq n \in \mathbb{N}$ we define $\gamma_{n+1}(G) = [\gamma_n(G), G] = [G,_n G]$. A group G is said to be nilpotent if there exist some t such that $\gamma_{t+1}(G) = 1$. The smallest t for which this fact holds is called the (nilpotency) class of G.

Another interesting series that we can define is the following: $\zeta_1(G) = Z(G)$ the centre of G, and for $n \geq 2$, $\zeta_n(G)$ is defined by

$$\zeta_n(G)/\zeta_{n-1}(G) = Z(G/\zeta_{n-1}(G))$$

It is important to recall that for $n \geq 1$, we get that $G = \zeta_n(G)$ if and only if $\gamma_{n+1}(G) = 1$.

Given a group G and two subgroups H and K such that $H \lhd K$ we will say that the factor K/H is a central section if $[K, G] \leq H$ or equivalently if $H \lhd G$ and $K/H \leq Z(G/H)$. Given G a group a series of G

$$1 = G_0 \leq G_1 \leq \ldots \leq G_{n-1} \leq G_n$$

is said to be central if for all $i : 1, \ldots, n$, $[G_i, G] \leq G_{i-1}$. In particular we get this result:

Lemma 1.1.3. *Let* $1 = G_0 \leq G_1 \leq \ldots \leq G_{n-1} \leq G_n$ *be a central series of the group* G*; then for all* $0 \leq i \leq n$,

$$\gamma_{n-i+1}(G) \leq G_i \leq \zeta_i(G).$$

1.2 Classes of groups

A group theoretical class, or a class of groups, $\mathfrak{X}$ is a class whose members are groups that satisfy the following properties:

1) The trivial group is in $\mathfrak{X}$;

2) If $H \simeq G$ and $G \in \mathfrak{X}$, then $H \in \mathfrak{X}$.

We will adopt $\mathcal{A}$, $\mathcal{N}$, $\mathcal{S}$, to denote respectively the class of abelian, the class of nilpotent and the class of solvable groups.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are classes of groups, $\mathfrak{X}\mathfrak{Y}$ denotes the class of groups G that contains a normal subgroup $N \lhd G$ such that $N \in \mathfrak{X}$ and $G/N \in \mathfrak{Y}$. The class $\mathfrak{X}\mathfrak{Y}$ is the class $\mathfrak{X}$-by-$\mathfrak{Y}$, e.g. $\mathcal{N}\mathcal{A}$ is the class of nilpotent-by-abelian groups, equivalently the groups with nilpotent derived group.

We will say that a class of groups $\mathfrak{X}$ is subgroups closed if for all $G \in \mathfrak{X}$ and $H \leq G$, then $H \in \mathfrak{X}$. Analogously, a class of groups $\mathfrak{X}$ is closed by homomorphic images or quotients if for all $G \in \mathfrak{X}$ and $N \lhd G$, then $G/N \in \mathfrak{X}$. Moreover we will say that $\mathfrak{X}$ is extension closed if given a group G and a normal subgroup N, if $N \in \mathfrak{X}$ and $G/N \in \mathfrak{X}$, then $G \in \mathfrak{X}$.

Let $\mathfrak{X}$ be a class of groups, we say that a group G is locally-$\mathfrak{X}$ if every finite subset of G is contained in a subgroup $H \leq G$ such that $H \in \mathfrak{X}$. In particular, we would like to stress the fact that, if $\mathfrak{X}$ is subgroups closed, then G is locally-$\mathfrak{X}$ if and only if every finitely generated subgroup of G is in $\mathfrak{X}$. This is the case of locally nilpotent group, a class that is central in this work.

1.3 Finiteness conditions

A finiteness condition is a property that is satisfied by all finite groups. For instance the most important finiteness conditions are the property of being periodic, locally finite and finitely generated. A periodic group is a group in which there are no elements of infinite order. A locally finite group is a group in which every finitely generated subgroup is finite. In particular every locally finite group is periodic; the inverse is not true (see for example, the Grigorchuk group [29]).

Among the most important finiteness conditions there are Max and Min, that are the maximal condition on subgroups and the minimal condition on subgroups.

Definition 1.3.1. *A group G is said to satisfy Max, the maximal condition on subgroups if for all*

$$H_0 \leq H_1 \leq H_2 \leq \dots$$

there exists $n \in \mathbb{N}$ such that $H_n = H_{n+1} = \dots$ or in other words every ascending chain of subgroups is finite.

Analogously, a group G is said to satisfy Min, the minimal condition on subgroups if for all

$$H_0 \geq H_1 \geq H_2 \geq \dots$$

there exists $n \in \mathbb{N}$ such that $H_n = H_{n+1} = \dots$ or in other words every descending chain of subgroups is finite.

In general, if $\mathcal{P}$ is a family of subgroups of the group G, then G is said to satisfy the maximal (minimal) condition on $\mathcal{P}$-subgroup if every ascending (descending) chain of $\mathcal{P}$-subgroups of G is finite.

Another couple of finiteness conditions that are less common are the weak minimal and maximal conditions on subgroups.

Definition 1.3.2. *A group G is said to satisfy the weak maximal condition, wmax, if there are no infinite ascending chains of subgroups*

$$H_0 \leq H_1 \leq \dots$$

in which each index $|G_{i+1} : G_i|$ is infinite. Replacing ascending with descending we obtain the weak minimal condition, wmin.

These chain conditions are strictly connected with some classes of groups that will be essential in the rest of the work. For this reason we decide to introduce them, stating the main definitions and results. We start with the groups that satisfy Min.

1.3.1 Min condition

Dealing with Min, it is natural to encounter Černikov groups. A group is a Černikov group if it is an extension of a direct product of a finite number of Prüfer groups by a finite group. They form a relevant class of groups that satisfy Min, but they are not the only ones, if we consider, for instance, Tarsky p-group. However for the solvable case this characterization holds (see [46, §5, p. 151]):

Theorem 1.3.3. *A solvable group satisfies Min if and only if it is a solvable Černikov group.*

A subgroup that is central in the study of groups with Min is the finite residual. So, given a class of groups $\mathfrak{X}$, we define what is the $\mathfrak{X}$-residual

Definition 1.3.4. *Let G be a group and $\mathfrak{X}$ a class of groups. The $\mathfrak{X}$-residual of G is the intersection of all normal subgroups whose factor groups in G belong to the class $\mathfrak{X}$. Needless to say, the finite residual is when $\mathfrak{X}$ is the class of finite groups.*

In the case of Černikov groups the finite residual itself has finite index. In fact:

Proposition 1.3.5. *Let the group G satisfy the minimal condition on normal subgroups. Then G admits a unique minimal subgroup of finite index, the finite residual, and it is characteristic.*

Sometimes it could be useful to deal with nilpotent Černikov groups or with Černikov p-groups. It is possible to prove that the former are central-by-finite and for the latter the following proposition holds.

Proposition 1.3.6. *Let p be a prime and G be a Černikov p-group. Then G is isomorphic to a subgroup of the wreath product $C_{p^\infty} \wr P$, where P is a suitable finite p-group.*

1.3.2 Max condition

A polycyclic group is a group that admits a finite series with cyclic factors. They are central concerning Max condition because of this theorem

Theorem 1.3.7. *A group is a solvable group satisfying Max if and only if it is polycyclic.*

A very important tool dealing with polycyclic groups is the following:

Definition 1.3.8. *In a polycyclic group G the number of infinite factors in a cyclic series is independent of the series and hence is an invariant, known as $h(G)$ the Hirsch length of G.*

Let us call a group poly-infinite cyclic if it has a finite series with infinite cyclic factors. These groups are polycyclic and torsion-free, but the converse is not true see example [46, §5, p. 152].

Proposition 1.3.9. *Every polycyclic group has a normal poly-infinite cyclic subgroup of finite index. Moreover, an infinite polycyclic group admits a non trivial torsion free abelian normal subgroup.*

When the group is nilpotent we can say more

Proposition 1.3.10. *Let G be a nilpotent group. The following conditions are equivalent:*

1) G is finitely generated;

2) G/G' is finitely generated;

3) G is polycyclic;

4) G satisfies Max.

Other two useful lemmas about finitely generated nilpotent groups are:

Lemma 1.3.11. *Let G be a group and X a system of generators. Then for all $n \geq 1$:*

$$\gamma_n(G) = \langle [x_1, \ldots, x_i] \mid i \geq n,\ x_1, \ldots, x_i \in X \rangle$$

For the proof of this latter lemma see [19, §5, p. 118].

Lemma 1.3.12. *Let G be a finitely generated nilpotent group, H a subgroup of G and π a set of primes. Let $x_1, \ldots, x_n$ be a set of generator of G and suppose that $x_i^{r_i} \in H$, for some positive π-number r_i, and $i : 1, \ldots, n$. Then each element of G has a positive π-power in H and $|G : H|$ is a finite π-number.*

For this lemma we refer to [40, §2, p. 39].

Another class of groups with Max that is intermediate between finitely generated nilpotent and polycyclic is the class of supersolvable groups. A group is said to be supersolvable if it has a finite normal series whose factors are cyclic. This class will have a key role in one of the final theorems; for that reason we will state some of the principal results, that involve it. The statements and the proofs, we refer to, are in [46, §5, p. 145-146]. We start with a theorem of Zappa:

Theorem 1.3.13. *If G is a supersolvable group, there exists a normal series*

$$1 = G_0 \leq G_1 \leq \ldots \leq G_n = G$$

in which each factor is cyclic of prime order or infinite order. Moreover the order of the factors from the left is this: odd factors in descending order of magnitude, infinite factors, factors of order two.

Corollary 1.3.14. *The elements of odd order in a supersolvable group form a characteristic subgroup.*

Another fundamental theorem is the following

Theorem 1.3.15. *Let G be a supersolvable group. Then $Fit(G)$ is nilpotent and $G/Fit(G)$ is a finite abelian group.*

A useful lemma we will refer to is:

Lemma 1.3.16. *If G is supersolvable then $1 \neq N \lhd G$ implies $1 \neq \langle x \rangle \lhd G$ for some $x \in N$.*

Now we state two theorems on polycyclic groups that give conditions to obtain in the first case nilpotency (see [46, §5, p. 149]), in the second case supersolvability [49].

Theorem 1.3.17. *(Hirsch) Let G be a polycyclic group. Then G is nilpotent if and only if every finite quotient of G is nilpotent.*

Theorem 1.3.18. *Let G be a polycyclic group. Then G is supersolvable if and only if every finite quotient of G is supersolvable.*

Although the structure of the statements of this two theorems is the same, it is necessary to stress the fact that the first has a fully group theory proof, the second needs some prerequisites in number theory.

1.4 Rank and Minimax groups

1.4.1 Rank in solvable groups

The term rank in algebra has many connotations. In solvable group theory it refers to the cardinality of a maximal linearly independent subset of some kind. The foundations of the theory of soluble groups of finite rank is to confer to [43].

We start with the abelian case:

Definition 1.4.1. *Let A be an abelian group. Then $r_0(A)$ is the cardinality of any maximal linearly independent subset of element of infinite order, and if p is a prime, $r_p(A)$ is the cardinality of any maximal linearly independent subset of elements of order p.*

From this definition it is easy to prove the following result.

Proposition 1.4.2. *Let A be an abelian group and T its torsion subgroup. Then:*

1) $r_0(A)$ *is finite if and only if* A/T *is isomorphic with a subgroup of the additive group of a finite dimensional rational vector space.*

2) If p *is a prime,* $r_p(A)$ *is finite if and only if the* p*-component* A_p *is the direct sum of finitely many cyclic or quasicyclic groups, i.e.* A_p *satisfies Min.*

For abelian groups we can define two further invariants:

the total rank

$$r(A) = r_0(A) + \sum_{p\ prime} r_p(A)$$

and the reduced rank

$$\tilde{r}(A) = r_0(A) + \underset{p\ prime}{Max}\,\{r_p(A)\}$$

From these definitions and keeping in mind that a solvable group admits a finite series with abelian factors, it is possible to define several classes of (solvable) groups with finite rank, in some sense. For a precise survey of the main classes see [40].

1.4.2 Minimax groups

The class we are interested in is the class of solvable minimax groups.

Definition 1.4.3. *A group is said to be minimax if it has a series of finite length for which each factor satisfies Max or Min.*

One of the main tools, we are going to use in the final part of this work, is the following

Definition 1.4.4. *Let* G *be a solvable minimax group. We know that there exists a series* $1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ *in which each factor satisfies max or min. Clearly it is possible to refine this series and obtain a series with cyclic factors (finite or infinite) and quasicyclic factors. By a routine application of Schreier refinement theorem it is possible to show that the number of infinite factors of that series is an invariant and we call it minimax length or minimality of* G*,* $m(G)$*. It can be thought as a generalization of Hirsch length for polycyclic groups.*

It is possible to prove the following lemma. We refer to [57].

Lemma 1.4.5. *Let* G *be a solvable minimax group and* $H \leq G$*. Then* $m(H) \leq m(G)$ *and the equality holds if and only if* $|G : H| < \infty$*.*

Let G be a solvable minimax group; for all $H \leq G$ we define

$$m(G : H) = m(G) - m(H)$$

So, $m(G : H) = 0$ if and only if $|G : H| < \infty$; moreover if $K \leq H \leq G$ we get by definition that

$$m(G : K) = m(G : H) + m(H : K).$$

Abelian minimax groups can be simply described (see [40, §5, p. 86]), in fact:

Proposition 1.4.6. *An abelian group A is minimax if and only if it has a finitely generated subgroup X such that A/X satisfies Min.*

A natural generalization can be stated for nilpotent minimax groups (see [47, §10, p. 168]).

Proposition 1.4.7. *Let G be a nilpotent minimax group with class c. Then there exists a series*

$$X = G_0 \lhd G_1 \lhd \ldots \lhd G_c = G$$

where X is finitely generated and G_{i+1}/G_i is an abelian group with Min for $i : 0, \ldots, c-1$.

Another interesting property is that the finite residual has a simple structure. To be more precise we state this theorem, admitting that for simplicity, we do not state it in the absolute generality. For the general statement see [47, §10, p. 169]

Theorem 1.4.8. *Let G be a solvable minimax group, let R be the subgroup generated by all the quasicyclic subgroups of G. Then R is the direct product of finitely many quasicyclic subgroups of G. Moreover R is the finite residual of G.*

Solvable minimax groups can be characterized in terms of the weak maximal and minimal conditions on subgroups, we introduced before. Baer in [10] established the following theorem:

Theorem 1.4.9. *Let G be a solvable group. the the following properties are equivalent:*

1) G is a minimax group;

2) G satisfies wmax;

3) G satisfies wmin.

1.5 Locally nilpotent groups

In this section we introduce a very important class of generalized nilpotent groups. This is an eloquent example of how some properties play a crucial role in finite group theory, but they are much weaker when applied to infinite groups. In view of the definition of locally-$\mathfrak{X}$ groups, it is straightforward to deduce the following:

Definition 1.5.1. *A locally nilpotent group is a group in which every finitely generated subgroup is nilpotent.*

We start this short presentation introducing the torsion subgroup. Given a locally nilpotent group G, the elements of finite order form a fully-invariant subgroup, that in general is indicated with $T(G)$, called the torsion subgroup of G. This group is the direct product of p-groups and the quotient G/T is torsion-free.

Now, we introduce the first important result about locally nilpotent groups.

Theorem 1.5.2. *(Hirsch-Plotkin) The product of a family of normal locally nilpotent subgroups of a group G is normal and locally nilpotent.*

The importance of this theorem resides in the fact that it gives rise to the following definition

Definition 1.5.3. *Given a group G, there is a unique maximal normal locally nilpotent subgroup containing all normal locally nilpotent subgroups of G. This group is called Hirsch-Plotkin radical.*

Another fundamental definition is that of ascendant subgroup. To give this definition it is probably necessary to introduce the concept of ascending series.

Definition 1.5.4. *Given a group G, we will call ascending series a set of subgroups $\{H_\alpha \mid \alpha \leq \beta \}$ indexed by ordinals less or equal to β, such that the following conditions hold:*

1) $H_{\alpha_1} \leq H_{\alpha_2}$, if $\alpha_1 \leq \alpha_2$;

2) $H_0 = 1$ and $H_\beta = G$;

3) $H_\alpha \lhd H_{\alpha+1}$;

4) $H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$ if λ is a limit ordinal.

Moreover the H_α are the terms, $H_{\alpha+1}/H_\alpha$ are the factors and β is the length or the ordinal type of the series.

With the previous definition we are ready to give also this:

Definition 1.5.5. *Given a group G, a subgroup that occurs in some ascending series of G is called ascendant subgroup.*

The first result about ascendant subgroups (see [46, §12, p. 344]) is the following:

Proposition 1.5.6. *Given G a group, the Hirsch-Plotkin radical contains all the ascendant locally nilpotent subgroups.*

Another important property of ascendant groups is related with ascendant abelian divisible subgroups (see [18, §1, p. 19]).

Proposition 1.5.7. *Let A be a periodic abelian divisible subgroup of the group G. If A is ascendant, then A^G is abelian divisible.*

Now we will give a concise presentation of the main subclasses of locally nilpotent groups that generalize nilpotency. The first class we introduce is the class of groups that satisfies the normalizer condition or, in other words, the class in which every proper subgroup is smaller than its normalizer. We know that in the finite group theory this condition is equivalent to the nilpotency of the group, but this is not the case. We now give a useful

Definition 1.5.8. *Let G be a group and H a subgroup. We define the series of successive normalizer H_α of H by the following rules:*

$$H_0 = H$$

$$H_{\alpha+1} = N_G(H_\alpha)$$

$$H_\beta = \bigcup_{\lambda<\beta} H_\lambda$$

for any successor ordinal α and limit ordinal β.

Having in mind this definition it is straightforward to deduce that in a group that satisfies the normalizer condition, there exists an ordinal α such that $H_\alpha = G$, and then by proposition 1.5.6, we conclude that those groups are locally nilpotent.

