A HOMOGENEOUS DECOMPOSITION THEOREM FOR VALUATIONS ON CONVEX FUNCTIONS

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ABSTRACT. The existence of a homogeneous decomposition for continuous and epi-translation invariant valuations on super-coercive functions is established. Continuous and epi-translation invariant valuations that are epi-homogeneous of degree n are classified. By duality, corresponding results are obtained for valuations on finite-valued convex functions.

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1. INTRODUCTION

Given a space of real-valued functions X, we consider real-valued valuations on X, that is, functionals $\mathrm{Z}\colon X \to \mathbb{R}$ such that

$$\mathrm{Z}(u \vee v) + \mathrm{Z}(u \wedge v) = \mathrm{Z}(u) + \mathrm{Z}(v) \tag{1}$$

for every $u, v \in X$ with $u \vee v$ and $u \wedge v \in X$, where $\vee$ and $\wedge$ denote the point-wise maximum and minimum, respectively. For X, the space of indicator functions of convex bodies (that is, compact convex sets) in $\mathbb{R}^n$, we obtain the classical notion of valuation on convex bodies. Here strong structure and classification theorems have been established over the last seventy years (see [1, 2, 6, 7, 19–21, 28] for some recent results and [22, 23, 36] for information on the classical theory). The aim of this article is to obtain such results also in the functional setting. In particular, we will establish a homogeneous decomposition result à la McMullen [30].

Valuations on function spaces have only recently started to attract attention. Classification results were obtained for L_p and Sobolev spaces [24–27, 29, 38, 39], spaces of quasi-convex functions [12, 13], of Lipschitz functions [17], of definable functions [4] and on Banach lattices [37]. Spaces of convex functions play a special role because of their close connection to convex bodies. Here classification results were obtained for $\mathrm{SL}(n)$ invariant and for monotone valuations in [8, 14, 15, 32–34] and the connection to valuations on convex bodies was explored by Alesker [3]. While the theory of translation invariant valuations is well developed for convex bodies, for convex functions the corresponding theory did not exist till now. We introduce the notion of *epi-translation invariance* to build such a theory. In particular, we will show that on the space of super-coercive convex functions there is a homogeneous decomposition for continuous and epi-translation invariant valuations and there exist non-trivial such valuations for each degree of epi-homogeneity while on the larger space of coercive convex functions all continuous and epi-translation invariant valuations are constant.

The general space of (extended real-valued) convex functions on $\mathbb{R}^n$ is defined as

$$\mathrm{Conv}(\mathbb{R}^n) = \{u\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \colon u \text{ is convex and lower semicontinuous, } u \not\equiv +\infty\}.$$

It is equipped with the topology induced by epi-convergence (see Section 2.2). Continuity of valuations defined on $\mathrm{Conv}(\mathbb{R}^n)$, or on subsets of $\mathrm{Conv}(\mathbb{R}^n)$, will be always with respect to this topology. The space $\mathrm{Conv}(\mathbb{R}^n)$ is a standard space in convex analysis (see [35]) and important in many applications. As we

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will show, $\operatorname{Conv}(\mathbb{R}^n)$ is too large for our purposes. We will be mainly interested in two of its subspaces. The first is formed by *coercive* functions,

$$\operatorname{Conv}_{\text{coe}}(\mathbb{R}^n) = \left\{ u \in \operatorname{Conv}(\mathbb{R}^n) \colon \lim_{|x|\to+\infty} u(x) = +\infty \right\},$$

where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$. The second is formed by *super-coercive* functions,

$$\operatorname{Conv}_{\text{sc}}(\mathbb{R}^n) = \left\{ u \in \operatorname{Conv}(\mathbb{R}^n) \colon \lim_{|x|\to+\infty} \frac{u(x)}{|x|} = +\infty \right\}.$$

The space of super-coercive convex functions is related to another subspace of $\operatorname{Conv}(\mathbb{R}^n)$, formed by convex functions with finite values,

$$\operatorname{Conv}(\mathbb{R}^n; \mathbb{R}) = \big\{ v \in \operatorname{Conv}(\mathbb{R}^n) \colon v(x) < +\infty \text{ for all } x \in \mathbb{R}^n \big\}.$$

Indeed, $v \in \operatorname{Conv}(\mathbb{R}^n;\mathbb{R})$ if and only if its standard conjugate or Legendre transform v^* belongs to $\operatorname{Conv}_{\text{sc}}(\mathbb{R}^n)$ (see Section 1.3).

1.1. One of the most important structural results for valuations on convex bodies is the existence of a homogeneous decomposition for translation invariant valuations. It was conjectured by Hadwiger and established by McMullen [30] (see Section 2.1). Our first aim is to establish such a result for valuations on convex functions. We define *epi-multiplication* by setting for $u \in \operatorname{Conv}(\mathbb{R}^n)$ and $\lambda > 0$,

$$\lambda \cdot u(x) = \lambda\, u\left(\frac{x}{\lambda}\right)$$

for $x \in \mathbb{R}^n$. From a geometric point of view, this operation has the following meaning: the epigraph of $\lambda \cdot u$ is obtained by rescaling the epigraph of u by the factor λ. We extend the definition of epi-multiplication to $0 \cdot u(x) = 0$ if $x = 0$ and $0 \cdot u(x) = +\infty$ if $x \neq 0$. It is easy to see that $u \in \operatorname{Conv}_{\text{sc}}(\mathbb{R}^n)$ implies $\lambda \cdot u \in \operatorname{Conv}_{\text{sc}}(\mathbb{R}^n)$ for $\lambda \geq 0$. A functional $\mathrm{Z} : \operatorname{Conv}_{\text{sc}}(\mathbb{R}^n) \to \mathbb{R}$ is called *epi-homogeneous* of degree $\alpha \in \mathbb{R}$ if

$$\mathrm{Z}(\lambda \cdot u) = \lambda^\alpha\, \mathrm{Z}(u)$$

for all $u \in \operatorname{Conv}_{\text{sc}}(\mathbb{R}^n)$ and $\lambda > 0$. Here and in the following corresponding definitions will be used for $\operatorname{Conv}(\mathbb{R}^n)$ and its subspaces.

We call $\mathrm{Z} : \operatorname{Conv}_{\text{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *translation invariant* if $\mathrm{Z}(u \circ \tau^{-1}) = \mathrm{Z}(u)$ for every $u \in \operatorname{Conv}_{\text{sc}}(\mathbb{R}^n)$ and every translation $\tau : \mathbb{R}^n \to \mathbb{R}^n$. If $u \in \operatorname{Conv}_{\text{sc}}(\mathbb{R}^n)$ then $u \circ \tau^{-1} \in \operatorname{Conv}_{\text{sc}}(\mathbb{R}^n)$ as well. We say that Z is *vertically translation invariant* if

$$\mathrm{Z}(u + \alpha) = \mathrm{Z}(u)$$

for all $u \in \operatorname{Conv}_{\text{sc}}(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}$. If Z is both translation invariant and vertically translation invariant, then Z is called *epi-translation invariant*. As we will see, the set of continuous, epi-translation invariant valuations on $\operatorname{Conv}_{\text{sc}}(\mathbb{R}^n)$ is non-empty. Note that a functional Z is epi-translation invariant if for all $u \in \operatorname{Conv}_{\text{sc}}(\mathbb{R}^n)$ the value $\mathrm{Z}(u)$ is not changed by translations of the epigraph of u.

The following result establishes a homogeneous decomposition for continuous and epi-translation invariant valuations on $\operatorname{Conv}_{\text{sc}}(\mathbb{R}^n)$.

Theorem 1. *If* $\mathrm{Z} : \operatorname{Conv}_{\text{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous and epi-translation invariant valuation, then there are continuous and epi-translation invariant valuations* $\mathrm{Z}_0, \dots, \mathrm{Z}_n : \operatorname{Conv}_{\text{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *such that* Z_i *is epi-homogeneous of degree* i *and* $\mathrm{Z} = \mathrm{Z}_0 + \cdots + \mathrm{Z}_n$.

We will see that this theorem is no longer true if we remove the condition of vertical translation invariance (see Section 8). We will also see that the set of continuous and epi-translation invariant valuations is trivial on the larger set of coercive convex functions (see Section 9). Hence the assumption of super-coercivity is in some sense necessary.

