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Geometric aspects of locally homogeneous Riemannian spaces

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Introduction

The subject of this thesis is the study of geometric properties of locally and globally homogeneous Riemannian spaces.

A smooth Riemannian manifold (M^m, g) is *globally homogeneous* if its isometry group $\mathrm{Iso}(M, g)$ acts transitively on it, i.e. if for any choice of $x, y \in M$ there exists an isometry $f_{x,y} : M \to M$ mapping x to y. Any globally homogeneous Riemannian space is complete (see [36, Ch IV, Thm 4.5]) and naturally equivariantly diffeomorphic to a (not necessarily unique) quotient space G/H. Here G is a Lie group which acts transitively, isometrically and almost-effectively on M, H is the isotropy of G at a distinguished point $x_{\mathrm{o}} \in M$ and the equivariant diffeomorphism $M \to \mathsf{G}/\mathsf{H}$ is given by $a\mathsf{H} \mapsto a \cdot x_{\mathrm{o}}$, where $\cdot$ denotes the left action of G on M. Any such quotient $(M = \mathsf{G}/\mathsf{H}, g)$ is reductive, i.e. the Lie subalgebra $\mathfrak{h} := \mathrm{Lie}(\mathsf{H})$ admits an $\mathrm{Ad}(\mathsf{H})$-invariant complement $\mathfrak{m}$ in $\mathfrak{g} := \mathrm{Lie}(\mathsf{G})$ (see [3, par 7.22]). The evaluation map $X \mapsto X^*_{x_{\mathrm{o}}} := \frac{d}{dt} \exp(tX) \cdot x_{\mathrm{o}}\big|_{t=0}$ gives rise to a canonical identification $\mathfrak{m} \simeq T_{x_{\mathrm{o}}}M$ and, more generally, to a canonical bijection

$$\big\{\mathsf{G}\text{-invariant tensor fields on } M\big\} \longleftrightarrow \big\{\mathrm{Ad}(\mathsf{H})\text{-invariant tensors on } \mathfrak{m}\big\} \ . \qquad (\diamond)$$

We stress that the relation $(\diamond)$ simplifies significantly the study of geometric problems on this kind of manifolds, making use of Linear Algebra and Representation Theory techniques. For this reason, globally homogeneous Riemannian spaces have been widely investigated in the literature. We mention, in particular, the problem of finding special metrics, e.g. Einstein metrics (see e.g. [87, 7, 12, 8]) and metrics with distinguished curvature conditions (see e.g. [84, 88]). Moreover, since by means of $(\diamond)$ any evolution PDE evolving a G-invariant tensor on M reduces to a dynamical system, globally homogeneous Riemannian spaces have been recently used to study the behavior of invariant geometric flows (see e.g. [42, 9, 40, 43, 11, 10]).

A larger class of manifolds, but perhaps less studied in the literature, is provided by the so called *locally homogeneous Riemannian spaces.* Namely, a smooth Riemannian manifold (M^m, g) is locally homogeneous if its pseudogroup of local isometries acts transitively on it, i.e. if for any choice of $x, y \in M$ there exist $\varepsilon = \varepsilon(x, y) > 0$ and a local isometry $f_{x,y} : \mathcal{B}_g(x, \varepsilon) \subset M \to \mathcal{B}_g(y, \varepsilon) \subset M$ mapping x to y. Here, $\mathcal{B}_g(x, \varepsilon)$ denotes the open ball of radius ε around x in M with respect to the Riemannian distance d_g induced by g. Clearly, any globally homogeneous space is, in particular, locally homogeneous. On the other hand, there exist explicit examples of incomplete locally homogeneous Riemannian spaces which are not locally isometric to any complete one, and hence not locally isometric to any globally homogeneous Riemannian space (see [52, 38]). We call them *strictly locally homogeneous.* Accordingly, we indicate by $\mathcal{H}_m^{\mathrm{loc}}$ the moduli spaces of the equivalence classes of m-dimensional locally homogeneous Riemannian spaces up to equivariant local isometries, and by $\mathcal{H}_m$ the moduli subspace of $\mathcal{H}_m^{\mathrm{loc}}$ given by the non-strictly locally homogeneous Riemannian spaces.

There are many reasons for investigating this larger class of manifolds.

Firstly, it often happens that one can explicitly construct compact non-strictly locally homogeneous Riemannian spaces with some additional properties, which do not admit any transitive isometric global Lie group action. For example, any compact hyperbolic manifold is, by definition, locally isometric to the real hyperbolic space $(\mathbb{R}H^m, g_{\mathrm{hyp}})$, and hence is locally homogeneous, but cannot admit globally non-trivial Killing vector fields by the Bochner Theorem. Also, there are numerous compact non-strictly locally homogeneous Hermitian surfaces, both Kähler and non-Kähler (see e.g. [85, 86]), while there are very few compact globally homogeneous Hermitian surfaces (see [81]).

Secondly, strictly locally homogeneous Riemannian manifolds come up naturally as limits of homogeneous spaces. Indeed, by the Cheeger-Gromov Compactness Theorem, a non-collapsing sequence of homogeneous spaces $(M^{(n)}, g^{(n)})$ with bounded geometry subconverges in the pointed $\mathcal{C}^\infty$-topology to a limit homogeneous space $(M^{(\infty)}, g^{(\infty)})$ (see e.g. [19, Ch 3] and [28, Sec 6.1]). Here, by *bounded geometry* we mean that for any integer $k \geq 0$ there exists $C > 0$, depending only on k, such that

$$\big|\operatorname{Rm}(g^{(n)})\big|_{g^{(n)}} + \big|\nabla^{g^{(n)}} \operatorname{Rm}(g^{(n)})\big|_{g^{(n)}} + \ldots + \big|(\nabla^{g^{(n)}})^k \operatorname{Rm}(g_{g^{(n)}})\big|_{g^{(n)}} \leq C$$

for any $n \in \mathbb{N}$. However, this framework excludes the study of collapsing sequences with bounded curvature (see [17, 18]). In order to overcome this issue, Glickenstein [24] and Lott [44] extended the Cheeger-Gromov Compactness Theorem to sequences of manifolds with no positive lower bound on the injectivity radii. Their key tool is to replace the Riemannian manifolds with the larger category of *Riemannian groupoids* (see [44] and references therein). Remarkably, applying this machinery to homogeneous spaces, the limit Riemannian groupoid one gets is an incomplete locally homogeneous space (see [44, Ex 5.7, Prop 5.9] and [11, Sec 5]) and it is often the case that one needs to study this completion for proving theorems about homogeneous spaces (see e.g. [10, 11]).

The main results of this thesis are contained in Chapter III, where a compactness theorem for the moduli space $\mathcal{H}_m^{\mathrm{loc}}$ is proved.

We recall that, in [41], Lauret developed a framework which allows him to parameterize the moduli space $\mathcal{H}_m$ by a quotient of a distinguished set of Lie algebras with an additional structure (see Section III.3.1). This parametrization associates to any element $\mu \in \mathcal{H}_m$ a pair $(\mathsf{G}_\mu/\mathsf{H}_\mu, g_\mu)$ given by the quotient of a simply connected Lie group G_μ by a closed connected subgroup $\mathsf{H}_\mu \subset \mathsf{G}_\mu$, and a G_μ-invariant metric g_μ. Furthermore, he endowed $\mathcal{H}_m$ with three different topologies and he discussed the relations among them. These are:

- the *pointed convergence topology*, that is the usual convergence in pointed Cheeger-Gromov topology of pointed Riemannian manifolds;
- the *infinitesimal convergence topology*, that is a weaker notion that involves only the germs of the metrics at a point;
- the *algebraic convergence topology*, which only takes into account the underlying algebraic structure.

Here we observe that some known results, which appeared in [77, 78], allow us to reformulate this construction for the entire moduli space $\mathcal{H}_m^{\mathrm{loc}}$ (see also [9, Sec 5]), and both the infinitesimal convergence and the algebraic convergence perfectly extend to this larger space. Actually, for what concerns the former, we introduce a weaker version of that, which we call *s-infintesimal convergence.* In detail

Definition (see Definition III.3.9). For any $s \in \mathbb{N} \cup \{\infty\}$ big enough, a sequence $(\mu^{(n)}) \subset \mathcal{H}_m^{\mathrm{loc}}$ *converges s-infinitesimally* to $\mu^{(\infty)} \in \mathcal{H}_m^{\mathrm{loc}}$ if the Riemannian curvature tensor and its first s covariant derivatives at the origin of $(\mathsf{G}_{\mu^{(n)}}/\mathsf{H}_{\mu^{(n)}}, g_{\mu^{(n)}})$ converge to those of $(\mathsf{G}_{\mu^{(\infty)}}/\mathsf{H}_{\mu^{(\infty)}}, g_{\mu^{(\infty)}})$.

In this definition, we require $s \geq \imath(m) + 2$, where $0 \leq \imath(m) < \frac{3}{2}m$ is defined as the maximum of the Singer invariants of m-dimensional locally homogeneous Riemannian spaces (see Formula (I.4.8)). This is motivated by the fact that any class $\mu \in \mathcal{H}_m^{\mathrm{loc}}$ is completely determined by the curvature tensor and its covariant derivatives at some point up to order $\imath(m)+2$ (see [76]). Note that Lauret's original definition is equivalent to ours in the case $s = \infty$ (see Proposition IV.5.2).

However, there is an issue that arises when one tries to extend the pointed convergence topology to $\mathcal{H}_m^{\mathrm{loc}}$. In fact, if $\mu \in \mathcal{H}_m^{\mathrm{loc}} \setminus \mathcal{H}_m$, i.e. it is strictly locally homogeneous, then the construction above determines a quotient $\mathsf{G}_\mu/\mathsf{H}_\mu$ by a non-closed subgroup $\mathsf{H}_\mu \subset \mathsf{G}_\mu$, which is not even Hausdorff. This issue is overcome by considering *local factor spaces*, which generalize the usual quotients of Lie groups (see [52, 78]). However, since a rigorous definition of local factor spaces depends on arbitrary choices of distinguished neighborhoods inside H_μ and G_μ (see Proposition II.4.1), this approach does not seem to fit well with the study of pointed convergence.

Generalizing a notion introduced in [11], we consider a special class of locally homogeneous Riemannian spaces, whose elements are called *geometric models.* More precisely

Definition (see Definition III.2.1). A *geometric model* is a smooth locally homogeneous Riemannian distance ball $(\mathcal{B}, \hat{g}) = (\mathcal{B}_{\hat{g}}(o, \pi), \hat{g})$ of radius π, centered at o, satisfying $|\sec(\hat{g})| \leq 1$ and $\mathrm{inj}_o(\mathcal{B}, \hat{g}) = \pi$.

A first useful fact about the geometric models is that they provide a parametrization for the moduli space $\mathcal{H}_m^{\mathrm{loc}}$ up to scaling. In fact, denoting by $\mathcal{H}_m^{\mathrm{loc}}(1) \subset \mathcal{H}_m^{\mathrm{loc}}$ the subspace given by the equivalence classes $\mu \in \mathcal{H}_m^{\mathrm{loc}}$ with bounded sectional curvature $|\sec(g_\mu)| \leq 1$, we prove the following

Theorem (see Theorem III.1.1). *For each $\mu \in \mathcal{H}_m^{\mathrm{loc}}(1)$, there exists a geometric model $(\mathcal{B}_\mu, \hat{g}_\mu) = (\mathcal{B}_{\hat{g}_\mu}(o_\mu, \pi), \hat{g}_\mu)$ which is equivariantly locally isometric to $(\mathsf{G}_\mu/\mathsf{H}_\mu, g_\mu)$, and it is unique up to a global equivariant isometry.*

As a consequence, this allows us to study pointed convergence in the moduli space $\mathcal{H}_m^{\mathrm{loc}}(1)$. The main result of this thesis is

Theorem (see Theorem III.1.2, Corollary III.1.3). *The moduli space $\mathcal{H}_m^{\mathrm{loc}}(1)$ is compact in the pointed $\mathcal{C}^{1,\alpha}$-topology for any $0 < \alpha < 1$.*

Notice that no assumption on the covariant derivatives of the curvature is imposed here, and hence this theorem generalizes Glickenstein and Lott's results for sequences of homogeneous spaces with bounded geometry. This also shows that the moduli space $\mathcal{H}_m^{\text{loc}}$ is the natural completion of the space $\mathcal{H}_m$ considered in [41], and that the geometric models provide the right theoretical framework to study convergence of homogeneous and locally homogeneous spaces from the geometric viewpoint. In this direction, we also proved

Theorem (see Theorem III.1.4). *Let $(\mu^{(n)}) \subset \mathcal{H}_m^{\text{loc}}(1)$ be a sequence, $\mu^{(\infty)} \in \mathcal{H}_m^{\text{loc}}(1)$ and $s \geq \imath(m)+2$ an integer.*

i) If $(\mathcal{B}_{\mu^{(n)}}, \hat{g}_{\mu^{(n)}})$ converges to $(\mathcal{B}_{\mu^{(\infty)}}, \hat{g}_{\mu^{(\infty)}})$ in the pointed $\mathcal{C}^{s+2}$-topology, then $(\mu^{(n)})$ converges s-infinitesimally to $\mu^{(\infty)}$.

ii) If $(\mu^{(n)})$ converges $(s+1)$-infinitesimally to $\mu^{(\infty)}$, then $(\mathcal{B}_{\mu^{(n)}}, \hat{g}_{\mu^{(n)}})$ converges to $(\mathcal{B}_{\mu^{(\infty)}}, \hat{g}_{\mu^{(\infty)}})$ in the pointed $\mathcal{C}^{s+2,\alpha}$-topology for any $0 < \alpha < 1$.

As observed in [41, Subsec 6.2] and [43, Subsec 3.4], in general the pointed convergence is stronger than the infinitesimal convergence. Indeed, Lauret exhibited an explicit sequence of Aloff–Wallach spaces $\big(W_{n,n+1}, g^{(n)}\big)$ converging infinitesimally to a limit Aloff–Wallach space $\big(W_{1,1}, g^{(\infty)}\big)$ (see [41, Ex 6.6]). Since $W_{1,1}$ is compact and $W_{n,n+1}$ are pairwise non-homeomorphic, it follows that there is no subsequence of $\big(W_{n,n+1}, g^{(n)}\big)$ converging to $\big(W_{1,1}, g^{(\infty)}\big)$ in the pointed $\mathcal{C}^\infty$-topology. Notice that this implies, in particular, that the injectivity radii along the sequence must tend to zero, otherwise one would have sub-convergence by the Cheeger-Gromov Compactness Theorem. However, up to a rescaling, we can assume that $|\sec(g^{(n)})| \leq 1$ and hence, by our previous theorem, the geometric models of $\big(W_{n,n+1}, g^{(n)}\big)$ converge to the geometric model of $\big(W_{1,1}, g^{(\infty)}\big)$ in the pointed $\mathcal{C}^\infty$-topology.

A key tool for the proof of our Theorem III.1.2 is a local version of the Myers-Steenrod Theorem, which is proved in Chapter II.

In this direction, we recall that a central problem in Lie theory, commonly known as the *Hilbert fifth problem*, has been to characterize Lie groups among all topological groups in terms of group theory and topology. This topic has remarkable consequences in Differential Geometry. In fact, it is often important to show that certain groups of differentiable transformations, on a given smooth manifold, can be turned into a Lie transformation group.

One of the most famed result of this type is due to Myers and Steenrod [53], who proved in 1939 that: *any closed group of isometries acting on a $\mathcal{C}^k$-Riemannian manifold, with $k \geq 2$, is a Lie group.* Afterwards, in 1952 Gleason, Montgomery and Zippin [23, 50] gave a complete answer to the Hilbert fifth problem. In particular, they proved that: *a locally compact topological group admits a Lie group structure if and only if it is locally Euclidean, and this occurs if and only if it has no small subgroups.*

It has to be said, however, that Lie groups were not originally conceived by Sophus Lie as global objects, but rather as what we call nowadays *local Lie groups*. In this regard, we notice that Myers and Steenrod themselves concluded their paper [53] with the open question: "*Is any locally compact group germ of local isometries a Lie group germ?*" About this issue, Mostow claimed in [52, p. 616]: "*It seems to the author that the arguments of Myers and Steenrod apply to closed local groups of isometries as well as to global groups.*". To the best of our knowledge, this statement has never been rigorously proved so far.

Recently Goldbring [25] solved the local version of the Hilbert fifth problem using techniques from non-standard Analysis. In particular, he proved a statement analogous to the one of Gleason, Montgomery and Zippin for local topological groups. Using this last result, we prove the following

Theorem (see Theorem II.1.1). *Any locally compact and effective local topological group of isometries acting on a pointed $\mathcal{C}^{k,\alpha}$-Riemannian manifold, with $k+\alpha>0$, is a local Lie group of isometries.*

which will be crucial in the proof of Theorem III.1.2.

In Chapter IV we carry out a deeper study of the s-infinitesimal convergence introduced in Chapter III. In particular, we construct an explicit 2-parameter family $\{\mu_\star(\varepsilon,\delta) : \varepsilon,\delta \in \mathbb{R}\,,\ \varepsilon>0\,,\ 0 \leq \delta < 1\} \subset \mathcal{H}_3$ with the following property: for any fixed integer $k \geq 0$, there are $(\varepsilon^{(n)}), (\delta^{(n)}) \subset \mathbb{R}$ and $C>0$ such that, letting $\hat{g}^{(n)} := \hat{g}_{\mu_\star(\varepsilon^{(n)},\delta^{(n)})}$,

$$\big|\operatorname{Rm}(\hat{g}^{(n)})\big|_{\hat{g}^{(n)}} + \big|\nabla^{\hat{g}^{(n)}}\operatorname{Rm}(\hat{g}^{(n)})\big|_{\hat{g}^{(n)}} + \ldots + \big|(\nabla^{\hat{g}^{(n)}})^k\operatorname{Rm}(g_{\hat{g}^{(n)}})\big|_{\hat{g}^{(n)}} \leq C\ ,$$
$$\big|(\nabla^{\hat{g}^{(n)}})^{k+1}\operatorname{Rm}(\hat{g}^{(n)})\big|_{\hat{g}^{(n)}} \to +\infty \quad \text{as } n \to +\infty\ .$$

The classe $\mu_\star(\varepsilon,\delta)$ comes from a slight modification of the well-known Berger spheres, arising from the canonical variation of the round metric with respect to the Hopf fibration $S^1 \to \mathsf{SU}(2) \to \mathbb{C}P^1$ (see [3, p. 252]), which correspond to the case $\delta = 0$. This allows us to prove the following

Theorem (see Theorem IV.1.1). *For any choice of $m, s \in \mathbb{N}$ such that $m \geq 3$ and $s \geq \imath(m) + 2$, the notion of s-infinitesimal convergence in $\mathcal{H}_m^{\mathrm{loc}}$ is strictly weaker than the one of $(s{+}1)$-infinitesimal convergence.*

In particular, this shows that keeping all the covariant derivatives of the curvature tensor bounded along a sequence of homogeneous spaces is a much more restrictive condition than just bound a finite number of them. This also leads to the construction of the first example of a sequence of smooth pointed locally homogeneous spaces converging to a smooth pointed locally homogeneous space in the pointed $\mathcal{C}^{k,\alpha}$-topology for any $0 < \alpha < 1$, and which does not admit any convergent subsequence in the pointed $\mathcal{C}^{k+1}$-topology. This phenomenon is for us somehow unexpected.

In the last part of the thesis, we deal with some problems concerning compact homogeneous spaces.

More precisely, in Chapter V we study the space $\mathcal{M}_1^{\mathsf{G}}$ of unit-volume G-invariant Riemannian metrics on a given compact, connected, almost-effective globally homogeneous space $M^m = \mathsf{G}/\mathsf{H}$. This space has been extensively investigated in the literature, especially in the context of the variational formulation of the Einstein equation. In fact, it is well known that G-invariant unit volume Einstein metrics on M can be characterized variationally as the critical points of the scalar curvature functional $\mathrm{scal} : \mathcal{M}_1^{\mathsf{G}} \to \mathbb{R}$. In [12], with the aim of studying homogeneous Einstein metrics from a variational viewpoint, the authors proved that the functional scal satisfies the *Palais-Smale condition* on the subsets $(\mathcal{M}_1^{\mathsf{G}})_\varepsilon := \{g \in \mathcal{M}_1^{\mathsf{G}} : \mathrm{scal}(g) \geq \varepsilon\}$, with $\varepsilon > 0$. Namely, if $(g^{(n)}) \subset \mathcal{M}_1^{\mathsf{G}}$ is a sequence with $\mathrm{scal}(g^{(n)}) \geq \varepsilon > 0$ and $\big|\operatorname{Ric}^{\mathrm{o}}(g^{(n)})\big|_{g^{(n)}} \to 0$, then one can extract a subsequence which converges in the $\mathcal{C}^\infty$-topology to an Einstein metric $g^{(\infty)} \in \mathcal{M}_1^{\mathsf{G}}$ with positive scalar curvature [12, Thm A]. Here, $\operatorname{Ric}^{\mathrm{o}}(g^{(n)})$ is the traceless Ricci tensor of $g^{(n)}$ and $|\cdot|_{g^{(n)}}$ is the norm induced by $g^{(n)}$ on the tensor bundle over M. As is well known, the traceless Ricci tensor is the negative gradient vector of the functional scal with respect to the standard L^2-metric $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathcal{M}_1^{\mathsf{G}}$. Let us stress that, in the general inhomogeneous setting, the Hilbert functional does not satisfy the Palais-Smale condition, even in the cohomogeneity one case (see [6]).

On the other hand, the Palais-Smale condition for the functional scal fails, in general, on the whole space $\mathcal{M}_1^{\mathsf{G}}$. In fact, sometimes there exist the so called 0-*Palais-Smale sequences*, i.e. $(g^{(n)}) \subset \mathcal{M}_1^{\mathsf{G}}$ such that $\mathrm{scal}(g^{(n)}) \to 0$ and $\big|\operatorname{Ric}^{\mathrm{o}}(g^{(n)})\big|_{g^{(n)}} \to 0$. Notice that, unlike the previous case, a 0-Palais-Smale sequence $(g^{(n)})$ cannot have convergent subsequences if M is not a torus. This means that $(g^{(n)})$ goes off to infinity on the set $\mathcal{M}_1^{\mathsf{G}}$ and, consequently, we say that such sequences are *divergent*. Remarkably, there are topological obstructions on the existence of 0-Palais-Smale sequences. Indeed, by [12, Thm 2.1], if M admits a 0-Palais-Smale sequence, then there exists a closed, connected intermediate subgroup $\mathsf{H}^{\mathrm{o}} \subsetneq \mathsf{K}^{\mathrm{o}} \subset \mathsf{G}^{\mathrm{o}}$ such that the quotient $\mathsf{K}^{\mathrm{o}}/\mathsf{H}^{\mathrm{o}}$ is a torus. Here, H^{o} and G^{o} denote the identity components of H and G, respectively.

This last theorem is optimal if the isotropy group H is connected. When H is disconnected, the authors conjectured that G/H is itself a homogeneous torus bundle [12, p. 697]. The main result proved in this chapter, for the purpose of generalizing [12, Thm 2.1], is the following

Theorem (see Theorem V.1.1, Theorem V.1.3). *Let $M^m = \mathsf{G}/\mathsf{H}$ be a compact, connected homogenous space and $(g^{(n)}) \subset \mathcal{M}_1^{\mathsf{G}}$ a diverging sequence of unit volume G-invariant metrics on M. If there exists $C > 0$ such that $|\sec(g^{(n)})| \leq C$ for any $n \in \mathbb{N}$, then there exists an intermediate closed subgroup $\mathsf{H} \subsetneq \mathsf{K} \subset \mathsf{G}$ such that the quotient K/H is a torus. If in addition $\mathrm{scal}(g^{(n)}) \geq \varepsilon > 0$, then there exists a second intermediate closed subgroup $\mathsf{K} \subsetneq \mathsf{K}' \subset \mathsf{G}$ such that the quotient K'/H is not a torus.*

Let us remark that in [11] the following estimate was proved: there exists a uniform constant $C > 0$, which depends only on the dimension $m \in \mathbb{N}$, such that

$$|\operatorname{Rm}(g)|_g \leq C|\operatorname{Ric}(g)|_g \quad \text{for any } g \in \mathcal{M}^{\mathsf{G}} \ ,$$

where $\operatorname{Rm}(g)$ denotes the curvature operator of g [11, Thm 4]. This implies, in particular, that any sequence $(g^{(n)}) \subset \mathcal{M}_1^{\mathsf{G}}$ with $\mathrm{scal}(g^{(n)}) \to \delta \geq 0$ and $\big|\operatorname{Ric}^{\mathrm{o}}(g^{(n)})\big|_{g^{(n)}} \to 0$ has bounded curvature and hence, assuming that M is not a torus, 0-Palais-Smale sequences are special examples of diverging sequences with bounded curvature. Consequently, since we require neither that the Lie groups H, G are connected, nor that the traceless Ricci goes to zero, our result generalizes [12, Thm 2.1]. Let us point out that this also proves the previously mentioned conjecture in [12, p. 697].

The original results presented in this thesis have been published in [61] or have been submitted for publication in [62, 63, 64].

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Chapter I

Preliminaries

I.1 Riemannian manifolds of low regularity

I.1.1 Notation

We indicate with $\langle \cdot, \cdot \rangle_{\mathrm{st}}$ the standard Euclidean inner product on $\mathbb{R}^m$ and with $|\cdot|_{\mathrm{st}}$ the induced norm. We also adopt the following standard notation for any pair $(k, \alpha) \in \big(\mathbb{Z}_{\geq 0} \times [0,1]\big) \cup \{(\infty, 0)\}$ and for any ball $B \subset\subset \mathbb{R}^m$ (see [22, p. 52]):

- if $\alpha = 0$, then $\mathcal{C}^{k,0}(\bar{B}) = \mathcal{C}^k(\bar{B})$ is the set of functions $f : B \to \mathbb{R}$ having all derivatives of order up to k continuous in B with continuous extensions to $\bar{B}$;
- if $\alpha \neq 0$, then $\mathcal{C}^{k,\alpha}(\bar{B})$ is the set of those functions in $\mathcal{C}^k(\bar{B})$ whose k-th order partial derivatives are uniformly α-Hölder continuous in B.

We also consider the total order relation

$$(k_1, \alpha_1) \leq (k_2, \alpha_2) \qquad \Longleftrightarrow \qquad k_1 < k_2 \;\text{ or }\; (k_1 = k_2) \wedge (\alpha_1 \leq \alpha_2)$$

so that $\mathcal{C}^{k_2,\alpha_2}(\bar{B}) \subsetneq \mathcal{C}^{k_1,\alpha_1}(\bar{B})$ for any $(k_1, \alpha_1) < (k_2, \alpha_2)$. For $(k, \alpha) \neq (\infty, 0)$ the space $\mathcal{C}^{k,\alpha}(\bar{B})$ is a Banach space with the norm

$$\|f\|_{\mathcal{C}^{k,\alpha}(\bar{B})} := \begin{cases} \|f\|_{\mathcal{C}^k(\bar{B})} & \text{if } \alpha = 0 \\ \|f\|_{\mathcal{C}^k(\bar{B})} + \max_{|j|=k} \|\partial^j f\|_{\alpha,B} & \text{if } \alpha \neq 0 \end{cases},$$

where

$$\|f\|_{\mathcal{C}^k(\bar{B})} := \sum_{s=0}^{k} \max_{|j|=s} \left\{ \sup_{x \in B} \big|(\partial^j f)(x)\big| \right\} , \quad \|u\|_{\alpha,B} := \sup_{x,y \in B} \frac{\big|u(x) - u(y)\big|}{|x-y|^\alpha} .$$

Here, $j = (j_1, \ldots, j_m)$ is a multi-index, $|j| := j_1 + \ldots + j_m$ and $\partial^j f := \frac{\partial^{|j|} f}{\partial^{j_1} x^1 \ldots \partial^{j_m} x^m}$.

We say that a function $F = (F^1, \ldots, F^q) : U \subset \mathbb{R}^m \to \mathbb{R}^q$ is *of class* $\mathcal{C}^{k,\alpha}$ if $F^i|_B \in \mathcal{C}^{k,\alpha}(\bar{B})$ for any $1 \leq i \leq q$ and for any ball $B \subset\subset U$. In what follows, *smooth* will always mean $\mathcal{C}^\infty$-smooth. We also say that F is *of class* $\mathcal{C}^\omega$ if $F^1, \ldots, F^q$ are real analytic.

A sequence $f^{(n)} : U^{(n)} \subset \mathbb{R}^m \to \mathbb{R}$ of functions of class $\mathcal{C}^{k,\alpha}$ *converges in the* $\mathcal{C}^{k,\alpha}$*-topology* to a function $f^{(\infty)} : U^{(\infty)} \subset \mathbb{R}^m \to \mathbb{R}$ of class $\mathcal{C}^{k,\alpha}$ if $\lim_{n \to +\infty} U^{(n)} = U^{(\infty)}$ and for any ball $B \subset\subset U^{(\infty)}$ it holds that

$$\left\| f^{(n)} - f^{(\infty)} \right\|_{\mathcal{C}^{k,\alpha}(\bar{B})} \to 0 \quad \text{as } n \to +\infty \quad .$$

A path $\gamma : I \subset \mathbb{R} \to \mathbb{R}^m$ is said to be *of class* $\mathcal{AC}$ if for any closed subinterval $[a,b] \subset I$, the restriction $\gamma|_{[a,b]}$ is absolutely continuous. We stress that if $\gamma : I \to \mathbb{R}^m$ is a path of class $\mathcal{AC}$, then the tangent vector $\dot{\gamma}(t) \in T_{\gamma(t)}\mathbb{R}^m = \mathbb{R}^m$ exists for almost all $t \in I$ and $\dot{\gamma} \in L^1([a,b]; \mathbb{R}^m)$ for every closed subinterval $[a,b] \subset I$.

I.1.2 Riemannian metrics of low regularity

Let M be a topological manifold. From now until the end of this section, every manifold is assumed to be connected. An atlas $\mathcal{A}$ on M is said to be a $\mathcal{C}^{k,\alpha}$*-atlas* if its overlap maps are of class $\mathcal{C}^{k,\alpha}$. A $\mathcal{C}^{k_1,\alpha_1}$-atlas $\mathcal{A}_1$ and a $\mathcal{C}^{k_2,\alpha_2}$-atlas $\mathcal{A}_2$ on M with $(k_1, \alpha_1) \leq (k_2, \alpha_2)$ are said to be *compatible* if their union $\mathcal{A}_1 \cup \mathcal{A}_2$ is a $\mathcal{C}^{k_1,\alpha_1}$-atlas on M. The following classical result guarantees the existence of smooth structures under far weaker hypotheses. More precisely

Theorem I.1.1 ([31], Thm 2.9). *Let M be a topological manifold and $\mathcal{A}$ a $\mathcal{C}^k$-atlas on M. If $k \geq 1$, then there exists a smooth atlas $\mathcal{A}_1$ on M compatible with $\mathcal{A}$. Moreover, if $\mathcal{A}_2$ is another smooth atlas on M compatible with $\mathcal{A}$, then $(M, \mathcal{A}_1)$ is smoothly diffeomorphic to $(M, \mathcal{A}_2)$.*

This theorem allows us to restrict our discussion, from now on, to the realm of smooth manifolds. On this regard, we recall the following standard definitions:

- a function $f : M_1 \to M_2$ between smooth manifolds is said to be *of class* $\mathcal{C}^{k,\alpha}$ if its expressions in local coordinates are of class $\mathcal{C}^{k,\alpha}$;
- a tensor field T is said to be *of class* $\mathcal{C}^{k,\alpha}$ if its components in local coordinates are of class $\mathcal{C}^{k,\alpha}$;

- a path $\gamma : I \subset \mathbb{R} \to M$ on a smooth manifold is said to be *of class* $\mathcal{AC}$ if its expressions in local coordinates are of class $\mathcal{AC}$.