The second subclass of locally nilpotent groups we will present is the class of hypercentral groups. An ascending series

$$1 = G_0 \lhd G_1 \lhd \ldots G_\beta = G$$

is said to be central if $G_\alpha \lhd G$ and the factor $G_{\alpha+1}/G_\alpha$ lies in $Z(G/G_\alpha)$ for every ordinal $\alpha < \beta$. A group is called hypercentral if it admits a central ascending series.

Given a group G, it is possible to define the upper central ascending series in this way: $\zeta_0(G) = 1$, $\zeta_\alpha(G)/\zeta_{\alpha-1}(G) = \zeta(G/\zeta_{\alpha-1}(G))$ if α is a successor ordinal and

$$\zeta_\lambda(G) = \bigcup_{\alpha<\lambda} \zeta_\alpha(G)$$

if λ is a limit ordinal. The terminal group of such a series is called the hypercentre of G. It is easy to understand that a group is hypercentral if and only if it coincides with its hypercentre. Moreover it is not difficult to deduce that given a central ascending series $\{G_\alpha\}$ of G, a subgroup H and observing that $HG_\alpha \lhd HG_{\alpha+1}$ we get that H is ascendant in G and hence the class of hypercentral groups is contained in the class of groups that satisfies the normalizer conditions. So, we resume these results with this scheme

nilpotent $\subset$ *hypercentral* $\subset$ *normalizer condition* $\subset$ *locally nilpotent*

We now introduce other subclasses of locally nilpotent groups, without giving any detail.

Definition 1.5.9. *A group is said to be a Gruenberg group if every cyclic subgroup is ascendant, and a Baer group if every cyclic subgroup is subnormal. In addition a group is said to be a Fitting group if every element is contained in a normal nilpotent subgroup. Finally the last class we define is the class N_1 of groups in which every subgroup is subnormal.*

Most of the inclusions we are going to show with the next schemes are straightforward

nilpotent $\subset$ N_1 $\subset$ *normalizer condition* $\subset$ *Gruenberg* $\subset$ *locally nilpotent*

nilpotent $\subset$ N_1 $\subset$ *Fitting* $\subset$ *Baer* $\subset$ *Gruenberg* $\subset$ *locally nilpotent*

with the non-negligible exception of the inclusion $N_1 \subset$ *Fitting* (see [17]) that is deeper to prove and beyond the objective of this short presentation.

Coming back to the class of hypercentral groups, which is the class we will examine more in depth among the classes of locally nilpotent groups, we will present other properties that are really typifying those groups. We will summarize them in a unique proposition

Proposition 1.5.10. *Let G be a group:*

1) If G is hypercentral and $1 \neq N \lhd G$, then $N \cap \zeta(G) \neq 1$;

2) (Baer) G is hypercentral if and only if every non trivial quotient of G has non trivial centre;

3) (Černikov) G is hypercentral if and only if for every countable sequence $g_1, g_2, \dots$ of elements of G, there exists $k \in \mathbb{N}$ such that $[g_1, g_2, \dots, g_k] = 1$

Other fundamental remarks we need to recall, dealing with locally nilpotent groups are about the already introduced chain conditions. In general, if we consider locally nilpotent groups with a finiteness condition it might be forced to become nilpotent. Here is a non trivial example that we will use (see [46, §12, p. 346-347]).

Theorem 1.5.11. *(McLain) Let G be a locally nilpotent group that satisfies the maximal condition on normal subgroups. Then G is a finitely generated nilpotent group.*

In other words this theorem affirms that Max and Max-n are the same for locally nilpotent groups. One may expect an analogue with Min-n, but we do not obtain nilpotency. Anyway we get a precise description:

Theorem 1.5.12. *(McLain) Let G be a locally nilpotent group. It satisfies the minimal condition on normal subgroups if and only if it is a direct product of finitely many Černikov p-groups, for various primes p. Moreover a locally nilpotent group that satisfies the minimal condition on normal subgroups is hypercentral.*

Gathering these two results we deduce this easy corollary:

Corollary 1.5.13. *Let G a locally nilpotent minimax group, then it is solvable.*

Another theorem we will use related to this topic is the following by Mal'cev (see [43]).

Theorem 1.5.14. *Let G be a locally nilpotent group. Then the abelian subgroup of G have finite torsion-free rank $r_0(G)$ if and only if $G/T(G)$ is a finite rank (torsion-free) nilpotent group.*

An immediate corollary is

Corollary 1.5.15. *Let G be a locally nilpotent minimax torsion-free group. Then it is nilpotent.*

1.6 Isolators

Going deeper in the study of locally nilpotent groups, in particular torsion-free locally nilpotent, a very effective concept is that of isolator of a subgroup. It has been introduced by Philip Hall in [31] and we suggest for a concise account [18].

Definition 1.6.1. *Given a group G, a subgroup $H \in G$ and a set of primes π. The π-isolator of H in G is the set:*

$$I_G^\pi(H) = \{g \in G \mid g^n \in H \ for \ some \ \pi - number \ n \geq 1\}.$$

If π is the set of all primes, we can speak about the isolator of H in G; thus

$$I_G(H) = \{g \in G \mid g^n \in H \ for \ some \ 1 \leq n \in \mathbb{N}\}$$

From now on, we will refer to isolators considering, in general, the set of all primes (our second definition). Nevertheless almost all the results we are going to state admit a local version by specializing some proofs.

In general (it is easy to find examples) isolators are subsets and non necessarily subgroups (e.g. the isolator of $\{1\}$ in D_∞). They gain the property to be subgroups in a wide family of groups: the family of locally nilpotent groups. In fact:

Proposition 1.6.2. *Let G be a locally nilpotent group. Then, for all $H \leq G$, $I_G(H)$ is a subgroup of G.*

Consider G a locally nilpotent group and H a subgroup. It is straightforward that $I_G(I_G(H)) = I_G(H)$ and that if $H \lhd G$ then $I_G(H) \lhd G$. Other important properties that are less obvious and they involve derived and central series are resumed in the next proposition.

Proposition 1.6.3. *Let G be a locally nilpotent group and let $H, K \leq G$. Then for every $1 \leq n \in \mathbb{N}$,*

1) $[G, I_G(H)] \leq I_G([G, H])$, thus if U/V is a central factor of G, then also $I_G(U)/I_G(V)$ is a central factor;

2) $\gamma_n(I_G(H)) \leq I_G(\gamma_n(H))$;

3) $I_G(H)^{(n)} \leq I_G(H^{(n)})$.

This result is an instance of a more general result proved by P. Hall in [31], that is the following

Theorem 1.6.4. *(P. Hall) Let G be a locally nilpotent group, $\theta(x_1, \dots, x_n)$ be a word in the n variables $x_1, \dots, x_n$, let π be a set of primes and $H_1, \dots, H_n \leq G$, then*

$$\theta(I_G^\pi(H_1), \dots, I_G^\pi(H_n)) \leq I_G^\pi(\theta(H_1, \dots, H_n)).$$

A corollary that is often useful is:

Corollary 1.6.5. *Let H, K subgroups of the locally nilpotent group G. Then*

$$[I_G(H), I_G(K)] \leq I_G([H, K]).$$

In particular $I_G(N_G(H)) \leq N_G(I_G(H))$.

For torsion-free locally nilpotent groups we have this stronger result:

Proposition 1.6.6. *Let G be a locally nilpotent torsion-free group and $H \leq G$. Then for every ordinal α*

$$\zeta_\alpha(I_G(H)) = I_G(\zeta_\alpha(H))$$

1.7 Engel groups

We conclude this part about generalized nilpotent groups presenting a class that is not locally nilpotent, that of Engel groups. We will state only the main definitions and results, for a detailed survey we suggest [54] of Traustason.

Definition 1.7.1. *Given a group G and an element $g \in G$, g is called a right Engel element if for each $x \in G$ there exists $n = n(g, x) \in \mathbb{N}$ such that $[g,_n x] = 1$. If n can be chosen independently of x, then g is a right n-Engel element or a bounded right element. The set of right Engel elements and the set of bounded right Engel elements are usually indicated respectively with*

$$R(G) \quad \text{and} \quad \bar{R}(G)$$

Similarly, given a group G and an element $g \in G$, g is called a left Engel element if for each $x \in G$ there exists $n = n(g, x) \in \mathbb{N}$ such that $[x,_n g] = 1$. If n can be chosen independently of x, then g is a left n-Engel element or a bounded left element. The set of left Engel elements and the set of bounded left Engel elements are indicated respectively with

$$L(G) \quad \text{and} \quad \bar{L}(G)$$

For any group G the equalities $G = L(G)$ and $G = R(G)$ are equivalent and a group with this property is called Engel group. As we are going to show in the next proposition, there is a strong link between, for example, $L(G)$ and the Hirsch-Plotkin radical of G, but anyway a famous example of Golod (see [28]) shows that there exist Engel groups that are not locally nilpotent.

We gather the main properties of those sets, derived from several theorems.

Proposition 1.7.2. *Let G be a group.*

1) *(Heineken)* $R(G)^{-1} \subseteq L(G)$ *and* $\bar{R}(G)^{-1} \subseteq \bar{L}(G)$*;*

2) $L(G)$ *contains the Hirsch-Plotkin radical of* G*;*

3) $R(G)$ *contains the hypercentre of* G*;*

4) *(Zorn) If* G *is finite and Engel then it is nilpotent.*

A theorem that deserves to be mentioned is the following due to Gruenberg [30], that shows us how the assumption of solvability simplifies the structure of an Engel Group.

Theorem 1.7.3. *(Gruenberg) Let* G *be a solvable group. The set of left Engel elements* $L(G)$ *coincides with the Hirsch-Plotkin radical of* G*. Then a solvable Engel group is a Gruenberg group.*

1.8 FC-groups

The last section of this introductive chapter is a very short presentation of FC-groups, a class of groups that generalizes both the class of finite groups and the class of abelian groups. For this class of groups we refer to [8] and [46, §14, p. 424-426]. Given a group G, an element $g \in G$ is said to be an FC-element if it has only a finite number of conjugates in G, or, in other words, if $|G : C_G(g)|$ is finite. The observation that gives these elements a special role is the following:

Proposition 1.8.1. *(Baer) In any group* G *the* FC*-elements form a characteristic subgroup.*

An FC-element can be thought as a generalization of an element of the centre that is, needless to say, an element with just a conjugate. For this reason the name of the subgroup of all the FC-elements is FC-centre. A group is said to be an FC-group if it equals its FC-centre.

For the sake of completeness we state a couple of important results.

Theorem 1.8.2. *(Baer) If* G *is an* FC*-group, then* $G/\zeta(G)$ *is a residually finite torsion group.*

Concerning torsion FC-groups we have:

Proposition 1.8.3. *A torsion group* G *is an* FC*-group if and only if each finite subset is contained in a finite normal subgroup.*

Chapter 2

Products of groups

The product of two subsets of a group is a topic that every student that attends a basic course of group theory has to deal with, even without full awareness of it. In fact, given a group G and a subgroup H, a coset xH is nothing else that the product of $\{x\}$ and H.

Besides the cosets, the other meaningful examples concern essentially the products of subgroups. A group G is said to be the product of its subgroups A and B, i. e. $G = AB$, if and only if

$$G = \{ab \mid a \in A,\ b \in B\}$$

We also say that G is factorized by A and B.

Given a group G and two subgroups A and B, a necessary and sufficient condition to have that AB is a subgroup as well is the following

Proposition 2.0.4. *Let G be a group and A and B two subgroups. Then AB is a subgroup of G if and only if $AB = BA$.*

To reaffirm the fact that the product of groups is in some sense more common and more natural than what in general one may imagine, it is sufficient to think of the well-known *Dedekind modular law*, that it is based on factorization of subgroups. Another more than obvious fact that should be sufficient to motivate this study is that the product we defined generalize, for instance, the direct product and the semidirect product, two fundamental concepts of group theory.

Anyway the literature of *general* groups product theory is not too extensive and regarding books to do a systematic study of that topic we suggest [1] for a general introduction and a more detailed attention to the infinite case and [12] for the finite case.

This chapter is a brief introduction to the groups product theory. Notation and results are mainly taken from [1].

2.1 Elementary properties

From the definition we have given of groups product, it is easy to see that every homomorphic image factorizes as follows

$$\frac{G}{N} = \frac{AN}{N}\frac{BN}{N}$$

Considering subgroups of factorized groups, they are not necessarily factorized. Anyway we can use this lemma

Lemma 2.1.1. *(Wielandt, 1958) Let the group $G = AB$ be the product of two subgroups A and B. For a subgroup S of G the following conditions are equivalent:*

1) If $ab \in S$, with $a \in A$ and $b \in B$, then $a \in S$;

2) $S = (A \cap S)(B \cap S)$ and $A \cap B \leq S$.

A subgroup S of a factorized group G is said to be factorized if it satisfies one of the equivalent conditions of the previous lemma.

We now state some lemmas to show the basic properties of product theory. We start with

Lemma 2.1.2. *Let the group $G = AB$ the product of two subgroups A and B. Then the following hold:*

1) the intersection of arbitrarily many factorized subgroups of G is factorized;

2) the subgroup generated by arbitrarily many factorized normal subgroup of G is factorized;

3) if N is a normal subgroup of the group G, a subgroup S/N of G/N is factorized if and only in S factorized in G.

According with this lemma the intersection $X(S)$ of all factorized subgroups of $G = AB$ containing the subgroup S, is the smallest factorized subgroup of G containing S. The subgroup $X(S)$ is called the factorizer of S in $G = AB$. The factorizer obtains an important role when the subgroup is normal. In fact we get an interesting and useful triple factorization.

Lemma 2.1.3. *Let the group $G = AB$ the product of two subgroups A and B, and let N be a normal subgroup of G. Then:*

1) $X(N) = AN \cap BN$;

2) $X(N) = (A \cap BN)N = (B \cap AN)N = (A \cap BN)(B \cap AN)$

As we have already said, subgroups of factorized groups are not necessarily factorized. This easy lemma, which we will use often, gives some sufficient conditions to factorize certain subgroups.

Lemma 2.1.4. *(Wielandt, 1958) Let the group $G = AB$ be the product of two subgroups A and B, and let A_0 and B_0 be normal subgroup respectively of A and B. Then, called $H = \langle A_0, B_0 \rangle$ and $L = A_0 \cap B_0$, the following hold:*

1) if one of the factor groups A/A_0 and B/B_0 is periodic, then

$$N_G(H) = N_A(H)N_B(H);$$

2) if one of the subgroups A and B is periodic, then

$$N_G(L) = N_A(L)N_B(L).$$

This lemma inspired us for a slight modification, that we proved mainly retracing the original proof.

Lemma 2.1.5. *Let the group $G = AB$ be the product of two subgroups A and B, and let A_0 and B_0 be normal subgroup respectively of A and B. Called $H = \langle A_0, B_0 \rangle$, if the condition $I_G(A \cap B) = G$ holds, then*

$$N_G(H) = N_A(H)N_B(H).$$

Proof. Let $g \in N_G(H)$; $G = AB$ so by definition $g = ab$, where $a \in A$ and $b \in B$. Clearly $H^g = H$ and $H^a = H^{b^{-1}}$. By the normality of A_0 and B_0, we have that $A_0 \leq H^a$ and $B_0 \leq H^{b^{-1}}$. This implies $H \leq H^a = H^{b^{-1}}$. Since $I_G(A \cap B) = G$, we get that there exists $m \in \mathbb{N}$ such that $a^m \in A \cap B$, so for this reason

$$H \leq H^a \leq H^{a^2} \leq \ldots \leq H^{a^m} = H$$

that means $H^a = H$ or, in other words, $a \in N_A(H)$, our thesis.

□

At this point it should be clear that normalizers of certain subgroups gained a crucial role in this field, because under not too restrictive hypotheses, they factorize. For instance, they provide examples of factorizers of subgroups, that often in literature permit to reduce to easier issues, e.g. to have a normal subgroup with perhaps a desired property.

In our case we use normalizers to *build* series of factorized subgroups, as those we are going to define in the next two lemmas

Lemma 2.1.6. *Let G be a group and $A, B \leq G$ such that $G = AB$. If A locally satisfies Max, then, called $M_0 = A \cap B$, $M_1 = N_B(M_0)$, ..., $M_{i+1} = N_B(M_i)$, we have that for all $i \geq 0$*

$$N_G(M_i) = N_A(M_i)M_{i+1}$$

Proof. We proceed by induction on i. If $i = 0$ we have that given $g \in N_G(M_0)$, there exist $a \in A$ and $b \in B$ such that $g = ab$ and then $M^{ab} = M$, from which $M^a = M^{b^{-1}} \leq A \cap B = M$. Now we have to prove the other inclusion, namely $M \leq M^a$. Consider $m \in M$; we know that $\langle m, a \rangle$ satisfy Max by hypothesis. So we are able to define

$$H = \langle m, m^a, \ldots, m^{a^t}, \ldots \rangle.$$

The sequence

$$H \leq H^{a^{-1}} \leq \ldots \leq H^{a^{-s}} \leq \ldots$$

is an ascendant chain of subgroup of $\langle m, a \rangle$, thus there exists t integer such that $H^{a^{-t}} = H^{a^{-t-1}}$, i.e. $H = H^a$ that implies $m \in M^a$ and then $M^a = M$. So, $a \in N_A(M)$ and we obtained the desired factorization $N_G(M) = N_A(M)N_B(M)$.

Now, let $i \geq 1$, consider $g = ab \in N_G(M_i)$. So,

$$M_i^a = M_i^{b^{-1}} \leq B$$

from which, in particular

$$M^a \leq M_i^a \cap A \leq B \cap A = M$$

that means, using the same techniques as before, $a \in N_A(M)$.