Milman and Rotem [31] discuss the problem to find a functional analog of Minkowski's mixed volume theorem. In particular, they point out that such a result is not possible on Conv$(\mathbb{R}^n)$ for inf-convolution as addition and the volume functional $u \mapsto \int_{\mathbb{R}^n} e^{-u(x)}\,\mathrm{d}x$. Instead, they define a new addition for convex functions to obtain a functional mixed volume theorem. A consequence of Theorem 1 is that continuous and epi-translation invariant valuations are multilinear on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ with respect to inf-convolution and epi-multiplication (see Theorem 21). Thus, for all such valuations, a functional analog of Minkowski's mixed volume theorem is obtained on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ with inf-convolution as addition.

1.2. The following result gives a characterization of continuous and epi-translation invariant valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, which are epi-homogeneous of degree n. For $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, we denote by $\mathrm{dom}(u)$ the set of points of $\mathbb{R}^n$ where u is finite and by ∇u the gradient of u. Note that by standard properties of convex functions, $\nabla u(x)$ is well defined for a.e. $x \in \mathrm{dom}(u)$. Let $C_c(\mathbb{R}^n)$ be the set of continuous functions with compact support on $\mathbb{R}^n$.

Theorem 2. *A functional* $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous and epi-translation invariant valuation that is epi-homogeneous of degree* n*, if and only if there exists* $\zeta \in C_c(\mathbb{R}^n)$ *such that*

$$\mathrm{Z}(u) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\,\mathrm{d}x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

We will also obtain a classification of continuous and epi-translation invariant valuations that are epi-homogeneous of degree 0. These are just constants. As a consequence of these results and Theorem 1, we obtain the following complete classification in dimension one.

Corollary 3. *A functional* $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}) \to \mathbb{R}$ *is a continuous and epi-translation invariant valuation, if and only if there exist a constant* $\zeta_0 \in \mathbb{R}$ *and a function* $\zeta_1 \in C_c(\mathbb{R})$ *such that*

$$\mathrm{Z}(u) = \zeta_0 + \int_{\mathrm{dom}(u)} \zeta_1(u'(x))\,\mathrm{d}x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R})$.

1.3. As mentioned before, there exists a bijection between $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ given by the standard conjugate, or Legendre transform, of convex functions. For $u \in \mathrm{Conv}(\mathbb{R}^n)$, we denote by u^* its conjugate, defined by

$$u^*(y) = \sup\nolimits_{x\in\mathbb{R}^n} \left(\langle x, y\rangle - u(x)\right)$$

for $y \in \mathbb{R}^n$, where $\langle x, y\rangle$ is the inner product of $x, y \in \mathbb{R}^n$. Note that $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ if and only if $u^* \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ (see, for example, [35, Theorem 11.8]).

Let Z be a continuous valuation on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. It was proved in [16] that $\mathrm{Z}^*\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$, defined by

$$\mathrm{Z}^*(u) = \mathrm{Z}(u^*),$$

is a continuous valuation as well. This fact permits to transfer results for valuations on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ to results valid for valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and vice versa. We call Z^* the dual valuation of Z.

A valuation Z on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ is called *homogeneous* if there exists $\alpha \in \mathbb{R}$ such that

$$\mathrm{Z}(\lambda v) = \lambda^\alpha\, \mathrm{Z}(v)$$

for all $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\lambda \geq 0$. We say that Z is *dually translation invariant* if for every linear function $\ell\colon \mathbb{R}^n \to \mathbb{R}$

$$\mathrm{Z}(v + \ell) = \mathrm{Z}(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. Let $\ell(y) = \langle y, x_0\rangle$ for $x_0, y \in \mathbb{R}^n$. As $(v+\ell)^*(x) = v^*(x - x_0)$ for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, we see that Z is dually translation invariant if and only if Z^* is translation invariant. We define vertical translation invariance for valuations on $\operatorname{Conv}(\mathbb{R}^n;\mathbb{R})$ in the same way as on $\operatorname{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. We say that Z is *dually epi-translation invariant* on $\operatorname{Conv}(\mathbb{R}^n;\mathbb{R})$ if it is vertically and dually translation invariant. Note that a functional Z is dually epi-translation invariant, if for all $v \in \operatorname{Conv}(\mathbb{R}^n;\mathbb{R})$, the value $\mathrm{Z}(v)$ is not changed by adding an affine function to v.

Let Z be a valuation on $\operatorname{Conv}(\mathbb{R}^n;\mathbb{R})$. We note the following simple facts. The valuation Z is vertically translation invariant if and only if Z^* has the same property. The valuation Z^* is epi-homogeneous of degree α if and only if Z is homogeneous of degree α.

Hence we obtain the following result as a consequence of Theorem 1.

Theorem 4. *If* $\mathrm{Z} : \operatorname{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ *is a continuous and dually epi-translation invariant valuation, then there are continuous and dually epi-translation invariant valuations* $\mathrm{Z}_0,\dots,\mathrm{Z}_n : \operatorname{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ *such that* Z_i *is homogeneous of degree* i *and* $\mathrm{Z} = \mathrm{Z}_0 + \cdots + \mathrm{Z}_n$.

Alesker [3] introduced the following class of valuations on $\operatorname{Conv}(\mathbb{R}^n;\mathbb{R})$. Given real symmetric $n\times n$ matrices $M_1,\dots,M_n$, denote by $\det(M_1,\dots,M_n)$ their mixed discriminant. Let $i \in \{1,\dots,n\}$ and write $\det(M[i], M_1,\dots,M_{n-i})$ for the mixed discriminant in which the matrix M is repeated i times. Let $A_1,\dots,A_{n-i}$ be continuous, symmetric $n\times n$ matrix-valued functions on $\mathbb{R}^n$ with compact support and $\zeta \in C_c(\mathbb{R}^n)$. Given a function $v \in \operatorname{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$, set

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n} \zeta(x) \det(\mathrm{D}^2 v(x)[i], A_1(x),\dots,A_{n-i}(x))\,\mathrm{d}x \tag{2}$$

where $\mathrm{D}^2 v$ is the Hessian matrix of v. Alesker [3] proved that Z can be extended to a continuous valuation on $\operatorname{Conv}(\mathbb{R}^n;\mathbb{R})$. Valuations of type (2) are homogeneous of degree i and dually epi-translation invariant. This implies in particular that the set of valuations with these properties is non-empty. Clearly, the dual functional Z^* is a continuous, epi-translation invariant, epi-homogeneous valuation on $\operatorname{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Next, we state the counterpart of Theorem 2 for valuations on $\operatorname{Conv}(\mathbb{R}^n;\mathbb{R})$. Let $\Theta_0(v,\cdot)$ be the Hessian measure of order 0 of a function $v \in \operatorname{Conv}(\mathbb{R}^n;\mathbb{R})$ (see Section 4 for the definition).

Theorem 5. *A functional* $\mathrm{Z}\colon \operatorname{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ *is a continuous and dually epi-translation invariant valuation that is homogeneous of degree* n*, if and only if there exists* $\zeta \in C_c(\mathbb{R}^n)$ *such that*

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \zeta(x)\,\mathrm{d}\Theta_0(v,(x,y))$$

for every $v \in \operatorname{Conv}(\mathbb{R}^n;\mathbb{R})$.

In the special case of dimension one, we obtain the following complete classification theorem.

Corollary 6. *A functional* $\mathrm{Z}\colon \operatorname{Conv}(\mathbb{R};\mathbb{R}) \to \mathbb{R}$ *is a continuous and dually epi-translation invariant valuation, if and only if there exist a constant* $\zeta_0 \in \mathbb{R}$ *and a function* $\zeta_1 \in C_c(\mathbb{R})$ *such that*

$$\mathrm{Z}(v) = \zeta_0 + \int_{\mathbb{R}\times\mathbb{R}} \zeta_1(x)\,\mathrm{d}\Theta_0(v,(x,y))$$

for every $v \in \operatorname{Conv}(\mathbb{R};\mathbb{R})$.

