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A-priori estimates for elliptic systems under general growth conditions

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Contents

1 A-priori higher integrability estimates under weak ellipticity assumptions 9

1.1 A-priori estimates 10

2 A-priori gradient estimates for elliptic systems under general growth conditions 24

2.1 Everywhere regularity for fast growth energies - the autonomous case 25

2.1.1 Ellipticity estimates 27

2.1.2 A-priori estimates 29

2.1.3 Approximation of the original problem with regular variational problems 33

2.2 Local minimizers with fast growth - the non autonomous case . . . 34

2.3 Regularity for elliptic systems with general growth - the autonomous case 37

2.3.1 Preliminary lemmata 39

2.3.2 A-priori estimates 44

2.3.3 Approximation of the original problem with regular variational problems 49

3 A-priori gradient estimates for elliptic systems under either slow or fast growth conditions 53

3.1 Some model examples 55

3.2 Preliminary lemmata 60

3.3 A-priori estimates 66

3.4 Ideas for approximation 80

Introduction

We are interested in the study of regularity properties of minimizers of integrals of the type

$$F(v) = \int_\Omega f(x, Dv(x))\, dx,$$

where, in general, $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is an open set, v is a Sobolev map of class $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ for some $p \geq 1$, $m \geq 1$, and $f(x,\xi) = f : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a Carathéodory function, i.e. a measurable function with respect to the spatial variable x and a continuous function with respect to the gradient variable. A local minimizer of F is a map $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ such that

$$F(u) \leq F(u + \varphi)$$

for every test function $\varphi \in W^{1,p}_0(\Omega, \mathbb{R}^m)$.

Under suitable regularity conditions on the integrand function f, every local minimizer of F solves the Euler-Lagrange system

$$\sum_{i=1}^{n} \sum_{\alpha=1}^{m} f_{\xi_i^\alpha \xi_j^\beta}(x, Du) = 0,$$

which is elliptic if f satisfies some convexity assumptions with respect to the gradient variable ξ.

In this thesis we study two different problems. First, we consider local minimizers of scalar integrals of the type

$$F(v) = \int_\Omega \sum_{i=1}^{n} a_i(x) \left|v_{x_i}\right|^{p_i}\, dx\,, \tag{1}$$

where a_i, $i = 1, \ldots, n$, are continuous positive functions and all the exponents p_i are greater than 1. In the mathematical literature the weak ellipticity has been first introduced by J.L Lions [43]. More recently, this condition has been considered by Cupini-Marcellini-Mascolo [20], [23], [24] and by L. Brasco *et al.* in a series of papers [10], [7], [9], [11].

The original idea of the study was to prove a Lipschitz continuity result for local minimizers of integral (1). Unfortunately, there are some technical issues that, by now, allow only to obtain an a-priori estimate for the L^p norms of the gradient of local minimizers. In particular, we can control the L^q norm of the gradient for every $q \geq 1$ in the case of all equal exponents $p_i = p$. The problem relies on the anisotropic structure of considered functionals, see Chapter 1 for details. However, the partial result in Lemma 1.1.1 is interesting and we believe that it could be a starting point to get an a-priori estimate in the full generality, both for different coefficients p_i and for the higher summability of minima.

The second problem that we face is the vectorial one with general growth conditions. It is well known that in the case $m \geq 2$ we do not expect everywhere regularity of minimizers of F or of weak solutions to the associated nonlinear differential system. Examples of non smooth solutions are originally due to De Giorgi [26], Giusti-Miranda [39], Nečas [56], and more recently to Šverák-Yan [58], De Silva-Savin [29], Mooney-Savin [55], Mooney [54]. A classical strategy to get everywhere regularity is to require that the integrand function f depends on the modulus of Du and not on the full gradient. In terms of f, we require that

$$f(x, \xi) = g(x, |\xi|)$$

where $g = g(x, t)$ is a suitable Carathéodory function.

In this *nonlinear context*, the first regularity result is due to Karen Uhlenbeck, obtained in her celebrated paper [60], published in 1977 and related to the energy-integral $f(x, \xi) = g(x, |\xi|) = |\xi|^p$ with exponents $p \geq 2$. Later Marcellini [46] in 1996 considered general energy-integrands $g(|\xi|)$ allowing *exponential growth* and Marcellini-Papi [51] in 2006 also some *slow growths*. Mascolo-Migliorini [52] studied some cases of integrands $g(x, |\xi|)$ which however ruled out the slow growth and power growth with exponents $p \in (1, 2)$. Only recently Beck-Mingione introduced in the integrand some $x-$dependence of the form $\int_\Omega \{g(|Du|) + h(x) \cdot u\}\, dx$ and they considered some sharp assumptions on the function $h(x)$, of the type $h \in L(n, 1)(\Omega; \mathbb{R}^m)$ in dimension $n > 2$ (i.e., $\int_0^{+\infty} \text{meas}\{x \in \Omega : |h(x)| > \lambda\}^{1/n}\, d\lambda < +\infty$; note that $L^{n+\varepsilon} \subset L(n, 1) \subset L^n$), or $h \in L^2 (\log L)^\alpha (\Omega; \mathbb{R}^m)$ for some $\alpha > 2$ when $n = 2$. Note that these assumptions on h are independent of the principal part g. Beck-Mingione obtained the local boundedness of the gradient Du of the local minimizer under some growth assumptions on $g(|\xi|)$, which however is assumed to be independent of x.

We see three works in particular. First, we report some results from Marcellini [46]. There, a Lipschitz regularity result is given without x-dependence and only for fast growth integrals. Then, we see that the x-dependence could be added, still with fast growth integrands, as done by Mascolo-Migliorini [52]. We do not show the details of the proofs, but this more generality is technically not easy to treat. Finally, we study the autonomous case of Marcellini-Papi [51]. We show in detail some preliminary lemmata used later in Chapter 3 and the approximation procedure as an idea of what should be done for our problem, even if the case shown is simpler since there is no x-dependence.

Results contained in Chapter 3 are essentially contained in [31]. We allow x-dependence in the principal part of the energy integrand, i.e., under the notation $|\xi| = t$, we consider a general integrand $g = g(x,t)$, which is a convex Carathéodory function, increasing with respect to $t \in [0,+\infty)$. We make assumptions that allow to consider both fast and slow growth of g. Model energy-integrals that we have in mind are, for instance, *exponential growth* with local Lipschitz continuous coefficients a, b $(a(x), b(x) \geq c > 0)$

$$\int_\Omega e^{a(x)|Du|^2}\, dx \quad \textit{or} \quad \int_\Omega b(x) \exp\left(\ldots \exp\left(a(x)|Du|^2\right)\right)\, dx\,;$$

variable exponents $(a, p \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, $a(x) \geq c > 0$ and $p(x) \geq p > 1)$

$$\int_\Omega a(x)|Du|^{p(x)}\, dx \quad \text{or} \quad \int_\Omega a(x)\left(1 + |Du|^2\right)^{p(x)/2}\, dx\,;$$

of course the classical $p-$*Laplacian* energy-integral, with a constant p strictly greater than 1 and integrand $f(x, Du) = a(x)|Du|^p$, is covered by the example (3.1.2): the theory considered here and Theorem 1 below apply to the $p-$Laplacian. Also *Orlicz-type* energy-integrals (see Chlebicka [14], Chlebicka *et al.* [15]), again with local Lipschitz continuous exponent $p(x) \geq p > 1$, of the type

$$\int_\Omega a(x)|Du|^{p(x)} \log(1 + |Du|)\, dx\,;$$

note that the a-priori estimate in Theorem 1 below holds also for some cases with *slow growth*, i.e. when $p(x) \geq 1$, in particular when $p(x)$ is identically equal to 1. See the details in Chapter 3. Moreover, we can consider a class of energy-integrals of the form

$$\int_\Omega h(a(x)|Du|)\, dx \quad \text{or} \quad \int_\Omega b(x)\, h(a(x)|Du|)\, dx\,,$$

with $a(x), b(x)$ locally Lipschitz continuous and nonnegative coefficients in Ω and $h : [0,+\infty) \to [0,+\infty)$ a convex increasing function of class $W^{2,\infty}_{\mathrm{loc}}([0,+\infty))$ as in the assumptions (2), (3) below. In addition, some $g(x,|\xi|)$ with *slow growth*, precisely *linear growth* as $t = |Du| \to +\infty$, such as, for $n = 2, 3$,

$$\int_\Omega \left\{|Du| - a(x)\sqrt{|Du|}\right\}\, dx\,,$$

with $a \in W^{1,\infty}_{\text{loc}}(\Omega)$, $a(x) \geq c > 0$ are included in the assumptions (here more precisely $t \to t - a(x)\sqrt{t}$ means a smooth convex function in $[0, +\infty)$, with derivative equal to zero at $t = 0$, which coincides with $t - a(x)\sqrt{t}$ for $t \geq t_0$, for a given $t_0 > 0$, and for $x \in \Omega$). Some of these examples are already covered by the regularity theories in literature; for instance, as already quoted, paper [60] by Uhlenbeck, Marcellini [46] and Marcellini-Papi [51], Mascolo-Migliorini [52], Beck-Mingione [4].

For completeness related to these researches we mention the *double phase problems*, recently intensively studied by Colombo-Mingione [18], [17] Baroni-Colombo-Mingione [1], [2], [3] and the *double phase* with *variable exponents* by Eleuteri-Marcellini-Mascolo [33], [32], [34]. See also Esposito-Leonetti-Mingione [35], Rădulescu-Zhang [57], Cencelja-Rădulescu-Repovš [13] and De Filippis [27]. For related recent references we quote Marcellini [45], [50], Cupini-Giannetti-Giova-Passarelli [19], Carozza-Giannetti-Leonetti-Passarelli [12], Cupini-Marcellini-Mascolo [24], [21], [22], Bousquet-Brasco [8], De Filippis-Mingione [28], Harjulehto-Hästö-Toivanen [40], Hästö-Ok [41], Mingione-Palatucci [53].

We show in Section 3.1 that the following assumptions cover the model examples. Precisely, we require the following growth conditions: let $t_0 > 0$ be fixed; for every open subset Ω' compactly contained in Ω there exist $\vartheta \geq 1$ and positive constants m and M_ϑ such that

$$\begin{cases} m h'(t) \leq g_t(x,t) \leq M_\vartheta \left[h'(t)\right]^\vartheta t^{1-\vartheta} \\ m h''(t) \leq g_{tt}(x,t) \leq M_\vartheta \left[h''(t)\right]^\vartheta \\ \left|g_{tx_k}(x,t)\right| \leq M_\vartheta \min\left\{g_t(x,t), t\, g_{tt}(x,t)\right\}^\vartheta \end{cases} \tag{2}$$

for every $t \geq t_0$ and for $x \in \Omega'$. The role of the parameter ϑ can be easily understood if we compare (2) with the above model examples; see Chapter 3 for details. Here, considering assumptions similar to those of [51], $h : [0, +\infty) \to [0, +\infty)$ is a convex increasing function of class $W^{2,\infty}_{\text{loc}}$ satisfying the following property: for some $\beta > \frac{1}{n}$ such that $(2\vartheta - 1)\vartheta < (1 - \beta)\frac{2^*}{2}$, and for every α such that $1 < \alpha \leq \frac{n}{n-1}$, there exist constants m_β and M_α such that

$$\frac{m_\beta}{t^{2\beta}}\left[\left(\frac{h'(t)}{t}\right)^{\frac{n-2}{n}} + \frac{h'(t)}{t}\right] \leq h''(t) \leq M_\alpha \left[\left(\frac{h'(t)}{t}\right)^{\alpha} + \frac{h'(t)}{t}\right] \tag{3}$$

for every $t \geq t_0$. We obtain the following a-priori gradient estimate.

Theorem 1 *Let us assume that assumptions (2) and (3),(3.0.3) hold. Then the gradient of any smooth local minimizer of the integral*

$$\int_\Omega g(x, |Dv(x)|)\, dx$$

is uniformly locally bounded in Ω. *Precisely, if* u *is a smooth local minimizer, then, there exists an exponent* $\omega > 1$ *and, for every* ρ, R, $0 < \rho < R$, *there exists a positive constant* C *such that*

$$\|Du\|_{L^\infty(B_\rho,\mathbb{R}^{m\times n})} \leq C\left\{\int_{B_R} (1 + g(x, |Du|))\, dx\right\}^\omega . \tag{0.0.1}$$

The exponent ω *depends on* ϑ, β, n, *while the constant* C *depends on* $\rho, R, n, \alpha, \beta, \vartheta, t_0$ *and* $\sup\{h''(t) :\ t \in [0, t_0]\}$.

The original idea of the work was to prove the Lipschitz continuity of local minimizers. Sadly, there are many difficulties that arise in the autonomous case when we allow both fast and slow growths. We have not proved a full regularity theorem yet, but only the a-priori estimate in Theorem 1. An approximation argument would give the local Lipschitz continuity of the minima. In fact, by applying the a-priori estimate to an approximating energy integrand $f_k(x, |\xi|)$ which converges to $f(x, |\xi|)$ as k goes to infinity and which satisfies standard growth conditions, we would obtain a sequence of smooth approximating solutions u_k with

$$\|Du_k\|_{L^\infty(B_\rho,\mathbb{R}^{m\times n})} \leq const,$$

for every fixed small radius ρ. The constant on the right hand side is independent of k. This approach is shown in Chapter 2, in particular in Section 2.3.3 where there are also some explicit computations. Some other good references in frameworks like ours are, for instance, Cupini-Marcellini-Mascolo [21], [22], [24], Eleuteri-Marcellini-Mascolo [32], [33], [34].

We conclude this introduction by a summary of the thesis. Chapter 1 is devoted to the study of an a-priori estimate for local minimizers of integrals with weak ellipticity assumptions.

In Chapter 2 we report some results in literature. In particular, in Section 2.1 we show some results of [46]. There, the author proves the local Lipschitz continuity without x-dependence and only for fast growth integrals. In Section 2.2, we refer to [52]. We see that the x-dependence can be added, only for integrals with fast growth. In Section 2.3 we study the autonomous case of Marcellini-Papi [51].

Finally, in Chapter 3, we show the results taken from [31].

Chapter 1

A-priori higher integrability estimates under weak ellipticity assumptions

In this chapter we are concerned with the study of local minimizers of integrals of the type

$$F(v) = \int_{\Omega} \sum_{i=1}^{n} a_i(x) |v_{x_i}|^{p_i} \, dx \,, \tag{1.0.1}$$

where a_i, $i = 1, \ldots, n$, are continuous positive functions and all the exponents p_i are greater than 1. If we denote by

$$f(x, Dv(x)) = \sum_{i=1}^{n} a_i(x) |v_{x_i}|^{p_i} \,,$$

then, under the notations $\xi = (\xi_i)_{i=1,\ldots,n}$ and $f = f(x, \xi) = \sum_{i=1}^{n} a_i(x) |\xi_i|^{p_i}$, we have $f_{\xi_i}(x, \xi) = p_i a_i(x) |\xi_i|^{p_i - 2} \xi_i$. For the quadratic form associated to the second derivatives of f we also get $f_{\xi_i \xi_i}(x, \xi) = p_i (p_i - 1) a_i(x) |\xi_i|^{p_i - 2}$ and thus

$$\sum_{i,j=1}^{n} f_{\xi_i \xi_j}(x, \xi) \lambda_i \lambda_j = \sum_{i=1}^{n} p_i (p_i - 1) a_i(x) |\xi_i|^{p_i - 2} \lambda_i^2 \,. \tag{1.0.2}$$

Therefore, for every set $\Omega' \subset\subset \Omega$, if we denote by $m = m(\Omega')$ the positive number

$$m = \min \left\{ p_i (p_i - 1) a_i(x) : \; i = 1, \ldots, n, \; x \in \overline{\Omega}' \right\} , \tag{1.0.3}$$

the ellipticity condition satisfied in this case is

$$\sum_{i,j=1}^{n} f_{\xi_i \xi_j}(x, \xi) \lambda_i \lambda_j \geq m \sum_{i=1}^{n} |\xi_i|^{p_i - 2} |\lambda|^2 \tag{1.0.4}$$

for every $x \in \Omega'$ and $\xi, \lambda \in \mathbb{R}^n$. We notice that, if all the exponents p_i are less than or equal to 2, then $p_i - 2 \leq 0$ and $|\xi_i|^{p_i-2} \geq |\xi|^{p_i-2}$ for all $i = 1, \ldots, n$; therefore in this case, at least if all the exponents p_i are equal to each other, say $p_i = p$ for all $i = 1, \ldots, n$, by denoting as before $m = m(\Omega')$ the positive number in (1.0.3), then

$$\sum_{i,j=1}^{n} f_{\xi_i \xi_j}(x, \xi) \lambda_i \lambda_j \geq m |\xi|^{p-2} |\lambda|^2, \tag{1.0.5}$$

for every $x \in \Omega'$ and for every $\xi, \lambda \in \mathbb{R}^n$. Inequality (1.0.5) is one of the usual forms of standard coercivity in Ω' of the function $f(x, \xi)$, or equivalently the usual form of ellipticity of Euler's first variation of the energy integral (1.0.1).

On the contrary, if for some index $i \in \{1, \ldots, n\}$ we have $p_i > 2$, then the quadratic form $\sum_{i,j=1}^{n} f_{\xi_i \xi_j}(x, \xi) \lambda_i \lambda_j$ in (1.0.2) is equal to zero on the nonzero vectors ξ of the form $\xi = (0, 0, \ldots, \xi_i, \ldots, 0)$. In this case the variational problem is not coercive in the usual sense and the Euler's first variation is a partial differential equation with some degenerate ellipticity; we call (1.0.4) *weak ellipticity.*

We consider an energy integral of the type

$$F(v) = \int_{\Omega} f(x, Dv(x))\, dx, \tag{1.0.6}$$

in an open set $\Omega \subset \mathbb{R}^n$ for $n \geq 2$ where $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is of class $W^{2,\infty}_{\text{loc}}(\Omega \times \mathbb{R}^n)$ and a Carathéodory function, i.e. a measurable function with respect to the variable x and continuous with respect to ξ. We assume the following growth conditions: there exist two positive constants m, M_1 such that

$$m \sum_i |\xi_i|^{p_i-2} \lambda_i^2 \leq \sum_{i,j} f_{\xi_i \xi_j}(x, \xi) \lambda_i \lambda_j \leq M \sum_i |\xi|^{p_i-2} |\lambda|^2 \tag{1.0.7}$$

for almost every $x \in \Omega$ and for every $\lambda, \xi \in \mathbb{R}^n$. Moreover there exists $M_2 > 0$ such that

$$|f_{\xi_i x_k}(x, \xi)| \leq M_2 h(x) |\xi|^{p_i-1}, \tag{1.0.8}$$

for every $\lambda, \xi \in \mathbb{R}^n$, where h is a locally bounded function in Ω.

1.1 A-priori estimates

We assume here some regularity assumptions. Precisely, we assume that there exist two positive constants c_1, c_2 such that for almost every $x \in \Omega$

$$c_1 |\lambda|^2 \leq \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(x, \xi) \lambda_i \lambda_j \leq c_2 |\lambda|^2, \tag{1.1.1}$$

for every $\lambda, \xi \in \mathbb{R}^n$. This assumption allows us to consider u as a function of class $W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap W^{2,2}_{\mathrm{loc}}(\Omega)$. We denote with B_ρ and B_R balls of radii respectively ρ and R contained in Ω and with the same center.

Lemma 1.1.1 *Let u be a local minimizer of integral (1.0.6). Suppose that the integrand function f satisfies (1.0.7), (1.0.8) and the supplementary assumption (1.1.1). Let $\eta \in C^1_0(\Omega)$. Then, for every $\beta \geq 0$ there exist constants $c_1, c_2, c_3, c_4 > 0$, not depending on the constants N and M of (1.1.1) such that*

$$\left(\int_\Omega \eta^{2^*} \sum_i \frac{c}{(\beta + p_i)^{2^*}} \left| u_{x_i} \right|^{\frac{\beta + p_i}{2} 2^*} dx \right)^{\frac{2}{2^*}} \tag{1.1.2}$$

$$\leq c_1 \int_\Omega \eta \, |D\eta| \, |Du|^{\beta+1} h(x) \sum_i \left| u_{x_i} \right|^{p_i - 1} dx$$

$$+ c_2 \int_\Omega \eta^2 \, (\beta + 1) \, |Du|^\beta \, h^2(x) \sum_i \left| u_{x_i} \right|^{p_i} dx$$

$$+ c_3 \int_\Omega |D\eta|^2 \, |Du|^{\beta+2} \sum_i \left| u_{x_i} \right|^{p_i - 2} dx$$

$$+ c_4 \int_\Omega |D\eta|^2 \sum_i \frac{1}{(\beta + p_i)^2} \left| u_{x_i} \right|^{\beta + p_i} dx.$$

Proof. Let u be a local minimizer of energy integral (1.0.6). By the right hand side of (1.1.1) u satisfies the Euler's first variation

$$\int_\Omega \sum_{i=1}^n f_{\xi_i}(x, Du) \, \varphi_{x_i}(x) \, dx = 0, \tag{1.1.3}$$

for every $\varphi \in W^{1,2}_0(\Omega)$. Using the technique of difference quotients we can prove that u admits second order weak partial derivatives, precisely that $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ and satisfies the second variation equation

$$\int_\Omega \left(\sum_{i,j=1}^n f_{\xi_i \xi_j}(x, Du) \, u_{x_k x_j} \varphi_{x_i} + \sum_{i=1}^n f_{\xi_i x_k}(x, Du) \, \varphi_{x_i} \right) dx = 0, \tag{1.1.4}$$

for every $k = 1, \ldots, n$ and for every $\varphi \in W^{1,2}_0(\Omega)$. Let $\eta \in C^1_0(\Omega)$, then for any fixed $k \in \{1, \ldots, n\}$ let us define

$$\varphi = \eta^2 u_{x_k} \sum_{h=1}^n \Phi\left(\left| u_{x_h} \right| \right),$$

where $\Phi : [0, +\infty) \to [0, +\infty)$ is a positive, increasing and locally Lipschitz continuous function. Then, almost everywhere in Ω, we have

$$\varphi_{x_i} = 2\eta\eta_{x_i} u_{x_k} \sum_h \Phi\left(|u_{x_h}|\right) + \eta^2 u_{x_k x_i} \sum_h \Phi\left(|u_{x_h}|\right) + \eta^2 u_{x_k} \sum_h \Phi'\left(|u_{x_h}|\right)\left(|u_{x_h}|\right)_{x_i}$$

and from equation (1.1.4) we deduce that for every $k = 1, \dots, n$

$$\begin{aligned} 0 = &\int_\Omega 2\eta \left(\sum_h \Phi\left(|u_{x_h}|\right)\right)\left(\sum_{i,j} f_{\xi_i\xi_j}\left(x, Du\right) u_{x_k x_j} \eta_{x_i} u_{x_k}\right) dx \\ &+ \int_\Omega \eta^2 \left(\sum_h \Phi\left(|u_{x_h}|\right)\right) \sum_{i,j} f_{\xi_i\xi_j}\left(x, Du\right) u_{x_k x_j} u_{x_k x_i}\, dx \\ &+ \int_\Omega \eta^2 \sum_{i,j} \left(\sum_h \Phi'\left(|u_{x_h}|\right)\left(|u_{x_h}|\right)_{x_i}\right) f_{\xi_i\xi_j}\left(x, Du\right) u_{x_k x_j} u_{x_k}\, dx \\ &+ \int_\Omega 2\eta \left(\sum_h \Phi\left(|u_{x_h}|\right)\right) \sum_i f_{\xi_i x_k}\left(x, Du\right) \eta_{x_i} u_{x_k}\, dx \\ &+ \int_\Omega \eta^2 \left(\sum_h \Phi\left(|u_{x_h}|\right)\right) \sum_i f_{\xi_i x_k}\left(x, Du\right) u_{x_k x_i}\, dx \\ &+ \int_\Omega \eta^2 \sum_i \left(\sum_h \Phi'\left(|u_{x_h}|\right)\left(|u_{x_h}|\right)_{x_i}\right) f_{\xi_i x_k}\left(x, Du\right) u_{x_k} \left(|Du|\right)_{x_i}\, dx \end{aligned} \tag{1.1.5}$$

We can estimate the first integral in (1.1.5) by using the Cauchy-Schwarz inequality and the inequality $2ab \leq \frac{1}{2}a^2 + 2b^2$:

$$\begin{aligned} &\left|\int_\Omega 2\eta \left(\sum_h \Phi\left(|u_{x_h}|\right)\right) \sum_{i,j} f_{\xi_i\xi_j}\left(x, Du\right) u_{x_k x_j} \eta_{x_i} u_{x_k}\, dx\right| \\ &\leq \int_\Omega 2 \left(\sum_h \Phi\left(|u_{x_h}|\right)\right)\left(\eta^2 \sum_{i,j} f_{\xi_i\xi_j}\left(x, Du\right) u_{x_k x_i} u_{x_k x_j}\right)^{\frac{1}{2}} \\ &\qquad \cdot \left(\sum_{i,j} f_{\xi_i\xi_j}\left(x, Du\right) \eta_{x_i} u_{x_k} \eta_{x_j} u_{x_k}\right)^{\frac{1}{2}} dx \\ &\leq \frac{1}{2} \int_\Omega \eta^2 \left(\sum_h \Phi\left(|u_{x_h}|\right)\right) \sum_{i,j} f_{\xi_i\xi_j}\left(x, Du\right) u_{x_k x_i} u_{x_k x_j}\, dx \end{aligned}$$

$$+2\int_{\Omega}\left(\sum_{h}\Phi\left(|u_{x_h}|\right)\right)\sum_{i,j}f_{\xi_i\xi_j}\left(x,Du\right)\eta_{x_i}\eta_{x_j}u_{x_k}^2\,dx.$$

From (1.1.5), we get

$$\frac{1}{2}\int_{\Omega}\eta^2\left(\sum_{h}\Phi\left(|u_{x_h}|\right)\right)\sum_{i,j}f_{\xi_i\xi_j}\left(x,Du\right)u_{x_kx_j}u_{x_kx_i}\,dx \tag{1.1.6}$$

$$+\int_{\Omega}\eta^2\sum_{i,j}\left(\sum_{h}\Phi'\left(|u_{x_h}|\right)\left(|u_{x_h}|\right)_{x_i}\right)f_{\xi_i\xi_j}\left(x,Du\right)u_{x_kx_j}u_{x_k}\,dx$$

$$+2\int_{\Omega}\eta\left(\sum_{h}\Phi\left(|u_{x_h}|\right)\right)\sum_{i}f_{\xi_ix_k}\left(x,Du\right)\eta_{x_i}u_{x_k}\,dx$$

$$+\int_{\Omega}\eta^2\left(\sum_{h}\Phi\left(|u_{x_h}|\right)\right)\sum_{i}f_{\xi_ix_k}\left(x,Du\right)u_{x_kx_i}\,dx$$

$$+\int_{\Omega}\eta^2\sum_{i}\left(\sum_{h}\Phi'\left(|u_{x_h}|\right)\left(|u_{x_h}|\right)_{x_i}\right)f_{\xi_ix_k}\left(x,Du\right)u_{x_k}\,dx$$

$$\leq 2\int_{\Omega}\left(\sum_{h}\Phi\left(|u_{x_h}|\right)\right)\sum_{i,j}f_{\xi_i\xi_j}\left(x,Du\right)\eta_{x_i}\eta_{x_j}u_{x_k}^2\,dx.$$

Since $\left(|Du|\right)_{x_i}=\frac{1}{|Du|}\sum_k u_{x_kx_i}u_{x_k}$ almost everywhere in Ω, it is natural to sum up with respect to k in (1.1.6) to obtain

$$\sum_{k}\sum_{i,j}f_{\xi_i\xi_j}\left(Du\right)u_{x_kx_j}u_{x_k}\left(|Du|\right)_{x_i}=|Du|\sum_{i,j}f_{\xi_i\xi_j}\left(Du\right)\left(|Du|\right)_{x_i}\left(|Du|\right)_{x_j}.$$

Therefore, from (1.1.6) we deduce the following estimate:

$$\int_{\Omega}\eta^2\,\Phi\left(|Du|\right)\sum_{i,j,k}f_{\xi_i\xi_j}\left(x,Du\right)u_{x_kx_i}u_{x_kx_j}\,dx \tag{1.1.7}$$

$$+2\int_{\Omega}\eta^2\,|Du|\,\Phi'\left(|Du|\right)\sum_{i,j}f_{\xi_i\xi_j}\left(x,Du\right)\left(|Du|\right)_{x_i}\left(|Du|\right)_{x_j}\,dx$$

$$+4\int_{\Omega}\eta\,\Phi\left(|Du|\right)\sum_{i,k}f_{\xi_ix_k}\left(x,Du\right)\eta_{x_i}u_{x_k}\,dx$$

$$+2\int_{\Omega}\eta^2\,\Phi\left(|Du|\right)\sum_{i,k}f_{\xi_ix_k}\left(x,Du\right)u_{x_kx_i}\,dx$$

$$+2\int_{\Omega}\eta^{2}\,\Phi'\left(|Du|\right)\sum_{i,k}f_{\xi_i x_k}\left(x,Du\right)u_{x_k}\left(|Du|\right)_{x_i}\,dx$$

$$\leq 4\int_{\Omega}|Du|^{2}\,\Phi\left(|Du|\right)\sum_{i,j}f_{\xi_i\xi_j}\left(x,Du\right)\eta_{x_i}\eta_{x_j}\,dx\,.$$