Let $0 \leq j \leq i$ be the greatest integer such that $a \in N_A(M_j)$. Suppose that $j < i$. We know that $\langle a, M_{j+1} \rangle \leq N_G(M_j)$, hence, by inductive hypothesis,

$$M_{j+1}^a \leq N_G(M_j) = N_A(M_j)M_{j+1}$$

Keeping in mind that $M_{j+1}^a \leq M_i^a = M_i^{b^{-1}} \leq B$, we obtain

$$M_{j+1}^a \leq N_A(M_j)M_{j+1} \cap B = M_{j+1}(N_A(M_j) \cap B) = M_{j+1}M = M_{j+1}$$

from which $M_{j+1}^a \leq M_{j+1}$.
Therefore,

$$M_{j+1} \leq M_{j+1}^{a^{-1}} \leq N_G(M_j)$$

and thus

$$M_{j+1}^{a^{-1}} = M_{j+1}^{a^{-1}} \cap M_{j+1}N_A(M_j) = (M_{j+1}^{a^{-1}} \cap N_A(M_j))M_{j+1} \leq (M_{j+1}^{a^{-1}} \cap A)M_{j+1} =$$

$$= (M_{j+1} \cap A)^{a^{-1}} M_{j+1} = M^{a^{-1}} M_{j+1} = M_{j+1}$$

from which $M_{j+1}^{a^{-1}} = M_{j+1}$ and then $a \in N_A(M_{j+1})$, a contradiction. So $j = i$, then $a \in N_A(M_i)$ and consequently our thesis.

□

Remark 2.1.7. *In this latter lemma we proved,* en passant*, also the following inclusions* $N_A(M_0) \geq N_A(M_1) \geq \ldots \geq N_A(M_j) \geq \ldots$

The following lemma is just an extension of the previous, using this time the ordinal numbers.

Lemma 2.1.8. *Let G be a group and $A, B \leq G$ such that $G = AB$. If A locally satisfies Max, $M_0 := A \cap B$ and (M_α) is the series of successive normalizers of M_0 in B, then called:*

$$Y_1 := N_G(M_0)$$

$$Y_\alpha := N_G(M_{\alpha-1})$$

if α is not a limit ordinal and

$$Y_\beta := \left(\bigcap_{\lambda<\beta} N_A(M_\lambda) \right) M_\beta$$

if β is a limit ordinal, we have that (Y_α) is a series of factorized subgroups of G and $(Y_\alpha \cap A)$ is a decreasing series of subgroup of A.

Proof. We proceed by transfinite induction. For $\alpha = 1$ is true for the previous lemma. If α is a limit ordinal we have only to prove that

$$Y_\alpha := \left(\bigcap_{\lambda<\alpha} N_A(M_\lambda) \right) M_\alpha$$

is a subgroup. To do that, we need only to observe that

$$\bigcap_{\lambda<\alpha} N_A(M_\lambda) \leq N_A(M_\alpha)$$

Suppose that $\alpha \geq 2$ is a successor ordinal. Consider $g \in N_G(M_{\alpha-1})$. By definition there exist $a \in A$ and $b \in B$ such that $g = ab$. So,

$$M_{\alpha-1}^a = M_{\alpha-1}^{b^{-1}} \leq B$$

from which, in particular

$$M_0^a \leq M_{\alpha-1}^a \cap A \leq B \cap A = M_0$$

that means, $a \in N_A(M_0)$.

Consider ρ such that $1 \leq \rho \leq \alpha$ and such that it is the smallest ordinal such that $a \notin N_A(M_\rho)$. If ρ is a successor ordinal, proceeding as in the previous lemma we get a contradiction. Otherwise, ρ is a limit ordinal. Then for all $\lambda < \rho$, $a \in N_A(M_\lambda)$, or, in other words

$$a \in \bigcap_{\lambda<\rho} N_A(M_\lambda) \leq N_A(M_\rho)$$

a contradiction. So ρ does not exist and we get our thesis.

□

To conclude this introductive part, we state three more useful lemmas.

Lemma 2.1.9. *Let the group $G = AB$ be factorized by $A, B \leq G$. Let the group D be such that $B \leq D \leq G$, and let a subgroup $Y \leq G$ be factorized in AD. Then it is factorized also in AB.*

Proof. By definition (see lemma 2.1.1), $Y = (A \cap Y)(D \cap Y)$ and $Y \geq A \cap D$. Applying twice Dedekind modular law, we get

$$D \cap Y = AB \cap D \cap Y = (A \cap D)B \cap Y = (A \cap D)(B \cap Y)$$

and for that

$$Y = (A \cap Y)(D \cap Y) = (A \cap Y)(A \cap D)(B \cap Y) = (A \cap Y)(B \cap Y)$$

□

Lemma 2.1.10. *(Amberg, 1973) Let the group $G = AB$ be the product of two subgroups A and B, and let A_0 and B_0 be two finite index subgroups of A and B respectively. Then, called $|A : A_0| = n$ and $|B : B_0| = m$, we have that $|G : \langle A_0, B_0 \rangle| \leq nm$.*

Lemma 2.1.11. *Let the group $G = AB$ be the product of two subgroups A and B. If $x, y \in G$, then $G = A^x B^y$. Moreover, there exists an element $z \in G$ such that $A^x = A^z$ and $B^y = B^z$.*

2.2 Chain conditions in factorized groups

In the first chapter we introduced some basic notions about chain conditions. Amberg in 1973 proved a lemma to show how the structure of a factorized group G in which the factors satisfy Max (Min) is influenced.

Lemma 2.2.1. *Let G be a group $A, B \leq G = AB$. If A, B satisfy the Max (Min), then G satisfies Max-n (Min-n).*

Using a similar strategy of that followed by Amberg, it is possible to prove an analogous lemma concerning the weak chain conditions on subgroups.

Lemma 2.2.2. *(Zaitsev) Let G be a group $A, B \leq G = AB$. If A, B satisfy the wmax (wmin), then G satisfies wmax-n (wmin-n).*

Essentially the first lemma affirms that the product of two subgroups that satisfy Max (Min) satisfies Max-n (Min-n). For what we saw reducing to locally nilpotent groups we get that Max-n (Min-n) gives us Max (Min). So, if the entire group is locally nilpotent, these properties, when owned by the factors, are owned by the whole group.

The second lemma affirms an analogous property related to wmax and wmin, but in this case if we reduce ourselves to the case of locally nilpotent we do not get the transmission of those properties from the factors to the entire group. For an example of this fact see [37] and [38]. In the same papers Kurdachenko proved the properties that can be deduced in locally nilpotent groups satisfying wmin-n. We will use them frequently in the final chapter to improve our results. For this reason a selection of those properties are summarized in the following:

Theorem 2.2.3. *(Kurdachenko) Let G be a locally nilpotent group. If it satisfies wmin-n, then the following hold:*

1) G is hypercentral and solvable;

2) $\mathfrak{F}(G)$ is periodic divisible and abelian;

3) $G/T(G)$ is minimax.

If it satisfies wmax-n, then

4) G is hypercentral if and only if it is solvable;

5) $\mathfrak{F}(G)$ is Černikov.

Here $\mathfrak{F}(G)$ is the $\mathfrak{F}$-perfect part of G. A group is said to be $\mathfrak{F}$-perfect if it has no proper normal subgroups of finite index (see for details [48, §9, p. 123]). Another important theorem, which we will use frequently in the final chapter to improve our results, is the theorem of Wilson (see [57]).

Theorem 2.2.4. *Let G be a solvable group, $A, B \leq G$ two minimax subgroups such that $G = AB$. Then G is minimax.*

2.3 Structure problem

It is quite natural to imagine that the structure of a factorized group $G = AB$ is influenced by the properties of its subgroups A and B. The point is how deep this influence is. There are trivial examples in which that influence is strong, for instance, if A and B are finite then G is finite; other examples that are not so straightforward, are, for instance, those considered in the previous section about Max, Min, wmax and wmin. Nevertheless this neat transmission of properties is not very common and it is actually a field of study in group product theory. On the other hand, it is also interesting to find a property $\mathcal{P}$ and a factorized group $G = AB$ such that the factors A and B satisfy $\mathcal{P}$ and G does not.

To give a couple of important examples, there exists $G = AB$ such that G is the product of two FC-groups, but it is not FC, see [47, §4, p. 125]; the second example is by Suchkov and is the following theorem (for the details see [1, §3, p. 42-46] or [52])

Theorem 2.3.1. *There exists a countable non-periodic group $G = AB$ that is factorized by two locally finite subgroups A and B.*

Returning to the main problem, we need to point out the most important results. The first is a true milestone in this theory. It is a result due to Itô [33], that making use of easy commutator calculation, proved:

Theorem 2.3.2. *(Itô, 1955) Let the group $G = AB$ the product of two abelian subgroups A and B. Then G is metabelian.*

In the same period, another relevant problem that drew the attention of many mathematicians (Huppert, Szép, Wielandt and others) was the product of finite nilpotent groups, that was first conjectured and then proved to be solvable.

Theorem 2.3.3. *(Kegel, Wielandt) Let the finite group $G = AB$ be the product of two nilpotent subgroups A and B. Then G is solvable.*

For the details of the proof we suggest [1, §2, p. 27-32]. We state also this

Corollary 2.3.4. *Let the finite group $G = G_1G_2 \dots G_r$ the product of pairwise permutable nilpotent subgroups, $G_1, \dots, G_r$. Then G is solvable.*

A very rich field of investigation is that of the abelian factorizations. In fact, the Itô theorem gives an important incentive guaranteeing solvability, and, as a matter of fact, Sesekin in [50] and [51] proved that if the factors are abelian with Min, so is the whole group; moreover if the factors satisfy max, the entire group is polycyclic. Investigating further the abelian factorization, Zaitsev obtained considerable results assuming finite rank conditions. In [58] he found for example, assuming finite $r_0(A)$ and $r_0(B)$, that $r_0(G) = r_0(A) + r_0(A) - r_0(A \cap B)$. In the same paper he proved an analogous result about finite minimax rank.

Another noteworthy field of investigation is factorization with finiteness conditions, in particular the so-called chain conditions. As we already mentioned, Amberg in [3] proved that the product of two groups that satisfy Max (Min), satisfies Max-n (Min-n). Zaitsev in [59] proved an analogous result about wmax and wmin. The most important result concerning groups factorized by two Černikov groups was proved by Černikov. The precise statement is the following:

Let $G = AB$ be a locally graded group where A and B are Černikov. Then G is Černikov.

This statement is really important also because it does not contain any *a priori* solvability condition, something that in most of the other demonstrations is necessary. To give some examples, Zaitsev in [60] and Lennox Rosebald in [41] proved that given a solvable group G factorized by its subgroups A and B, G is polycyclic if and only if A and B are polycyclic. An analogous result has been proved for solvable factorized groups if one of the two factors is nilpotent by Amberg and Robinson in [6], and if both the factors are locally nilpotent by Zaitsev in [59].

To complete this short overview, we cite an open question.

Problem 2.3.5. *Let the group $G = AB$ be the product of the abelian-by-finite subgroups A and B. Is it solvable-by-finite?*

We still do not know if this statement is true or not, but it was proved in several particular cases. For a precise summary of those cases see [4]. We recall only one instance among the other, proved by Černikov (1981).

Theorem 2.3.6. *Let the group $G = AB$ be the product of two central-by-finite subgroups A and B. Then G is solvable-by-finite.*

Chapter 3

$\mathcal{C}$-connection between subgroups

Let $\mathcal{C}$ be a class of groups, let G be a group and consider $A, B \leq G$ two subgroups. We will say that the subgroups A and B are $\mathcal{C}$-connected if for all $a \in A$ and $b \in B$ the subgroup $\langle a, b \rangle$ belongs to $\mathcal{C}$. This definition, that is central in this work, was formulated for the first time by Carocca in [14].

Although this definition is clear and immediately comprehensible, it is not easy to make any remarks without choosing any specific class of groups. So far, the literature about $\mathcal{C}$-connection shows us some results regarding these specific classes of groups:

1) the class $\mathcal{N}$ of nilpotent groups and, given $k \in \mathbb{N}$, the class $\mathcal{N}_k$ of groups with nilpotency class at most k;

and a bit more marginally

2) the class $\mathcal{S}$ of solvable groups;

3) the class $\mathcal{U}$ of supersolvable groups;

4) the class $\mathcal{NA}$ of nilpotent-by-abelian groups;

5) the class $\mathcal{N}^2$ of metanilpotent groups.

3.1 Total permutability and Carocca's theorems

To trace the origin of the definition of $\mathcal{C}$-connection, it is probably necessary to state the following theorem by Asaad and Shaalan [7]

Theorem 3.1.1. *Let $G = AB$ be a finite group, such that A and B are totally permutable supersolvable groups. Then G is supersolvable.*

And the generalization Carocca gave in [14],

Theorem 3.1.2. *Let $G = G_1 G_2 \dots G_r$ be a finite group such that $G_1, G_2, \dots, G_r$ are pairwise totally permutable subgroups of G. Let $\mathcal{F}$ be a saturated formation which contains the class of supersolvable groups. If for all $i : 1, \dots, r$ the subgroups G_i are in $\mathcal{F}$, then $G \in \mathcal{F}$.*

Where given a group $G = AB$ we say that A and B are totally permutable if and only if every subgroup of A is permutable with every subgroup of B.

To better understand the reason why these two results gave rise to idea of the $\mathcal{C}$-connection, we need to stress the fact that if $G = AB$ and A, B are totally permutable, then for all $a \in A$ and $b \in B$ we get

$$\langle a, b\rangle = \langle a\rangle\langle b\rangle = \langle b\rangle\langle a\rangle$$

and a well known result by Douglas [22] and Itô [34] says that the product of two cyclic groups is supersolvable and abelian-by-finite, i.e. total permutability implies $\mathcal{U}$-connection. Observe that the other implication is not true: for example the dihedral group D_8 is nilpotent (and then supersolvable), but is not hard to find two non totally permutable subgroups.

Anyway, it is not possible to generalize theorem 3.1.2 with the weaker condition of $\mathcal{U}$-connection instead of total permutability. In fact

Example 3.1.3. *(Peterson, 1973) Consider $H := C_5 \times C_5$ and call x, y two generators of those cyclic groups; let a, b be the following automorphisms*

$$x^a = x^2,\ y^a = y^3,\ x^b = y,\ y^b = x$$

Call $A := \langle a, b\rangle$, that is isomorphic to D_8, and $G := H \rtimes A$. This group G is an example of a finite group that is a product of two supersolvable, $\mathcal{U}$-connected subgroups, but it is not supersolvable. In fact it is not difficult to show that for every element $c \in A$, the subgroups $\langle c, H\rangle$ are all supersolvable, fact that guarantees the $\mathcal{U}$-connection, while G' is not nilpotent and then G is not supersolvable.

Even though in this case $\mathcal{U}$-connection does not work very well, Carocca succeeded in proving in [14]

Theorem 3.1.4. *Let $G = G_1 G_2 \dots G_r$ be a finite group such that $G_1, \dots, G_r$ are pairwise permutable subgroups of G. Let $\mathcal{F}$ be a saturated formation such that $\mathcal{N} \subseteq \mathcal{F}$. If for every pair $i, j \in \{1, \dots, r\}$, $i \neq j$, the subgroups G_i and G_j are $\mathcal{N}$-connected $\mathcal{F}$-groups, then $G \in \mathcal{F}$.*

Some years later in [15], he proved, using the classification theorems of finite simple groups, the following:

Theorem 3.1.5. *Let $G = G_1 G_2 \ldots G_r$ be a finite group such that $G_1, \ldots, G_r$ are solvable subgroups of G. If they are pairwise permutable and $\mathcal{S}$-connected subgroups, then G is solvable.*

Considering once again example 3.1.3, we can remark that is not even possible to generalize theorem 3.1.4, with $\mathcal{U}$-connection.

In conclusion we would like to underline a fact concerning the $\mathcal{U}$-connection. Although apparently there is no literature about it, the fact that the total permutability implies $\mathcal{U}$-connection allows us to include the literature about total permutability in the $\mathcal{U}$-connection literature. Nevertheless, we will not examine it in depth, but we cite, for instance, [12] for interested readers.

3.2 Cosubnormality and $\mathcal{N}$-connection

After we mentioned the relationship that holds between $\mathcal{U}$-connection and total permutability, it may be interesting to show the relationships that occur between cosubnormality and $\mathcal{N}$-connection. Given a finite group G, we say that $A, B \leq G$ are cosubnormal, and we write $AcsB$, if A, B are subnormal in their joint $\langle A, B \rangle$. Moreover we say that $A, B \leq G$ are strongly cosubnormal, $AscsB$ if every subgroup of A is cosubnormal with every subgroup of B.

Remark 3.2.1. *If G is finite, A and B are $\mathcal{N}$-connected is equivalent to say that every cyclic subgroup of A is cosubnormal with every cyclic subgroup of B. In other words in finite groups strong cosubnormality implies $\mathcal{N}$-connection.*

This property has been deeply discussed in [11], where the authors proved the following characterization theorem:

Theorem 3.2.2. *Let G a finite group and $A, B \leq G = \langle A, B \rangle$. The following statements are equivalent:*

1) $AscsB$;

2) $AcsB$ and A, B are $\mathcal{N}$-connected;

3) $[A, B] \leq Z_\infty(G)$.

Corollary 3.2.3. *Let G be a finite group and A, B two $\mathcal{N}$-connected subgroups of G. Then $[A, B]$ is nilpotent if and only if A and B are subnormal in $\langle A, B \rangle$.*

This theorem shows us that cosubnormality and $\mathcal{N}$-connection are closely related concepts, even though they are not equivalent. In fact: consider a non nilpotent group G. If we put $G = A = B$, then A and B are trivially cosubnormal, but they are not $\mathcal{N}$-connected. For the vice versa, the authors construct an example in [11] or alternatively we propose the following example in [24]

Example 3.2.4. *(Fumagalli) Let G be the symmetric group of degree 8. Consider*

$$h := (12)(34)(56)(78), \quad a := (23)(45)(67), \quad b := (24)(35)(67)$$

Called $H := \langle h \rangle$ and $A := \langle a, b \rangle$, it is possible to show that $G = \langle A, H \rangle$ and A and H are $\mathcal{N}_3$-connected subgroups, but for example H is not subnormal in G.