The plan for this paper is as follows. In Section 2, we collect results on convex bodies and functions needed for the proofs of the main results. In Section 3, an inclusion-exclusion principle is established for valuations on convex functions and in Section 4, the existence and properties of the valuations in Theorem 2 and Theorem 5 are deduced by using results on Hessian valuations. Theorem 1 is proved in Section 5. As a consequence the polynomiality of epi-translation invariant valuations is obtained and a connection to the valuations introduced by Alesker is established in Section 6. The proof of Theorem 2 is given in Section 7. In the final sections, the necessity of the assumptions in Theorem 1 is demonstrated.

2. Preliminaries

We work in n-dimensional Euclidean space $\mathbb{R}^n$, with $n \geq 1$, endowed with the Euclidean norm $|\cdot|$ and the usual scalar product $\langle \cdot, \cdot \rangle$.

2.1. A *convex body* is a nonempty, compact and convex subset of $\mathbb{R}^n$. The family of all convex bodies is denoted by $\mathcal{K}^n$. A *polytope* is the convex hull of finitely many points in $\mathbb{R}^n$. The set of polytopes, denoted by $\mathcal{P}^n$, is contained in $\mathcal{K}^n$. We equip both $\mathcal{K}^n$ and $\mathcal{P}^n$ with the topology coming from the Hausdorff metric.

A functional $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ is a valuation if

$$\mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L) = \mathrm{Z}(K) + \mathrm{Z}(L)$$

for every $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$. We say that Z is translation invariant if $\mathrm{Z}(\tau K) = \mathrm{Z}(K)$ for all translations $\tau : \mathbb{R}^n \to \mathbb{R}^n$ and $K \in \mathcal{K}^n$. It is homogeneous of degree $\alpha \in \mathbb{R}$, if $\mathrm{Z}(\lambda\, K) = \lambda^\alpha\, \mathrm{Z}(K)$ for all $K \in \mathcal{K}^n$ and $\lambda \geq 0$.

The following result by McMullen [30] establishes a homogeneous decomposition for continuous and translation invariant valuations on $\mathcal{K}^n$.

Theorem 7 (McMullen)**.** *If $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ is a continuous and translation invariant valuation, then there are continuous and translation invariant valuations $\mathrm{Z}_0, \dots, \mathrm{Z}_n : \mathcal{K}^n \to \mathbb{R}$ such that Z_i is homogeneous of degree i and $\mathrm{Z} = \mathrm{Z}_0 + \cdots + \mathrm{Z}_n$.*

We recall two classification results for valuations on convex bodies. First, we note that it is easy to see that every continuous and translation invariant valuation that is homogeneous of degree 0 is constant. The classification of continuous and translation invariant valuations that are n-homogeneous is due to Hadwiger [22]. Let V_n denote n-dimensional volume (that is, n-dimensional Lebesgue measure).

Theorem 8 (Hadwiger)**.** *A functional $\mathrm{Z}\colon \mathcal{K}^n \to \mathbb{R}$ is a continuous and translation invariant valuation that is homogeneous of degree n, if and only if there exists $\alpha \in \mathbb{R}$ such that $\mathrm{Z} = \alpha\, V_n$.*

2.2. Given a subset $A \subset \mathbb{R}^n$, let $\mathbf{I}_A \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ denote the (convex) indicatrix function of A,

$$\mathbf{I}_A(x) = \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{if } x \notin A. \end{cases}$$

Note that for a convex body K, we have $\mathbf{I}_K \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

We equip $\mathrm{Conv}(\mathbb{R}^n)$ with the topology associated to epi-convergence. Here a sequence $u_k \in \mathrm{Conv}(\mathbb{R}^n)$ is *epi-convergent* to $u \in \mathrm{Conv}(\mathbb{R}^n)$ if for all $x \in \mathbb{R}^n$ the following conditions hold:

(i) For every sequence x_k that converges to x, we have $u(x) \leq \liminf_{k\to\infty} u_k(x_k)$.
(ii) There exists a sequence x_k that converges to x such that $u(x) = \lim_{k\to\infty} u_k(x_k)$.

The following result can be found in [35, Theorem 11.34].

Proposition 9. *A sequence u_k of functions from $\mathrm{Conv}(\mathbb{R}^n)$ epi-converges to $u \in \mathrm{Conv}(\mathbb{R}^n)$ if and only if the sequence u_k^* epi-converges to u^*.*

If $u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$, then for $t \in \mathbb{R}$ the sublevel sets $\{u \leq t\} = \{x \in \mathbb{R}^n \colon u(x) \leq t\}$ are either empty or in $\mathcal{K}^n$. The next result, which follows from [15, Lemma 5] and [5, Theorem 3.1], shows that on $\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$ epi-convergence is equivalent to Hausdorff convergence of sublevel sets, where we say that $\{u_k \leq t\} \to \emptyset$ as $k \to \infty$ if there exists $k_0 \in \mathbb{N}$ such that $\{u_k \leq t\} = \emptyset$ for $k \geq k_0$.

Lemma 10. *If $u_k, u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$, then u_k epi-converges to u if and only if $\{u_k \leq t\} \to \{u \leq t\}$ for every $t \in \mathbb{R}$ with $t \neq \min_{x\in\mathbb{R}^n} u(x)$.*

2.3. A function $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ is called *piecewise affine* if there exist finitely many affine functions $w_1,\dots,w_m:\mathbb{R}^n\to\mathbb{R}$ such that

$$v=\bigvee_{i=1}^m w_i. \tag{3}$$

The set of piecewise affine functions will be denoted by $\mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n;\mathbb{R})$. It is a subset of $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

We recall that epi-convergence in $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ is equivalent to uniform convergence on compact sets (see, for example, [35, Theorem 7.17]). Hence the following proposition follows from standard approximation results for convex functions.

Proposition 11. *For every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$, there exists a sequence in $\mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n;\mathbb{R})$ which epi-converges to v.*

We also need to introduce a counterpart of $\mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n;\mathbb{R})$ in $\mathrm{Conv}_{\text{sc}}(\mathbb{R}^n)$. For given polytopes $P,P_1,\dots,P_m\in\mathcal{P}^n$, the collection $\{P_1,\dots,P_m\}$ is called a *polytopal partition* of P if $P=\bigcup_{i=1}^m P_i$ and the P_i's have pairwise disjoint interiors. A function $u\in\mathrm{Conv}_{\text{sc}}(\mathbb{R}^n)$ belongs to $\mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n)$ if there exists a polytope P and a polytopal partition $\{P_1,\dots,P_m\}$ of P such that

$$u=\bigwedge_{i=1}^m (w_i+\mathbf{I}_{P_i})$$

where $w_1,\dots,w_m:\mathbb{R}^n\to\mathbb{R}$ are affine.

By [35, Theorem 11.14], a function u is in $\mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n)$ if and only if u^* is in $\mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n;\mathbb{R})$. Hence, we obtain the following consequence of Proposition 9 and Proposition 11.

Corollary 12. *For every $u\in\mathrm{Conv}_{\text{sc}}(\mathbb{R}^n)$, there exists a sequence in $\mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n)$ which epi-converges to u.*

Since $\mathrm{Conv}_{\text{sc}}(\mathbb{R}^n)$ is a dense subset of $\mathrm{Conv}_{\text{coe}}(\mathbb{R}^n)$, it is easy to see that the statement of Corollary 12 also holds if $\mathrm{Conv}_{\text{sc}}(\mathbb{R}^n)$ is replaced by $\mathrm{Conv}_{\text{coe}}(\mathbb{R}^n)$.

3. The Inclusion-Exclusion Principle

It is often useful to extend the valuation property (1) to several convex functions. For valuations on convex bodies, this is an important tool and a consequence of Groemer's extension theorem [18]. For $m\geq 1$ and $u_1,\dots,u_m\in\mathrm{Conv}(\mathbb{R}^n)$, we set $u_J=\bigvee_{j\in J}u_j$ for $\emptyset\neq J\subset\{1,\dots,m\}$. Let $|J|$ denote the number of elements in J.

Theorem 13. *If $\mathrm{Z}:\mathrm{Conv}(\mathbb{R}^n)\to\mathbb{R}$ is a continuous valuation, then*

$$\mathrm{Z}(u_1\wedge\cdots\wedge u_m)=\sum_{\emptyset\neq J\subset\{1,\dots,m\}}(-1)^{|J|-1}\,\mathrm{Z}(u_J) \tag{4}$$

for all $u_1,\dots,u_m\in\mathrm{Conv}(\mathbb{R}^n)$ and $m\in\mathbb{N}$ whenever $u_1\wedge\cdots\wedge u_m\in\mathrm{Conv}(\mathbb{R}^n)$.