We use the left hand side of ellipticity assumption (1.0.7) to get

$$\int_{\Omega}\eta^{2}\left(\sum_{h}\Phi\left(|u_{x_h}|\right)\right)\sum_{i,j,k}f_{\xi_i\xi_j}\left(x,Du\right)u_{x_k x_i}u_{x_k x_j}\,dx$$

$$\geq m\int_{\Omega}\eta^{2}\left(\sum_{h}\Phi\left(|u_{x_h}|\right)\right)\sum_{i,k}|u_{x_i}|^{p_i-2}\,u_{x_k x_i}^{2}\,dx$$

and

$$\int_{\Omega}\eta^{2}\,|Du|\,\Phi'\left(|Du|\right)\sum_{i,j}f_{\xi_i\xi_j}\left(x,Du\right)\left(|Du|\right)_{x_i}\left(|Du|\right)_{x_j}\,dx$$

$$\geq m\int_{\Omega}\eta^{2}\,|Du|\,\Phi'\left(|Du|\right)\sum_{i}|u_{x_i}|^{p_i-2}\left(|Du|\right)_{x_i}^{2}\,.$$

Thus

$$\begin{aligned} & m\int_{\Omega}\eta^{2}\left(\sum_{h}\Phi\left(|u_{x_h}|\right)\right)\sum_{i,k}|u_{x_i}|^{p_i-2}\,u_{x_k x_i}^{2}\,dx \\ & +2m\int_{\Omega}\eta^{2}\,|Du|\,\Phi'\left(|Du|\right)\sum_{i}|u_{x_i}|^{p_i-2}\left(|Du|\right)_{x_i}^{2} \\ \leq\; & 4\left|\int_{\Omega}\eta\left(\sum_{h}\Phi\left(|u_{x_h}|\right)\right)\sum_{i,k}f_{\xi_i x_k}\left(x,Du\right)\eta_{x_i}u_{x_k}\,dx\right| \\ & +2\left|\int_{\Omega}\eta^{2}\left(\sum_{h}\Phi\left(|u_{x_h}|\right)\right)\sum_{i,k}f_{\xi_i x_k}\left(x,Du\right)u_{x_k x_i}\,dx\right| \\ & +2\left|\int_{\Omega}\eta^{2}\sum_{i,k}\left(\sum_{h}\Phi'\left(|u_{x_h}|\right)\left(|u_{x_h}|\right)_{x_i}\right)f_{\xi_i x_k}\left(x,Du\right)u_{x_k}\,dx\right| \\ & +4\int_{\Omega}|Du|^{2}\,\Phi\left(|Du|\right)\sum_{i,j}f_{\xi_i\xi_j}\left(x,Du\right)\eta_{x_i}\eta_{x_j}\,dx\;=I_1+I_2+I_3+I_4. \end{aligned}\tag{1.1.8}$$

We deal with the first integral in the right hand side of (1.1.8), denoted by I_1:

$$I_1 = \left| 4 \int_\Omega \eta \left(\sum_h \Phi\left(|u_{x_h}|\right) \right) \sum_{i,k} f_{\xi_i x_k}\left(x, Du\right) u_{x_k} \eta_{x_i} \, dx \right| \tag{1.1.9}$$

$$\leq 4M_2 \int_\Omega \eta \left(\sum_h \Phi\left(|u_{x_h}|\right) \right) h(x) \sum_{i,k} |u_{x_i}|^{p_i - 1} \left| \eta_{x_i} u_{x_k} \right| \, dx$$

$$\leq 4M_2 \int_\Omega \eta \, |D\eta| \left(\sum_h \Phi\left(|u_{x_h}|\right) \right) h(x) \left(\sum_i |u_{x_i}|^{p_i - 1} \right) \left(\sum_k |u_{x_k}| \right) \, dx$$

$$\leq 4M_2 \int_\Omega \eta \, |D\eta| \, |Du| \left(\sum_h \Phi\left(|u_{x_h}|\right) \right) h(x) \sum_i |u_{x_i}|^{p_i - 1} \, dx,$$

by (1.0.8). Again, by using (1.0.8) and the Young inequality we estimate I_2:

$$I_2 = 2 \left| \int_\Omega \eta^2 \left(\sum_h \Phi\left(|u_{x_h}|\right) \right) \sum_{i,k} f_{\xi_i x_k}\left(x, Du\right) u_{x_k x_i} \, dx \right| \tag{1.1.10}$$

$$\leq 2M_2 \int_\Omega \eta^2 \left(\sum_h \Phi\left(|u_{x_h}|\right) \right) h(x) \sum_{i,k} |u_{x_i}|^{p_i - 1} \left| u_{x_k x_i} \right| \, dx$$

$$\leq 2M_2 \sum_{i,k} \int_\Omega \left(\eta^2 \left(\sum_h \Phi\left(|u_{x_h}|\right) \right) |u_{x_i}|^{p_i - 2} u_{x_k x_i}^2 \right)^{1/2}$$

$$\cdot \left(\eta^2 \left(\sum_h \Phi\left(|u_{x_h}|\right) \right) h^2(x) \, |u_{x_i}|^{p_i} \right)^{1/2} dx$$

$$\leq 2M_2 \varepsilon \int_\Omega \eta^2 \left(\sum_h \Phi\left(|u_{x_h}|\right) \right) \sum_{i,k} |u_{x_i}|^{p_i - 2} u_{x_k x_i}^2 \, dx$$

$$+ 2M_2 C_\varepsilon \int_\Omega \eta^2 \left(\sum_h \Phi\left(|u_{x_h}|\right) \right) h^2(x) \sum_i |u_{x_i}|^{p_i} \, dx.$$

In a similar way, we estimate I_3:

$$I_3 = 2 \left| \int_\Omega \eta^2 \, \Phi'\left(|Du|\right) \sum_{i,k} f_{\xi_i x_k}\left(x, Du\right) u_{x_k} \left(|Du|\right)_{x_i} \, dx \right| \tag{1.1.11}$$

$$\leq 2M_3 \int_\Omega \eta^2 \, \Phi'\left(|Du|\right) h(x) \sum_{i,k} |u_{x_i}|^{p_i - 1} \left(|Du|\right)_{x_i} u_{x_k} \, dx$$

$$\leq c \int_\Omega \eta^2 \, \Phi' \left(|Du| \right) h(x) \, |Du| \sum_i |u_{x_i}|^{p_i - 1} \left(|Du| \right)_{x_i} \, dx$$

$$= \sum_i c \int_\Omega \left[\eta^2 \, \Phi' \left(|Du| \right) h^2(x) \, |Du| \, |u_{x_i}|^{p_i} \right]^{1/2} \cdot$$

$$\cdot \left[\eta^2 \, \Phi' \left(|Du| \right) |Du| \, |u_{x_i}|^{p_i - 2} \left(|Du| \right)_{x_i}^2 \right]^{1/2} \, dx$$

$$\leq C_\varepsilon \int_\Omega \eta^2 \, \Phi' \left(|Du| \right) h^2(x) \, |Du| \sum_i |u_{x_i}|^{p_i} \, dx$$

$$+ \varepsilon \int_\Omega \eta^2 \, \Phi' \left(|Du| \right) |Du| \sum_i |u_{x_i}|^{p_i - 2} \left(|Du| \right)_{x_i}^2 \, dx$$

and then, choosing ε sufficiently small

$$\int_\Omega \eta^2 \, \Phi \left(|Du| \right) \sum_{i,k} |u_{x_i}|^{p_i - 2} \, u_{x_k x_i}^2 \, dx \tag{1.1.12}$$

$$+ \int_\Omega \eta^2 \, |Du| \, \Phi' \left(|Du| \right) \sum_i |u_{x_i}|^{p_i - 2} \left(|Du| \right)_{x_i}^2$$

$$\leq c_1 \int_\Omega \eta \, |D\eta| \, |Du| \, \Phi \left(|Du| \right) \sum_i |u_{x_i}|^{p_i - 1} \, dx$$

$$+ c_2 \int_\Omega \eta^2 \, \left[\Phi \left(|Du| \right) + |Du| \, \Phi' \left(|Du| \right) \right] h^2(x) \sum_i |u_{x_i}|^{p_i} \, dx$$

$$+ 4 \int_\Omega |Du|^2 \, \Phi \left(|Du| \right) \sum_{i,j} f_{\xi_i \xi_j} \left(x, Du \right) \eta_{x_i} \eta_{x_j} \, dx.$$

Then, since the second term in the left hand side of (1.1.12) is positive, we get

$$\int_\Omega \eta^2 \, \Phi \left(|Du| \right) \sum_{i,k} |u_{x_i}|^{p_i - 2} \, u_{x_k x_i}^2 \, dx \tag{1.1.13}$$

$$\leq c_1 \int_\Omega \eta \, |D\eta| \, |Du| \, \Phi \left(|Du| \right) h(x) \sum_i |u_{x_i}|^{p_i - 1} \, dx$$

$$+ c_2 \int_\Omega \eta^2 \, \left[\Phi \left(|Du| \right) + |Du| \, \Phi' \left(|Du| \right) \right] h^2(x) \sum_i |u_{x_i}|^{p_i} \, dx$$

$$+ 4 \int_\Omega |Du|^2 \, \Phi \left(|Du| \right) \sum_{i,j} f_{\xi_i \xi_j} \left(x, Du \right) \eta_{x_i} \eta_{x_j} \, dx.$$

By the right hand side of (1.0.7) we finally get

$$\int_{\Omega} \eta^2 \, \Phi\left(|Du|\right) \sum_{i,k} |u_{x_i}|^{p_i-2} \, u_{x_k x_i}^2 \, dx \tag{1.1.14}$$

$$\leq c_1 \int_{\Omega} \eta \, |D\eta| \, |Du| \, \Phi\left(|Du|\right) h(x) \sum_i |u_{x_i}|^{p_i-1} \, dx$$

$$+c_2 \int_{\Omega} \eta^2 \, \left[\Phi\left(|Du|\right) + |Du| \, \Phi'\left(|Du|\right)\right] h^2(x) \sum_i |u_{x_i}|^{p_i} \, dx$$

$$+c_3 \int_{\Omega} |Du|^2 \, \Phi\left(|Du|\right) |D\eta|^2 \sum_i |u_{x_i}|^{p_i-2} \, dx.$$

Now, let us take $\Phi\left(t\right) = |t|^{\beta}$, from which we deduce that $\Phi'\left(|Du|\right) = \beta \, |Du|^{\beta-1}$. Thus,

$$\int_{\Omega} \eta^2 \, |Du|^{\beta} \sum_{i,k} |u_{x_i}|^{p_i-2} \, u_{x_k x_i}^2 \, dx \tag{1.1.15}$$

$$\leq c_1 \int_{\Omega} \eta \, |D\eta| \, |Du|^{\beta+1} \, h(x) \sum_i |u_{x_i}|^{p_i-1} \, dx$$

$$+c_2 \int_{\Omega} \eta^2 \, \left(\beta+1\right) |Du|^{\beta} \, h^2(x) \sum_i |u_{x_i}|^{p_i} \, dx$$

$$+c_3 \int_{\Omega} |D\eta|^2 \, |Du|^{\beta+2} \sum_i |u_{x_i}|^{p_i-2} \, dx.$$

First, we observe that the sum over i and k in the left hand side of (1.1.15) can be omitted. Then we observe that

$$\left| \frac{\partial}{\partial x_k} \, |u_{x_i}|^{\frac{\beta+p_i}{2}} \right|^2 = \frac{\left(\beta+p_i\right)^2}{4} \, |u_{x_i}|^{\beta+p_i-2} \, u_{x_i x_k}^2 .$$

By the inequality

$$\left| \frac{\partial}{\partial x_k} \left(\eta v\right) \right|^2 \leq 2 \left(\eta^2 \, |v_{x_k}|^2 + v^2 \, \left|\eta_{x_k}\right|^2 \right) \leq 2 \left(\eta^2 \, |v_{x_k}|^2 + v^2 \, |D\eta|^2 \right),$$

with $v = |u_{x_i}|^{\frac{\beta+p_i}{2}}$, we then get

$$\frac{c}{\left(\beta+p_j\right)^2} \int_{\Omega} \left| \frac{\partial}{\partial x_k} \left(\eta \left|u_{x_j}\right|^{\frac{\beta+p_j}{2}} \right) \right|^2 dx \tag{1.1.16}$$

$$\leq c_1 \int_\Omega \eta \, |D\eta| \, |Du|^{\beta+1} \, h(x) \sum_i |u_{x_i}|^{p_i-1} \, dx$$

$$+c_2 \int_\Omega \eta^2 \, (\beta+1) \, |Du|^\beta \, h^2(x) \sum_i |u_{x_i}|^{p_i} \, dx$$

$$+c_3 \int_\Omega |D\eta|^2 \, |Du|^{\beta+2} \sum_i |u_{x_i}|^{p_i-2} \, dx$$

$$+c_4 \int_\Omega |D\eta|^2 \sum_i \frac{1}{(\beta+p_i)^2} \, |u_{x_i}|^{\beta+p_i} \, dx$$

for some constants c_1, c_2, c_3, c_4 and for every $j, k = 1, \dots n$.

We multiply in with respect to k in (1.1.16) and we use the Sobolev inequality for products to get

$$\frac{c}{(\beta+p_j)^2} \left(\int_\Omega \left(\eta \left| u_{x_j} \right|^{\frac{\beta+p_j}{2}} \right)^{2^*} \, dx \right)^{\frac{2}{2^*}} \tag{1.1.17}$$

$$\leq c_1 \int_\Omega \eta \, |D\eta| \, |Du|^{\beta+1} \, h(x) \sum_i |u_{x_i}|^{p_i-1} \, dx$$

$$+c_2 \int_\Omega \eta^2 \, (\beta+1) \, |Du|^\beta \, h^2(x) \sum_i |u_{x_i}|^{p_i} \, dx$$

$$+c_3 \int_\Omega |D\eta|^2 \, |Du|^{\beta+2} \sum_i |u_{x_i}|^{p_i-2} \, dx$$

$$+c_4 \int_\Omega |D\eta|^2 \sum_i \frac{1}{(\beta+p_i)^2} \, |u_{x_i}|^{\beta+p_i} \, dx,$$

for every $j = 1, \dots, n$. Finally, if we sum with respect to $j = 1, \dots, n$, we get

$$\sum_j \frac{c}{(\beta+p_j)^2} \left(\int_\Omega \left(\eta \left| u_{x_j} \right|^{\frac{\beta+p_j}{2}} \right)^{2^*} \, dx \right)^{\frac{2}{2^*}} \tag{1.1.18}$$

$$\leq c_1 \int_\Omega \eta \, |D\eta| \, |Du|^{\beta+1} \, h(x) \sum_i |u_{x_i}|^{p_i-1} \, dx$$

$$+c_2 \int_\Omega \eta^2 \, (\beta+1) \, |Du|^\beta \, h^2(x) \sum_i |u_{x_i}|^{p_i} \, dx$$

$$+c_3 \int_\Omega |D\eta|^2 \, |Du|^{\beta+2} \sum_i |u_{x_i}|^{p_i-2} \, dx$$

$$+c_4 \int_\Omega |D\eta|^2 \sum_i \frac{1}{(\beta+p_i)^2} |u_{x_i}|^{\beta+p_i} \, dx.$$

By using the inequality $\sum_{i=1}^n y_i^a \leq \left(\sum_{i=1}^n y_i\right)^a$ for $a > 0$ and the Minkowski inequality, we can move the sum on the left side of (1.1.18) inside the integral and get

$$\left(\int_\Omega \eta^{2^*} \sum_i \frac{c}{(\beta+p_i)^{2^*}} |u_{x_i}|^{\frac{\beta+p_i}{2}2^*} \, dx\right)^{\frac{2}{2^*}} \tag{1.1.19}$$

$$\leq c_1 \int_\Omega \eta \, |D\eta| \, |Du|^{\beta+1} h(x) \sum_i |u_{x_i}|^{p_i-1} \, dx$$

$$+c_2 \int_\Omega \eta^2 \, (\beta+1) \, |Du|^\beta \, h^2(x) \sum_i |u_{x_i}|^{p_i} \, dx$$

$$+c_3 \int_\Omega |D\eta|^2 \, |Du|^{\beta+2} \sum_i |u_{x_i}|^{p_i-2} \, dx$$

$$+c_4 \int_\Omega |D\eta|^2 \sum_i \frac{1}{(\beta+p_i)^2} |u_{x_i}|^{\beta+p_i} \, dx.$$

■

In order to proceed with the estimates, another (and more restrictive) assumption is needed. We suppose that all the exponents $p_i = p$. We observe that integrals in the right hand side of (1.1.2) have homogeneous exponents, in the sense that the sum of all the exponents of the derivatives of u involved in each integral equals $p + \beta$. Now the problem is that we do not have an inequality that let us reduce the right hand side to only one integral. Before we go on with the estimate, we report here an useful algebraic lemma, taken from [47].

Lemma 1.1.2 *Let $y_i \geq 0$ for every $i = 1, \ldots, n$ and let $a, b > 0$. Then*

$$\sum_{i=1}^n y_i^a \cdot \sum_{i=1}^n y_i^b \leq \left(1 + \frac{n(n-1)}{2}\right) \sum_{i=1}^n y_s^{a+b}.$$

Proof.

$$\begin{aligned}
\sum_{i=1}^n y_i^a \cdot \sum_{i=1}^n y_i^b &= \sum_{i=1}^n y_i^{a+b} + \sum_{i \neq j} \left[y_i^a y_j^b + y_j^a y_i^b\right] \\
&\leq \sum_{i=1}^n y_i^{a+b} + \sum_{i \neq j} \left[\left(\frac{a}{a+b} y_i^{a+b} + \frac{b}{a+b} y_j^{a+b}\right) + \left(\frac{a}{a+b} y_j^{a+b} + \frac{b}{a+b} y_i^{a+b}\right)\right] \\
&= \sum_{i=1}^n y_i^{a+b} + \binom{n}{2} \sum_{i=1}^n y_i^{a+b}.
\end{aligned} \tag{1.1.20}$$

■

Lemma 1.1.3 *Let u be a local minimizer of integral (1.0.6). Suppose that the integrand function f satisfies (1.0.7), (1.0.8) and the supplementary assumption (1.1.1) and moreover let us assume that $p_i = p$ for every $i = 1, \dots, n$.*

Then for every $k \geq 0$ and ρ, R ($\rho < R$) there exists a constant c, depending on ρ, R, k, but not on the constant m and M_1 of (1.1.1), such that

$$\left(\int_{B_{R_k}} \sum_i |u_{x_i}|^{\beta_k + p} \, dx \right)^{\frac{1}{\beta_k + p}} \leq c(k) \left(\int_{B_{R_0}} \sum_i |u_{x_i}|^p \, dx \right)^{\frac{1}{p}} \tag{1.1.21}$$

where $\beta_k = p \left(\left(\frac{2^*}{2}\right)^{k-1} - 1 \right)$.

Proof. We observe that

$$|Du|^\beta = \left(\sum_{i=1}^n u_{x_i}^2 \right)^{\frac{\beta}{2}} \leq \left(\sum_{i=1}^n |u_{x_i}| \right)^\beta .$$

Since

$$\left(\sum_{i=1}^n |u_{x_i}| \right)^\beta \leq 2^{\beta(n-1)} \sum_{i=1}^n |u_{x_i}|^\beta \tag{1.1.22}$$

for every $\beta \geq 0$, by using (1.1.22) with β, $\beta + 1$ and $\beta + 2$ as exponents, we can derive from formula (1.1.19) the following inequality:

$$\left(\int_\Omega \eta^{2^*} \sum_i \frac{c}{(\beta + p)^{2^*}} |u_{x_i}|^{\frac{\beta+p}{2} 2^*} \, dx \right)^{\frac{2}{2^*}} \tag{1.1.23}$$

$$\leq c_1 2^{\beta(n-1)} \int_\Omega \eta \, |D\eta| \, h(x) \left(\sum_i |u_{x_i}|^{\beta+1} \right) \left(\sum_i |u_{x_i}|^{p-1} \right) dx$$

$$+ c_2 2^{\beta(n-1)} \int_\Omega \eta^2 \, (\beta + 1) \, h^2(x) \left(\sum_i |u_{x_i}|^{\beta} \right) \left(\sum_i |u_{x_i}|^{p} \right) dx$$

$$+ c_3 2^{\beta(n-1)} \int_\Omega |D\eta|^2 \left(\sum_i |u_{x_i}|^{\beta+2} \right) \left(\sum_i |u_{x_i}|^{p-2} \right) dx$$

$$+ c_4 \int_\Omega |D\eta|^2 \sum_i \frac{1}{(\beta + p_i)^2} |u_{x_i}|^{\beta + p_i} \, dx.$$

By Lemma 1.1.2, (1.1.23) becomes

$$\left(\int_\Omega \eta^{2^*} \sum_i \frac{c}{(\beta+p)^{2^*}} |u_{x_i}|^{\frac{\beta+p}{2}2^*} \, dx\right)^{\frac{2}{2^*}}$$
$$\leq c2^{\beta(n-1)}(\beta+1)\int_\Omega \left(|\eta|^2+|D\eta|^2\right)\left(1+h^2(x)\right)\sum_i |u_{x_i}|^{\beta+p} \, dx.$$

Moreover, since h is bounded we have

$$\left(\int_\Omega \eta^{2^*} \sum_i \frac{1}{(\beta+p)^{2^*}} |u_{x_i}|^{\frac{\beta+p}{2}2^*} \, dx\right)^{\frac{2}{2^*}} \tag{1.1.24}$$
$$\leq c2^{\beta(n-1)}(\beta+1)\int_\Omega \left(|\eta|^2+|D\eta|^2\right)\sum_i |u_{x_i}|^{\beta+p} \, dx$$

and then

$$\left(\int_\Omega \eta^{2^*} \sum_i |u_{x_i}|^{\frac{\beta+p}{2}2^*} \, dx\right)^{\frac{2}{2^*}} \tag{1.1.25}$$
$$\leq c2^{\beta(n-1)}(\beta+p)^3\int_\Omega \left(|\eta|^2+|D\eta|^2\right)\sum_i |u_{x_i}|^{\beta+p} \, dx.$$

We take $\eta \in C_0^1(B_R)$ such that $\eta \geq 0$ in B_R, $\eta = 1$ on B_ρ and $|D\eta| \leq \frac{2}{R-\rho}$. Then

$$\left(\int_{B_\rho} \sum_{i=1}^n |u_{x_i}|^{\frac{\beta+p}{2}2^*} \, dx\right)^{\frac{2}{2^*}} \leq \frac{c(\beta+p)^3 2^{\beta(n-1)}}{(R-\rho)^2}\int_{B_R}\sum_{i=1}^n |u_{x_i}|^{\beta+p} \, dx. \tag{1.1.26}$$

We define by induction a sequence β_k in the following way:

$$\begin{cases} \beta_0 = 0, \\ \beta_{k+1} = (\beta_k+p)\frac{2^*}{2} - p \quad \forall k \geq 0. \end{cases} \tag{1.1.27}$$

Then

$$\beta_k = p\left(\frac{2^*}{2}-1\right)\sum_{i=0}^{k-1}\left(\frac{2^*}{2}\right)^i \tag{1.1.28}$$

for every $k \geq 1$ and

$$\beta_k = p\left(\left(\frac{2^*}{2}\right)^k - 1\right) \tag{1.1.29}$$

for every $k \geq 0$. Of course, for every $k \geq 1$ the representation formulas (1.1.28) and (1.1.29) are equivalent to each other. For $k = 1$ the right hand side of (1.1.28) is equal to $p\,(2^*/2 - 1)$ like in (1.1.27). If we assume that (1.1.28) holds for some k, then by (1.1.27) we have

$$\beta_{k+1} = (\beta_k + p)\,\frac{2^*}{2} - p = \left(p\left(\frac{2^*}{2} - 1\right)\sum_{i=0}^{k-1}\left(\frac{2^*}{2}\right)^i + p\right)\frac{2^*}{2} - p$$

$$= p\left(\frac{2^*}{2} - 1\right)\frac{2^*}{2}\sum_{i=0}^{k-1}\left(\frac{2^*}{2}\right)^i + p\left(\frac{2^*}{2} - 1\right)$$

$$= p\left(\frac{2^*}{2} - 1\right)\left(\sum_{i=0}^{k-1}\left(\frac{2^*}{2}\right)^{i+1} + 1\right) = p\left(\frac{2^*}{2} - 1\right)\sum_{i=0}^{k}\left(\frac{2^*}{2}\right)^i.$$

Let us fix $\rho_0 < R_0$ and let us also define

$$R_k = \rho_0 + (R_0 - \rho_0)\,2^{-k}$$

for every $k \geq 0$. Let us insert $R = R_k$, $\rho = R_{k+1}$ and $\beta = \beta_k$ in (1.1.26). If we define

$$A_k = \left(\int_{B_{R_k}} \sum_i |u_{x_i}|^{\beta_k + p}\, dx\right)^{\frac{1}{\beta_k + p}}$$

then, under these notations, (1.1.26) can be written in the form

$$A_{k+1} \leq \left[\frac{c\,(\beta_k + p)^3\, 2^{\beta_k(n-1)} 4^{k+1}}{(R_0 - \rho_0)^2}\right]^{\frac{1}{\beta_k + p}} A_k. \tag{1.1.30}$$

By iterating (1.1.30) we obtain

$$A_{k+1} \leq \prod_{i=0}^{k}\left[\frac{c\,(\beta_i + p)^3\, 2^{\beta_i(n-1)} 4^{i+1}}{(R_0 - \rho_0)^2}\right]^{\frac{1}{\beta_i + p}} \left(\int_{B_{R_0}} \sum_i |u_{x_i}|^p\, dx\right)^{\frac{1}{p}}.$$

■

In this context, the following step would be to pass to the limit as $k \to \infty$ estimate (1.1.21). Our estimate has a problem in this sense. The quantity denoted with $c(k)$ in the right hand side does not allow a finite quantity in the limit. To be more precise, the fact that the term $2^{\beta_i(n-1)}$ appears in the product does not give the finiteness in the limit. Anyway, we have an estimate for every k of the $L^{\beta_k + p}$ norm of the gradient of u and it is a natural consequence to think about the higher integrability of the gradient of u. We use the following well known interpolation inequality

Lemma 1.1.4 *Let* $v \in L^r(\Omega)$ *and let* $p \le q \le r$. *Define* $\sigma \in (0,1)$ *such that* $= \frac{\sigma q}{p} + \frac{(1-\sigma)q}{r} = 1$, *then*

$$\|v\|_{L^q(\Omega)} \le \|v\|^{\lambda}_{L^p(\Omega)} + \|v\|^{1-\lambda}_{L^r(\Omega)}. \tag{1.1.31}$$

Proof. We have

$$\|v\|^q_{L^q(\Omega)} = \int_\Omega |v|^q \, dx = \int_\Omega |v|^{\lambda q} |v|^{(1-\lambda)q} \, dx.$$

We apply the Hölder inequality and we get

$$\|v|_{L^q(\Omega)} \le \left(\int_\Omega (|v|^{\sigma q})^{\frac{p}{\sigma q}} \, dx \right)^{\frac{\sigma q}{p}} \left(\int_\Omega \left(|v|^{(1-\sigma)q}\right)^{\frac{r}{(1-\sigma)q}} \, dx \right)^{\frac{(1-\sigma)q}{r}}$$

$$= \left(\int_\Omega |v|^p \, dx \right)^{\frac{\sigma q}{p}} \left(\int_\Omega |v|^r \, dx \right)^{\frac{(1-\sigma)q}{r}} = \left(\|v\|^{\sigma}_{L^p(\Omega)} \|v|^{\sigma}_{L^r(\Omega)} \right)^q.$$

■

Now, the fact that $\beta_0 = 0$ and $\beta_k = p\left(\left(\frac{2^*}{2}\right)^k - 1\right)$ ensures that for every $q \ge 0$ there exists $k \in \mathbb{N}$ such that $\beta_k \le q \le \beta_{k+1}$. Then, by using the interpolation inequality (1.1.31) with $p = \beta_k$ and $r = \beta_{k+1}$ and by using the estimate (1.1.21) we have proved the following a-priori estimate on the L^q-norm of Du for every positive q.