In the important case of products the authors proved that $\mathcal{N}$-connection and strong cosubnormality coincide.

Theorem 3.2.5. *Let G be a finite group and let $A, B \leq G = AB$. If A and B are $\mathcal{N}$-connected, the they are strongly cosubnormal.*

3.3 Generalizing Carocca theorems

The work of Carocca on $\mathcal{N}$-connection of products of groups was taken further first in the finite solvable universe in [13], for products of two $\mathcal{N}$-connected groups with a particular accent on the formation theory; after it was extended in the finite universe and to the products of finitely many factors in [32]. The results we are going to state are proved in these two articles. We start by showing a proposition that collects the basic properties of a product of pairwise $\mathcal{N}$-connected and permutable groups.

Proposition 3.3.1. *Let the finite group $G = G_1 \dots G_r$ be the product of the pairwise permutable $\mathcal{N}$-connected subgroups $G_1, \dots, G_r$. Then the following properties holds:*

1) $[G_i^{\mathcal{N}}, G_j] = 1$ for all $i, j : 1, \dots, r$. In particular $G_i^{\mathcal{N}} \lhd G$ for all $i : 1, \dots, r$;

2) $G_i \lhd \lhd G$ for all $i : 1, \dots, r$;

3) If $(G_i)_p \in Syl_p(G_i)$ for each $i : 1, \dots, r$ and prime p, then $(G_i)_p(G_j)_p = (G_j)_p(G_i)_p \in Syl_p(G_iG_j)$;

4) If X_i is a p-subgroup of G_i for each $i : 1, \dots, r$ and prime p, then $\langle X_1, \dots, X_r \rangle$ is a p-subgroup of G;

5) If X_i is a nilpotent subgroup of G_i for each $i : 1, \dots, r$, then $\langle X_1, \dots, X_r \rangle$ is a nilpotent subgroup of G;

6) $G^{\mathcal{N}} = G_1^{\mathcal{N}} \dots G_r^{\mathcal{N}}$;

7) If $I, J \subseteq \{1, \dots, r\}$ and $I \cap J = \emptyset$, then $\prod_{i \in I} G_i$ and $\prod_{j \in J} G_j$ are $\mathcal{N}$-connected;

8) If $I, J \subseteq \{1, \dots, r\}$ and $I \cap J = \emptyset$, then

$$[\prod_{i \in I} G_i, \prod_{j \in J} G_j] \leq Z_\infty(G)$$

are $\mathcal{N}$-connected; Moreover

$$\prod_{i \in I} G_i \cap \prod_{j \in J} G_j \leq Z_\infty(G);$$

9) If X_i is a π-subgroup of G_i for each $i : 1, \dots, r$ and set of primes π, then $\langle X_1, \dots, X_r \rangle$ is a π-subgroup of G.

Now we report some generalizations of Theorem 3.1.4. The first one is more a consideration about the fact that the hypothesis are not sharp. In fact in the Carocca's original statement the saturated formation $\mathcal{F}$ is assumed to contain $\mathcal{N}$, but the same proof shows that this hypothesis is not necessary. For this reason in [32] we have the following restatement

Theorem 3.3.2. *Let $G = G_1 G_2 \dots G_r$ be a finite group such that $G_1, \dots, G_r$ are pairwise permutable subgroups of G. Let $\mathcal{F}$ be a saturated formation. If for every pair $i, j \in \{1, \dots, r\}$, $i \neq j$, the subgroups G_i and G_j are $\mathcal{N}$-connected $\mathcal{F}$-groups, then $G \in \mathcal{F}$.*

The second generalization has been proved in [13] for solvable groups and then extended in [32] to a more general result that is the following

Lemma 3.3.3. *Let $G = G_1 G_2 \dots G_r$ be a finite group such that $G_1, \dots, G_r$ are pairwise permutable subgroups of G. Let $\mathcal{F}$ be a formation such that $\mathcal{N} \subseteq \mathcal{F}$. If for every pair $i, j \in \{1, \dots, r\}$, $i \neq j$, the subgroups G_i and G_j are $\mathcal{N}$-connected $\mathcal{F}$-groups, then $G \in \mathcal{F}$.*

The authors noticed that in the case of general formation the hypothesis $\mathcal{N} \subseteq \mathcal{F}$ is required.

Example 3.3.4. *Consider the formation $\mathcal{F}$ of all elementary abelian p-groups, for p prime. Let $G = \mathbb{Z}_p \wr \mathbb{Z}_p = B \rtimes \mathbb{Z}_p$, where B is the base group following the standard notations. Obviously $B, \mathbb{Z}_p \in \mathcal{F}$ and they are $\mathcal{N}$-connected, but $G \notin \mathcal{F}$.*

Another interesting result is the following

Proposition 3.3.5. *Let $G = G_1 G_2 \dots G_r$ be a finite group such that $G_1, \dots, G_r$ are pairwise permutable $\mathcal{N}$-connected subgroups of G. Then*

$$Z_\infty(G) = Z_\infty(G_1) \dots Z_\infty(G_r)$$

Among other results presented, we will cite one about $\mathcal{F}$-projector and another one about $\mathcal{F}$-residual, where $\mathcal{F}$ is, as usual, a formation.

Theorem 3.3.6. *Let $\mathcal{F}$ be a saturated formation and let $G = G_1 \dots G_r$ be the product of pairwise $\mathcal{N}$-connected and permutable subgroups $G_1, \dots, G_r$. If $X_i \in Proj_{\mathcal{F}}(G_i)$, far every $i : 1, \dots, r$ then $X_1 \dots X_r$ is a pairwise permutable product of the subgroups $X_1, \dots, X_r$ and $X_1 \dots X_r \in Proj_{\mathcal{F}}$. Moreover, if G has a unique conjugacy class of $\mathcal{F}$-projectors,then every $\mathcal{F}$-projector of G has this form.*

Proposition 3.3.7. *Let $\mathcal{F}$ be a formation and let the group $G = G_1 \dots G_r$ be the product of the pairwise $\mathcal{N}$-connected and permutable subgroups $G_1, \dots, G_r$. If either $\mathcal{F}$ is saturated or $\mathcal{N} \subseteq \mathcal{F} \subseteq S$, then*

$$G^{\mathcal{F}} = G_1^{\mathcal{F}} \dots G_r^{\mathcal{F}}$$

In particular, if $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$, for all $i : 1, \dots, r$.

3.4 $\mathcal{C}$-connection with $\mathcal{C} \in \{\mathcal{N}^2, \mathcal{N}\mathcal{A}, \mathcal{N}_2\}$

More recently M. P. Gállego, P. Hauck, M. D. Pérez-Ramos introduced the study of three different kinds of $\mathcal{C}$-connection. The first one is the connection where $\mathcal{C} = \mathcal{N}^2$ the class of metanilpotent groups, the second is the connection where $\mathcal{C} = \mathcal{N}\mathcal{A}$ the class of nilpotent-by-abelian groups and the third is the connection with $\mathcal{C} = \mathcal{N}_2$ the class of nilpotent groups with nilpotency class at most 2.

3.4.1 $\mathcal{N}^2$-connection

We start with the case of metanilpotent connection. All the results we are going to mention are in [25]. The main result is the following

Theorem 3.4.1. *Let G be a finite solvable group, $A, B \leq G = AB$ and suppose that A and B are $\mathcal{N}^2$-connected. Then*

$$\frac{G}{F(G)} = \frac{AF(G)}{F(G)} \frac{BF(G)}{F(G)}$$

is an $\mathcal{N}$-connected product of the two factors.

Although the statement of this theorem is very simple, the demonstration is rather complex and a bit technical. We will only comment on some remarks the authors make.

Remark 3.4.2. *It is not possible to generalize theorem 3.4.1 to the case of a product of more then two factors. For instance, let G be the symmetric group of degree 4. Choose*

$$A := \langle (12) \rangle \quad B := \langle (123) \rangle \quad C := \langle (12)(34), (13)(24) \rangle$$

Then $G = ABC$ is the product of these pairwise permutable $\mathcal{N}^2$-connected subgroups, but $AF(G)/F(G)$ and $BF(G)/F(G)$ are not $\mathcal{N}$-connected.

Remark 3.4.3. *It is not even possible to remove the hypothesis for G to be the product of A and B. In fact, considering the group $G := A_4 \wr C_2$, called $A := A_4 \times 1$ and $B := C_2$ we get $G = \langle A, B \rangle$ with A and B $\mathcal{N}^2$-connected subgroups, but $AF(G)/F(G)$ and $BF(G)/F(G)$ are not $\mathcal{N}$-connected.*

3.4.2 $\mathcal{NA}$-connection

The second connection we are going to present in this section is the $\mathcal{F}$-connection with respect to all saturated formations $\mathcal{F} \subseteq \mathcal{NA}$. We will cite only a few of the results that can be found in [26]. The main theorem contains some statements both for $\mathcal{F}$-connected subset and subgroups. We resume it stating only the subgroups instances.

Theorem 3.4.4. *Let $\mathcal{F}$ a saturated formation such that $\mathcal{F} \subseteq \mathcal{NA}$. Let A and B two subgroups of the finite group $G = \langle A, B \rangle$. The following condition are equivalent:*

1) A and B are $\mathcal{F}$-connected and $[A, B] \leq F(G)$;

2) A and B^g are $\mathcal{F}$-connected for all $g \in G$, and $[A, B] \leq F(G)$;

3) $[A, B]\langle g \rangle \in \mathcal{F}$ for all $g \in G$, and $[A, B] \leq F(G)$;

4) A^G and B^G are $\mathcal{F}$-connected.

Another interesting result the authors proved, recall the one on $\mathcal{N}^2$-connection.

Theorem 3.4.5. *Let G be a finite group, $G = AB$ where A and B are subgroups. Then A and B are $\mathcal{NA}$-connected if and only if $[A, B] \leq F(G)$ or equivalently $AF(G)/F(G)$ and $BF(G)/F(G)$ are $\mathcal{A}$-connected.*

For this theorem we can deduce a nice corollary, that has been already proved in [16]. It is the following:

Corollary 3.4.6. *Any saturated formation $\mathcal{F} \subseteq \mathcal{NA}$ is 2-recognizable.*

Among the other results we cite the following characterization:

Proposition 3.4.7. *Let $\mathcal{F}$ be a saturated formation such that $\mathcal{F} \subseteq \mathcal{N}\mathcal{A}$. The following conditions are equivalent:*

1) $\mathcal{F} \subseteq \mathcal{N}\mathcal{E}_2$, where $\mathcal{E}_2$ is the class of elementary abelian 2-groups;

2) If a product $G = AB$ is the product of $\mathcal{F}$-connected subgroups A and B, then $A, B \in \mathcal{F}$ implies $G \in \mathcal{F}$.

3.4.3 $\mathcal{N}_2$-connection

As far as the $\mathcal{N}_2$-connection is concerned, the main results are to be found in [27]. This connection is in some sense the strongest we have seen until now and one of the few in literature that is treated without assuming the finiteness of the involved groups. With a little calculation the authors proved the following:

Proposition 3.4.8. *Let G be a group and A and B two $\mathcal{N}_2$-connected subgroups of G. Then:*

1) $\langle a^B \rangle$ and $\langle b^A \rangle$ are abelian for all $a \in A$ and $b \in B$;

2) $[A, B]$ centralizes A' and B';

3) $\langle a^B \rangle$ and B are $\mathcal{N}_2$-connected.

This proposition calls to mind the results obtained by Levi in [42] about 2-Engel groups; in fact it is the case if we assume $G = A = B$.

Some other properties that are investigated are those of the subgroup $[A, B]$ and those related to elements of order 2 which assume an important role. To give an example it is proved that the group $[A, B]'$ can be generated by elements of order two. There are however lots of results to show the structure of those groups and examples to underline that these results are sharp. We state only one theorem among the others:

Theorem 3.4.9. *Let A and B be $\mathcal{N}_2$-connected subgroups of $G = \langle A, B \rangle$. Then:*

1) A^2 is subnormal in G of defect at most 3;

2) A' is subnormal in G of defect at most 2;

3) $\gamma_n(A)$ and $A^{(m)}$ are normal in G for all $n \geq 3$ and $m \geq 2$.

3.5 $\mathcal{N}_k$-connection and FE_k-property

The most recent development ascribable to the theory of $\mathcal{C}$-connection is in the universe of infinite groups and regards the FE_k-property. If k is a positive integer, a group G is said to have the FE_k-property if for every element $g \in G$ there exists a normal subgroup X_g such that $|G : X_g|$ is finite and the subgroup $\langle x, g \rangle$ is nilpotent of class at most k for all $x \in X_g$. Clearly this is equivalent to saying that every cyclic subgroup of G is $\mathcal{N}_k$-connected with a (normal) subgroup of finite index. The aim of theory of FE_k-groups is to extend the properties of FC-groups, as we will see. All the results we are going to show appear in [21].

Here is a selection of the main results provided by the authors. We start with a lemma

Lemma 3.5.1. *Let G be a locally graded FE_k-group, where k is a positive integer. Then the finite residual of G is locally nilpotent.*

Now, we present the main theorem

Theorem 3.5.2. *Let G be a group. Let H and F be respectively the Hirsch-Plotkin radical and the Fitting subgroup.*

1) *If G is a solvable-by-finite FE_k-group, for some $1 \leq k \in \mathbb{N}$, then H is a Bear group and G/H is a periodic FC-group.*

2) *If G is an FE_2-group, then G/F is a FC-group and G/H is periodic.*

And in conclusion here is another theorem.

Theorem 3.5.3. *Let G be a finitely generated solvable-by-finite group and let k be a positive integer . If G is an FE_k-group then there exists $m \in \mathbb{N}$ depending only on k, such that the group $G/\zeta_m(G)$ is finite.*

Chapter 4

$\mathcal{N}$-connection

As we saw in the previous chapter, $\mathcal{C}$-connection in product gained, little by little, the interest of many authors. However the results that come from the literature are essentially in the universe of finite groups and this fact could be a boundary to the development of that theory. One of the purposes, maybe secondary, of this work is to include also infinite groups and to show some properties in which $\mathcal{C}$-connection and, in particular, $\mathcal{N}$-connection are in some sense decisive. In the following pages we try to show some cases in which $\mathcal{N}$-connection is determinating in the factorization of certain subgroups or in proving some specific properties.

4.1 Torsion subgroups and $\mathcal{N}$-connection

The set of the elements of finite order of a group G is often indicated with $T(G)$ and in general it is not a subgroup (think for instance of the infinite dihedral group). Nevertheless, for some classes of groups, this is true.

So, probably, it could be interesting to show how the $\mathcal{N}$-connection contributes to facilitate the factorization of the torsion group in a product of groups.

Lemma 4.1.1. *Let $G = \langle a, b\rangle$ be a two generated nilpotent group. If $|a| = \infty$ and $|b| < \infty$ then there exists $n \in \mathbb{N}$ such that $\langle a^n\rangle = Z(G)$ and, in other words, G is central-by-finite.*

Proof. Called $T := T(G)$, it is not difficult to prove that $|T(G)| < \infty$ and $G = \langle a\rangle T$. So, there exists $n \in \mathbb{N}$ such that $a^n \in C_G(T)$, and then $\langle a^n\rangle = Z(G)$ □

Proposition 4.1.2. *Let $G = AB$ be the product of the $\mathcal{N}$-connected subgroups A and B. If $T(A)$ and $T(B)$ are subgroups and $A \cap B$ is periodic then $T(G)$ is a subgroup of G and $T(G) = T(A)T(B)$.*

Proof. By definition and $\mathcal{N}$-connection we have $T(A)T(B) \subseteq T(G)$ and $T(B)T(A) \subseteq T(G)$. Consider $g \in T(G)$. There exist $a \in A$ and $b \in B$ such that $g = ab$. Concerning the order of a and b, we can divide the problem into 3 cases:

1) $|a|$,$|b| < \infty$;

2) $|a| = \infty$, $|b| < \infty$ (or $|a| < \infty$, $|b| = \infty$);

3) $|a| = \infty$, $|b| = \infty$.

Case 2) is not possible, in fact $\langle a, b\rangle$ is nilpotent and clearly

$$\langle a, b\rangle = \langle ab, b\rangle \leq T(\langle a, b\rangle)$$

that is a finite group, but $|a| = \infty$, a contradiction.

Case 3) is not possible too. In fact, stressing the fact that

$$\langle a, b\rangle = \langle ab, b\rangle = \langle a, ab\rangle$$

by lemma 4.1.1, we have that there exist $n, m \in \mathbb{N}$ such that $\langle a^n\rangle = Z(\langle a, b\rangle)$ and $\langle b^m\rangle = Z(\langle a, b\rangle)$, i.e. $\langle a^n\rangle = \langle b^m\rangle$. Thus, $A \cap B$ contains an element of infinite order, a contradiction.

So, the only possibility is case 1) that means $T(G) \subseteq T(A)T(B)$ and $T(G) \subseteq T(B)T(A)$. That leads us to affirm:

$$T(G) = T(A)T(B) = T(B)T(A),$$

our thesis.

□

Let us point out by an example the reason why the hypothesis of periodicity of the group $A \cap B$ is essential.

Example 4.1.3. *Consider $A = \langle a\rangle$ where $|a| = \infty$ and $X = \langle x\rangle$ with $|x| = n < \infty$. Consider now $G = A \times X$. We call $b = a^{-1}x$ and $B = \langle b\rangle$. Then $G = AB$, A and B are trivially $\mathcal{N}$-connected and*

$$b^n = (a^{-1}x)^n = (a^{-1})^n x^n = (a^{-1})^n$$

i.e. $T(A) = 1$, $T(B) = 1$, but $T(G) = X \neq 1$

On the other hand it is possible to prove that also by removing the $\mathcal{N}$-connection proposition 4.1.2 is no longer true. For an example see the theorem of Suchkov 2.3.1.