Note that $\mathrm{Conv}_{\text{coe}}(\mathbb{R}^n)$ and $\mathrm{Conv}_{\text{sc}}(\mathbb{R}^n)$ are closed under the operation of taking maxima. Hence Theorem 13 also holds with $\mathrm{Conv}(\mathbb{R}^n)$ replaced by one of these spaces.

Let $\bigwedge \mathrm{Conv}(\mathbb{R}^n)$ denote the set of finite minima of convex functions from $\mathrm{Conv}(\mathbb{R}^n)$. It is easy to see that $\bigwedge \mathrm{Conv}(\mathbb{R}^n)$ is a lattice. If Z is a valuation on a lattice, a simple induction argument shows that the inclusion-exclusion principle (4) holds. Hence Theorem 13 is a consequence of the following extension result.

Theorem 14. *A continuous valuation on* $\mathrm{Conv}(\mathbb{R}^n)$ *admits a unique extension to a valuation on the lattice* $\bigwedge \mathrm{Conv}(\mathbb{R}^n)$.

We identify a convex function with its epigraph. Let $\mathcal{C}^{n+1}_{\text{epi}}$ be the set of closed convex sets in $\mathbb{R}^{n+1}$ that are epigraphs of functions in $\mathrm{Conv}(\mathbb{R}^n)$ and equip this set with the Painlevé-Kuratowski topology, which corresponds to the topology induced by epi-convergence (see, for example, [35, Definition 7.1]). A slight modification of Groemer's extension theorem [18] (or see [36, Theorem 6.2.3] or [23]) shows that the following statement is true (we omit the proof). Here $\bigcup \mathcal{C}^{n+1}_{\text{epi}}$ is the set of all finite unions of elements from $\mathcal{C}^{n+1}_{\text{epi}}$. Theorem 14 is equivalent to Theorem 15.

Theorem 15. *A continuous valuation on* $\mathcal{C}^{n+1}_{epi}$ *admits a unique extension to a valuation on the lattice* $\bigcup \mathcal{C}^{n+1}_{epi}$.

We require the following simple consequence of the inclusion-exclusion principle, Theorem 13 and of Corollary 12.

Lemma 16. *Let* Z *be a continuous valuation on* $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ *(or on* $\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$*). If*

$$\mathrm{Z}(w + \mathbf{I}_P) = 0 \tag{5}$$

for every affine function $w : \mathbb{R}^n \to \mathbb{R}$ *and for every polytope* P*, then* $\mathrm{Z} \equiv 0$.

Proof. By Corollary 12 (and the remark following it), it suffices to prove that $\mathrm{Z}(u) = 0$ for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ (or $u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$) that is piecewise affine. So, let $u = \bigwedge_{i=1}^m (w_i + \mathbf{I}_{P_i})$ with $w_1, \dots, w_m$ affine and $P_1, \dots, P_m \in \mathcal{P}^n$. By Theorem 13 (and the remark following it), it is enough to show that

$$\mathrm{Z}\left(\bigvee_{j \in J} (w_j + \mathbf{I}_{P_j})\right) = 0$$

for every $\emptyset \neq J \subset \{1, \dots, m\}$. This follows from (5) as $\bigvee_{j \in J} (w_j + \mathbf{I}_{P_j})$ is a piecewise affine function restricted to a polytope. $\square$

4. Hessian measures and valuations

For $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we denote by $\partial u(x)$ the subgradient of u at x, that is,

$$\partial u(x) = \{y \in \mathbb{R}^n \colon u(z) \geq u(x) + \langle z - x, y \rangle \text{ for all } z \in \mathbb{R}^n\}.$$

We set

$$\Gamma_u = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \colon y \in \partial u(x)\}.$$

In other words, Γ_u is the generalized graph of ∂u.

Next, we recall the notion of Hessian measures of a function $u \in \mathrm{Conv}(\mathbb{R}^n)$. These are non-negative Borel measures defined on the Borel subsets of $\mathbb{R}^n \times \mathbb{R}^n$, which we will denote by $\Theta_i(u, \cdot)$ with $i = 0, \dots, n$. Their definition can be given as follows (see also [10, 11, 16]). Let $\eta \subset \mathbb{R}^n \times \mathbb{R}^n$ be a Borel set and $s > 0$. Consider the following set

$$P_s(u, \eta) = \{x + sy \colon (x, y) \in \Gamma_u \cap \eta\}.$$

It can be proven (see Theorem 7.1 in [16]) that $P_s(u,\eta)$ is measurable and that its measure is a polynomial in the variable s, that is, there exists $(n+1)$ non-negative coefficients $\Theta_i(u,\eta)$ such that

$$\mathcal{H}^n(P_s(u,\eta)) = \sum_{i=0}^{n} \binom{n}{i} s^i \Theta_{n-i}(u,\eta).$$

Here $\mathcal{H}^n$ is the n-dimensional Hausdorff measure in $\mathbb{R}^n$, normalized so that it coincides with the Lebesgue measure in $\mathbb{R}^n$. The previous formula defines the Hessian measures of u; for more details we refer the reader to [10, 11, 16].

According to Theorem 8.2 in [16], for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and for every Borel subset η of $\mathbb{R}^n \times \mathbb{R}^n$

$$\Theta_i(v,\eta) = \Theta_{n-i}(v^*,\hat{\eta}), \tag{6}$$

where $\hat{\eta} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \colon (y,x) \in \eta\}$.

We require the following statement for Hessian valuations for $i=0$. As the proof is the same for all indices i, we give the more general statement. Let $[\mathrm{D}^2 v(x)]_i$ be the i-th elementary symmetric function of the eigenvalues of the Hessian matrix $\mathrm{D}^2 v$.

Theorem 17. *Let $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ have compact support with respect to the second variable. For $i \in \{0,1,\dots,n\}$,*

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta(v(x),x,y)\,\mathrm{d}\Theta_i(v,(x,y)) \tag{7}$$

is well defined for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and defines a continuous valuation on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. Moreover,

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n} \zeta(v(x),x,\nabla v(x))\,[\mathrm{D}^2 v(x)]_{n-i}\,\mathrm{d}x \tag{8}$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$.

We use the following result.

Theorem 18 ([16], Theorem 1.1)**.** *Let $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ have compact support with respect to the second and third variables. For every $i \in \{0,1,\dots,n\}$, the functional defined by*

$$v \mapsto \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta(v(x),x,y)\,\mathrm{d}\Theta_i(v,(x,y))$$

defines a continuous valuation on $\mathrm{Conv}(\mathbb{R}^n)$. Moreover,

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n} \zeta(v(x),x,\nabla v(x))\,[\mathrm{D}^2 v(x)]_{n-i}\,\mathrm{d}x \tag{9}$$

for $v \in \mathrm{Conv}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$.

Proof of Theorem 17. Since ζ has compact support with respect to the second variable, there is $r>0$ such that $\zeta(t,x,y)=0$ for every $y \in \mathbb{R}^n$ with $|y| \geq r$ and $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. Let $v, v_k \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ be such that v_k epi-converges to v. Since the functions are convex and finite this implies uniform convergence on compact sets, in particular on $B_r := \{x \in \mathbb{R}^n : |x| \leq r\}$. Moreover, the sequence v_k is uniformly bounded on B_r and uniformly Lipschitz. Hence, there exists $c>0$ such that

$$|v_k(x)| \leq c,\ |v(x)| \leq c,\ |y| \leq c$$

for all $k \in \mathbb{N}$, $x \in B_r$ and $y \in \partial v_k(x) \cup \partial v(x)$.