Lemma 1.1.5 *Let* $q > 0$ *and let suppose that* u *is a local minimizer of integral* (1.0.6) *with all the hypotheses of Lemma 1.1.3. Then, for every* $0 < \rho < R$ *there exist* $\sigma \in (0,1)$ *and a constant* c *depending on* ρ, R, q, *but not on the constant* m *and* M_1 *of* (1.1.1), *such that*

$$\|Du\|_{L^q(B_\rho)} \le c \|Du\|^{\sigma}_{L^p(B_R)}. \tag{1.1.32}$$

Chapter 2

A-priori gradient estimates for elliptic systems under general growth conditions

In this chapter we focus on the general problem of regularity of minimizers of integrals $\int_\Omega f\,dx$ both in the case of autonomous systems, $f = f\,(Du(x))$, and non autonomous ones, $f = f(x, Du(x))$. As already said in the introduction, we always assume the so called Uhlenbeck structure on f, i.e. $f(x,\xi) = g(x,|\xi|)$ for every $\xi \in \mathbb{R}^{m\times n}$, or $f(\xi) = g(|\xi|)$ in the autonomous case, and the assumptions will be directly made on the function g. We will see three results already existing in literature. In Section 2.1, we report what is contained in Marcellini [46]. We see the approach to the autonomous case for energy integrals $g = g(t)$ with fast growth. In particular, the quotient function $g'(t)/t$ is assumed to be increasing and this restricts the model problems to those with at least quadratic growth. In Section 2.2 we report the improvement of the techniques used in the autonomous case to the non-autonomous one; we refer to Mascolo-Migliorini [52]. There we continue to have at least quadratic growth since the quotient $g_t(x,t)/t$ is again assumed to be increasing. This monotonicity assumption is removed in Marcellini-Papi [51], but only in the non autonomous case. The results of this paper are contained in Section 2.3, where we show in detail also the approximating process.

2.1 Everywhere regularity for fast growth energies - the autonomous case

Let us consider the autonomous case

$$F(v) = \int_{\Omega} g\left(|Dv|\right)\, dx. \tag{2.1.1}$$

Here $\Omega \subset \mathbb{R}^n$ is an open set and $g : [0, +\infty) \to [0, +\infty)$ is a convex function of class $C^2\left([0, +\infty)\right)$. Here, by a local minimizer of integral (2.1.1) we mean a function $u \in W^{1,2}_{\text{loc}}\left(\Omega, \mathbb{R}^m\right)$ such that $g\left(|Du|\right) \in L^1_{\text{loc}}\left(\Omega\right)$ with the property that $F(u) \leq F(u + \varphi)$ for every $\varphi \in C^1_0\left(\Omega, \mathbb{R}^m\right)$. No growth conditions are assumed. Instead, some non-oscillatory assumptions are made:

$$\begin{array}{l} \text{(i) the function } a(t) = \frac{g'(t)}{t} : [0, +\infty) \to [0, +\infty) \text{ is increasing;} \\ \text{(ii) for every } \alpha > 1, \ \lim\limits_{t \to \infty} \frac{a'(t)}{[a(t)]^{\alpha}} \text{ exists.} \end{array} \tag{2.1.2}$$

Of course, assumption (2.1.2) implies some growth conditions: i) implies that $a(t) \geq a(1)$ for every $t \geq 1$ so that g has at least quadratic growth. Moreover, the derivative $a'(t)$ can be bounded in terms of the α-power of $a(t)$ (see Lemma 2.1.8 below). Assumption (2.1.2) allows to consider a class of elliptic problems under general growth conditions, including the slow exponential growth such as the case

$$f(\xi) = \exp\left(|\xi|^{\alpha}\right) \text{ as } |\xi| \to +\infty \tag{2.1.3}$$

with $\alpha \in \mathbb{R}^+$ a small parameter depending on n. These growth restrictions are weak enough to be satisfied, for example, not only by the family of functions with exponential growth in (2.1.3) with a small exponent, but also with exponent $\alpha \geq 2$ or even by a finite composition of functions of exponential type, as for example

$$f(\xi) = (\exp(\ldots(\exp(\exp|\xi|^2)^{\alpha_1})^{\alpha_2})\ldots)^{\alpha_k}, \tag{2.1.4}$$

with $\alpha_i \geq 1$ for every $i = 1, \ldots, k$.

Moreover, also non-power growth are compatible with (2.1.2) such as

$$f(\xi) = \begin{cases} |\xi|^{p+1-\sin\log\log|\xi|}, & \text{if } |\xi| \in (e, +\infty) \\ e\,|\xi|^p, & \text{if } |\xi| \in [0, e) \end{cases} \tag{2.1.5}$$

with $p \geq 3$. The condition $p \geq 2$ is sufficient for the convexity of f in (2.1.5), while $p \geq 3$ is sufficient for the monotonicity of $a(t)$. The regularity result of [46] is contained in the following theorem. There, B_ρ and B_R denote, respectively, balls of radii ρ and R, contained in Ω and with the same center.

Theorem 2.1.1 *Let u be a local minimizer of (2.1.1). Suppose that the non-oscillatory condition (2.1.2) holds, then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and for every $\varepsilon > 0$ and $0 < \rho < R$, there exists a constant $c = c(\varepsilon, n, \rho, R)$ such that*

$$\|Du\|^2_{L^\infty(B_\rho,\mathbb{R}^{m\times n})} \leq c \left[\int_{B_R} (1 + g(|Du|))\, dx \right]^{1+\varepsilon} \tag{2.1.6}$$

Remark 2.1.2 *If a does not satisfy condition (2.1.2), but does satisfy the growth condition*

$$0 \leq a'(t)\, t \leq c\, a(t), \quad \forall t \geq 0$$

which, in terms of g, is equivalent to

$$0 \leq g'(t) \leq g''(t)\, t \leq c\, g'(t), \quad \forall t \geq 0,$$

then (2.1.6) in Theorem 2.1.1 still holds even with $\varepsilon = 0$. In this case, system (2.1.1) is uniformly elliptic and then problems with integrand functions with exponential growth $\alpha \geq 2$ or as in (2.1.4) are ruled out.

Once we have an estimate for the L^∞-norm of the gradient such as (2.1.6), the behaviour as $t \to \infty$ of a becomes irrelevant to obtain further regularity and well known results with assumptions on the behaviour of a as $t \to 0^+$ can be applied. In particular, papers [30], [36], [42], [44] treat the scalar case $m = 1$, while [16], [38], [59], [60] treat the vectorial one with $m \geq 1$. We get the following corollary.

Corollary 2.1.3 *If $a \in C^1(0, +\infty)$ satisfies the non-oscillatory condition (2.1.2) and if there exist $p \geq 2$ and positive constants m, M such that*

$$mt^{p-2} \leq a(t) \leq a(t) + t\, a'(t) \leq Mt^{p-2} \tag{2.1.7}$$

for every $t \in (0, 1]$, then every local minimizer of (2.1.1) is of class $C^{1,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$.

Observe that condition (2.1.7) reads as

$$mt^{p-2} \leq \frac{g'(t)}{t} \leq g''(t) \leq Mt^{p-2}$$

for every $t \in (0, 1]$, in terms of $g \in C^2(0, +\infty)$. If $a(0) > 0$ then the problem is uniformly elliptic as $|\xi| \to 0$ since (2.1.7) holds with $p = 2$. Thus, since $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, u_{x_k} is a weak solution to a system with Hölder continuous coefficients (the second variation system, see (2.1.16) below). Then, the regularity theory for linear elliptic systems with smooth coefficients applies as, for example, in [37].

Corollary 2.1.4 *If $a \in C^{k-1,\alpha}([0,+\infty))$, for some $k \geq 2$, satisfies the non-oscillatory condition (2.1.2) and $a(0) > 0$, then every local minimizer of (2.1.1) is of class $C^{k,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$.*

The strategy to prove Theorem 2.1.1 is the classical one regarding regularity results in this field of the calculus of variations. More regularity is assumed in order to obtain an a-priori estimate. Then, the original problem is approximated by a sequence of more regular ones.

2.1.1 Ellipticity estimates

Before we proceed with the very a-priori estimate, we need to enunciate some preliminary lemmata in order to better understand conditions (2.1.2). In terms of the integrand function g, condition (2.1.2) reads as

$$\begin{aligned}&g : [0,+\infty) \to [0,+\infty) \text{ is a convex function of class } C^2([0,+\infty)),\\ &\text{with } g'(t)/t \text{ increasing in } (0,+\infty).\end{aligned} \tag{2.1.8}$$

Since $f(\xi) = g(|\xi|)$, for every $\xi \in \mathbb{R}^{m\times n}$, we have

$$\begin{cases} f_{\xi_i^\alpha} = g'(|\xi|)\frac{\xi_i^\alpha}{|\xi|}, \\ \\ f_{\xi_i^\alpha \xi_j^\beta} = \left(\frac{g''(|\xi|)}{|\xi|^2} - \frac{g'(|\xi|)}{|\xi|^3}\right)\xi_i^\alpha \xi_j^\beta + \frac{g'(|\xi|)}{|\xi|}\delta_{\xi_i^\alpha \xi_j^\beta}. \end{cases} \tag{2.1.9}$$

The following lemma shows the link between the first and the second derivative of g and highlights the ellipticity conditions on f given by assumption (2.1.8).

Lemma 2.1.5 *Under notation (2.1.1) and assumptions (2.1.8) on f and g, the following conditions hold:*

$$\frac{g'(t)}{t} \leq g''(t),$$

$$\frac{g'(|\xi|)}{|\xi|}|\lambda|^2 \leq \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(\xi)\lambda_i^\alpha \lambda_j^\beta \leq g''(|\xi|)|\lambda|^2, \tag{2.1.10}$$

for every $\lambda, \xi \in \mathbb{R}^{m\times n}$.

Proof. Since $g'(t)/t$ is increasing,

$$0 \leq \frac{d}{dt}\left[\frac{g'(t)}{t}\right] = \frac{g''(t)t - g'(t)}{t^2}$$

for every $t > 0$, from which the first statement of the Lemma. Besides, from (2.1.9) we deduce

$$\min\left\{g''(|\xi|), \frac{g'(|\xi|)}{|\xi|}\right\}|\lambda|^2 \le \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(\xi)\lambda_i^\alpha \lambda_j^\beta$$
$$\le \max\left\{g''(|\xi|), \frac{g'(|\xi|)}{|\xi|}\right\}|\lambda|^2 .$$

Again, by the assumptions, the latter formula reduces to (2.1.10). ■

Remark 2.1.6 *Observe that since $a(t) = g'(t)/t$ is increasing, $g'(0) = 0$. By adding a constant to g, we can assume without loss of generality that $g(0) = 0$. Finally, not to consider a trivial situation, g is not identically equal to 0 and, up to rescaling, $g(1) > 0$. Furthermore, the assumption that $g'(t)/t$ is an increasing function is an intermediate condition between the convexity of g and the convexity of g'. In fact, it implies that g' is increasing too, while if g' is convex and $g'(0) = 0$, then we have $0 = g'(0) \ge g'(t) + g''(t)\,(-t)$. Thus, since*

$$\frac{d}{dt}\frac{g'(t)}{t} = \frac{g''(t)\,t - g'(t)}{t^2},$$

$g'(t)/t$ is increasing.

Assumptions (2.1.2) and (2.1.8) are linked by the following lemmata.

Lemma 2.1.7 *If g satisfies assumption (2.1.8), then the following conditions are equivalent:*

i) for every $\alpha > 1$ the limit $\lim_{t\to\infty} \frac{a'(t)t}{[a(t)]^\alpha}$ exists;

ii) for every $\alpha > 1$ the limit $\lim_{t\to\infty} \frac{g''(t)t^\alpha}{[g'(t)]^\alpha}$ exists;

iii) for every $\alpha > 1$, $\lim_{t\to\infty} \frac{g'(t)\,t^{2\alpha-1}}{[g(t)]^\alpha}, \lim_{t\to\infty} \frac{g''(t)\,t^\alpha}{[g(t)]^\alpha} < \infty$.

Lemma 2.1.8 *The following conditions are consequence of any of conditions of Lemma (2.1.7) and they are equivalent to each other:*

iv) for every $\alpha > 1$ there exists a constant $c = c(\alpha)$ such that

$$\begin{cases} g'(t)\,t^{2\alpha-1} \le c\,[g(t)]^\alpha\,, \\ & \text{for every } t \ge 1 \\ g''(t)\,t^\alpha \le c\,[g'(t)]^\alpha\,. \end{cases}$$

v) for every $\alpha > 1$ there exists a constant $c = c(\alpha)$ such that

$$g''(t)t^{\alpha} \leq c\,[g(t)]^{\alpha}\,, \;\; \textit{for every } t \geq 1.$$

Remark 2.1.9 *If $g'(t)/t \to \infty$, which is the most interesting case, then either iv) or v) are equivalent to any of the conditions of Lemma 2.1.7. Conversely, if $g'(t)/t$ has a finite limit as $t \to \infty$, then the condition $\alpha > 1$ becomes irrelevant and it can be more convenient to consider iv) with $\alpha = 1$ as an assumption, instead of (2.1.8ii). In this case we can assume that the conclusion of (2.1.6) holds with $\varepsilon = 0$ too.*

In terms of the convex function g, the non-oscillatory conditions in (2.1.2) regarding the function a correspond respectively to the monotonicity assumption (2.1.8) and to any of the conditions of Lemma 2.1.8. For this reason, the conclusion of Theorem 2.1.6 holds under the assumption that g satisfies (2.1.8) and any of the conditions of Lemma 2.1.8.

The following two lemmata are used to treat the left hand side in the proof of the a-priori estimate in Lemma 2.1.11 below.

Lemma 2.1.10 *If g satisfies (2.1.8) and the conditions stated in Lemma 2.1.8 then for every $\alpha > 1$ there exists a constant $c_1 = c_1(\alpha)$ such that*

$$1 + g''(t)\,t^2 \leq c_1(1 + g(t))^{\alpha} \tag{2.1.11}$$

for every $t \geq 0$. Moreover, for every $\beta \geq 2$ there exists a constant $c_2 = c_2(\beta)$ such that

$$\left[1 + \int_0^t s^{\gamma}\sqrt{\frac{g'(s)}{s}}\,ds\right]^{\beta} \geq c_2\left[1 + \left(\frac{t^{\gamma+1}}{\gamma+1}\right)^{\beta} g''(t)\right] \tag{2.1.12}$$

for every $t \geq 0$ and for every $\gamma \geq 0$.

Here the constants c_1 and c_2 depend also on the constants appearing in iv) and v) of Lemma 2.1.8, on $g(1), g'(1)$, $g''(1)$ and on a lower bound for $\int_0^1 \sqrt{g'(s)/s}$.

The proof of this Lemma is strongly related to the assumption that $g'(t)/t$ is increasing. When we will drop this condition, proofs of lemmata like this will be more complicated. See, for instance, Lemma 2.3.4.

2.1.2 A-priori estimates

We make the following supplementary assumption: there exist two positive constants N and M such that

$$N|\lambda|^2 \leq \sum_{i,j,\alpha,\beta} f_{\xi_i^{\alpha}\xi_j^{\beta}}(\xi)\lambda_i^{\alpha}\lambda_j^{\beta} \leq M|\lambda|^2 \tag{2.1.13}$$

for every $\lambda, \xi \in \mathbb{R}^{m\times n}$, or equivalently, in term of g, $N \leq g'(t)/t \leq g''(t) \leq M$ for every $t > 0$.

This assumption is needed only for the a-priori estimate and allows to use the second variational weak equation (2.1.16) below. Successively it will be removed by approximating the original problem with regular variational ones. This is possible because the two constants N and M do not enter explicitly in the bound for the L^∞-norm of the gradient.

Lemma 2.1.11 *Under assumptions (2.1.2) and (2.1.13), let* $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ *be a minimizer of the integral (2.1.1). Then* $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ *and there exists a constant* c*, depending on* n *and on the constant of Lemma 2.1.8 but not on* N *and* M*, such that*

$$\|Du\|^2_{L^\infty(B_\rho, \mathbb{R}^{m\times n})} \leq \frac{c}{(R-\rho)^n} \int_{B_R} \left(1 + |Du|^2 g''(|Du|)\right) dx \tag{2.1.14}$$

for every $0 < \rho < R$.

We only give a sketch of the proof in order to underline some crucial differences with a-priori estimates of the following sections and of Chapter 3. By the left hand side of (2.1.13), $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ and, by the right hand side of (2.1.13) it satisfies the Euler's first variation:

$$\int_\Omega \sum_{i,\alpha} f_{\xi_i^\alpha}(Du)\, \varphi^\alpha_{x_i}\, dx = 0, \tag{2.1.15}$$

for every $\varphi \in W^{1,2}(\Omega, \mathbb{R}^m)$. Again, by using some known techniques (see [5], [37], [49], [47], [48]) we can prove that u admits second order weak partial derivatives. Precisely, $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and satisfies the second variation

$$\int_\Omega \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} \varphi^\alpha_{x_i}\, dx = 0 \tag{2.1.16}$$

for every $k = 1, \ldots, n$ and every $\varphi \in W^{1,2}_0(\Omega, \mathbb{R}^m)$. Fixed $k \in \{1, \ldots, n\}$, let $\eta \in C^1_0(\Omega)$ and $\varphi^\alpha = \eta^2 u^\alpha_{x_i} \Phi(|Du|)$ for every $\alpha = 1, \ldots, m$. Here Φ is a positive, increasing, bounded and Lipschitz continuous function in $[0, +\infty)$. In particular Φ and Φ' are bounded in $[0, +\infty)$ so that $u \in W^{1,2}_0(\Omega, \mathbb{R}^m)$. Then

$$\varphi^\alpha_{x_i} = 2\eta\eta_{x_i} u^\alpha_{x_k} \Phi(|Du|) + \eta^2 u^\alpha_{x_i x_k} \Phi(|Du|) + \eta^2 u^\alpha_{x_k} \Phi'(|Du|)(|Du|)_{x_i}$$

and from (2.1.16) we deduce

$$\int_\Omega 2\eta\Phi(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} \eta_{x_i} u^\alpha_{x_k}\, dx$$

$$+\int_\Omega \eta^2\Phi(|Du|)\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(Du)u^\beta_{x_jx_k}u^\alpha_{x_ix_k}\,dx \tag{2.1.17}$$

$$+\int_\Omega \eta^2\Phi'(|Du|)\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(Du)u^\beta_{x_jx_k}u^\alpha_{x_k}(|Du|)_{x_i}\,dx = 0.$$

We proceed with the estimate of the third integral in (2.1.17). Here we use the assumption that $g'(t)/t$ is increasing. In Section 2.3 and in Chapter 3 it will be removed and another technique will be used to treat this integral. By using (2.1.9) we obtain

$$\begin{aligned}
&\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(Du)u^\beta_{x_jx_k}u^\alpha_{x_k}(|Du|)_{x_i}\\
&= \left(\frac{g''(|Du|)}{|Du|^2}-\frac{g'(|Du|)}{|Du|^3}\right)\sum_{i,j,\alpha,\beta} u^\alpha_{x_i}u^\beta_{x_j}u^\beta_{x_jx_k}u^\alpha_{x_k}(|Du|)_{x_i}\\
&\quad +\frac{g'(|Du|)}{|Du|}\sum_{i,\alpha} u^\alpha_{x_ix_k}u^\alpha_{x_k}(|Du|)_{x_i}.
\end{aligned}$$

Since

$$(|Du|)_{x_i} = \frac{1}{|Du|}\sum_{k,\alpha} u^\alpha_{x_ix_k}u^\alpha_{x_k}, \tag{2.1.18}$$

it is natural to sum up with respect to k:

$$\begin{aligned}
&\sum_k\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(Du)u^\beta_{x_jx_k}u^\alpha_{x_k}(|Du|)_{x_i}\\
&= \left(\frac{g''(|Du|)}{|Du|}-\frac{g'(|Du|)}{|Du|^2}\right)\sum_{i,k,\alpha} u^\alpha_{x_i}(|Du|)_{x_k}u^\alpha_{x_k}(|Du|)_{x_i}\\
&\quad +g'(|Du|)\sum_i\left((|Du|)_{x_i}\right)^2 \qquad (2.1.19)\\
&= \left(\frac{g''(|Du|)}{|Du|}-\frac{g'(|Du|)}{|Du|^2}\right)\left(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\right)^2\\
&\quad +g'(|Du|)\sum_i\left((|Du|)_{x_i}\right)^2 \ge 0
\end{aligned}$$

since, by Lemma 2.1.5, $g'(|Du|)\le |Du|\,g''(|Du|)$.

The other integrals in (2.1.17) shall be treated in a similar way in sections 2.3 and 3.3. By using the Cauchy-Schwarz inequality, the inequality $2ab\le \frac{1}{2}a^2+2b^2$ and by using (2.1.19) we can get the following estimate

$$\int_\Omega \eta^2\Phi(|Du|)\frac{g'(|Du)}{|Du|}|D(|Du|)|^2 \tag{2.1.20}$$

$$\le 4 \int_\Omega |D\eta|^2 \Phi(|Du|) g''(|Du|)\, |Du|^2\, dx.$$

We define $\Phi(t) = t^{2\gamma}$ with $\gamma \ge 0$, then by using some techniques that we shall see in the following sections and by applying Lemma 2.1.10, from inequality (2.1.20) we obtain

$$\left[\int_{B_\rho} \left(1 + |Du|^{\delta \frac{2^*}{2}} g''(|Du|) \right) dx \right]^{2/2^*} \tag{2.1.21}$$

$$\le c_3 \left(\frac{4\delta}{R-\rho} \right)^2 \int_{B_R} \left(1 + |Du|^\delta g''(|Du|) \right) dx$$

where we have defined $\delta = 2(\gamma+1) \ge 2$ and 2^* is the usual Sobolev exponent, i.e. $\frac{2n}{n-2}$ if $n \ge 3$ and any fixed real number greater than 2 if $n = 2$.

Fixed $0 < \rho_0 < R_0$ and defined $\rho_i = \rho_0 + \frac{R_0 - \rho_0}{2^i}$, for every $i \in \mathbb{N}$, we write (2.3.15) with $R = \rho_{i-1}$ and $\rho = \rho_i$. Moreover, we put $\delta = 2, 2\left(\frac{2^*}{2}\right), 2\left(\frac{2^*}{2}\right)^2, \ldots$ and we iterate (2.1.21). We get

$$\left\{ \int_{B_{\rho_i}} \left(1 + |Du|^{2\left(\frac{2^*}{2}\right)^i} g''(|Du|) \right) dx \right\}^{(2/2^*)^i} \tag{2.1.22}$$

$$\le c_4 \int_{B_{R_0}} \left(1 + |Du|^2 g''(|Du|) \right) dx,$$

for some c_4 depending on ρ_0, R_0, n. By using the fact that $g'(t)/t$ and $g'(t)$ are increasing, by Lemma 2.1.5 $g'(1) \le g'(t)/t \le g''(t)$ for every $t \ge 1$. Thus, if $t \ge 0$ and $\alpha > 0$ we have $g''(t)\, t^\alpha + 1 \ge t^\alpha$. By going to the limit as $i \to \infty$ in (2.1.22) we obtain

$$\sup\left\{ |Du(x)|^2 : x \in B_{\rho_0} \right\} = \lim_{i\to\infty} \left[\int_{B_{\rho_0}} |Du|^{2\left(\frac{2^*}{2}\right)^i} dx \right]^{\left(\frac{2}{2^*}\right)^i}$$

$$\le \limsup_{i\to\infty} \left[\frac{1}{c} \int_{B_{\rho_i}} \left(1 + |Du|^{2\left(\frac{2^*}{2}\right)^i} g''(|Du|) \right) dx \right]^{2\left(\frac{2}{2^*}\right)^i}$$

$$\le c \int_{B_{R_0}} \left(1 + |Du|^2 g''(|Du|) \right) dx,$$

which concludes the proof of (2.1.14).

Observe that estimate (2.1.14) has the term $g''(|Du|)$ on the right hand side and not $g(|Du|)$. To complete the proof we need to control the right hand side of (2.1.14) with something independent of the supplementary assumption (2.1.13). The following lemma allows to write everything referring to the function g and not on its derivatives.

Lemma 2.1.12 *Under the assumptions of Lemma 2.1.11, let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ be a minimizer of integral (2.1.1). Then, for every $\varepsilon > 0$ and for every $0 < \rho < R$ there exists a constant c, depending on ε, ρ, R, n and on the constant of Lemma 2.1.8, but not on the constants m and M in the supplementary (2.1.13), such that*

$$\int_{B_\rho} \left(1 + |Du|^2 g''(|Du|)\right) dx \le c \left\{ \int_{B_R} (1 + g(|Du|))\, dx \right\}^{1+\varepsilon}, \qquad (2.1.23)$$

which, combined with (2.1.14), gives

$$\|Du\|^2_{L^\infty(B_\rho, \mathbb{R}^{m\times n})} \le c \left\{ \int_{B_R} (1 + g(|Du|))\, dx \right\}^{1+\varepsilon}. \qquad (2.1.24)$$

2.1.3 Approximation of the original problem with regular variational problems

We see here only a sketch of the proof of the approximation. Later, in Section 2.3, we will see the approximation procedure in the autonomous case for both fast and slow growth.

In order to get the regularity result (2.1.24) for the original problem, assumption (2.1.13) needs to be removed. We define an increasing sequence $g_k(t)$ approximating $g(t)$ such that for every $k \in \mathbb{N}$, an uniform ellipticity condition holds:

$$m_k \le \frac{g_k'(t)}{t} \le g_k''(t) \le M_k$$

for some $m_k, M_k > 0$ and such that for some constant c independent of k,

$$g_k(t) \le c\,(1 + g(t)),$$

for every $k \in \mathbb{N}$ and for every $t \ge 0$. Moreover, such g_k satisfies the assumption of Lemma 2.1.8, again with constants independent of k. Then, for every $k \in \mathbb{N}$ we consider the integral functional

$$F_k(v) = \int_\Omega g_k(|Dv|)\, dx \qquad (2.1.25)$$

and we take a minimizer $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ of $F(v)$ in (2.1.1), such that $f(Dv) \in L^1_{\mathrm{loc}}(\Omega)$. Moreover, we define a sequence $\{u_k\} \subseteq W^{1,2}(\Omega, \mathbb{R}^m)$ of minimizers of

(2.1.25), with the Dirichlet condition $u_k = u$ on the boundary ∂B_R of B_R. Here $R > 0$ is such that the ball B_R is contained in Ω. By applying the a-priori estimate to g_k, we can prove that for every $\varepsilon > 0$ and for every $\rho < R$ there exists c

$$\|Du_k\|^2_{L^\infty(B_\rho,\mathbb{R}^{m\times n})} \le c\left\{\int_{B_R} (1+g(|Du|))\ dx\right\}^{1+\varepsilon} \tag{2.1.26}$$

for every $k \in \mathbb{N}$. By (2.1.26), the sequence u_k converges in the weak-* topology of $W^{1,\infty}(B_\rho, \mathbb{R}^m)$ to a function w, that results to be a minimizer of $F(v)$. Then

$$\|Dw\|^2_{L^\infty(B_\rho,\mathbb{R}^{m\times n})} \le c\left\{\int_{B_R} (1+g(|Du|))\ dx\right\}^{1+\varepsilon}. \tag{2.1.27}$$

Under the assumptions on f, uniqueness of the minimizers for the Dirichlet problem is not guaranteed. Anyway, since $g''(t)$ and $g'(t)\,t$ are positive for any $t \ge t_0$, $f(\xi) = g(|\xi|)$ is locally strictly convex for $|\xi| > 1$. Thus, $|Dw(x)| = |Du(x)|$ for almost every $x \in B_R$ such that $|Du(x)| > 1$ and this implies that also Du satisfies (2.1.27), giving finally (2.1.6).

2.2 Local minimizers with fast growth - the non autonomous case

In this section we collect some results presented in [52] by Mascolo-Migliorini. There, the authors give a regularity result for minimizers of vectorial functionals

$$F(v) = \int_\Omega g(x, |Dv(x)|)\ dx \tag{2.2.1}$$

that depend also on x, with g satisfying the key condition that $g_t(x,t)/t$ is increasing with respect to the variable t. As in the preceding section, by a minimizer of (2.2.1), we mean a $W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^m)$ function u such that $g(x, |Du|) \in L^1_{\text{loc}}(\Omega)$ and $F(u) \le F(u+\varphi)$ for every $\varphi \in C^1_0(\Omega, \mathbb{R}^m)$.

The following conditions on g are required to prove regularity:

i) for almost every $x \in \Omega$, $g(x,\cdot)$ is a positive convex function of class $C^2([0,+\infty))$ with $g_t(x,t)/t$ positive and increasing with respect to t for almost every $x \in \Omega$;

ii) for every $\Omega' \subset\subset \Omega$ there exists a positive constant Λ depending Ω' such that

$$g_{tt}(x,t) \le \Lambda$$

for every $t \in [0,1]$ and almost every $x \in \Omega'$ and there exist $t_0 \in (0,1)$ and $\lambda = \lambda(\Omega') > 0$ such that

$$g(x,t_0) \geq \lambda$$

almost everywhere in Ω;

iii) for every $\Omega' \subset\subset \Omega$ and $\alpha > 1$ there exists a positive constant c_1 such that

$$g_{tt}(x,t)t^{2\alpha} \leq c_1 \left[g(x,t)\right]^{\alpha}$$

for every $t \geq 1$ and almost every $x \in \Omega$;

iv) for every $t \in [0,+\infty)$, $g_t(x,t)$ admits weak derivatives $g_{tx_k}(x,t)$ for every $s = 1,\ldots,n$ which are Carathéodory functions in $\Omega \times [0,+\infty)$ and locally integrable in Ω. Moreover, for every $\Omega' \subset\subset \Omega$ and $\alpha > 1$ there exists a positive constant c_2 such that

$$|g_{tx_s}(x,t)| \leq c_2 g_t(x,t)\left[1 + g_t^{\alpha-1}(x,t)\right]$$

for every $t \geq 0$ and almost every $x \in \Omega'$;

v) for every $\Omega' \subset\subset \Omega$ and Q' compact subset of $[1,+\infty)$, $g_{tt}(x,t) \in L^{\infty}(\Omega' \times Q')$.