4.2 Isolators and $\mathcal{N}$-connection

In chapter 1, we introduced a very useful tool in the theory of locally nilpotent groups: the isolator of a subgroup. We only recall the fact that given a locally nilpotent group G, for every subgroup $H \leq G$, the isolator $I_G(H)$ is a subgroup of G.

It is easy to show that, if G is a factorized group, $H \leq G$ and we consider $I_G(H)$, that the isolator is not necessarily factorized (see for instance the previous example 4.1.3). This fact should not be surprising because the factorization of a subgroup, as we already observed, is not common. However, in the following Proposition we show that adding a few hypotheses, we easily get some factorizations.

Proposition 4.2.1. *Let G be a group and $A, B \leq G = AB$ such that they are $\mathcal{N}$-connected. Suppose that for all $C \leq A$ and $D \leq B$, $I_A(C)$ and $I_B(D)$ are subgroups (e.g. A and B are locally nilpotent). If $S, T \leq G$ are such that $A \leq S$ and $B \leq T$, then*

1) $I_G(S) = AI_B(S \cap B)$ and $I_G(T) = I_A(A \cap T)B$;

2) $I_G(S) \cap I_G(T) = I_G(S \cap T)$.

Moreover if we have $H, K \leq G$ such that $A \cap B \leq H \leq A$ and $A \cap B \leq K \leq B$, then:

3) $I_G(H) = I_A(H)I_B(A \cap B)$ and $I_G(K) = I_A(A \cap B)I_B(K)$.

Proof. 1) Consider $g \in I_G(S)$. By definition there exists $t \in \mathbb{N}$ such that $g^t \in S$. By hypothesis there exist $a \in A$ and $b \in B$ such that $g = ab$ and for the $\mathcal{N}$-connection $\langle a, b\rangle$ is nilpotent; hence by lemma 1.3.12, $|\langle a, g\rangle : \langle a, g^t\rangle| < \infty$ so there exists $m \in \mathbb{N}$ such that $b^m \in \langle a, g^t\rangle \leq S$, in other words $b \in I_B(S \cap B)$. For this reason

$$I_G(S) \subseteq AI_B(S \cap B)$$

Analogously this $I_G(S) \subseteq I_B(S \cap B)A$ holds, and observing the obvious inclusion A, $I_B(S \cap B) \subseteq I_G(S)$ we have that $I_G(S)$ is a group and that the required factorization holds.

2) Holds by definition.

3) Consider $g \in I_G(H)$. By definition there exists $t \in \mathbb{N}$ such that $g^t \in H$. By hypothesis there exist $a \in A$ and $b \in B$ such that $g = ab$ and for the $\mathcal{N}$-connection $\langle a, b\rangle$ is nilpotent; hence by lemma 1.3.12, $|\langle a, g\rangle : \langle a, g^t\rangle| < \infty$ so there exists $m \in \mathbb{N}$ such that $b^m \in \langle a, g^t\rangle \leq A$, in other words $b \in I_B(A \cap B)$. Similarly

$\langle g^t, b^m \rangle$ has finite index in $\langle g, b \rangle$, so there exists $n \in \mathbb{N}$ such that $a^n \in \langle g^t, b^m \rangle \leq H$, i.e. $a \in I_A(H)$. For $I_G(K) = I_A(A \cap B) I_B(K)$ we can apply the same method.

□

Even in this case $\mathcal{N}$-connection plays a crucial role, and also in this situation Theorem 2.3.1 provides an example in which by removing that hypothesis, the proposition is no longer true.

4.3 Hypercentre and FC-hypercentre

In this paragraph, given the factorized group $G = AB$ where A and B are locally nilpotent $\mathcal{N}$-connected subgroups, we will see a strong link between the hypercentre and the FC-hypercentre of G.

What we are going to present is a slight generalization of the result of McLain about the hypercentre and FC-hypercentre of a locally nilpotent group G, for details see [47, §4, p. 130]. In fact, assuming in our case $A = B = G$, we get trivially the local nilpotence of G.

The statements and the observations we are going to present are essentially analogous to those of McLain, but the main demonstration is different and deserves to be proved. We start with a lemma:

Lemma 4.3.1. *Let $G = AB$ be a group factorized by two locally nilpotent subgroups A and B. Let N be a normal subgroup of G. If $|G : C_G(N)| < \infty$, then the following hold:*

1) the subgroup N is solvable;

Moreover if A and B are $\mathcal{N}$-connected

2) the subgroup N is nilpotent.

Proof. 1) Consider $C_G(N)$, which is a normal subgroup of finite index in G. The factor group $G/C_G(N)$ is the product of two finite nilpotent subgroups, hence for 2.3.3 it is solvable. Thus also the subgroup $NC_G(N)/C_G(N)$ is solvable, and since

$$\frac{NC_G(N)}{C_G(N)} \simeq \frac{N}{C_G(N) \cap N} \simeq \frac{N}{Z(N)},$$

that implies that N is solvable.

2) In this case $G/C_G(N)$ is nilpotent for 3.1.4 and then also $N/Z(N)$ is nilpotent.

□

Now we need to introduce some definitions and notations. In any group it is possible to form the upper FC-central series

Definition 4.3.2. *The upper FC-central series is the series $\{F_\alpha\}$ defined by the following rules $F_0 = 1$, $F_{\alpha+1}/F_\alpha$ = FC-centre of G/F_α and if λ is a limit ordinal*

$$F_\lambda = \bigcup_{\beta<\lambda} F_\beta$$

The limit of the upper FC-central series is the FC-hypercentre. A group G is said to be FC-hypercentral if it coincides with its FC-hypercentre. For the rest of the section we will use Z_α and F_α respectively for the αth term of the upper central series and of the upper FC-central series of the involved group. We are now ready for the following:

Proposition 4.3.3. *Let $G = AB$ be a group factorized by A and B that are two locally nilpotent $\mathcal{N}$-connected subgroups. Then*

$$FC(G) \leq Z_\omega(G)$$

Proof. Let $x \in FC(G)$. We divide the proof in two cases:

Case 1) $|x| < \infty$. Without loss of generality we can assume that x is a p-element, for a prime p. By Dicman lemma, $\langle x\rangle^G$ is finite. By the previous lemma, we get that $\langle x\rangle^G$ is nilpotent. If we consider P the Sylow p-group, we get that $x \in P char \langle x\rangle^G \lhd G$, i.e. $P = \langle x\rangle^G$. Now,

$$AP = AP \cap AB = (AP \cap B)A$$

is a product of locally nilpotent $\mathcal{N}$-connected subgroups. This implies that every finite homomorphic image is nilpotent. We deduce that $A/C_A(P)$ is a finite p-group. Analogously $B/C_B(P)$ is a finite p-group and by $\mathcal{N}$-connection and lemma 2.1.10, we get that $G/C_G(P)$ is a finite p-group, then $P \leq Z_n(G)$ for some $1 \leq n \in \mathbb{N}$.

Case 2) $|x| = \infty$. Let $K = \langle x\rangle^G$; notice that $G/C_G(K)$ is a finite group. Now, for the previous lemma K is a nilpotent, centre-by-finite subgroup. Thus K' is finite for a known theorem. Clearly $K' \lhd G$ then, using case 1), we can reduce ourselves to $K' = 1$. So, we have that K is a finitely generated abelian group. By case 1) we can reduce to the torsion free case. Let p be a prime such that it does not divide $|G/C_G(K)|$. By case 1) K/K^p is contained in the hypercentre of G/K^p; however, stressing the fact that $G/C_G(K)$ is a p'-group, we conclude that $K/K^p \leq Z(G/K^p)$, that is $[K, G] \leq K^p$. This fact holds for an infinite number of primes p and then we get $[K, G] = 1$ or in other words $K \leq Z(G)$.

□

Remark 4.3.4. *Following the same strategy of McLain in [45], it is possible to extend 4.3.3 and prove the following fact:*

Let $G = AB$ be a group factorized by A and B that are two locally nilpotent $\mathcal{N}$-connected subgroups. Then $Z_\alpha \leq F_\alpha \leq Z_{\omega\alpha}$, for each ordinal α.

Corollary 4.3.5. *Let $G = AB$ be a group factorized by A and B that are two locally nilpotent $\mathcal{N}$-connected subgroups. Then the hypercentre and the FC-hypercentre coincide.*

4.4 Examples of $\mathcal{N}$-connected subgroups

In this last section we present some examples of $\mathcal{N}$-connected subgroups of a group.

Example 4.4.1. *The first examples that come to mind are the locally nilpotent groups. In fact, if G is a locally nilpotent group, for all couples of subgroups A and B, they are $\mathcal{N}$-connected. Anyway, it is clear that there is, at least, a wider class of groups that contains locally nilpotent groups and in which all couples of subgroups are $\mathcal{N}$-connected: the class of weakly nilpotent groups.*

A group G is said to be weakly nilpotent if and only if each couple of elements $x, y \in G$ generates a nilpotent group, i.e. $\langle x, y\rangle \in \mathcal{N}$. Keeping in mind the definition of $\mathcal{N}$-connection, it is possible to say that a group is weakly nilpotent if and only if it is $\mathcal{N}$-connected with itself. Nevertheless we have few important examples of groups that are weakly nilpotent but not locally nilpotent. Essentially they are represented by the Golod groups [28]. The construction of these groups, although very interesting, is beyond the scope of this work. So we only state the following

Theorem 4.4.2. *(Golod) Let p be a prime, and $d \geq 2$. then there exists a d-generated infinite p-group such that every its $(d-1)$-generated subgroup is finite.*

Assuming $d \geq 3$, we get this

Corollary 4.4.3. *For every prime p there exists a finitely generated weakly nilpotent p-group which is not nilpotent.*

Another example of a group in which it is possible to select two $\mathcal{N}$-connected subgroups is the first Grigorchuk group. This group was constructed by Grigorchuk in [29] in 1980, and it represents a counterexample in many interesting cases. Among the various items, we cite the fact that it is a finitely generated periodic and infinite group, it has an intermediate growth and it is amenable, but not elementary amenable. Now we give a concise presentation, for obvious reason, focused on the $\mathcal{N}$-connection.

Example 4.4.4. *The first Grigorchuk group is defined as a subgroup of the automorphisms group of a binary rooted tree, T_2. In particular given $I = \{0, 1\}$ an alphabet, the root will be the empty word, and the other vertices will be finite length words in the alphabet and the adjacency relations are represented by the addition (or the subtraction) of a letter on the right-hand side. Let us define the automorphism τ like the automorphism that inverts the first letter of all the vertices of the tree T_2, roughly speaking it inverts the two main branches. For the other three automorphisms we need, we will give recurrence definitions.*

To be more precise, given an automorphism $x = (y, z)$, it is defined in this way: it fixes the first letter of the word that represents a vertex, and it will play the role of the automorphism y on the second letter, if the first one was 0, otherwise it will play the role of z. So now we are ready to define the other automorphisms we need, which are

$$a = (b, \tau) \quad b = (c, \tau) \quad c = (a, id)$$

where id is the trivial automorphism.

Now we are able to define the first Grigorchuk group $\mathcal{G}$, that is

$$\mathcal{G} = \langle a, b, c, \tau \rangle$$

To understand which are the subgroups that are $\mathcal{N}$-connected we state a lemma with the basic properties.

Lemma 4.4.5. *Let $\mathcal{G}$ be the first Grigorchuk group, with the notation above the following hold:*

1) $|a| = |b| = |c| = |\tau| = 2$ and $ab = c$, i.e. $\langle a, b \rangle \simeq C_2 \times C_2$;

2) $\langle c, \tau \rangle \simeq D_8$, $\langle b, \tau \rangle \simeq D_{16}$ and $\langle a, \tau \rangle \simeq D_{32}$;

3) every element of $\mathcal{G}$ has order a power of 2, but $\mathcal{G}$ has infinite exponent.

From this lemma it is straightforward to deduce this corollary

Corollary 4.4.6. *The group $A = \langle a, b \rangle$ and the group $T = \langle \tau \rangle$ are $\mathcal{N}_4$-connected.*

The last example of this section belongs to the finite universe. We have just seen that given a group G that is generated by two $\mathcal{N}$-connected subgroups, we can not hope for, in some senses, interesting properties. So, if one imposes the finiteness of G, hoping for a stronger result (compared with the infinite case), one is disappointed. To be more precise we show

Example 4.4.7. *Consider G to be the symmetric group on a set of 8 elements S_8. As we already mentioned Fumagalli, in [24], observed that considering*

$$h := (12)(34)(56)(78), \quad a := (23)(45)(67), \quad b := (24)(35)(67)$$

and calling $H := \langle h \rangle$ *and* $A := \langle a, b \rangle$*, we obtain that* $G = \langle A, H \rangle$*. With a little calculation we proved that* A *and* H *are* $\mathcal{N}_3$*-connected subgroups, i.e. we found an example of non solvable group that is generated by two abelian subgroups satisfying* $\mathcal{N}_3$*-connection.*

Chapter 5

$\mathcal{N}$-connection in products of groups

This chapter is the central core of the work. We will try to give some answers to the already mentioned structure problem adding the $\mathcal{N}$-connection, that is:

Given $G = AB$ a factorized group where A and B are $\mathcal{N}$-connected subgroups; suppose that A and B satisfies a certain property $\mathcal{P}$, what can be said about the whole structure of the group G?

Needless to say, this problem is broad, so we do not have any intention of completeness, but we have tried to present a neat exposition and to give some non-trivial results. The properties that, in our case, turned out to be effective and fruitful are the chain conditions, which we introduced in the preliminaries.

The first section is about two results where G is solvable. In the first case we do not assume solvability but we get it by the theorem of Itô. In the second case we assume it in the hypotheses, but anyway the result we proved turned out to be fundamental in some of the successive demonstrations.

The second section concerns products of supersolvable $\mathcal{N}$-connected subgroups. The class of supersolvable groups is a wide subclass of the class of polycyclic groups, or in other terms the class of solvable groups that satisfies Max. We will prove that the product of supersolvable $\mathcal{N}$-connected subgroups is supersolvable.

The third section concerns products of Černikov subgroups. This class arises in the framework of groups that satisfies Min. For a long time they were considered the only examples of those groups and however they are the only examples if we assume solvability. We will give a proof of the fact that the product of Černikov $\mathcal{N}$-connected subgroups is Černikov.

The fourth section treats the case of minimax subgroups. We will prove that the product of minimax hypercentral $\mathcal{N}$-connected subgroups is minimax hypercentral. To prove this statement, we will use both the result on solvable subgroups and the result on Černikov subgroups.

In the last section we present a proposition that is to be considered more like an example or an exercise in which we show what can be said about the product of e nilpotent periodic group and a nilpotent group with Max. We report it because the proof is short and it also covers a product of subgroups that does not belong to any of the previous cases. After that, we give two examples of products of $\mathcal{N}$-connected subgroups to conclude.

5.1 The solvable case

As we mentioned before, one of the most investigated problems in groups product theory is the one we called the structure problem. An interesting problem that is possible to examine in depth is the product of abelian groups, even though the theorem of Itô narrowed down the research area. Beyond other results we have, for example, the fact that all the elements of the upper central series factorize (see [1, §2]); the fact that we have some links with the ring theory and in particular with the class of radical rings is interesting. This class of rings permits the construction of a certain number of examples (and counterexamples). We will not analyze these links in depth, but we refer to the results of Sysak (see for instance [1, §6, p. 137] or [53]), who constructed examples of countable torsion-free groups with a triple abelian factorization and that are not locally polycyclic. This is to motivate the following easy proposition about the product of $\mathcal{N}$-connected abelian subgroups, that is always locally nilpotent, without the addition of any further hypotheses on the factors.

Proposition 5.1.1. *Let G be group and let $A, B \leq G = AB$ where A and B are $\mathcal{N}$-connected abelian subgroups, then G is a Gruenberg group.*

Proof. The proof consists essentially in proving, with a little calculation, that $A, B \subseteq L(G)$. Let $g \in G$, since $G = AB$ is a product we know that there exist $\alpha \in A$ and $\beta \in B$ such that $g = \alpha\beta$. For all $a \in A$ and $n \geq 1$, we get:

$$[g,_n a] = [\alpha\beta,_n a] = [\alpha\beta, a,_{n-1} a] = [[\alpha, a]^\beta [\beta, a],_{n-1} a] = [\beta,_n a]$$

So, choosing n equal to the nilpotency class of the group $\langle a, \beta\rangle$, we have that $[g,_n a] = 1$ and $A \subseteq L(G)$. Similarly $B \subseteq L(G)$.

Now, G is solvable (metabelian) by Itô theorem 2.3.2, then by Gruenberg theorem, called H the Hirsch-Plotkin radical of G, we get $G = L(G) = H$, the thesis. □

A natural question that arises after this proposition is how to generalize it to nilpotent, hypercentral or other classes strictly connected with abelianity. But we get immediately two limitations. First of all, we have to underline that the thesis of the previous proposition cannot be easily improved, in fact we have the following

Example 5.1.2. *Let H be a Prüfer 2-group and τ the inversion automorphism on H. Then $G = H \rtimes \langle \tau \rangle$ is a product of two abelian groups, that are $\mathcal{N}$-connected (for instance it is not difficult to prove that G is hypercentral), but G is not a Baer group.*

Example 5.1.3. *Let C_p be a cyclic group of order p, and P an infinite elementary abelian p-group. Consider $G = C_p \wr P$. Clearly G is a product of the base group, that is abelian, and P. It is easy to prove that G it is a solvable $(p+1)$-Engel group, so it is Gruenberg, but it does not satisfy the normalizer condition.*

Secondly, a problem that arises is that we lose the solvability of G, because the theorem of Itô may not hold anymore. However, if we assume solvability among the hypotheses, we get a non-trivial result that will be an essential tool for most of the following demonstrations.

Proposition 5.1.4. *Let G be a solvable group and let $A, B \leq G$ such that $G = AB$, where A, B are $\mathcal{N}$-connected subgroups. Suppose that A is hypercentral and B locally nilpotent; then B is contained in $HP(G)$, the Hirsch-Plotkin radical of G.*

Proof. Write $H := HP(G)$, the Hirsch-Plotkin radical of G.