A $\mathcal{C}^{k,\alpha}$*-Riemannian manifold* (M,g) is the datum of a smooth manifold M together with a Riemannian metric g on M of class $\mathcal{C}^{k,\alpha}$, that is $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ are of class $\mathcal{C}^{k,\alpha}$ for any choice of coordinate vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}$. Let us define

$$\begin{gathered}\mathcal{I} := \big\{\gamma : [0, T_\gamma] \to M \text{ path of class } \mathcal{AC}\ \big\}\ , \\ \ell_g : \mathcal{I} \to \mathbb{R}\ , \quad \ell_g(\gamma) := \int_0^{T_\gamma} \big|\dot\gamma(t)\big|_g\, dt\ , \\ \mathtt{d}_g : M \times M \to \mathbb{R}\ , \quad \mathtt{d}_g(x,y) := \inf\big\{\ell_g(\gamma) : \gamma \in \mathcal{I},\, \gamma(0) = x,\, \gamma(T_\gamma) = y\big\}\ .\end{gathered} \tag{I.1.1}$$

Proposition I.1.2 ([14])**.** *Let* (M,g) *be a* $\mathcal{C}^0$*-Riemannian manifold and* $(\mathcal{I}, \ell_g, \mathtt{d}_g)$ *as in* (I.1.1).

i) The map ℓ_g *is additive with respect to concatenation, continuous on segments and invariant under reparametrizations.*

ii) The map $\mathtt{d}_g$ *is a distance function and it determines the same topology of* M*.*

iii) Given a path $\gamma \in \mathcal{I}$*, the following equalities hold:*

$$|\dot\gamma(t)|_g = \lim_{\delta\to 0} \frac{\mathtt{d}_g(\gamma(t+\delta), \gamma(t))}{\delta} \qquad \textit{for any } 0 < t < T_\gamma \textit{ such that } \dot\gamma(t) \textit{ exists}\ ,$$

$$\ell_g(\gamma) = \sup\left\{\sum_{i=1}^N \mathtt{d}_g(\gamma(t_{i-1}), \gamma(t_i)) : N \in \mathbb{N},\ 0 = t_0 < t_1 < \ldots < t_N = T_\gamma\right\}\ .$$

This last result shows that the triple $(\mathcal{I}, \ell_g, \mathtt{d}_g)$, defined in (I.1.1), turns a $\mathcal{C}^{k,\alpha}$-Riemannian manifold (M,g) into a separable, locally compact length space. From now on, we will use the notation $\mathcal{B}_g(p,r)$ to denote the metric ball centered at $p \in M$ of radius $r > 0$ in $(M, \mathtt{d}_g)$.

We recall that path $\gamma \in \mathcal{I}$ is said to be a *geodesic* if there exists $\lambda_\gamma > 0$ such that, for any $t_\mathrm{o} \in [0, T_\gamma]$, there exists $\varepsilon_\mathrm{o} > 0$ such that $\mathtt{d}_g(\gamma(t_1), \gamma(t_2)) = \lambda_\gamma |t_1 - t_2|$ for any $t_1, t_2 \in (t_\mathrm{o} - \varepsilon_\mathrm{o}, t_\mathrm{o} + \varepsilon_\mathrm{o}) \cap [0, T_\gamma]$. A geodesic γ is said to be *minimizing* if $\mathtt{d}_g(\gamma(t_1), \gamma(t_2)) = \lambda_\gamma |t_1 - t_2|$ for any $t_1, t_2 \in [0, T_\gamma]$. Furthermore, (M,g) is said to be *complete* if the metric space $(M, \mathtt{d}_g)$ is complete. See [13, Thm 2.5.28] for a generalization of the classical Hopf-Rinow Theorem in this low regularity category.

If $(k,\alpha) \geq (1,0)$, then, given any local chart (U,ξ) on M, one can consider the *Christoffel symbols of* g *in the coordinates* (U,ξ)

$$\Gamma^{r\,(U,\xi)}_{ij} := \tfrac{1}{2}((\xi^{-1})^* g)^{sr}\big(((\xi^{-1})^* g)_{ir,j} + ((\xi^{-1})^* g)_{jr,i} - ((\xi^{-1})^* g)_{ij,r}\big) ,$$

which are functions of class $\mathcal{C}^{k-1,\alpha}$. Here, the comma indicates the ordinary partial derivative.

We say that a path $\gamma \in \mathcal{I}$ *satisfies the geodesic equation* if it is of class $\mathcal{C}^2$ and, for any local chart (U,ξ) on M such that $\gamma((0,T_\gamma)) \cap U \neq 0$, it holds that

$$\big(\tfrac{d^2(\xi\circ\gamma)}{dt^2}\big)^r + \big(\tfrac{d(\xi\circ\gamma)}{dt}\big)^i \big(\tfrac{d(\xi\circ\gamma)}{dt}\big)^j \Gamma^{r\,(U,\xi)}_{ij}(\xi\circ\gamma) = 0 \quad \text{in } \gamma^{-1}(\gamma((0,T_\gamma)) \cap U). \quad \text{(I.1.2)}$$

From the Peano Theorem, equation (I.1.2) admits a local solution for any initial condition $(x,v) \in TM$. Furthermore, if $(k,\alpha) \geq (1,1)$, then this solution is unique by means of Picard-Lindelöf Theorem. Notice also that any solution of (I.1.2) is necessarily of class $\mathcal{C}^{k+1,\alpha}$. The following proposition resumes some results about the regularity of geodesics and their relation with the geodesic equation.

Proposition I.1.3 ([16, 45, 73])**.** *Let* (M,g) *be a* $\mathcal{C}^{k,\alpha}$*-Riemannian manifold.*

- *If* $(0,0) < (k,\alpha) \leq (0,1)$*, then any geodesic is of class* $\mathcal{C}^{1,\frac{\alpha}{2-\alpha}}$.
- *If* $(1,0) \leq (k,\alpha) < (1,1)$*, then the solutions of the geodesic equation* (I.1.2) *need not be geodesics. Conversely, any geodesic is of class* $\mathcal{C}^{2,\alpha}$ *and it satisfies the geodesic equation.*
- *If* $(k,\alpha) \geq (1,1)$*, then any path* $\gamma \in \mathcal{I}$ *is a geodesic if and only if it is of class* $\mathcal{C}^{k+1,\alpha}$ *and satisfies the geodesic equation.*

This last claim brings us to introduce the *Riemannian exponential map.* More concretely

Proposition I.1.4 ([39, 48])**.** *Let* (M,g) *be a* $\mathcal{C}^{k,\alpha}$*-Riemannian manifold with* $(k,\alpha) \geq (1,1)$*. Then, there exist a maximal open set* $\mathcal{U} \subset TM$ *and a map* $\mathrm{Exp}(g) : \mathcal{U} \to M$ *of class* $\mathcal{C}^{k-1,\alpha}$ *with the following properties.*

i) *For any* $x \in M$*, the set* $\mathcal{U} \cap T_xM$ *is non-empty and star-shaped with respect to the origin.*

ii) *For any* $(x,v) \in \mathcal{U}$*, the path* $[0,1] \ni t \mapsto \mathrm{Exp}(g)(x,tv)$ *is the geodesic starting from* $x \in M$ *and tangent to* $v \in T_xM$*.*

iii) For any $x \in M$, there exists a neighborhood $\mathcal{U}_x \subset \mathcal{U} \cap T_xM$ of the origin such that the restriction $\mathrm{Exp}(g)(x,\cdot)|_{\mathcal{U}_x}$ is a $\mathcal{C}^{k-1,\alpha}$-diffeomorphism with its image.

Notice that the Riemannian exponential is not defined if $(0,0) \leq (k,\alpha) < (1,1)$ and, if $(k,\alpha) = (1,1)$, it is just a $\mathcal{C}^{0,1}$-homeomorphism. Furthermore, if $(1,1) < (k,\alpha) < (\infty,0)$, then the atlas of normal coordinates on (M,g) is just of class $\mathcal{C}^{k-1,\alpha}$ and hence the components of $\mathrm{Exp}(g)(x,\cdot)^*g$ are merely of class $\mathcal{C}^{k-2,\alpha}$. Nevertheless, we mention that, for $(k,\alpha) \geq (1,0)$, there exists a distinguished $\mathcal{C}^{k+1,\alpha}$-atlas, which consists of *harmonic coordinate charts*, that is the best possible choice in terms of regularity of g. We refer to the seminal work [21] for more details.

Given two $\mathcal{C}^{k,\alpha}$-Riemannian manifolds (M_1,g_1) and (M_2,g_2), a function $f : M_1 \to M_2$ is said to be a *metric isometry* if it is surjective and distance preserving, i.e. $\mathtt{d}_{g_1}(x,y) = \mathtt{d}_{g_2}(f(x),f(y))$ for any $x,y \in M_1$. It is straightforward to observe that any metric isometry is a $\mathcal{C}^{0,1}$-homeomorphism and that the inverse of a metric isometry is itself a metric isometry. On the other hand, a map $f : M_1 \to M_2$ is called a *Riemannian isometry* if it is a $\mathcal{C}^{k+1,\alpha}$-diffeomorphisms between M_1 and M_2 such that $f^*g_2 = g_1$. Notice that any Riemannian isometry is, in particular, a metric isometry. Remarkably, the following weaker converse assertion holds.

Theorem I.1.5 ([16, 68, 71, 74],[80])**.** *Let $f : (M_1,g_1) \to (M_2,g_2)$ be a metric isometry between $\mathcal{C}^{k,\alpha}$-Riemannian manifolds. If $(k,\alpha){>}(0,0)$, then f is of class $\mathcal{C}^{k+1,\alpha}$ and it is a Riemannian isometry.*

From now on, we will use the term *isometry* just to indicate a metric isometry. By means of Theorem I.1.5, this coincides with the notion of Riemannian isometry with only one exception, namely the pathological case $(k,\alpha) = (0,0)$. Furthermore, the full isometry group of a $\mathcal{C}^{k,\alpha}$-Riemannian manifold (M,g) will be denoted by $\mathrm{Iso}(M,g)$.

Finally, we list some notation. Given a sufficiently regular $\mathcal{C}^{k,\alpha}$-Riemannian manifold (M,g), we denote by ∇^g its Levi-Civita covariant derivative and by $\mathrm{Rm}(g)(X{\wedge}Y) := \nabla^g_{[X,Y]} - [\nabla^g_X, \nabla^g_Y]$ its Riemannian curvature operator. Moreover, we denote the sectional curvature by $\sec(g)$, the Ricci curvature by $\mathrm{Ric}(g)$, the injectivity radius at $p \in M$ by $\mathrm{inj}_p(M,g)$ and for any integer $k \geq 0$ we set

$$\begin{aligned} \mathrm{Rm}^k(g) &: \textstyle\bigotimes^k TM \otimes \Lambda^2 TM \to \mathfrak{so}(TM,g) \ , \\ \mathrm{Rm}^k(g)(X_1,\ldots,X_k|Y_1{\wedge}Y_2) &:= \big((\nabla^g)^k_{X_1,\ldots,X_k} \mathrm{Rm}(g)\big)(Y_1{\wedge}Y_2) \ . \end{aligned} \tag{I.1.3}$$

Remark I.1.6. From now on, when the regularity of a tensor field is not specified, it is intended to be smooth.

I.2 Convergence of Riemannian manifolds

In this section, we recall the notion of Gromov-Hausdorff convergence for compact metric spaces and we give the definition of pointed $\mathcal{C}^{k,\alpha}$-convergence for pointed Riemannian manifolds. This last definition is usually stated for complete Riemannian manifolds (see e.g. [19, Ch 3]), but here we present a version which holds also in the non-complete setting.

I.2.1 Gromov-Hausdorff convergence of compact metric spaces

Let $Z = (Z, \mathtt{d}_Z)$ be a metric space. For any choice of compact subsets $K_1, K_2 \subset Z$, the *Hausdorff distance between K_1 and K_2* is defined by

$$\mathtt{dist}^Z_{\mathrm{H}}(K_1, K_2) := \inf\big\{\varepsilon > 0 : K_1 \subset U_\varepsilon(K_2)\,,\ K_2 \subset U_\varepsilon(K_1)\big\}\ ,$$

where $U_\varepsilon(K) := \{x \in Z : \mathtt{d}_Z(x, K) < \varepsilon\}$ is the ε-tube around K in Z. The pair $\big(\{K \subset Z \text{ compact}\}, \mathtt{dist}^Z_{\mathrm{H}}\big)$ is itself a metric space and it is compact if and only if Z is compact as well (see e.g. [70, p. 195]).

Let now $X = (X, \mathtt{d}_X)$ and $Y = (Y, \mathtt{d}_Y)$ be two compact metric spaces. The *Gromov-Hausdorff distance between X and Y* is defined as

$$\begin{aligned}\mathtt{dist}_{\mathrm{GH}}(X, Y) := \inf\Big\{&\mathtt{dist}^Z_{\mathrm{H}}\big(\phi_1(X), \phi_2(Y)\big) : Z \text{ is a metric space}\,,\\ &\phi_1 : X \to Z \text{ and } \phi_2 : Y \to Z \text{ are isometric embeddings}\Big\}\ .\end{aligned}$$

Letting $\mathcal{X}$ denote the set of isometric classes of compact metric spaces, it turns out that $(\mathcal{X}, \mathtt{dist}_{\mathrm{GH}})$ is a complete metric space (see e.g. [70, Prop 1.1.4]). Therefore, a sequence $(X^{(n)}) \subset \mathcal{X}$ of compact metric spaces is said to *converge in the Gromov-Hausdorff topology to a compact metric space $X^{(\infty)}$* if $\lim_{n\to+\infty} \mathtt{dist}_{\mathrm{GH}}(X^{(n)}, X^{(\infty)}) = 0$.

We stress now that, given two compact metric space $X, Y \in \mathcal{X}$, in general it is difficult to compute the explicit value of $\mathtt{dist}_{\mathrm{GH}}(X, Y)$. To this purpose, we recall that a *Gromov-Hausdorff ε-approximation between X and Y* is a pair (φ, ψ)

of not necessarily continuous maps $\varphi : X \to Y$ and $\psi : Y \to X$ satisfying for any $x, x' \in X$, $y, y' \in Y$

$$\begin{aligned} \left|\mathtt{d}_X(x,x') - \mathtt{d}_Y(\varphi(x),\varphi(x'))\right| < \varepsilon \ , \quad & \mathtt{d}_X(x,(\psi\circ\varphi)(x)) < \varepsilon \ , \\ \left|\mathtt{d}_Y(y,y') - \mathtt{d}_X(\psi(y),\psi(y'))\right| < \varepsilon \ , \quad & \mathtt{d}_Y(y,(\varphi\circ\psi)(y)) < \varepsilon \ . \end{aligned}$$

Remarkably, the following fact holds: if there exists a Gromov-Hausdorff ε-approximation between X and Y, then $\mathtt{dist}_{\mathrm{GH}}(X,Y) \leq \frac{3}{2}\varepsilon$ (see e.g. [70, Lemma 1.3.3]). In particular, it comes that a sequence $(X^{(n)}) \subset \mathcal{X}$ converges to $X^{(\infty)}$ in the Gromov-Hausdorff topology if and only if, for any $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that, for any $n > N$, there exists a Gromov-Hausdorff ε-approximation $(\varphi^{(n)}, \psi^{(n)})$ between $X^{(n)}$ and $X^{(\infty)}$.

I.2.2 Convergence of Riemannian metric tensors

Let M be a smooth manifold. We recall that a sequence $(T^{(n)})$ of tensor fields on M of class $\mathcal{C}^{k,\alpha}$ *converges in the* $\mathcal{C}^{k,\alpha}$*-topology* to a tensor field T on M of class $\mathcal{C}^{k,\alpha}$ of the same type if for any local chart (U,ξ) on M, the components of $(\xi^{-1})^*T^{(n)}$ converge to the components of $(\xi^{-1})^*T$ in the $\mathcal{C}^{k,\alpha}$-topology.

Let us review some properties related to the convergence of Riemannian metrics. For this purpose, for any fixed $p, q \in M$ we consider the subset

$$\mathcal{I}_{p,q} := \left\{\gamma \in \mathcal{I} : \gamma(0) = p \,,\ \gamma(T_\gamma) = q\right\} \ ,$$

where $\mathcal{I}$ has been defined in I.1.1.

Proposition I.2.1. *Let $(g^{(n)})$ be a sequence of $\mathcal{C}^0$-Riemannian metrics on M which converges to a $\mathcal{C}^0$-Riemannian metric $(g^{(\infty)})$ in the $\mathcal{C}^0$-topology. Then*
i) $\ell_{g^{(n)}}(\gamma) \to \ell_{g^{(\infty)}}(\gamma)$ *as* $n \to +\infty$, *for any* $\gamma \in \mathcal{I}_{p,q}$;
ii) $\limsup_{n\to+\infty} \mathtt{d}_{g^{(n)}}(p,q) \leq \mathtt{d}_{g^{(\infty)}}(p,q)$.

Proof. In order to prove claim (i), fix $\gamma \in \mathcal{I}_{p,q}$. We can assume, without loss of generality, that $M = \mathbb{R}^m$ and $T_\gamma = 1$. Then

$$\ell_{g^{(n)}}(\gamma) = \int_0^1 \sqrt{g^{(n)}_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)}\, dt \ .$$

By assumption, $\big\|g^{(n)}_{ij}(\gamma(\cdot)) - g^{(\infty)}_{ij}(\gamma(\cdot))\big\|_{\mathcal{C}^0([0,1])} \to 0$ for any $1 \leq i,j \leq m$. Hence:

- $\sqrt{g^{(n)}_{ij}(\gamma(\cdot))\dot\gamma^i\dot\gamma^j}$ converges almost-everywhere to $\sqrt{g^{(\infty)}_{ij}(\gamma(\cdot))\dot\gamma^i\dot\gamma^j}$ in $[0,1]$;
- there exists $F \in L^1([0,1];\mathbb{R})$ such that $\sqrt{g^{(n)}_{ij}(\gamma(\cdot))\dot\gamma^i\dot\gamma^j} \leq F$ almost-everywhere in $[0,1]$.

Therefore, claim (i) follows from the Lebesgue Dominated Convergence Theorem.

For claim (ii), fix $\varepsilon > 0$ and let $\gamma \in \mathcal{I}_{p,q}$ such that $\ell_{g^{(\infty)}}(\gamma) < \mathtt{d}_{g^{(\infty)}}(p,q) + \frac{\varepsilon}{2}$. Since $\ell_{g^{(n)}}(\gamma) \to \ell_{g^{(\infty)}}(\gamma)$ as $n \to +\infty$, there exists $\bar n = \bar n(\varepsilon) > 0$ such that $\ell_{g^{(\infty)}}(\gamma) - \frac{\varepsilon}{2} < \ell_{g^{(n)}}(\gamma) < \ell_{g^{(\infty)}} + \frac{\varepsilon}{2}$ for any $n > \bar n$. Then

$$\mathtt{d}_{g^{(n)}}(p,q) \leq \ell_{g^{(n)}}(\gamma) < \ell_{g^{(\infty)}}(\gamma) + \tfrac{\varepsilon}{2} < \mathtt{d}_{g^{(\infty)}}(p,q) + \varepsilon \qquad \text{for any } n > \bar n$$

and hence the thesis follows. □

As the next example shows, this result cannot be improved, not even for the $\mathcal{C}^\infty$-convergence.

Example I.2.2. Let $M := \big\{(x^1,x^2) \in \mathbb{R}^2 : |x^2| < 1\big\}$, g_{st} the Euclidean metric on M and $(a^{(n)}) \subset (0,1]$ a monotone sequence such that $a^{(n)} \to 0$. For any $n \in \mathbb{N}$, we define

$$R^{(n)} := \big\{(x^1,x^2) \in \mathbb{R}^2 : |x^1| < n, |x^2| < 1 - \tfrac{1}{n}\big\} \subset M$$

and we consider a smooth function $f^{(n)} : M \to \mathbb{R}$ such that

- $f^{(n)} = 1$ in $\overline{R^{(n)}}$,
- $(a^{(n)})^2 < f^{(n)} < 1$ in $R^{(n+1)} \setminus \overline{R^{(n)}}$,
- $f^{(n)} = (a^{(n)})^2$ in $M \setminus R^{(n+1)}$.

Let $g^{(n)} := f^{(n)} g_{\mathrm{st}}$, fix the points $p := (0, \frac{2}{3}), q := (0, -\frac{2}{3}) \in M$ and consider for any $n \in \mathbb{N}$ the piecewise linear path $\gamma^{(n)} : [0,1] \to M$ with vertices

$$\begin{gathered} p^{(n)}_0 := p \ , \quad p^{(n)}_1 := \big(0, 1 - \tfrac{1}{n+1}\big) \ , \quad p^{(n)}_2 := \big(n+1, 1 - \tfrac{1}{n+1}\big) \ , \\ p^{(n)}_3 := \big(n+1, -1 + \tfrac{1}{n+1}\big) \ , \quad p^{(n)}_4 := \big(0, -1 + \tfrac{1}{n+1}\big) \ , \quad p^{(n)}_5 := q \ . \end{gathered}$$

Therefore

$$\mathtt{d}_{g^{(n)}}(p,q) \leq \ell_{g^{(n)}}(\gamma^{(n)}) < \tfrac{1}{3} + (2(n+1)+2)a^{(n)} + \tfrac{1}{3} = \tfrac{2}{3} + 2(n+2)a^{(n)}$$

and hence, by choosing $a^{(n)} := \frac{1}{6(n+2)}$, we get $\limsup_{n\to+\infty} \mathtt{d}_{g^{(n)}}(p,q) \leq 1$. On the other hand, $(g^{(n)})$ converges in the $\mathcal{C}^\infty$-topology to g_{st} on M and $\mathtt{d}_{\mathrm{st}}(p,q) = \frac{4}{3} > 1$.

Let us indicate now by $\mathcal{M}$ the space of smooth Riemannian metrics on M and by $\mathrm{Diff}(M)\backslash\mathcal{M}$ the moduli space of smooth metrics up to smooth diffeomorphism. Then, as the next example shows, the quotient topology on $\mathrm{Diff}(M)\backslash\mathcal{M}$, inherited from the $\mathcal{C}^\infty$-convergence in $\mathcal{M}$, is not Hausdorff.

Example I.2.3. Let g_o be a non-flat Riemannian metric on $\mathbb{R}^m$ which coincides with the Euclidean metric g_st outside the closed ball $\overline{B_\mathrm{st}(0,1)}$ and fix a vector $v \in \mathbb{R}^m$, $|v|_\mathrm{st} = 1$. We consider now the sequence $(\phi^{(n)}) \subset \mathrm{Diff}(\mathbb{R}^m)$ of diffeomorphisms defined by $\phi^{(n)}(x) := x + nv$. Since $(d\phi^{(n)})_x = \mathrm{Id}_{\mathbb{R}^m}$ and $|\phi^{(n)}(x)|_\mathrm{st} \geq n - |x|_\mathrm{st}$ for every $x \in \mathbb{R}^m$, it follows that $(\phi^{(n)})^* g_\mathrm{o}$ converges to g_st in the $\mathcal{C}^\infty$-topology, which is not isometric to g_o. Therefore, g_st and g_o cannot be separated by $\mathrm{Diff}(\mathbb{R}^m)$-invariant neighborhoods in $\mathcal{M}$.

I.2.3 Pointed convergence of Riemannian manifolds

Let now (M,g) be a $\mathcal{C}^{k,\alpha}$-Riemannian manifold. The *distance of a point* $p \in M$ *from the boundary of* (M,g) is defined as the supremum

$$\mathtt{dist}(p, \partial(M,g)) := \sup\left\{ r \geq 0 : \mathcal{B}_g(p,r) \subset\subset M \right\} .$$

Notice that, by the Hopf-Rinow-Cohn-Vossen Theorem, the manifold (M,g) is complete if and only if $\mathtt{dist}(p, \partial(M,g)) = +\infty$ for some, and hence for any, $p \in M$ (see [13, Thm 2.5.28]).

Definition I.2.4. Let $(M^{(n)}, g^{(n)}, p^{(n)})$ be a sequence of pointed $\mathcal{C}^{k,\alpha}$-Riemannian m-manifolds, $\delta^{(n)} := \mathtt{dist}(p^{(n)}, \partial(M^{(n)}, g^{(n)}))$ and assume that $\delta^{(n)} \to \delta^{(\infty)} \in (0, +\infty]$. The sequence $(M^{(n)}, g^{(n)}, p^{(n)})$ is said to *converge in the pointed* $\mathcal{C}^{k,\alpha}$-*topology* to a pointed $\mathcal{C}^{k,\alpha}$-Riemannian manifold $(M^{(\infty)}, g^{(\infty)}, p^{(\infty)})$ if there exist:

i) an exhaustion $(U^{(n)})$ of $M^{(\infty)}$ by relatively compact open sets containing the point $p^{(\infty)}$;
ii) a sequence $(\tilde{\delta}^{(n)}) \subset \mathbb{R}$ such that $0 < \tilde{\delta}^{(n)} \leq \delta^{(n)}$ and $\tilde{\delta}^{(n)} \to \delta^{(\infty)}$;
iii) a sequence of $\mathcal{C}^{k+1,\alpha}$-embeddings $\phi^{(n)} : U^{(n)} \to M^{(n)}$ such that $\phi^{(n)}(p^{(\infty)}) = p^{(n)}$;

so that the following conditions are satisfied:

- $\mathcal{B}_{g^{(n)}}(p^{(n)}, \tilde{\delta}^{(n)}) \subset \phi^{(n)}(U^{(n)}) \subset \mathcal{B}_{g^{(n)}}(p^{(n)}, \delta^{(n)})$;
- $\phi^{(n)*} g^{(n)}$ converges to $g^{(\infty)}$ in the $\mathcal{C}^{k,\alpha}$-topology.

Notice that this type of convergence implies that, for any fixed $0 < r < \delta^{(\infty)}$, there exists an integer $\bar{n} = \bar{n}(r) \in \mathbb{N}$ such that the Riemannian distance ball $\mathcal{B}_{g^{(n)}}(p^{(n)}, r)$ is compactly contained in $M^{(n)}$, for any $n \geq \bar{n}$, and the sequence of compact balls $\big(\overline{\mathcal{B}_{g^{(n)}}(p^{(n)}, r)}, \mathtt{d}_{g^{(n)}}\big)_{n \geq \bar{n}}$ converges in the Gromov-Hausdorff topology to $\big(\overline{\mathcal{B}_{g^{(\infty)}}(p^{(\infty)}, r)}, \mathtt{d}_{g^{(\infty)}}\big)$ (see [66, p. 415] and [13, Ex 8.1.3]). Moreover:

- if $\delta^{(\infty)}$ is finite, then the limit space $(M^{(\infty)}, g^{(\infty)}, p^{(\infty)})$ is an incomplete Riemannian distance ball of radius $\delta^{(\infty)}$, centered at $p^{(\infty)}$, and also $\mathtt{dist}(p^{(\infty)}, \partial(M^{(\infty)}, g^{(\infty)})) = \delta^{(\infty)}$;
- if $\delta^{(\infty)} = +\infty$, then the limit space $(M^{(\infty)}, g^{(\infty)}, p^{(\infty)})$ is complete.

Notice also that, if $\delta^{(n)} = +\infty$ for any $n \in \mathbb{N}$, then we get back to the usual definition of pointed convergence for complete Riemannian manifolds (see e.g. [66, p. 415]).

Example I.2.5. Let us consider the sequence of pointed m-dimensional spheres $(M^{(n)}, g^{(n)}, p^{(n)}) := (S^m, n^2 g_{\text{round}}, o)$, where g_{round} is the round metric of radius 1 on S^m and $o := (+1, 0, \ldots, 0)$ is the North pole. Obviously in this case $\delta^{(n)} = +\infty$ for any $n \in \mathbb{N}$. Let us define

$$\phi^{(n)} : \mathbb{R}^m \to S^m , \quad \phi^{(n)}(y) := \left(\frac{4n^2 - |y|^2_{\text{st}}}{4n^2 + |y|^2_{\text{st}}}, \frac{4n\, y^1}{4n^2 + |y|^2_{\text{st}}}, \ldots, \frac{4n\, y^m}{4n^2 + |y|^2_{\text{st}}} \right) .$$

One can directly check that each map $\phi^{(n)}$ is a smooth diffeomorphism of $\mathbb{R}^m$ into $S^m \setminus \{(-1, 0, \ldots, 0)\}$, with inverse given by

$$(\phi^{(n)})^{-1} : S^m \setminus \{(-1, 0, \ldots, 0)\} \to \mathbb{R}^m , \quad (\phi^{(n)})^{-1}(x) := \left(\frac{2n\, x^1}{1 + x^0}, \ldots, \frac{2n\, x^m}{1 + x^0} \right) .$$

Moreover

$$\phi^{(n)}(0) = o , \quad \phi^{(n)*} g^{(n)} = \left(\frac{4n^2}{4n^2 + |y|^2_{\text{st}}} \right)^2 g_{\text{st}} \quad \text{for any } n \in \mathbb{N} ,$$

where we recall that g_{st} is the standard Euclidean metric on $\mathbb{R}^m$. So, we conclude that $(S^m, n^2 g_{\text{round}}, o)$ converges to $(\mathbb{R}^m, g_{\text{st}}, 0)$ in the pointed $\mathcal{C}^\infty$-topology.

Remark I.2.6. Let us stress that the pointed convergence is significantly different from the mere convergence of metric tensors that we saw in Section I.2.2, even when we are considering a sequence of Riemannian metrics on the same manifold.

- Referring to Example I.2.3, take a sequence of points $(p^{(n)}) \subset \mathbb{R}^m$. Then, it is easy to check that $(\mathbb{R}^m, g_{\mathrm{o}}, p^{(n)})$ converges to $(\mathbb{R}^m, g_{\mathrm{o}}, p^{(\infty)})$ in the pointed $\mathcal{C}^\infty$-topology if $p^{(n)} \to p^{(\infty)}$ for some $p^{(\infty)} \in \mathbb{R}^m$, while it converges to $(\mathbb{R}^m, g_{\mathrm{st}}, 0)$ if $p^{(n)} \to +\infty$.
- Referring to Example I.2.5, the metrics $g^{(n)} = n^2 g_{\mathrm{round}}$ on S^m do not converge to any metric on S^m in the $\mathcal{C}^\infty$-topology, while $(S^m, g^{(n)}, o)$ converges to $(\mathbb{R}^m, g_{\mathrm{st}}, 0)$ in the pointed $\mathcal{C}^\infty$-topology.

We mention here that the Cheeger-Gromov Precompactness Theorem states that given two constants $D_{\mathrm{o}}, v_{\mathrm{o}} > 0$, the space of smooth compact Riemannian m-manifolds with bounded curvature, diameter at most D_{o} and volume at least v_{o} is precompact in the pointed $\mathcal{C}^{1,\alpha}$-topology. Various versions of this classical result are known, e.g. for complete non-compact but pointed Riemannian manifolds or for bounded domains in possibly incomplete pointed Riemannian manifolds. We refer to the survey paper [29] and references therein for more details.

We state below some well-known results that will be needed in Chapter III.

Definition I.2.7 ([1], Sec 2)**.** Let (M^m, g) be a smooth Riemannian manifold, $p \in M$. The *$\mathcal{C}^{k,\alpha}$-harmonic radius of (M^m, g) at p*, which we denote by $\mathrm{har}_p^{k,\alpha}(M, g)$, is the largest $r \geq 0$ such that there exists a local chart $\xi = (\xi^1, \dots, \xi^m) : \mathcal{B}_g\big(p, \sqrt{2}r\big) \to \mathbb{R}^m$ such that

i) $\xi(p) = 0$ and $((\xi^{-1})^* g)_{ij}(0) = \delta_{ij}$ for any $1 \leq i, j \leq m$;
ii) $\Delta_g \xi^i = 0$ for any $1 \leq i \leq m$;
iii) $\frac{1}{2}|v|^2_{\mathrm{st}} \leq ((\xi^{-1})^* g)_{ij}(x) v^i v^j \leq 2|v|^2_{\mathrm{st}}$ for any $v \in \mathbb{R}^m$ and $x \in \xi\big(\mathcal{B}_g\big(p, \sqrt{2}r\big)\big)$;
iv) $r^{k+\alpha} \cdot \|((\xi^{-1})^* g)_{ij}\|_{\mathcal{C}^{k,\alpha}(\overline{B_{\mathrm{st}(0,r)}})} \leq 1$ for any $1 \leq i, j \leq m$.