Next, let $\eta : \mathbb{R}^n \to \mathbb{R}$ be smooth with compact support such that $\eta(y) = 1$ for all $y \in \mathbb{R}^n$ with $|y| \le c$ and define $\tilde{\zeta} \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ by

$$\tilde{\zeta}(t, x, y) = \zeta(t, x, y)\, \eta(y).$$

The function $\tilde{\zeta}$ satisfies the conditions of Theorem 18 and $\zeta(v(x), x, y) = \tilde{\zeta}(v(x), x, y)$ for all $x \in \mathbb{R}^n$, $y \in \partial v(x)$ and $\zeta(v_k(x), x, y) = \tilde{\zeta}(v_k(x), x, y)$ for all $x \in \mathbb{R}^n$, $y \in \partial v_k(x)$ and $k \in \mathbb{N}$. Hence, by Theorem 18,

$$\begin{aligned}\int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta(v_k(x), x, y) \,\mathrm{d}\Theta_i(v_k, (x, y)) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \tilde{\zeta}(v_k(x), x, y) \,\mathrm{d}\Theta_i(v_k, (x, y)) \\ &\longrightarrow \int_{\mathbb{R}^n \times \mathbb{R}^n} \tilde{\zeta}(v(x), x, y) \,\mathrm{d}\Theta_i(v, (x, y)) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta(v(x), x, y) \,\mathrm{d}\Theta_i(v, (x, y))\end{aligned}$$

as $k \to \infty$. Since v and v_k were arbitrary this shows that (7) is well defined and continuous. Since such a function $\tilde{\zeta}$ can especially be found for any finite number of functions in $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, this also proves the valuation property. Property (8) follows from (9). $\square$

As a simple consequence of Theorem 17 we obtain the following statement.

Proposition 19. *For $\zeta \in C_c(\mathbb{R}^n)$, the functional $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) \to \mathbb{R}$, defined by*

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta(x) \,\mathrm{d}\Theta_0(v, (x, y)), \tag{10}$$

is a continuous, dually epi-translation invariant valuation which is is homogeneous of degree n.

Proof. By Theorem 17 the map defined by (10) is a continuous valuation on on $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$. It remains to show dually epi-translation invariance. For $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) \cap C^2(\mathbb{R}^n)$ it follows from (8) that

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n} \zeta(x)\ \det(\mathrm{D}^2 v(x)) \,\mathrm{d}x$$

which is clearly invariant under the addition of constants and linear terms. The statement now easily follows for general $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ by approximation. $\square$

By the considerations presented in Section 1.3, (6) and Proposition 19 lead to the following result.

Proposition 20. *For $\zeta \in C_c(\mathbb{R}^n)$, the functional $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$, defined by*

$$\mathrm{Z}(u) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta(y) \,\mathrm{d}\Theta_n(u, (x, y)),$$

is a continuous and epi-translation invariant valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ which is epi-homogeneous of degree n.

Note, that if Z is as in Proposition 20, then

$$\mathrm{Z}(u) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta(y) \,\mathrm{d}\Theta_n(u, (x, y)) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x)) \,\mathrm{d}x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. See also [16, Section 10.4].

5. Proof of Theorem 1

For $y \in \mathbb{R}^n$, define the linear function $\ell_y \colon \mathbb{R}^n \to \mathbb{R}$ as

$$\ell_y(x) = \langle x, y\rangle.$$

For $K \in \mathcal{K}^n$, the function $\ell_y + \mathbf{I}_K$ belongs to $\operatorname{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Claim. *The functional $\tilde{\mathrm{Z}}_y \colon \mathcal{K}^n \to \mathbb{R}$, defined by*

$$\tilde{\mathrm{Z}}_y(K) = \mathrm{Z}(\ell_y + \mathbf{I}_K),$$

is a continuous and translation invariant valuation.

Proof. i) The valuation property. Let $K, L \in \mathcal{K}^n$ be such that $K \cup L \in \mathcal{K}^n$. Note that

$$(\ell_y + \mathbf{I}_K) \vee (\ell_y + \mathbf{I}_L) = \ell_y + \mathbf{I}_{K\cap L}; \quad (\ell_y + \mathbf{I}_K) \wedge (\ell_y + \mathbf{I}_L) = \ell_y + \mathbf{I}_{K\cup L}.$$

Hence the valuation property of Z implies that $\tilde{\mathrm{Z}}_y$ is a valuation.

ii) Translation invariance. Let $x_0 \in \mathbb{R}^n$. For every $x \in \mathbb{R}^n$ we have

$$\begin{aligned} \ell_y(x) + \mathbf{I}_{K+x_0}(x) &= \langle x, y\rangle + \mathbf{I}_K(x - x_0) \\ &= \langle x - x_0, y\rangle + \mathbf{I}_K(x - x_0) + \langle x_0, y\rangle \\ &= \ell_y(x - x_0) + \mathbf{I}_K(x - x_0) + \langle x_0, y\rangle. \end{aligned}$$

In other words, the functions $\ell_y + \mathbf{I}_{K+x_0}$ and $\ell_y + \mathbf{I}_K$ differ only by a translation of the variable and by an additive constant. Using the epi-translation invariance of Z we get

$$\tilde{\mathrm{Z}}_y(K + x_0) = \mathrm{Z}(\ell_y + \mathbf{I}_{K+x_0}) = \mathrm{Z}(\ell_y + \mathbf{I}_K) = \tilde{\mathrm{Z}}_y(K).$$

iii) Continuity. By Lemma 10, a sequence of convex bodies K_i converges to K if and only if $\ell_y + \mathbf{I}_{K_i}$ epi-converges to $\ell_y + \mathbf{I}_K$. Hence the continuity of Z implies that of $\tilde{\mathrm{Z}}_y$. $\square$

Let $y \in \mathbb{R}^n$ be fixed. By the previous claim and Theorem 7, there exist continuous and translation invariant valuations $\tilde{\mathrm{Z}}_{y,0}, \dots, \tilde{\mathrm{Z}}_{y,n}$ on $\mathcal{K}^n$ such that $\tilde{\mathrm{Z}}_{y,j}$ is j-homogeneous and

$$\tilde{\mathrm{Z}}_y = \sum_{j=0}^n \tilde{\mathrm{Z}}_{y,j}\,.$$

Let $K \in \mathcal{K}^n$. For $\lambda \geq 0$, we have $\lambda \cdot (\ell_y + \mathbf{I}_K) = \ell_y + \mathbf{I}_{\lambda K}$. Therefore we obtain, for every $\lambda \geq 0$,

$$\mathrm{Z}(\lambda \cdot (\ell_y + \mathbf{I}_K)) = \sum_{j=0}^n \tilde{\mathrm{Z}}_{y,j}(K)\lambda^j.$$

We consider the system of equations,

$$\mathrm{Z}(k \cdot (\ell_y + \mathbf{I}_K)) = \sum_{j=0}^n \tilde{\mathrm{Z}}_{y,j}(K)k^j, \quad k = 0, 1, \dots, n. \tag{11}$$

Its associated matrix is a Vandermonde matrix and invertible. Hence there are coefficients α_{ij} for $i, j = 0, \dots n$, such that

$$\tilde{\mathrm{Z}}_{y,i}(K) = \sum_{j=0}^n \alpha_{ij}\, \mathrm{Z}(k \cdot (\ell_y + \mathbf{I}_K)), \quad i = 0, \dots, n.$$

Note that the coefficients α_{ij} are independent of y and K.

For $i = 0, \dots, n$, we define $\mathrm{Z}_i \colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ as

$$\mathrm{Z}_i(u) = \sum_{j=0}^{n} \alpha_{ij}\, \mathrm{Z}(j \,\mathbin{\vcenter{\hbox{\scriptsize$\blacksquare$}}}\, u).$$

In general, if Z is a continuous, epi-translation invariant valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $\lambda \geq 0$, then the functional $u \mapsto \mathrm{Z}(\lambda \,\mathbin{\vcenter{\hbox{\scriptsize$\blacksquare$}}}\, u)$ is a continuous, epi-translation valuation as well. Hence Z_i is a continuous, epi-translation invariant valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, for every $i = 0, \dots, n$.