Let $g \in G$; since $G = AB$ is a product, we know that there exist $\alpha \in A$ and $\beta \in B$ such that $g = \alpha\beta$. For each $a \in Z(A)$, by $\mathcal{N}$-connection, there exists $n \geq 1$ such that $[\beta,_n a] = 1$, whence,

$$[g,_n a] = [\alpha\beta,_n a] = [\alpha\beta, a,_{n-1} a] = [[\alpha, a]^\beta [\beta, a],_{n-1} a] = [\beta,_n a] = 1.$$

Thus, $Z(A) \subseteq L(G)$ and, since G is solvable, $Z(A) \leq H$ by Gruenberg Theorem.

Let the ordinal λ be the hypercentral length of A and, for every ordinal $\sigma \leq \lambda$, let

$$K_\sigma = Z_\sigma(A)^G = Z_\sigma(A)^B = \langle a^b \mid a \in Z_\sigma(A), b \in B \rangle.$$

Fix $b \in B$. We prove, by induction on σ that $b \in L(K_\sigma \langle b \rangle)$.

For the first step, observe that $K := K_1 = Z(A)^B$ is contained in H. Now, every $g \in K$ can be written as a finite product of elements of $Z(A)$ conjugated by elements of B, so there exist $a_1, \dots, a_k \in Z(A)$ and $b_1, \dots, b_k \in B$ such that $g \in S := \langle a_1^{b_1}, \dots, a_k^{b_k} \rangle$. Thus, S is finitely generated and nilpotent, since it is contained in H. By $\mathcal{N}$-connection the groups $\langle a_i^{b_i}, b \rangle$ are nilpotent and finitely generated, for all $i \in \{1, \dots, k\}$, and clearly the same holds for the groups $\langle a_i^{b_i} \rangle^{\langle b \rangle}$. Hence

$$R := S^{\langle b \rangle} = \langle \langle a_1^{b_1} \rangle^{\langle b \rangle}, \dots, \langle a_k^{b_k} \rangle^{\langle b \rangle} \rangle$$

is finitely generated; more specifically, there exists $t \geq 1$ such that

$$X = \{a_i^{b_i b^j} \mid 1 \leq i \leq k,\ 0 \leq j \leq t\}$$

is a set of generators of R. By lemma 1.3.11, we have

$$R' = \langle [x_1, x_2, \dots, x_i] \mid\ i \geq 2,\ x_j \in X \ for\ all\ 1 \leq j \leq, i \rangle.$$

In particular for every $x_u, x_v \in X$,

$$[x_u x_v, b]R' = [x_u, b][x_v, b][x_u, b, x_v]R' = [x_u, b][x_v, b]R'.$$

Repeated use of this last observation let us conclude that there exists $n \in \mathbb{N}$ such that $[R,_n b] \leq R'$. Hence, both $R\langle b\rangle/R'$ and R are nilpotent, and so, by a well known nilpotency criterion of Philip Hall (see [46, §5, p. 129-130]), $R\langle b\rangle$ is nilpotent. In particular there exists $t \in \mathbb{N}$ such that $[g,_t b] = 1$, thus proving that b is a left Engel element of $K\langle b\rangle = K_1\langle b\rangle$.

Now, suppose we have shown that b is a left Engel element of $K_\tau\langle b\rangle$ for every ordinal τ, with $\tau < \sigma \leq \lambda$. If σ is a limit ordinal, then clearly b is left Engel in $K_\sigma\langle b\rangle$. Thus, assume $\sigma = \tau + 1$ is a successor ordinal. Since K_τ is normal in G,

$$\frac{K_{\tau+1}}{K_\tau} \leq Z\left(\frac{G}{K_\tau}\right)^{G/K_\tau},$$

and

$$\frac{G}{K_\tau} = \frac{AM_\tau}{M_\tau}\frac{BM_\tau}{M_\tau}$$

(where the two factors are $\mathcal{N}$-connected), the same argument of the previous step yields that bK_τ is left Engel in $K_{\tau+1}\langle b\rangle/K_\tau$. Hence, for every $g \in K_{\tau+1}\langle b\rangle$ there exists $s \geq 1$ such that

$$[g,_s b] \in K_\tau.$$

On the other hand, by inductive assumption there exists $r \geq 1$ such that

$$[g,_{s+r} b] = [g,_s b,_r b] = 1,$$

thus proving that b is a left Engel element in $K_{\tau+1}\langle b\rangle$. This completes our inductive argument, and finally proves that b is left Engel in $Z_\lambda\langle b\rangle = A^G\langle b\rangle$.

To conclude, let, as before, $b \in B$ and let $g \in G$. Since,

$$\frac{G}{A^G} \simeq \frac{B}{B \cap A^G}$$

is locally nilpotent, there exists $n \geq 1$ such that $[g,_n b] \in A^G$. By what we have shown, there also exists $m \geq 1$ such that

$$[g,_{n+m} b] = [[g,_n b],_m b] = 1.$$

Hence b is a left Engel element of G. Therefore, $B \subseteq L(G)$, and so, by Gruenberg Theorem $B \leq H = HP(G)$.

□

Corollary 5.1.5. *Let G be a solvable group and let $A, B \leq G$ such that $G = AB$. If A, B are hypercentral and $\mathcal{N}$-connected, then G is locally nilpotent.*

5.2 Product of supersolvable subgroups

The class of supersolvable groups is one of the most interesting among the classes of groups that satisfy Max. The literature about the product of supersolvable subgroups counts several results. We already cited that the product of two cyclic groups is supersolvable (see [34] and [22]), and there are many results in particular dealing with triple factorizations, by de Giovanni and others (see for instance [1, §6, p. 158-168], [5] and [23]).

Here we prove that the product of supersolvable $\mathcal{N}$-connected subgroups is supersolvable. To reach this result we use some fundamental ingredients. The first one is an invariant inherited by polycyclic group, the Hirsch length which is introduced in the preliminaries and we use to make induction. The others are specific properties that are inherent to supersolvable groups. For instance, we use the fact that if a polycyclic group has all finite homomorphic images that are supersolvable, then it is supersolvable; moreover we use the property that every infinite supersolvable group admits a normal infinite cyclic subgroup, or, for example, the fact that it is virtually nilpotent.

Lemma 5.2.1. *Let G be a group $A, B \leq G$ such that $G = AB$ and A, B are $\mathcal{N}$-connected. If A and B are supersolvable (f.g. nilpotent) then G is polycyclic if and only if it is supersolvable (f.g. nilpotent).*

Proof. If G is supersolvable (f.g nilpotent) the result is clear. Suppose that G is polycyclic and A and B supersolvable. Consider a normal subgroup of finite index in G. We have

$$\frac{G}{N} = \frac{AN}{N}\frac{BN}{N}$$

that is a product of finite supersolvable $\mathcal{N}$-connected subgroups. The class of finite supersolvable groups is a saturated formation that contains the class of finite nilpotent groups. So, we can apply Carocca's theorem 3.1.4 and we get that G/N is supersolvable. Thus, we have a polycyclic group in which every finite homomorphic image is supersolvable. Applying 1.3.18, G is supersolvable. The same holds when A and B are finitely generated nilpotent subgroups, using, in this case, theorem 1.3.17.

□

Before the statement of the main theorem of this section, we need to point out some technical properties we will use in the inductive process.

Lemma 5.2.2. *Let G be a group and $H, K \leq G$ such that $G = \langle H, K \rangle$. If $|G : K| < \infty$, then $|H : H \cap K| < \infty$*

Lemma 5.2.3. *Let G be a finitely generated nilpotent group and $H \leq G$ such that $N_G(H)/H$ is a finite group. Then we have $|G : H| < \infty$*

Corollary 5.2.4. *Let G be a supersolvable group, $F = Fit(G)$ and $H \leq F$ such that $N_G(H)/H$ is a finite group. Then we have $|G : H| < \infty$*

Proof. Consider G and H as in the hypotheses. Clearly $|N_F(H) : H| < \infty$. Applying 5.2.3, we have $|F : H| < \infty$ and together with the fact that $|G : F| < \infty$ by theorem 1.3.15, we have the thesis.

□

Lemma 5.2.5. *Let G be a group, $A, B \leq G$ such that $G = AB$. If A satisfies Max, then G satisfies the maximal condition on the subgroups that contain B.*

Proof. Consider $S_1 \leq S_2 \leq \ldots$ an ascending chain of subgroups that contain B. Each S_i is factorized, with the following factorization

$$S_i = (S_i \cap A)B.$$

The chain $S_1 \cap A \leq S_2 \cap A \leq \ldots$ is an ascending chain in A, which is a group that satisfies Max. Thus, there exists $t \in \mathbb{N}$ such that $S_t = S_{t+j}$, for all $j \geq 0$. Hence

$$S_t = (S_t \cap A)B = (S_{t+j} \cap A)B = S_{t+j}$$

for all $j \geq 0$, that is the thesis.

□

Theorem 5.2.6. *Let G be a group and $A, B \leq G = AB$ where A and B are supersolvable $\mathcal{N}$-connected subgroups. Then G is supersolvable.*

Proof. We proceed by induction on $h := h(A) + h(B)$ where $h(X)$ is the Hirsch length of the group X. If A and B are finite the result holds by 3.1.4. If A (or B) is finite, then B is a subgroup of finite index in G, hence, being B_G a normal supersolvable subgroup and the factor G/B_G a supersolvable finite group, we conclude using lemma 5.2.1. Thus, we can assume that both $h(A), h(B) \geq 1$. We define these two sets:

$$\mathcal{A} = \{S \mid A \leq S \leq G,\ S\ supersolvable,\ h(S \cap B) \geq 1\}$$

$$\mathcal{B} = \{T \mid B \leq T \leq G,\ T\ supersolvable,\ h(T \cap A) \geq 1\}$$

We prove that either $\mathcal{A}$ or $\mathcal{B}$ is non empty. In fact, if $h(A \cap B) \geq 1$, both $\mathcal{A}$ and $\mathcal{B}$ are non empty. Suppose that $h(A \cap B) = 0$, i.e. $A \cap B$ is a finite group; by Lemma 1.3.16 we can choose $a \in A$, $b \in B$ such that $|a| = \infty = |b|$ and $\langle a \rangle \lhd A$, $\langle b \rangle \lhd B$. By $\mathcal{N}$-connection, we know that $\langle a, b \rangle$ is an infinite nilpotent group and for this reason we have that

$$Z(\langle a, b \rangle) \leq C_G(a) \cap C_G(b) \leq N_G(\langle a \rangle) \cap N_G(\langle b \rangle).$$

Thus, it is easy to prove that one among $N_A(\langle b \rangle)$ and $N_B(\langle a \rangle)$ is infinite. Suppose that $N_A(\langle b \rangle)$ is infinite and call $T = N_G(\langle b \rangle)$, that contains B.

Now, if $|A : N_A(\langle b \rangle)| = \infty$, then T is supersolvable by inductive hypothesis, and $\mathcal{B} \neq \emptyset$. Otherwise, $|A : N_A(\langle b \rangle)| < \infty$, then consider

$$\frac{T}{\langle b \rangle} = \frac{N_A(\langle b \rangle)\langle b \rangle}{\langle b \rangle}\frac{B}{\langle b \rangle}.$$

Observing that $h(\langle b \rangle) \geq 1$, by inductive hypothesis we have that $T/\langle b \rangle$ is supersolvable, and, by Lemma 5.2.1, T is supersolvable. Hence, $\mathcal{B} \neq \emptyset$.
Clearly, by Lemma 5.2.5, $\mathcal{B}$ admits a maximal element D.

Now, D factorizes: in fact $D = (D \cap A)B$. If $h(A \cap D) = h(A)$, then $|G : D| < \infty$ and D_G is a supersolvable normal subgroup of G. By the Theorem of Carocca 3.1.4 G/D_G is supersolvable and, by Lemma 5.2.1, G is supersolvable. Otherwise, $|A : D \cap A| = \infty$. In this case we call $F_A = Fit(A)$ and $F_D = Fit(D)$.

We define the following chain of subgroups

$$M_0 = F_A \cap F_D, \quad M_1 = N_{F_D}(M_0), \quad \ldots \quad M_i = N_{F_D}(M_{i-1})$$

We know that F_D is nilpotent and normal in D, so we can deduce that there exists $t \in \mathbb{N}$ such that $M_t = N_{F_D}(M_{t-1}) = F_D$. Thus, we show that the following factorization holds:

$$N_G(M_i) = N_A(M_i)N_D(M_i)$$

for all $i : 1 \ldots t$. We proceed by induction on i. If $i = 0$, consider $g \in N_G(M_0)$; then, $g = ab$ for certain $a \in A$ and $b \in D$, and from $M_0^g = M_0$, we deduce

$M_0^a = M_0^{b^{-1}} \leq F_A \cap F_D = M_0$ and, by the nilpotency of M_0, $a \in N_A(M_0)$ that is the thesis.

Suppose that $i \geq 1$ and consider, $g \in N_G(M_i)$. As before, we can affirm that $g = ab$ and $M_i^a = M_i^{b^{-1}}$. It is clear that

$$M_0^a \leq F_A \cap M_i^a = F_A \cap M_i^{b^{-1}} \leq F_A \cap F_D,$$

that is $M_0^a \leq M_0$, i.e. $a \in N_A(M_0)$. Suppose that $0 \leq j \leq i$ is the greatest index such that $a \in N_A(M_j)$, and suppose that $j \leq i-1$. Knowing that

$$M_{j+1}^a \leq M_i^a \leq M_i^{b^{-1}} \leq F_D$$

and $\langle a, M_{j+1} \rangle \leq N_G(M_j)$, we deduce

$$M_{j+1}^a \leq N_G(M_j) \cap F_D = N_{F_D}(M_j) = M_{j+1}$$

that is, $M_{j+1}^a \leq M_{j+1}$. For inductive hypothesis $N_G(M_j) = N_A(M_j)N_D(M_j)$ and, by lemma 2.1.9, $N_G(M_j) = N_A(M_j)N_B(M_j)$. Passing to the quotient

$$\frac{N_G(M_j)}{M_j} = \frac{N_A(M_j)M_j}{M_j}\frac{N_B(M_j)M_j}{M_j}$$

These two last factors are $\mathcal{N}$-connected supersolvable and, using the properties of Hirsch length and the fact that $h(M_j) \geq h(M_0) \geq 1$, we can affirm that $N_G(M_j)/M_j$ is supersolvable and hence, by lemma 5.2.1, $N_G(M_j)$ is supersolvable. So, $\langle a, M_{j+1} \rangle \leq N_G(M_j)$ satisfies Max and then $M_{j+1}^a = M_{j+1}$. This contradicts the assumption on j. Thus $i = j$, and the statement is proved. Moreover we observe that $N_G(F_D) = N_A(F_D)N_D(F_D) = N_A(F_D)D$ holds as well.

Now, we prove that $N_G(F_D)$ is supersolvable. In fact, we know that $N_G(F_D) = N_A(F_D)N_D(F_D)$ holds; by Lemma 2.1.9 we have $N_G(F_D) = N_A(F_D)N_B(F_D)$. Passing to the quotient

$$\frac{N_G(F_D)}{F_D} = \frac{N_A(F_D)F_D}{F_D}\frac{N_B(F_D)F_D}{F_D},$$

that is a product of supersolvable $\mathcal{N}$-connected subgroups and $h(F_D) \geq 1$, so we can apply the inductive hypothesis and conclude by Lemma 5.2.1, that $N_G(F_D)$ is supersolvable.

The last fact we have to prove is that $N_G(F_D) > D$. We prove a stronger fact, that is

$$|N_G(M_i) : M_{i+1}| = \infty$$

for all $i : 0, \ldots, t-1$. Proceed by induction on i. If $i = 0$, we know by corollary 5.2.4 that $|N_A(M_0) : M_0| = \infty$. By lemma 5.2.2, $|N_G(M_0) : N_D(M_0)| = \infty$ and then $|N_G(M_0) : M_1| = \infty$. Suppose that $i \geq 1$. By inductive hypothesis we know that

$$|N_G(M_{i-1}) : M_i| = \infty$$

Keeping in mind that M_i is nilpotent, we want to prove that

$$|N_{N_G(M_{i-1})}(M_i) : M_i| = \infty.$$

Call $W = N_G(M_{i-1})$, $F_W = Fit(W)$ and $K = F_W \cap F_D$. Clearly $K \leq M_i$. If $K = M_i$, then, by corollary 5.2.4, we get the thesis. Otherwise, consider $N_W(K)$. By corollary 5.2.4, we have that $|N_W(K) : K| = \infty$. Being $K \lhd N_D(M_{i-1})$, we have that $N_W(K) \geq N_D(M_{i-1})$. So, for this reason, this factorization

$$N_W(K) = N_A(K)N_D(M_{i-1})$$

holds. Consider $R = N_A(K)K$, $S = R_{N_W(K)}$ (the core of R in $N_W(K)$) and $T = S \cap F_W$. By construction, $K \leq T \lhd N_W(K)$. Define $L = TN_D(M_{i-1})$. Considering the quotient

$$\frac{L}{K} = \frac{T}{K}\frac{N_D(M_{i-1})}{K},$$

we have that T/K is nilpotent and by Theorem 1.3.15, $N_D(M_{i-1})/K$ is abelian.

In particular L/K is supersolvable and it is the product of two nilpotent $\mathcal{N}$-connected subgroups, hence, by Lemma 5.2.1, L/K is nilpotent. We know that $|L/K : M_i/K| = \infty$, so, by lemma 5.2.3, we get

$$|N_{L/K}(M_i/K) : M_i/K| = \infty,$$

that implies

$$|N_L(M_i) : M_i| = \infty,$$

and then

$$|N_{N_G(M_i)}(M_i) : M_i| = \infty.$$

Thus, by lemma 5.2.2,

$$|N_G(M_i) : N_D(M_i)| = \infty$$

and then the thesis $|N_G(M_i) : M_{i+1}| = \infty$.