Here, Δ_g is the Laplace-Beltrami operator of (M, g). Notice that, by (i) and (iii), it comes that $B_{\mathrm{st}}(0, r) \subset \xi\big(\mathcal{B}_g\big(p, \sqrt{2}r\big)\big) \subset B_{\mathrm{st}}(0, 2r)$ in $\mathbb{R}^m$. Moreover, from (ii), (iv) and the classical Schauder interior estimates (see [22, Thm 6.2 and (4.17)]), there exists a constant $C = C(m, k, \alpha, r) > 0$ such that

$$\|f \circ \xi^{-1}\|_{\mathcal{C}^{k+1,\alpha}(\overline{B_{\mathrm{st}}(0,\frac{r}{2})})} \leq C \sup_{\xi^{-1}(B_{\mathrm{st}}(0,r))} |f|$$

for any Δ_g-harmonic function $f : \xi^{-1}(B_{\mathrm{st}}(0, r)) \subset M \to \mathbb{R}$ (see also [34, Sec 1]).

Harmonic coordinates and the harmonic radius play a central role in convergence theory of Riemannian manifolds. We recall the following important result.

Theorem I.2.8 (see e.g. [29], Thm 6)**.** *Let* (M^m, g) *be a smooth, not necessarily complete, Riemannian manifold,* $U \subset M$ *an open subset and* $\delta > 0$ *such that the following conditions are satisfied.*

a) There exist an integer $k \geq 0$ *and a constant* $C > 0$ *such that*

$$\sup\Big\{\big|\operatorname{Rm}^j(g)_x\big|_g : x \in M\,,\ \mathtt{d}_g(U,x) < \delta\Big\} \leq C \qquad \textit{for any } 0 \leq j \leq k\ .$$

b) There exists a constant $i_{\mathrm{o}} \in \mathbb{R}$ *such that*

$$\inf\Big\{\operatorname{inj}_x(M,g) : x \in M\,,\ \mathtt{d}_g(U,x) < \delta\Big\} \geq i_{\mathrm{o}} > 0\ .$$

Then, for any $0 < \alpha < 1$ *there exists a positive constant* $r_{\mathrm{o}} = r_{\mathrm{o}}(m,\delta,k,\alpha,C,i_{\mathrm{o}})$ *such that* $\operatorname{har}^{k+1,\alpha}_x(M,g) \geq r_{\mathrm{o}}$ *for any* $x \in U$*.*

which in turn implies

Theorem I.2.9 (Cheeger-Gromov precompactness Theorem)**.** *Let us consider a sequence* $(M^{(n)}, g^{(n)}, p^{(n)})$ *of pointed smooth complete Riemannian m-manifolds and suppose that the following conditions are satisfied.*

a) There exist an integer $k \geq 0$ *and a constant* $C > 0$ *such that*

$$\sup\Big\{\big|\operatorname{Rm}^j(g^{(n)})_x\big|_{g^{(n)}} : x \in M^{(n)}\Big\} \leq C \qquad \textit{for any } 0 \leq j \leq k\ .$$

b) There exists a constant $i_{\mathrm{o}} \in \mathbb{R}$ *such that*

$$\operatorname{inj}_{p^{(n)}}(M^{(n)}, g^{(n)}) \geq i_{\mathrm{o}} > 0\ .$$

Then, for any $0 < \alpha < 1$ *there exists a pointed complete* $\mathcal{C}^{k+1,\alpha}$*-Riemannian m-manifold* $(M^{(\infty)}, g^{(\infty)}, p^{(\infty)})$ *such that the sequence* $(M^{(n)}, g^{(n)}, p^{(n)})$ *converges, up to a subsequence, to* $(M^{(\infty)}, g^{(\infty)}, p^{(\infty)})$ *in the pointed* $\mathcal{C}^{k+1,\alpha}$*-topology.*

We also mention that one way to get convergence results, when the sequence of Riemannian manifolds being considered is not complete, is to use local mollifications in the sense of Hochard [32]. More concretely, we have

Lemma I.2.10 ([32], Lemma 6.2 and [75], Lemma 4.3)**.** *Let* (M, g) *be a smooth, not necessarily complete, Riemannian manifold and* $V \subset M$ *an open set. Assume that for some* $0 < \rho \leq 1$ *it holds*

$$\big|\operatorname{Rm}(g)_x\big|_g \leq \rho^{-2}\ , \quad \operatorname{inj}_x(M,g) \geq \rho \qquad \textit{for any } x \in V \quad .$$

Then, there exists a constant $c = c(m) \geq 1$, an open subset $\tilde{V} \subset V$ and a smooth Riemannian metric $\tilde{g}$ defined on $\tilde{V}$ such that $(\tilde{V}, \tilde{g})$ is a complete Riemannian manifold satisfying

- $\sup \big\{ \big| \operatorname{Rm}(\tilde{g})_x \big|_{\tilde{g}} : x \in \tilde{V} \big\} \leq c\rho^{-2}$;
- $V_{(2\rho)} \subset \tilde{V}_{(\rho)} \subset V_{(\rho)} \subset \tilde{V} \subset V$;
- $\tilde{g} = g$ *on* $\tilde{V}_{(\rho)}$;

where we used the notation $U_{(r)} := \{x \in U : \mathcal{B}_g(x, r) \subset\subset U\}$ for any open subset $U \subset M$.

This last result will be used in Chapter III.

I.3 Groups of transformations

In this subsection, we collect some basic facts on global and local groups of transformations. For more details, we refer to [56, Ch 1], [59], [67].

I.3.1 Global groups of transformations

We recall that a *topological group* is a Hausdorff topological space equipped with a continuous group structure. A topological group is a *(real analytic) Lie group* if it is endowed with a smooth (resp. real analytic) manifold structure with respect to which the group operations are smooth (resp. real analytic). It is well known that the category of real analytic Lie groups is equivalent to the category of smooth Lie groups via the forgetful functor (see e.g. [36, p. 43]). The following characterization of Lie groups is the above mentioned solution to the Hilbert fifth problem.

Theorem I.3.1 ([23, 50])**.** *For any connected and locally compact topological group G, the following conditions are equivalent.*

a) G *is locally Euclidean, i.e. there is a neighborhood of $e \in \mathsf{G}$ homeomorphic to an open ball of some $\mathbb{R}^N$.*

b) G *has no small subgroups (NSS for short), i.e. there exists a neighborhood of $e \in \mathsf{G}$ containing no nontrivial subgroups of G.*

c) G *admits a unique smooth manifold structure which turns it into a Lie group.*

Let M be a smooth manifold. Given $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, a *topological group of $\mathcal{C}^k$-transformations $G = (\mathsf{G}, \Theta)$ on M* is the datum of a topological group G

together with a continuous action $\Theta : \mathsf{G} \times M \to M$ on M such that the map $\Theta(a) := \Theta(a, \cdot) : M \to M$ is of class $\mathcal{C}^k$ for any $a \in \mathsf{G}$. We recall that the correspondence $a \mapsto \Theta(a)$ determines a group homeomorphism $\mathsf{G} \to \mathrm{Diff}^k(M)$ from G to the group of $\mathcal{C}^k$-diffeomorphisms of M, and that $G = (\mathsf{G}, \Theta)$ is called *effective* (resp. *almost-effective*) if the kernel of $\mathsf{G} \to \mathrm{Diff}^k(M)$ is trivial (resp. discrete). Furthermore, we say that $G = (\mathsf{G}, \Theta)$ is *closed* if it is effective and $\Theta(\mathsf{G})$ is closed in $\mathrm{Diff}^k(M)$.

A topological group of $\mathcal{C}^k$-transformations $G = (\mathsf{G}, \Theta)$ on M is called *Lie group of $\mathcal{C}^k$-transformations* if G is a Lie group and the function $\Theta : \mathsf{G} \times M \to M$ is of class $\mathcal{C}^k$.

Remark I.3.2. By [4, Thm 4], the second condition above is redundant. Namely, if G is a Lie group and each map $\Theta(a) : M \to M$ is of class $\mathcal{C}^k$, then the map $\Theta : \mathsf{G} \times M \to M$ is automatically of class $\mathcal{C}^k$.

If M is equipped with a $\mathcal{C}^{k,\alpha}$-Riemannian metric g, then $G = (\mathsf{G}, \Theta)$ is called *topological (resp. Lie) group of isometries* if each map $\Theta(a) : M \to M$ is an isometry of (M, g). By the classical Myers-Steenrod Theorem, it is known that *any closed topological group of isometries of a $\mathcal{C}^k$-Riemannian manifold, with $k \geq 2$, is a Lie group of isometries.*

I.3.2 Local topological groups and local Lie groups

A *local topological group* is a tuple $(\mathsf{G}, e, \mathcal{J}(\mathsf{G}), \mathcal{D}(\mathsf{G}), \jmath, \nu)$ given by:

i) a Hausdorff topological space G with a distinguished element $e \in \mathsf{G}$ called *unit*;

ii) a neighborhood $\mathcal{J}(\mathsf{G}) \subset \mathsf{G}$ of e and an open subset $\mathcal{D}(\mathsf{G}) \subset \mathsf{G} \times \mathsf{G}$ which contains both $\mathsf{G} \times \{e\}, \{e\} \times \mathsf{G}$;

iii) two continuous maps $\jmath : \mathcal{J}(\mathsf{G}) \to \mathsf{G}$, $\nu : \mathcal{D}(\mathsf{G}) \to \mathsf{G}$;

so that, for any choice of $a, a_1, a_2 \in \mathsf{G}$ and $b \in \mathcal{J}(\mathsf{G})$:

- $\nu(a, e) = \nu(e, a) = a$;
- if $(a_1, a), (a, a_2), (a_1, \nu(a, a_2)), (\nu(a_1, a), a_2) \in \mathcal{D}(\mathsf{G})$, then $\nu(a_1, \nu(a, a_2)) = \nu(\nu(a_1, a), a_2)$;
- $(b, \jmath(b)), (\jmath(b), b) \in \mathcal{D}(\mathsf{G})$ and $\nu(b, \jmath(b)) = \nu(\jmath(b), b) = e$.

From now on, we adopt the usual notation $a_1 \cdot a_2 := \nu(a_1, a_2)$, $a^{-1} := \jmath(a)$ and we will indicate any local topological group $(\mathsf{G}, e, \mathcal{J}(\mathsf{G}), \mathcal{D}(\mathsf{G}), \jmath, \nu)$ simply by G.

Given a local topological group G, every neighborhood $\mathcal{U}$ of the unit $e \in \mathsf{G}$ inherits a structure of local topological group induced by G. In fact, if we set

$$\mathcal{J}(\mathcal{U}) := \mathcal{J}(\mathsf{G}) \cap \mathcal{U} \cap \jmath^{-1}(\mathcal{U}) , \quad \mathcal{D}(\mathcal{U}) := \mathcal{D}(\mathsf{G}) \cap (\mathcal{U} \times \mathcal{U}) \cap \nu^{-1}(\mathcal{U}) ,$$

then one can directly check that $(\mathcal{U}, e, \mathcal{J}(\mathcal{U}), \mathcal{D}(\mathcal{U}), \jmath|_{\mathcal{J}(\mathcal{U})}, \nu|_{\mathcal{D}(\mathcal{U})})$ is itself a local topological group. In this case, we say that $\mathcal{U}$ *is a restriction of* G. We remark that G can be restricted to a neighborhood $\mathcal{U}$ of the unit which is *symmetric*, i.e. $\mathcal{U} = \mathcal{J}(\mathcal{U})$, and *cancellative*, i.e. for any $a, a_1, a_2 \in \mathcal{U}$ it holds that:

- if $(a, a_1), (a, a_2) \in \mathcal{D}(\mathcal{U})$ and $a \cdot a_1 = a \cdot a_2$, then $a_1 = a_2$;
- if $(a_1, a), (a_2, a) \in \mathcal{D}(\mathcal{U})$ and $a_1 \cdot a = a_2 \cdot a$, then $a_1 = a_2$;
- if $(a_1, a_2) \in \mathcal{D}(\mathcal{U})$, then $(a_2^{-1}, a_1^{-1}) \in \mathcal{D}(\mathcal{U})$ and $(a_1 \cdot a_2)^{-1} = a_2^{-1} \cdot a_1^{-1}$

(see e.g. [79, Sec 1.5.6], [25, Cor 2.17]). In particular, this implies that $(\jmath|_{\mathcal{U}} \circ \jmath|_{\mathcal{U}}) = \mathrm{Id}_{\mathcal{U}}$. From now on, any local topological group G and any neighborhood $\mathcal{U} \subset \mathsf{G}$ of the unit are assumed to be symmetric and cancellative.

Example I.3.3 ([57], Ex 7)**.** Let $\mathsf{G} = (\mathsf{G}, e, \mathcal{J}(\mathsf{G}), \mathcal{D}(\mathsf{G}), \jmath, \nu)$ be defined by the following:

$$\mathsf{G} := \mathbb{R} , \quad \mathcal{J}(\mathsf{G}) := \mathbb{R} \setminus \{\tfrac{1}{2}, 1\} , \quad \mathcal{D}(\mathsf{G}) := \{(x, y) \in \mathbb{R}^2 : |xy| \neq 1\} ,$$
$$e := 0 , \quad \jmath(x) := \frac{-x}{1 - 2x} , \quad \nu(x, y) := \frac{x + y - 2xy}{1 - xy} .$$

Here, we are considering $\mathbb{R}$ with the standard Euclidean topology. Then, one can directly check that G is a local topological group. Moreover, for any $0 < t \leq \frac{1}{2}$, the interval $\mathcal{U}_t := \left(\frac{-t}{1-2t}, t\right)$ is symmetric and cancellative.

A subset $\mathsf{H} \subset \mathsf{G}$ which contains the unit $e \in \mathsf{G}$ is said to be a *sub-local group*, if there exists a neighborhood $\mathcal{V}$ of H such that for any $a \in \mathsf{H}$, $(a_1, a_2) \in (\mathsf{H} \times \mathsf{H}) \cap \mathcal{D}(\mathsf{G})$ it holds

$$a^{-1} \in \mathcal{V} \implies a^{-1} \in \mathsf{H} , \qquad a_1 \cdot a_2 \in \mathcal{V} \implies a_1 \cdot a_2 \in \mathsf{H} .$$

Any such an open subset $\mathcal{V} \subset \mathsf{G}$ is called *associated neighborhood for* H.

A sub-local group H such that $\mathsf{H} \times \mathsf{H} \subset \mathcal{D}(\mathsf{G})$ and with $\mathcal{V} = \mathsf{G}$ is called a *subgroup*. Notice that, by such hypothesis, $a \cdot b \in \mathsf{H}$, $a^{-1} \in \mathsf{H}$ for any $a, b \in \mathsf{H}$ and therefore H is a topological group in the usual sense. The local topological group G is said *to have no small subgroups* (NSS) if there exists a neighborhood of the unit with no nontrivial subgroups.

Given two local topological groups G and G', a *local homomorphism from* G *to* G' is a pair $(\mathcal{U}, \varphi)$ given by a neighborhood $\mathcal{U} \subset \mathsf{G}$ of the unit and a continuous function $\varphi : \mathcal{U} \to \mathsf{G}'$ such that

- $\varphi(e) = e'$ and $\varphi(\mathcal{D}(\mathcal{U})) \subset \mathcal{D}(\mathsf{G}')$,
- $\varphi(a^{-1}) = \varphi(a)^{-1}$ and $\varphi(a_1 \cdot a_2) = \varphi(a_1) \cdot \varphi(a_2)$ for any $a \in \mathcal{U}$, $(a_1, a_2) \in \mathcal{D}(\mathcal{U})$.

Two local homomorphisms $(\mathcal{U}_1, \varphi_1)$, $(\mathcal{U}_2, \varphi_2)$ are *equivalent* if there exists a neighborhood $\tilde{\mathcal{U}} \subset \mathcal{U}_1 \cap \mathcal{U}_2$ of the unit such that $\varphi_1|_{\tilde{\mathcal{U}}} = \varphi_2|_{\tilde{\mathcal{U}}}$. For the sake of shortness, we will simply write $\varphi : \mathsf{G} \to \mathsf{G}'$ to denote a local homomorphism $(\mathcal{U}, \varphi)$, determined up to an equivalence. The composition $\varphi' \circ \varphi$ of two local homomorphisms is defined in an obvious way and a local homomorphism $\varphi : \mathsf{G} \to \mathsf{G}'$ is called a *local isomorphism* if there exists a local homomorphism $\psi : \mathsf{G}' \to \mathsf{G}$ such that $\psi \circ \varphi = \mathrm{Id}_{\mathsf{G}}$ and $\varphi \circ \psi = \mathrm{Id}_{\mathsf{G}'}$, where the equalities are up to equivalence.

Example I.3.4. Let G be the local topological group defined in Example I.3.3 and $\mathbb{R} = (\mathbb{R}, +)$ the additive group of real numbers. For any $0 < t \leq \frac{1}{2}$, the pair $(\mathcal{U}_t, \varphi_t)$ given by

$$\mathcal{U}_t := \left(\tfrac{-t}{1-2t}, t\right) , \quad \varphi_t(x) := \frac{x}{x-1}$$

is a local homomorphism from G to $\mathbb{R}$. However, the group G is not *globalizable*, i.e. there exists no local homomorphism $\tilde{\varphi}$ defined on the whole G to a (global) topological group $\tilde{\mathsf{G}} = (\tilde{\mathsf{G}}, \star)$. Indeed, if we assume by contradiction that such $\tilde{\varphi}$ exists, then for any $x \in \mathsf{G}$ we would get

$$\tilde{\varphi}(1) = \tilde{\varphi}(\nu(1, x)) = \tilde{\varphi}(1) \star \tilde{\varphi}(x) \quad \Longrightarrow \quad \tilde{\varphi}(x) = e_{\tilde{\mathsf{G}}} ,$$

which is not possible.

A *local Lie group* is a local topological group that is also a smooth manifold in such a way that the local group operations $\jmath : \mathsf{G} \to \mathsf{G}$ and $\nu : \mathcal{D}(\mathsf{G}) \to \mathsf{G}$ are smooth. Just like in the global Lie groups theory, one can associate a Lie algebra $\mathfrak{g}$ of left invariant vector fields to any local Lie group G. Analogues of Lie's three fundamental theorems hold also for the local Lie groups ([5, Ch 3]). In particular, it turns out that every local Lie group is locally isomorphic to some Lie group by means of a smooth local isomorphism. We resume in the following theorem the solution of the Hilbert fifth problem for local topological groups provided by Goldbring. We refer to [25] for the proof and more details.

Theorem I.3.5 ([25]). *For any locally compact local topological group* G*, the conditions listed below are equivalent.*

a) G *is locally Euclidean.*

b) G *is NSS.*

c) G *is locally isomorphic to a Lie group.*

I.3.3 Local (topological and Lie) groups of transformations

Let (M,p) be a pointed smooth manifold. A *local topological group of* $\mathcal{C}^k$*-transformations on* (M,p) is a tuple $G = (\mathsf{G}, \mathcal{U}_{\mathsf{G}}, \Omega_p, \mathcal{W}, \Theta)$ formed by:

i) a (local) topological group G and a neighborhood $\mathcal{U}_{\mathsf{G}} \subset \mathsf{G}$ of the unit;

ii) a neighborhood $\Omega_p \subset M$ of p;

iii) an open subset $\mathcal{W} \subset \mathcal{U}_{\mathsf{G}} \times \Omega_p$ which contains both $\mathcal{U}_{\mathsf{G}} \times \{p\}$, $\{e\} \times \Omega_p$ and a continuous map $\Theta : \mathcal{W} \to \Omega_p$;

such that the following hold:

- for any $(a,b) \in (\mathcal{U}_{\mathsf{G}} \times \mathcal{U}_{\mathsf{G}}) \cap \mathcal{D}(\mathsf{G})$ and $x \in \Omega_p$ it holds that

$$\Theta(a, \Theta(b,x)) = \Theta(a \cdot b, x)$$

  provided that $(b,x), (a \cdot b, x) \in \mathcal{W}$ and $(a, \Theta(b,x)) \in \mathcal{W}$;
- for any $a \in \mathcal{U}_{\mathsf{G}}$, the map $\Theta(a) := \Theta(a, \cdot)$, defined on the open subset $\mathcal{W}(a) := \{x : (a,x) \in \mathcal{W}\} \subset \Omega_p$, is of class $\mathcal{C}^k$;
- $\Theta(e) = \mathrm{Id}_{\Omega_p}$, i.e. $\Theta(e,x) = x$ for any $x \in \Omega_p$.

From the definition, it follows that, for any $a \in \mathcal{U}_{\mathsf{G}}$, there exist a neighborhood $U \subset \Omega_p$ of p and a neighborhood $V \subset \Omega_p$ of $\Theta(a,p)$ such that $\Theta(a)|_U : U \to V$ is a $\mathcal{C}^k$-diffeomorphism, with inverse given by $\big(\Theta(a)|_U\big)^{-1} = \Theta(a^{-1})|_V$. We say that G is *almost-effective* (resp. *effective*) if the set

$$\big\{a \in \mathcal{U}_{\mathsf{G}} : \Theta(a) \text{ fixes a neighborhood of } p\big\}$$

is discrete (resp. equal to {e}). We also say that $G = (\mathsf{G}, \mathcal{U}_{\mathsf{G}}, \Omega_p, \mathcal{W}, \Theta)$ is *locally compact* if G is locally compact. We will tacitly assume that Ω_p, $\mathcal{W}$ are connected and that $\mathcal{W}(a)$ is connected for any $a \in \mathcal{U}_{\mathsf{G}}$. The *orbit of* G *through* p is the set

$$\begin{aligned} G(p) := \Big\{ &\big(\Theta(a_1) \circ \ldots \circ \Theta(a_N)\big)(p) : N \geq 1\,,\ a_i \in \mathcal{U}_{\mathsf{G}} \text{ for any } 1 \leq i \leq N\,, \\ &\big(\Theta(a_{j+1}) \circ \ldots \circ \Theta(a_N)\big)(p) \in \mathcal{W}(a_j) \text{ for any } 1 \leq j \leq N-1 \Big\}\,. \end{aligned}$$

Motivated by the terminology for Lie algebra actions, we say that G is *transitive* if $G(p)$ contains a neighborhood of the point p.

Two local topological groups $G_i = (\mathsf{G}_i, \mathcal{U}_{\mathsf{G}_i}, \Omega_{p_i}, \mathcal{W}_i, \Theta_i)$ of $\mathcal{C}^k$-transformations acting on (M_i, p_i), with $i = 1, 2$, are said to be *locally $\mathcal{C}^k$-equivalent* if there exist

i) a neighborhood $\mathcal{U}_o \subset \mathcal{U}_{\mathsf{G}_1}$ of the unit and a local isomorphism $\varphi : \mathsf{G}_1 \to \mathsf{G}_2$ defined on $\mathcal{U}_o$ with $\varphi(\mathcal{U}_o) \subset \mathcal{U}_{\mathsf{G}_2}$;

ii) two nested neighborhoods $U_o \subset U \subset \Omega_{p_1}$ of p_1 and an open $\mathcal{C}^k$-embedding $f : U \to \Omega_{p_2}$ with $f(p_1) = p_2$;

such that the following hold:

- $\mathcal{U}_o \times U_o \subset \mathcal{W}_1$, $\Theta_1(\mathcal{U}_o \times U_o) \subset U$ and $\varphi(\mathcal{U}_o) \times f(U_o) \subset \mathcal{W}_2$;
- for any $(a, x) \in \mathcal{U}_o \times U_o$, it holds that $f\big(\Theta_1(a, x)\big) = \Theta_2\big(\varphi(a), f(x)\big)$.

A local topological group of $\mathcal{C}^k$-transformations $G = (\mathsf{G}, \mathcal{U}_{\mathsf{G}}, \Omega_p, \mathcal{W}, \Theta)$ on (M, p) is called *local Lie group of $\mathcal{C}^k$-transformations* if G is a Lie group and the map Θ is of class $\mathcal{C}^k$.

Remark I.3.6. In perfect analogy with what occurs for global groups of transformations, by [4, Thm 4] the second condition is redundant here as well. Namely, if the (local) topological group G is a Lie group and each map $\Theta(a) : \mathcal{W}(a) \to \Omega_p$ is of class $\mathcal{C}^k$, then the map $\Theta : \mathcal{W} \to \Omega_p$ is automatically of class $\mathcal{C}^k$.

If (M, p) is equipped with a $\mathcal{C}^k$-Riemannian metric g, then $G = (\mathsf{G}, \mathcal{U}_{\mathsf{G}}, \Omega_p, \mathcal{W}, \Theta)$ is called *local topological (resp. Lie) group of isometries* if each map $\Theta(a) : \mathcal{W}(a) \to \Omega_p$ is a local isometry of (M, g).

Finally, consider an almost-effective local Lie group $G = (\mathsf{G}, \mathcal{U}_{\mathsf{G}}, \Omega_p, \mathcal{W}, \Theta)$ of $\mathcal{C}^k$-transformations on (M, p) and suppose that $k \geq 2$. Let also $\mathfrak{g} := \mathrm{Lie}(\mathsf{G})$ and $\exp : \mathfrak{g} \to \mathsf{G}$ the Lie exponential of G. For any $X \in \mathfrak{g}$, we consider the open set

$$\mathcal{W}^X := \{(t, x) \in \mathbb{R} \times \Omega_p : (\exp(tX), x) \in \mathcal{W}\}$$

and the map of class $\mathcal{C}^k$

$$\Theta^X : \mathcal{W}^X \to M \ , \quad \Theta^X(t, x) := \Theta\big(\exp(tX), x\big) \ .$$

This allows us to consider the differential

$$\Theta_* : \mathfrak{g} \to \mathcal{C}^{k-1}(\Omega_p; TM|_{\Omega_p}) \ , \quad \Theta_*(X)_x := \frac{d}{dt}\Theta^X(t, x)\Big|_{t=0} \ . \tag{I.3.1}$$

In full analogy with the theory of Lie group actions, one can prove that the map Θ_* is $\mathbb{R}$-linear, injective and, for any $X, Y \in \mathfrak{g}$

$$\Theta_*\big(\mathrm{Ad}(a)X\big)_{\Theta(a,x)} = d(\Theta(a))_x\big(\Theta_*(X)\big)_x \quad \text{for any } (a,x) \in \mathcal{W} ,$$
$$\Theta_*\big([X,Y]\big) = -[\Theta_*(X), \Theta_*(Y)] \ . \tag{I.3.2}$$

The vector fields in $\Theta_*(\mathfrak{g})$ are called *infinitesimal generator of* G. If G acts by isometries, then its infinitesimal generators are Killing vector fields. Notice also that G is transitive if and only if the set $\{\Theta_*(X)|_p : X \in \mathfrak{g}\}$ coincides with the whole tangent space T_pM.

I.4 Basics on locally homogeneous spaces

We recall that a $\mathcal{C}^{k,\alpha}$-Riemannian manifold (M,g) is *locally homogeneous* if the pseudogroup of local isometries of (M,g) acts transitively on M, i.e. if for any $x, y \in M$ there exist $\varepsilon = \varepsilon(x,y) > 0$ a local isometry $f : \mathcal{B}_g(x,\varepsilon) \to \mathcal{B}_g(y,\varepsilon)$ such that $f(x) = y$. From now on, we use the term *locally homogeneous space* to denote locally homogeneous Riemannian manifolds which are smooth, and hence real analytic (see e.g. [77, Thm 2.2], [11, Lemma 1.1]).

I.4.1 Nomizu algebras

Given a locally homogeneous space (M,g) and a distinguished point $p \in M$, it is known that there exists a local Lie group of isometries which acts transitively on (M,g,p) according to the definitions in Section I.3. Such local Lie group is constructed as follows. Consider the *Killing generators at* p, that is the pairs $(v,A) \in T_pM \oplus \mathfrak{so}(T_pM, g_p)$ such that

$$A \cdot g_p = 0 \ , \quad v \lrcorner \mathrm{Rm}^{k+1}(g)_p + A \cdot \mathrm{Rm}^k(g)_p = 0 \quad \text{for any } k \in \mathbb{Z}_{\geq 0} \, , \tag{I.4.1}$$

where $\mathfrak{so}(T_pM, g_p)$ acts on the tensor algebra of T_pM as a derivation. The space of Killing generators at p is denoted by $\mathfrak{kill}^g$. This notation is due to the following fact. For any Killing vector field X of (M,g) defined in a neighborhood of p, the pair $\big(X_p, -(\nabla^g X)_p\big)$ is a Killing generator of (M,g) at p. Conversely, being (M,g) real analytic, by [55, Thm 2] there exists a neighborhood $\Omega_p \subset M$ of p such that, for any $(v,A) \in \mathfrak{kill}^g$, there exists a Killing vector field X on Ω_p with $X_p = v$ and $-(\nabla^g X)_p = A$.