The regularity result obtained is the following:

Theorem 2.2.1 *Let $g = g(x,t) : \Omega \times [0,\infty) \to [0,\infty)$ be a function of class C^2 that satisfies conditions i)-v) above, then every local minimizer u of (2.2.1) is of class $W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ and there exist two positive constants c,σ such that for every $0 < \rho < R$*

$$\|Du\|_{L^{\infty}(B_\rho,\mathbb{R}^{m\times n})} \leq c\left\{\int_{B_R}\left(1 + g(x,|Du|)\right)dx\right\}^{1+\sigma}. \tag{2.2.2}$$

Remark 2.2.2 *Observe that since $\frac{g_t(x,t)}{t}$ is assumed to be increasing, then necessarily $g_t(x,0) = 0$ for almost every $x \in \Omega$. Moreover, without loss of generality, by adding a bounded function of x to g, we can reduce to the case $g(x,0) = 0$ for almost every $x \in \Omega$. We deduce from i) that*

$$0 \leq g(x,t) \leq g_t(x,t)t,$$

$$0 \leq g_t(x,t) \leq g_{tt}(x,t)t$$

for almost every $x \in \Omega$ and for every $t > 0$.

Condition iii) is a non-oscillatory condition which is similar to the one introduced in Section 2.1 when there is no x-dependence.

With calculations similar to those of the preceding section (precisely (2.1.9) and Lemma 2.1.5), we could see that the following ellipticity estimate holds:

$$\frac{g_t(x,|\xi|)}{|\xi|}|\lambda|^2 \leq \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,\xi)\,\lambda_i^\alpha \lambda_j^\beta \leq g_{tt}(x,|\xi|)\,|\lambda|^2 \tag{2.2.3}$$

for almost every $x \in \Omega$ and for every $\xi, \lambda \in \mathbb{R}^{m\times n}$.

Conditions *i)-v)* are enough to cover integrand functionals with fast growth as, for instance,

$$g(x,|\xi|) = a(x)h\left(|\xi|\right)^{p(x)}$$

where $W^{1,\infty}_{\mathrm{loc}}(\Omega) \ni a(x), p(x) \geq c > 0$ for almost every $x \in \Omega$ and h is a C^2 function strictly increasing such that $h'(t)/t$ is positive and increasing and satisfying the non-oscillatory condition at the infinity given by *iii)*. In other words, the dependence on $|\xi|$ is driven by a function satisfying the conditions given in Section 2.1. For example, functions like $h(t) = t^m$ or $h(t) = t^m \ln(t+1)$ satisfy all the assumptions provided $mp(x) \geq 2$ for almost every $x \in \Omega$. On the other hand, exponential growths such as e^{t^m} or $t^{\ln(t)}$ as $t \to \infty$ or, moreover, e^{t^m} with $m \geq 2$ are allowed. Furthermore, exponentials like

$$g(x,t) = \exp\left(t^{p(x)}\right)$$

as $t \to \infty$, or, in analogy with (2.1.4), finite composition of exponentials as

$$g(x,t) = (\exp(\dots(\exp t^{p_1(x)})^{p_2(x)})\dots)^{p_k(x)}$$

with $p_i(x) \geq 2$ satisfy conditions *i)-v)*.

The strategy used to prove Theorem 2.2.1 is the same used to prove regularity in Section 2.1. Here, x-dependence gives many technical problems in the proof. In the first instance, we assume, as in (2.1.13), that there exist positive constants N, M_1, depending on $\Omega' \subset\subset \Omega$ such that

$$N|\lambda|^2 \leq \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,\xi)\lambda_i^\alpha \lambda_j^\beta \leq M_1|\lambda|^2$$

that is equivalent to

$$N \leq \frac{g_t(x,t)}{t} \leq g_{tt}(x,t) \leq M_1$$

for every $t > 0$ and almost every $x \in \Omega'$. Moreover we make a supplementary assumption on the mixed derivative: there exists a constant M_2 such that

$$|g_{tx_k}(x,t)| \leq M_2\left(1+t^2\right)^{\frac{1}{2}},$$

that gives

$$|f_{\xi_i^\alpha x_k}(x,\xi)| \le M_2\left(1+|\xi|^2\right)^{\frac{1}{2}}$$

for every $\xi \in \mathbb{R}^{m\times n}$ and almost every $x \in \Omega'$. With these supplementary assumptions, we can prove (2.2.2). The problem in the calculations rises from Euler's second variation equation. When we derive, we have three more terms presenting the mixed second derivatives g_{tx_k}. We drop some technicalities that we shall see in Chapter 3, then, as usual in this context, via an approximation argument, Theorem 2.2.1 is proved without the supplementary restrictions.

2.3 Regularity for elliptic systems with general growth - the autonomous case

The results in this section have been established in Marcellini-Papi [51]. Here the function $g'(t)/t$ is not assumed to be increasing. In Section 2.3 we have seen that this restriction allows to consider integrands with exponential growth of the type (2.1.4). However, some interesting low growth model integrands are not included. For example, in the case of $g(t)=t^p$ this restriction implies $p\ge 2$.

By a local minimizer of (2.2.1) we mean a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ such that $g(x,|Du|) \in L^1_{\mathrm{loc}}(\Omega)$ with the property that $F(u)\le F(u+\varphi)$ for every $\varphi \in C^1_0(\Omega,\mathbb{R}^m)$. We consider a minimizer of (2.1.1), where g is an increasing convex function without growth assumptions when t goes to ∞. The following condition allows to consider growths moving between linear and exponential functions:

let $t_0, H>0$ and $\beta \in \left(\frac{1}{n},\frac{2}{n}\right)$; for every $\alpha \in \left(1,\frac{n}{n-1}\right]$ there exists a constant $K=K(\alpha)$ such that

$$Ht^{-2\beta}\left[\left(\frac{g'(t)}{t}\right)^{\frac{n-2}{n}}+\frac{g'(t)}{t}\right] \le g''(t) \le K\left[\frac{g'(t)}{t}+\left(\frac{g'(t)}{t}\right)^{\alpha}\right] \qquad (2.3.1)$$

for every $t\ge t_0$.

The parameter α in the right hand side of (2.3.1) is a parameter used to test more functions g in order to cover more examples. As in Theorem 2.1.1, functions with exponential growth like (2.1.4) are included in assumption (2.3.1). Moreover, the condition in the left hand side allows to achieve functions with second derivative vanishing as $t^{-\gamma}$ (i.e. γ-elliptic) with γ not too large related to the dimension n, in particular $\gamma < 1+\frac{2}{n}$. Examples of γ-elliptic linear integrands that satisfy assumption (2.3.1) are given by

$$g_\gamma(t)=\int_0^t\int_0^s\left(1+z^2\right)^{-\gamma/2}\,dzds \qquad (2.3.2)$$

for $t \geq 0$. For $\gamma = 1$, g_γ behaves like $t\log(1+t)$ and in the limit case $\gamma = 3$, it becomes $(1+t^2)^{1/2}$.

Other examples in the linear case include $g(t) = 1+t-\sqrt{t}$ for $t \geq 1$ and $n < 4$, or more in general, for $r \in (0,1)$

$$g_r(t) = h(t) - t^r,$$

for $t \geq 1$ and $n < \frac{2}{1-r}$ and also

$$g_r(t) = h(t) + (1-t^r)^{1/r}$$

for $t \geq 1$ and $n < \frac{2}{r}$ where h is a convex function such that

$$c_1(1+t) \leq h(t) \leq c_2(1+t)$$

for some suitable constants c_1 and c_2.

Furthermore, condition (2.3.1) is satisfied if we consider functions that satisfy the so called p, q-growth conditions, studied by Marcellini in [47], [48], without any restriction on the exponents p, q $(p \leq q)$. For example, fixed $1 < p < q$, consider the following oscillating function, treated in [25],

$$g(t) = \begin{cases} t^p & \text{if } t \leq \tau_0, \\ \\ t^{\frac{p+q}{2} + \frac{q-p}{2}\sin\log\log\log t} & \text{if } t > \tau_0, \end{cases} \tag{2.3.3}$$

where τ_0 is such that $\sin\log\log\log\tau_0 = -1$. Observe that g oscillates between t^p, to which it is tangent in τ_n when $\sin\log\log\log\tau_n = -1$, and t^q, to which it is tangent in σ_n with $\sin\log\log\log\sigma_n = 1$. By direct computations it is possible to see that one can choose τ_0 and t_0 large enough such that g is convex and satisfies (2.3.1).

Therefore condition (2.3.1) unifies and generalizes many cases in the literature for the integral (2.1.1), including in particular the linear case studied in [6], the non standard p, q-growth condition, the exponential growth considered in Theorem 2.1.1 and the oscillating function in (2.3.3).

The following two theorems are the main results. The first one is valid under general growth condition, while the second one is specific for the linear case.

Theorem 2.3.1 *(General growth) Let $g : [0,\infty) \to [0,\infty)$ be a convex function of class $W^{2,\infty}$ with $g(0) = g'(0) = 0$ satisfying the general growth condition (2.3.1). Let $u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^m)$ a local minimizer of integral (2.1.1), then $u \in W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^m)$.*

Moreover, the following estimate holds: for every $\varepsilon > 0$ and $0 < \rho < R$, there exists a constant C, depending on $\varepsilon, n, \rho, R, H, K$ and $\sup_{0\leq t\leq t_0} g''(t)$ such that

$$\|Du\|^{2-\beta n}_{L^\infty(B_\rho, \mathbb{R}^{m\times n})} \leq C\left\{\int_{B_R} (1+g(|Du|))\,dx\right\}^{\frac{1}{1-\beta}+\varepsilon}. \tag{2.3.4}$$

Theorem 2.3.2 *(Linear growth) Let $g : [0, \infty) \to [0, \infty)$ be a convex function of class $W^{2,\infty}$ with $g(0) = g'(0) = 0$. Assume that g has a linear behaviour at the infinity, driven by*

$$\lim_{t \to \infty} \frac{g(t)}{t} = l \in (0, +\infty) \tag{2.3.5}$$

and that, for every $t \geq t_0$, g'' satisfies the inequalities

$$H \frac{1}{t^\gamma} \leq g''(t) \leq K \frac{1}{t}, \tag{2.3.6}$$

for some positive constants H, K, t_0 and some $\gamma \in \left[1, 1 + \frac{2}{n}\right)$. Then every local minimizer u of (2.1.1) is of class $W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and, for every $0 < \rho < R$ the following estimate is satisfied:

$$\|Du\|_{L^\infty(B_\rho, \mathbb{R}^{m \times n})} \leq C \int_{B_R} (1 + g(|Du|)) \, dx, \tag{2.3.7}$$

where $\beta = \frac{\gamma}{2} - \frac{n-2}{2n}$ and C depends on n, ρ, R, l, H, K and $\sup_{0 \leq t \leq t_0} g''(t)$.

Remark 2.3.3 *In the case of Theorem 2.3.2, $2 - \beta n \in (0, 1]$ since $\gamma \in \left[1, 1 + \frac{2}{n}\right)$. Moreover, if we reduce condition (2.3.1) to linear growth, we can see that the estimate (2.3.7) is sharper than the estimate that would come from (2.3.4). Theorem 2.3.2, then, is not a particular case of Theorem 2.3.1.*

The procedure to prove Theorem 2.3.1 is similar to the one used in Section 2.1. Some technical lemmata regarding the different growth conditions required here and the approximation procedure are the main differences with the proof of Theorem 2.1.1. The starting point is the second variational weak equation derived with the supplementary assumption that both $g''(t)$ and $\frac{g'(t)}{t}$ are bounded by constants N and M for every $t > 0$. An estimate like (2.3.4) is derived for this more regular class of problems. Then, via an approximation argument, Theorem 2.3.1 is proved in its full generality.

2.3.1 Preliminary lemmata

We see here some technical lemmata useful to derive the a-priori estimate. We also show some proofs that will be used to prove some lemmata in Section 3.2. First of all, the following ellipticity estimate holds:

$$\min \left\{ g''(|\xi|), \frac{g'(|\xi|)}{|\xi|} \right\} \leq \frac{\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} \lambda_i^\alpha \lambda_j^\beta}{|\lambda|^2} \tag{2.3.8}$$

$$\leq \max \left\{ g''(|\xi|), \frac{g'(|\xi|)}{|\xi|} \right\},$$

for every $\lambda, \xi \in \mathbb{R}^{m\times n}$. The difference with (2.1.10) is that the function $g'(t)/t$ is not increasing. Later on we will use the quantity on the right hand side of (2.3.8) that we denote by

$$\mathcal{H}(t) = \max \left\{ g''(t), \frac{g'(t)}{t} \right\}. \tag{2.3.9}$$

Since $g'(t) = \int_0^t g''(s)\, ds \leq M_T t$ for every $t \leq T$, $g'(t)/t$ is bounded on $[0, T]$ for every $T > 0$, and so is $\mathcal{H}$. In analogy with Theorem 2.1.1, where $g'(0) = 0$, we do not assume $g'(0) > 0$, but more generally g and g' could be equal to 0 in an interval $[0, \bar{t}]$ for some $\bar{t} > 0$.

The following lemma is crucial and it will be used in Section 3.2 to prove an analogous result.

Lemma 2.3.4 *Let g be as in Theorem 2.3.1. Then for every $\delta \in [\frac{2\alpha}{2-\alpha}, 2^*]$ and for every $\gamma \geq 0$ there exists a constant C such that*

$$1 + \int_0^t s^\gamma \sqrt{g''(s)}\, ds \geq C \left[1 + \left(\frac{t^{\gamma+1-\beta}}{\gamma+1} \right)^\delta \mathcal{H}(t) \right]^{\frac{1}{\delta}} \tag{2.3.10}$$

for every $t \geq 0$.

Proof. Without any lost of generality, we can suppose that $t_0 = 1$ and that $g(t_0) > 0$. We observe that

$$\left[1 + \left(\frac{t^{\gamma+1-\beta}}{\gamma+1} \right)^\delta \mathcal{H}(t) \right]^{\frac{1}{\delta}} \leq \left[1 + \left(\frac{t^{\gamma+1-\beta}}{\gamma+1} \right) \mathcal{H}(t)^{\frac{1}{\delta}} \right] \tag{2.3.11}$$

for every $t > 0$ and for every $\gamma \geq 0$. We denote with

$$F_1(t, \gamma) = 1 + \int_0^t s^\gamma \sqrt{g''(s)}\, ds,$$

and

$$F_2(t, \gamma) = 1 + \left(\frac{t^{\gamma+1-\beta}}{\gamma+1} \right) \mathcal{H}(t)^{\frac{1}{\delta}}.$$

The quotient F_1/F_2 results to be lower bounded in the strip $[0, 1] \times [0, +\infty]$ by the constant $C = \left(1 + \max_{0 \leq t \leq 1} [g'(t)/t, g''(t)]^{1/\delta}\right)^{-1}$. The conclusion (2.3.10) follows for $0 \leq t \leq 1$. Let $t \geq 1$. It is clear that

$$\mathcal{H}(t) \leq g''(t) + \frac{g'(t)}{t}$$

and by the right hand side of (2.3.1)

$$\mathcal{H}(t) \leq (K+1)\left[\frac{g'(t)}{t} + \left(\frac{g'(t)}{t}\right)^{\alpha}\right]. \tag{2.3.12}$$

Instead of proving (2.3.10) we can prove the following

$$1 + \int_0^t s^{\gamma}\sqrt{g''(s)}\,ds \geq C\left[1 + \frac{t^{\gamma+1-\beta}}{\gamma+1}\left[\frac{g'(t)}{t} + \left(\frac{g'(t)}{t}\right)^{\alpha}\right]^{1/\delta}\right] \tag{2.3.13}$$

for every $t \geq 1$. It is sufficient to show the inequality between the derivatives side to side with respect to t of (2.3.13):

$$\begin{aligned} g''(t) \geq Ct^{-\beta}\Bigg[\left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}} + \left(\frac{g'(t)}{t}\right)^{\frac{\alpha}{\delta}} + \left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}-1} g''(t) \\ + \left(\frac{g'(t)}{t}\right)^{\frac{\alpha}{\delta}-1} g''(t)\Bigg], \end{aligned} \tag{2.3.14}$$

that, since $\alpha > 1$, can be read as

$$\begin{cases} g''(t) \geq Ct^{-\beta}\left[\left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}} + \left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}-1} g''(t)\right] & \text{if } \frac{g'(t)}{t} \leq 1, \\ \\ g''(t) \geq Ct^{-\beta}\left[\left(\frac{g'(t)}{t}\right)^{\frac{\alpha}{\delta}} + \left(\frac{g'(t)}{t}\right)^{\frac{\alpha}{\delta}-1} g''(t)\right] & \text{if } \frac{g'(t)}{t} \geq 1. \end{cases}$$

In the first case, by the right hand side of (2.3.1) we get

$$\sqrt{g''(t)} \geq \sqrt{H}t^{-\beta}\left(\frac{g'(t)}{t}\right)^{\frac{1}{2^*}} \geq \sqrt{H}t^{-\beta}\left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}} \tag{2.3.15}$$

since $\delta \leq 2^*$. Moreover, by the right hand side of (2.3.1)

$$g''(t)\left(\frac{g'(t)}{t}\right)^{-1} \leq 2K.$$

Thus

$$\sqrt{g''(t)} \geq \frac{\sqrt{H}}{2K}t^{-\beta}\left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}} 2K \geq \frac{\sqrt{H}}{2K}t^{-\beta}\left(\frac{g'(t)}{t}\right)^{\frac{1}{\delta}-1} g''(t). \tag{2.3.16}$$

Putting together (2.3.15) and (2.3.16) we get

$$\sqrt{g''(t)} \geq \frac{\sqrt{H}}{4K} t^{-\beta} \left[\left(\frac{g'(t)}{t} \right)^{\frac{1}{\delta}} + \left(\frac{g'(t)}{t} \right)^{\frac{1}{\delta}-1} g''(t) \right] \tag{2.3.17}$$

if $\frac{g'(t)}{t} \leq 1$. With similar arguments, if $\frac{g'(t)}{t} \geq 1$, we get

$$\sqrt{g''(t)} \geq \sqrt{H} t^{-\beta} \sqrt{\frac{g'(t)}{t}}$$

and since $\delta \geq \frac{2\alpha}{2-\alpha}$ we get

$$\sqrt{g''(t)} \geq \sqrt{H} t^{-\beta} \left(\frac{g'(t)}{t} \right)^{\frac{\alpha}{\delta}}. \tag{2.3.18}$$

Moreover, by the right hand side of (2.3.1)

$$\sqrt{g''(t)} \leq \sqrt{2K} \left(\frac{g'(t)}{t} \right)^{\frac{\alpha}{2}},$$

which is equivalent to

$$g''(t) \leq \sqrt{2K} \left(\frac{g'(t)}{t} \right)^{\frac{\alpha}{2}} \sqrt{g''(t)}.$$

Furthermore, we can write

$$\sqrt{g''(t)} \geq \frac{1}{\sqrt{2K}} \left(\frac{g'(t)}{t} \right)^{-\frac{\alpha}{2}} g''(t) \geq \frac{1}{\sqrt{2K}} t^{-\beta} \left(\frac{g'(t)}{t} \right)^{-\frac{\alpha}{2}} g''(t).$$

Since $\delta \geq \frac{2\alpha}{2-\alpha}$, we have $\frac{\alpha}{\delta} - 1 \leq -\frac{\alpha}{2}$ and hence

$$\sqrt{g''(t)} \geq \frac{1}{2K} t^{-\beta} \left(\frac{g'(t)}{t} \right)^{\frac{\alpha}{\delta}-1} g''(t). \tag{2.3.19}$$

Therefore, in the case $\frac{g'(t)}{t} \geq 1$, we obtain

$$g''(t) \geq \min\left\{ \frac{\sqrt{H}}{2}, \frac{1}{2\sqrt{2K}} \right\} t^{-\beta} \left[\left(\frac{g'(t)}{t} \right)^{\frac{\alpha}{\delta}} + \left(\frac{g'(t)}{t} \right)^{\frac{\alpha}{\delta}-1} g''(t) \right]. \tag{2.3.20}$$

from (2.3.18) and (2.3.19). Finally, (2.3.14) holds for $t \geq 1$ too and Lemma 2.3.4 is proved. ∎

Lemma 2.3.5 *Let g be as in Theorem 2.3.1 and satisfying the right hand side of condition (2.3.1). Then, there exists a constant C, depending on $K, g'(t_0), t_0$ and α such that*

$$g'(t)t \leq C(1+g(t))^{\frac{1}{2-\alpha}} \tag{2.3.21}$$

for every $t \geq 0$.

Proof. Let $t \geq t_0$; after a multiplication for t and a side to side integration between t_0 and t in the right hand side of (2.3.1), we get

$$\int_{t_0}^{t} sg''(s)\, s \leq K \int_{t_0}^{t} g'(s)\, ds + K \int_{t_0}^{t} s \left(\frac{g'(s)}{s}\right)^{\alpha} ds.$$

Thus, integrating by parts the left hand side

$$g'(t)t \leq g'(t_0)t_0 + (K+1)\int_{t_0}^{t} g'(s)\, ds + K\int_{t_0}^{t} s^{2-2\alpha} g'(s)\left(g'(s)s\right)^{\alpha-1} ds$$

$$\leq g'(t_0)t_0 + (K+1)g(t) + Kt_0^{2-2\alpha}\left(g'(t)t\right)^{\alpha-1} g(t).$$

By dividing both sides for $(g'(t)t)^{\alpha-1}$ we have

$$(g'(t)t)^{2-\alpha} \leq (g'(t_0)t_0)^{2-\alpha} + \left(\frac{K+1}{(g'(t_0)t_0)^{\alpha-1}} + Kt_0^{2-2\alpha}\right) g(t).$$

Then, for all $t \geq t_0$,

$$g'(t)t \leq C_1(1+g(t))$$

with C_1 depending on $K, g'(t_0), t_0$ and α. Then, if $C \geq C_1$ inequality (2.3.21) holds for every $t \geq 0$ since $g'(t)t \leq g'(t_0)t_0$ for every $t \leq t_0$ being g' increasing. ■

Lemma 2.3.6 *Let g be as in Theorem 2.3.1 and let $\mathcal{H}$ be the function defined in (2.3.9). Suppose that g satisfies the right hand side of (2.3.1), then there exists a constant C such that for any $1 < \eta \leq \frac{3n}{3n-4}$,*

$$1 + \mathcal{H}(t)t^2 \leq C(1+g(t))^{\eta} \tag{2.3.22}$$

for every $t \geq 0$, where $\eta = \eta(\alpha) = \frac{\alpha}{2-\alpha}$ and the constant C depends on $K, \sup_{0 \leq t \leq t_0} g''(t)$ and α.

Proof. By definition (2.3.9), we have that $\mathcal{H}(t)t^2 \leq g'(t)t + g''(t)t^2$ for every $t \geq 0$. Let $t \geq t_0 \geq 1$, then by the right hand side of (2.3.1) and by Lemma 2.3.5 we obtain

$$g''(t)t^2 \leq KC(1+g(t))^{\frac{1}{2-\alpha}} + KC^{\alpha}(1+g(t))^{\frac{\alpha}{2-\alpha}} t^{2-2\alpha} \tag{2.3.23}$$

$$\leq 2C_1(1+g(t))^{\frac{\alpha}{2-\alpha}}.$$

If $t \leq t_0$ we have

$$g''(t)t^2 \leq \sup_{0\leq t\leq t_0} g''(t)t^2 \leq t_0^2 \sup_{0\leq t\leq t_0} g''(t) \leq C_{t_0}. \tag{2.3.24}$$

by putting together (2.3.23), (2.3.24) and Lemma 2.3.5 we have proved (2.3.22). ■

2.3.2 A-priori estimates

We make the following supplementary assumption: there exist two positive constants N, M such that

$$N|\lambda|^2 \leq \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(\xi)\lambda_i^\alpha \lambda_j^\beta \leq M|\lambda|^2 \tag{2.3.25}$$

for every $\lambda, \xi \in \mathbb{R}^{m\times n}$.

This is equivalent to ask that both $g'(t)/t$ and $g''(t)$ are bounded by constants N, M for every $t > 0$. This assumption is the analogous of (2.1.13), but now, we remember, $g'(t)/t$ is not necessarily increasing. The strategy to prove an a-priori estimate is similar to the one used to prove (2.1.11). Condition (2.3.25) leads to the following equation:

$$A_k + B_k + C_k = 0 \tag{2.3.26}$$

where we have defined

$$A_k = \int_\Omega 2\eta\Phi(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} u^\alpha_{x_k} \eta_{x_i}\, dx,$$

$$B_k = \int_\Omega \eta^2\Phi(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} u^\alpha_{x_i x_k}\, dx,$$

$$C_k = \int_\Omega \eta^2\Phi'(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} u^\alpha_{x_k} \left(|Du|\right)_{x_i}\, dx.$$

As in the proof of Lemma 2.1.11, we estimate integral A_k with the inequality $2ab \leq \frac{1}{2}a^2 + 2b^2$:

$$|A_k| \leq \int_\Omega 2\Phi \left[\eta^2 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} u^\alpha_{x_i x_k}\right]^{\frac{1}{2}}$$

$$\cdot \left[\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du)\eta_{x_i} u^\alpha_{x_k} \eta_{x_j} u^\beta_{x_k} \right]^{\frac{1}{2}} dx \tag{2.3.27}$$

$$\leq \int_\Omega \Phi \left[\frac{\eta^2}{2} \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} u^\alpha_{x_i x_k} \right.$$

$$\left. +2 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du)\eta_{x_i} u^\alpha_{x_k} \eta_{x_j} u^\beta_{x_k} \right] dx.$$

As in Lemma 2.1.5, since (2.1.18) holds, it is natural to sum up with respect to k to obtain

$$\sum_k C_k = \int_\Omega \eta^2 \Phi'(|Du|) \left[\left(\frac{g''(|Du|)}{|Du|^2} - \frac{g'(|Du|)}{|Du|^3} \right) \right. \tag{2.3.28}$$

$$\left. \cdot \sum_\alpha \left(\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right)^2 + g'(|Du|)|D(|Du|)|^2 \right] dx$$

$$= \int_\Omega \eta^2 \Phi'(|Du|) \left(\frac{g''(|Du|)}{|Du|^2} \sum_\alpha \left(\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right)^2 + g'(|Du|)|D(|Du|)|^2 \right.$$

$$\left. - \frac{g'(|Du|)}{|Du|^2} \sum_\alpha \left(\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right)^2 \right) dx.$$

Since, by Cauchy-Schwarz inequality we have

$$\sum_\alpha \left(\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right)^2 \leq \sum_{i,\alpha} (u^\alpha_{x_i})^2 \sum_i (|Du|)^2_{x_i} \leq |Du|^2 |D(|Du|)|^2,$$

then

$$\sum_k C_k \geq \int_\Omega \eta^2 \Phi'(|Du|) \frac{g''(|Du|)}{|Du|} \sum_\alpha \left(\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right)^2 dx \geq 0. \tag{2.3.29}$$

Furthermore,

$$\sum_k B_k = \int_\Omega \eta^2 \Phi(|Du|) \left[\left(g''(|Du|) - \frac{g'(|Du|)}{|Du|} \right) |D(|Du|)|^2 \right. \tag{2.3.30}$$

$$\left. + \frac{g'(|Du|)}{|Du|} |D^2 u|^2 \right] dx.$$

By (2.1.18) and by the Cauchy-Schwarz inequality we have

$$|D(|Du|)|^2 \leq |D^2u|^2$$

from which we deduce that

$$\sum_k B_k \geq \int_\Omega \eta^2 \Phi(|Du|) g''(|Du|) |D(|Du|)|^2 \, dx. \tag{2.3.31}$$

Then, from (2.3.26), by using the inequalities (2.3.29) and (2.3.31), we get

$$\frac{1}{2} \int_\Omega \eta^2 \Phi(|Du|) g''(|Du|) g''(|Du|) |D(|Du|)|^2 \, dx$$

$$\leq \frac{1}{2} \sum_k B_k + \sum_k C_k \tag{2.3.32}$$

$$\leq 2 \int_\Omega \Phi(|Du|) \sum_{i,j,\alpha,\beta,k} f_{\xi_i^\alpha \xi_j^\beta}(Du) \eta_{x_i} u^\alpha_{x_k} \eta_{x_j} u^\beta_{x_k} \, dx.$$

By the right hand side of (2.3.8), we obtain

$$\int_\Omega \eta^2 \Phi(|Du|) g''(|Du|) |D(|Du|)|^2 \, dx \tag{2.3.33}$$

$$\leq 4 \int_\Omega \Phi(|Du|) \mathcal{H}(|Du|) |D\eta|^2 |Du|^2 \, dx$$

for every $\Phi : [0, +\infty) \to [0, +\infty)$ increasing, local Lipschitz continuous function with Φ and Φ' bounded on $[0, +\infty)$. With the same considerations of Lemma 2.1.11, we can allow more general functions Φ and consider a positive, increasing and local Lipschitz continuous function in $[0, +\infty)$. In the same way we also define

$$G(t) = 1 + \int_0^t \sqrt{\Phi(s) g''(s)} \, ds$$

for every $t \geq 0$. As in Lemma 2.1.11, we use the Sobolev's inequality to find that

$$\left[\int_\Omega \left(\eta G\left(|Du|\right) \right)^{2^*} \, dx \right]^{\frac{2}{2^*}} \leq C_1 \int_\Omega |D\eta|^2 \left(1 + 3\Phi(|Du|) \mathcal{H}(|Du|) |Du|^2 \right) \, dx.$$

As in the proof of Lemma 2.1.11 we define $\Phi(t) = t^{2\gamma}$ with $\gamma \geq 0$ and we use Lemma 2.3.4 with $\delta = 2^*$ to state that there exists a constant C_2 such that

$$\left[\int_\Omega \eta^{2^*} \left(1 + |Du|^{(\gamma+1-\beta)2^*} \mathcal{H}(|Du|) \right) \, dx \right]^{\frac{2}{2^*}} \tag{2.3.34}$$

$$\leq C_2(\gamma+1)^2 \int_\Omega |D\eta|^2 \left(1+|Du|^{2\gamma}\mathcal{H}(|Du|)|Du|^2\right).$$

Let η be equal to 1 in B_ρ, with support contained in B_R and such that $|D\eta| \leq \frac{2}{R-\rho}$. Fixed R_0 and ρ_0, with $\rho_0 < R_0$, we define the decreasing sequence of radii

$$\rho_i = \rho_0 + \frac{R_0-\rho_0}{2^i}$$

and the increasing sequence of exponents

$$\delta_{i+1} = (\delta_i - 2\beta)\frac{2^*}{2}$$

with $\delta_0 = 2$. Then, (2.3.25) becomes

$$\left[\int_{B_{\rho_{i+1}}} \left(1+|Du|^{\delta_{i+1}}\mathcal{H}(|Du|)\right)\,dx\right]^{\frac{2}{2^*}} \tag{2.3.35}$$

$$\leq C_2\left(\frac{\delta_i 2^{i+1}}{R_0-\rho_0}\right)^2 \int_{B_{\rho_i}} \left(1+|Du|^{\delta_i}\mathcal{H}(|Du|)\right)\,dx.$$

By iterating (2.3.35) we get

$$\left[\int_{B_{\rho_{i+1}}} \left(1+|Du|^{(2-\beta n)\left(\frac{2^*}{2}\right)^{i+1}+\beta n}\mathcal{H}(|Du|)\right)\,dx\right]^{\left(\frac{2}{2^*}\right)^{i+1}} \tag{2.3.36}$$

$$\leq C\int_{B_{R_0}} \left(1+|Du|^2\mathcal{H}(|Du|)\right)\,dx$$

where the exponent in the first integral comes from the computation

$$\delta_{i+1} = 2\left(\frac{2^*}{2}\right)^{i+1} - 2\beta\sum_{k=1}^{i+1}\left(\frac{2^*}{2}\right)^k = (2-\beta n)\left(\frac{2^*}{2}\right)^{i+1} + \beta n$$

and the constant in the right hand side is given by

$$C \leq \prod_{k=0}^{\infty}\left[\frac{8C_2}{(R_0-\rho_0)^2}(2^*)^{2k}\right]^{\left(\frac{2}{2^*}\right)^k} = \left[\left(\frac{C_3}{(R_0-\rho_0)^2}\right)^{\sum_{k=0}^\infty\left(\frac{2}{2^*}\right)^k}\right](2^*)^{\sum_{k=0}^\infty k\left(\frac{2}{2^*}\right)^k}$$

$$= \left(\frac{C_3}{(R_0-\rho_0)^2}\right)^{\frac{n}{2}}(2^*)^{\frac{n(n-2)}{n}} = \frac{C_4}{(R_0-\rho_0)^n}$$

for every $n \geq 3$, while if $n = 2$ then for every $\varepsilon > 0$ we can choose 2^* so that $C = \frac{C_4}{(R_0-\rho_0)^{2+\varepsilon}}$.