By (11) and the definition of Z_i, for every $y \in \mathbb{R}^n$ and $K \in \mathcal{K}^n$ we may write

$$\mathrm{Z}_i(\ell_y + \mathbf{I}_K) = \tilde{\mathrm{Z}}_{y,i}(K).$$

Therefore

$$\mathrm{Z}(\ell_y + \mathbf{I}_K) = \sum_{i=0}^{n} \mathrm{Z}_i(\ell_y + \mathbf{I}_K).$$

Moreover, by the homogeneity of the $\mathrm{Z}_{y,i}$ we have, for $\lambda \geq 0$,

$$\mathrm{Z}_i(\lambda \,\mathbin{\vcenter{\hbox{\scriptsize$\blacksquare$}}}\, (\ell_y + \mathbf{I}_K)) = \tilde{\mathrm{Z}}_{y,i}(\lambda K) = \lambda^i\, \tilde{\mathrm{Z}}_{y,i}(K) = \lambda^i\, \mathrm{Z}_i(\ell_y + \mathbf{I}_K).$$

As a conclusion, we have the following statement: there exist continuous and epi-translation invariant valuations $\mathrm{Z}_0, \dots, \mathrm{Z}_n$ on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ such that, for every $y \in \mathbb{R}^n$ and for every $K \in \mathcal{K}^n$, setting $u = \ell_y + \mathbf{I}_K$, we have

$$\mathrm{Z}(u) = \sum_{i=0}^{n} \mathrm{Z}_i(u),$$

and, for every $\lambda \geq 0$,

$$\mathrm{Z}_i(\lambda \,\mathbin{\vcenter{\hbox{\scriptsize$\blacksquare$}}}\, u) = \lambda^i\, \mathrm{Z}_i(u).$$

The same statement holds if we replace $u = \ell_y + \mathbf{I}_K$ by $u = \ell_y + \mathbf{I}_K + \alpha$, for any constant $\alpha \in \mathbb{R}$ as all valuations involved are vertically translation invariant.

If we apply Lemma 16 to

$$\mathrm{Z} - \sum_{i=0}^{n} \mathrm{Z}_i,$$

we get that this valuation vanishes on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, so that

$$\mathrm{Z}(u) = \sum_{i=0}^{n} \mathrm{Z}_i(u)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. For $\lambda \geq 0$, the same lemma applied to the valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ defined by

$$u \mapsto \mathrm{Z}_i(\lambda \,\mathbin{\vcenter{\hbox{\scriptsize$\blacksquare$}}}\, u) - \lambda^i\, \mathrm{Z}_i(u),$$

shows that this must be identically zero as well, that is, Z_i is epi-homogeneous of degree i. The proof is complete.

6. Polynomiality

In this section we establish the polynomial behavior of continuous and epi-translation invariant valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. This corresponds to the polynomiality of translation invariant valuations on convex bodies stated by Hadwiger and proved by McMullen [30]. We start by recalling the definition of inf-convolution (see, for example, [35, 36]). For $u, v \in \mathrm{Conv}(\mathbb{R}^n)$, we define the function $u \,\square\, v \colon \mathbb{R}^n \to [-\infty, +\infty]$ by

$$u \,\square\, v(z) = \inf\{u(x) + v(y) \colon x, y \in \mathbb{R}^n,\, x + y = z\}$$

for $z \in \mathbb{R}^n$. This operation can be extended to more than two functions with corresponding coefficients. The inf-convolution has a straightforward geometric meaning: the epigraph of $u \,\square\, v$ is the Minkowski sum of the epigraphs of u and v.

By [36, Section 1.6], for every $\alpha, \beta > 0$ and for every $u, v \in \mathrm{Conv}(\mathbb{R}^n)$, we have $\alpha \cdot u \,\square\, \beta \cdot u \in \mathrm{Conv}(\mathbb{R}^n)$, if this function does not attain $-\infty$. Moreover, in this case we have the following relation (see for instance [9, Proposition 2.1]):

$$(\alpha \cdot u \,\square\, \beta \cdot v)^* = (\alpha u^* + \beta v^*).$$

This shows in particular that if $u, v \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ then $\alpha \cdot u \,\square\, \beta \cdot v \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. Indeed, in this case u^* and v^* belong to $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ and so does their usual sum. Consequently, its conjugate belongs to $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. We say that Z is *epi-additive* if

$$\mathrm{Z}(\alpha \cdot u \,\square\, \beta \cdot v) = \alpha\, \mathrm{Z}(u) + \beta\, \mathrm{Z}(v)$$

for all $\alpha, \beta > 0$ and $u, v \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Let $\mathrm{Z} \colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ be a continuous, epi-translation invariant valuation that is epi-homogeneous of degree $m \in \{1, \ldots, n\}$. For $u_1 \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, we consider the functional $\mathrm{Z}_{u_1} \colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$\mathrm{Z}_{u_1}(u) = \mathrm{Z}(u \,\square\, u_1).$$

The functional Z_{u_1} is a continuous and epi-translation invariant valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. Indeed, the valuation property, continuity and vertical translation invariance follow immediately from the corresponding properties of Z. As for translation invariance, let $x_0 \in \mathbb{R}^n$ and $\tau : \mathbb{R}^n \to \mathbb{R}^n$ be the translation by x_0, that is, $\tau(x) = x + x_0$. We have

$$(u \circ \tau^{-1}) \,\square\, u_1 \;=\; \big((u \circ \tau^{-1})^* + u_1^*\big)^* = (u^* + \langle \cdot, x_0 \rangle + u_1^*)^* = (u \,\square\, u_1) \circ \tau^{-1}.$$

Hence the epi-translation invariance of Z_{u_1} follows from the epi-translation invariance of Z. Therefore, we may apply Theorem 1 to obtain a polynomial expansion

$$\mathrm{Z}((\lambda \cdot u) \,\square\, u_1) = \mathrm{Z}_{u_1}(\lambda \cdot u) = \sum_{i=0}^{n} \lambda^i\, \mathrm{Z}_{u_1,i}(u)$$

for $\lambda \geq 0$ and $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, where the functionals $\mathrm{Z}_{u_1,i}$ are continuous, epi-translation invariant valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ that are epi-homogeneous of degree $i \in \{0, \ldots, n\}$.

Similarly, for fixed $\bar{u} \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ one can show that $v \mapsto \mathrm{Z}_{v,i}(\bar{u})$ defines a continuous and epi-translation invariant valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. Hence, as in the proof of Theorem 6.3.4 in [36], we may repeat this argument to obtain the following statement.

Theorem 21. *Let* $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *be a continuous and epi-translation invariant valuation that is epi-homogeneous of degree* $m \in \{1,\dots,n\}$*. There exists a symmetric function* $\bar{\mathrm{Z}} : (\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n))^m \to \mathbb{R}$ *such that for* $k \in \mathbb{N}$*,* $u_1,\dots,u_k \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ *and* $\lambda_1,\dots,\lambda_k \geq 0$*,*

$$\mathrm{Z}(\lambda_1 \cdot u_1 \,\square \cdots \square\, \lambda_k \cdot u_k) = \sum_{\substack{i_1,\dots,i_k \in \{0,\dots,m\}\\ i_1+\cdots+i_k=m}} \binom{m}{i_1 \cdots i_k} \lambda_1^{i_1} \cdots \lambda_k^{i_k}\, \bar{\mathrm{Z}}(u_1[i_1],\dots,u_k[i_k]),$$

where $u_j[i_j]$ *means that the argument* u_j *is repeated* i_j *times. Moreover, the function* $\bar{\mathrm{Z}}$ *is epi-additive in each variable. For* $i \in \{1,\dots,m\}$ *and* $u_{i+1},\dots,u_m \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$*, the map* $u \mapsto \bar{\mathrm{Z}}(u[i],u_{i+1},\dots,u_m)$ *is a continuous, epi-translation invariant valuation on* $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ *that is epi-homogeneous of degree* i*.*

The special case $m = 1$ in the previous result leads to the following result.