Lemma I.4.1. *Let X, Y be Killing vector fields defined in a neighborhood $\Omega_p \subset M$ of p and set $(v, A) := \big(X_p, -(\nabla^g X)_p\big)$, $(w, B) := \big(Y_p, -(\nabla^g Y)_p\big)$. Then*

$$[X, Y]_p = A(w) - B(v) \ , \quad -(\nabla^g [X, Y])_p = [A, B] + \mathrm{Rm}(g)_p(v \wedge w) \ . \qquad \text{(I.4.2)}$$

Proof. Since ∇^g is torsion-free, we get

$$[X, Y]_p = (\nabla^g_X Y)_p - (\nabla^g_Y X)_p = A(w) - B(v) \ . \qquad \text{(I.4.3)}$$

Let us write now $-\nabla^g X = \mathcal{L}_X - \nabla^g_X$, where $\mathcal{L}_X$ indicates the Lie derivative along X. From the Jacobi Identity it comes that

$$\begin{aligned}
\mathcal{L}_{[X,Y]} &= [\mathcal{L}_X, \mathcal{L}_Y] \\
\nabla^g_{[X,Y]} - \nabla^g[X, Y] &= [\nabla^g_X - \nabla^g X, \nabla^g_Y - \nabla^g Y] \\
\nabla^g_{[X,Y]} - \nabla^g[X, Y] &= [\nabla^g_X, \nabla^g_Y] - [\nabla^g_X, \nabla^g Y] + [\nabla^g_Y, \nabla^g X] + [\nabla^g X, \nabla^g Y] \\
-\nabla^g[X, Y] &= [\nabla^g X, \nabla^g Y] - \mathrm{Rm}(g)(X \wedge Y) - [\nabla^g_X, \nabla^g Y] + [\nabla^g_Y, \nabla^g X]
\end{aligned}$$

and hence

$$\begin{aligned}
-(\nabla^g[X, Y])_p = [A, B] + \mathrm{Rm}(g)_p(v \wedge w) + \big([\nabla^g_X, -\nabla^g Y] - \mathrm{Rm}(g)(X \wedge Y)\big)_p - \\
- \big([\nabla^g_Y, -\nabla^g X] - \mathrm{Rm}(g)(Y \wedge X)\big)_p \ . \qquad \text{(I.4.4)}
\end{aligned}$$

Let us consider now, for any Killing vector field V defined in the neighborhood Ω_p of p, the tensor $\alpha_V \in \bigotimes^3 T^*M|_{\Omega_p}$ given by

$$\alpha_V(Z_1, Z_2, Z_3) := g\big([\nabla^g_{Z_1}, -\nabla^g V] Z_2 - \mathrm{Rm}(g)(Z_1 \wedge V) Z_2, Z_3\big) \ .$$

Then, α_V is symmetric with respect to Z_1 and Z_2, since

$$\begin{aligned}
\alpha_V(Z_1,&Z_2, Z_3) - \alpha_V(Z_2, Z_1, Z_3) = \\
&= g\big([\nabla^g_{Z_1}, -\nabla^g V] Z_2, Z_3\big) - g\big(\mathrm{Rm}(g)(Z_1 \wedge V) Z_2, Z_3\big) - \\
&\qquad - g\big([\nabla^g_{Z_2}, -\nabla^g V] Z_1, Z_3\big) + g\big(\mathrm{Rm}(g)(Z_2 \wedge V) Z_1, Z_3\big) \\
&= -g(\nabla^g_{Z_1} \nabla^g_{Z_2} V, Z_3) + g(\nabla^g_{\nabla^g_{Z_1} Z_2} V, Z_3) - \mathrm{Rm}(g)(Z_1 \wedge V, Z_2 \wedge Z_3) + \\
&\qquad + g(\nabla^g_{Z_2} \nabla^g_{Z_1} V, Z_3) - g(\nabla^g_{\nabla^g_{Z_2} Z_1} V, Z_3) + \mathrm{Rm}(g)(Z_2 \wedge V, Z_1 \wedge Z_3) \\
&= \mathrm{Rm}(g)(Z_1 \wedge Z_2, V \wedge Z_3) + \mathrm{Rm}(g)(V \wedge Z_1, Z_2 \wedge Z_3) + \mathrm{Rm}(g)(Z_2 \wedge V, Z_1 \wedge Z_3) \\
&= 0 \ .
\end{aligned}$$

On the other hand, α_V is skew-symmetric with respect to Z_2 and Z_3. Indeed

$$\begin{aligned} g([\nabla^g_{Z_1}, -\nabla^g V]Z_2, Z_3) &= -g(\nabla^g_{Z_1}\nabla^g_{Z_2}V, Z_3) + g(\nabla^g_{\nabla^g_{Z_1}Z_2}V, Z_3) \\ &= -Z_1(g(\nabla^g_{Z_2}V, Z_3)) + g(\nabla^g_{Z_2}V, \nabla^g_{Z_1}Z_3) - \\ &\qquad\qquad - g(\nabla^g_{Z_3}V, \nabla^g_{Z_1}Z_2) \\ &= Z_1(g(\nabla^g_{Z_3}V, Z_2)) - g(\nabla^g_{Z_3}V, \nabla^g_{Z_1}Z_2) + \\ &\qquad\qquad + g(\nabla^g_{Z_2}V, \nabla^g_{Z_1}Z_3) \\ &= g(\nabla^g_{Z_1}\nabla^g_{Z_3}V, Z_2) - g(\nabla^g_{\nabla^g_{Z_1}Z_3}V, Z_2) \\ &= -g([\nabla^g_{Z_1}, -\nabla^g V]Z_3, Z_2)\ . \end{aligned}$$

Therefore, it comes necessarily that $\alpha_V = 0$ and so

$$\big([\nabla^g_X, -\nabla^g Y] - \mathrm{Rm}(g)(X\wedge Y)\big)_p = \big([\nabla^g_Y, -\nabla^g X] - \mathrm{Rm}(g)(Y\wedge X)\big)_p = 0\ . \quad \text{(I.4.5)}$$

The thesis comes from (I.4.3), (I.4.4) and (I.4.5). □

From (I.4.2), it follows that the space of Killing generators $\mathfrak{kill}^g$ of (M,g) at p can be endowed with a Lie algebra structures by setting

$$\big[(v,A),(w,B)\big] := \big(A(w) - B(v), [A,B] + \mathrm{Rm}(g)_p(v\wedge w)\big)\ . \quad \text{(I.4.6)}$$

The pair $\mathfrak{kill}^g = (\mathfrak{kill}^g, [\,,])$ is called the *Nomizu algebra of* (M,g,p) and it is isomorphic to the Lie algebra of local Killing vector fields of (M,g) defined in a neighborhood of p. By [59, Thm XI] (see also [10, Thm A.4]) there exists a local Lie group of isometries whose infinitesimal generators are precisely the Killing vector fields in $\mathfrak{kill}^g$.

I.4.2 Orthogonal transitive Lie algebras

In this subsection, we briefly summarize some algebraic tools which are useful to study locally homogeneous Riemannian manifolds. We refer to [77, 78] and [41] for more details.

Firstly, as we already mentioned in the Introduction, we introduce the following notation.

Definition I.4.2. Let (M, g) be a locally homogeneous space. Then, (M, g) is said to be *strictly locally homogeneous* if it is not locally isometric to some homogeneous space, otherwise it is said to be *non-strictly locally homogeneous.*

Notice that a complete locally homogeneous space (M, g) is always non-strictly locally homogeneous. In fact, by hypothesis its Riemannian universal cover $(\widetilde{M}, g)$ is a simply connected, complete locally homogeneous space and hence, by [76], it comes that $(\widetilde{M}, g)$ is homogeneous. Here, with a slight abuse of notation, we are denoting by g both the Riemannian metric on M and its pullback on the universal cover $\widetilde{M}$. Moreover, by [52, Sec 7] we know that any locally homogeneous space (M^m, g) of dimension $m \leq 4$ is non-strictly locally homogeneous.

Secondly, we introduce the following definition, which plays an important role in comparing strictly and non-strictly locally homogeneous spaces.

Definition I.4.3 ([58], p. 51)**.** Let G be a connected Lie group, $\mathfrak{h} \subset \mathfrak{g} := \mathrm{Lie}(\mathsf{G})$ a Lie subalgebra and $\mathsf{H} \subset \mathsf{G}$ the connected Lie subgroup with $\mathrm{Lie}(\mathsf{H}) = \mathfrak{h}$. The *Malcev-closure of* $\mathfrak{h}$ *in* G is the Lie algebra $\overline{\mathfrak{h}}^{\mathsf{G}} \subset \mathfrak{g}$ of the closure $\overline{\mathsf{H}}$ of H in G. The Lie algebra $\mathfrak{h}$ is said to be *Malcev-closed in* G if $\mathfrak{h} = \overline{\mathfrak{h}}^{\mathsf{G}}$.

Notice that the notion of Malcev-closure depends on the Lie group G, i.e. it is possible that $\overline{\mathfrak{h}}^{\mathsf{G}} \neq \overline{\mathfrak{h}}^{\mathsf{G}'}$ for two connected Lie groups with $\mathrm{Lie}(\mathsf{G}) = \mathrm{Lie}(\mathsf{G}') = \mathfrak{g}$.

Example I.4.4. Let $\mathfrak{g} := \mathbb{R}^2$ and $\mathsf{G} := \mathbb{R}^2$, $\mathsf{G}' := \mathsf{T}^2 = \mathbb{Z}^2 \backslash \mathbb{R}^2$. We consider the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ spanned by the vector $(1, \sqrt{2})^t \in \mathbb{R}^2$. Then it follows that $\mathfrak{h} = \overline{\mathfrak{h}}^{\mathsf{G}} \subsetneq \overline{\mathfrak{h}}^{\mathsf{G}'} = \mathfrak{g}$ (see Lemma III.3.5).

Then, we recall the following

Definition I.4.5. Let $m, q \in \mathbb{Z}_{\geq 0}$. An *orthogonal transitive Lie algebra* $(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \langle\, ,\rangle)$ *of rank* (m, q) is the datum of

- a $(q + m)$-dimensional Lie algebra $\mathfrak{g}$;
- a q-dimensional Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which does not contain any non-trivial ideal of $\mathfrak{g}$;
- an $\mathrm{ad}(\mathfrak{h})$-invariant complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$;
- an $\mathrm{ad}(\mathfrak{h})$-invariant Euclidean product $\langle\, ,\rangle$ on $\mathfrak{m}$.

An orthogonal transitive Lie algebra $(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \langle\, ,\rangle)$ is said to be *regular* if $\mathfrak{h}$ is Malcev-closed in the simply connected Lie group G with $\mathrm{Lie}(\mathsf{G}) = \mathfrak{g}$ (see Definition I.4.3), *non-regular* otherwise.

Let $(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \langle\,,\rangle)$ be an orthogonal transitive Lie algebra of rank (m, q). Since there are no ideals of $\mathfrak{g}$ in $\mathfrak{h}$, the adjoint action of $\mathfrak{h}$ on $\mathfrak{m}$ is a faithful representation in $\mathfrak{so}(\mathfrak{m}, \langle\,,\rangle)$ and so $q \leq \frac{m(m-1)}{2}$. An *adapted frame* is a basis $u = (e_1, \ldots, e_{q+m}) : \mathbb{R}^{q+m} \to \mathfrak{g}$ such that $\mathfrak{h} = \mathrm{span}(e_1, \ldots, e_q)$, $\mathfrak{m} = \mathrm{span}(e_{q+1}, \ldots, e_{q+m})$ and $\langle e_{q+i}, e_{q+j}\rangle = \delta_{ij}$. Given two orthogonal transitive Lie algebras $(\mathfrak{g}_i = \mathfrak{h}_i + \mathfrak{m}_i, \langle\,,\rangle_i)$, we call *orthogonal transitive Lie algebras isomorphism* (*isomorphism* for short) any Lie algebra isomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\varphi(\mathfrak{h}_1) = \mathfrak{h}_2$, $\varphi(\mathfrak{m}_1) = \mathfrak{m}_2$ and $\langle\,,\rangle_1 = (\varphi|_{\mathfrak{m}_1})^*\langle\,,\rangle_2$.

Remark I.4.6. Let $(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \langle\,,\rangle)$ be an orthogonal transitive Lie algebra. By [78, Lemma 3.1], one can extend $\langle\,,\rangle$ to an $\mathrm{ad}(\mathfrak{h})$-invariant Euclidean product $\langle\,,\rangle'$ on the whole $\mathfrak{g}$ in such a way that $\langle\mathfrak{h}, \mathfrak{m}\rangle' = 0$ and $\langle\,,\rangle'|_{\mathfrak{h}\otimes\mathfrak{h}}$ coincides with the Cartan-Killing form of $\mathfrak{so}(\mathfrak{m}, \langle\,,\rangle)$.

An important class of examples of orthogonal transitive Lie algebras are provided by the Nomizu algebras. In fact, let (M, g) be a locally homogeneous space, $p \in M$ a distinguished point and $\mathfrak{kill}^g = \{$ Killing generators (v, A) at p $\}$ the Nomizu algebra of (M, g, p). Consider the scalar product on $\mathfrak{kill}^g$ given by

$$\langle\!\langle (v, A), (w, B)\rangle\!\rangle_g := g_p(v, w) - \mathrm{Tr}(AB)\ ,$$

set $\mathfrak{kill}^g_0 := \{(0, A) \in \mathfrak{kill}^g\}$ and let $\mathfrak{m}$ be the $\langle\!\langle\,,\rangle\!\rangle_g$-orthogonal complement of $\mathfrak{kill}^g_0$ in $\mathfrak{kill}^g$. Since (M, g) is locally homogeneous, this gives rise to a linear isomorphism $\mathfrak{m} \simeq T_pM$, which allow us to define a scalar product $\langle\,,\rangle_g$ on $\mathfrak{m}$ induced by the metric tensor on M. Then, $(\mathfrak{kill}^g = \mathfrak{kill}^g_0 + \mathfrak{m}, \langle\,,\rangle_g)$ is an orthogonal transitive Lie algebra. We stress that the Nomizu algebra, modulo isomorphism, does not depend on the choice of the point p. Furthermore, two locally homogeneous spaces are locally isometric if and only if their Nomizu algebras are isomorphic.

The notion of Malcev-closure is related to the strictly locally homogeneous spaces by the following result.

Theorem I.4.7 ([77], Lemma 3.5 and Prop 4.4)**.** *Let (M, g) be a locally homogeneous space, $p \in M$ a distinguished point and $(\mathfrak{kill}^g = \mathfrak{kill}^g_0 + \mathfrak{m}, \langle\,,\rangle_g)$ be the Nomizu algebra of (M, g, p). Then, $(\mathfrak{kill}^g = \mathfrak{kill}^g_0 + \mathfrak{m}, \langle\,,\rangle_g)$ is regular if and only if (M, g) is non-strictly locally homogeneous.*

I.4.3 Ambrose-Singer connections and the Singer invariant

Let us recall here the following classical result.

Theorem I.4.8 ([82], Thm 2.1)**.** *A smooth Riemannian manifold (M, g) is a locally homogeneous space if and only if it admits a metric connection with parallel torsion and curvature. Any such connection is called an Ambrose-Singer connection.*

In general, Ambrose-Singer connections are far from being unique, but there is always a canonical one, which is characterized as follows. Fix $p \in M$ and, for any $k \geq 0$, set

$$\mathfrak{i}(k) := \big\{A \in \mathfrak{so}(T_pM, g_p) : \, A \cdot \mathrm{Rm}^r(g)_p = 0 \;, \quad 0 \leq r \leq k\big\} \;. \tag{I.4.7}$$

Since $\big(\mathfrak{i}(k)\big)_{k \in \mathbb{Z}_{\geq 0}}$ is a filtration of the finite dimensional Lie algebra $\mathfrak{so}(T_pM, g_p)$, there exists a first integer k_g such that $\mathfrak{i}(k_g) = \mathfrak{i}(k_g+1)$, which is called the *Singer invariant of* (M, g). It actually holds that $\mathfrak{i}(k) = \mathfrak{i}(k_g)$ for any integer $k \geq k_g$.

Theorem I.4.9 ([82])**.** *Let (M, g) be a locally homogeneous space and $p \in M$ a distinguished point. Then there exists a unique Ambrose-Singer connection D^g on (M, g) such that $S^g_p \in \mathfrak{i}(k_g)^{\perp}$, where $S^g := D^g - \nabla^g$ and $\mathfrak{i}(k_g)^{\perp}$ is the orthogonal complement of $\mathfrak{i}(k_g)$ in $\mathfrak{so}(T_pM, g_p)$ with respect to the Cartan-Killing form. Moreover $\mathfrak{i}(k) = \mathfrak{i}(k_g)$ for any integer $k \geq k_g$ and the pairs $(v, S^g_p(v))$, $v \in T_pM$, are Killing generator at p.*

Remark I.4.10. Notice that for non-strictly locally homogeneous spaces, the canonical Ambrose-Singer connection coincides with its *canonical connection* according to [37, Sec X.2].

By the results in [76, 54], it is known that a locally homogeneous space (M, g) with Singer invariant k_g is completely determined by the curvature and its covariant derivatives up to order $k_g + 2$ at a single point. It is also known that $k_g < \frac{3}{2} \dim M$ ([26, p. 165]), but in general this estimate is not sharp. For later purposes, for any integer $m \geq 1$ we set

$$\imath(m) := \max\{k_g : (M, g) \text{ locally homogeneous, } \dim M \leq m\} \;. \tag{I.4.8}$$

Notice that $m \mapsto \imath(m)$ is not decreasing and $0 \leq \imath(m) < \frac{3}{2}m$. Moreover, $\imath(1) = \imath(2) = 0$ and the main result in [35] implies that $\imath(3) = \imath(4) = 1$. We also mention that from the main theorem in [47] we get $\lim_{m\to+\infty} \imath(m) = +\infty$.

For any $m, s \in \mathbb{N}$ with $m \geq 1$ and $s \geq \imath(m) + 2$, we define $\widetilde{\mathcal{R}}^s(m)$ to be the set of all the $(s{+}1)$-tuples

$$(R^0, R^1, \ldots, R^s) \in W^s(m) := \bigoplus_{k=0}^{s} \Big(\bigotimes\nolimits^k (\mathbb{R}^m)^* \otimes \Lambda^2(\mathbb{R}^m)^* \otimes \mathfrak{so}(m) \Big)$$

satisfying the following conditions (R1) and (R2).

(R1) The following identities hold:

$$\begin{aligned}
&\langle R^0(Y_1{\wedge}Y_2)V_1, V_2\rangle_{\mathrm{st}} = \langle R^0(V_1{\wedge}V_2)Y_1, Y_2\rangle_{\mathrm{st}} \ , \\
&\mathfrak{S}_{Y_1,Y_2,V_1} \langle R^0(Y_1{\wedge}Y_2)V_1, V_2\rangle_{\mathrm{st}} = 0 \ , \\
&\langle R^1(X_1|Y_1{\wedge}Y_2)V_1, V_2\rangle_{\mathrm{st}} = \langle R^1(X_1|V_1{\wedge}V_2)Y_1, Y_2\rangle_{\mathrm{st}} \ , \\
&\mathfrak{S}_{Y_1,Y_2,V_1} \langle R^1(X_1|Y_1{\wedge}Y_2)V_1, V_2\rangle_{\mathrm{st}} = 0 \ , \\
&\mathfrak{S}_{X_1,Y_1,Y_2} \langle R^1(X_1|Y_1{\wedge}Y_2)V_1, V_2\rangle_{\mathrm{st}} = 0 \ , \\
&R^{k+2}(X_1, X_2, X_3, \ldots X_{k+2}|Y_1{\wedge}Y_2) - R^{k+2}(X_2, X_1, X_3, \ldots X_{k+2}|Y_1{\wedge}Y_2) = \\
&\qquad = -\big(R^0(X_1{\wedge}X_2)\cdot R^k\big)(X_3, \ldots X_{k+2}|Y_1{\wedge}Y_2) \quad \text{for any } 0 \leq k \leq s-2 \ ,
\end{aligned}$$

where $\mathfrak{S}_{A,B,C}$ denotes the sum over the cyclic permutations of A, B, C and $\mathfrak{so}(m)$ acts on the tensor algebra on $\mathbb{R}^m$ by derivation.

(R2) For any $1 \leq k \leq s$, the maps

$$\begin{aligned}
\alpha^k : \mathfrak{so}(m) \to W^k(m) \ , &\quad \alpha^k(A) := (A\cdot R^0, A\cdot R^1, \ldots, A\cdot R^i) \ , \\
\beta^k : \mathbb{R}^m \to W^{k-1}(m) \ , &\quad \beta^i(X) := (X \lrcorner R^1, X \lrcorner R^2, \ldots, X \lrcorner R^i)
\end{aligned}$$

are such that

$$\begin{aligned}
\beta^k(\mathbb{R}^m) \subset \alpha^{k-1}(\mathfrak{so}(m)) &\quad \text{for any } \imath(m)+2 \leq k \leq s \ , \\
\ker(\alpha^k) = \ker(\alpha^{k+1}) &\quad \text{for any } \imath(m) \leq k \leq s-1 \ .
\end{aligned}$$

Note that $\widetilde{\mathcal{R}}^s(m)$ is invariant under the standard left action of $\mathsf{O}(m)$. Hence, we introduce the following

Definition I.4.11. Let $m, s \in \mathbb{N}$ with $m \geq 1$, $s \geq \imath(m) + 2$. We call *Riemannian s-tuples of rank m* the elements of the quotient $\mathcal{R}^s(m) := \mathsf{O}(m) \backslash \widetilde{\mathcal{R}}^s(m)$.

This definition is motivated by the following

Theorem I.4.12 ([54])**.** *Let (M^m, g) be a locally homogeneous space. For any point $p \in M$, $u : \mathbb{R}^n \to T_pM$ orthonormal and $s \geq \imath(m) + 2$, it holds that*

$$\big(u^* \operatorname{Rm}^0(g)_p, u^* \operatorname{Rm}^1(g)_p, \dots, u^* \operatorname{Rm}^s(g)_p\big) \in \widetilde{\mathcal{R}}^s(m)$$

and the corresponding Riemannian s-tuple $\rho^s \in \mathcal{R}^s(m)$ is independent of p and u.

Conversely, for any Riemannian s-tuple $\rho^s \in \mathcal{R}^s(m)$, there exists a locally homogeneous space (M^m, g), which is uniquely determined up to local isometry, such that $\big(u^* \operatorname{Rm}(g)_p, u^* \operatorname{Rm}^1(g)_p, \dots, u^* \operatorname{Rm}^s(g)_p\big)$, for some point $p \in M$ and $u : \mathbb{R}^n \to T_pM$ orthonormal, is a representative of ρ^s.

I.5 Compact homogeneous spaces

I.5.1 The space of invariant metrics

Let $M = \mathsf{G}/\mathsf{H}$ be a compact, connected and almost effective m-dimensional homogeneous space, with G and H compact Lie groups. We fix once and for all an $\operatorname{Ad}(\mathsf{G})$-invariant Euclidean inner product Q on the Lie algebra $\mathfrak{g} := \operatorname{Lie}(\mathsf{G})$ and we indicate with $\mathfrak{m}$ the Q-orthogonal complement of $\mathfrak{h} := \operatorname{Lie}(\mathsf{H})$ in $\mathfrak{g}$. From now on, we will always identify any G-invariant tensor field on M with the corresponding $\operatorname{Ad}(\mathsf{H})$-invariant tensor on $\mathfrak{m}$ by the natural evaluation map at the point $e\mathsf{H} \in M$. The restriction $Q_{\mathfrak{m}} := Q|_{\mathfrak{m} \otimes \mathfrak{m}}$ of Q on the complement $\mathfrak{m}$ defines a normal G-invariant metric on M. Up to a normalization, we can assume that $\operatorname{vol}(Q_{\mathfrak{m}}) = 1$. We denote by $\mathcal{M}^{\mathsf{G}}$ the set of G-invariant metrics on M and by $\mathcal{M}^{\mathsf{G}}_1$ the subset of unit volume ones.

The set of inner products on $\mathfrak{m}$, which we indicate with $P(\mathfrak{m})$, is an open cone in the space $\operatorname{Sym}(\mathfrak{m}, Q_{\mathfrak{m}})$ of symmetric endomorphism of $(\mathfrak{m}, Q_{\mathfrak{m}})$ by means of the embedding

$$g \longmapsto A_g \ , \quad g = Q_{\mathfrak{m}}(A_g \,\cdot\, , \,\cdot\,) \tag{I.5.1}$$

and it is acted transitively by $\mathsf{GL}(\mathfrak{m})$, with isotropy in $Q_{\mathfrak{m}}$ isomorphic to $\mathsf{O}(\mathfrak{m}, Q_{\mathfrak{m}})$. So, it admits the coset space presentation $P(\mathfrak{m}) = \mathsf{GL}(\mathfrak{m})/\mathsf{O}(\mathfrak{m}, Q_{\mathfrak{m}})$. It can also be endowed with the standard $\mathsf{GL}(\mathfrak{m})$-invariant Riemannian metric defined by

$$\langle A_1, A_2\rangle_g := \operatorname{Tr}(A_g^{-1}A_1A_g^{-1}A_2) \quad \text{for any } A_1, A_2 \in T_gP(\mathfrak{m}) \simeq \operatorname{Sym}(\mathfrak{m}, Q_{\mathfrak{m}}) \ . \tag{I.5.2}$$

Since the map $a \mapsto (a^T)^{-1}$ is an involutive automorphism of $\mathsf{GL}(\mathfrak{m})$ with fixed point set $\mathsf{O}(\mathfrak{m}, Q_{\mathfrak{m}})$, $P(\mathfrak{m})$ is a Riemannian symmetric space. The space $\mathcal{M}^{\mathsf{G}}$ is nothing but the fixed point set of the isometric action of H on $P(\mathfrak{m})$ given by

$$A_g \longmapsto (\operatorname{Ad}(h)|_{\mathfrak{m}})A_g(\operatorname{Ad}(h)|_{\mathfrak{m}})^T \ , \quad h \in \mathsf{H}\,,\ g \in P(\mathfrak{m}) \tag{I.5.3}$$

and so $\mathcal{M}^{\mathsf{G}}$ is a totally geodesic submanifold of $P(\mathfrak{m})$. Since $P(\mathfrak{m})$ splits isometrically as $\mathbb{R} \times \mathsf{SL}(\mathfrak{m})/\mathsf{SO}(\mathfrak{m}, Q_{\mathfrak{m}})$, and $\mathsf{SL}(\mathfrak{m})/\mathsf{SO}(\mathfrak{m}, Q_{\mathfrak{m}})$ is a symmetric space of non-compact type, we conclude that $\mathcal{M}^{\mathsf{G}}$, endowed with the restriction of (I.5.2), is a Riemannian symmetric space with non-positive sectional curvature.

We consider now a $Q_{\mathfrak{m}}$-orthogonal, $\operatorname{Ad}(\mathsf{H})$-invariant irreducible decomposition

$$\mathfrak{m} = \mathfrak{m}_1 + \ldots + \mathfrak{m}_\ell \ . \tag{I.5.4}$$

If the adjoint representation of H is *monotypic*, i.e. $\mathfrak{m}_i \not\simeq \mathfrak{m}_j$ for any $1 \le i < j \le \ell$, the decomposition (I.5.4) is unique up to ordering and by the Schur Lemma any invariant metric $g \in \mathcal{M}^{\mathsf{G}}$ can be uniquely written as

$$g = \lambda_1 Q_{\mathfrak{m}_1} + \cdots + \lambda_\ell Q_{\mathfrak{m}_\ell} \ , \tag{I.5.5}$$

where $Q_{\mathfrak{m}_i} := Q|_{\mathfrak{m}_i \otimes \mathfrak{m}_i}$ and $\lambda_1, \ldots, \lambda_\ell \in \mathbb{R}$ are positive coefficients. In general, the decomposition (I.5.4) is not unique if some modules $\mathfrak{m}_i$ are equivalent to each other and the invariant metrics need not to be diagonal anymore. We denote by $\mathcal{F}^{\mathsf{G}}$ the space of ordered, $Q_{\mathfrak{m}}$-orthogonal, $\operatorname{Ad}(\mathsf{H})$-invariant, irreducible decompositions of $\mathfrak{m}$, which is itself a compact homogeneous space (see [7, Lemma 4.19]).

The space $\mathcal{M}^{\mathsf{G}}$ can be described in terms of any fixed decomposition $\varphi \in \mathcal{F}^{\mathsf{G}}$. Instead of using such approach, we will allow the decomposition of $\mathfrak{m}$ to vary in the space $\mathcal{F}^{\mathsf{G}}$. Indeed, it is known that for any $g \in \mathcal{M}^{\mathsf{G}}$, there exists $\varphi = (\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell) \in \mathcal{F}^{\mathsf{G}}$ with respect to which g is diagonal, i.e. takes the form (I.5.5) (see see e.g. [87, Sec 1]). Any such φ will be called a *good decomposition for* g. Notice that a metric g may admit more than one good decomposition.

Since $\mathcal{M}^{\mathsf{G}}$ is a symmetric space with non-positive sectional curvature, by the Cartan-Hadamard Theorem, its Riemannian exponential map is surjective. Moreover, by (I.5.1) and (I.5.3), we get

$$\begin{aligned} T_{Q_{\mathfrak{m}}}\mathcal{M}^{\mathsf{G}} &= \mathrm{Sym}(\mathfrak{m}, Q_{\mathfrak{m}})^{\mathrm{Ad}(\mathsf{H})} \\ &= \Big\{ v \in \mathrm{Sym}(\mathfrak{m}, Q_{\mathfrak{m}}) : (\mathrm{Ad}(h)|_{\mathfrak{m}})\cdot v\cdot(\mathrm{Ad}(h)|_{\mathfrak{m}})^T = v \text{ for any } h \in \mathsf{H} \Big\} . \end{aligned}$$

For any fixed $v \in T_{Q_{\mathfrak{m}}}\mathcal{M}^{\mathsf{G}}$, there exists a decomposition $\varphi = (\mathfrak{m}_1, \dots, \mathfrak{m}_\ell) \in \mathcal{F}^{\mathsf{G}}$ such that

$$v = v_1 Q_{\mathfrak{m}_1} + \ldots + v_\ell Q_{\mathfrak{m}_\ell} \quad \text{for some } \ v_1, \dots, v_\ell \in \mathbb{R} .$$

By [30, p. 226], the geodesic $\gamma_v(t)$ in $\mathcal{M}^{\mathsf{G}}$ starting from $Q_{\mathfrak{m}}$ and tangent to $v \in T_{Q_{\mathfrak{m}}}\mathcal{M}^{\mathsf{G}}$, with respect to the same decomposition φ, takes the form

$$\gamma_v(t) = e^{tv_1} Q_{\mathfrak{m}_1} + \ldots + e^{tv_\ell} Q_{\mathfrak{m}_\ell} . \tag{I.5.6}$$

Any such decomposition will be called *good decomposition for v*. Notice that the eigenvalues v_i do not depend on the choice of the good decomposition. Since $\mathrm{vol}(\gamma_v(t)) = \exp(t\,\mathrm{Tr}(v))$, it follows that $\mathcal{M}_1^{\mathsf{G}}$ is a totally geodesic submanifold of $\mathcal{M}^{\mathsf{G}}$. In particular, we consider the unit tangent sphere

$$\Sigma := \Big\{ v \in \mathrm{Sym}(\mathfrak{m}, Q_{\mathfrak{m}})^{\mathrm{Ad}(\mathsf{H})} : \ \mathrm{Tr}(v^2) = 1 \,, \ \mathrm{Tr}(v) = 0 \Big\} \tag{I.5.7}$$

so that

$$\mathcal{M}_1^{\mathsf{G}} = \{Q_{\mathfrak{m}}\} \cup \{\gamma_v(t) : v \in \Sigma \,, \ t > 0\} .$$

Notice that the space $\mathcal{M}_1^{\mathsf{G}}$ is a singleton if and only if G/H is isotropy irreducible. In that case, $\Sigma = \emptyset$.

I.5.2 Curvature of compact homogeneous spaces

Let us fix a decomposition $\varphi = (\mathfrak{m}_1, \dots, \mathfrak{m}_\ell) \in \mathcal{F}^{\mathsf{G}}$ for the reductive complement $\mathfrak{m}$ and set $I := \{1, \dots, \ell\}$. Notice that the number ℓ of irreducible invariant submodules does not depend on the choice of the decomposition φ. We set $d_i := \dim(\mathfrak{m}_i)$ which are again, up to ordering, independent of φ. A basis (e_α) for $\mathfrak{m}$ is said to be *φ-adapted* if

$$e_1, \dots, e_{d_1} \in \mathfrak{m}_1 \ , \quad e_{d_1+1}, \dots, e_{d_1+d_2} \in \mathfrak{m}_2 \ , \quad \ldots \quad , \quad e_{d_1+\ldots+d_{\ell-1}+1}, \dots, e_n \in \mathfrak{m}_\ell \ .$$

For any subset $I' \subset I$, we set

$$\mathfrak{m}_{I'} := \sum_{i \in I'} \mathfrak{m}_i \ , \quad d_{I'} := \sum_{i \in I'} d_i \ . \tag{I.5.8}$$

Moreover, for any $I_1, I_2, I_3 \subset I$, we define

$$[I_1 I_2 I_3]_\varphi := \sum_{\substack{e_\alpha \in \mathfrak{m}_{I_1} \\ e_\beta \in \mathfrak{m}_{I_2} \\ e_\gamma \in \mathfrak{m}_{I_3}}} Q([e_\alpha, e_\beta], e_\gamma)^2 \ , \tag{I.5.9}$$

where (e_α) is a $Q_{\mathfrak{m}}$-orthonormal φ-adapted basis for $\mathfrak{m}$. If at least one of the three index sets is a singleton, say e.g. $I_1 = \{i\}$, we will shortly write $[iI_2I_3]_\varphi$ instead of $[\{i\}I_2I_3]_\varphi$. Notice that $[I_1I_2I_3]_\varphi$ is symmetric in all three entries and it does not depend on the choice of the $Q_{\mathfrak{m}}$-orthonormal basis (e_α). Furthermore, $[I_1I_2I_3]_\varphi \geq 0$ and $[I_1I_2I_3]_\varphi = 0$ if and only if $[\mathfrak{m}_{I_1}, \mathfrak{m}_{I_2}] \cap \mathfrak{m}_{I_3} = \{0\}$. Finally, though the coefficients $[I_1I_2I_3]_\varphi$ do depend on the choice of φ, the correspondence $\varphi \to [I_1I_2I_3]_\varphi$ is a continuous function on $\mathcal{F}^{\mathsf{G}}$ (see [7, Sec 4.3]).