In particular $N_G(F_D)$ is a supersolvable group, $N_G(F_D) > D$ and this fact contradicts the maximality of D. So, $|G : D| < \infty$ and then G is supersolvable.

□

Corollary 5.2.7. *Let G be a group and $A, B \leq G = AB$ where A and B are finitely generated nilpotent $\mathcal{N}$-connected subgroups. Then G is a finitely generated nilpotent group.*

Proof. The result is an easy application of theorem 5.2.6 and theorem 1.3.17 □

Corollary 5.2.8. *Let G be a group, $G = G_1 G_2 \dots G_n$ where $G_1, G_2, \dots, G_n$ are pairwise permutable supersolvable (finitely generated nilpotent) subgroups of G. If for all $i, j : 1, \dots, n$ such that $i \neq j$ we have that G_i and G_j are $\mathcal{N}$-connected, then G is supersolvable (finitely generated nilpotent).*

Dealing with products of groups with Max, we can state an open problem:

Problem 5.2.9. *Let G be a group and $A, B \leq G = AB$ where A and B are polycyclic $\mathcal{N}$-connected subgroups. Is it true that G is a polycyclic group?*

5.3 Product of Černikov subgroups

For a long time, the only known examples of groups satisfying Min on subgroups were the Černikov groups. In 1979 Ol'shanskii proved the existence of Tarski groups, which are infinite groups in which all proper non-trivial subgroups have prime order p, with p a sufficiently large prime. It is clear that the existence of Tarski groups excludes the coincidence between Černikov groups and groups with Min. However the class of Černikov groups is quite important and it has been proved that in the solvable case it corresponds with the family of groups with Min. As we have already mentioned in the introduction, Černikov groups are extensions of a direct product of a finite number of quasi cyclic groups by a finite group, for this reason the product of groups of this kind is an issue of the open problem of the product of abelian-by-finite subgroups. In literature there are some partial results about those products. The most important is by Černikov (see [20]) and it regards locally graded groups. This should confirm that our investigation adding the $\mathcal{N}$-connection is reasonably justified.

Now, we introduce some technical lemmas that are necessary for the stated theorem.

Considering a product of Černikov groups, it is possible to describe somehow the finite residual in terms of the finite residuals of the factors, in other words we are able to state the following:

Lemma 5.3.1. *Let $G = AB$ be a Černikov group; then if we call A_0, B_0 and G_0 the finite residuals of, respectively, A, B and G, the following holds:*

$$G_0 = \langle A_0, B_0 \rangle.$$

Proof. By a lemma of Amberg [2] or 2.1.10, the group $\langle A_0, B_0 \rangle$ has finite index in G and therefore by definition we know that $G_0 \leq \langle A_0, B_0 \rangle$. The other inclusion is obvious considering the definition of finite residual.

□

Before going further, we need a technical lemma.

Lemma 5.3.2. *Let G be a periodic group and $1 \neq A \leq G$ a divisible abelian subgroup. Let $a \in A$ and $u \in C_G(x)$ for all $x \in A$ such that $a \in \langle x \rangle$. Then $u \in C_G(A)$.*

Proof. Let $h \in A$ and let $|h| = m$. Consider $a_1 \in A$ such that $a_1^m = a$. Therefore $(a_1 h)^m = a_1^m = a$. Hence the following holds

$$u \in C_G(a_1) \cap C_G(a_1 h) \leq C_G(h).$$

□

Notation 5.3.3. *Given a group G such that $G = AB$, where $A, B \leq G$ and an element $g \in G$, we call:*

$$\Pi_A(g) = \{\ a \in A \mid \textit{there exists } b \in B \textit{ s.t. } ab = g\ \}$$

and for the subset X of G, we call:

$$\Pi_A(X) = \bigcup_{x \in X} \Pi_A(x).$$

Now, we are ready to state and prove the main theorem of this section. The proof is divided in two cases to simplify the reading: firstly the case in which we assume that A and B are p-groups and then the general case.

Theorem 5.3.4. *Let G be a group and let A, B be two subgroups of G such that $G = AB$. Assume that A and B are $\mathcal{N}$-connected and Černikov p-groups, where p is a prime number. Then G is a Černikov p-group.*

Proof. Since A e B are Černikov p-groups, we know that there exist $A_0 \lhd A$ and $B_0 \lhd B$ normal subgroups with finite index respectively in A and B both of those are a direct product of a finite number of copies of the Prüfer p-group. From now on, we divide the proof in 4 steps. In the first three we assume, in addition, that $A_0 \cap B_0 = 1$. The fourth step is without this assumption.

Step 1.

Consider $a \in A_0$ and $b \in B_0$ and call $H = \langle a, b \rangle$, which we know to be nilpotent by the $\mathcal{N}$-connection. We have that if $C_G(H)$ is infinite, then the set $C_{A_0}(H) \cup C_{B_0}(H)$ is infinite. We prove this fact. Using the notation introduced above, we call $A^* =$

$\Pi_A(C_G(H))$ and $B^* = \Pi_B(C_G(H))$ that are subset of G. Since $C_G(H) \subseteq A^*B^*$, we have that $A^* \cup B^*$ is infinite. Without loss of generality, we can suppose that A^* is infinite. For all $x \in A^*$ there exist a $y \in B$ such that $xy \in C_G(H)$. In particular $b^{xy} = b$, from which,

$$b^x = b^{y^{-1}} \in b^B$$

where b^B is a conjugacy class in B. Since $b \in B_0$, we have that b^B is finite, thus there exists an infinite subset $X \subseteq A^*$ such that, fixed $x_0 \in X$, it holds $b^x = b^{x_0}$ for all $x \in X$. Consequently $\{x_0x^{-1} | x \in X\} \subseteq C_A(b)$ is infinite. Therefore $C_{A_0}(b)$ is infinite too and since $a \in A_0$ is abelian we can conclude, by observing that

$$C_{A_0}(b) = C_{A_0}(\langle a, b\rangle) = C_{A_0}(H).$$

Step 2.

Suppose that $Z(G) = 1$: we claim that there exist $a \in A_0$ and $b \in B_0$ such that $C_{B_0}(a) = 1 = C_{A_0}(b)$.
In fact, consider the set $\{C_{B_0}(a) \mid a \in A_0\}$; it, clearly, admits a minimal element. Let $C = C_{B_0}(a)$ be such minimal element. If $C \neq 1$, thus by divisibility and minimality, we have that for all x such that $a \in \langle x\rangle$, then $C_{B_0}(\langle x\rangle) = C_{B_0}(\langle a\rangle)$. Therefore, we can apply 5.3.2 and conclude that $C_{B_0}(a) = C_{B_0}(A_0)$. So, $C_G(C) \geq \langle A_0, B_0\rangle$ and hence it has finite index. Since G is a p-group, this implies that $Z(G) \neq 1$. Thus, if $Z(G) = 1$, the minimal element of the set $\{C_{B_0}(a) \mid a \in A_0\}$ is exactly $\{1\}$. The same holds for $\{C_{A_0}(b) \mid b \in B_0\}$.

Step 3.

Now our objective is to prove that $Z(G) \neq 1$.
Suppose that $Z(G) = 1$. Then, by step 2, there exist $a \in A_0$ and $b \in B_0$ such that $C_{B_0}(a) = 1 = C_{A_0}(b)$. By step 1, $C_G(\langle a, b\rangle)$ is finite. Moreover by $\mathcal{N}$-connection we can deduce that

$$1 \neq Z(\langle a, b\rangle) \leq C_G(\langle a, b\rangle)$$

namely $C_G(\langle a, b\rangle)$ is nontrivial and finite. We can choose a and b such that $|C_G(\langle a, b\rangle)|$ is minimal. Let $1 \neq u \in C_G(\langle a, b\rangle)$ and let $x \in A_0$ where $a \in \langle x\rangle$; thus $C_G(\langle x, b\rangle) \leq C_G(\langle a, b\rangle)$ and hence the equality holds and, in particular, $u \in C_G(\langle x, b\rangle) \leq C_G(\langle x\rangle)$. By 5.3.2 $u \in C_G(A_0)$. Analogously, we can demonstrate that $u \in C_G(B_0)$ and then $C_G(u) \geq \langle A_0, B_0\rangle$, letting us deduce, as before, $Z(G) \neq 1$, .

Step 4

We claim that G is a Černikov group. First of all, we want to demonstrate that G is hypercentral proving the fact that every nontrivial quotient group of G has non trivial centre (proposition 1.5.10). In fact, since all the hypotheses we assumed hold for all quotient groups, it is sufficient to show that $Z(G) \neq 1$. So, if $A_0 \cap B_0 = 1$ then we obtain our thesis by step 3. Otherwise, called $K := A_0 \cap B_0 \neq 1$ it is true that $C_G(K) \geq \langle A_0, B_0 \rangle$ has finite index. Hence, as we observed in the previous steps, $Z(G) \neq 1$. In conclusion, since a hypercentral p-group with Min-n, is a Černikov p-group by 1.5.12, we reached our objective.

□

Now, using $\mathcal{N}$-connection and proceeding by induction, we are ready to prove the general case that is the following:

Theorem 5.3.5. *Let G be a group and let A, B be two subgroups of G such that $G = AB$. Assume that A and B are $\mathcal{N}$-connected and Černikov groups. Then G is a Černikov group.*

Proof. Since A and B are Černikov groups, we know that there exist $A_0 \lhd A$ and $B_0 \lhd B$ subgroups of finite index, respectively, in A and B, with the following structure

$$A_0 = \mathop{\times}_{i=1}^{s} P_i \quad B_0 = \mathop{\times}_{j=1}^{t} Q_j$$

where in A_0 each P_i is a direct product of h_i copies of Prüfer p_i-groups, in which p_i are distinct primes for $i : 1, \ldots s$; analogously in B_0 each Q_j is the direct product of k_j copies of Prüfer q_j-groups, in which q_j are distinct primes for $j : 1, \ldots t$.

In terms of minimality, this means $h_i = m(P_i)$ and $k_j = m(Q_j)$. So, we have

$$m(A) + m(B) = m(A_0) + m(B_0) = \sum_{i=1}^{s} m(P_i) + \sum_{j=1}^{t} m(Q_j)$$

We proceed by induction on $m(A) + m(B)$. If $m(A) = 0$ (or $m(B) = 0$), then B is a subgroup of finite index in G, hence B_G is a Černikov normal subgroup of finite index in G; but clearly a vitually Černikov group is a Černikov group, so we have the thesis. Otherwise, we can assume that $m(A) \geq 1$ e $m(B) \geq 1$. For the previous theorem, we can suppose that A_0 and B_0 are not both p-groups. Hence, without loss of generality we can also assume that $h_1, h_2 \geq 1$ and $k_1 \geq 1$. If $p_i \neq q_j$ for all i and j, then $\langle A_0, B_0 \rangle$ is abelian by $\mathcal{N}$-connection, so nothing is lost if we assume $p_1 = q_1$.

Call $P := P_1$. Since $P char A_0$, we have that $P \lhd A$. Consider $N_G(P)$. By Wielandt lemma 2.1.4, we know that the following factorization holds

$$N_G(P) = N_A(P)N_B(P) = AN_B(P)$$

Moreover we know:

$$Q_2 \times \ldots \times Q_t \leq C_B(P) \leq N_B(P)$$

Now, we can split in two different cases:

Case 1)

If also $Q_1 \leq N_B(P)$, then $|G : N_G(P)| < \infty$, so we can reduce ourselves to prove that $N_G(P)$ is Černikov. Now, P is Černikov, $N_G(P)/P$ is Černikov by inductive hypothesis, hence, by extension, $N_G(P)$ is Černikov.

Case 2)

Suppose that $Q_1 \not\leq N_B(P)$, then called $B^* := N_B(P)$ we have that $B_0 \cap B^* < B_0$ and $|B^* : B_0 \cap B^*| < \infty$, so, called B_0^* the finite residual of B^*, we deduce that:

$$B_0^* \leq B_0 \cap B^* < B_0.$$

Hence, B_0^* and B_0 are both direct products of a finite number of Prüfer groups and $B_0^* < B_0$, B_0 has no proper subgroups of finite index, then this fact allow us to apply the inductive hypothesis on $N_G(P)$ and to affirm that it is Černikov. By 5.3.1 we deduce that the finite residual of $N_G(P)$ is exactly $\langle A_0, B_0^* \rangle$ from which we have $[P_i, Q_j] = 1$ for all $i : 1, \ldots, s$ and for all $j : 2, \ldots, t$. Now, considering P_2 and repeating the same argument, if case 1 holds the proof is done, otherwise $[P_i, Q_j] = 1$ must hold for all $i : 1, \ldots, s$ and for all $j : 1, 3, \ldots, t$. Gathering all these informations, we conclude $[A_0, B_0] = 1$, from which $\langle A_0, B_0 \rangle$ is abelian. Finally knowing that product of groups with Min satisfy Min-n 2.2.1 or [2] and knowing that if a group satisfy Min-n each finite index subgroup does [46, §3, p. 64] or [56], we conclude that G is Černikov.

□

We know state two easy corollaries of the latter theorem.

Corollary 5.3.6. *Let G be a group and let A, B be two subgroups of G such that $G = AB$. Assume that A and B are $\mathcal{N}$-connected solvable groups with Min. Then G is a solvable group with Min.*

Proof. By theorem 1.3.3, A and B are Černikov groups and by theorem 5.3.5 also G is Černikov. We need only to prove the solvability of G, but this follows from the fact that, if G_0 is the finite residual of G, G_0 is abelian and G/G_0 is solvable by 3.1.4.

□

Corollary 5.3.7. *Let G be a group, $G = G_1G_2 \dots G_n$ where $G_1, G_2, \dots, G_n$ are pairwise permutable Černikov (solvable with Min) subgroups of G. If for all $i, j : 1, \dots, n$ such that $i \neq j$ we have that G_i and G_j are $\mathcal{N}$-connected, then G is Chernikov (solvable with Min).*

5.4 Product of minimax hypercentral subgroups

Minimax groups are those groups that admit a series of finite length for which each factor satisfies Max or Min. The result we are going to show is that the product of hypercentral minimax $\mathcal{N}$-connected subgroups is hypercentral minimax. In literature, the product of minimax subgroups or, more in general, the product of finite rank subgroups is a topic that has attracted many authors. Anyway, the most interesting results are often obtained assuming a solvability condition on the whole group. Wilson proved, in [57], that the solvable product of two minimax subgroups is minimax. In our case instead of the solvability requirements (such as for instance the whole group is locally solvable or solvable-by-finite), we add a nilpotence relation between the factor, i.e. the $\mathcal{N}$-connection.

This result on the product of minimax subgroups is the most relevant result of this work, and to prove it we employ different ingredients that are, in some sense, particular to the product of $\mathcal{N}$-connected subgroups. We will use the results on the factorization of certain torsion subgroups and on certain isolator subgroups without the *a priori* assumption of local nilpotence.

The proof we are going to show is divided in two main parts. The first part concerns the case in which we assume that the set of elements of finite order is the trivial group. Before starting, we need a preliminary lemma. This preliminary lemma is an improvement of Corollary 5.1.5 in the case of a (solvable) product of two minimax hypercentral group, using the theorem of Kurdachenko. In fact we get:

Lemma 5.4.1. *Let G be a solvable group and let $A, B \leq G$ such that $G = AB$ and A, B are $\mathcal{N}$-connected hypercentral minimax subgroups, then G is minimax hypercentral; moreover, if G is torsion-free, then G is nilpotent.*

Proof. G is minimax by the theorem 2.2.4. Using Corollary 5.1.5 we get that G is locally nilpotent. By 1.4.9 A and B satisfy both wmax and wmin; by 2.2.1 G satisfies both wmin-n and wmax-n, and by the theorem of Kurdachenko 2.2.3, G is hypercentral.

Moreover if G is torsion-free by the corollary 1.5.15, G is nilpotent.

□

Proposition 5.4.2. *Let the group $G = AB$, where A and B are $\mathcal{N}$-connected, minimax, nilpotent subgroups. Assume further that $T(G) = 1$, then G is a minimax nilpotent group.*

Proof. We proceed by induction on $m := m(A) + m(B)$. If $m(A) = 0$ (or $m(B) = 0$), then $A = 1$ (or $B = 1$) and the proposition is trivially true. Thus, we can assume that both $m(A)$ and $m(B) \geq 1$. We define these two sets:

$$\mathcal{A} = \{S \mid A \leq S \leq G,\ S \text{ minimax nilpotent},\ m(S \cap B) \geq 1\}$$

$$\mathcal{B} = \{T \mid B \leq T \leq G,\ T \text{ minimax nilpotent},\ m(T \cap A) \geq 1\}$$

We prove that either $\mathcal{A}$ or $\mathcal{B}$ is non empty. In fact, if $h(A \cap B) \geq 1$, both $\mathcal{A}$ and $\mathcal{B}$ are non empty. Suppose that $h(A \cap B) = 0$, i.e. $A \cap B = 1$, so choosing $1 \neq a \in Z(A)$, $1 \neq b \in Z(B)$ we get, by $\mathcal{N}$-connection, that $\langle a, b\rangle$ is an infinite nilpotent group and, for this reason, we have that

$$Z(\langle a, b\rangle) \leq C_G(a) \cap C_G(b).$$

Thus, it is easy to prove that one among $C_A(b)$ and $C_B(a)$ is infinite. Suppose that $C_A(b)$ is infinite and call $T = C_G(b)$; clearly $T \supseteq B$.

Now, if $|A : C_A(b)| = \infty$, then T is nilpotent minimax by inductive hypothesis, and $\mathcal{B} \neq \emptyset$. Otherwise, $|A : C_A(b)| < \infty$, then consider

$$\frac{T}{\langle b\rangle} = \frac{C_A(b)\langle b\rangle}{\langle b\rangle}\frac{B}{\langle b\rangle}.$$

Observing that $m(\langle b\rangle) \geq 1$, by inductive hypothesis we have that $T/\langle b\rangle$ nilpotent minimax, and, by Lemma 5.4.1, T is nilpotent minimax. Hence, $\mathcal{B} \neq \emptyset$.