Corollary 22. *If* $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous and epi-translation invariant valuation that is epi-homogeneous of degree 1, then* Z *is epi-additive.*

Finally, we also obtain the dual statements. We say that a functional $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is *additive* if $\mathrm{Z}(\alpha\, v + \beta\, w) = \alpha\, \mathrm{Z}(v) + \beta\, \mathrm{Z}(w)$ for all $\alpha, \beta \geq 0$ and $v, w \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

Theorem 23. *Let* $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ *be a continuous, dually epi-translation invariant valuation that is homogeneous of degree* $m \in \{1,\dots,n\}$*. There exists a symmetric function* $\bar{\mathrm{Z}} : (\mathrm{Conv}(\mathbb{R}^n;\mathbb{R}))^m \to \mathbb{R}$ *such that for* $k \in \mathbb{N}$*,* $v_1,\dots,v_k \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ *and* $\lambda_1,\dots,\lambda_k \geq 0$*,*

$$\mathrm{Z}(\lambda_1\, v_1 + \cdots + \lambda_k\, v_k) = \sum_{\substack{i_1,\dots,i_k \in \{0,\dots,m\}\\ i_1+\cdots+i_k=m}} \binom{m}{i_1 \cdots i_k} \lambda_1^{i_1} \cdots \lambda_k^{i_k}\, \bar{\mathrm{Z}}(v_1[i_1],\dots,v_k[i_k]).$$

Moreover, the function $\bar{\mathrm{Z}}$ *is additive in each variable. For* $i \in \{1,\dots,m\}$ *and* $v_{i+1},\dots,v_m \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$*, the map* $v \mapsto \bar{\mathrm{Z}}(v[i],v_{i+1},\dots,v_m)$ *is a continuous and dually epi-translation invariant valuation on* $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ *that is homogeneous of degree* i*.*

The special case $m = 1$ in the previous result leads to the following result.

Corollary 24. *If* $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ *is a continuous and dually epi-translation invariant valuation that is homogeneous of degree 1, then* Z *is additive.*

Let $\zeta \in C_c(\mathbb{R}^n)$. By Proposition 19, the functional

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta(x)\, \mathrm{d}\Theta_0(v,(x,y))$$

defines a continuous, dually epi-translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ that is homogeneous of degree n. Hence, by Theorem 23, for $v_1,\dots,v_k \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\lambda_1,\dots,\lambda_k \geq 0$, there exists a symmetric function $\bar{\mathrm{Z}} : (\mathrm{Conv}(\mathbb{R}^n;\mathbb{R}))^n \to \mathbb{R}$ such that

$$\mathrm{Z}(\lambda_1\, v_1 + \cdots + \lambda_k\, v_k) = \sum_{\substack{i_1,\dots,i_k \in \{0,\dots,n\}\\ i_1+\cdots+i_k=n}} \binom{n}{i_1 \cdots i_k} \lambda_1^{i_1} \cdots \lambda_k^{i_k}\, \bar{\mathrm{Z}}(v_1[i_1],\dots,v_k[i_k]).$$

If we assume in addition that $v_1,\dots,v_k \in C^2(\mathbb{R}^n)$, then by (8) and properties of the mixed discriminant, we can also write

$$\begin{aligned}\mathrm{Z}(\lambda_1\, v_1 + \cdots + \lambda_k\, v_k) &= \int_{\mathbb{R}^n} \zeta(x)\, \det(\mathrm{D}^2(\lambda_1\, v_1 + \cdots + \lambda_k\, v_k)(x))\, \mathrm{d}x\\ &= \sum_{i_1,\dots,i_n=1}^{k} \lambda_{i_1} \cdots \lambda_{i_n} \int_{\mathbb{R}^n} \zeta(x)\, \det(\mathrm{D}^2 v_{i_1}(x),\dots,\mathrm{D}^2 v_{i_n}(x))\, \mathrm{d}x.\end{aligned}$$

It is now easy to see that for such functions $v_1, \dots, v_k$ and $i_1, \dots, i_k \in \{0, \dots, n\}$ with $i_1 + \cdots + i_k = n$,

$$\bar{\mathrm{Z}}(v_1[i_1], \dots, v_k[i_k]) = \int_{\mathbb{R}^n} \zeta(x)\, \det(\mathrm{D}^2 v_1(x)[i_1], \dots, \mathrm{D}^2 v_k[i_k])\,\mathrm{d}x.$$

Note that this is a special case of (2).

7. Classification Theorems

The classification of valuations that are epi-homogenous of degree 0 is straightforward.

Theorem 25. *A functional* $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous and epi-translation invariant valuation that is epi-homogeneous of degree* 0*, if and only if* Z *is constant.*

Proof. Let $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ be a continuous and epi-translation invariant valuation that is epi-homogeneous of degree zero. We show that Z is constant. Indeed, for given $y \in \mathbb{R}^n$, the functional $\tilde{\mathrm{Z}}_y\colon \mathcal{K}^n \to \mathbb{R}$ defined by

$$\tilde{\mathrm{Z}}_y(K) = \mathrm{Z}(\ell_y + \mathbf{I}_K).$$

is a zero-homogeneous, continuous and translation invariant valuation on $\mathcal{K}^n$ and therefore constant. Such a constant cannot depend on y, as, choosing $K = \{0\}$, we obtain

$$\mathbf{I}_{\{0\}} + \ell_y = \mathbf{I}_{\{0\}} + \ell_{y_0}$$

for all $y, y_0 \in \mathbb{R}^n$. Hence there exists $\alpha \in \mathbb{R}$ such that

$$\mathrm{Z}(\mathbf{I}_K + \ell_y) = \alpha$$

for all $K \in \mathcal{K}^n$ and $y \in \mathbb{R}^n$. Thus the statement follows from applying Lemma 16 to $\mathrm{Z} - \alpha$. □

By duality, we also obtain the following result.

Theorem 26. *A functional* $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) \to \mathbb{R}$ *is a continuous and dually epi-translation invariant valuation that is homogeneous of degree* 0*, if and only if* Z *is constant.*

Next, we prove Theorem 2. The "if" part of the proof follows from Proposition 20 and the subsequent remark. The proof of the theorem is completed by the next statement.

Proposition 27. *If* $\mathrm{Z} : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, epi-translation invariant valuation, that is epi-homogeneous of degree* n*, then there exists* $\zeta \in C_c(\mathbb{R}^n)$ *such that*

$$\mathrm{Z}(u) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\,\mathrm{d}x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Proof. For $y \in \mathbb{R}^n$, we consider the map $\tilde{\mathrm{Z}}_y\colon \mathcal{K}^n \to \mathbb{R}$ defined by

$$\tilde{\mathrm{Z}}_y(K) = \mathrm{Z}(\ell_y + \mathbf{I}_K).$$

We know from the proof of Theorem 1 that $\tilde{\mathrm{Z}}_y$ is a continuous and translation invariant valuation on $\mathcal{K}^n$. Moreover, as the functional Z is epi-homogeneous of degree n, the functional $\tilde{\mathrm{Z}}_y$ is homogeneous of degree n. By Theorem 8, for each $y \in \mathbb{R}^n$, there exists a constant, that we denote by $\zeta(y)$, such that

$$\tilde{\mathrm{Z}}(K) = \zeta(y)\, V_n(K) \tag{12}$$

for every $K \in \mathcal{K}^n$. As Z is continuous, the function $\zeta : \mathbb{R}^n \to \mathbb{R}$ is continuous. We prove, by contradiction, that ζ has compact support. Assume that there exists a sequence $y_k \in \mathbb{R}^n$, such that

$$\lim_{k\to\infty} |y_k| = +\infty \tag{13}$$

and $\zeta(y_k) \neq 0$ for every k. Without loss of generality, we may assume that

$$\lim_{k\to\infty} \frac{y_k}{|y_k|} = e_n \tag{14}$$

where e_n is the n-th element of the canonical basis of $\mathbb{R}^n$.

Let

$$B_k = \{x \in y_k^\perp \colon |x| \le 1\}, \quad B_\infty = \{x \in e_n^\perp \colon |x| \le 1\}.$$

Define the cylinder

$$C_k = \left\{ x + t y_k \colon x \in B_k, t \in \left[0, \frac{1}{\zeta(y_k)}\right] \right\}.$$

We have

$$V_n(C_k) = \frac{\kappa_{n-1}}{\zeta(y_k)},$$

where κ_{n-1} is the $(n-1)$-dimensional volume of the unit ball in $\mathbb{R}^{n-1}$.