We introduce now the Casimir operator

$$C_{Q_{\mathfrak{h}}} : \mathfrak{m} \to \mathfrak{m} \ , \quad C_{Q_{\mathfrak{h}}} := -\sum_i \operatorname{ad}(z_i) \circ \operatorname{ad}(z_i) \ ,$$

where $Q_{\mathfrak{h}} := Q|_{\mathfrak{h} \otimes \mathfrak{h}}$ and (z_i) is any $Q_{\mathfrak{h}}$-orthonormal basis for $\mathfrak{h}$. Then, the following conditions hold:

$$C_{Q_{\mathfrak{h}}}|_{\mathfrak{m}_i} = c_i \operatorname{Id}_{\mathfrak{m}_i} \ , \tag{I.5.10}$$

with $c_i \geq 0$ and $c_i = 0$ if and only if $[\mathfrak{h}, \mathfrak{m}_i] = \{0\}$ (see [87, Sec 1]). We also define the coefficients $b_1, \dots, b_\ell \in \mathbb{R}$ by setting

$$(-B)|_{\mathfrak{m}_i \otimes \mathfrak{m}_i} = b_i Q_{\mathfrak{m}_i} \ , \tag{I.5.11}$$

where B is the Cartan-Killing form of $\mathfrak{g}$. Since $\mathfrak{g}$ is compact, it follows that $b_i \geq 0$ and $b_i = 0$ if and only if $\mathfrak{m}_i \subset \mathfrak{z}(\mathfrak{g})$. If G is semisimple, then one can choose $Q = -B$, so that $b_i = 1$ for any i.

Notice that both the coefficients c_i and b_i do depend on the choice of φ, while

$$b_{\mathsf{G}/\mathsf{H}} := \operatorname{Tr}_{Q_{\mathfrak{m}}}(-B) = \sum_{i \in I} d_i b_i \tag{I.5.12}$$

does not. Moreover, they are related by the following useful relation (see [87, Lemma 1.5]):

$$d_i b_i = 2 d_i c_i + \sum_{j,k\in I} [ijk]_\varphi \quad \text{for any } i \in I \ . \tag{I.5.13}$$

Let now $g \in \mathcal{M}^{\mathsf{G}}$ be a diagonal metric as in (I.5.5) with respect to φ. The next proposition gives explicit formulas for the sectional curvature $\sec(g)$ of g along φ-adapted 2-planes in $\mathfrak{m}$. Notice that one could obtain (I.5.14) and (I.5.15) from [27, Cor 1.13] where the authors proved a more general formula for the sectional curvature of diagonal cohomogeneity one metrics.

Proposition I.5.1. *Let $X, Y \in \mathfrak{m}$ be $Q_{\mathfrak{m}}$-orthonormal vectors. If $X \in \mathfrak{m}_i$ and $Y \in \mathfrak{m}_j$ for some $i, j \in I$, then the sectional curvature of g along $X \wedge Y$ is given by the following formulas.*

- *if $i = j$, then*

$$\sec(g)(X\wedge Y) = \frac{1}{\lambda_i}\big|[X,Y]_{\mathfrak{h}}\big|_Q^2 + \sum_{k\in I}\frac{4\lambda_i - 3\lambda_k}{4\lambda_i^2}\big|[X,Y]_{\mathfrak{m}_k}\big|_Q^2 \ , \tag{I.5.14}$$

- *if $i \neq j$, then*

$$\sec(g)(X\wedge Y) = \sum_{k\in I}\frac{\lambda_i^2 + \lambda_j^2 - 3\lambda_k^2 - 2\lambda_i\lambda_j + 2\lambda_i\lambda_k + 2\lambda_j\lambda_k}{4\lambda_i\lambda_j\lambda_k}\big|[X,Y]_{\mathfrak{m}_k}\big|_Q^2 \ . \tag{I.5.15}$$

Proof. We let $\tilde{X} := \frac{1}{\sqrt{\lambda_i}}X$, $\tilde{Y} := \frac{1}{\sqrt{\lambda_j}}Y$. By [3, Thm 7.30] it holds that

$$\begin{aligned}\sec(X\wedge Y) = -\tfrac{3}{4}\big|[\tilde{X},\tilde{Y}]_{\mathfrak{m}}\big|_g^2 - \tfrac{1}{2}g\big([\tilde{X},[\tilde{X},\tilde{Y}]]_{\mathfrak{m}},\tilde{Y}\big) - \tfrac{1}{2}g\big([\tilde{Y},[\tilde{Y},\tilde{X}]]_{\mathfrak{m}},\tilde{X}\big) + \\ + \big|U^g(\tilde{X},\tilde{Y})\big|_g^2 - g\big(U^g(\tilde{X},\tilde{X}),U^g(\tilde{Y},\tilde{Y})\big) \ ,\end{aligned} \tag{I.5.16}$$

where $U^g : \mathfrak{m}\otimes\mathfrak{m} \to \mathfrak{m}$ is the symmetric tensor uniquely defined by

$$2g(U^g(X,Y),Z) := g([Z,X]_{\mathfrak{m}},Y) + g([Z,Y]_{\mathfrak{m}},X) \ . \tag{I.5.17}$$

We observe that

$$\begin{aligned}\big|[\tilde{X},\tilde{Y}]_{\mathfrak{m}}\big|_g^2 &= \sum_{k\in I}\frac{\lambda_k}{\lambda_i\lambda_j}\big|[X,Y]_{\mathfrak{m}_k}\big|_Q^2 \ , \\ g([\tilde{X},[\tilde{X},\tilde{Y}]]_{\mathfrak{m}},\tilde{Y}) &= \frac{1}{\lambda_i}Q([X,[X,Y]],Y) = -\frac{1}{\lambda_i}\big|[X,Y]\big|_Q^2 \ , \\ g([\tilde{Y},[\tilde{Y},\tilde{X}]]_{\mathfrak{m}},\tilde{X}) &= \frac{1}{\lambda_j}Q([Y,[Y,X]],X) = -\frac{1}{\lambda_j}\big|[X,Y]\big|_Q^2 \ .\end{aligned} \tag{I.5.18}$$

Let now (e_α) be a φ-adapted $Q_{\mathfrak{m}}$-orthonormal basis for $\mathfrak{m}$. Then

$$g(U^g(\tilde{X},\tilde{X}),e_\alpha) = g([e_\alpha,\tilde{X}],\tilde{X}) = \tfrac{1}{\lambda_i}Q([X,X],e_\alpha) = 0$$

and so

$$U^g(\tilde{X},\tilde{X}) = U^g(\tilde{Y},\tilde{Y}) = 0\ . \tag{I.5.19}$$

Finally

$$\begin{aligned}
|U^g(\tilde{X},\tilde{Y})|_g^2 &= \sum_{k\in I}\sum_{e_\alpha\in\mathfrak{m}_k} g(U^g(\tilde{X},\tilde{Y}),\tfrac{1}{\sqrt{\lambda_k}}e_\alpha)^2 \\
&= \sum_{k\in I}\sum_{e_\alpha\in\mathfrak{m}_k} \frac{1}{4\lambda_i\lambda_j\lambda_k}\Big(g([e_\alpha,X],Y)+g([e_\alpha,Y],X)\Big)^2 \qquad \text{(I.5.20)}\\
&= \sum_{k\in I}\frac{|\lambda_i-\lambda_j|^2}{4\lambda_i\lambda_j\lambda_k}\big|[X,Y]_{\mathfrak{m}_k}\big|_Q^2\ .
\end{aligned}$$

By (I.5.18), (I.5.19) and (I.5.20), formula (I.5.16) becomes

$$\begin{aligned}
&\sec(g)(X\wedge Y) = \\
&= -\sum_{k\in I}\frac{3\lambda_k}{4\lambda_i\lambda_j}\big|[X,Y]_{\mathfrak{m}_k}\big|_Q^2 + \frac{1}{2}\Big(\frac{1}{\lambda_i}+\frac{1}{\lambda_j}\Big)\big|[X,Y]\big|_Q^2 + \sum_{k\in I}\frac{|\lambda_i-\lambda_j|^2}{4\lambda_i\lambda_j\lambda_k}\big|[X,Y]_{\mathfrak{m}_k}\big|_Q^2 \\
&= \frac{\delta_{ij}}{\lambda_i}\big|[X,Y]_{\mathfrak{h}}\big|_Q^2 + \sum_{k\in I}\frac{2\lambda_i+2\lambda_j-3\lambda_k}{4\lambda_i\lambda_j}\big|[X,Y]_{\mathfrak{m}_k}\big|_Q^2 + \sum_{k\in I}\frac{|\lambda_i-\lambda_j|^2}{4\lambda_i\lambda_j\lambda_k}\big|[X,Y]_{\mathfrak{m}_k}\big|_Q^2 \\
&= \frac{\delta_{ij}}{\lambda_i}\big|[X,Y]_{\mathfrak{h}}\big|_Q^2 + \sum_{k\in I}\frac{\lambda_i^2+\lambda_j^2-3\lambda_k^2-2\lambda_i\lambda_j+2\lambda_i\lambda_k+2\lambda_j\lambda_k}{4\lambda_i\lambda_j\lambda_k}\big|[X,Y]_{\mathfrak{m}_k}\big|_Q^2
\end{aligned}$$

and so both (I.5.14) and (I.5.15) follow. □

As far as it concerns the Ricci tensor $\mathrm{Ric}(g) : \mathfrak{m}\otimes\mathfrak{m}\to\mathbb{R}$, the following lemma holds true (see also [60, Lemma 1.1]).

Lemma I.5.2. *For any* $1\leq i\leq \ell$ *it holds that*

$$\mathrm{Ric}(g)|_{\mathfrak{m}_i\otimes\mathfrak{m}_i} = \lambda_i\,\mathrm{ric}_i(g)\,Q_{\mathfrak{m}_i}\ ,$$

where

$$\mathrm{ric}_i(g) := \frac{b_i}{2\lambda_i} - \frac{1}{2d_i}\sum_{j,k\in I}[ijk]_\varphi\frac{\lambda_k}{\lambda_i\lambda_j} + \frac{1}{4d_i}\sum_{j,k\in I}[ijk]_\varphi\frac{\lambda_i}{\lambda_j\lambda_k}\ . \tag{I.5.21}$$

If the adjoint representation of H *on* $\mathfrak{m}$ *is monotypic, then the Ricci tensor decomposes as*

$$\mathrm{Ric}(g) = \lambda_1 \,\mathrm{ric}_1(g) Q_{\mathfrak{m}_1} + \ldots + \lambda_\ell \,\mathrm{ric}_\ell(g) Q_{\mathfrak{m}_\ell} \ .$$

Proof. By the Schur Lemma, for any $1 \leq i \leq \ell$ there exist $x_i \in \mathbb{R}$ such that $\mathrm{Ric}(g)|_{\mathfrak{m}_i \otimes \mathfrak{m}_i} = x_i Q_{\mathfrak{m}_i}$. Then, letting (e_α) be a φ-adapted $Q_{\mathfrak{m}}$-orthonormal basis for $\mathfrak{m}$, it necessarily holds that

$$\mathrm{ric}_i(g) = \frac{x_i}{\lambda_i} = \frac{1}{d_i \lambda_i} \sum_{e_\alpha \in \mathfrak{m}_i} \mathrm{Ric}(g)(e_\alpha, e_\alpha) = \frac{1}{d_i} \sum_{e_\alpha \in \mathfrak{m}_i} \mathrm{Ric}(g)\big(\tfrac{e_\alpha}{\sqrt{\lambda_i}}, \tfrac{e_\alpha}{\sqrt{\lambda_i}}\big) \ . \tag{I.5.22}$$

Notice that, from (I.5.9), (I.5.10) and the $\mathrm{Ad}(\mathsf{G})$-invariance of Q, we directly obtain that

$$\sum_{\substack{e_\alpha \in \mathfrak{m}_i \\ e_\beta \in \mathfrak{m}_j}} \big|[e_\alpha, e_\beta]_{\mathfrak{h}}\big|_Q^2 = \delta_{ij} d_i c_i \ , \quad \sum_{\substack{e_\alpha \in \mathfrak{m}_i \\ e_\beta \in \mathfrak{m}_j}} \big|[e_\alpha, e_\beta]_{\mathfrak{m}_k}\big|_Q^2 = [ijk]_\varphi \ . \tag{I.5.23}$$

Therefore, for any fixed $i \in I$, we get

$$\begin{aligned}
\sum_{j \in I} \sum_{\substack{e_\alpha \in \mathfrak{m}_i \\ e_\beta \in \mathfrak{m}_j}} & \sec(g)(e_\alpha \wedge e_\beta) = \\
&= \sum_{j \in I} \sum_{\substack{e_\alpha \in \mathfrak{m}_i \\ e_\beta \in \mathfrak{m}_j}} \frac{\delta_{ij}}{\lambda_i} \big|[e_\alpha, e_\beta]_{\mathfrak{h}}\big|_Q^2 + \\
&\qquad + \sum_{j,k \in I} \sum_{\substack{e_\alpha \in \mathfrak{m}_i \\ e_\beta \in \mathfrak{m}_j}} \frac{\lambda_i^2 + \lambda_j^2 - 3\lambda_k^2 - 2\lambda_i\lambda_j + 2\lambda_i\lambda_k + 2\lambda_j\lambda_k}{4\lambda_i\lambda_j\lambda_k} \big|[e_\alpha, e_\beta]_{\mathfrak{m}_k}\big|_Q^2 \\
&\overset{(\mathrm{I.5.23})}{=} \frac{d_i c_i}{\lambda_i} + \frac{1}{4} \sum_{j,k \in I} [ijk]_\varphi \frac{\lambda_i^2 - (\lambda_j - \lambda_k)^2}{\lambda_i \lambda_j \lambda_k} \\
&\overset{(\mathrm{I.5.13})}{=} \frac{d_i b_i}{2\lambda_i} + \frac{1}{4} \sum_{j,k \in I} [ijk]_\varphi \Big(-\frac{\lambda_j^2 + \lambda_k^2}{\lambda_i \lambda_j \lambda_k} + \frac{\lambda_i}{\lambda_j \lambda_k} \Big) \\
&= \frac{d_i b_i}{2\lambda_i} - \frac{1}{2} \sum_{j,k \in I} [ijk]_\varphi \frac{\lambda_k}{\lambda_i \lambda_j} + \frac{1}{4} \sum_{j,k \in I} [ijk]_\varphi \frac{\lambda_i}{\lambda_j \lambda_k} \ .
\end{aligned} \tag{I.5.24}$$

Finally, from (I.5.22) and (I.5.24) we conclude that

$$\begin{aligned}
\mathrm{ric}_i(g) &= \frac{1}{d_i} \sum_{e_\alpha \in \mathfrak{m}_i} \mathrm{Ric}(g)\big(\tfrac{e_\alpha}{\sqrt{\lambda_i}}, \tfrac{e_\alpha}{\sqrt{\lambda_i}}\big) \\
&= \frac{1}{d_i} \sum_{j \in I} \sum_{\substack{e_\alpha \in \mathfrak{m}_i \\ e_\beta \in \mathfrak{m}_j}} \sec(g)(e_\alpha \wedge e_\beta) \\
&= \frac{b_i}{2\lambda_i} - \frac{1}{2d_i} \sum_{j,k \in I} [ijk]_\varphi \frac{\lambda_k}{\lambda_i \lambda_j} + \frac{1}{4d_i} \sum_{j,k \in I} [ijk]_\varphi \frac{\lambda_i}{\lambda_j \lambda_k} \ .
\end{aligned}$$

The last claim follows directly by applying the Schur Lemma. □

Notice that the coefficients $\mathrm{ric}_i(g)$ defined in (I.5.21) are precisely the diagonal terms of the Ricci operator $\mathrm{Ric}(g) : \mathfrak{m} \to \mathfrak{m}$ given by the relation $\mathrm{Ric}(g)(X, Y) = g\big(\mathrm{Ric}(g)(X), Y\big)$.

Finally, by (I.5.21), it comes that the scalar curvature of g is given by (see also [87, Sec 1])

$$\mathrm{scal}(g) = \sum_{i \in I} d_i \, \mathrm{ric}_i(g) = \frac{1}{2} \sum_{i \in I} \frac{d_i b_i}{\lambda_i} - \frac{1}{4} \sum_{i,j,k \in I} [ijk]_\varphi \frac{\lambda_i}{\lambda_j \lambda_k} \ . \tag{I.5.25}$$

Chapter II

A local version of the Myers-Steenrod Theorem

II.1 Statement of results

In this chapter we give a characterization of local groups of isometries that admit structures of local Lie transformation groups. More precisely, we prove the following

Theorem II.1.1. *Any locally compact and effective local topological group of isometries acting on a pointed $\mathcal{C}^{k,\alpha}$-Riemannian manifold, with $k + \alpha > 0$, is a local Lie group of isometries.*

Our result can be considered as a local version of the Myers-Steenrod Theorem [53]. We recall that the most enhanced version of this result is actually a consequence of the celebrated Gleason, Montgomery and Zippin solution to Hilbert's fifth problem [23, 50]:

(H5) *A locally compact topological group admits a Lie group structure if and only if it is locally Euclidean, and this occurs if and only if it has no small subgroups.*

Note that (H5) is a characterization of Lie groups among all topological groups in terms of just group theory and topology. It was thus natural to expect that a similar property holds for local Lie groups too. However, such a result was proved only recently by Goldbring in [25] using techniques from non-standard Analysis. The proof of our Theorem II.1.1 is strongly based on Goldbring's Theorem.

For a better understanding of our result, it is convenient to briefly review the relations between the solution to the Hilbert's fifth problem (H5) and the various known versions of the Myers-Steenrod Theorem. We start by recalling that the original paper [53] contains the following two results:

(MS1) *Any distance preserving map between $\mathcal{C}^k$-Riemannian manifolds, with $k \geq 2$, is of class $\mathcal{C}^{k-1}$;*

(MS2) *Any closed group of isometries acting on a $\mathcal{C}^k$-Riemannian manifold, with $k \geq 2$, is a Lie transformation group.*

Subsequently, the works by Calabi and Hartman [16], Rešetnjak [68], Sabitov [71] and Shefel' [74] allowed to obtain the following stronger version of (MS1):

(MS1') *Any distance preserving map between $\mathcal{C}^{k,\alpha}$-Riemannian manifolds, with $k + \alpha > 0$, is of class $\mathcal{C}^{k+1,\alpha}$.*

Now, claims (MS1') and (H5) imply a strengthened version of (MS2), which holds under much lower regularity assumptions. Namely

(MS2') *Any closed group of isometries acting on a $\mathcal{C}^{k,\alpha}$-Riemannian manifold, with $k + \alpha > 0$, is a Lie transformation group.*

To the best of our knowledge, this is the strongest version of the Myers-Steenrod Theorem which can be obtained using the so far known results. For the reader convenience, we provide a proof in Section II.2 below. Our Theorem II.1.1 is obtained under the same regularity assumptions of such stronger version of (MS2) and can therefore be considered as a perfect analogue of it in the category of local groups of transformations.

We would like to point out that, as many authors have predicted the existence of a local version of (H5), also the contents of our Theorem II.1.1 were expected to be true (see [52, p. 616]). On the other hand, its proof remained an open problem since the very first appearance of the Myers-Steenrod Theorem in [53], where the authors themselves ended the paper asking explicitly whether any locally compact group germ of local isometries were a Lie group germ or not. We guess that the lapse of time that passed between the statement of the problem and the finding of the solution presented here was caused by the lack of specific technical tools for dealing with local Lie groups, a gap which was finally filled in the previous quoted paper by Goldbring.

As a by-product, we also obtain a useful regularity result for locally homogeneous Riemannian metrics, on which the main results of Chapter III are based.

Namely

Theorem II.1.2. *Let (M, g) be a locally homogeneous $\mathcal{C}^1$-Riemannian manifold. If there exist a point $p \in M$ and a locally compact, effective local topological group of isometries which acts transitively on (M, g, p), then (M, g) is real analytic.*

II.2 The Myers-Steenrod Theorem in low regularity

We provide here a proof of the version (MS2') of the Myers-Steenrod Theorem. We also show how it yields a useful regularity property for homogeneous Riemannian manifolds. First, we recall the following crucial result, which is a consequence of Theorem I.3.1.

Theorem II.2.1 ([51] Thm 2, p. 208)**.** *Let $G = (\mathsf{G}, \Theta)$ be a topological group of $\mathcal{C}^k$-transformations on a smooth manifold M, with $k \geq 1$. If G is effective and locally compact, then G is a Lie group of $\mathcal{C}^k$-transformations.*

We also need the following property, which is essentially due to van Dantzig and van der Waerden [20]. Let (M, g) be a $\mathcal{C}^{k,\alpha}$-Riemannian manifold and $\mathrm{Iso}(M, g)$ its full isometry group. We recall that the compact-open topology τ_{co} on $\mathrm{Iso}(M, g)$ is generated by the subbasis formed by the sets

$$(f; K; \varepsilon) := \{h \in \mathrm{Iso}(M, g) : \mathtt{d}_g(f(x), h(x)) < \varepsilon \text{ for any } x \in K\} ,$$

with $f \in \mathrm{Iso}(M, g)$, $K \subset M$ compact, $\varepsilon > 0$. On the other hand, the point-open topology τ_{po} on $\mathrm{Iso}(M, g)$ is generated by the subbasis formed by the sets

$$(f; x; \varepsilon) := \{h \in \mathrm{Iso}(M, g) : \mathtt{d}_g(f(x), h(x)) < \varepsilon\} ,$$

with $f \in \mathrm{Iso}(M, g)$, $x \in M$, $\varepsilon > 0$.

Lemma II.2.2. *On* $\mathrm{Iso}(M, g)$*, the compact-open topology coincides with the point-open topology. This topology is Hausdorff, it makes the group operations continuous and it is the coarsest topology with respect to which the action of* $\mathrm{Iso}(M, g)$ *on* M *is continuous. Furthermore, with respect to such topology,* $\mathrm{Iso}(M, g)$ *is locally compact and its action on* M *is proper.*

Proof. We set $G := \mathrm{Iso}(M, g)$ for short. Let us fix $f \in G$, $K \subset M$ compact, $\varepsilon > 0$ and let $x_1, \dots, x_N \in K$ be such that $K \subset \mathcal{B}_g(x_1, \frac{\varepsilon}{3}) \cup \ldots \cup \mathcal{B}_g(x_N, \frac{\varepsilon}{3})$. We have to show that $A := \bigcap_{1 \le i \le N}(f; x_i; \frac{\varepsilon}{3})$ is contained in $(f; K; \varepsilon)$. So, let us consider $h \in A$ and $x \in K$. By construction, there exists $1 \le i \le N$ such that $\mathtt{d}_g(x, x_i) < \frac{\varepsilon}{3}$. But then

$$\mathtt{d}_g(f(x), h(x)) \le \mathtt{d}_g(f(x), f(x_i)) + \mathtt{d}_g(f(x_i), h(x_i)) + \mathtt{d}_g(h(x_i), h(x)) < \varepsilon$$

and hence $\tau_{co} \subset \tau_{po}$. Since the other inclusion is obvious, we conclude that $\tau_{co} = \tau_{po}$. The second claim is just a collection of some well known properties of the compact-open topology. We refer to the main theorem of [46] for the last claim. □

We are now ready to prove the following

Corollary II.2.3 (Enhanced version of the Myers-Steenrod Theorem)**.** *Any closed group of isometries of a $\mathcal{C}^{k,\alpha}$-Riemannian manifold, with $k + \alpha > 0$, is a Lie group of isometries.*

Proof. Let (M, g) be a $\mathcal{C}^{k,\alpha}$-Riemannian manifold, with $k + \alpha > 0$, and consider its full isometry group $G = \mathrm{Iso}(M, g)$. Then, by means of Theorem I.1.5 and Lemma II.2.2, G is an effective group of $\mathcal{C}^{k+1}$-transformation and G is locally compact. Then, by Theorem II.2.1, it is a Lie group of isometries and the thesis follows. □

This corollary yields the following improvement of a well known property of homogeneous Riemannian manifolds. As usual, a $\mathcal{C}^{k,\alpha}$-Riemannian manifold (M, g) is called *homogeneous* if it admits a closed, transitive group of isometries.

Theorem II.2.4. *Any homogeneous $\mathcal{C}^{0,\alpha}$-Riemannian manifold, with $\alpha > 0$, is real analytic.*

Proof. Let (M, g) be a $\mathcal{C}^{0,\alpha}$-Riemannian manifold, with $\alpha > 0$, and $G = (\mathsf{G}, \Theta)$ a closed, transitive topological group of isometries acting on (M, g). Pick a distinguished point $x_o \in M$ and consider the *isotropy subgroup of G at x_o*, i.e. $\mathsf{H} := \{a \in \mathsf{G} : \Theta(a, x_o) = x_o\}$. From Corollary II.2.3, it follows that G is a Lie group of isometries and, by means of Theorem I.1.5, the map $\Theta : \mathsf{G} \times M \to M$

is of class $\mathcal{C}^1$. Then, H is an embedded Lie subgroup of G and we get the $\mathcal{C}^1$-diffeomorphism

$$\vartheta_{x_o} : \mathsf{G}/\mathsf{H} \to M \ , \quad \vartheta_{x_o}(a\mathsf{H}) := \Theta(a, x_o) \ . \tag{II.2.1}$$

Since G acts by isometries, there exists a unique invariant $\mathcal{C}^\omega$-Riemannian metric $\tilde{g}$ on G/H which makes the map $\vartheta_{x_o} : (\mathsf{G}/\mathsf{H}, \tilde{g}) \to (M, g)$ an isometry. From this the thesis follows. □

II.3 Proof of Theorem II.1.1

The purpose of this section is to give the proof of a local analogue of Theorem II.2.1, namely

Theorem II.3.1. *Let $G = (\mathsf{G}, \mathcal{U}_\mathsf{G}, \Omega_p, \mathcal{W}, \Theta)$ be a local topological group of $\mathcal{C}^k$-transformations on a pointed smooth manifold (M, p), with $k \geq 1$. If G is locally compact and effective, then G is a local Lie group of $\mathcal{C}^k$-transformations.*

of which Theorem II.1.1 is an immediate consequence.

First, we need a preparatory lemma. For its statement, we introduce the following definition. Let G be a local topological group. For any integer $N \geq 1$ and for any $a_1, \ldots, a_N, b \in \mathsf{G}$, we say that *the element $a_1 \cdot a_2 \cdot \ldots \cdot a_N$ is well defined and equal to b*, for short $a_1 \cdot a_2 \cdot \ldots \cdot a_N = b$, if the following condition defined by induction on N is satisfied: for any $1 \leq i \leq N$ there exist $b_i, b'_i \in \mathsf{G}$ such that $a_1 \cdot \ldots \cdot a_i = b_i$, $a_{i+1} \cdot \ldots \cdot a_N = b'_i$, $(b'_i, b''_i) \in \mathcal{D}(\mathsf{G})$ and $b_i \cdot b'_i = b$. If $\mathcal{U} \subset \mathsf{G}$ is a neighborhood of the unit such that $a_1 \cdot \ldots \cdot a_N$ is well defined for any choice of $a_1, \ldots, a_N \in \mathcal{U}$, we set $\mathcal{U}^N := \big\{a_1 \cdot \ldots \cdot a_N : a_1, \ldots, a_N \in \mathcal{U}\big\}$.

Lemma II.3.2. *Let $G = (\mathsf{G}, \mathcal{U}_\mathsf{G}, \Omega_p, \mathcal{W}, \Theta)$ be a local topological group of $\mathcal{C}^k$-transformations on a pointed smooth manifold (M, p). Then:*

i) For any compact set $K \subset \Omega_p$, there exists a neighborhood $\mathcal{U} \subset \mathcal{U}_\mathsf{G}$ of the unit such that $\mathcal{U} \times K \subset \mathcal{W}$.

ii) For any fixed $N \in \mathbb{N}$, there exists a neighborhood $\mathcal{W}_N$ of $\{e\} \times \Omega_p$ in $\mathcal{W}$ such that for any

$$(a_1, x), \ldots, (a_N, x) \in \mathcal{W}_N$$

the element $a_1 \cdot \ldots \cdot a_N$ is well defined and $(b_i \cdot b'_i, x), (b_i, \Theta(b'_i, x)) \in \mathcal{W}$ for any $1 \leq i \leq N$, where $b_i := a_1 \cdot \ldots \cdot a_i$ and $b'_i := a_{i+1} \cdot \ldots \cdot a_N$.

Proof. To prove the first claim, it is sufficient to observe that, since $\{e\} \times K$ is compact, there exists an finite open cover $\{\mathcal{I}_1, \dots, \mathcal{I}_\ell\}$ of $\{e\} \times K$ inside $\mathcal{W}$, where the open sets $\mathcal{I}_i$ have the form $\mathcal{I}_i = \mathcal{U}_i \times U_i$ for any $1 \leq i \leq \ell$. Then $\mathcal{U} \subset \bigcap_{1 \leq i \leq \ell} \mathcal{U}_i$ satisfies (i).

We now recall that there exists a sequence of nested neighborhoods

$$\{e\} \subset \ldots \subset \widetilde{\mathcal{D}}_N(\mathsf{G}) \subset \widetilde{\mathcal{D}}_{N-1}(\mathsf{G}) \subset \ldots \subset \widetilde{\mathcal{D}}_2(\mathsf{G}) \subset \mathsf{G}$$

of the unit such that, for any $N \geq 2$ and for any choice of N elements $a_1, \dots, a_N \in \widetilde{\mathcal{D}}_N(\mathsf{G})$, the product $a_1 \cdot \ldots \cdot a_N$ is well defined (see e.g. [25, Lemma 2.5]).

Fix $N \in \mathbb{N}$. By (i) we can consider N exhaustions $\{U_1^{(n)}\}, \dots, \{U_N^{(n)}\}$ of Ω_p by relatively compact open sets and two sequences $\{\mathcal{U}^{(n)}\}$, $\{\mathcal{U}'^{(n)}\}$ of neighborhoods of the unit in $\mathcal{U}_\mathsf{G} \subset \mathsf{G}$ such that:

- $U_1^{(n)} \subset\subset U_2^{(n)} \subset\subset \cdots \subset\subset U_N^{(n)} \subset\subset U_1^{(n+1)}$ and $\mathcal{U}^{(n+1)} \subset \mathcal{U}'^{(n+1)} \subset \mathcal{U}^{(n)}$,
- $\mathcal{U}'^{(n)} \times U_N^{(n)} \subset \big(\widetilde{\mathcal{D}}_N(\mathsf{G}) \times M\big) \cap \mathcal{W}$,
- $\big(\mathcal{U}^{(n)}\big)^N \subset \mathcal{U}'^{(n)}$ and $\Theta\big(\mathcal{U}^{(n)} \times U_i^{(n)}\big) \subset U_{i+1}^{(n)}$ for any $1 \leq i \leq N-1$.

It is immediate now to realize that for any $(a_1, x), \dots, (a_N, x) \in \mathcal{U}^{(n)} \times U_1^{(n)}$ it holds that $a_1 \cdot \ldots \cdot a_N$ is well defined and that $(b_i \cdot b'_i, x), (b_i, \Theta(b'_i, x)) \in \mathcal{W}$ for any $1 \leq i \leq N$, with $a_1 \cdot \ldots \cdot a_i = b_i$ and $a_{i+1} \cdot \ldots \cdot a_N = b'_i$. Therefore, if we define $\mathcal{W}_N := \bigcup_{k \in \mathbb{N}} \mathcal{U}^{(n)} \times U_1^{(n)}$, then claim (ii) follows. □

We observe that Theorem II.3.1 involves only local objects. Hence, without loss of generality, we may assume that $(M, p) = (\mathbb{R}^m, 0)$. However, for the sake of clarity, in what follows we will still use the symbols p and Ω_p for 0 and the distinguished neighborhood of 0, respectively.