Clearly, $\mathcal{B}$ is a partially ordered set by inclusion; we would like to prove that it is inductive, i.e. every totally ordered subset of $\mathcal{B}$ has an upper bound in $\mathcal{B}$. Thus, consider $(S_i)_{i\in I}$ a chain in $\mathcal{B}$. We need to prove that the upper bound $U = \bigcup_{i\in I} S_i$ belongs to $\mathcal{B}$. Each S_i is nilpotent minimax, then the union U is locally nilpotent. But $U = (U \cap A)B$ is a product of nilpotent minimax groups, then it satisfies wmin-n by 2.2.2 and, by the Kurdachenko theorem 2.2.3, it is hypercentral and solvable. Furthermore, by 2.2.4, U is minimax and by 1.5.15 nilpotent, that is our thesis. Thus, we can apply Zorn Lemma and affirm that $\mathcal{B}$ admits a maximal element. We call this maximal element D.
If $D = G$, we are done. Otherwise $|A : A \cap D| > 1$.

We define the following sequence $N_0 = A \cap D$ and $N_j = N_D(N_{j-1})$ for all $j \geq 1$. D is nilpotent, thus there exists $t \in \mathbb{N}$ such that $D = N_D(N_{t-1})$. By lemma 2.1.6 the factorizations

$$N_G(N_j) = N_A(N_j)N_D(N_j)$$

hold for all $j : 0, \ldots, t-1$. Moreover, by Lemma 2.1.9, also

$$N_G(N_j) = N_A(N_j)N_B(N_j)$$

hold for all $j : 0, \ldots, t-1$. Passing to the quotient we get

$$G^* := \frac{N_G(N_j)}{N_j} = \frac{N_A(N_j)N_j}{N_j}\frac{N_B(N_j)N_j}{N_j} =: A^*B^*$$

Now, $A^* \cap B^* = 1$, then by Lemma 4.1.2, $T(G^*) = T(A^*)T(B^*)$, so, by Theorem 5.3.5, $T(G^*)$ is a solvable Černikov group. Moreover, knowing that $1 \leq m(N_0) \leq m(N_j)$ and considering $G^*/T(G^*)$, we can affirm that it is nilpotent minimax by inductive hypothesis. So, by Lemma 5.4.1, $N_G(N_j)$ is nilpotent for all $j : 1, \ldots, t-1$. In particular $N_G(N_{t-1}) = N_A(N_{t-1})D$ is nilpotent.
Finally, we prove that $|N_G(N_{t-1}) : D| > 1$. To do that, we prove that

$$|N_A(N_j) : N_0| > 1$$

proceeding by induction on j. If $j = 0$, we have that $|N_A(N_0) : N_0| > 1$, since A is nilpotent. Suppose it is true for $j-1$, then

$$|N_A(N_{j-1}) : N_0| > 1$$

that implies

$$|N_G(N_{j-1}) : N_j| > 1$$

and, for the nilpotency of $N_G(N_{j-1})$, we get

$$|N_{N_G(N_{j-1})}(N_j) : N_j| > 1$$

and then

$$|N_G(N_j) : N_{j+1}| > 1,$$

that means

$$|N_A(N_j) : N_0| > 1.$$

So, $N_G(N_{t-1})$ is nilpotent minimax and includes properly D, contradicting the maximality of D.

□

The second part, on the other hand, is the case in which we assume that the isolator of the intersection $A \cap B$ is the whole group. The demonstration is based on a simple observation, that is: assuming $G = I_G(A \cap B)$, both A and B have to satisfy the minimal condition on the subgroups that contain $A \cap B$, that is essentially what we prove in our preliminary results. After that we are able to conclude.

Lemma 5.4.3. *Let G be a group and H, K and N such that $H \leq K \leq G$ and $N \lhd G$. If $HN = KN$ and $H \cap N = K \cap N$, then $H = K$.*

Proposition 5.4.4. *Let G be a solvable minimax group and $H \leq G$ a subgroup such that $I_G(H) = G$. Then G satisfies the minimal condition on the subgroups that contain H.*

Proof. We proceed by induction on the derived length N of G. If $n = 1$ then G is abelian and $H \lhd G$. Consider G/H, it is periodic minimax, in other words it satisfies Min. Suppose that $n \geq 2$. Then consider $A = G^{(n-1)}$. Clearly $I_A(A \cap H) = A$ and $I_{G/A}(HA/A) = G/A$. Consider

$$M_0 \geq M_1 \geq M_2 \geq \dots$$

a descending chain of subgroups containing H. Consider also

$$M_0 \cap A \geq M_1 \cap A \geq M_2 \cap A \geq \dots$$

that is a descending chain in A containing $H \cap A$, and

$$M_0A/A \geq M_1A/A \geq M_2A/A \geq \dots$$

a descending chain in G/A containing HA/A. By inductive hypothesis there exist $r, s \in \mathbb{N}$ such that

$$M_r \cap A = M_{r+1} \cap A = M_{r+2} \cap A \dots$$

and

$$M_sA/A = M_{s+1}A/A = M_{s+2}A/A = \dots$$

i.e. $M_sA = M_{s+1}A = M_{s+2}A = \dots$.

Choosing $t = max\{r, s\}$ and applying the previous lemma we get the thesis.

□

Proposition 5.4.5. *Let G be a group $A, B \leq G = AB$ such that A and B are hypercentral minimax $\mathcal{N}$-connected subgroups. If $G = I_G(A \cap B)$, then G is hypercentral minimax.*

Proof. Consider the set $\mathcal{A} = \{S \mid B \leq S \leq G,\ S\ minimax\ and\ hypercentral\}$. Clearly $B \in \mathcal{A}$, so $\mathcal{A}$ is not empty. The set $\mathcal{A}$ is clearly a partially ordered set by inclusion; we would like to prove that it is inductive, i.e. every totally ordered subset of $\mathcal{A}$ has an upper bound in $\mathcal{A}$. Thus, consider $(S_i)_{i \in I}$ a chain in $\mathcal{A}$. We need to prove that the upper bound $U = \bigcup_{i \in I} S_i$ belongs to $\mathcal{A}$. Each S_i is hypercentral, then the union U is locally nilpotent. But $U = (U \cap A)B$ is a product of (solvable) minimax groups, then it satisfies wmin-n by 2.2.2 and by the Kurdachenko theorem 2.2.3, it is hypercentral and solvable. Furthermore by 2.2.4 it is minimax, that is our thesis. Thus, we can apply Zorn Lemma and affirm that $\mathcal{A}$ admits a maximal element, and we call it D.

If $D \geq A$, i.e. $D = G$ we are done. Otherwise, let $M_0 = A \cap D$, clearly $M_0 < A$ and call (M_α) the series of successive normalizers of M_0 in D. Then called:

$$Y_1 := N_G(M_0);$$

$$Y_\alpha := N_G(M_{\alpha-1})$$

if α is a successor ordinal, and

$$Y_\beta := \left(\bigcap_{\lambda<\beta} N_A(M_\lambda) \right) M_\beta$$

if β is a a limit ordinal, we have by lemma 2.1.8, that (Y_α) is a series of factorized subgroups of G and $(Y_\alpha \cap A)$ is a decreasing series of subgroup of A. By proposition 5.4.4, A satisfies Min on the subgroups that contain $A \cap B$. So in particular, if we consider the series $(Y_\alpha \cap A)$, there exists $t \in \mathbb{N}$ such that $Y_t \cap A = Y_\lambda \cap A$ for all ordinals $\lambda \geq t$.

We now prove that each Y_α is hypercentral. Proceed by induction on α. If $\alpha = 1$, we consider $N_G(M_0) = N_A(M_0)N_D(M_0) = N_A(M_0)N_B(M_0)$ by lemma 2.1.9; passing to the quotient we get

$$\frac{N_G(M_0)}{M_0} = \frac{N_A(M_0)}{M_0}\frac{N_B(M_0)}{M_0}$$

that is a product of $\mathcal{N}$-connected Chernikov groups, then by theorem 5.3.5 $N_G(M_0)/M_0$ is Chernikov. Then $N_G(M_0)$ is solvable and using 5.4.1, we get the desired hypercentrality. Moreover by 2.2.4 $N_G(M_0)$ is minimax.

So let $\alpha \geq 2$, then if α is a successor ordinal, we have

$$N_G(M_{\alpha-1}) = N_A(M_{\alpha-1})N_D(M_{\alpha-1}) = N_A(M_{\alpha-1})N_B(M_{\alpha-1})$$

passing to the quotient

$$\frac{N_G(M_{\alpha-1})}{M_{\alpha-1}} = \frac{N_A(M_{\alpha-1})M_{\alpha-1}}{M_{\alpha-1}}\frac{N_B(M_{\alpha-1})M_{\alpha-1}}{M_{\alpha-1}},$$

that is a product of $\mathcal{N}$-connected Chernikov groups and, as we did before, we conclude that Y_α is hypercentral minimax. Suppose now that α is a limit ordinal. We just proved that $N_A(M_t) = N_A(M_\lambda) =: A^*$ for all $\lambda \geq t$. It is needless to say that $\alpha \geq t$, hence by definition we have that

$$Y_\alpha := \left(\bigcap_{\lambda<\alpha} N_A(M_\lambda) \right) M_\alpha = A^* M_\alpha$$

Now, keeping in mind the definition of M_α, we obtain that

$$Y_\alpha = A^* \left(\bigcup_{\lambda<\alpha} M_\lambda \right) = \left(\bigcup_{\lambda<\alpha} A^* M_\lambda \right) = \bigcup_{\lambda<\alpha} Y_\lambda$$

Each Y_λ is locally nilpotent for this reason the union Y_α is locally nilpotent. By the theorem of Kurdachenko 2.2.3 Y_α is solvable and hypercentral and, by theorem 2.2.4, it is minimax.

The final step is to prove that $|Y_\alpha \cap A : D \cap A| > 1$, for all α. By the observation we did before, we have only to prove that $|Y_t \cap A : D \cap A| > 1$. We proceed by induction on $j : 1, \ldots, t$. If $j = 1$ we have $Y_1 = N_G(M_0)$ and then knowing that A is hypercentral, we get $|N_A(M_0) : M_0| > 1$. Suppose that $j \geq 2$. By inductive hypothesis, we have

$$|N_A(M_{j-1}) : D \cap A| > 1;$$

that implies $|N_G(M_{j-1}) : M_j| > 1$. Then stressing the fact that $Y_j = N_G(M_{j-1})$ is hypercentral, we have that

$$|N_{N_G(M_{j-1})}(M_j) : M_j| > 1,$$

and then

$$|N_G(M_j) : M_{j+1}| > 1,$$

that means

$$|N_A(M_j) : D \cap A| > 1.$$

Now, D is hypercentral, so there exists an ordinal ρ such that $M_\rho = D$ and then $Y_\rho = (Y_\rho \cap A)D$. Moreover Y_ρ is hypercentral, minimax and it includes properly D, contradicting the maximality of D.

□

Theorem 5.4.6. *Let $G = AB$ where A and B are hypercentral minimax $\mathcal{N}$-connected subgroups, then G is hypercentral minimax.*

Proof. Let $H = A \cap B$. Observe that $I_A(H)$ and $I_B(H)$ are subgroups, so by 4.2.1, $I_G(H)$ is a subgroup and it satisfies the hypothesis of lemma 5.4.5. For this reason we can affirm that $I_G(H)$ is hypercentral minimax. Now, by definition

$$T(G) \subseteq I_G(H) = I_A(H) I_B(H)$$

Thus $T(G)$ is a subgroup, it is normal in G and satisfies Min. The group $G/T(G)$ satisfies proposition 5.4.2 and we get the thesis from lemma 5.4.1.

□

Corollary 5.4.7. *Let G be a group, $G = G_1G_2 \dots G_n$ where $G_1, G_2, \dots, G_n$ are pairwise permutable hypercentral minimax subgroups of G. If for all $i, j : 1, \dots, n$ such that $i \neq j$, we have that G_i and G_j are $\mathcal{N}$-connected, then G is hypercentral minimax.*

5.5 Examples

This last section starts with a result that should be considered just like an example or an exercise. We include it mainly for the sake of completeness and because it covers a marginal case that is not proved in the previous sections: the product of a factor that is a nilpotent periodic group (without assumption of Min condition) and the other that is nilpotent and satisfies Max. The proof is easy and it is based on the following observation that we state as a lemma:

Lemma 5.5.1. *Let G be a group and $A, B \leq G = \langle A, B\rangle$ such that A and B are $\mathcal{N}$-connected . Suppose that there exist $A_0 \leq A$ and $B_0 \leq B$ such that:*

(1) A_0 admits a system of generators $a_1, \dots, a_n$ such that $|a_i| < \infty$ for all $i : 1, \dots, n$;

(2) B_0 is finitely generated nilpotent group.

Then there exists $B_M \leq B_0$ with finite index in B_0 such that

$$[A_0, B_M] = 1.$$

Proof. We know that every finitely generated nilpotent group contains a torsion-free (finitely generated nilpotent) subgroup of finite index, so nothing is lost if we assume that B_0 is torsion-free. Consider $A_0 = \langle a_1, \dots, a_n\rangle$, where $|a_i| < \infty$ for all $i : 1, \dots, n$ and $B_0 = \langle b_1, \dots, b_m\rangle$. Consider the groups $\langle a_i, b_j\rangle$ with $i : 1, \dots, n$ and $j : 1, \dots, m$. They are nilpotent by $\mathcal{N}$-connection. By lemma 4.1.1 there exist $m_{ij} \in \mathbb{N}$ such that $b_j^{m_{ij}} \in Z(\langle a_i, b_j\rangle)$, for all i, j. Considering

$$M_j = \underset{i:1,\dots,n}{l.c.m.} \{m_{ij}\},$$

we obtain that $b_j^{M_j} \in C_G(A_0)$, so, called

$$B_M = \langle b_1^{M_1}, b_2^{M_2}, \dots, b_d^{M_d}\rangle$$

we have that B_M has finite index in B_0 by 1.3.12 and $B_M \leq C_G(A_0)$ that means $[A_0, B_M] = 1$.

□

Proposition 5.5.2. *Let G be a group $A, B \leq G = AB$ where A and B are $\mathcal{N}$-connected, A is a periodic nilpotent subgroup and B is a finitely generated nilpotent subgroup. Then G is locally nilpotent.*

Proof. Consider $g_1, \ldots, g_n \in G$. We know that for each g_i there exist $a_i \in A$ and $b_i \in B$ such that $g_i = a_i b_i$. So we will prove that there exists a solvable subgroup that contains $\langle g_1, \ldots, g_n \rangle$. Consider $A_0 = \langle a_1, \ldots, a_n \rangle$. By the previous lemma there exists $B_M \leq B$ with finite index such that $[A_0, B_M] = 1$. Nothing is lost if we assume that $B_M \lhd B$, otherwise it is sufficient to take the core. Thus $N_G(B_M) = N_A(B_M)B \geq \langle A_0, B \rangle$. Now,

$$\frac{N_G(B_M)}{B_M} = \frac{N_A(B_M)B_M}{B_M} \frac{B}{B_M}$$

where the second factor is finite nilpotent, while the first

$$\frac{N_A(B_M)B_M}{B_M} = \frac{N_A(B_M)}{N_A(B_M) \cap B_M} \simeq N_A(B_M)$$

is nilpotent and it has finite index in $N_G(B_M)/B_M$. This implies that $N_G(B_M)$ is (metanilpotent) solvable as desired. Consider

$$\langle g_1, \ldots, g_n \rangle \leq \langle a_1, \ldots, a_n, B \rangle \leq N_G(B_M) = N_A(B_M)B.$$

The group $S = \langle a_1, \ldots, a_n, B \rangle$ contains B, so $S = (S \cap A)B$. Applying Corollary 5.1.5 we obtain that S is locally nilpotent. But it is finitely generated, hence it is nilpotent. This fact means that G is locally nilpotent.

□

We, now, show a couple of examples of factorized groups by $\mathcal{N}$-connected subgroups.

Example 5.5.3. *Let p be a prime and $A = C_{p^\infty}$ the Prüfer p-group. Consider the endomorphism x defined by*

$$a^x = a^{1+p+p^2+\cdots}$$

for all $a \in A$. It is not difficult to show that it is an automorphism of A and that $|x|$ is infinite. Let $G = A \rtimes \langle x \rangle$ and call $\langle x \rangle = B$.

A is abelian and satisfies Min, B is abelian and satisfies Max. We prove that A and B are $\mathcal{N}$-connected, in fact for all $a \in A$, $[a, x] = a^p$, thus, by induction, if $|a| = p^k$ then $\langle a, x \rangle$ is nilpotent of class at most k. Applying the theorem G is hypercentral minimax.

Clearly, in this case, it could have been easier to check directly the hypercentrality, anyway in this context it could make sense also proceeding with $\mathcal{N}$-connection.

Example 5.5.4. *This example taken from the Marconi example [1, §7, p. 208-209], [44], that constructed a group satisfying the following:*

Theorem 5.5.5. *(Marconi) There exists a periodic unsolvable group* $G = ABC$ *which is the product of three pairwise permutable abelian subgroups* A, B *and* C *with Prüfer rank 1.*

We will not do a detailed discussion of the construction the author gave, we would only like to underline certain properties. Call $BC = H$*; by the theorem of Itô* H *is metabelian. However it is possible to prove:*

Proposition 5.5.6. *Let* G *the group constructed in the previous theorem. Then the following hold:*

1) G *is hypercentral;*

2) *the Prüfer rank* $\tilde{r}(H)$ *is finite;*

So we have that $G = AH$ *is a product of two* $\mathcal{N}$*-connected subgroups,* A *is abelian and* H *is hypercentral metabelian, both have finite Prüfer rank, but the group is unsolvable. So weakening the hypothesis on the rank, i.e. passing from finite minimax rank to finite Prüfer rank, we lose the solvability.*

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