We observe that for every $t > 0$, $1 + t^\alpha \mathcal{H}(t) \geq 1 + t^{\alpha-1} g'(t)$ by the definition of $\mathcal{H}$. Moreover, if $t \geq 1$, since $g'(t)$ is increasing, $1 + t^{\alpha-1} g'(t) \geq t^{\alpha-1} g'(1)$ and, if $t \leq 1$ we have $1 + t^{\alpha-1} g'(t) \geq 1 \geq t^{\alpha-1}$. Therefore

$$\left[\int_{B_{\rho_0}} |Du|^{\left(2-\beta n+\frac{\beta n-1}{\left(\frac{2^*}{2}\right)^{i+1}}\right)\left(\frac{2^*}{2}\right)^{i+1}} dx \right] \leq C \int_{B_{R_0}} \left(1 + |Du|^2 \mathcal{H}(|Du|)\right) dx.$$

Finally, we pass to the limit as $i \to \infty$ and we find that

$$\sup \left\{ |Du(x)|^{2-\beta n} : x \in B_{\rho_0} \right\} \leq C \int_{B_{R_0}} \left(1 + |Du|^2 \mathcal{H}(|Du|)\right) dx.$$

This concludes the proof of the following Lemma, which is the equivalent of Lemma 2.1.11 in Section 2.1.

Lemma 2.3.7 *Let g be as in Theorem 2.1.1 and satisfy the supplementary assumption (2.3.25). Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ be a minimizer of integral (2.1.1). Then there exists a constant C depending on R, ρ, n, but not depending on N and M of (2.3.25), such that*

$$\|Du\|^{2-\beta n}_{L^\infty(B_\rho, \mathbb{R}^{m\times n})} \leq C \int_{B_R} \left(1 + |Du|^2 \mathcal{H}(|Du|)\right) dx.$$

Remark 2.3.8 *We observe that the strategy used to prove that $\sum_k C_k \geq 0$ could also be used in the proof of (2.1.19) in Section 2.1. There, we have used the important hypothesis that $g'(t)/t$ is increasing which could be dropped for this particular inequality but not for other fundamental estimates for the a-priori estimate of Lemma 2.1.11.*

The analogous of Lemma 2.1.12 is the following lemma, which allows to reduce the estimate of the L^∞-norm of the local minimizer u only to the function g and, in this case, not to the function $\mathcal{H}$, which, recall, is the maximum between $g'(t)/t$ and $g''(t)$. It is proved by using Lemma 2.3.5. We do not report here the proof of this lemma since it follows from Lemma 2.3.7 as Lemma 3.3.2 in Chapter 3 follows from Lemma 3.3.1.

Lemma 2.3.9 *Let g be as in Theorem 2.1.1 and satisfy the supplementary assumption (2.3.25). Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ be a minimizer of integral (2.1.1). Then* *for every $\varepsilon > 0$ and for every $0 < \rho < R$ there exists a constant C depending on n, ε, ρ, R such that*

$$\int_{B_\rho} \left(1 + |Du|^2 \mathcal{H}(|Du|)\right) dx \leq C \left\{ \int_{B_R} (1 + g(|Du|)) \, dx \right\}^{\frac{1}{1-\beta}+\varepsilon}. \tag{2.3.37}$$

The constant C depends also on $g(t_0)$, $g'(t_0)$, K, H, $\sup_{0 \leq t \leq t_0} g''(t)$, $\inf_{0 \leq t \leq t_0} g''(t)$, but not on the constants N and M of assumption (2.3.25).

Together, Lemmata 2.3.9 and 2.3.7 give the following

Theorem 2.3.10 *Let g be as in Theorem 2.1.1 and satisfy the supplementary assumption (2.3.25). Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ be a minimizer of integral (2.1.1). Then there exists a constant C depending on R, ρ, n, but not depending on N and M of (2.3.25), such that*

$$\|Du\|^{2-\beta n}_{L^\infty(B_\rho, \mathbb{R}^{m \times n})} \leq C \left\{ \int_{B_R} (1 + g(|Du|)) \, dx \right\}^{\frac{1}{1-\beta}+\varepsilon}.$$

2.3.3 Approximation of the original problem with regular variational problems

The approximation argument in the setting of Lemma 2.1.11 is a bit more complicated than the one sketched in Section 2.1 since the technicality of the hypotheses increases the difficulty of the estimates. We consider g as in Theorem 2.3.1. One and only one of the following cases holds:

i) there exists a sequence $\{t_n\}$ such that $\lim_{t\to\infty} t_n = \infty$ and $\frac{g'(t)}{t} = 1$,

ii) there exists T such that $\frac{g'(t)}{t} > 1$ for every $t \geq T$,

iii) there exists T such that $\frac{g'(t)}{t} < 1$ for every $t \geq T$.

Let $\bar{t} = \inf \left\{ t > 0 : \frac{g'(t)}{t} > 0 \right\}$. Up to rescaling we can assume $0 \leq \bar{t} < 1 \leq t_0$. We consider a vanishing sequence ε_n in the following way. If i) holds we choose $\varepsilon_n = \frac{1}{t_n}$, while if we are either in case i) or iii) we consider any vanishing sequence such that $\frac{1}{\varepsilon_n} \geq T$. It is obvious that one can choose n large enough to have $\bar{t} + \varepsilon_n < 1$ and $\frac{1}{\varepsilon_n} \geq \max\{T, \bar{t} + \varepsilon_n\}$. We define the approximating function by defining its derivative:

$$g'_{\varepsilon_n}(t) = \begin{cases} \frac{g'(\bar{t}+\varepsilon_n)}{\bar{t}+\varepsilon_n} t & \text{if } 0 \leq t \leq \bar{t} + \varepsilon_n, \\ g'(t) & \text{if } \bar{t} + \varepsilon_n < t \leq \frac{1}{\varepsilon_n}, \\ \min\left\{ \varepsilon_n g'\left(\frac{1}{\varepsilon_n}\right) t, \, g'(t) + \varepsilon_n t - 1 \right\} & \text{if } t > \frac{1}{\varepsilon_n}. \end{cases} \tag{2.3.38}$$

Then g_ε is defined by

$$g_{\varepsilon_n}(t) = \int_0^t g'_{\varepsilon_n}(s)\,ds \tag{2.3.39}$$

which results to be a convex function of class C^1 in $[0,+\infty)$ satisfying condition of Lemma 2.1.11 and the supplementary assumption (2.3.25) with suitable constants $N(\varepsilon_n)$ and $M(\varepsilon_n)$.

Lemma 2.3.11 *Let g be as in Theorem 2.3.1 and satisfying the left hand side of (2.3.1). Let g_{ε_n} as defined in (2.3.39). Then there exists a positive constant H_1 such that*

$$H_1 t^{-2\beta}\left[\left(\frac{g'_{\varepsilon_n}(t)}{t}\right)^{\frac{2}{2^*}} + \frac{g'_{\varepsilon_n}(t)}{t}\right] \le g''_{\varepsilon_n}(t) \tag{2.3.40}$$

for every $t \ge t_0$.

Lemma 2.3.12 *Let g be as in Theorem 2.3.1 and satisfying the right hand side of (2.3.1). Let g_{ε_n} as defined in (2.3.39). There exists a positive constant K_1 such that for any $\alpha > 1$ we have*

$$g''_{\varepsilon_n}(t) \le K_1\left[\frac{g'_{\varepsilon_n}(t)}{t} + \left(\frac{g'_{\varepsilon_n}(t)}{t}\right)^{\alpha}\right] \tag{2.3.41}$$

for every $t \ge t_0$.

The two lemmata above ensure that the approximating function g_{ε_n} satisfies the main assumption (2.3.1). The following lemma is used to pass to the limit as $\varepsilon_n \to 0$ and in its proof it is used the definition of g_{ε_n} for $t > \frac{1}{\varepsilon_n}$.

Lemma 2.3.13 *Let g be as in Theorem 2.3.1 and let g_{ε_n} as defined in (2.3.39). Then there exists a constant C such that*

$$g_{\varepsilon_n}(t) \le C\,(1 + g(t)) + \varepsilon_n t^2 \tag{2.3.42}$$

for every $t \ge 0$.

We define the approximating integral and we pass to the limit as $\varepsilon_n \to 0$. Let us consider the sequence of integral functionals

$$F_{\varepsilon_n}(v) = \int_\Omega g_{\varepsilon_n}(|Dv|)\,dx. \tag{2.3.43}$$

Let $u \in W^{1,1}_{\text{loc}}(\Omega,\mathbb{R}^m)$ be a local minimizer of (2.1.1). We denote by u_σ a sequence of smooth functions defined from u by means of standard mollifiers, then

$u_\sigma \in W^{1,2}(B_R, \mathbb{R}^m)$. Moreover we take a minimizer $u_{\varepsilon_n,\sigma}$ of F_{ε_n} that satisfies the Dirichlet condition $u_{\varepsilon_n,\sigma} = u_\sigma$ on the boundary ∂B_R, that is, since F_{ε_n} has quadratic growth,

$$\int_{B_R} g_{\varepsilon_n}(|Du_{\varepsilon_n,\sigma}|)\, dx \leq \int_{B_R} g_\varepsilon(|Dv|)\, dx$$

for every $v \in u_\sigma + W_0^{1,2}(B_R, \mathbb{R}^m)$. Then, for every ε_n, g_{ε_n} satisfies conditions (2.3.1), (2.3.25) (for some constants $N(\varepsilon_n), M(\varepsilon_n)$) with constants H and K not depending on ε_n. Therefore, we can apply the a-priori estimate of Theorem 2.3.10 to get that for every ε_n and for every ρ ($\rho < R$) there exists a constant C_1, independent of $N, M, \varepsilon_n, \sigma$ such that for some $\beta \in \left(\frac{1}{n}, \frac{2}{n}\right)$ we have

$$\|Du_{\varepsilon_n,\sigma}\|^{2-\beta n}_{L^\infty(B_\rho, \mathbb{R}^{m\times n})} \leq C_1 \left\{ \int_{B_R} 1 + g_{\varepsilon_n}(|Du_{\varepsilon_n,\sigma}|)\, dx \right\}^{\frac{1}{1-\beta}+\varepsilon}.$$

By the minimality of $u_{\varepsilon_n,\sigma}$ we can write that

$$\int_{B_R} g_{\varepsilon_n}(|Du_{\varepsilon_n,\sigma}|)\, dx \leq \int_{B_R} g_{\varepsilon_n}(|Du_\sigma|)\, dx \tag{2.3.44}$$

and by Lemma (2.3.42) and the properties of mollifiers

$$\int_{B_R} g_{\varepsilon_n}(|Du_\sigma|)\, dx \tag{2.3.45}$$

$$\leq C_2 \left[\int_{B_R} (1 + g(|Du_\sigma|))\, dx + \varepsilon_n \int_{B_R} |Du_\sigma|^2\, dx \right]$$

$$\leq C_2 \left[\int_{B_R} (1 + g(|Du|))\, dx + \varepsilon_n \int_{B_R} |Du_\sigma|^2\, dx \right] \leq C_3$$

with C_3 depending on σ. As a consequence

$$\|Du_{\varepsilon_n,\sigma}\|^{2-\beta n}_{L^\infty(B_\rho, \mathbb{R}^{m\times n})} \tag{2.3.46}$$

$$\leq C_4 \left[\int_{B_R} (1 + g(|Du|))\, dx + \varepsilon_n \int_{B_R} |Du_\sigma|^2\, dx \right]^{\frac{1}{1-\beta}+\varepsilon} \leq C_5(\sigma).$$

Then, for every fixed σ, $|Du_{\varepsilon,\sigma}|$ is equibounded with respect to ε_n. Up to a subsequence, $u_{\varepsilon_n,\sigma}$ converges to some function w_σ in the weak-* topology of $W^{1,\infty}(B_\rho, \mathbb{R}^{m\times n})$. Going to the limit for $\varepsilon_n \to 0$ we obtain

$$\|Dw_\sigma\|^{2-\beta n}_{L^\infty(B_\rho, \mathbb{R}^{m\times n})} \leq C_4 \left[\int_{B_{R+\sigma}} (1 + g(|Du|))\, dx \right]^{\frac{1}{1-\beta}+\varepsilon}. \tag{2.3.47}$$

Hence, also $|Dw_\sigma|$ is equibounded in $L^\infty\left(B_\rho, \mathbb{R}^{m\times n}\right)$ and we can take a subsequence which converges in the weak-* topology of $L^\infty\left(B_\rho, \mathbb{R}^{m\times n}\right)$ to Dw for some function w. By the lower semicontinuity and by the convexity of g, we could prove that $u = w$. In particular, we consider ε_n sufficiently small in dependence of σ: fixed σ, we consider $\varepsilon_n \leq \bar{\varepsilon}_n(\sigma)$, with $\bar{\varepsilon}_n(\sigma)$ such that $\frac{1}{\bar{\varepsilon}_n(\sigma)} > |C_5(\sigma)|^{\frac{1}{2-\beta n}}$. Then, by (2.3.46), $|Du_{\varepsilon_n,\sigma}| < \frac{1}{\varepsilon_n}$ and by the definition of g_{ε_n}, we have

$$g_{\varepsilon_n}(t) = \begin{cases} \frac{g'(\bar{t}+\varepsilon_n)}{\bar{t}+\varepsilon_n}\frac{t^2}{2} & \text{if } 0 \leq t \leq \bar{t}+\varepsilon_n, \\ g(t) - g(\bar{t}+\varepsilon_n) + \frac{g'(\bar{t}+\varepsilon_n)(\bar{t}+\varepsilon_n)}{2} & \text{if } \bar{t}+\varepsilon_n < t \leq \frac{1}{\varepsilon_n}, \end{cases}$$

and hence we can write

$$g(t) \leq g(\bar{t}+\varepsilon_n) + g_{\varepsilon_n}(t),$$

when $\bar{t}+\varepsilon_n \leq t \leq \frac{1}{\varepsilon_n}$. Hence, by the lower semicontinuity we obtain

$$\int_{B_\rho} g\left(|Dw_\sigma|\right)\, dx \leq \liminf_{\varepsilon_n\to 0} \int_{B_R} g\left(|Du_{\varepsilon_n,\sigma}|\right)\, dx$$

$$\leq \liminf_{\varepsilon_n\to 0} \int_{B_R} g_{\varepsilon_n}\left(|Du_{\varepsilon_n,\sigma}|\right)\, dx.$$

By (2.3.44) and Lemma 2.3.13 we can deduce that $g_{\varepsilon_n}\left(|Du_{\varepsilon_n,\sigma}\right)$ is equibounded with respect to ε_n and then, by the dominated convergence theorem

$$\liminf_{\varepsilon_n\to 0} \int_{B_R} g_{\varepsilon_n}\left(|Du_{\varepsilon_n,\sigma}|\right)\, dx \leq \int_{B_R} g\left(|Du_\sigma|\right)\, dx \leq \int_{B_{R+\sigma}} g\left(|Du|\right)\, dx.$$

Thus, for every $\rho < R$,

$$\int_{B_\rho} g\left(|Dw_\sigma|\right)\, dx \leq \int_{B_{R+\sigma}} g\left(|Du|\right)\, dx.$$

By the lower semicontinuity, we have

$$\int_{B_R} g\left(|Dw|\right)\, dx \leq \liminf_{\sigma\to 0} \int_{B_R} g\left(|Dw_\sigma|\right)\, dx \leq \int_{B_R} g\left(|Du|\right)\, dx.$$

The assumptions on g do not give uniqueness of the minimizer for the Dirichlet problem. Anyway, since $g(|\xi|)$ is locally strictly convex for $|\xi| > 1$, for almost every $x \in B_R$ we have $|Dw(x)| = |Du(x)|$ and thus (2.3.47) holds for Du too, concluding the proof of Theorem 2.3.1.

Chapter 3

A-priori gradient estimates for elliptic systems under either slow or fast growth conditions

We are interested in the regularity of *local minimizers* of energy-integrals of the calculus of variations of the form

$$F(v) = \int_{\Omega} f(x, Dv)\, dx\,, \tag{3.0.1}$$

where Ω is an open set of $\mathbb{R}^n$ for some $n \geq 2$ and Dv is the $m \times n$ gradient-matrix of a map $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ with $m \geq 1$. Here $f : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a convex *Carathéodory integrand*; i.e., $f = f(x, \xi)$ is measurable with respect to $x \in \mathbb{R}^n$ and it is a convex function with respect to $\xi \in \mathbb{R}^{m\times n}$. A *local minimizer* of the energy-functional F in (3.0.1) is a map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ of class $W^{1,1}_{\mathrm{loc}}$ such that $f(x, Du) \in L^1_{\mathrm{loc}}(\Omega)$ and satisfying the inequality $F(u) \leq F(u + \varphi)$ for every test function φ with compact support in Ω; i.e. $\varphi \in C^1_0(\Omega; \mathbb{R}^m)$. As usual in the vectorial setting, we require the Uhlenbeck condition on f, i.e. $f(x, \xi) = g(x, |\xi|)$. As anticipated in the Introduction, we recall that we allow x-dependence in the principal part of the energy integrand. We consider a general integrand of the form $g = g(x, t)$, where $g : \Omega \times [0, \infty) \to [0, \infty)$ is a Carathéodory function, convex and increasing with respect to $t \in [0, \infty)$. Our assumptions, stated below in (3.0.2), allow us to consider *both fast and slow growth* on the integrand $g(x, |Du|)$.

Without loss of generality, by changing $g(x, t)$ with $g(x, t) - g(x, 0)$ if necessary, we can reduce ourselves to the case $g(x, 0) = 0$ for almost every $x \in \Omega$. We assume that the partial derivatives g_t, g_{tt}, g_{tx_k} exist (for every $k = 1, 2, \ldots n$) and that they are Carathéodory functions too, with $g_t(x, 0) = 0$.

In the next section we show that the following assumptions (3.0.2), (3.0.3) cover examples of functional with both fast growth and slow growth. Precisely, we require the following growth conditions: let $t_0 > 0$ be fixed; for every open subset Ω' compactly contained in Ω, there exist $\vartheta \geq 1$ and positive constants m and M_ϑ such that

$$\begin{cases} mh'(t) \leq g_t(x,t) \leq M_\vartheta \left[h'(t)\right]^\vartheta t^{1-\vartheta} \\ mh''(t) \leq g_{tt}(x,t) \leq M_\vartheta \left[h''(t)\right]^\vartheta \\ \left|g_{tx_k}(x,t)\right| \leq M_\vartheta \min\left\{g_t(x,t), t\, g_{tt}(x,t)\right\}^\vartheta \end{cases} \tag{3.0.2}$$

for every $t \geq t_0$ and for $x \in \Omega'$. The role of the parameter ϑ can be easily understood if we compare (3.0.2) with the model examples (see the next section). Here, $h : [0,+\infty) \to [0,+\infty)$ is a convex increasing function of class $W^{2,\infty}_{\text{loc}}$ satisfying the following property: for some $\beta > \frac{1}{n}$ such that $(2\vartheta-1)\vartheta < (1-\beta)\frac{2^*}{2}$, and for every α such that $1 < \alpha \leq \frac{n}{n-1}$, there exist constants m_β and M_α such that

$$\frac{m_\beta}{t^{2\beta}}\left[\left(\frac{h'(t)}{t}\right)^{\frac{n-2}{n}} + \frac{h'(t)}{t}\right] \leq h''(t) \leq M_\alpha\left[\left(\frac{h'(t)}{t}\right)^\alpha + \frac{h'(t)}{t}\right] \tag{3.0.3}$$

for every $t \geq t_0$. We obtain the following a-priori gradient estimate.

Theorem 3.0.1 *Let us assume that conditions (3.0.2),(3.0.3) hold. Then the gradient of any smooth local minimizer of the integral (3.0.1) is uniformly locally bounded in Ω. Precisely, if u is such a local minimizer, then there exists an exponent $\omega > 1$ and, for every ρ, R, $0 < \rho < R$, there exists a positive constant C such that,*

$$\|Du\|_{L^\infty(B_\rho,\mathbb{R}^{m\times n})} \leq C\left\{\int_{B_R}(1+g(x,|Du|))\,dx\right\}^\omega. \tag{3.0.4}$$

The exponent ω depends on ϑ, β, n while the constant C depends on $\rho, R, n\alpha, \beta, \vartheta, t_0$ and $\sup\{h''(t) : t \in [0,t_0]\}$.

As described above, Theorem 3.0.1 gives an *a-priori local gradient bound.* In the regularity theory for weak solutions this is the main step to get the *local Lipschitz continuity* of solutions, since the minimizer is assumed to be smooth enough for the validity of the Euler's first and second variations (see also the statement of Theorem 3.3.3), but the constants in the bound (3.0.4) do not depend on this smoothness. An approximation argument gives this *local Lipschitz continuity* property. In fact, by applying the a-priori gradient estimate to an approximating energy integrand $f_k(x,|\xi|)$ which converges to $f(x,|\xi|)$ as $k \to +\infty$ and which satisfies standard growth conditions, we obtain a sequence of smooth approximating solutions u_k with

$$\|Du_k\|_{L^\infty(B_\rho,\mathbb{R}^{m\times n})} \leq const\,,$$

for every fixed small radius ρ, and the constant is independent of k. In the limit as $k \to +\infty$ also the solution u to the original variational problem, related to the energy integrand $f(x,|\xi|)$, comes out to have locally bounded gradient and thus it is local Lipschitz continuous in Ω. As described in Chapter 2, we can pass to the limit as $k \to \infty$ in order to prove local Lipschitz continuity in Ω of the solution u of the original variational problem. In the next Section we analyze some examples, in particular the model examples from (3.1.1) to (3.1.5), and we show the role of the parameters ϑ, β, α in the assumptions (3.0.2), (3.0.3). Then, after some preliminary results proposed in Section 3.2, in Section 3.3 we give the *a-priori estimate* as in (3.0.4) and we conclude the proof of Theorem 3.0.1.

3.1 Some model examples

We recall some of the examples that are covered by the assumptions of Theorem 3.0.1. We can consider both fast and slow growth on the integrand $g(x,|Du|)$. Model energy-integrals that we have in mind are, for instance, *exponential growth* with local Lipschitz continuous coefficients a, b $(a(x), b(x) \geq c > 0)$

$$\int_\Omega e^{a(x)|Du|^2}\,dx \quad \textit{or} \quad \int_\Omega b(x)\exp\left(\ldots\exp\left(a(x)|Du|^2\right)\right)\,dx\,; \tag{3.1.1}$$

variable exponents $(a, p \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, $a(x) \geq c > 0$ and $p(x) \geq p > 1)$

$$\int_\Omega a(x)|Du|^{p(x)}\,dx \quad \text{or} \quad \int_\Omega a(x)\left(1+|Du|^2\right)^{p(x)/2}\,dx\,; \tag{3.1.2}$$

of course the classical $p-$*Laplacian* energy-integral, with a constant p strictly greater than 1 and integrand $f(x,Du) = a(x)|Du|^p$, is covered by the example (3.1.2): the theory considered here and the Theorem 3.0.1 below apply to the $p-$Laplacian. Also *Orlicz-type* energy-integrals (see Chlebicka [14], Chlebicka *et al.* [15]), again with local Lipschitz continuous exponent $p(x) \geq p > 1$, of the type

$$\int_\Omega a(x)|Du|^{p(x)}\log(1+|Du|)\,dx\,; \tag{3.1.3}$$

note that the a-priori estimate in Theorem 3.0.1 below holds also for some cases with *slow growth*, i.e., when $p(x) \geq 1$, in particular when $p(x)$ is identically equal to 1. See (3.1.10) and the details in the next section. A class of energy-integrals of the form

$$\int_\Omega h(a(x)|Du|)\,dx \quad \text{or} \quad \int_\Omega b(x)\,h(a(x)|Du|)\,dx\,, \tag{3.1.4}$$

with $a(x), b(x)$ locally Lipschitz continuous and nonnegative coefficients in Ω and $h : [0,+\infty) \to [0,+\infty)$ a convex increasing function of class $W^{2,\infty}_{\mathrm{loc}}([0,+\infty))$ as in (3.0.3) below. Also some $g(x, |\xi|)$ with *slow growth*, precisely *linear growth* as $t = |Du| \to +\infty$, such as, for $n = 2, 3$,

$$\int_\Omega \left\{ |Du| - a(x) \sqrt{|Du|} \right\} dx \,, \tag{3.1.5}$$

with $a \in W^{1,\infty}_{\text{loc}}(\Omega)$, $a(x) \geq c > 0$ (here more precisely $t \to t - a(x)\sqrt{t}$ means a smooth convex function in $[0, +\infty)$, with derivative equal to zero at $t = 0$, which coincide with $t - a(x)\sqrt{t}$ for $t \geq t_0$, for a given $t_0 > 0$, and for all $x \in \Omega$). As we have shown in the previous chapter, in particular in Section 2.2, integrals like (3.1.1), (3.1.2) and (3.1.4) with $p(x) \geq 2$ have been proved by Mascolo-Migliorini [52]. We show in this section that also the *sub-quadratic* cases with $p(x) \geq p > 1$ satisfy assumptions (3.0.2) and (3.0.3). Moreover, also integrals with slow growth as (3.1.5) are covered by (3.0.2) and (3.0.3), that, although technical, can be considered general enough to cover the cases from (3.1.1) to (3.1.5) and ensure the gradient bound in (3.0.4).