Proposition II.3.3. *Let $k \geq 1$ and $G = (\mathsf{G}, \mathcal{U}_\mathsf{G}, \Omega_p, \mathcal{W}, \Theta)$ a locally compact local group of $\mathcal{C}^k$-transformations on $(M, p) = (\mathbb{R}^m, 0)$. Then, there exist a relatively compact neighborhood $\mathcal{V} \subset \mathcal{U}_\mathsf{G}$ of the unit and a ball $B \subset \Omega_p$ centered at p which satisfy the following property: if H is a subgroup of G entirely contained in $\mathcal{V}$, then there exists a neighborhood $V_o \subset B$ of the origin such that $\Theta(a)|_{V_o} = \mathrm{Id}_{V_o}$ for any $a \in \mathsf{H}$.*

Proof. By [49, Thm 1] and [4, p. 685], given $(a, x) \in \mathcal{W}$, for any neighborhood $\mathcal{V}_a \subset \mathcal{U}_\mathsf{G}$ of a and for any ball $B \subset \Omega_p$ centered at x such that $\mathcal{V}_a \times B \subset \mathcal{W}$, the following holds: *every partial derivative of the function $\Theta(b)|_B : B \to \mathbb{R}^m$ up to order k is continuous with respect to $b \in \mathcal{V}_a$.*

Since $\Theta(e)$ is the identity map of $\Omega_p \subset \mathbb{R}^m$, from Lemma IV.4.1 it follows that there exist a relatively compact neighborhood $\mathcal{V} \subset \mathsf{G}$ of the unit and a ball $B \subset\subset \Omega_p$ of the origin such that $\bar{\mathcal{V}} \times \bar{B} \subset \mathcal{W}_N$, with $N \geq 2$ big enough, and the family of functions $\left\{(\Theta(a) - \mathrm{Id})|_B : B \to \mathbb{R}^m\right\}_{a \in \mathcal{V}}$ is uniformly bounded in the Banach space $\mathcal{C}^k(\bar{B})$ by a positive constant $C \in \mathbb{R}$, which can be taken as small as one likes by restricting $\mathcal{V}$. Let now H be a subgroup of G entirely contained in $\mathcal{V}$. By taking the closure, one can suppose that H is closed and hence compact. We define the map

$$T : B \to \mathbb{R}^m \ , \quad T(x) := \int_{\mathsf{H}} \Theta(a, x)\, d\lambda(a) \ ,$$

where λ is the Haar measure of H, normalized in such a way that $\lambda(\mathsf{H}) = 1$. By differentiating under the integral sign, it follows that T is of class $\mathcal{C}^k$. Moreover

$$\|T - \mathrm{Id}\,\|_{\mathcal{C}^k(\bar{B})} \leq \int_{\mathsf{H}} \|\Theta(a) - \mathrm{Id}\,\|_{\mathcal{C}^k(\bar{B})}\, d\lambda(a) \leq C \ .$$

By the Inverse Function Theorem, there exists an open neighborhood $V \subset B$ of the origin such that the restriction $T|_V : V \to \mathbb{R}^m$ is an open $\mathcal{C}^k$-embedding and $T(V) \subset B$. On the other hand, we can choose a sufficiently small neighborhood $V_{\mathrm{o}} \subset V$ of the origin such that $\Theta(a)(V_{\mathrm{o}}) \subset V$ for any $a \in \mathcal{V}$. Then, from the bi-invariance of the Haar measure, for any $b \in \mathsf{H}$ and for any $x \in V_{\mathrm{o}}$ it follows that

$$\begin{aligned} \big(T \circ \Theta(b)\big)(x) &= \int_{\mathsf{H}} \big(\Theta(a) \circ \Theta(b)\big)(x)\, d\lambda(a) \\ &= \int_{\mathsf{H}} \Theta(a \cdot b)(x)\, d\lambda(a) = \int_{\mathsf{H}} \Theta(a)(x)\, d\lambda(a) = T(x) \ . \end{aligned}$$

Since T is invertible in V, we get $\Theta(b)|_{V_{\mathrm{o}}} = \mathrm{Id}_{V_{\mathrm{o}}}$ for any $b \in \mathsf{H}$. □

We are now able to conclude the proof of Theorem II.3.1. Suppose that G is a locally compact and effective local topological group of $\mathcal{C}^k$-transformations on $(\mathbb{R}^m, 0)$. From Proposition II.3.3, we directly get that the abstract (local) group of G is NSS. By Theorem I.3.5 and Remark I.3.6, we get the thesis.

II.4 Proof of Theorem II.1.2

Since we deal with locally homogeneous manifolds, in order to prove Theorem II.1.2 we need to define rigorously a *local analogous* of the usual quotient of Lie groups. Namely

Proposition II.4.1. *Let G be a Lie group and $\mathsf{H} \subset \mathsf{G}$ be a (not necessarily closed) Lie subgroup.*

a) There exist a neighborhood $\mathcal{U}_{\mathsf{H}} \subset \mathsf{H}$ of the unit in the manifold topology of H and two neighborhoods $\mathcal{U}, \mathcal{V} \subset \mathsf{G}$ of the identity such that: $\mathcal{U}_{\mathsf{H}}$ is a sub-local group of G with associated neighborhood $\mathcal{V}$, $\mathcal{U}_{\mathsf{H}}$ is closed in $\mathcal{V}$ and $\mathcal{U}^6 \subset \mathcal{V}$.

b) The binary relation on $\mathcal{U}$ defined by

$$a \sim b \overset{\text{def}}{\iff} a^{-1} \cdot b \in \mathcal{U}_{\mathsf{H}}$$

is an equivalence relation on $\mathcal{U}$ and the equivalence class $[a]_\sim$ of $a \in \mathcal{U}$ verifies $[a]_\sim = (a\mathcal{U}_{\mathsf{H}}) \cap \mathcal{U}$.

c) The quotient space $(\mathsf{G}/\mathsf{H})_{(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})} := \mathcal{U}/\!\sim\, = \big\{(a\mathcal{U}_{\mathsf{H}}) \cap \mathcal{U} : a \in \mathcal{U}\big\}$ is a topological manifold and it admits a real analytic structure, which is unique up to $\mathcal{C}^\omega$-diffeomorphism, with respect to which the following conditions hold:

- *the canonical projection $\pi_{(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})} : \mathcal{U} \to (\mathsf{G}/\mathsf{H})_{(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})}$ is a $\mathcal{C}^\omega$-submersion;*
- *the tuple $G_{(\mathsf{G},\mathsf{H}),(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})} := (\mathsf{G}, \mathcal{U}, (\mathsf{G}/\mathsf{H})_{(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})}, \mathcal{W}, \Theta)$ with*

$$\mathcal{W} := \big\{\big(a, (b\mathcal{U}_{\mathsf{H}}) \cap \mathcal{U}\big) : a \in \mathcal{U},\, b \in \mathcal{U},\, a \cdot b \in \mathcal{U}\big\}\ ,$$
$$\Theta : \mathcal{W} \to (\mathsf{G}/\mathsf{H})_{\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V}}\ , \quad \Theta(a)\big((b\mathcal{U}_{\mathsf{H}}) \cap \mathcal{U}\big) := ((a \cdot b)\mathcal{U}_{\mathsf{H}}) \cap \mathcal{U}$$

is a local Lie group of $\mathcal{C}^\omega$-transformations acting transitively on $\big((\mathsf{G}/\mathsf{H})_{(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})}, (e\mathcal{U}_{\mathsf{H}}) \cap \mathcal{U}\big)$.

d) If $(\mathcal{U}_{\mathsf{H}}, \mathcal{U}, \mathcal{V})$ and $(\mathcal{U}'_{\mathsf{H}}, \mathcal{U}', \mathcal{V}')$ are two triples both satisfying all conditions in (a), then $G_{(\mathsf{G},\mathsf{H}),(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})}$ is locally $\mathcal{C}^\omega$-equivalent to $G_{(\mathsf{G},\mathsf{H}),(\mathcal{U}'_{\mathsf{H}},\mathcal{U}',\mathcal{V}')}$.

Proof. The proof of (a) is straightforward, while (b) is the statement of [25, Lemma 2.13]. To prove (c), one can easily adapt the well known proof of the corresponding statement for the quotient of a Lie group with respect to a closed subgroup (see e.g. [30, Ch II, Sec 4]). Finally, to prove (d), let us consider two neighborhoods $\mathcal{U}_1, \mathcal{U}_2 \subset \mathsf{G}$ of the unit such that $(\mathcal{U}_1)^2 \subset \mathcal{U}_2$ and $\mathcal{U}_{\mathsf{H}} \cap \mathcal{U}_2 = \mathcal{U}'_{\mathsf{H}} \cap \mathcal{U}_2$. Then let us pick a neighborhood $\mathcal{U}_o \subset \mathcal{U} \cap \mathcal{U}' \cap \mathcal{U}_1$ of the unit in G. One can directly check that the map

$$\pi_{(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})}(\mathcal{U}_o) \to \pi_{(\mathcal{U}'_{\mathsf{H}},\mathcal{U}',\mathcal{V}')}(\mathcal{U}_o)\ , \quad (a\mathcal{U}_{\mathsf{H}}) \cap \mathcal{U} \mapsto (a\mathcal{U}'_{\mathsf{H}}) \cap \mathcal{U}'$$

is a $\mathcal{C}^\omega$-diffeomorphism. $\square$

Given a Lie group G together with a Lie subgroup H, we call *admissible triple for* H *in* G any choice of $(\mathcal{U}_{\mathsf{H}}, \mathcal{U}, \mathcal{V})$ as in (a) and *local factor space of* G *modulo* H any quotient $(\mathsf{G}/\mathsf{H})_{(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})}$ as in (c). Notice that H is closed in G if and only if $(\mathsf{H}, \mathsf{G}, \mathsf{G})$ is an admissible triple for H in G and, in that case, $(\mathsf{G}/\mathsf{H})_{(\mathsf{H},\mathsf{G},\mathsf{G})} = \mathsf{G}/\mathsf{H}$. For other details concerning local factor spaces and locally homogeneous metrics, we refer to [52] and [78].

Let now $G = (\mathsf{G}, \mathcal{U}_{\mathsf{G}}, \Omega_p, \mathcal{W}, \Theta)$ be an almost-effective local Lie group of $\mathcal{C}^k$-transformations on (M,p), with $k \geq 2$, and consider its infinitesimal generators $\Theta_* : \mathfrak{g} \to \mathcal{C}^{k-1}(\Omega_p; TM|_{\Omega_p})$. We define the subset $\mathfrak{h} := \big\{X \in \mathfrak{g} : \Theta_*(X)_p = 0\big\}$. By (I.3.2), it follows that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and so we can consider the unique connected Lie subgroup H of G such that $\mathrm{Lie}(\mathsf{H}) = \mathfrak{h}$. We call it the *abstract isotropy subgroup of G at p.* Notice that G is almost-effective if and only if $\mathfrak{h}$ does not contain any non-trivial ideal of $\mathfrak{g}$, while G is effective if and only if H does not contain any non-trivial normal subgroup of G. As expected, the following proposition holds.

Proposition II.4.2. *For any admissible triple $(\mathcal{U}_{\mathsf{H}}, \mathcal{U}, \mathcal{V})$ for H in G, G is locally $\mathcal{C}^k$-equivalent to the local Lie group of $\mathcal{C}^\omega$-transformations $G_{(\mathsf{G},\mathsf{H}),(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})}$ defined in (c) of Proposition II.4.1.*

Proof. Let $(\mathcal{U}_{\mathsf{H}}, \mathcal{U}, \mathcal{V})$ be an admissible triple for H in G and choose a sufficiently small neighborhood of the unit $\mathcal{U}_{\mathrm{o}} \subset \mathcal{U} \cap \mathcal{U}_{\mathsf{G}}$. Then, the identity map $\mathrm{Id}_{\mathsf{G}} : \mathsf{G} \to \mathsf{G}$ and the map

$$\vartheta_{x_{\mathrm{o}}} : \pi_{(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})}(\mathcal{U}_{\mathrm{o}}) \to M \ , \quad \vartheta_{x_{\mathrm{o}}}((a\mathcal{U}_{\mathsf{H}}) \cap \mathcal{U}) := \Theta(a,p)$$

give rise to a local $\mathcal{C}^k$-equivalence between G and $G_{(\mathsf{G},\mathsf{H}),(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})}$. □

Let now (M,g) be a locally homogeneous $\mathcal{C}^1$-Riemannian manifold and assume that there exist a point $p \in M$ and a locally compact, effective local topological group of isometries $G = (\mathsf{G}, \mathcal{U}_{\mathsf{G}}, \Omega_p, \mathcal{W}, \Theta)$ which acts transitively on (M,g,p).

Lemma II.4.3. *For any fixed $x_{\mathrm{o}} \in M$, there exists a neighborhood $U_{x_{\mathrm{o}}} \subset M$ of x_{o} and an open $\mathcal{C}^2$-embedding $\varphi_{x_{\mathrm{o}}} : U_{x_{\mathrm{o}}} \to \mathbb{R}^m$ such that the pulled-back metric $(\varphi_{x_{\mathrm{o}}}^{-1})^*g$ on the open set $\varphi_{x_{\mathrm{o}}}(U_{x_{\mathrm{o}}}) \subset \mathbb{R}^m$ is real analytic.*

Proof. Since (M,g) is locally homogeneous, it is sufficient to prove the claim for $x_{\mathrm{o}} = p$. By means of Theorem II.1.1 and Theorem I.1.5, G is a local Lie group of isometries and the map Θ is of class $\mathcal{C}^2$. Then, let H be the abstract isotropy of G at p and pick an admissible triple $(\mathcal{U}_{\mathsf{H}}, \mathcal{U}, \mathcal{V})$ for H in G. By means of Proposition II.4.2, G is locally $\mathcal{C}^2$-equivalent to the local Lie group $G_{(\mathsf{G},\mathsf{H}),(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})}$ of $\mathcal{C}^\omega$-transformations on the local factor space $(\mathsf{G}/\mathsf{H})_{(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})}$. Since G acts on (M, g, p) by isometries, there exists a unique $\mathcal{C}^\omega$-Riemannian metric $\tilde{g}$ on $(\mathsf{G}/\mathsf{H})_{(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})}$ which makes the map $\vartheta_{x_\mathrm{o}} : \pi_{(\mathcal{U}_{\mathsf{H}},\mathcal{U},\mathcal{V})}(\mathcal{U}_\mathrm{o}) \to M$ defined in the proof of Proposition II.4.2 a local isometry. $\square$

We may now conclude the proof of Theorem II.1.2. By Lemma II.4.3, there exists a $\mathcal{C}^2$-atlas $\mathcal{A} = \{(U_\alpha, \xi_\alpha)\}$ on M such that the metric g is real analytic with respect to each coordinate chart $(U_\alpha, \xi_\alpha) \in \mathcal{A}$. But then, by [21, Lemma 1.2], there exists a $\mathcal{C}^\omega$-atlas $\mathcal{A}'$ on M which is compatible with $\mathcal{A}$ and with respect to which g is real analytic.

Chapter III

A compactness theorem for locally homogeneous spaces

III.1 Statement of results

In this chapter we prove the main results of the thesis, which concern the compactness of some moduli spaces of locally homogeneous Riemannian manifolds.

Generalizing a notion introduced in [11], we consider a distinguished set of locally homogeneous spaces $(\mathcal{B}, \hat{g})$ called *geometric models* (see Definition III.2.1). These objects provide a useful parametrization for the subspace $\mathcal{H}_m^{\mathrm{loc}}(1) \subset \mathcal{H}_m^{\mathrm{loc}}$ of those equivalence classes $\mu \in \mathcal{H}_m^{\mathrm{loc}}$ with bounded sectional curvature $|\sec(g_\mu)| \leq 1$. Indeed, we prove

Theorem III.1.1. *In any class $\mu \in \mathcal{H}_m^{\mathrm{loc}}(1)$ there exists a geometric model $(\mathcal{B}_\mu, \hat{g}_\mu) = (\mathcal{B}_{\hat{g}_\mu}(o_\mu, \pi), \hat{g}_\mu)$, which is unique up to a global equivariant isometry.*

The proof of Theorem III.1.1 is based on the following two preliminary results. Firstly, for any equivariant local isometry class $\mu \in \mathcal{H}_m(1) := \mathcal{H}_m^{\mathrm{loc}}(1) \cap \mathcal{H}_m$ the existence of a geometric model is a direct consequence of the Rauch Comparison Theorem (see Remark III.2.4). Secondly, for any $\mu \in \mathcal{H}_m^{\mathrm{loc}} \setminus \mathcal{H}_m$ there exists a sequence $(\mu^{(n)}) \subset \mathcal{H}_m$ which converges to μ in an appropriate topology (see Theorem III.3.4).

By means of these facts, Theorem III.1.1 is a consequence of the next

Theorem III.1.2. *The space of geometric models is compact in the pointed $\mathcal{C}^{1,\alpha}$-topology for any $0 < \alpha < 1$.*

We would like to point out that our Theorem III.1.2 can be considered as a generalization of a previous result by Böhm, Lafuente and Simon. In fact, they proved the following in [11, Thm 1.6]: *a sequence of geometric models $(\mathcal{B}^{(n)}, \hat{g}^{(n)})$ with $\mathrm{Ric}(\hat{g}^{(n)}) \to 0$ converges, up to a subsequence, to a smooth flat Riemannian manifold (M, g) in the pointed $\mathcal{C}^{1,\alpha}$-topology for any $0 < \alpha < 1$.* Indeed, both theorems rely on a Cheeger-Gromov-type precompactness theorem for incomplete Riemannian manifolds and in both proofs there is the need of showing that the limit space, which is a priori just a $\mathcal{C}^{1,\alpha}$-Riemannian manifold, is indeed smooth. In [11, Thm 1.6] such required regularity is achieved by means of the additional assumption on the Ricci tensor, while in our result we use Theorem II.1.2 proved in Chapter II.

Notice that Theorem III.1.1 and Theorem III.1.2 directly imply the following

Corollary III.1.3. *The moduli space $\mathcal{H}^{\mathrm{loc}}_m(1)$ is compact in the pointed $\mathcal{C}^{1,\alpha}$-topology for any $0 < \alpha < 1$.*

In order to use the compactness result of Theorem III.1.2 in the proof of Theorem III.1.1, one needs to take care of some special issues on the convergence of locally homogeneous spaces. In particular, we use the fact that *the homogenous spaces are dense in the class of locally homogenous spaces with respect to the topology of the algebraic convergence.* Moreover, the pointed $\mathcal{C}^{1,\alpha}$-convergence of a sequence of geometric models, combined with the additional assumption of algebraic convergence, implies the pointed $\mathcal{C}^{\infty}$-convergence of the geometric models. This is the crucial step for concluding the proof of Theorem III.1.1.

As a by-product of our analysis of the various notions of convergence and of Theorem III.1.1, we also obtain

Theorem III.1.4. *Let $(\mu^{(n)}) \subset \mathcal{H}^{\mathrm{loc}}_m(1)$ be a sequence, $\mu^{(\infty)} \in \mathcal{H}^{\mathrm{loc}}_m(1)$ and $s \geq \imath(m) + 2$ an integer.*

i) If $(\mathcal{B}_{\mu^{(n)}}, \hat{g}_{\mu^{(n)}})$ converges to $(\mathcal{B}_{\mu^{(\infty)}}, \hat{g}_{\mu^{(\infty)}})$ in the pointed $\mathcal{C}^{s+2}$-topology, then $(\mu^{(n)})$ converges s-infinitesimally to $\mu^{(\infty)}$.

ii) If $(\mu^{(n)})$ converges $(s+1)$-infinitesimally to $\mu^{(\infty)}$, then $(\mathcal{B}_{\mu^{(n)}}, \hat{g}_{\mu^{(n)}})$ converges to $(\mathcal{B}_{\mu^{(\infty)}}, \hat{g}_{\mu^{(\infty)}})$ in the pointed $\mathcal{C}^{s+2,\alpha}$-topology for any $0 < \alpha < 1$.

which in turn immediately implies

Corollary III.1.5. *Let $(\mu^{(n)}) \subset \mathcal{H}_m^{\mathrm{loc}}(1)$ be a sequence and $\mu^{(\infty)} \in \mathcal{H}_m^{\mathrm{loc}}(1)$.*

i) The geometric models $(\mathcal{B}_{\mu^{(n)}}, \hat{g}_{\mu^{(n)}})$ converge to $(\mathcal{B}_{\mu^{(\infty)}}, \hat{g}_{\mu^{(\infty)}})$ in the pointed $\mathcal{C}^\infty$-topology if and only if $(\mu^{(n)})$ converges infinitesimally to $\mu^{(\infty)}$.

ii) If $(\mu^{(n)})$ converges algebraically to $\mu^{(\infty)}$, then $(\mathcal{B}_{\mu^{(n)}}, \hat{g}_{\mu^{(n)}})$ converges to $(\mathcal{B}_{\mu^{(\infty)}}, \hat{g}_{\mu^{(\infty)}})$ in the pointed $\mathcal{C}^\infty$-topology.

Let us remark that here, as we pointed out in the Introduction, the role of the geometric models is crucial. Indeed, in general pointed convergence is much stronger than the infinitesimal convergence (see [41, Ex 6.6]). For reader's convenience, we summarize in the following diagram the relations among the three notions of convergence.

$$\begin{tikzcd}
\begin{array}{c}\text{Pointed } \mathcal{C}^\infty\text{-convergence} \\ \text{of the geometric models}\end{array} \arrow[rr, Rightarrow, shift left] \arrow[dr, Rightarrow, "/" marking] & & \begin{array}{c}\text{Infinitesimal} \\ \text{convergence}\end{array} \arrow[ll, Rightarrow, shift left] \arrow[dl, Rightarrow, "/" marking] \\
& \begin{array}{c}\text{Algebraic} \\ \text{convergence}\end{array} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

III.2 The space of geometric models

III.2.1 Geometric models

We begin this section by introducing the following

Definition III.2.1. A *geometric model* is a smooth locally homogeneous Riemannian distance ball $(\mathcal{B}, \hat{g}) = (\mathcal{B}_{\hat{g}}(o, \pi), \hat{g})$ of radius π satisfying $|\sec(\hat{g})| \leq 1$ and $\mathrm{inj}_o(\mathcal{B}, \hat{g}) = \pi$.

From now until the end of Section III.2, up to pulling back the metric via the Riemannian exponential map, any geometric model will be always assumed to be of the form $(B^m, \hat{g})$, where $B^m := B_{\mathrm{st}}(0, \pi) \subset \mathbb{R}^m$ is the m-dimensional Euclidean ball of radius π, and the standard coordinates of B^m will be always assumed to be normal for $\hat{g}$ at 0. Therefore the geodesics starting from $0 \in B^m$ are precisely the straight lines and $\mathtt{d}_{\hat{g}}(0,x) = |x|_{\mathrm{st}}$ for any $x \in B^m$. Hence, $\mathcal{B}_{\hat{g}}(0,r) = B_{\mathrm{st}}(0,r)$ for any $0 < r \leq \pi$. Moreover, by [11, Lemma 1.3] it comes that

$$\mathrm{inj}_x(B^m, \hat{g}) = \pi - |x|_{\mathrm{st}} \quad \text{for any } x \in B^m \ . \tag{III.2.1}$$

Since $(B^m, \hat{g})$ is real analytic, any local isometry can be extended uniformly, i.e.

Lemma III.2.2 ([11], Lemma 1.4)**.** *Let $x, y \in B^m$ and set $\rho_{x,y} := \pi - \max\{|x|_{\mathrm{st}}, |y|_{\mathrm{st}}\}$. Then, for any isometry $f : \mathcal{B}_{\hat{g}}(x,\varepsilon) \to \mathcal{B}_{\hat{g}}(y,\varepsilon)$ such that $f(x) = y$ and $0 < \varepsilon \leq \rho_{x,y}$, there exists an isometry $\tilde{f} : \mathcal{B}_{\hat{g}}(x,\rho_{x,y}) \to \mathcal{B}_{\hat{g}}(y,\rho_{x,y})$ such that $\tilde{f}|_{\mathcal{B}_{\hat{g}}(x,\varepsilon)} = f$.*

Proof. Let $0 < \varepsilon \leq \rho_{x,y}$ and $f : \mathcal{B}_{\hat{g}}(x,\varepsilon) \to \mathcal{B}_{\hat{g}}(y,\varepsilon)$ a local isometry such that $f(x) = y$. By (III.2.1), the Riemannian exponential maps

$$\mathrm{Exp}(\hat{g})_x : B_{\hat{g}_x}(0_x, \rho_{x,y}) \to \mathcal{B}_{\hat{g}}(x, \rho_{x,y}) \ , \quad \mathrm{Exp}(\hat{g})_y : B_{\hat{g}_y}(0_y, \rho_{x,y}) \to \mathcal{B}_{\hat{g}}(y, \rho_{x,y})$$

are $\mathcal{C}^\omega$-diffeomorphisms. Let us define

$$\tilde{f} : \mathcal{B}_{\hat{g}}(x, \rho_{x,y}) \to \mathcal{B}_{\hat{g}}(y, \rho_{x,y}) \ , \quad \tilde{f} := \mathrm{Exp}(\hat{g})_y \circ df|_x \circ \mathrm{Exp}(\hat{g})_x^{-1} \ .$$

Since f is an isometry, it preserves geodesics and hence $\tilde{f}|_{\mathcal{B}_{\hat{g}}(x,\varepsilon)} = f$. Moreover, the analytic functions $(\tilde{f}^*\hat{g})_{ij}$ and $\hat{g}_{ij}|_{\mathcal{B}_{\hat{g}}(x,\rho_{x,y})}$ coincide in the open ball $\mathcal{B}_{\hat{g}}(x,\varepsilon)$. Therefore $(\tilde{f}^*\hat{g})_{ij} = \hat{g}_{ij}|_{\mathcal{B}_{\hat{g}}(x,\rho_{x,y})}$ and hence $\tilde{f}$ is an isometry. $\square$

By repeating the same argument, one can also prove the following

Lemma III.2.3. *Let $(B^m, \hat{g}_1)$ and $(B^m, \hat{g}_2)$ be two geometric models. Then, any isometry*

$$f : (B_{\mathrm{st}}(0,\varepsilon), \hat{g}_1) \to (B_{\mathrm{st}}(0,\varepsilon), \hat{g}_2) \quad \text{with } 0 < \varepsilon < \pi$$

can be uniquely extended to an isometry $\tilde{f} : (B^m, \hat{g}_1) \to (B^m, \hat{g}_2)$.

Remark III.2.4. As pointed out in [11, Sec 1], any m-dimensional homogeneous space (M, g) verifying $|\sec(g)| \leq 1$ is locally isometric to a geometric model $(B^m, \hat{g})$, which is unique up to isometry by Lemma III.2.3. In fact, (M, g) is complete (see [36, Ch IV, Thm 4.5]) and, fixed a point $p \in M$, by the Rauch comparison theorem (see e.g. [33, Sec 6.5]), the differential of the Riemannian exponential $\mathrm{Exp}(g)_p : T_pM \to M$ of (M, g) at p is injective at every point of $B_{g_p}(0_p, \pi) \subset T_pM$. By choosing an orthonormal frame $u : \mathbb{R}^m \to T_pM$, one can consider the pulled-back metric $\hat{g} := (\mathrm{Exp}(g)_p \circ u)^* g$ on $B^m = B_{\mathrm{st}}(0,\pi) \subset \mathbb{R}^m$. Then, it is easy to check that $(B^m, \hat{g})$ is a geometric model, which is clearly locally isometric to (M, g).

Notice that the argument in Remark III.2.4 cannot be used to prove that any m-dimensional, possibly incomplete, locally homogeneous space (M, g) verifying $|\sec(g)| \leq 1$ is locally isometric to a geometric model $(B^m, \hat{g})$. Nonetheless, this claim is true by means of Theorem III.1.1.

III.2.2 Proof of Theorem III.1.2

Let $(B^m, \hat{g}^{(n)})$ be a sequence of geometric models. The main purpose of this section is to prove the following

Theorem III.2.5. *The sequence $(B^m, \hat{g}^{(n)})$ subconverges to a limit geometric model $(B^m, \hat{g}^{(\infty)})$ in the pointed $\mathcal{C}^{1,\alpha}$-topology for any $0 < \alpha < 1$.*

which proves Theorem III.1.2. We begin with the following

Proposition III.2.6. *The sequence $(B^m, \hat{g}^{(n)})$ subconverges to an incomplete pointed $\mathcal{C}^{1,\alpha}$-Riemannian manifold $\big(M^{(\infty)}, g^{(\infty)}, o\big)$ in the pointed $\mathcal{C}^{1,\alpha}$-topology.*

Proof. Fix a sequence $(\varepsilon_k) \subset \mathbb{R}$ with $0 < \varepsilon_k << 1$ and $\varepsilon_k \to 0$. We use local mollifications in the sense of Hochard [32]. More concretely, by Lemma I.2.10 and (III.2.1) there exists a uniform constant $c = c(m) \geq 1$ and, for any fixed $k \in \mathbb{N}$, there exist open sets

$$B_{\mathrm{st}}(0, \pi - 2\varepsilon_k) \subset M_k^{(n)} \subset B_{\mathrm{st}}(0, \pi - \varepsilon_k)$$

and smooth Riemannian metrics $g_k^{(n)}$ on $M_k^{(n)}$ such that:

i) $\big(M_k^{(n)}, g_k^{(n)}\big)$ are complete smooth Riemannian manifolds,

ii) $g_k^{(n)} = \hat{g}^{(n)}$ on $B_{\mathrm{st}}(0, \pi - 3\varepsilon_k)$,

iii) $\sup \big\{ \big| \mathrm{Rm}(g_k^{(n)})_x \big|_{\hat{g}_k^{(n)}} : x \in M_k^{(n)} \big\} < c\, \varepsilon_k^{-2}$.

From (ii) it follows that $\mathrm{inj}_0\big(M_k^{(n)}, g_k^{(n)}\big) \geq \pi - 3\varepsilon_k$. Therefore by (i), (iii) and Theorem I.2.9 we can extract a subsequence in such a way that $\big(M_1^{(n)}, g_1^{(n)}, 0\big)$ converges in the pointed $\mathcal{C}^{1,\alpha}$-topology to a pointed $\mathcal{C}^{1,\alpha}$-Riemannian manifold $\big(M_1^{(\infty)}, g_1^{(\infty)}, o_1\big)$ as $n \to +\infty$. Iterating this construction for any $k \in \mathbb{N}$ and using a Cantor diagonal procedure, we can extract a subsequence in such a way that $\big(M^{(n)}_k, g^{(n)}_k, 0\big)$ converges in the pointed $\mathcal{C}^{1,\alpha}$-topology to a pointed $\mathcal{C}^{1,\alpha}$-Riemannian manifold $\big(M^{(\infty)}_k, g^{(\infty)}_k, o_k\big)$ as $n \to +\infty$ for any fixed $k \in \mathbb{N}$.

In particular, for any $k \in \mathbb{N}$ we have a sequence of $\mathcal{C}^{2,\alpha}$-embeddings

$$\psi^{(n)}_k : \mathcal{B}_{g^{(\infty)}_k}(o_k, \pi - 4\varepsilon_k) \subset M^{(\infty)}_k \to M^{(n)}_k$$

such that $\psi^{(n)}_k(o_k) = 0$ and $(\psi^{(n)}_k)^* g^{(n)}_k$ converges in the $\mathcal{C}^{1,\alpha}$-topology to $g^{(\infty)}_k$ in $\mathcal{B}_{g^{(\infty)}_k}(o_k, \pi - 4\varepsilon_k)$ as $n \to +\infty$. This implies that $\psi^{(n)}_k\big(\mathcal{B}_{g_k}(o_k, \pi - 4\varepsilon_k)\big) \subset B_{\mathrm{st}}(0, \pi - 3\varepsilon_k)$ for $n \in \mathbb{N}$ sufficiently large and therefore by (ii) it comes that $(\psi^{(n)}_k)^* \hat{g}^{(n)}$ converges in the $\mathcal{C}^{1,\alpha}$-topology to $g^{(\infty)}_k$ in $\mathcal{B}_{g^{(\infty)}_k}(o_k, \pi - 4\varepsilon_k)$ as $n \to +\infty$. Hence, there exist pointed isometric embeddings

$$\varphi_k : \big(\mathcal{B}_{g^{(\infty)}_k}(o_k, \pi - 4\varepsilon_k), g^{(\infty)}_k, o_k\big) \to \big(\mathcal{B}_{g^{(\infty)}_{k+1}}(o_{k+1}, \pi - 4\varepsilon_{k+1}), g^{(\infty)}_{k+1}, o_{k+1}\big)$$

for any $k \in \mathbb{N}$. Therefore, we may consider the direct limit

$$\big(M^{(\infty)}, g^{(\infty)}, o\big) := \varinjlim \Big\{\big(\mathcal{B}_{g^{(\infty)}_k}(o_k, \pi - 4\varepsilon_k), g^{(\infty)}_k, o_k\big), \varphi_k\Big\}_{k\in\mathbb{N}} .$$

The triple $\big(M^{(\infty)}, g^{(\infty)}, o\big)$ is a pointed $\mathcal{C}^{1,\alpha}$-Riemannian manifold (see e.g. [19, Ch 4, Sec 2.3]). Moreover, by construction we get the thesis. $\square$

Up to passing to a subsequence of $(B^m, \hat{g}^{(n)})$, we can assume that there exist an exhaustion $(U^{(n)})$ of $M^{(\infty)}$ by relatively compact open sets centered at o and a sequence of $\mathcal{C}^{2,\alpha}$-diffeomorphisms $\phi^{(n)} : U^{(n)} \to B_{\mathrm{st}}\big(0, \pi - \frac{1}{2^n}\big)$ such that $\phi^{(n)}(o) = 0$ and $g^{(n)} := \phi^{(n)*}\hat{g}^{(n)}$ converges in the $\mathcal{C}^{1,\alpha}$-topology to $g^{(\infty)}$.