For $k \in \mathbb{N}$, we consider the function

$$u_k = \ell_{y_k} + \mathbf{I}_{C_k}.$$

This is a sequence of functions in $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$; using (13) and (14), it follows from Lemma 10 that u_k epi-converges to

$$u_\infty = \mathbf{I}_{B_\infty}.$$

In particular, by the continuity of Z and (12) we get

$$0 = \mathrm{Z}(u_\infty) = \lim_{k\to\infty} \mathrm{Z}(u_k).$$

On the other hand, by the definition of u_k and (12),

$$\mathrm{Z}(u_k) = \zeta(y_k)\, V_n(C_k) = \kappa_{n-1} > 0.$$

This completes the proof. $\square$

8. Valuations without Vertical Translation Invariance

In this part we see that Theorems 1 and 4 are no longer true if we remove the assumption of vertical translation invariance. To do so, on the base of Theorem 17 we construct the following example. For $\eta \in C_c(\mathbb{R}^n)$ and $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$, define

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n} e^{v(x) - \langle \nabla v(x), x\rangle}\, \eta(x) \det(\mathrm{D}^2 v(x))\, \mathrm{d}x. \tag{15}$$

By Theorem 17, the functional defined in (15) can be extended to a continuous valuation on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. It is dually translation invariant but not vertically translation invariant. We choose $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ as

$$v(x) = \tfrac{1}{2}|x|^2.$$

Note that the Hessian matrix of v is everywhere equal to the identity matrix. Hence $\det(\mathrm{D}^2 v) = 1$ on $\mathbb{R}^n$. For $\lambda \ge 0$ we have

$$\mathrm{Z}(\lambda v) = \lambda^n \int_{\mathbb{R}^n} \eta(x) e^{\lambda \frac{|x|^2}{2}}\, \mathrm{d}x.$$

If η is non-negative and $\eta(x) \ge 1$ for every x such that $1 \le |x| \le 2$, then

$$\mathrm{Z}(\lambda v) \ge c \lambda^n e^{\lambda/2}$$

for a suitable constant $c > 0$ and for every $\lambda \ge 0$. Hence $\mathrm{Z}(\lambda v)$ does not have polynomial growth as λ tends to ∞.

Theorem 28. *There exist continuous, dually translation invariant valuations on* $\operatorname{Conv}(\mathbb{R}^n;\mathbb{R})$ *which cannot be written as finite sums of homogeneous valuations.*

As a consequence we also have the following dual statement.

Theorem 29. *There exist continuous, translation invariant valuations on* $\operatorname{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ *which cannot be written as finite sums of epi-homogeneous valuations.*

9. Epi-translation Invariant Valuations on Coercive Functions

In this part we prove that every continuous and epi-translation invariant valuation on $\operatorname{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$ is trivial.

Theorem 30. *Every continuous, epi-translation invariant valuation* $\mathrm{Z} : \operatorname{Conv}_{\mathrm{coe}}(\mathbb{R}^n) \to \mathbb{R}$ *is constant.*

Proof. Let $\mathrm{Z} : \operatorname{Conv}_{\mathrm{coe}}(\mathbb{R}^n) \to \mathbb{R}$ be a continuous, epi-translation invariant valuation. We need to show that there exists $\alpha \in \mathbb{R}$ such that $\mathrm{Z}(u) = \alpha$ for every $u \in \operatorname{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$. As in the proof of Theorem 1 define for $y \in \mathbb{R}^n \backslash \{0\}$ the map $\tilde{\mathrm{Z}}_y : \mathcal{K}^n \to \mathbb{R}$ by

$$\tilde{\mathrm{Z}}_y(K) = \mathrm{Z}(\ell_y + \mathbf{I}_K)$$

for every $K \in \mathcal{K}^n$. Since $\tilde{\mathrm{Z}}_y$ is a continuous and translation invariant valuation, by Theorem 7 it admits a homogeneous decomposition

$$\tilde{\mathrm{Z}}_y = \sum_{j=0}^{n} \tilde{\mathrm{Z}}_{y,j},$$

where each $\tilde{\mathrm{Z}}_{y,j}$ is a continuous, translation invariant valuation on $\mathcal{K}^n$ that is homogeneous of degree j.

Next, we will show that $\tilde{\mathrm{Z}}_{y,j} \equiv 0$ for all $1 \leq j \leq n$. Since

$$\tilde{\mathrm{Z}}_{y,0}(K) = \lim_{\lambda \to 0} \tilde{\mathrm{Z}}_{y,0}(\lambda K) = \tilde{\mathrm{Z}}_{y,0}(\{0\})$$

for every $K \in \mathcal{K}^n$, this will then imply that $\tilde{\mathrm{Z}}_y$ is constant. By continuity it is enough to show that $\tilde{\mathrm{Z}}_{y,j}$ vanishes on polytopes for all $1 \leq j \leq n$. Since $\tilde{\mathrm{Z}}_y$ is continuous, it is enough to restrict to polytopes with no facet parallel to $y^\perp$. Therefore, fix such a polytope $P \in \mathcal{P}^n$ of dimension at least one. By translation invariance we can assume that the origin is one of the vertices of P and that P lies in the half-space $\{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0\}$. In particular, this gives $P \cap y^\perp = \{0\}$, $\langle x, y \rangle > 0$ for all $x \in P \backslash \{0\}$ and moreover $\langle x, y \rangle > 0$ and for all $x \in \lambda P \backslash \{0\}$ for all $\lambda > 0$. Due to the choice of P we obtain that $\ell_y + \mathbf{I}_{\lambda P}$ is epi-convergent to $\ell_y + \mathbf{I}_C$ as $\lambda \to \infty$ where C is the infinite cone over P with apex at the origin, that is C is the positive hull of P. Furthermore $\ell_y + \mathbf{I}_C \in \operatorname{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$ since $y \neq 0$. By continuity this gives

$$\mathrm{Z}(\ell_y + \mathbf{I}_C) = \lim_{\lambda \to \infty} \mathrm{Z}(\ell_y + \mathbf{I}_{\lambda P}) = \lim_{\lambda \to \infty} \tilde{\mathrm{Z}}_y(\lambda P) = \lim_{\lambda \to \infty} \sum_{j=0}^{n} \lambda^j \tilde{\mathrm{Z}}_{y,j}(P).$$

Since the left side of this equation is finite, we have $\tilde{\mathrm{Z}}_{y,n}(P) = 0$. Otherwise, the right side would be $\pm\infty$, depending on the sign of $\tilde{\mathrm{Z}}_{y,n}(P)$. Since P was arbitrary, we obtain that $\tilde{\mathrm{Z}}_{y,n}$ vanishes on all compact convex polytopes of dimension greater or equal than 1 and by continuity $\tilde{\mathrm{Z}}_{y,n} \equiv 0$. Similarly, one can now show by induction that also $\tilde{\mathrm{Z}}_{y,j} \equiv 0$ for all $1 \leq j \leq n-1$.

We have proven so far that for every $y \in \mathbb{R}^n \backslash \{0\}$ there exists a constant $\alpha(y) \in \mathbb{R}$ such that $\tilde{\mathrm{Z}}_y \equiv \alpha(y)$. Since

$$\alpha(y) = \tilde{\mathrm{Z}}_y(\{0\}) = \mathrm{Z}(\mathbf{I}_{\{0\}}),$$

we obtain that $\alpha(y)$ is in fact independent of y, that is, there exists $\alpha \in \mathbb{R}$ such that $\tilde{\mathrm{Z}}_y \equiv \alpha$ for every $y \in \mathbb{R}^n$. By the definition of $\tilde{\mathrm{Z}}_y$ and the vertical translation invariance of $\tilde{\mathrm{Z}}$ this gives $\mathrm{Z}(\ell_y + \mathbf{I}_K + \beta) = \alpha$ for every $K \in \mathcal{K}^n$, $y \in \mathbb{R}^n \backslash \{0\}$ and $\beta \in \mathbb{R}$. The claim now follows from Lemma 16. $\square$

If $u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$, then its conjugate $u^* \in \mathrm{Conv}(\mathbb{R}^n)$ and the origin is an interior point of its domain (see, for example, [35, Theorem 11.8]). Let $\mathrm{Conv}_{\mathrm{od}}(\mathbb{R}^n)$ be the set of functions in $\mathrm{Conv}(\mathbb{R}^n)$ with the origin in the interior of its domain. Theorem 30 has the following dual.

Theorem 31. *Every continuous, dually epi-translation invariant valuation* $\mathrm{Z} : \mathrm{Conv}_{\mathrm{od}}(\mathbb{R}^n) \to \mathbb{R}$ *is constant.*

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