We first note that the parameter β has not a relevant role when the variational problem has *fast growth*, i.e. if $\frac{h'(t)}{t} \to \infty$ when $t \to \infty$. In this case, for instance, we can choose $\beta = \frac{3}{2n}$, the intermediate point of the interval $\left(\frac{1}{n}, \frac{2}{n}\right)$, and for ϑ any real number greater than or equal to 1 (in some cases ϑ strictly greater than 1, see the details below) such that $(2\vartheta - 1)\vartheta < \frac{n - \frac{3}{2}}{n-2}$.

We start with the example (3.1.1), with $g(x, t) = e^{a(x)t^2}$ and the positive coefficient $a(x)$ locally Lipschitz continuous in Ω. In order to prove the local L^∞ bound of the gradient of a local minimizer we first fix a ball B compactly contained in Ω such that the oscillation of $a(x)$ is small in $\overline{B}$; precisely, under the notation

$$a_M = \max\left\{a(x) : \ x \in \overline{B}\right\}, \quad a_m = \min\left\{a(x) : \ x \in \overline{B}\right\}, \tag{3.1.6}$$

given $\vartheta > 1$ we choose the radius of the ball B small enough such that $a_M \leq \vartheta a_m$. Then, if we define $h(t) =: e^{a_m t^2}$, we have

$$h(t) = e^{a_m t^2} \leq e^{a(x)t^2} = g(x, t) \leq e^{a_M t^2} \leq e^{\vartheta a_m t^2} = [h(t)]^\vartheta$$

for every $t \geq 0$ and $x \in \overline{B}$. Similarly for the derivatives $g_t(x, t) = 2a(x)\, t e^{a(x)t^2}$ and $h'(t) = 2a_m t e^{a_m t^2}$ we obtain

$$\frac{h'(t)}{t} \leq \frac{g_t(x, t)}{t} \leq 2a_M e^{a_M t^2} \leq 2\vartheta a_m e^{\vartheta a_m t^2}$$

$$= (2a_m)^\vartheta (2a_m)^{1-\vartheta} \vartheta \left(e^{a_m t^2}\right)^\vartheta = \vartheta (2a_m)^{1-\vartheta} \left[\frac{h'(t)}{t}\right]^\vartheta$$

for every $t > 0$ and $x \in \overline{B}$. For the second derivatives with respect to t we have similar estimates:

$$h''(t) \leq g_{tt}(x,t) \leq 2a_m \vartheta e^{\vartheta a_m t^2} \left(1 + 2\vartheta a_m t^2\right)$$

$$\leq (2a_m)^{\vartheta} (2a_m)^{1-\vartheta} \vartheta^2 \left[e^{a_m t^2} \left(1 + 2a_m t^2\right)\right]^{\vartheta} = (2a_m)^{1-\vartheta} \vartheta^2 \left[h''(t)\right]^{\vartheta}.$$

Then, if we call $M_\vartheta = \max\left\{\vartheta\,(2a_m)^{1-\vartheta}, \vartheta^2\,(2a_m)^{1-\vartheta}\right\}$, the first two conditions of (3.0.2) hold.

From $g_t\,(x,t) = 2a\,(x)\,te^{a(x)t^2}$ we estimate the mixed derivative g_{tx_k}. In this case

$$\min\left\{g_t(x,t), t\,g_{tt}\,(x,t)\right\} = g_t(x,t)$$

and then we consider the quotient

$$\frac{|g_{tx_k}(x,t)|}{g_t^{\vartheta}\,(x,t)} = \frac{\left|2a_{x_k}\,(x)\,te^{a(x)t^2}\,(1 + a\,(x)\,t^2)\right|}{(2a(x)te^{a(x)t^2})^{\vartheta}};$$

if we denote by L the Lipschitz constant of the coefficient $a\,(x)$ in B, we have

$$\frac{|g_{tx_k}(x,t)|}{g_t^{\vartheta}\,(x,t)} \leq \frac{2L}{(2a_m)^{\vartheta}} \frac{1 + a_M t^2}{(te^{a(x)t^2})^{\vartheta-1}}$$

and, for every $\vartheta > 1$, the right hand side is bounded in $\overline{B}$ for every $t \geq 1$. Therefore also the last condition in (3.0.2) is satisfied.

It remains to verify the condition (3.0.3) for the function $h\,(t) = e^{a_m t^2}$. Here the parameter $\alpha > 0$ plays a crucial role since $h'(t) = 2a_m te^{a_m t^2}$ and $h''(t) = 2a_m e^{a_m t^2} + (2a_m t)^2\, e^{a_m t^2}$ and we cannot bound $h''(t)$ in terms of $\frac{h'(t)}{t}$. On the contrary, for every $\alpha > 1$ there exists a constant M_α such that the following bound, which implies the bound required in (3.0.3), holds

$$h''\,(t) \leq M_\alpha \left(\frac{h'\,(t)}{t}\right)^{\alpha}, \qquad \forall\, t > 0.$$

The left hand side inequality in (3.0.3) is satisfied since in this case $h''\,(t) \geq \frac{h'(t)}{t}$ for every $t > 0$ and the quantity $\left(\frac{h'(t)}{t}\right)^{\frac{n-2}{n}}$ goes to $+\infty$ slower than $\frac{h'(t)}{t}$. Moreover $t^{-2\beta} \to 0$ as $t \to +\infty$.

The example (3.1.1), by changing $a\,(x)$ with $a^2\,(x)$, can be equivalently written in the form

$$\int_\Omega e^{a^2(x)|Du|^2}\,dx = \int_\Omega e^{(a(x)|Du|)^2}\,dx$$

and can be considered an example in the class of energy-integrals as in (3.1.4), of the type

$$\int_{\Omega} h\left(a\left(x\right)\left|Du\right|\right)\,dx \tag{3.1.7}$$

where $h:[0,+\infty)\to[0,+\infty)$ is a convex increasing function of class $W^{2,\infty}_{\mathrm{loc}}\left([0,+\infty)\right)$ satisfying (3.0.3). If $a\left(x\right)$ is a positive continuous coefficient in Ω we can use a similar argument as above and obtain, by Theorem 3.0.1, the a-priori estimate also of minima of the integral (3.1.7).

The example (3.1.4) is similar to (3.1.1). In this case we have to test the conditions in (3.0.2). Under the notation $g\left(x,t\right)=t^{p(x)}$ (for simplicity we consider here $a\left(x\right)$ identically equal to 1), we have $g_t\left(x,t\right)=p\left(x\right)t^{p(x)-1}$ and

$$\begin{aligned} g_{tx_k}\left(x,t\right) &= p_{x_k}\left(x\right)t^{p(x)-1}+p\left(x\right)\frac{\partial}{\partial x_k}\left[e^{(p(x)-1)\log t}\right] \\ &= p_{x_k}\left(x\right)t^{p(x)-1}\left[1+p\left(x\right)\log t\right]\,. \end{aligned}$$

If we denote by L the Lipschitz constant of $p\left(x\right)$ on a fixed open subset Ω' whose closure is contained in Ω, then

$$\frac{\left|g_{tx_k}(x,t)\right|}{\left(g_{tt}\left(x,t\right)t\right)^{\vartheta}}\leq L\frac{1+p\left(x\right)\log t}{p^{\vartheta}\left(x\right)\left(p\left(x\right)-1\right)^{\vartheta}t^{(\vartheta-1)(p(x)-1)}}$$

and thus the quotient is bounded for $t\in[1,+\infty)$ and $x\in\Omega'$ if $p\left(x\right)>1$ is locally Lipschitz continuous in Ω (i.e., also being $p\left(x\right)\geq c>1$ for some constant $c=c\left(\Omega'\right)$) and $\vartheta>1$. Also here note the role of the parameter ϑ strictly greater than 1. Since $g_{tt}\left(x,t\right)t$ and $g_t\left(x,t\right)$ are of the same order as $t\to+\infty$, similarly

$$\frac{\left|g_{tx_k}(x,t)\right|}{g_t^{\vartheta}\left(x,t\right)}\leq L\frac{1+p(x)\log(t)}{p^{\vartheta}(x)t^{(\vartheta-1)(p(x)-1)}}.$$

The other conditions in (3.0.2) can be tested as before.

Similar computations can be carried out for the example (3.1.3), with $g\left(x,t\right)=a\left(x\right)t^{p(x)}\log(1+t)$, under the assumption that the coefficient $a\left(x\right)$ and the exponent $p\left(x\right)$ are locally Lipschitz continuous in Ω and that, for every Ω' compactly contained in Ω, there exists a constant $c>1$ such that $p\left(x\right)\geq c$ for every $x\in\Omega'$. Also the limit case enters in this regularity theory, when the exponent $p\left(x\right)\geq1$ for every $x\in\Omega'$, however by assuming in this special case that $a\left(x\right)$ is identically equal to 1; see the details below in (3.1.10).

We now consider the *slow growth* example (3.1.5). Here

$$g\left(x,t\right)=t-a\left(x\right)\sqrt{t}\,. \tag{3.1.8}$$

As already mentioned, $g(x,t)$ in (3.1.8) means a smooth convex function in $[0,+\infty)$, with derivative equal to zero at $t=0$, which coincides with $t-a(x)\sqrt{t}$ for $t\geq t_0$, for a given $t_0>0$ and for $x\in\Omega$. Again, we use the notation in (3.1.6), precisely given $\Omega'\subset\subset\Omega$,

$$a_M=\max\{a(x):\ x\in\Omega'\},\qquad a_m=\min\{a(x):\ x\in\Omega'\},$$

and we require a_m to be positive. Then, for $x\in\Omega'$ and $t\geq t_0$,

$$\begin{cases} g_t(x,t)=1-\frac{1}{2}a(x)\,t^{-\frac{1}{2}}\geq 1-\frac{1}{2}a_M t^{-\frac{1}{2}} \\ t\,g_{tt}(x,t)=\frac{1}{4}a(x)\,t^{-\frac{1}{2}}\leq \frac{1}{4}a_M t^{-\frac{1}{2}} \end{cases}$$

and for large t we have

$$\min\{g_t(x,t),t\,g_{tt}(x,t)\}=t\,g_{tt}(x,t).$$

If we denote by L the Lipschitz constants of $a(x)$ on $\Omega'\subset\subset\Omega$, we fix $\vartheta=1$ and we obtain the bounded quotient for $t\geq t_0$

$$\frac{|g_{tx_k}(x,t)|}{\min\{g_t(x,t),t\,g_{tt}(x,t)\}}=\frac{|g_{tx_k}(x,t)|}{t\,g_{tt}(x,t)}\leq\frac{\frac{L}{2}t^{-\frac{1}{2}}}{\frac{1}{4}a_m t^{-\frac{1}{2}}}=\frac{2L}{a_m}.$$

In order to test the other conditions in (3.0.2), we define $h(t)=:t-\sqrt{t}$ and, for $(x,t)\in\Omega'\times(0,+\infty)$, we have

$$\min\{1;a_M\}\,h'(t)\leq g_t(x,t)\leq\max\{1;a_m\}\,h'(t)\ ,$$

$$a_m h''(t)\leq g_{tt}(x,t)\leq a_M h''(t)\ .$$

Then (3.0.2) are satisfied with $\vartheta=1$. Finally the convex function $h(t)=:t-\sqrt{t}$ satisfies (3.0.3). In fact, since $h''(t)=\frac{1}{4}t^{-\frac{3}{2}}$ and $\frac{h'(t)}{t}=t^{-1}-\frac{1}{2}t^{-\frac{3}{2}}$, then as $t\to+\infty$, $h''(t)$ goes to zero faster than $\frac{h'(t)}{t}$ and for every $\alpha>1$ there exists a constant M_α such that

$$h''(t)\leq M_\alpha\left[\left(\frac{h'(t)}{t}\right)^\alpha+\frac{h'(t)}{t}\right],\qquad\forall\,t\geq 1.$$

On the other side, since as $t\to+\infty$ the quantity $\frac{h'(t)}{t}\to 0$ and $\frac{n-2}{n}<1$, then $\frac{h'(t)}{t}$ goes to zero faster than $\left(\frac{h'(t)}{t}\right)^{\frac{n-2}{n}}$. Therefore we can equivalently test the condition

$$\frac{m_\beta}{t^{2\beta}}\left(\frac{h'(t)}{t}\right)^{\frac{n-2}{n}}\leq h''(t)\ ,\qquad\forall\,t\geq 1. \tag{3.1.9}$$

The order of infinitesimal of the left hand side in (3.1.9) is $\frac{1}{t^{2\beta+\frac{n-2}{n}}}$, while the order of infinitesimal of the right hand side is $\frac{1}{t^{\frac{3}{2}}}$. Therefore condition (3.1.9) is satisfied for some constant m_β if

$$2\beta + \frac{n-2}{n} \geq \frac{3}{2}.$$

Under the condition $(2\vartheta - 1)\vartheta < (1-\beta)\frac{2^*}{2}$, since $\vartheta \geq 1$ a simple computation shows that $\beta < \frac{2}{n}$ too. Then, we get the following condition for β:

$$\frac{1}{4} + \frac{1}{n} \leq \beta < \frac{2}{n}\,,$$

which is compatible if $n = 2, 3$.

With a similar computation we can treat the example (3.1.3) for general locally Lipschitz continuous coefficients $a\,(x)$ and exponents $p\,(x)$, by assuming that there exists a constant $c > 1$ such that $p\,(x) \geq c$ for almost every $x \in \Omega$. While, under the more general assumption $p\,(x) \geq 1$ for every $x \in \Omega$, we need to assume $a\,(x)$ identically equal to 1. I.e., for instance, if $p\,(x)$ is identically equal to 1, then we consider the energy integral

$$\int_\Omega |Du| \log(1 + |Du|)\, dx\,; \tag{3.1.10}$$

here the function h is defined by

$$h\,(t) = t \log\,(1+t)$$

and comes out that the right inequality in (3.0.3) is satisfied for some constant M:

$$h''\,(t) = \frac{2+t}{(1+t)^2} \leq M\frac{h'\,(t)}{t} = M\left(\frac{\log\,(1+t)}{t} + \frac{1}{1+t}\right), \qquad \forall\, t \geq 1.$$

While in the left hand side of (3.0.3), since as $t \to +\infty$ the quantity $\frac{h'(t)}{t}$ converges to 0, we test (3.1.9) and the problem is to compare the order of infinitesimal of the left hand side in (3.1.9), which is $\frac{\log(1+t)}{t^{2\beta+\frac{n-2}{n}}}$, with the order of infinitesimal of the right hand side, equal to $\frac{1}{t}$. A sufficient condition in this case is $2\beta + \frac{n-2}{n} > 1$ (with the strict sign inequality), which is compatible with $\frac{1}{n} < \beta < \frac{2}{n}$ for every $n \in \mathbb{N}$, $n \geq 2$.

3.2 Preliminary lemmata

In the following we consider test maps $\varphi = (\varphi^\alpha)_{\alpha=1,2,\dots,n}$ with components of the form $\varphi^\alpha = \eta^2 u^\alpha_{x_k} \Phi\,(|Du|)$, where $\eta \in C^1_0\,(\Omega)$ and $\Phi = \Phi\,(t)$ is a real nonnegative function, defined for $t \in [0, +\infty)$. We consider $\Phi(t)$ depending on a real parameter $\gamma \geq 0$ (in general without denoting explicitly this dependence) by separating two cases: the first one with γ large and the second one when γ is small. In order to simplify the proofs, throughout the paper we will assume that $t_0 = 1$ and $g(x, 1) > 0$ for almost every $x \in \Omega$. Precisely, if $\gamma > 1$ we define

$$\Phi(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1, \\ (t-1)^{\gamma} & \text{if } t > 1. \end{cases} \tag{3.2.1}$$

The following simple inequality holds.

Lemma 3.2.1 *For every $\gamma \in (1, +\infty)$ the function $\Phi(t)$ defined in (3.2.1) satisfies the inequality*

$$0 \leq \Phi'(t)\, t \leq \gamma\left(1 + 2\Phi(t)\right), \quad \forall\, t \geq 0. \tag{3.2.2}$$

Proof. The inequality (3.2.2) is trivial when $t \in [0, 1]$. If $t > 1$ then

$$\Phi'(t)\, t = \gamma (t-1)^{\gamma-1} t = \gamma (t-1)^{\gamma} + \gamma (t-1)^{\gamma-1} = \gamma\Phi(t) + \gamma (t-1)^{\gamma-1}. \tag{3.2.3}$$

Since $(t-1)^{\gamma-1} \leq (t-1)^{\gamma}$ when $t \geq 2$, while $(t-1)^{\gamma-1} \leq 1$ if $t \in [1, 2]$, then in any case

$$(t-1)^{\gamma-1} \leq 1 + (t-1)^{\gamma} = 1 + \Phi(t), \quad \forall\, t \geq 1. \tag{3.2.4}$$

From (3.2.3),(3.2.4) we get the conclusion (3.2.2). ■

For $\gamma \in [0, 1]$ we define

$$\Phi(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1, \\ (t-1)^2\, t^{\gamma-2} & \text{if } t > 1. \end{cases} \tag{3.2.5}$$

Lemma 3.2.2 *For every $\gamma \in [0, 1]$ the function $\Phi(t)$ defined in (3.2.5) satisfies the inequality*

$$0 \leq \Phi'(t)\, t \leq 2 + (\gamma + 2)\, \Phi(t), \quad \forall\, t \geq 0. \tag{3.2.6}$$

Proof. The inequality (3.2.2) is satisfied when $t \in [0, 1]$. If $t > 1$ then

$$\begin{aligned} \Phi'(t) &= 2(t-1)\, t^{\gamma-2} + (\gamma - 2)(t-1)^2\, t^{\gamma-3} \\ &= (t-1)\, t^{\gamma-3}\left[2t + (\gamma-2)(t-1)\right] = (t-1)\, t^{\gamma-3}\left[\gamma(t-1) + 2\right] \end{aligned}$$

and thus $\Phi'(t) \geq 0$ and

$$\Phi'(t)\, t = \gamma (t-1)^2\, t^{\gamma-2} + 2(t-1)\, t^{\gamma-2} = \gamma\Phi(t) + 2(t-1)\, t^{\gamma-2}. \tag{3.2.7}$$

Since $(t-1)\, t^{\gamma-2} \leq (t-1)^2\, t^{\gamma-2}$ when $t \geq 2$, while if $t \in [1, 2]$ then $(t-1)\, t^{\gamma-2} \leq t^{\gamma-2} \leq 1$ since $\gamma - 2 \leq 0$; therefore in any case

$$(t-1)\, t^{\gamma-2} \leq (t-1)^2\, t^{\gamma-2} + 1 = 1 + \Phi(t), \quad \forall\, t \geq 1. \tag{3.2.8}$$

From (3.2.7),(3.2.8) we get the thesis (3.2.6). ■

Remark 3.2.3 *In the next section we consider real nonnegative functions $\Phi = \Phi(t)$ as in (3.2.1) when $\gamma \in (1, +\infty)$ or as in (3.2.5) when $\gamma \in [0, 1]$. As consequence of Lemmata 3.2.1 and 3.2.2, for every $\gamma \in [0, +\infty)$ we are allowed to consider functions $\Phi_\gamma : [0, +\infty) \to [0, +\infty)$ (later we do not denote explicitly the dependence on the parameter γ), which are increasing and of class C^1 in $[0, +\infty)$, identically equal to zero when $t \in [0, 1]$ and satisfy the growth conditions*

$$\begin{cases} 0 \leq \Phi_\gamma(t) \leq t^\gamma \\ 0 \leq \Phi'_\gamma(t)\, t \leq \max\{2; \gamma\} + \max\{2\gamma; \gamma + 2\}\, \Phi_\gamma(t) \end{cases}, \quad \forall\, t \geq 0, \ \ \forall\, \gamma \geq 0, \tag{3.2.9}$$

and the second inequality is implied and can be also simply written, for instance, in the form

$$0 \leq \Phi'_\gamma(t)\, t \leq (2\gamma + 2)\left(1 + \Phi_\gamma(t)\right), \quad \forall\, t \geq 0, \ \ \forall\, \gamma \geq 0. \tag{3.2.10}$$

In what follows, we also use the functions

$$\mathcal{K}_M(t) = \max\left\{h''(t), \frac{h'(t)}{t}\right\} \tag{3.2.11}$$

and

$$\mathcal{K}_m(t) = \min\left\{h''(t), \frac{h'(t)}{t}\right\} \tag{3.2.12}$$

related to the function h defined in (3.0.3). We will use the following lemma when $\gamma \geq 1$.

Lemma 3.2.4 *Let h satisfy (3.0.3) and let $\mathcal{K}_m, \mathcal{K}_M$ be the functions defined in (3.2.11), (3.2.12). Then, for every σ with $\frac{2\alpha}{2^*(2-\alpha)} \leq \sigma \leq 1$ and for every $\gamma \geq 1$ there exists a constant C (depending on α) such that*

$$1 + \int_1^t (s-1)^\gamma \sqrt{\mathcal{K}_m(s)}\, ds \geq C \left[1 + \left(\frac{(t-1)^{\gamma+1} t^{-\beta}}{\gamma + 1}\right)^{2^*} \mathcal{K}_M^{\frac{1}{\sigma}}(t)\right]^{\frac{1}{2^*}}, \tag{3.2.13}$$

for every $t \geq 1$.

Proof. Let us define $\delta = 2^*\sigma$, then we observe that

$$\left[1 + \left(\frac{(t-1)^{\gamma+1} t^{-\beta}}{\gamma + 1}\right)^{2^*} \mathcal{K}_M^{\frac{1}{\sigma}}(t)\right]^{\frac{1}{2^*}} \leq \left[1 + \frac{(t-1)^{\gamma+1} t^{-\beta}}{\gamma + 1} \mathcal{K}_M^{\frac{1}{\delta}}(t)\right],$$

for every $t \geq 1$ and for every $\gamma \geq 1$. By definition of $\mathcal{K}$ we get

$$\mathcal{K}_M(t) \leq \frac{h'(t)}{t} + h''(t)$$

and, by the right hand side of (3.0.3) we can write

$$\mathcal{K}_M(t) \leq (m_\alpha + 1)\left[\frac{h'(t)}{t} + \left(\frac{h'(t)}{t}\right)^\alpha\right].$$

From these, instead of proving (3.2.13) we can prove that

$$\begin{aligned} 1 + \int_1^t (s-1)^\gamma \sqrt{\mathcal{K}_m(s)}\,ds \qquad (3.2.14) \\ \geq C\left[1 + \frac{(t-1)^{\gamma+1}t^{-\beta}}{\gamma+1}\left(\frac{h'(t)}{t} + \left(\frac{h'(t)}{t}\right)^\alpha\right)^{\frac{1}{\delta}}\right] \end{aligned}$$

for every $t \geq 1$. At this end it is sufficient to show the inequality between the derivatives side to side with respect to t of (3.2.14), i.e., since $\frac{\gamma+1-\frac{t-1}{t}\beta}{\gamma+1}$, $\frac{1}{\gamma+1}$, $\frac{t-1}{t} < 1$,

$$\begin{aligned} \sqrt{\mathcal{K}_m(t)} \geq Ct^{-\beta}\Bigg[\left(\frac{h'(t)}{t}\right)^{\frac{1}{\delta}} + \left(\frac{h'(t)}{t}\right)^{\frac{\alpha}{\delta}} + \left(\frac{h'(t)}{t}\right)^{\frac{1}{\delta}-1} h''(t) \\ + \left(\frac{h'(t)}{t}\right)^{\frac{\alpha}{\delta}-1} h''(t)\Bigg], \end{aligned}$$

where we still denote by C the new constant. If $\mathcal{K}_m(t) = h''(t)$, then we can conclude by arguing as in Lemma 2.3.4. If otherwise $\mathcal{K}_m(t) = \frac{h'(t)}{t}$, then it is sufficient to show that

$$\sqrt{\frac{h'(t)}{t}} \geq c\left((h''(t))^{\frac{\alpha}{\delta}} + (h''(t))^{\frac{1}{\delta}}\right) \qquad (3.2.15)$$

which holds by the assumption on h (3.0.3). In fact, if $h''(t) \geq 1$, then (3.2.15) is equivalent to

$$\sqrt{\frac{h'(t)}{t}} \geq c(h''(t))^{\frac{\alpha}{\delta}}$$

and since (3.0.3) holds and $h'(t)/t \leq h''(t)$ there exists a constant c such that $h''(t) \leq c\left(\frac{h'(t)}{t}\right)^\alpha$ for every $\alpha > 1$. Since $\frac{2\alpha}{\delta} > 1$, (3.2.15) holds. The other case, $h''(t) \leq 1$ can be treated with a similar argument. Therefore, (3.2.14) is proved and then (3.2.13) is proved too. ■

Remark 3.2.5 *We will also use the following inequality, which is implied by (3.2.13):*

$$1+\int_1^t (s-1)^\gamma \sqrt{\mathcal{K}_m(s)}\,ds \geq C\left[1+\left(\frac{(t-1)^{\gamma+1}t^{-\beta}}{\gamma+1}\right)^{2^*\sigma}\mathcal{K}_M(t)\right]^{\frac{1}{2^*\sigma}}, \quad (3.2.16)$$

for every σ with $\frac{2\alpha}{2^*(2-\alpha)} \leq \sigma \leq 1$ and for every $\gamma \geq 1$.

Let us now treat the case of $0 \leq \gamma \leq 1$.

Lemma 3.2.6 *Let h satisfy (3.0.3) and let $\mathcal{K}_M, \mathcal{K}_m$ be the functions defined in (3.2.11), (3.2.12). Then, for every σ with $\frac{2\alpha}{2^*(2-\alpha)} \leq \sigma \leq 1$ and for every $\gamma \in [0,1]$ there exists a constant C (depending on α) such that, for every $t \geq 1$,*

$$1+\int_1^t (s-1)s^{\gamma-1} \sqrt{\mathcal{K}_m(s)}\,ds \geq C\left[1+\left(\frac{(t-1)^{\gamma+1}t^{-\beta}}{\gamma+1}\right)^{2^*}\mathcal{K}_M^{\frac{1}{\sigma}}(t)\right]^{\frac{1}{2^*}}. \quad (3.2.17)$$

Proof. By arguing as in Lemma 3.2.4, all we need to prove is the following:

$$\begin{aligned} &1+\int_1^t (s-1)s^{\gamma-1}\sqrt{\mathcal{K}_m(s)}\,ds \qquad (3.2.18)\\ &\geq C\left[1+\frac{(t-1)^{\gamma+1}t^{-\beta}}{\gamma+1}\left(\frac{h'(t)}{t}+\left(\frac{h'(t)}{t}\right)^\alpha\right)^{\frac{1}{\delta}}\right].\end{aligned}$$

Moreover, since $\gamma < 1$ and $t \geq 1$, we have

$$\begin{aligned} &1+\frac{(t-1)^{\gamma+1}t^{-\beta}}{\gamma+1}\left(\frac{h'(t)}{t}+\left(\frac{h'(t)}{t}\right)^\alpha\right)^{\frac{1}{\delta}}\\ &\leq C\left(1+\frac{(t-1)^2 t^{\gamma-\beta-1}}{\gamma+1}\left(\frac{h'(t)}{t}+\left(\frac{h'(t)}{t}\right)^\alpha\right)^{\frac{1}{\delta}}\right),\end{aligned}$$

where C does not depend on γ. Then, it is sufficient to prove that

$$1+\int_1^t (s-1)s^{\gamma-1}\sqrt{\mathcal{K}_m(s)}\,ds \geq C\left(1+\frac{(t-1)^2 t^{\gamma-\beta-1}}{\gamma+1}\left(\frac{h'(t)}{t}+\left(\frac{h'(t)}{t}\right)^\alpha\right)^{\frac{1}{\delta}}\right).$$

As before, it is sufficient to show the inequality between the derivatives side to side with respect to t, i.e., since $\frac{t-1}{t} < 1$ and $(\gamma-\beta-1)\frac{t-1}{t}+2 < \gamma+1$,

$$\sqrt{\mathcal{K}_m(t)} \geq Ct^{-\beta}\left[\left(\frac{h'(t)}{t}\right)^{\frac{1}{\delta}}+\left(\frac{h'(t)}{t}\right)^{\frac{\alpha}{\delta}}+\left(\frac{h'(t)}{t}\right)^{\frac{1}{\delta}-1}h''(t)\right.$$

$$+\left(\frac{h'(t)}{t}\right)^{\frac{\alpha}{\delta}-1} h''(t)\Bigg].$$

Again, by arguing as in Lemma 3.2.4 we can conclude the proof. ■

Remark 3.2.7 *We can argue as in Remark 3.2.5 to obtain*

$$1+\int_1^t (s-1)s^{\gamma-1}\sqrt{\mathcal{K}_m(s)}\,ds \geq C\left[1+\left(\frac{(t-1)^{\gamma+1}t^{-\beta}}{\gamma+1}\right)^{2^*\sigma}\mathcal{K}_M(t)\right]^{\frac{1}{2^*\sigma}}, \quad (3.2.19)$$

for every σ *with* $\frac{2\alpha}{2^*(2-\alpha)} \leq \sigma \leq 1$ *and for every* $0 \leq \gamma \leq 1$.