Let us define the subsets

$$\mathcal{U}^{(n)} := \bigcup_{p\in U^{(n)}} \{p\} \times B_{g^{(n)}_p}\big(0_p, \pi - \tfrac{1}{2^n} - |\phi^{(n)}(p)|_{\mathrm{st}}\big) \subset TM^{(\infty)} ,$$

$$\mathcal{U}^{(\infty)} := \bigcup_{p\in M^{(\infty)}} \{p\} \times B_{g^{(\infty)}_p}\big(0_p, \pi - \mathtt{d}_{g^{(\infty)}}(o,p)\big) \subset TM^{(\infty)} .$$

and the subsets

$$\mathcal{V}^{(n)} := \bigcup_{p\in U^{(n)}} \{p\} \times \mathcal{B}_{g^{(n)}}\big(p, \pi - \tfrac{1}{2^n} - |\phi^{(n)}(p)|_{\mathrm{st}}\big) \subset M^{(\infty)} \times M^{(\infty)} ,$$

$$\mathcal{V}^{(\infty)} := \bigcup_{p\in M^{(\infty)}} \{p\} \times \mathcal{B}_{g^{(\infty)}}\big(p, \pi - \mathtt{d}_{g^{(\infty)}}(o,p)\big) \subset M^{(\infty)} \times M^{(\infty)} .$$

Notice that the metrics $g^{(n)}$ on $U^{(n)}$ are merely of class $\mathcal{C}^{1,\alpha}$. Nonetheless, we can consider

$$E^{(n)}:\mathcal{U}^{(n)}\to U^{(n)}\ ,\quad E^{(n)}(p,v):=(\phi^{(n)})^{-1}\Big(\operatorname{Exp}(\hat{g}^{(n)})\big(\phi^{(n)}(p),(d\phi^{(n)})_p(v)\big)\Big)\ .$$

By construction, the maps

$$\check{E}^{(n)}:\mathcal{U}^{(n)}\to\mathcal{V}^{(n)}\ ,\quad \check{E}^{(n)}(p,v)=\big(p,E^{(n)}(p,v)\big)$$

are $\mathcal{C}^{2,\alpha}$-diffeomorphisms and we indicate their inverses by $\check{L}^{(n)}:\mathcal{V}^{(n)}\to\mathcal{U}^{(n)}$. Notice that the maps $\check{L}^{(n)}$ are necessarily of the form $\check{L}^{(n)}(p,q)=\big(p,L^{(n)}(p,q)\big)$, with $E^{(n)}\big(p,L^{(n)}(p,q)\big)=q$ for any $(p,q)\in\mathcal{V}^{(n)}$ and $L^{(n)}\big(p,E^{(n)}(p,v)\big)=v$ for any $(p,v)\in\mathcal{U}^{(n)}$.

Proposition III.2.7. *The sequence $(\check{E}^{(n)})$ subconverge uniformly on compact sets to a $\mathcal{C}^{0,1}$-homeomorphism*

$$\check{E}^{(\infty)}:\mathcal{U}^{(\infty)}\to\mathcal{V}^{(\infty)}\ ,\quad \check{E}^{(\infty)}(p,v)=\big(p,E^{(\infty)}(p,v)\big)\ . \tag{III.2.2}$$

Proof. Let us compute the differential of $E^{(n)}$ at a point $(p,v)\in\mathcal{U}^{(n)}$. For this purpose, pick two vectors $w_1\in T_pM^{(\infty)}$ and $w_2\in T_v(T_pM^{(\infty)})=T_pM^{(\infty)}$, consider the parallel transports of the pushforwards $(d\phi^{(n)})_p(v),(d\phi^{(n)})_p(w_2)\in T_{\phi^{(n)}(p)}B^m$ along the $\hat{g}^{(n)}$-geodesic $s\mapsto\operatorname{Exp}(\hat{g}^{(n)})(\phi^{(n)}(p),s(d\phi^{(n)})_p(w_1))$ and pull them back by using $\phi^{(n)}$. We indicate such paths with $v^{(n)}(w_1;s)$ and $w_2^{(n)}(w_1;s)$, respectively. Moreover, consider the $\hat{g}^{(n)}$-Jacobi field along the $\hat{g}^{(n)}$-geodesic $t\mapsto\operatorname{Exp}(\hat{g}^{(n)})(\phi^{(n)}(p),t(d\phi^{(n)})_p(v))$ with initial conditions $(d\phi^{(n)})_p(w_1),(d\phi^{(n)})_p(w_2)\in T_{\phi^{(n)}(p)}B^m$ and pull it back by using $\phi^{(n)}$. We indicate it with $J^{(n)}_{w_1,w_2}(v;t)$. Then, it is straightforward to check that the differential

$$\big(dE^{(n)}\big)_{(p,v)}:T_pM^{(\infty)}\oplus T_pM^{(\infty)}\to T_{E^{(n)}(p,v)}M^{(\infty)}$$

is given by

$$\begin{aligned}\big(dE^{(n)}\big)_{(p,v)}(w_1,w_2)&=\frac{\partial}{\partial s}E^{(n)}\Big(E^{(n)}(p,sw_1),t\big(v^{(n)}(w_1;s)+s\,w_2^{(n)}(w_1;s)\big)\Big)\Big|_{\substack{s=0\\t=1}}\\&=J^{(n)}_{w_1,w_2}(v;1)\ .\end{aligned} \tag{III.2.3}$$

We recall that, since $|\sec(g^{(n)})| \leq 1$ by assumption, from the Rauch comparison Theorem (see [33, Thm 6.5.1, Thm 6.5.2]) it follows that for any $t \in [0,1]$

$$\begin{aligned} \big|J^{(n)}_{w_1,0}(v;t)\big|_{g^{(n)}} &\leq \cosh\big(t|v|_{g^{(n)}}\big)|w_1|_{g^{(n)}} \ , \\ \frac{\sin\big(t|v|_{g^{(n)}}\big)}{|v|_{g^{(n)}}}|w_2|_{g^{(n)}} \leq \big|J^{(n)}_{0,w_2}(v;t)\big|_{g^{(n)}} &\leq \frac{\sinh\big(t|v|_{g^{(n)}}\big)}{|v|_{g^{(n)}}}|w_2|_{g^{(n)}} \ . \end{aligned} \tag{III.2.4}$$

By (III.2.3), the differential

$$\big(d\check{E}^{(n)}\big)_{(p,v)} : T_pM^{(\infty)} \oplus T_pM^{(\infty)} \to T_pM^{(\infty)} \oplus T_{E^{(n)}(p,v)}M^{(\infty)}$$

of the map $\check{E}^{(n)}$ is given by

$$\big(d\check{E}^{(n)}\big)_{(p,v)}(w_1,w_2) = \big(w_1, J^{(n)}_{w_1,w_2}(v;1)\big) \ .$$

By (III.2.4) it comes that

$$\begin{aligned} \big|\big(d\check{E}^{(n)}\big)_{(p,v)}(w_1,w_2)\big|_{g^{(n)}} &\leq |w_1|_{g^{(n)}} + \big|J^{(n)}_{w_1,w_2}(v;1)\big|_{g^{(n)}} \\ &\leq |w_1|_{g^{(n)}} + \big|J^{(n)}_{w_1,0}(v;1)\big|_{g^{(n)}} + \big|J^{(n)}_{0,w_2}(v;1)\big|_{g^{(n)}} \\ &\leq \left(1+\cosh\big(|v|_{g^{(n)}}\big)+\frac{\sinh\big(|v|_{g^{(n)}}\big)}{|v|_{g^{(n)}}}\right)\big(|w_1|_{g^{(n)}}+|w_2|_{g^{(n)}}\big) \end{aligned} \tag{III.2.5}$$

and also

$$\begin{aligned} \sqrt{2}\,\big|\big(d\check{E}^{(n)}\big)_{(p,v)}(w_1,w_2)\big|_{g^{(n)}} &\geq |w_1|_{g^{(n)}} + \tfrac{1}{12}\big|J^{(n)}_{w_1,w_2}(v;1)\big|_{g^{(n)}} \\ &\geq |w_1|_{g^{(n)}} - \tfrac{1}{12}\big|J^{(n)}_{w_1,0}(v;1)\big|_{g^{(n)}} + \tfrac{1}{12}\big|J^{(n)}_{0,w_2}(v;1)\big|_{g^{(n)}} \\ &\geq \big(1-\tfrac{1}{12}\cosh\big(|v|_{g^{(n)}}\big)\big)|w_1|_{g^{(n)}}+\frac{\sin\big(t|v|_{g^{(n)}}\big)}{12|v|_{g^{(n)}}}|w_2|_{g^{(n)}} \ . \end{aligned} \tag{III.2.6}$$

From (III.2.3),(III.2.5) and (III.2.6), since $g^{(n)}$ converges in the $\mathcal{C}^{1,\alpha}$-topology to $g^{(\infty)}$, for any compact set $K \subset M$ there exists $\bar{\delta} = \bar{\delta}(K) > 0$ such that, for any fixed $0 < \delta < \bar{\delta}$, there exist $\bar{n} = \bar{n}(K,\delta) \in \mathbb{N}$ and $C = C(K,\delta), c = c(K,\delta) > 0$ such that for any $p \in K$ and $n \geq \bar{n}$

$$\emptyset \neq B_{g^{(\infty)}_p}\big(0_p, \pi - \delta - \mathtt{d}_{g^{(\infty)}}(o,p)\big) \subset B_{g^{(n)}_p}\big(0_p, \pi - \tfrac{1}{2^n} - |\phi^{(n)}(p)|_{\mathrm{st}}\big) \subset T_pM^{(\infty)}$$

and for any $v \in T_pM^{(\infty)}$ with $|v|_{g^{(\infty)}} \leq \pi - \delta - \mathtt{d}_{g^{(\infty)}}(o,p)$ it holds

$$\mathtt{d}_{g^{(\infty)}}\big(\check{E}^{(n)}(p,v),(o,o)\big) \leq C < \sqrt{2}\pi \ ,$$
$$e^{-c}(|w_1|_{g^{(\infty)}} + |w_2|_{g^{(\infty)}}) < \big|\big(d\check{E}^{(n)}\big)_{(p,v)}(w_1,w_2)\big|_{g^{(\infty)}} < e^{c}(|w_1|_{g^{(\infty)}} + |w_2|_{g^{(\infty)}}) \ . \tag{III.2.7}$$

Here, we considered on the product $M^{(\infty)} \times M^{(\infty)}$ the distance

$$\mathtt{d}_{g^{(\infty)}}\big((p_1,q_1),(p_2,q_2)\big) := \sqrt{\mathtt{d}_{g^{(\infty)}}(p_1,p_2)^2 + \mathtt{d}_{g^{(\infty)}}(q_1,q_2)^2} \ .$$

Therefore, (III.2.7) implies that the sequence of diffeomorphism $(\check{E}^{(n)})$ is uniformly locally bi-Lipschitz (see e.g. [39, Lemma 2.10]). Let us fix a compact set $K \subset M$, let $\bar{\delta} = \bar{\delta}(K)$ be as above and consider a sequence $(\delta_i) \subset \mathbb{R}$ such that $0 < \delta_{i+1} < \delta_i < \bar{\delta}$ and $\delta_i \to 0$. By combining the Ascoli-Arzelà Theorem with a Cantor diagonal procedure, from (III.2.7) it comes that the restriction $\check{E}^{(n)}|_{\mathcal{U}^{(n)}_K}$ subconverges uniformly on compact sets of $\mathcal{U}^{(\infty)}_K$, where we have set $\mathcal{U}^{(n)}_K := \mathcal{U}^{(n)} \cap TM^{(\infty)}|_K$ and $\mathcal{U}^{(\infty)}_K := \mathcal{U}^{(\infty)} \cap TM^{(\infty)}|_K$. By considering now an exhaustion of $M^{(\infty)}$ by compact sets K_j and by applying again a Cantor diagonal procedure, we get that $(\check{E}^{(n)})$ subconverges uniformly on compact sets to a $\mathcal{C}^{0,1}$-homeomorphisms

$$\check{E}^{(\infty)} : \mathcal{U}^{(\infty)} \to \mathcal{V}^{(\infty)} \ , \quad \check{E}^{(\infty)}(p,v) = \big(p, E^{(\infty)}(p,v)\big)$$

and this completes the proof. □

By means of the proposition above, we pass to a subsequence of $(B^m, \hat{g}^{(n)})$ in such a way that the maps $\check{E}^{(n)}$ converge uniformly on compact sets to a $\mathcal{C}^{0,1}$-homeomorphism $\check{E}^{(\infty)}$, which is necessarily of the form (III.2.2). We denote its inverse by

$$\check{L}^{(\infty)} : \mathcal{V}^{(\infty)} \to \mathcal{U}^{(\infty)} \ , \quad \check{L}^{(\infty)}(p,q) := \big(p, L^{(\infty)}(p,q)\big) \ .$$

Up to pass to a further subsequence, we can assume that $(\check{L}^{(n)})$ converges uniformly on compact sets to $\check{L}^{(\infty)}$. For the sake of shortness, we set

$$E^{(n)}_p := E^{(n)}(p,\cdot) \ , \quad E^{(\infty)}_p := E^{(\infty)}(p,\cdot) \ ,$$
$$L^{(n)}_p := L^{(n)}(p,\cdot) \ , \quad L^{(\infty)}_p := L^{(\infty)}(p,\cdot) \ .$$

Clearly it holds that $L^{(n)}_p = \big(E^{(n)}_p\big)^{-1}$ and $L^{(\infty)}_p = \big(E^{(\infty)}_p\big)^{-1}$.

In the next theorem we will construct explicitly a local topological group of isometries acting on the limit manifold $(M^{(\infty)}, g^{(\infty)}, o)$. This will allow us to apply Theorem II.1.2 afterwards. Firstly, we denote by $\mathrm{O}_{g^{(\infty)}}\big(M^{(\infty)}\big) \to M^{(\infty)}$ the orthonormal frame bundle of $(M^{(\infty)}, g^{(\infty)})$. Secondly, letting u_{st} be the standard orthonormal frame of $\big(T_0 B^m, \hat{g}^{(n)}_0\big) = \big(\mathbb{R}^m, \langle\cdot,\cdot\rangle_{\mathrm{st}}\big)$, we can assume, up to passing to a further subsequence, that $u^{(n)}_{\mathrm{st}} := ((d\phi^{(n)})_o)^{-1}(u_{\mathrm{st}})$ converges to an orthonormal frame $u^{(\infty)}_{\mathrm{st}}$ for $\big(T_o M^{(\infty)}, g^{(\infty)}_o\big)$. This yields an identification

$$\begin{aligned}\mathrm{O}_{g^{(\infty)}}\big(M^{(\infty)}\big) = \Big\{&(p,a) : p \in M^{(\infty)}\,,\\ &a : \big(T_o M^{(\infty)}, g^{(\infty)}_o\big) \to \big(T_p M^{(\infty)}, g^{(\infty)}_p\big) \text{ linear isometry } \Big\}\,.\end{aligned}$$

Theorem III.2.8. *There exists a locally compact and effective local topological group of isometries acting transitively on the pointed $\mathcal{C}^{1,\alpha}$-Riemannian manifold $(M^{(\infty)}, g^{(\infty)}, o)$.*

Proof. Since $(B^m, \hat{g}^{(n)})$ are locally homogeneous and smooth, we can pick for any $n \in \mathbb{N}$ an effective local Lie group of isometries $G^{(n)}$ acting transitively on $(B^m, \hat{g}^{(n)}, 0)$. By Lemma III.2.2, any local isometry $f^{(n)} \in G^{(n)}$ admits a unique analytic extension

$$f^{(n)} : B_{\mathrm{st}}\big(0, \pi - |f^{(n)}(0)|_{\mathrm{st}}\big) \to \mathcal{B}_{\hat{g}^{(n)}}\big(f^{(n)}(0), \pi - |f^{(n)}(0)|_{\mathrm{st}}\big)\ .$$

For any $n \in \mathbb{N}$ and for any $f^{(n)} \in G^{(n)}$ such that $|f^{(n)}(0)|_{\mathrm{st}} < \frac{\pi}{2}$, we define

$$\tilde{f}^{(n)} : \mathrm{Dom}(\tilde{f}^{(n)}) \to \mathrm{Im}(\tilde{f}^{(n)})\ , \quad \tilde{f}^{(n)} := \phi^{(n)-1} \circ f^{(n)} \circ \big(\phi^{(n)}|_{\mathrm{Dom}(\tilde{f}^{(n)})}\big)\ ,$$

where $\mathrm{Dom}(\tilde{f}^{(n)}), \mathrm{Im}(\tilde{f}^{(n)}) \subset U^{(n)}$ are the open subsets given by

$$\begin{aligned}\mathrm{Dom}(\tilde{f}^{(n)}) &:= \big(f^{(n)} \circ \phi^{(n)}\big)^{-1}\Big(B_{\mathrm{st}}\big(0, \pi - \tfrac{1}{2^n}\big) \cap \mathcal{B}_{\hat{g}^{(n)}}\big(f^{(n)}(0), \pi - |f^{(n)}(0)|_{\mathrm{st}}\big)\Big)\ ,\\ \mathrm{Im}(\tilde{f}^{(n)}) &:= \big(\phi^{(n)}\big)^{-1}\Big(B_{\mathrm{st}}\big(0, \pi - \tfrac{1}{2^n}\big) \cap \mathcal{B}_{\hat{g}^{(n)}}\big(f^{(n)}(0), \pi - |f^{(n)}(0)|_{\mathrm{st}}\big)\Big)\ .\end{aligned}$$

Hence $\tilde{G}^{(n)} := \big\{\tilde{f}^{(n)} : f^{(n)} \in G^{(n)},\ |f^{(n)}(0)|_{\mathrm{st}} < \frac{\pi}{2}\big\}$ is a transitive, effective local Lie group of isometries on $(M^{(\infty)}, g^{(n)}, o)$. We are going to describe explicitly below the structure of local group of $\tilde{G}^{(n)}$.

The multiplication

$$\tilde{\nu}^{(n)} : \mathcal{D}(\tilde{G}^{(n)}) \to \tilde{G}^{(n)} \tag{III.2.8}$$

is defined in the following way. First, we set

$$\mathcal{D}(\tilde{G}^{(n)}) := \big\{ \big(\tilde{f}_1^{(n)}, \tilde{f}_2^{(n)}\big) \in \tilde{G}^{(n)} \times \tilde{G}^{(n)} : \big| f_1^{(n)}(f_2^{(n)}(0)) \big|_{\mathrm{st}} < \tfrac{\pi}{2} \big\} \ .$$

Then, given $\big(\tilde{f}_1^{(n)}, \tilde{f}_2^{(n)}\big) \in \mathcal{D}(\tilde{G}^{(n)})$, we consider a neighborhood V of $0 \in B^m$ such that

$$V \subset B_{\mathrm{st}}\big(0, \pi - |f_2^{(n)}(0)|_{\mathrm{st}}\big) \ , \quad f_2(V) \subset B_{\mathrm{st}}\big(0, \pi - |f_1^{(n)}(0)|_{\mathrm{st}}\big)$$

and we set $f_3^{(n)} := f_1^{(n)} \circ (f_2^{(n)}|_V)$. Then, we consider the analytic extension of $f_3^{(n)}$ as above and we set $\tilde{\nu}^{(n)}\big(\tilde{f}_1^{(n)}, \tilde{f}_2^{(n)}\big) := \tilde{f}_3^{(n)}$. In the same fashion, the inversion map

$$\tilde{\jmath}^{(n)} : \tilde{G}^{(n)} \to \tilde{G}^{(n)} \tag{III.2.9}$$

is defined by choosing, for any given $\tilde{f}^{(n)} \in \tilde{G}^{(n)}$, a neighborhood V of $0 \in B^m$ such that

$$V \subset B_{\mathrm{st}}\big(0, \pi - |f^{(n)}(0)|_{\mathrm{st}}\big) \ , \quad 0 \in f^{(n)}(V)$$

and defining $f'^{(n)} := (f^{(n)}|_V)^{-1}$. Then, we consider the analytic extension of $f'^{(n)}$ and we set $\tilde{\jmath}^{(n)}(\tilde{f}^{(n)}) := \tilde{f}'^{(n)}$.

We stress the fact that any $\tilde{f}^{(n)} \in \tilde{G}^{(n)}$ verifies

$$\tilde{f}^{(n)}(x) = \big(E^{(n)}_{\tilde{f}^{(n)}(p)} \circ \big(d\tilde{f}^{(n)}|_p\big) \circ L_p^{(n)}\big)(x) \tag{III.2.10}$$

for any $x \in \mathrm{Dom}(\tilde{f}^{(n)}) \cap \mathcal{B}_{g^{(n)}}\big(p, \pi - \frac{1}{2^n} - |\phi^{(n)}(p)|\big)$, for any $p \in \mathcal{B}_{g^{(\infty)}}(o, \frac{\pi}{2})$.

Let us consider now a dense and countable subset $S \subset \mathcal{B}_{g^{(\infty)}}(o, \frac{\pi}{2})$. By combining the Ascoli-Arzelà Theorem with a Cantor diagonal procedure (see Step 1 of the proof of [28, Thm 6.6]), up to pass to a subsequence the following claim holds true: for any $p \in S$, there exists a sequence $\tilde{f}^{(n)} \in \tilde{G}^{(n)}$ which converges in the $\mathcal{C}^1$-topology to a $g^{(\infty)}$-isometry $\tilde{f}^{(\infty)} : \mathcal{B}_{g^{(\infty)}}(o, \pi - \mathtt{d}_{g^{(\infty)}}(o,p)) \to \mathcal{B}_{g^{(\infty)}}(p, \pi - \mathtt{d}_{g^{(\infty)}}(o,p))$, with $\tilde{f}^{(\infty)}(o) = p$. For the sake of brevity, we just write $\tilde{f}^{(\infty)} = \lim_{\mathcal{C}^1} \tilde{f}^{(n)}$ and

$$\mathrm{Dom}(\tilde{f}^{(\infty)}) := \mathcal{B}_{g^{(\infty)}}\big(o, \pi - \mathtt{d}_{g^{(\infty)}}(o, \tilde{f}^{(\infty)}(o))\big) \ . \tag{III.2.11}$$

Let us set

$$\tilde{G}^{(\infty)} := \big\{ \tilde{f}^{(\infty)} : \text{ there exists a sequence } \tilde{f}^{(n)} \in \tilde{G}^{(n)} \text{ s.t. } \tilde{f}^{(\infty)} = \lim\nolimits_{\mathcal{C}^1} \tilde{f}^{(n)} \ \big\} \ .$$

Then, $\tilde{G}^{(\infty)}$ can be endowed with a structure of local group in the following way.

Firstly we define the subset $\mathcal{D}(\tilde{G}^{(\infty)}) \subset \tilde{G}^{(\infty)} \times \tilde{G}^{(\infty)}$ of those pairs $(\tilde{f}_1^{(\infty)}, \tilde{f}_2^{(\infty)})$ for which there exist $\tilde{f}_1^{(n)}, \tilde{f}_2^{(n)} \in \tilde{G}^{(n)}$ such that $\big(\tilde{f}_1^{(n)}, \tilde{f}_2^{(n)}\big) \in \mathcal{D}(\tilde{G}^{(n)})$ for any $n \in \mathbb{N}$ large enough and $\tilde{f}_i^{(\infty)} = \lim_{\mathcal{C}^1} \tilde{f}_i^{(n)}$. Notice that the definition of $\mathcal{D}(\tilde{G}^{(\infty)})$ does not depend on the choice of the sequences $(\tilde{f}_i^{(n)})$, $i = 1, 2$. Then, we set

$$\tilde{\nu}^{(\infty)} : \mathcal{D}(\tilde{G}^{(\infty)}) \to \tilde{G}^{(\infty)} \ , \quad \tilde{\nu}^{(\infty)}(\tilde{f}_1^{(\infty)}, \tilde{f}_2^{(\infty)}) := \lim_{\mathcal{C}^1} \tilde{\nu}^{(n)}(\tilde{f}_1^{(n)}, \tilde{f}_2^{(n)}) \ , \tag{III.2.12}$$

where $\tilde{\nu}^{(n)}$ was defined in (III.2.8). Analogously, we set

$$\tilde{\jmath}^{(\infty)} : \tilde{G}^{(\infty)} \to \tilde{G}^{(\infty)} \ , \quad \tilde{\jmath}^{(\infty)}(\tilde{f}^{(\infty)}) := \lim_{\mathcal{C}^1} \tilde{\jmath}^{(n)}(\tilde{f}^{(n)}) \ , \tag{III.2.13}$$

where $\tilde{\jmath}^{(n)}$ was defined in (III.2.9). One can directly check that both (III.2.12) and (III.2.13) are well defined.

From Proposition III.2.7 and (III.2.10) it comes that any $\tilde{f}^{(\infty)} \in \tilde{G}^{(\infty)}$ verifies

$$\tilde{f}^{(\infty)}(x) = \big(E_{\tilde{f}(p)}^{(\infty)} \circ \big(d\tilde{f}^{(\infty)}|_p\big) \circ L_p^{(\infty)}\big)(x) \tag{III.2.14}$$

for any $x \in \mathrm{Dom}(\tilde{f}^{(n)}) \cap \mathcal{B}_{g^{(\infty)}}\big(p, \pi - \mathtt{d}_{g^{(\infty)}}(o,p)\big)$, for any $p \in \mathcal{B}_{g^{(\infty)}}(o, \frac{\pi}{2})$.

Let us consider now the map

$$\sigma : \tilde{G}^{(\infty)} \to \mathrm{O}_{g^{(\infty)}}\big(M^{(\infty)}\big) \ , \quad \sigma(\tilde{f}^{(\infty)}) := \big(\tilde{f}^{(\infty)}(o), d\tilde{f}^{(\infty)}|_o\big) \ . \tag{III.2.15}$$

From (III.2.14) it comes directly that σ is injective. Then, we indicate with $G^{(\infty)}$ the topological closure of $\sigma(\tilde{G}^{(\infty)})$ inside $\mathrm{O}_{g^{(\infty)}}\big(M^{(\infty)}\big)$. One can define a structure of local topological group of $G^{(\infty)}$ by defining an open set $\mathcal{D}(G^{(\infty)}) \subset G^{(\infty)} \times G^{(\infty)}$ and two maps $\nu^{(\infty)} : \mathcal{D}(G^{(\infty)}) \to G^{(\infty)}$, $\jmath^{(\infty)} : G^{(\infty)} \to G^{(\infty)}$ by extending (III.2.12), (III.2.13) in the following way.

First, let $\mathcal{D}(G^{(\infty)})$ be the subset of those pairs $((p_1, a_1), (p_2, a_2)) \in G^{(\infty)} \times G^{(\infty)}$ for which there exist $(\tilde{f}_{1,k}^{(\infty)}), (\tilde{f}_{2,k}^{(\infty)}) \subset \tilde{G}^{(\infty)}$ such that $\big(\tilde{f}_{1,k}^{(\infty)}, \tilde{f}_{2,k}^{(\infty)}\big) \in \mathcal{D}(\tilde{G}^{(\infty)})$ for any $k \in \mathbb{N}$ large enough and $(p_i, a_i) = \lim_{k \to +\infty} \sigma(\tilde{f}_{i,k}^{(\infty)})$. Then, we define

$$\begin{gathered} \nu^{(\infty)} : \mathcal{D}(G^{(\infty)}) \to G^{(\infty)} \ , \\ \nu^{(\infty)}((p_1, a_1), (p_2, a_2)) := \lim_{k \to +\infty} \sigma\big(\tilde{\nu}^{(\infty)}(\tilde{f}_{1,k}^{(\infty)}, \tilde{f}_{2,k}^{(\infty)})\big) \ . \end{gathered} \tag{III.2.16}$$

Notice that, if we set $p_{i,k} := \tilde{f}^{(\infty)}_{i,k}(o)$ and $a_{i,k} := d\tilde{f}^{(\infty)}_{i,k}|_o$ we get

$$\begin{aligned}
&\tilde{\nu}^{(\infty)}(\widetilde{f^{(\infty)}}_{1,k}, \tilde{f}^{(\infty)}_{2,k})(o) = \big(E^{(\infty)}_{p_{1,k}} \circ a_{1,k} \circ L^{(\infty)}_o \circ E^{(\infty)}_{p_{2,k}} \circ a_{2,k}\big)(0_o) \ , \\
&d\big(\tilde{\nu}^{(\infty)}(\tilde{f}^{(\infty)}_{1,k}, \tilde{f}^{(\infty)}_{2,k})\big)|_o = \\
&\qquad = L^{(\infty)}_{(E^{(\infty)}_{p_{1,k}} \circ a_{1,k} \circ L^{(\infty)}_o \circ E^{(\infty)}_{p_{2,k}} \circ a_{2,k})(0_o)} \circ E^{(\infty)}_{p_{1,k}} \circ a_{1,k} \circ L^{(\infty)}_o \circ E^{(\infty)}_{p_{2,k}} \circ a_{2,k}
\end{aligned}$$

and hence the limit (III.2.16) exists, it does not depend on the choice of $(\tilde{f}^{(\infty)}_{1,k}, \tilde{f}^{(\infty)}_{2,k})$ and the map $\nu^{(\infty)}$ is continuous. Analogously, we define

$$\jmath^{(\infty)} : G^{(\infty)} \to G^{(\infty)} \ , \quad \jmath^{(\infty)}(p,a) := \lim_{k \to +\infty} \sigma\big(\tilde{\jmath}^{(\infty)}(\tilde{f}^{(\infty)}_k)\big) \ , \tag{III.2.17}$$

where $(\tilde{f}^{(\infty)}_k) \subset \tilde{G}^{(\infty)}$ is any sequence such that $(p,a) = \lim_{k \to +\infty} \sigma(\tilde{f}^{(\infty)}_k)$. Then, setting $p_k := \tilde{f}^{(\infty)}_k(o)$ and $a_k := d\tilde{f}^{(\infty)}|_o$ we get

$$\begin{aligned}
\tilde{\jmath}(\tilde{f}^{(\infty)}_k)(o) &= \big(E^{(\infty)}_o \circ a_k^{-1} \circ L^{(\infty)}_{p_k}\big)(o) \ , \\
d\big(\tilde{\jmath}^{(\infty)}(\tilde{f}^{(\infty)}_k)\big)|_o &= L^{(\infty)}_{(E^{(\infty)}_o \circ a_k^{-1} \circ L^{(\infty)}_{p_k})(o)} \circ E^{(\infty)}_o \circ a_k^{-1} \circ L^{(\infty)}_{p_k} \circ E^{(\infty)}_o
\end{aligned}$$

and hence the limit (III.2.17) exists, it does not depend on the choice of $(\tilde{f}^{(\infty)}_k)$ and the map $\jmath^{(\infty)}$ is continuous. Finally, we define the open set

$$\mathcal{W}^{(\infty)} := \bigcup_{(p,a) \in G^{(\infty)}} \{(p,a)\} \times \mathcal{B}_{g^{(\infty)}}(o, \pi - \mathtt{d}_{g^{(\infty)}}(o,p)) \subset G^{(\infty)} \times M^{(\infty)} \tag{III.2.18}$$

and the map

$$\Theta^{(\infty)} : \mathcal{W}^{(\infty)} \to M^{(\infty)} \ , \quad \Theta^{(\infty)}((p,a),x) := \lim_{k \to +\infty} \tilde{f}^{(\infty)}_k(x) \ , \tag{III.2.19}$$

where $(\tilde{f}^{(\infty)}_k) \subset \tilde{G}^{(\infty)}$ is any sequence such that $(p,a) = \lim_{k \to +\infty} \sigma(\tilde{f}^{(\infty)}_k)$. Again, the limit (III.2.19) exists, it does not depend on $(\tilde{f}^{(\infty)}_k)$ and the map $\Theta^{(\infty)}$ is continuous. From now on, we identify any pair $(p,a) \in G^{(\infty)}$ with the corresponding map $f^{(\infty)} := \Theta^{(\infty)}((p,a),\cdot)$ and we just write "$f^{(\infty)} \in G^{(\infty)}$". Notice that the map $f^{(\infty)} \in G^{(\infty)}$ corresponding to a pair (p,a) is given by

$$f^{(\infty)} = E^{(\infty)}_p \circ a \circ L^{(\infty)}_o|_{\mathrm{Dom}(f^{(\infty)})} \ , \quad \mathrm{Dom}(f^{(\infty)}) := \mathcal{B}_{g^{(\infty)}}\big(o, \pi - \mathtt{d}_{g^{(\infty)}}(o,p)\big) \ . \tag{III.2.20}$$

From (III.2.19) it follows that each map $f^{(\infty)} \in G^{(\infty)}$ is an isometry, and hence $G^{(\infty)}$ is a local topological group of isometries on $(M^{(\infty)}, g^{(\infty)}, o)$. Since it is a closed subset of $\mathrm{O}_{g^{(\infty)}}\big(M^{(\infty)}\big)$, it is locally compact. Moreover, by (III.2.20) it is effective. Finally, since by construction the orbit $G^{(\infty)}(o)$ contains the whole ball $\mathcal{B}_{g^{(\infty)}}(o, \frac{\pi}{2})$, we conclude that $G^{(\infty)}$ is transitive. $\square$

Finally, we are ready to prove Theorem III.2.5.