We can resume Lemmata 3.2.4 and 3.2.6 in the following lemma, where Φ is the function defined in (3.2.1) if $\gamma \geq 1$ and the function defined in (3.2.5) if $0 \leq \gamma \leq 1$.

Lemma 3.2.8 *Let* h *satisfy (3.0.3) and let* $\mathcal{K}_M, \mathcal{K}_m$ *be the functions defined in (3.2.11), (3.2.12). Then, for every* σ *with* $\frac{2\alpha}{2^*(2-\alpha)} \leq \sigma \leq 1$ *and for every* $\gamma \geq 0$ *there exists a constant* C *(depending on* α*) such that, for every* $t \geq 1$,

$$1+\int_1^t \sqrt{\Phi(s)\mathcal{K}_m(s)}\,ds \geq C\left[1+\left(\frac{(t-1)^{\frac{\gamma}{2}+1}t^{-\beta}}{\gamma+1}\right)^{2^*}\mathcal{K}_M^{\frac{1}{\sigma}}(t)\right]^{\frac{1}{2^*}}. \quad (3.2.20)$$

We will use two consequences of (3.2.20) in section 3.3. The first one is the particular case of $\sigma = \frac{1}{\vartheta}$:

$$1+\int_1^t \sqrt{\Phi(s)\mathcal{K}_m(s)}\,ds \geq C\left[1+\left(\frac{(t-1)^{\frac{\gamma}{2}+1}t^{-\beta}}{\gamma+1}\right)^{2^*}\mathcal{K}_M^{\vartheta}(t)\right]^{\frac{1}{2^*}}. \quad (3.2.21)$$

The second one is essentially the content of Remarks 3.2.5 and 3.2.7 and it is resumed in the following:

$$1+\int_1^t \sqrt{\Phi(s)\mathcal{K}_m(s)}\,ds \geq C\left[1+\left(\frac{(t-1)^{\frac{\gamma}{2}+1}t^{-\beta}}{\gamma+1}\right)^{2^*\sigma}\mathcal{K}_M(t)\right]^{\frac{1}{2^*\sigma}}, \quad (3.2.22)$$

for any $\frac{2\alpha}{2^*(2-\alpha)} \leq \sigma \leq 1$, $\gamma \geq 0$ and every $t \geq 1$.

Next Lemma 3.2.9 is Lemma 2.3.5 that we write again here with g replaced by h, while Lemma 3.2.10 is the generalization of Lemma 2.3.6 with $\vartheta \geq 1$.

Lemma 3.2.9 *Let* h *satisfy the right hand side of (3.0.3). Then there exists a constant* C*, depending on* m_α, $h'(t_0)$, t_0, α *such that, for every* $t \geq 1$,

$$h'(t)t \leq C\left(1+h(t)\right)^{\frac{1}{2-\alpha}}. \quad (3.2.23)$$

Lemma 3.2.10 *Let h satisfy the right hand side of (3.0.3) and let $\mathcal{K}_M$ be the functions defined in (3.2.11). Then, for every $1 \leq \tau < \frac{2^*(2-\alpha)}{2\alpha}$, there exists a constant C such that for any η, $1 < \eta \leq \frac{n}{n-2}$,*

$$1 + \mathcal{K}_M^\tau(t)t^{2\tau} \leq C\,(1 + h(t))^\eta\,, \tag{3.2.24}$$

for every $t \geq 1$, where $\eta = \eta(\alpha) = \frac{\alpha}{2-\alpha}$ and the constant C depends only on m_α, $\sup_{0\leq t\leq 1} h''(t)$, *$\alpha$.*

Proof. By the definition of $\mathcal{K}_M$ we have that we have that

$$\mathcal{K}_M^\vartheta(t)t^{2\vartheta} \leq \left(\frac{h'(t)}{t}\right)^\vartheta t^{2\vartheta} + (h''(t))^\vartheta\, t^{2\vartheta} = (h'(t)t)^\vartheta + \left(h''(t)t^2\right)^\vartheta \tag{3.2.25}$$

for every $t \geq 1$. By the right hand side of (3.0.3) and by Lemma 3.2.9 we obtain

$$\begin{aligned} h''(t)t^2 &\leq m_\alpha C\,(1 + h(t))^{\frac{1}{2-\alpha}} + m_\alpha C^\alpha\,(1 + h(t))^{\frac{\alpha}{2-\alpha}}\, t^{2-2\alpha} \\ &\leq C\,(1 + h(t))^{\frac{\alpha}{2-\alpha}}\,. \end{aligned} \tag{3.2.26}$$

By putting together (3.2.25) and (3.2.26) we obtain the result. ■

3.3 A-priori estimates

By the representation $f\,(x,\xi) = g\,(x,|\xi|)$, we have

$$f_{\xi_i^\alpha}\,(x,\xi) = g_t\,(x,|\xi|)\,\frac{\xi_i^\alpha}{|\xi|}, \tag{3.3.1}$$

$$f_{\xi_i^\alpha \xi_j^\beta}\,(x,\xi) = \left(\frac{g_{tt}\,(x,|\xi|)}{|\xi|} - \frac{g_t\,(x,|\xi|)}{|\xi|^2}\right)\xi_i^\alpha \xi_j^\beta + \frac{g_t\,(x,|\xi|)}{|\xi|}\delta_{\xi_i^\alpha \xi_j^\beta}. \tag{3.3.2}$$

Thus, the following ellipticity estimates hold:

$$\begin{aligned} \min\left\{g_{tt}\,(x,|\xi|)\,, \frac{g_t\,(x,|\xi|)}{|\xi|}\right\}|\lambda|^2 &\leq \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}\,(x,\xi)\,\lambda_i^\alpha \lambda_j^\beta \\ &\leq \max\left\{g_{tt}\,(x,|\xi|)\,, \frac{g_t\,(x,|\xi|)}{|\xi|}\right\}|\lambda|^2\,, \end{aligned} \tag{3.3.3}$$

for every λ, $\xi \in \mathbb{R}^{m\times n}$. Let us define

$$\mathcal{H}_m\,(x,t) = \min\left\{g_{tt}\,(x,t)\,, \frac{g_t\,(x,t)}{t}\right\} \tag{3.3.4}$$

and

$$\mathcal{H}_M(x,t) = \max\left\{g_{tt}(x,t), \frac{g_t(x,t)}{t}\right\}, \tag{3.3.5}$$

then (3.3.3) becomes

$$\mathcal{H}_m(x,|\xi|)|\lambda|^2 \leq \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,\xi)\lambda_i^\alpha \lambda_j^\beta \leq \mathcal{H}_M(x,|\xi|)|\lambda|^2, \tag{3.3.6}$$

for every λ, $\xi \in \mathbb{R}^{m\times n}$.

We make the following supplementary assumption, which could be later removed with an approximating procedure, for instance as in Section 5 of [46] and in Section 6 of [51]: there exist two positive constants N, M such that

$$N|\lambda|^2 \leq \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,\xi)\lambda_i^\alpha \lambda_j^\beta \leq M|\lambda|^2, \tag{3.3.7}$$

for every $\lambda, \xi \in \mathbb{R}^{m\times n}$ and for almost every $x \in \Omega$. This is equivalent to say that both $\frac{g_t}{t}$ and g_{tt} are bounded by constants N, M for every $t > 0$ and for almost every $x \in \Omega$. This assumption allows to consider u as a function of class $W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^m) \cap W^{2,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$. We denote B_ρ and B_R balls of radii, respectively, ρ and R ($\rho < R$) contained in Ω and with the same center. In what follows, we will denote by

$$\tilde{B}_R = B_R \cap \{x : |Du(x)| \geq 1\}.$$

Lemma 3.3.1 *Let g, h respectively satisfy (3.0.2) and (3.0.3). Suppose that the supplementary condition (3.3.7) is satisfied. Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ be a local minimizer of (3.0.1). Then, under the notation $\tau = (2\vartheta - 1)\vartheta$, for every ρ, R ($0 < \rho < R$) there exists a constant C, not depending on N, and M, such that*

$$\|Du\|^{\left(1-\beta-\frac{2}{2^*}\tau\right)n}_{L^\infty\left(\tilde{B}_\rho,\mathbb{R}^{m\times n}\right)} \leq \frac{C}{(R-\rho)^n}\int_{\tilde{B}_R}\left(1+|Du|^{2\tau}\mathcal{K}_M^\tau(|Du|)\right)dx. \tag{3.3.8}$$

The constant C depends on $n, \vartheta, \beta, \alpha$.

Proof. Let u be a local minimizer of (3.0.1). We denote by $u = (u^\alpha)_{\alpha=1,\ldots,n}$ its components. By the left hand side of (3.3.7), $u \in W^{1,2}(\Omega,\mathbb{R}^m)$ and by the right hand side of (3.3.7) u satisfies the Euler's first variation:

$$\int_\Omega \sum_{i,\alpha} f_{\xi_i^\alpha}(x,Du)\varphi^\alpha_{x_i}\,dx = 0,$$

for every $\varphi = (\varphi^{\alpha}) \in W_0^{1,2}(\Omega, \mathbb{R}^m)$. Using the technique of difference quotients we can prove that u admits second order weak partial derivatives, precisely that $u \in W_{\mathrm{loc}}^{2,2}(\Omega, \mathbb{R}^m)$ and satisfies the second variation

$$\int_{\Omega} \left(\sum_{i,j,\alpha,\beta} f_{\xi_i^{\alpha} \xi_j^{\beta}}(x, Du)\, \varphi_{x_i}^{\alpha} u_{x_j x_k}^{\beta} + \sum_{i,\alpha} f_{\xi_i^{\alpha} x_k}(x, Du)\, \varphi_{x_i}^{\alpha} \right) dx = 0, \tag{3.3.9}$$

for every $k = 1, \ldots, n$ and for every $\varphi = (\varphi^{\alpha}) \in W_0^{1,2}(\Omega, \mathbb{R}^m)$.

Let $R > 0$ and $\eta \in C_0^1(B_R)$. Fixed a positive integer $k \leq n$ we consider a test function $\varphi = (\varphi)_{\alpha=1,\ldots,n}$ with components defined by

$$\varphi^{\alpha} = \eta^2 u_{x_k}^{\alpha} \Phi(|Du|),$$

where $\Phi : [0, +\infty) \mapsto [0, +\infty)$ is an increasing bounded Lipschitz continuous function, such that there exists a constant $c_{\Phi} \geq 0$ such that

$$\Phi'(t) t \leq c_{\Phi} (1 + \Phi(t)) \tag{3.3.10}$$

for every $t \geq 1$ and such that $\Phi(t) = 0$ if $t \in [0, 1]$. Then, for the partial derivatives of φ, we have

$$\varphi_{x_i}^{\alpha} = 2\eta \eta_{x_i} u_{x_k}^{\alpha} \Phi(|Du|) + \eta^2 u_{x_k x_i}^{\alpha} \Phi(|Du|) + \eta^2 u_{x_k}^{\alpha} \Phi'(|Du|)\, (|Du|)_{x_i}.$$

From (3.3.9), we obtain

$$\begin{aligned} 0 = &\int_{\tilde{B}_R} 2\eta \Phi(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^{\alpha} \xi_j^{\beta}}(x, Du)\, \eta_{x_i} u_{x_j x_k}^{\beta} u_{x_k}^{\alpha}\, dx \\ &+ \int_{\tilde{B}_R} \eta^2 \Phi(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^{\alpha} \xi_j^{\beta}}(x, Du)\, u_{x_j x_k}^{\beta} u_{x_i x_k}^{\alpha}\, dx \\ &+ \int_{\tilde{B}_R} \eta^2 \Phi'(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^{\alpha} \xi_j^{\beta}}(x, Du)\, u_{x_k}^{\alpha} u_{x_j x_k}^{\beta} (|Du|)_{x_i}\, dx \\ &+ \int_{\tilde{B}_R} 2\eta \Phi(|Du|) \sum_{i,\alpha} f_{\xi_i^{\alpha} x_k}(x, Du)\, \eta_{x_i} u_{x_k}^{\alpha}\, dx \\ &+ \int_{\tilde{B}_R} \eta^2 \Phi(|Du|) \sum_{i,\alpha} f_{\xi_i^{\alpha} x_k}(x, Du)\, u_{x_i x_k}^{\alpha}\, dx \\ &+ \int_{\tilde{B}_R} \eta^2 \Phi'(|Du|) \sum_{i,\alpha} f_{\xi_i^{\alpha} x_k}(x, Du)\, u_{x_k}^{\alpha} (|Du|)_{x_i}\, dx \end{aligned} \tag{3.3.11}$$

$$=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

We start estimating I_1 in (3.3.11) with the Cauchy-Schwarz inequality and Young inequality $2ab \le \frac{1}{2}a^2 + 2b^2$:

$$|I_1| \le \int_{\tilde{B}_R} 2\Phi(|Du|) \left[\eta^2 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, u^\beta_{x_j x_k} u^\alpha_{x_i x_k}\right]^{1/2} \tag{3.3.12}$$

$$\cdot \left[\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, \eta_{x_i}\eta_{x_j} u^\alpha_{x_k} u^\beta_{x_k}\right]^{1/2} dx$$

$$\le \int_{\tilde{B}_R} \Phi(|Du|) \left[\frac{1}{2}\eta^2 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, u^\beta_{x_j x_k} u^\alpha_{x_i x_k}\right.$$

$$\left. +2 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} \eta_{x_i}\eta_{x_j} u^\alpha_{x_k} u^\beta_{x_k}\right] dx.$$

From (3.3.11) and (3.3.12) we obtain

$$\frac{1}{2} I_2 + I_3 + I_4 + I_5 + I_6 \le 2 \int_{\tilde{B}_R} \Phi(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} \eta_{x_i}\eta_{x_j} u^\alpha_{x_k} u^\beta_{x_k}\, dx. \tag{3.3.13}$$

We use the expression of the second derivatives of f to estimate I_3. Since

$$(|Du|)_{x_i} = \frac{1}{|Du|} \sum_{\alpha,k} u^\alpha_{x_i x_k} u^\alpha_{x_k}, \tag{3.3.14}$$

it is natural to sum over k and we observe that

$$\sum_k \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, u^\alpha_{x_k} u^\beta_{x_j x_k} (|Du|)_{x_i} \tag{3.3.15}$$

$$= \left(\frac{g_{tt}(x, |Du|)}{|Du|^2} - \frac{g_t(x, |Du|)}{|Du|^3}\right) \sum_{i,j,k,\alpha,\beta} u^\alpha_{x_i} u^\beta_{x_j} u^\beta_{x_j x_k} u^\alpha_{x_k} (|Du|)_{x_i}$$

$$+ \frac{g_t(x, |Du|)}{|Du|} \sum_{i,k,\alpha} u^\alpha_{x_i x_k} u^\alpha_{x_k} (|Du|)_{x_i}$$

$$= \left(\frac{g_{tt}(x, |Du|)}{|Du|} - \frac{g_t(x, |Du|)}{|Du|^2}\right) \sum_{i,k,\alpha} u^\alpha_{x_i} (|Du|)_{x_i} u^\alpha_{x_k} (|Du|)_{x_k}$$

$$+ g_t(x, |Du|) \sum_i (|Du|)^2_{x_i}$$

$$= \left(\frac{g_{tt}(x,|Du|)}{|Du|} - \frac{g_t(x,|Du|)}{|Du|^2} \right) \sum_\alpha \left[\sum_i u^\alpha_{x_i} |Du|_{x_i} \right]^2$$
$$+ g_t(x,|Du|) |D(|Du|)|^2 .$$

Now, if we denote with $\tilde{I}_s$ the sum over k of I_s, for $s = 1, \ldots, 6$, we have that

$$\tilde{I}_3 = \int_{\tilde{B}_R} \eta^2 \Phi'(|Du|) \left[\left(\frac{g_{tt}(x,|Du|)}{|Du|} - \frac{g_t(x,|Du|)}{|Du|^2} \right) \right. \tag{3.3.16}$$
$$\left. \cdot \sum_\alpha \left[\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right]^2 + g_t(x,|Du|) |D(|Du|)|^2 \right] dx$$
$$= \int_{\tilde{B}_R} \eta^2 \Phi'(|Du|) \left[\frac{g_{tt}(x,|Du|)}{|Du|} \sum_\alpha \left(\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right)^2 + g_t(x,|Du|) |D(|Du|)|^2 \right.$$
$$\left. - \frac{g_t(x,|Du|)}{|Du|^2} \sum_\alpha \left(\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right)^2 \right] dx.$$

Since, by Cauchy-Schwarz inequality we get

$$\sum_\alpha \left(\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right)^2 \leq \sum_{i,\alpha} \left(u^\alpha_{x_i} \right)^2 \sum_i (|Du|)^2_{x_i} \leq |Du|^2 |D(|Du|)|^2 , \tag{3.3.17}$$

then we can conclude that

$$\tilde{I}_3 \geq \int_{\tilde{B}_R} \eta^2 \Phi'(|Du|) \frac{g_{tt}(x,|Du|)}{|Du|} \sum_\alpha \left(\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right)^2 dx \geq 0. \tag{3.3.18}$$

Now, we consider the term $\frac{1}{2} I_2$ in inequality (3.3.13). From the ellipticity condition (3.3.6)

$$|\tilde{I}_2| \geq \int_{\tilde{B}_R} \eta^2 \Phi(|Du|) \mathcal{H}_m(x,|Du|) \left| D^2 u \right|^2 dx. \tag{3.3.19}$$

By using (3.3.18), (3.3.19) and by summing over k in formula (3.3.13), we obtain

$$\frac{1}{2} \int_{\tilde{B}_R} \eta^2 \Phi(|Du|) \mathcal{H}_m(x,|Du|) \left| D^2 u \right|^2 dx \tag{3.3.20}$$
$$\leq |\tilde{I}_4| + |\tilde{I}_5| + |\tilde{I}_6| + 2 \int_{\tilde{B}_R} \Phi(|Du|) \sum_{i,j,k,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j} \eta_{x_i} \eta_{x_j} u^\alpha_{x_k} u^\beta_{x_k} dx.$$

Consider now $\tilde{I}_4$. Since

$$|f_{\xi_i x_k}(x,\xi)| \leq |g_{t x_k}(x,|\xi|)| \tag{3.3.21}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^{m\times n}$, then, by the third assumption of (3.0.2),

$$\begin{aligned} |\tilde{I}_4| &= \left| \int_{\tilde{B}_R} 2\eta\Phi(|Du|) \sum_{i,k,\alpha} f_{\xi_i^\alpha x_k}(x,|Du|)\, \eta_{x_i} u^\alpha_{x_k}\, dx \right| \\ &\leq \int_{\tilde{B}_R} 2\eta\,|D\eta|\,\Phi(|Du|) \sum_i |g_{tx_k}(x,|Du|)|\,|Du|\, dx \\ &\leq c\int_{\tilde{B}_R} 2\eta\,|D\eta|\,\Phi(|Du|)\,\mathcal{H}_m^\vartheta(x,|Du|)\,|Du|^{1+\vartheta}\, dx \\ &\leq c\int_{\tilde{B}_R} 2\eta\,|D\eta|\,\Phi(|Du|)\,\mathcal{H}_M^\vartheta(x,|Du|)\,|Du|^{1+\vartheta}\, dx. \end{aligned} \tag{3.3.22}$$

By Young inequality and again the third condition of (3.0.2), we obtain

$$\begin{aligned} |\tilde{I}_5| &= \left| \int_{\tilde{B}_R} \eta^2\Phi(|Du|) \sum_{i,k,\alpha} f_{\xi_i^\alpha x_k}(x,|Du|)\, u^\alpha_{x_i x_k}\, dx \right| \\ &\leq c\int_{\tilde{B}_R} \eta^2\Phi(|Du|) \sum_k |g_{tx_k}(x,|Du|)|\,\left|D^2u\right| dx \\ &\leq c\int_{\tilde{B}_R} \eta^2\Phi(|Du|)\,\mathcal{H}_m^\vartheta(x,|Du|)\,|Du|^\vartheta \left|D^2u\right| dx \\ &\leq c\int_{\tilde{B}_R} \eta^2\Phi(|Du|)\left(\mathcal{H}_m(x,|Du|)\left|D^2u\right|^2\right)^{1/2}\left(\mathcal{H}_m^{2\vartheta-1}(x,|Du|)\,|Du|^{2\vartheta}\right)^{1/2} dx \\ &\leq c\varepsilon\int_{\tilde{B}_R} \eta^2\Phi(|Du|)\,\mathcal{H}_m(x,|Du|)\left|D^2u\right|^2 dx \\ &\quad + \frac{c}{4\varepsilon}\int_{\tilde{B}_R} \eta^2\Phi(|Du|)\,\mathcal{H}_M^{2\vartheta-1}(x,|Du|)\,|Du|^{2\vartheta}\, dx. \end{aligned} \tag{3.3.23}$$

We choose ε sufficiently small to absorb the first integral in the last inequality of formula (3.3.23) in the left hand side of (3.3.20). Similarly

$$|\tilde{I}_6| = \left| \int_{\tilde{B}_R} \eta^2\Phi'(|Du|) \sum_{i,k,\alpha} f_{\xi_i^\alpha x_k}(x,|Du|)\, u^\alpha_{x_k}\,(|Du|)_{x_i}\, dx \right| \tag{3.3.24}$$

$$\leq c \int_{\tilde{B}_R} \eta^2 \Phi'(|Du|) \sum_{i,k} |g_{tx_k}(x,|Du|)|\, |Du|\, (|Du|)_{x_i}\; dx$$

$$\leq c \int_{\tilde{B}_R} \eta^2 \Phi'(|Du|)\, |Du| \left(\mathcal{H}_m(x,|Du|) \sum_i (|Du|)_{x_i}^2 \right)^{1/2}$$

$$\cdot \left(\mathcal{H}_m^{2\vartheta-1}(x,|Du|)\, |Du|^{2\vartheta} \right)^{1/2} dx$$

$$\leq c\varepsilon \int_{\tilde{B}_R} \eta^2 \Phi'(|Du|)\, |Du|\, \mathcal{H}_m(x,|Du|) \sum_i (|Du|)_{x_i}^2\; dx$$

$$+\frac{c}{4\varepsilon} \int_{\tilde{B}_R} \eta^2 \Phi'(|Du|)\, |Du|\, \mathcal{H}_m^{2\vartheta-1}(x,|Du|)\, |Du|^{2\vartheta}\; dx.$$

$$\leq c_\Phi \varepsilon \int_{\tilde{B}_R} \eta^2 (1+\Phi(|Du|))\, \mathcal{H}_m(x,|Du|) \left|D^2u\right|^2 dx$$

$$+\frac{c_\Phi}{4\varepsilon} \int_{\tilde{B}_R} \eta^2 (1+\Phi(|Du|))\, \mathcal{H}_M^{2\vartheta-1}(x,|Du|)\, |Du|^{2\vartheta}\; dx$$

where in the last inequality we have used (3.3.10) and (3.3.17).

We add in both sides of (3.3.20) the quantity

$$\int_{\tilde{B}_R} \eta^2 \mathcal{H}_m(x,|Du|) \left|D^2u\right|^2 dx$$

and we get

$$\int_{\tilde{B}_R} \eta^2 (1+\Phi(|Du|))\, \mathcal{H}_m(x,|Du|) \left|D^2u\right|^2 dx \tag{3.3.25}$$

$$\leq c \int_{\tilde{B}_R} 2\eta\, |D\eta|\, \Phi(|Du|)\, \mathcal{H}_M^{\vartheta}(x,|Du|)\, |Du|^{1+\vartheta}\; dx$$

$$+c \int_{\tilde{B}_R} \eta^2 \Phi(|Du|)\, \mathcal{H}_M^{2\vartheta-1}(x,|Du|)\, |Du|^{2\vartheta}\; dx$$

$$+c_\Phi \varepsilon \int_{\tilde{B}_R} \eta^2 (1+\Phi(|Du|))\, \mathcal{H}_m(x,|Du|) \left|D^2u\right|^2 dx$$

$$+\frac{c_\Phi}{4\varepsilon} \int_{\tilde{B}_R} \eta^2 (1+\Phi(|Du|))\, \mathcal{H}_M^{2\vartheta-1}(x,|Du|)\, |Du|^{2\vartheta}\; dx.$$

$$+c \int_{\tilde{B}_R} \Phi(|Du|) \sum_{i,j,k,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} \eta_{x_i} \eta_{x_j} u_{x_k}^\alpha u_{x_k}^\beta\, dx.$$

Then, choosing ε sufficiently small and using the ellipticity condition (3.3.6) in (3.3.25) and by also using the Cauchy-Schwarz inequality, we get

$$\int_{\tilde{B}_R} \eta^2 \Phi\left(|Du|\right) \mathcal{H}_m\left(x, |Du|\right) \left|D\left(|Du|\right)\right|^2 \, dx \tag{3.3.26}$$

$$\leq c \int_{\tilde{B}_R} 2\eta \left|D\eta\right| \Phi\left(|Du|\right) \mathcal{H}_M^{\vartheta}\left(x, |Du|\right) |Du|^{1+\vartheta} \, dx$$

$$+ c(1 + c_\Phi) \int_{\tilde{B}_R} \eta^2 \left(1 + \Phi\left(|Du|\right)\right) \mathcal{H}_M^{2\vartheta-1}\left(x, |Du|\right) |Du|^{2\vartheta} \, dx$$

$$+ c \int_{\tilde{B}_R} |D\eta|^2 \Phi\left(|Du|\right) \mathcal{H}_M^{2\vartheta-1}\left(x, |Du|\right) |Du|^2 \, dx.$$

Then, since $|Du| \geq 1$ in $\tilde{B}_R$, (3.3.26) becomes

$$\int\limits_{\tilde{B}_R} \eta^2 \Phi\left(|Du|\right) \mathcal{H}_m\left(x, |Du|\right) \left|D\left(|Du|\right)\right|^2 \, dx \tag{3.3.27}$$

$$\leq c\left(1 + c_\Phi\right) \int_{\tilde{B}_R} \left(\eta^2 + |D\eta|^2\right) \left(1 + \Phi\left(|Du|\right)\right) \left(\mathcal{H}_M^{\vartheta}\left(x, |Du|\right)\right.$$

$$\left. + \mathcal{H}_M^{2\vartheta-1}\left(x, |Du|\right)\right) |Du|^{2\vartheta} \, dx.$$

Now we deal with the problem of a non bounded and non globally Lipschitz continuous Φ. We can approximate Φ with a sequence of functions Φ_r, each of them equal to Φ in the interval $[0, r]$ and extended continuously to $[r, +\infty)$ with the constant value $\Phi(r)$. We insert Φ_r in (3.3.27) and we pass to the limit as $r \to \infty$ by the monotone convergence theorem. So we obtain that (3.3.27) is true for every Φ positive, increasing and locally Lipschitz continuous function in $[0, +\infty)$.

We apply (3.0.2) and we find that the function $\mathcal{K}_M$ defined in (3.2.11) satisfies the condition

$$\mathcal{H}_M(x, t) \leq M_\vartheta \mathcal{K}_M^{\vartheta}(t)$$

for every $t \geq 1$ and for almost every $x \in B_R$. Similarly, the function $\mathcal{K}_m$ defined in (3.2.12) satisfies

$$\mathcal{H}_m(x, t) \geq m \mathcal{K}_m(t)$$

for every $t \geq 1$ and for almost every $x \in B_R$. We define

$$G(t) = 1 + \int_0^t \sqrt{\Phi(s) \mathcal{K}_m(s)} \, ds.$$

By the Hölder inequality, since Φ is increasing and $h'(0) = 0$, we get the following inequalities concerning G and its derivatives:

$$G^2(t) = \left(1 + \int_0^t \sqrt{\Phi(s)\mathcal{K}_m(s)}\, ds\right)^2 \le 2 + 2\Phi(t)t \int_0^t h''(s)\, ds \le 2 + 2\Phi(t)th'(t)$$
$$\le 2\left(1 + \Phi(t)\mathcal{K}_M(t)t^2\right)$$

and

$$|D(\eta G(|Du|))|^2 \le 2|D\eta|^2 |G(|Du|)|^2 + 2\eta^2 |G'(|Du|)|^2 |D(|Du|)|^2 .$$

We get

$$\int_{\tilde{B}_R} |D(\eta G(|Du|))|^2\, dx \tag{3.3.28}$$
$$\le c(1 + c_\Phi) \int_{\tilde{B}_R} \left(\eta^2 + |D\eta|^2\right)(1 + \Phi(|Du|))\left(\mathcal{K}_M^{\vartheta^2}(x, |Du|)\right.$$
$$\left.+\mathcal{K}_M^{(2\vartheta-1)\vartheta}(|Du|)\right)|Du|^{2\vartheta}\, dx \;+\; c_2 \int_{\tilde{B}_R} |D\eta|^2 \left(1 + \Phi(|Du|)\mathcal{K}_M(|Du|)|Du|^2\right) dx.$$

We recall that ϑ satisfies the following conditions

$$1 \le (2\vartheta - 1)\vartheta < (1 - \beta)\frac{2^*}{2}$$

and we introduce the notation

$$\tau = (2\vartheta - 1)\vartheta. \tag{3.3.29}$$

Therefore $1 \le \tau < (1-\beta)\frac{2^*}{2}$ and $\tau \ge \vartheta^2$. Then, (3.3.28) leads to

$$\int_{\tilde{B}_R} |D(\eta G(|Du|))|^2\, dx \tag{3.3.30}$$
$$\le c(1 + c_\Phi) \int_{\tilde{B}_R} \left(\eta^2 + |D\eta|^2\right)\left[1 + \Phi(|Du|)\mathcal{K}_M^\tau(|Du|)|Du|^{2\tau}\right] dx.$$

We apply the Sobolev inequality to get

$$\left[\int_{\tilde{B}_R} |\eta G(|Du|)|^{2^*}\, dx\right]^{2/2^*} \tag{3.3.31}$$
$$\le c(1 + c_\Phi) \int_{\tilde{B}_R} \left(\eta^2 + |D\eta|^2\right)\left(1 + \Phi(|Du|)\mathcal{K}_M^\tau(|Du|)|Du|^{2\tau}\right) dx.$$

We choose Φ equal to the function defined in (3.2.1) if $\gamma \ge 1$. Then, since $t^{\gamma-2} \le (t-1)^{\gamma-2}$ for every $\gamma \in [0, 2]$ and $t \ge 1$, by Lemmata 3.2.1 and 3.2.2, (3.3.31) becomes

$$\left[\int_{\tilde{B}_R} |\eta G(|Du|)|^{2^*} \, dx\right]^{2/2^*}$$
$$\leq c(1+\gamma)\int_{\tilde{B}_R} \left(\eta^2 + |D\eta|^2\right)\left(1 + (|Du|-1)^{\gamma} \mathcal{K}_M^{\tau}(|Du|)\,|Du|^{2\tau}\right) dx.$$

Since $\mathcal{K}_m$ satisfies (3.2.21), we get

$$\left[\int_{\tilde{B}_R} \eta^{2^*}\left(1 + (|Du|-1)^{\frac{2^*}{2}(\gamma+2)}\,|Du|^{-2^*\beta}\,\mathcal{K}_M^{\tau}(|Du|)\right) dx\right]^{2/2^*} \tag{3.3.32}$$
$$\leq c(\gamma+1)^3\int_{\tilde{B}_R} \left(\eta^2 + |D\eta|^2\right)\left(1 + (|Du|-1)^{\gamma} \mathcal{K}_M^{\tau}(|Du|)\,|Du|^{2\tau}\right) dx.$$

Now, since

$$\begin{aligned} &1 + (t-1)^{\frac{2^*}{2}(\gamma+2)}\, t^{-2^*\beta}\mathcal{K}_M^{\tau}(t) \\ &= 1 + (t-1)^{\frac{2^*}{2}(\gamma+2)-(2\tau+2^*\beta)}\,(t-1)^{2\tau+2^*\beta}\, t^{-2^*\beta}\mathcal{K}_M^{\tau}(t) \\ &\geq C\left(1 + (t-1)^{\frac{2^*}{2}(\gamma+2)-(2\tau+2^*\beta)}\, t^{2\tau}\mathcal{K}_M^{\tau}(t)\right), \end{aligned}$$

with C not depending on γ, (3.3.32) becomes

$$\left[\int_{\tilde{B}_R} \eta^{2^*}\left(1 + (|Du|-1)^{\frac{2^*}{2}(\gamma+2)-(2\tau+2^*\beta)}\,\mathcal{K}_M^{\tau}(|Du|)\,|Du|^{2\tau}\right) dx\right]^{2/2^*} \tag{3.3.33}$$
$$\leq c(\gamma+1)^3\int_{\tilde{B}_R} \left(\eta^2 + |D\eta|^2\right)\left(1 + (|Du|-1)^{\gamma} \mathcal{K}_M^{\tau}(|Du|)\,|Du|^{2\tau}\right) dx.$$

Let us fix $0 < \rho < R$ and take $\eta \equiv 1$ in $\tilde{B}_\rho$ and $|D\eta| \leq \frac{2}{R-\rho}$. Then, fixed $\overline{\rho} < \overline{R}$, let us also define the decreasing sequence of radii $\{\rho_i\}$, defined by

$$\rho_i = \overline{\rho} + \frac{\overline{R} - \overline{\rho}}{2^i},$$

for every $i \geq 0$. We define the sequence $\{\gamma_i\}$ defined by the recurrence $\gamma_0 = 0$,

$$\gamma_{i+1} = \frac{2^*}{2}(\gamma_i + 2) - (2\tau + 2^*\beta),$$

which is non decreasing by the properties of β and τ. Then for every $i \geq 0$

$$\left[\int_{\tilde{B}_{\rho_{i+1}}} \left(1 + (|Du|-1)^{\gamma_{i+1}} \mathcal{K}_M^{\tau}(|Du|)\,|Du|^{2\tau}\right) dx\right]^{2/2^*} \tag{3.3.34}$$

$$\leq c(\gamma_i+1)^3\left(\frac{2^{i+1}}{\overline{R}-\overline{\rho}}\right)^2\int_{\tilde{B}_{\rho_i}}\left(1+(|Du|-1)^{\gamma_i}\,\mathcal{K}_M^{\tau}\left(|Du|\right)|Du|^{2\tau}\right)\,dx.$$

By iterating (3.3.34), we get

$$\left[\int_{\tilde{B}_{\rho_{i+1}}}\left(1+(|Du|-1)^{k_i(2^*/2)^{i+1}}\,\mathcal{K}_M^{\tau}\left(|Du|\right)|Du|^{2\tau}\right)\,dx\right]^{(2/2^*)^{i+1}} \tag{3.3.35}$$

$$\leq C\int_{\tilde{B}_{\overline{R}}}\left(1+|Du|^{2\tau}\,\mathcal{K}_M^{\tau}\left(|Du|\right)\right)\,dx$$

where we have denoted with $k_i=\left(1-\beta-\frac{2}{2^*}\tau\right)\left(1-\frac{1}{(2^*/2)^{i+1}}\right)n$. The exponent in the first integral is given by computing

$$\gamma_{i+1}=\gamma_0\left(\frac{2^*}{2}\right)^{i+1}-2\left(\beta+\frac{2}{2^*}\tau-1\right)\sum_{k=1}^{i+1}\left(\frac{2^*}{2}\right)^k$$

$$=\left(1-\beta-\frac{2}{2^*}\tau\right)\left[\left(\frac{2^*}{2}\right)^{i+1}-1\right]n.$$

Observe that the quantity $1-\beta-\frac{2}{2^*}\tau>0$ by the restrictions on τ. The constant C in (3.3.35) is such that

$$C\leq\prod_{k=0}^{\infty}\left(\frac{c(2^*)^{3k}}{\overline{R}-\overline{\rho}}\right)^{\left(\frac{2}{2^*}\right)^k}=\left(\frac{c}{\left(\overline{R}-\overline{\rho}\right)^2}\right)^{\sum_{k=0}^{\infty}\left(\frac{2}{2^*}\right)^k}\cdot(2^*)^{\sum_{k=0}^{\infty}k\left(\frac{2}{2^*}\right)^k}$$

$$=\frac{c}{\left(\overline{R}-\overline{\rho}\right)^n},$$

for every $n\geq 3$; otherwise, if $n=2$, then for every $\varepsilon>0$ we can choose $2^*>2$ so that $C=\frac{C}{(\overline{R}-\overline{\rho})^{2+\varepsilon}}$.

We observe that the function $1+(t-1)^{\alpha}\mathcal{K}_M^{\tau}(t)t^{2\tau}\geq 1+(t-1)^{\alpha}t^{\tau}\left(h'(t)\right)^{\tau}$, since $\mathcal{K}_M(t)\geq\frac{h'(t)}{t}$ for every $t\geq 1$. Moreover, since h' is increasing and $\tau\geq 1$ we have that $1+(t-1)^{\alpha}t^{\tau}\left(h'(t)\right)^{\tau}\geq 1+(t-1)^{\alpha+\tau}\left(h'(1)\right)^{\tau}$. Then, for every $i\geq 0$ we have

$$\left[\int_{\tilde{B}_{\rho_{i+1}}}(|Du|-1)^{\left(k_i+\frac{\tau}{(2^*/2)^{i+1}}\right)(2^*/2)^{i+1}}\,dx\right]^{(2/2^*)^{i+1}} \tag{3.3.36}$$

$$\leq C\int_{\tilde{B}_{\overline{R}}}\left(1+|Du|^{2\tau}\,\mathcal{K}_M^{\tau}\left(|Du|\right)\right)\,dx.$$

Finally we pass to the limit as $i \to \infty$ and we obtain

$$\sup\left\{(|Du(x)|-1)^{\left(1-\beta-\frac{2}{2^*}\tau\right)n} \,:\, x \in \tilde{B}_{\bar{\rho}}\right\}$$

$$= \lim_{i\to\infty}\left[\int_{\tilde{B}_{\rho_{i+1}}} (|Du|-1)^{\left(k_i+\frac{\tau}{(2^*/2)^{i+1}}\right)(2^*/2)^{i+1}}\, dx\right]^{(2/2^*)^{i+1}}$$

$$\le \frac{C}{(\bar{R}-\bar{\rho})^n}\int_{\tilde{B}_{\bar{R}}} \left(1+|Du|^{2\tau}\mathcal{K}_M^\tau(|Du|)\right) dx.$$

■

Lemma 3.3.2 *Let g, h respectively satisfy (3.0.2) and (3.0.3). Suppose that the supplementary condition (3.3.7) is satisfied. Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ be a local minimizer of (3.0.1). Then, for every $\varepsilon > 0$ and for every ρ, R $(0 < \rho < R)$ there exists a constant C such that*

$$\int_{\tilde{B}_\rho} \left(1+|Du|^{2\tau}\mathcal{K}_M^\tau(|Du|)\right) dx \le C\left[\int_{\tilde{B}_R} (1+h(|Du|))\, dx\right]^{\frac{\tau}{1-\beta}+\varepsilon}. \qquad (3.3.37)$$

The constant C depends on $n, \varepsilon, \vartheta, \rho, R, \beta, \alpha$ and $\sup\{h''(t) :\ t \in [0,1]\}$.

Proof. In Lemma 3.2.8 we have considered parameters α and γ such that $\alpha \in \left(1, \frac{2n}{2n-1}\right]$ and $\gamma \ge 0$. Here we restrict ourselves to the case $1 < \alpha \le \frac{2n\tau}{n(1+\tau)-1}$ and $\gamma = 0$. Then, Lemma 3.2.8 holds for any $\nu \in \left[1, \frac{2^*(2-\alpha)}{2\alpha}\right]$. Since $\tau < \frac{2^*}{2}(1-\beta)$, we have that $1 < (1-\beta)\frac{2^*}{2\tau}$, therefore it is possible to limit ν to satisfy the condition $1 < \nu < (1-\beta)\frac{2^*}{2\tau}$. Finally, since $\beta > \frac{1}{n}$, we have $\alpha \le \frac{2n\tau}{n(1+\tau)-1} < \frac{2\tau}{1-\beta+\tau}$ which implies $1-\beta < \frac{2-\alpha}{\alpha}\tau$. Thus,

$$\nu \in \left[1, (1-\beta)\frac{2^*}{2\tau}\right] \subseteq \left[1, 2^*\frac{2-\alpha}{2\alpha}\right]$$

whence we obtain that the conditions of Lemma 3.2.8 are satisfied. Therefore there exists a constant c such that

$$[G(t)]^{2^*} = \left[\left(1+\int_0^t \sqrt{\Phi(s)\mathcal{K}_m(s)}\, ds\right)^{\frac{2^*}{\nu}}\right]^\nu \ge c\left(1+\left((t-1)t^{-\beta}\right)^{\frac{2^*}{\nu}}\mathcal{K}_M^\tau(t)\right)^\nu.$$

Under the notations of lemma 3.3.1, formula (3.3.33) with $\gamma = 0$ becomes

$$\left[\int_{\tilde{B}_R} \eta^{2^*}\left(1+(|Du|-1)^{(2^*-(2\tau+2^*\beta))\frac{1}{\nu}}\mathcal{K}_M^\tau(|Du|)\,|Du|^{\frac{2\tau}{\nu}}\right)^\nu dx\right]^{\frac{2}{2^*}} \qquad (3.3.38)$$

$$\leq c \int_{\tilde{B}_R} \left(\eta^2 + |D\eta|^2\right) \left(1 + \mathcal{K}_M^\tau(|Du|)\,|Du|^{2\tau}\right)\, dx.$$

Moreover, since there exists a constant C_1 such that

$$\left[\int_{\tilde{B}_R} \eta^{2^*} \left(1 + (|Du|-1)^{(2^*-(2\tau+2^*\beta))\frac{1}{\nu}}\, \mathcal{K}_M^\tau(|Du|)\,|Du|^{\frac{2\tau}{\nu}}\right)^\nu dx\right]^{\frac{2}{2^*}}$$

$$\geq C_1 \left[\int_{\tilde{B}_R} \eta^{2^*} \left(1 + |Du|^{2^*\frac{1-\beta}{\nu}}\, \mathcal{K}_M^\tau(|Du|)\right)^\nu dx\right]^{\frac{2}{2^*}},$$

(3.3.38) gives

$$\left[\int_{\tilde{B}_R} \eta^{2^*} \left(1 + |Du|^{2^*\frac{1-\beta}{\nu}}\, \mathcal{K}_M^\tau(|Du|)\right)^\nu dx\right]^{\frac{2}{2^*}} \tag{3.3.39}$$

$$\leq c \int_{\tilde{B}_R} \left(\eta^2 + |D\eta|^2\right) \left(1 + \mathcal{K}_M^\tau(|Du|)\,|Du|^{2\tau}\right)\, dx.$$

Since $\nu < (1-\beta)\frac{2^*}{2\tau}$, we have $2^*\frac{1-\beta}{\nu} > 2\tau$ and then, if we define $V = V(x) = 1 + |Du|^{2\tau}\, \mathcal{K}_M^\tau(|Du|)$, (3.3.39) becomes

$$\left[\int_{\tilde{B}_R} \eta^{2^*} V^\nu\, dx\right]^{\frac{2}{2^*}} \leq c \int_{\tilde{B}_R} \left(\eta^2 + |D\eta|^2\right) V\, dx. \tag{3.3.40}$$

As in the previous Lemma 3.3.1, we consider a test function η equal to 1 in $\tilde{B}_\rho$ with $|D\eta| \leq \frac{2}{R-\rho}$ and we obtain

$$\left(\int_{\tilde{B}_\rho} V^\nu\, dx\right)^{\frac{2}{2^*}} \leq \frac{c}{(R-\rho)^2} \int_{\tilde{B}_R} V\, dx. \tag{3.3.41}$$

Let $\mu > \frac{2^*}{2}$, then by the Hölder inequality we have

$$\left(\int_{\tilde{B}_\rho} V^\nu\, dx\right)^{\frac{2}{2^*}} \leq \frac{c}{(R-\rho)^2} \int_{\tilde{B}_R} V^{\frac{\nu}{\mu}} V^{1-\frac{\nu}{\mu}}\, dx \tag{3.3.42}$$

$$\leq \left(\int_{\tilde{B}_R} V^\nu\, dx\right)^{\frac{1}{\mu}} \left(\int_{\tilde{B}_R} V^{\frac{\mu-\nu}{\mu-1}}\, dx\right)^{\frac{\mu-1}{\mu}}.$$

Let R_0 and ρ_0 be fixed. For any $i \in \mathbb{N}$ we consider (3.3.42) with $R = \rho_i$ and $\rho = \rho_{i-1}$, where $\rho_i = R_0 - \frac{R_0-\rho_0}{2^i}$. By iterating (3.3.42), since $R - \rho = \frac{R_0-\rho_0}{2^i}$, similarly to the computation in [46] and [51] we can write

$$\int_{\tilde{B}_{\rho_0}} V^\nu\, dx \leq c \left(\frac{1}{(R_0-\rho_0)^2}\right)^{\left(\frac{2^*\mu}{2\mu-2}\right)^i} \left(\int_{\tilde{B}_{\rho_i}} V^\nu\, dx\right)^{\left(\frac{2^*}{2\mu}\right)^i} \tag{3.3.43}$$

$$\cdot \left(\int_{\tilde{B}_{\rho_0}} V^{\frac{\mu-\nu}{\mu-1}}\, dx \right)^{\frac{2^*(\mu-1)}{2\mu-2^*}} .$$

Since $\frac{\mu-\nu}{\mu-1} < 1$ we can apply Lemma 3.2.10 and obtain

$$\int_{\tilde{B}_{\rho_0}} V^{\nu}\, dx \leq c \left(\frac{1}{(R_0-\rho_0)^2} \right)^{\frac{2^*\mu}{2\mu-2^*}} \left(\int_{B_{\rho_i}} V^{\frac{1}{\sigma}}\, dx \right)^{\left(\frac{2^*}{2\mu}\right)^i}$$

$$\cdot \left(\int_{\tilde{B}_{\rho_0}} [1+h(|Du|)]\, dx \right)^{\frac{2^*(\mu-1)}{2\mu-2^*}} .$$

In the limit as $i \to \infty$ we get

$$\int_{\tilde{B}_{\rho_0}} V^{\nu}\, dx \leq c \left(\frac{1}{(R_0-\rho_0)^2} \right)^{\frac{2^*\mu}{2\mu-2^*}} \left(\int_{\tilde{B}_{\rho_0}} [1+h(|Du|)]\, dx \right)^{\frac{2^*(\mu-1)}{2\mu-2^*}} .$$

Finally,

$$\int_{B_{\rho_0}} V\, dx \leq |\tilde{B}_{\rho_0}|^{1-\frac{1}{\nu}} \left(\int_{\tilde{B}_{\rho_0}} V^{\nu}\, dx \right)^{\frac{1}{\nu}} \tag{3.3.44}$$

$$\leq c \left(\frac{1}{(R_0-\rho_0)^2} \right)^{\frac{2^*\mu}{(2\mu-2^*)\nu}} \left(\int_{\tilde{B}_{R_0}} [1+h(|Du|)]\, dx \right)^{\frac{2^*(\mu-1)}{(2\mu-2^*)\nu}} .$$

As $\nu \to \frac{2^*(1-\beta)}{2\tau}$ and $\mu \to \infty$ the two exponents in (3.3.44) converge to $\frac{\tau}{1-\beta}$ and we have the result. ■

By combining Lemmata 3.3.1, 3.3.2 and by using again (3.0.2), we obtain

$$\|Du\|_{L^\infty\left(\tilde{B}_\rho,\mathbb{R}^{m\times n}\right)}^{\left(1-\beta-\frac{2}{2^*}\tau\right)n} \leq C \int_{\tilde{B}_R} \left(1+|Du|^{2\tau}\, \mathcal{K}_M^{\tau}(|Du|)\right) dx \tag{3.3.45}$$

$$\leq C' \left[\int_{\tilde{B}_R} (1+h(|Du|))\, dx \right]^{\frac{\tau}{1-\beta}+\varepsilon} \leq C' \left[\int_{\tilde{B}_R} (1+g(x,|Du|))\, dx \right]^{\frac{\tau}{1-\beta}+\varepsilon} ,$$

since, by (3.0.2) and the fact that $h(0) = 0 = g(x,0)$, $h(t) = \int_0^t h'(s)\, ds \leq \int_0^t g_t(x,s)\, ds = g(x,t)$. In order to go from $\tilde{B}_\rho, \tilde{B}_R$ to B_ρ, B_R we observe that $\|Du\|_{L^\infty\left(\tilde{B}_\rho,\mathbb{R}^{m\times n}\right)} \leq 1 + \|Du\|_{L^\infty(B_\rho,\mathbb{R}^{m\times n})}$ and from (3.3.45) we also get

$$\|Du\|_{L^\infty(B_\rho,\mathbb{R}^{m\times n})}^{\left(1-\beta-\frac{2}{2^*}\tau\right)n} \leq C'' \left[\int_{B_R} (1+g(x,|Du|))\, dx \right]^{\frac{\tau}{1-\beta}+\varepsilon} .$$

We summarize in the next statement the a-priori estimate that we have proved.

Theorem 3.3.3 *Let g, h respectively satisfy (3.0.2) and (3.0.3). Suppose that the supplementary condition (3.3.7) is satisfied. Let $u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^m)$ be a local minimizer of (3.0.1). Then, for every $\varepsilon > 0$ and for every ρ, R $(0 < \rho < R)$, there exists a constant C such that*

$$\|Du\|_{L^\infty(B_\rho, \mathbb{R}^{m\times n})}^{\left(1-\beta-\frac{2}{2^*}\tau\right)n} \leq C \left[\int_{B_R} (1 + g(x, |Du|))\, dx\right]^{\frac{\tau}{1-\beta}+\varepsilon}, \tag{3.3.46}$$

where $\tau = (2\vartheta-1)\vartheta$. The constant C depends on $n, \varepsilon, \vartheta, \rho, R, t_0, \beta, \alpha$ and $\sup\{h''(t) : t \in [0, t_0]\}$, but is independent of the constants in supplementary condition (3.3.7).

We note that $1-\beta-\frac{2}{2^*}\tau > 0$ since $\tau < (1-\beta)\frac{2^*}{2}$. Note also that $\left(1-\beta-\frac{2}{2^*}\tau\right)n < 1$; in fact, since $\tau \geq 1$ and $\beta > \frac{1}{n}$,

$$\left(1-\beta-\frac{2}{2^*}\tau\right)n < \left(1-\frac{1}{n}-\frac{2}{2^*}\right)n = 1.$$

Moreover $\frac{\tau}{1-\beta}+\varepsilon > 1$ and thus (3.3.46) gives the final representation of the a-priori estimate as stated in (3.0.4), with

$$\omega := \frac{\frac{\tau}{1-\beta}+\varepsilon}{\left(1-\beta-\frac{2}{2^*}\tau\right)n} > 1\,.$$

3.4 Ideas for approximation

In order to get a regularity result for local minimizers of integrals as seen in this chapter, we should apply an approximation argument similar to the one seen in Chapter 2, in particular in sections 2.1.3 and 2.3.3. In this section we show some possible approaches. The first one follows the idea shown in Section 2.3.3, while the other one implies a complete change of strategy also in the proof of the a-priori estimate done in preceding sections, in particular it does not require an additional function h.

The most important problem in the setting within this chapter (in particular conditions (3.0.2) and (3.0.3)) is that we have two functions that need to be approximated. We could pick, for instance, functions h_{ε_n} and g_{ε_n}, in the spirit of (2.3.38), by defining

$$g_t^{\varepsilon_n}(x,t) = \begin{cases} g_t(x,t) & \text{if } 1 \leq t \leq \frac{1}{\varepsilon_n}, \\ \min\left\{\varepsilon_n g_t\left(x, \frac{1}{\varepsilon_n}\right)t, g_t(x,t) + \varepsilon_n t - 1\right\} & \text{if } t > \frac{1}{\varepsilon_n} \end{cases}$$

and

$$h'_{\varepsilon_n}(t) = \begin{cases} h'(t) & \text{if } 1 \le t \le \frac{1}{\varepsilon_n}, \\ \min\left\{\varepsilon_n h'\left(\frac{1}{\varepsilon_n}\right) t,\, h'(t) + \varepsilon_n t - 1\right\} & \text{if } t > \frac{1}{\varepsilon_n}, \end{cases}$$

where ε_n is a vanishing sequence such that $\frac{1}{\varepsilon_n} \ge 1$. Then, we could define

$$g^{\varepsilon_n}(x,t) = \int_0^t g_t^{\varepsilon_n}(x,s)\, ds \tag{3.4.1}$$

and

$$h_{\varepsilon_n}(t) = \int_0^t h'_{\varepsilon_n}(s)\, ds.$$

There are no challenges in proving that conditions (3.0.3) are satisfied also by h_{ε_n} by using Lemmata 2.3.11 and 2.3.12. Moreover, by Lemma 2.3.13 we can prove that

$$h_{\varepsilon_n}(t) \le C\left(1 + h(t)\right) + \varepsilon_n t^2$$

for every $t \ge 1$. First of all, by using lemmata in Section 2.3 (precisely Lemmata 2.3.11, 2.3.12 and 2.3.13), the function h_{ε_n} satisfies condition (3.0.3). The critical issue in this approach relies on the link between h and g. If $1 \le t \le \frac{1}{\varepsilon_n}$ then there are no problems in proving conditions (3.0.2) for g^{ε_n} and h_{ε_n}. On the other hand, when $t > \frac{1}{\varepsilon_n}$ we do not know if g^{ε_n} and h_{ε_n} are *well coupled*, i.e. if both assumes simultaneously the first or the second value inside the minimum. To be more precise, if for instance $g_t^{\varepsilon_n}(x,t) = \varepsilon_n g_t(x, \frac{1}{\varepsilon_n})t$ and $h'_{\varepsilon_n} = \varepsilon_n h'(\frac{1}{\varepsilon_n})t$, then we could prove (3.0.2) by using same hypotheses on g and h. The same argument applies in the other well coupled configuration. If, for example, $g_t^{\varepsilon_n}(x,t) = \varepsilon_n g_t(x, \frac{1}{\varepsilon_n})t$ and $h'_{\varepsilon_n}(t) = h'(t) + \varepsilon_n t - 1$, then it is not possible to prove the second condition of (3.0.2). In fact, in this case we have $g_{tt}^{\varepsilon_n}(x,t) = \varepsilon_n g_t(x, \frac{1}{\varepsilon_n})$ and $h''_{\varepsilon_n}(t) = h''(t) + \varepsilon_n$ and we have not found a way to control h'' with g_t and g_t with a power of h'' yet. We hope that by using conditions (3.0.3) we could manage this issue in the future. Another approach would be to approximate the function g with another sequence, but we have not found the correct one yet.

As mentioned at the beginning of this section, we could change the proof approach by not requiring an additional function h. In particular, we could assume for g a condition similar to (3.0.3): for some β, $\frac{1}{n} < \beta < \frac{2}{n}$, and for every α such that $1 < \alpha \le \frac{n}{n-1}$ there exist constants m and M_α such that

$$\frac{m}{t^{2\beta}}\left[\left(\frac{g_t(x,t)}{t}\right)^{\frac{n-2}{n}} + \frac{g_t(x,t)}{t}\right] \le g_{tt}(x,t) \le M_\alpha\left[\left(\frac{g_t(x,t)}{t}\right)^{\alpha} + \frac{g_t(x,t)}{t}\right]$$

for every $t \ge t_0$ and for almost every $x \in \Omega$. For what concerns the condition on the mixed derivative g_{tx_k}, we could keep the last assumption in (3.0.2), which, we recall, is the following: there exist $\vartheta \geq 1$ and a positive constant M_ϑ such that

$$|g_{tx_k}(x,t)| \leq M_\vartheta \min\{g_t(x,t), tg_{tt}(x,t)\}^\vartheta$$

for every $t \geq t_0$ and for almost every $x \in \Omega$. In this way, the proof of the a-priori estimate works until formula (3.3.30). There, we have estimated the derivative $|D(\eta G(x,|Du|))|$ where, in this new setting,

$$G(x,t) = 1 + \int_0^t \sqrt{\Phi(s)\mathcal{H}_m(x,s)}\, ds.$$

Now,

$$\begin{aligned} &|D(\eta G(x,|Du|))|^2 \\ &\leq c|D\eta|^2 G^2(x,|Du|) + c\eta^2 [G_t(x,|Du|) D(|Du|)]^2 + c\eta^2 [G_x(x,|Du|)]^2 . \end{aligned}$$

We observe that the last term does not appear in the calculations above since the function h depends only on t. Instead, it occurs in the a-priori estimates of [52], mentioned in Section 2.2. In that case the key assumption that $g_t(x,t)/t$ is increasing is made and then it is possible to estimate the quantity $[G_x(x,|Du|)]^2$ from above. Since, as already said in Section 2.1 (precisely in Lemma 2.1.5), $g_t(x,t)/t \leq g_{tt}(x,t)$, in that particular case we have

$$\left[\frac{\partial G}{\partial x_i}(x,t)\right]^2 = \left[\int_0^t \sqrt{\frac{\Phi(s)}{s}} \frac{g_{tx_i}(x,s)}{2\sqrt{g_t(x,s)}}\, ds\right]^2$$

and this could be treated by using the condition on the mixed second derivatives. If we drop the key assumption, then in the same estimate we obtain

$$\left[\frac{\partial G}{\partial x_i}(x,t)\right]^2 = \left[\int_0^t \sqrt{\Phi(s)} \frac{\frac{\partial \mathcal{H}_m}{\partial x_i}(x,s)}{2\sqrt{\mathcal{H}_m(x,s)}}\, ds\right]^2 .$$

Now we do not know which is the value assumed by $\mathcal{H}_m$ and when $\mathcal{H}_m(x,s) = g_{tt}(x,s)$ we have to estimate the third derivative g_{ttx_i} and we need to add some assumptions also on those terms (in addition to some Sobolev properties for the higher derivatives) such as, for instance

$$|g_{ttx_i}(x,t)| \leq M_\vartheta \min\{g_t(x,t), g_{tt}(x,t)t\}^\vartheta ,$$

for some $\vartheta \geq 1$ and $M_\vartheta > 0$. With these conditions we could go on by defining g_{ε_n} as in (3.4.1), by using Lemmata 2.3.11, 2.3.12 and 2.3.13 and then by following the limit procedure seen in Subsection 2.3.3.

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