Proof of Theorem III.2.5. Since each $(B^m, \hat{g}^{(n)})$ is locally homogeneous, the limit space $\big(M^{(\infty)}, g^{(\infty)}\big)$ is locally homogeneous (see [11, proof of Thm 1.6]). Hence, by Theorem II.1.2 and Theorem III.2.8 it follows that $\big(M^{(\infty)}, g^{(\infty)}\big)$ is a locally homogeneous $\mathcal{C}^\omega$-Riemannian manifold. Since convergence in pointed $\mathcal{C}^{1,\alpha}$-topology implies convergence in pointed Gromov-Hausdorff topology, it holds that $\sec(g^{(\infty)}) \geq -1$. Moreover, by [66, Thm. 6.4.8] we get a positive uniform lower bound on the convexity radius $\mathrm{conv}_0(B^m, \hat{g}^{(n)})$ along the sequence, and hence by [15, Thm. 5.1] we get $\sec(g^{(\infty)}) \leq 1$. Furthermore by [72, Lemma 1.5] it necessarily holds that $E^{(\infty)} = \mathrm{Exp}(g^{(\infty)})$ (see also the proof of [65, Thm 4.4]). Therefore, fixed a $g^{(\infty)}$-orthonormal frame $u : \mathbb{R}^m \to T_oM$, one can consider the pulled-back metric $\hat{g}^{(\infty)} := (\mathrm{Exp}(g^{(\infty)})_o \circ u)^* g^{(\infty)}$ on $B^m = B_{\mathrm{st}}(0, \pi) \subset \mathbb{R}^m$. It is easy to realize that $(B^m, \hat{g}^{(\infty)})$ is a geometric model and, since it is isometric to $(M^{(\infty)}, g^{(\infty)})$, we conclude that $(B^m, \hat{g}^{(n)}, 0)$ converges to the geometric model $(B^m, \hat{g}^{(\infty)}, 0)$ in the pointed $\mathcal{C}^{1,\alpha}$-topology. $\square$

From Theorem III.2.5 we also get the following

Corollary III.2.9. *Let $(B^m, \hat{g}^{(n)})$ be a sequence of geometric models and assume that there exist an integer $k \geq 0$ and a constant $C > 0$ such that*

$$\big|\, \mathrm{Rm}^j(\hat{g}^{(n)}) \big|_{\hat{g}^{(n)}} \leq C \ , \quad 0 \leq j \leq k \ .$$

Then the sequence $(B^m, \hat{g}^{(n)})$ subconverges to a geometric model $(B^m, \hat{g}^{(\infty)})$ in the pointed $\mathcal{C}^{k+1,\alpha}$-topology for any $0 < \alpha < 1$.

Proof. By means of Theorem III.2.5, we can assume that $(B^m, \hat{g}^{(n)})$ converges to a geometric model $(B^m, \hat{g}^{(\infty)})$ in the pointed $\mathcal{C}^{1,\alpha}$-topology as $n \to +\infty$. Fix $0 < \delta < \pi$ and set $\Omega_\delta := B_{\mathrm{st}}(0, \pi - \delta) \subset B^m$. Then, by Theorem I.2.8, we get

$$\mathrm{har}_x^{k+1,\alpha}\big(B^m, \hat{g}^{(n)}\big) \geq r_{\mathrm{o}} > 0 \quad \text{for any } x \in \Omega_\delta \ .$$

Following [34, proof of Thm A] and [1, Lemma 2.1], it is possible to construct, up to passing to a subsequence, smooth embeddings $\psi^{(n)} : \Omega_\delta \to \mathbb{R}^N$, for some $N >> m$, such that $\psi^{(n)}(0) = 0$ and $\Omega_\delta^{(n)} := \psi^{(n)}(\Omega_\delta) \subset \mathbb{R}^N$ are locally represented as graphs of smooth functions over $B_{\mathrm{st}}\big(0, \frac{r_o}{2}\big) \subset \mathbb{R}^m$, uniformly bounded in $\mathcal{C}^{k+2,\alpha}\big(\overline{B_{\mathrm{st}}\big(0, \frac{r_o}{2}\big)}\big)$. Then, up to passing to a subsequence, $\Omega_\delta^{(n)}$ converges in the $\mathcal{C}^{k+2,\alpha}$-topology as submanifolds of $\mathbb{R}^N$ to an embedded $\mathcal{C}^{k+2,\alpha}$-submanifold $\Omega_\delta^{(\infty)}$ of $\mathbb{R}^N$. Set $g^{(n)} := ((\psi^{(n)})^{-1})^* \hat{g}^{(n)}$. Up to pass to a further subsequence, the projection along the normals of $\Omega_\delta^{(\infty)}$ induces a sequences of $\mathcal{C}^{k+2,\alpha}$-diffeomorphisms $\mathrm{pr}^{(n)} : \Omega_\delta^{(n)} \to \Omega_\delta^{(\infty)}$ and the metrics $((\mathrm{pr}^{(n)})^{-1})^* g^{(n)}$ converges in the $\mathcal{C}^{k+1,\alpha}$-topology to a $\mathcal{C}^{k+1,\alpha}$-Riemannian $g^{(\infty)}$ on $\Omega_\delta^{(\infty)}$.

Since we assumed that $(B^m, \hat{g}^{(n)})$ converges in the pointed $\mathcal{C}^{1,\alpha}$-topology to $(B^m, \hat{g}^{(\infty)})$, it comes that there exists an isometric embedding $\varphi : (\Omega_\delta^{(\infty)}, g^{(\infty)}) \to (B^m, \hat{g}^{(\infty)})$ with $\varphi(0) = 0$. Moreover, given any compact set $K \subset B^m$, one can take δ small enough in such a way that $K \subset \varphi\big(\Omega_\delta^{(\infty)}\big)$. This completes the proof. $\square$

III.3 Algebraic aspects of locally homogeneous spaces

III.3.1 The space of locally homogeneous spaces

For any $m, q \in \mathbb{Z}$, with $m \geq 1$ and $0 \leq q \leq \frac{m(m-1)}{2}$, we denote by $\mathcal{H}_{q,m}^{\mathrm{loc}}$ the moduli space of transitive orthogonal Lie algebras $(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \langle\, , \rangle)$ of rank (q, m) up to isomorphisms, and by $\mathcal{H}_{q,m} \subset \mathcal{H}_{q,m}^{\mathrm{loc}}$ the subset of moduli space of regular ones (see Section I.4.5). Moreover, we define

$$\mathcal{V}_{q,m} := \big(\mathsf{GL}(q) \times \mathsf{O}(m)\big) \backslash \big(\Lambda^2(\mathbb{R}^{q+m})^* \otimes \mathbb{R}^{q+m}\big) ,$$

where we considered a fixed decomposition $\mathbb{R}^{q+m} = \mathbb{R}^q \oplus \mathbb{R}^m$ and the diagonal embedding of $\mathsf{GL}(q) \times \mathsf{O}(m)$ into $\mathsf{GL}(q+m)$, which acts on $\Lambda^2(\mathbb{R}^{q+m})^* \otimes \mathbb{R}^{q+m}$ on the left by change of basis:

$$(a \cdot \mu)(X, Y) := a\, \mu(a^{-1} X, a^{-1} Y) .$$

Following the same argument as [41], one can show that the map

$$\Phi_{q,m} : \mathcal{H}_{q,m}^{\mathrm{loc}} \to \mathcal{V}_{q,m} , \quad (\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \langle\, , \rangle) \mapsto \mu := u^*\big([\cdot, \cdot]_{\mathfrak{g}}\big) ,$$

where $u : \mathbb{R}^{m+q} \to \mathfrak{g}$ is any adapted linear frame for $(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \langle\, , \rangle)$, is well defined, injective and that its image contains precisely the elements $\mu \in \mathcal{V}_{q,m}$ which verify the following conditions:

(h1) μ satisfies the Jacobi Identity, $\mu(\mathbb{R}^q, \mathbb{R}^q) \subset \mathbb{R}^q$ and $\mu(\mathbb{R}^q, \mathbb{R}^m) \subset \mathbb{R}^m$;

(h2) $\langle \mu(Z, X), Y\rangle_{\mathrm{st}} = \langle X, \mu(Z, Y)\rangle_{\mathrm{st}}$ for any $X, Y \in \mathbb{R}^m$, $Z \in \mathbb{R}^q$;

(h3) $\big\{Z \in \mathbb{R}^q : \mu(Z, \mathbb{R}^m) = \{0\}\big\} = \{0\}$.

From now on, we identify $\mathcal{H}^{\mathrm{loc}}_{q,m}$ with its image through $\Phi_{q,m}$, i.e. we think $\mathcal{H}^{\mathrm{loc}}_{q,m} \subset \mathcal{V}_{q,m}$. For the sake of clarity, for any $\mu \in \Phi_{q,m}(\mathcal{H}^{\mathrm{loc}}_{q,m}) \simeq \mathcal{H}^{\mathrm{loc}}_{q,m}$ we set

$$\mathfrak{g}_\mu := (\mathbb{R}^{q+m}, \mu) \ , \quad \mathfrak{h}_\mu := (\mathbb{R}^q, \mu|_{\mathbb{R}^q \times \mathbb{R}^q})$$

so that $(\mathfrak{g}_\mu = \mathfrak{h}_\mu + \mathbb{R}^m, \langle\, , \rangle_{\mathrm{st}})$ is the orthogonal transitive Lie algebra uniquely associated to the bracket μ.

Remark III.3.1. Conditions (h1) and (h2) above are closed, while (h3) is open. Nonetheless, as pointed out in [41], given any sequence $(\mu^{(n)}) \subset \mathcal{H}^{\mathrm{loc}}_{q,m}$ which converges to an element $\tilde{\mu} \in \mathcal{V}_{q,m} \setminus \mathcal{H}^{\mathrm{loc}}_{q,m}$ in the standard topology of $\mathcal{V}_{q,m}$, there exists a decomposition $\mathbb{R}^q = \mathbb{R}^{q-q'} \oplus \mathbb{R}^{q'}$ for some $0 \le q' < q$ such that

$$\mathbb{R}^{q-q'} = \big\{Z \in \mathbb{R}^q : \mu(Z, \mathbb{R}^m) = \{0\}\big\}$$

and the restriction

$$(\tilde{\mu})_{|q',m} := \mathrm{pr}_{\mathbb{R}^{q'+m}} \circ (\tilde{\mu}|_{\mathbb{R}^{q'+m} \times \mathbb{R}^{q'+m}}) \ , \tag{III.3.1}$$

satisfies $(\tilde{\mu})_{|q',m} \in \mathcal{H}^{\mathrm{loc}}_{q',m}$. Here, $\mathbb{R}^{q'+m} = \mathbb{R}^{q'} \oplus \mathbb{R}^m$ and $\mathrm{pr}_{\mathbb{R}^{q'+m}} : \mathbb{R}^{q+m} \to \mathbb{R}^{q'+m}$ denotes the projection with respect to the direct sum decomposition $\mathbb{R}^{q+m} = \mathbb{R}^{q-q'} \oplus \mathbb{R}^{q'+m}$.

Moreover, for later purposes, we consider the orthogonal decomposition (see Remark I.4.6)

$$\mu = \mu|_{\mathfrak{h}_\mu \wedge \mathfrak{g}_\mu} + \mu_{\mathfrak{h}_\mu} + \mu_{\mathbb{R}^m} \ , \quad \text{with} \quad \mu_{\mathfrak{h}_\mu} : \mathbb{R}^m \wedge \mathbb{R}^m \to \mathfrak{h}_\mu \ , \quad \mu_{\mathbb{R}^m} : \mathbb{R}^m \wedge \mathbb{R}^m \to \mathbb{R}^m \tag{III.3.2}$$

and we also set

$$\mathcal{H}^{\mathrm{loc}}_m := \bigcup_{q=0}^{\frac{m(m-1)}{2}} \mathcal{H}^{\mathrm{loc}}_{q,m} \ , \quad \mathcal{H}_m := \bigcup_{q=0}^{\frac{m(m-1)}{2}} \mathcal{H}_{q,m} \ . \tag{III.3.3}$$

The set $\mathcal{H}^{\mathrm{loc}}_m$ parametrizes the moduli space of the equivalence classes of m-dimensional smooth locally homogeneous Riemannian manifolds, up to equivariant local isometries, in the following way.

Theorem III.3.2 ([77], Lemma 3.5 and Prop 4.4)**.** *For any $\mu \in \mathcal{H}^{\mathrm{loc}}_m$, there exist a pointed locally homogeneous space (M, g, p) and an injective Lie algebra homomorphism $\varphi : \mathfrak{g}_\mu \to \mathfrak{kill}^g$ such that $\varphi(\mathfrak{h}_\mu) \subset \mathfrak{kill}^g_0$ and $(\varphi(\mathbb{R}^m), (\varphi^{-1})^*\langle\, ,\rangle_{\mathrm{st}}) = (T_pM, g_p)$. The space (M, g, p) is uniquely determined up to an equivariant local isometry and it is non-strictly locally homogeneous if and only if μ is regular.*

Given an element $\mu \in \mathcal{H}^{\mathrm{loc}}_m$, the locally homogeneous space uniquely associated to μ as in Theorem III.3.2 is constructed in the following way. Let G_μ be the only simply connected Lie group with $\mathrm{Lie}(\mathsf{G}_\mu) = \mathfrak{g}_\mu$ and H_μ the Lie subgroup of G_μ with $\mathrm{Lie}(\mathsf{H}_\mu) = \mathfrak{h}_\mu$, which is closed in G_μ if and only if $\mu \in \mathcal{H}_m$. Then we consider the local factor space $\mathsf{G}_\mu/\mathsf{H}_\mu$, which admits a unique suitable real analytic manifold structure (see Proposition II.4.1). Moreover, by means of the standard local action of G_μ on $\mathsf{G}_\mu/\mathsf{H}_\mu$, one can construct a uniquely determined invariant Riemannian metric g_μ on $\mathsf{G}_\mu/\mathsf{H}_\mu$ such that $(\mathbb{R}^m, \langle\, ,\rangle_{\mathrm{st}}) \simeq (T_{e_\mu\mathsf{H}_\mu}\mathsf{G}_\mu/\mathsf{H}_\mu, g_\mu|_{e_\mu\mathsf{H}_\mu})$.

III.3.2 Curvature of locally homogeneous spaces

For the sake of shortness, we also indicate any geometric quantity related to $(\mathsf{G}_\mu/\mathsf{H}_\mu, g_\mu)$ just by writing μ in the place of g_μ, e.g. we write $\mathrm{Rm}^k(\mu)$ instead of $\mathrm{Rm}^k(g_\mu)$. We will also denote by $\mathfrak{kill}(\mu)$ the Nomizu algebra of $(\mathsf{G}_\mu/\mathsf{H}_\mu, g_\mu)$.

For any $\mu \in \mathcal{H}^{\mathrm{loc}}_m$, the canonical Ambrose-Singer connection D^μ (see Section I.4.3) is uniquely determined by the $(1,2)$-tensor $S^\mu := D^\mu - \nabla^\mu$, which can be identified with the linear map $S^\mu : \mathbb{R}^m \to \mathfrak{so}(m)$ defined by (see [37, Thm 3.3, Ch X])

$$-2\langle S^\mu(X)Y, Z\rangle_{\mathrm{st}} = \langle \mu_{\mathbb{R}^m}(X,Y), Z\rangle_{\mathrm{st}} + \langle \mu_{\mathbb{R}^m}(Z,X), Y\rangle_{\mathrm{st}} + \langle \mu_{\mathbb{R}^m}(Z,Y), X\rangle_{\mathrm{st}}\ . \tag{III.3.4}$$

Then, by [37, Thm 2.3, Ch X] it comes that

$$\mathrm{Rm}^0(\mu)(X\wedge Y) = \mathrm{ad}_\mu\big(\mu_{\mathfrak{h}_\mu}(X,Y)\big) - [S^\mu(X), S^\mu(Y)] - S^\mu(\mu_{\mathbb{R}^m}(X,Y))\ , \tag{III.3.5}$$

where $\mu_{\mathfrak{h}_\mu}$ and $\mu_{\mathbb{R}^m}$ were defined in (III.3.2). Moreover, since $D^\mu\,\mathrm{Rm}(\mu) = 0$ we obtain

$$X \lrcorner\, \mathrm{Rm}^{k+1}(\mu) = -S^\mu(X)\cdot \mathrm{Rm}^k(\mu) \quad \text{for any } k \geq 0\ . \tag{III.3.6}$$

We also stress that for any $k \geq 0$

$$\begin{aligned}\mathrm{Rm}^{k+2}(X_1, X_2, X_3, \dots, X_{k+2}|Y_1 \wedge Y_2) - \mathrm{Rm}^{k+2}(X_2, X_1, X_3, \dots, X_{k+2}|Y_1 \wedge Y_2) = \\ = -\big(\mathrm{Rm}^0(X_1, X_2) \cdot \mathrm{Rm}^k\big)(X_3, \dots, X_{k+2}|Y_1 \wedge Y_2) \ . \quad \text{(III.3.7)}\end{aligned}$$

Notice that in both (III.3.6) and (III.3.7), given a linear map $P : \otimes^q \mathbb{R}^m \to \mathfrak{so}(m)$ and an element $A \in \mathfrak{so}(m)$, the linear map $(A \cdot P) : \otimes^q \mathbb{R}^m \to \mathfrak{so}(m)$ is defined by

$$(A \cdot P)(V_1, \dots, V_q) := [A, P(V_1, \dots, V_q)] - \sum_{i=1}^{q} P\big(V_1, \dots, A \cdot V_i, \dots, V_q\big) \ . \quad \text{(III.3.8)}$$

Finally, for the sake of notation we denote by

$$\mathcal{H}^{\mathrm{loc}}_{q,m}(1) := \{\mu \in \mathcal{H}^{\mathrm{loc}}_{q,m} : |\sec(\mu)| \leq 1\} \ , \quad \mathcal{H}_{q,m}(1) := \mathcal{H}^{\mathrm{loc}}_{q,m}(1) \cap \mathcal{H}_{q,m}$$

and we define the spaces $\mathcal{H}^{\mathrm{loc}}_{m}(1)$ and $\mathcal{H}_{m}(1)$ in the same fashion as in (III.3.3).

Remark III.3.3. We stress that for any $\mu \in \mathcal{H}^{\mathrm{loc}}_{q,m}$ there exists $R > 0$ such that $R \cdot \mu \in \mathcal{H}^{\mathrm{loc}}_{q,m}(1)$, where the bracket $R \cdot \mu$ is defined by

$$(R \cdot \mu)|_{\mathfrak{h}_\mu \wedge \mathfrak{g}_\mu} := \mu|_{\mathfrak{h}_\mu \wedge \mathfrak{g}_\mu} \ , \quad (R \cdot \mu)_{\mathfrak{h}_\mu} := \tfrac{1}{R^2} \mu_{\mathfrak{h}_\mu} \ , \quad (R \cdot \mu)_{\mathbb{R}^m} := \tfrac{1}{R} \mu_{\mathfrak{h}_\mu} \quad \text{(III.3.9)}$$

(see III.3.2). Indeed, the locally homogeneous space $(\mathsf{G}_{R\cdot\mu}/\mathsf{H}_{R\cdot\mu}, g_{R\cdot\mu})$ is locally equivariantly isometric to $(\mathsf{G}_\mu/\mathsf{H}_\mu, R^2 g_\mu)$ and hence it follows that $\sec(R \cdot \mu) = \frac{1}{R} \sec(\mu)$.

III.3.3 The subset $\mathcal{H}_{q,m}$ is dense in $\mathcal{H}^{\mathrm{loc}}_{q,m}$

In this subsection we are going to show that the inclusion $\mathcal{H}^{\mathrm{loc}}_{q,m} \subset \mathcal{V}_{q,m}$ induces a topology on the set $\mathcal{H}^{\mathrm{loc}}_{q,m}$ with respect to which the subset $\mathcal{H}_{q,m}$ is dense. More precisely (see also [83, Thm 4.1])

Theorem III.3.4. *The set $\mathcal{H}_{q,m}$ is dense in $\mathcal{H}^{\mathrm{loc}}_{q,m}$ with respect to the standard topology induced by $\mathcal{V}_{q,m}$.*

In order to prove Theorem III.3.4, we need the following

Lemma III.3.5. *Let $\mathsf{T}^m = \mathbb{Z}^m \backslash \mathbb{R}^m$ be the m-torus and let L_v be the 1-parameter subgroup of T^m generated by a fixed element $v = (v^1, \dots, v^m) \in \mathbb{R}^m = \mathrm{Lie}(\mathsf{T}^m)$. Then*

$$\dim \overline{\mathsf{L}_v} = \dim_{\mathbb{Q}} \mathrm{span}_{\mathbb{Q}}(v^1, \dots, v^m) \ , \tag{III.3.10}$$

where $\mathrm{span}_{\mathbb{Q}}(v^1, \dots, v^m) \subset \mathbb{R}$ *is the $\mathbb{Q}$-subspace of $\mathbb{R}$ generated by the components of v.*

Proof. We recall that the closed subgroups of a locally compact topological group are characterized as being those which are intersections of kernels of continuous characters (see [69, Rem 1.20]). Moreover, the continuous characters of T^m are precisely the maps

$$\chi_r : \mathsf{T}^m \to S^1 \ , \quad \chi_r(\mathbb{Z}^m \cdot x) := e^{2\pi\sqrt{-1}\langle x, r\rangle_{\mathrm{st}}} \quad \text{with } r \in \mathbb{Z}^m \ .$$

Since $\mathsf{L}_v \subset \ker(\chi_r)$ if and only if $(\chi_r)_*(tv) \in \mathbb{Z}$ for any $t \in \mathbb{R}$, i.e. if and only if $\langle v, r\rangle_{\mathrm{st}} = 0$, we get

$$\overline{\mathsf{L}_v} = \bigcap_{\substack{r \in \mathbb{Z}^n \\ \langle v, r\rangle_{\mathrm{st}} = 0}} \ker(\chi_r) \ .$$

So by a straightforward computation it comes that

$$\dim \overline{\mathsf{L}_v} = \dim_{\mathbb{R}} \bigcap_{\substack{r \in \mathbb{Z}^m \\ \langle v, r\rangle_{\mathrm{st}} = 0}} \ker(\chi_r)_* = m - \mathrm{rank}_{\mathbb{Z}} F \ ,$$

where we denoted by F the free $\mathbb{Z}$-module $F := \{r \in \mathbb{Z}^m : \langle v, r\rangle_{\mathrm{st}} = 0\}$. Since the rank of F is

$$\mathrm{rank}_{\mathbb{Z}} F = m - \dim_{\mathbb{Q}} \mathrm{span}_{\mathbb{Q}}(v^1, \dots, v^m) \ ,$$

the thesis follows. □

Corollary III.3.6. *Let $\mathsf{T}^m = \mathbb{Z}^m \backslash \mathbb{R}^m$ be the m-torus, $\mathrm{Gr}_s(\mathfrak{t})$ the Grassmannian of s-planes inside $\mathfrak{t} := \mathrm{Lie}(\mathsf{T}^m)$ and $\mathrm{Gr}^{\star}_s(\mathfrak{t})$ the subset of those $\mathfrak{a} \in \mathrm{Gr}_s(\mathfrak{t})$ such that $\overline{\mathfrak{a}}^{\mathsf{T}^m} = \mathfrak{a}$. Then $\mathrm{Gr}^{\star}_s(\mathfrak{t})$ is dense in $\mathrm{Gr}_s(\mathfrak{t})$ for any integer $0 \leq s \leq m$.*

Finally, we are ready to prove Theorem III.3.4.

Proof of Theorem III.3.4. Fix $\mu \in \mathcal{H}^{\mathrm{loc}}_{q,m} \setminus \mathcal{H}_{q,m}$. Let $\langle\,,\rangle'_\mu$ be the $\mathrm{ad}(\mathfrak{h}_\mu)$-invariant Euclidean product on $\mathfrak{g}_\mu$ defined in Remark I.4.6 and G_μ the simply connected Lie group with $\mathrm{Lie}(\mathsf{G}_\mu) = \mathfrak{g}_\mu$. Then, the Malcev-closure $\overline{\mathfrak{h}}^{\mathsf{G}_\mu}_\mu$ of $\mathfrak{h}_\mu$ in G_μ turns out to be faithfully represented by its adjoint action as a subalgebra of $\mathfrak{so}(\mathfrak{g}_\mu, \langle\,,\rangle'_\mu)$ (see [78, Sec 3]) and hence it is reductive. Therefore by [58, Thm 3, p. 52] it follows that

$$\overline{\mathfrak{h}}^{\mathsf{G}_\mu}_\mu = [\mathfrak{h}_\mu, \mathfrak{h}_\mu] + \mathfrak{t}_\mu \ , \qquad \text{with } \mathfrak{t}_\mu \subset \mathfrak{g}_\mu \text{ abelian such that } \overline{\mathfrak{z}(\mathfrak{h}_\mu)}^{\mathsf{G}_\mu} = \mathfrak{t}_\mu \ .$$

By Corollary III.3.6, we can pick a sequence of subalgebras $\mathfrak{a}^{(n)} \subset \mathfrak{t}_\mu$ which converges to $\mathfrak{z}(\mathfrak{h}_\mu)$ with respect to the standard Euclidean topology such that $\overline{\mathfrak{a}^{(n)}}^{\mathsf{G}_\mu} = \mathfrak{a}^{(n)}$. Then we define $\mathfrak{h}^{(n)} := [\mathfrak{h}_\mu, \mathfrak{h}_\mu] + \mathfrak{a}^{(n)}$, $\mathfrak{m}^{(n)}$ as the $\langle,\rangle'$-orthogonal complement of $\mathfrak{h}^{(n)}$ inside $\mathfrak{g}_\mu$ and $\langle,\rangle^{(n)} := \langle,\rangle'|_{\mathfrak{m}^{(n)} \times \mathfrak{m}^{(n)}}$. It is easy to check that $(\mathfrak{g}_\mu = \mathfrak{h}^{(n)} + \mathfrak{m}^{(n)}, \langle,\rangle^{(n)})$ is a regular orthogonal transitive Lie algebra, and so it corresponds uniquely to an element $\mu^{(n)} \in \mathcal{H}_{q,m}$. Finally, by the very construction, we can conclude that $\mu^{(n)} \to \mu$ as $n \to +\infty$ in the standard topology induced by $\mathcal{V}_{q,m}$. □

Remark III.3.7. We recall again that, by [52, Sec 7], it follows that $\mathcal{H}_{q,m} = \mathcal{H}^{\mathrm{loc}}_{q,m}$ for any $m \leq 4$.

III.3.4 Algebraic convergence and infinitesimal convergence

We begin by recalling the following

Definition III.3.8. A sequence $(\mu^{(n)}) \subset \mathcal{H}^{\mathrm{loc}}_{q,m}$ is said to *converge algebraically to* $\mu^{(\infty)} \in \mathcal{H}^{\mathrm{loc}}_m$ if one of the two mutually exclusive conditions below is satisfied:

i) $\mu^{(\infty)} \in \mathcal{H}^{\mathrm{loc}}_{q,m}$ and $\mu^{(n)} \to \mu^{(\infty)}$ in the standard topology induced by $\mathcal{V}_{q,m}$;

ii) $\mu^{(\infty)} \in \mathcal{H}^{\mathrm{loc}}_{q',m}$ for some $0 \leq q' < q$ and there exists $\tilde{\mu} \in \mathcal{V}_{q,m} \setminus \mathcal{H}^{\mathrm{loc}}_{q,m}$ such that $\mu^{(n)} \to \tilde{\mu}^{(\infty)}$ in the standard topology of $\mathcal{V}_{q,m}$ and $(\tilde{\mu}^{(\infty)})_{|q',m} = \mu^{(\infty)}$ as in Remark III.3.1.

We introduce also a second notion of convergence. Firstly, let us notice that Theorem I.4.12 and Theorem III.3.2 allow us to consider the map

$$\mathcal{H}^{\mathrm{loc}}_m \to \mathcal{R}^s(m) \ , \quad \mu \mapsto \rho^s(\mu) = \big(\mathrm{Rm}^0(\mu), \mathrm{Rm}^1(\mu), \ldots, \mathrm{Rm}^s(\mu)\big)$$

which associates to any $\mu \in \mathcal{H}^{\mathrm{loc}}_m$ the Riemannian s-tuple $\rho^s(\mu)$ of $(\mathsf{G}_\mu/\mathsf{H}_\mu, g_\mu)$, for any integer $s \geq \imath(m) + 2$. This map is surjective but not injective. In particular, by Theorem I.4.12 and Theorem III.3.2 it holds that $\rho^s(\mu_1) = \rho^s(\mu_2)$ for some, and hence for any, $s \geq \imath(m) + 2$ if and only if $\mathfrak{kill}(\mu_1) = \mathfrak{kill}(\mu_2)$.

Definition III.3.9. Let $m, s \in \mathbb{N}$ with $s \geq \imath(m)+2$. A sequence $(\mu^{(n)}) \subset \mathcal{H}^{\mathrm{loc}}_m$ is said to *converge s-infinitesimally to* $\mu^{(\infty)} \in \mathcal{H}^{\mathrm{loc}}_m$ if $\rho^s(\mu^{(n)}) \to \rho^s(\mu^{(\infty)})$ as $n \to +\infty$ in the standard Euclidean topology of $\mathcal{R}^s(m)$. If $(\mu^{(n)})$ converges s-infinitesimally to $\mu^{(\infty)} \in \mathcal{H}^{\mathrm{loc}}_m$ for any integer $s \geq \imath(m) + 2$, then $(\mu^{(n)})$ is said to *converge infinitesimally to* $\mu^{(\infty)}$.

Notice that by the previous observation, if a sequence $(\mu^{(n)}) \subset \mathcal{H}^{\mathrm{loc}}_m$ converges s_1-infinitesimally to $\mu^{(\infty)}_1$ and s_2-infinitesimally to $\mu^{(\infty)}_2$ for some integers $s_2 \geq s_1 \geq \imath(m) + 2$, then $\mathfrak{kill}(\mu^{(\infty)}_1) = \mathfrak{kill}(\mu^{(\infty)}_2)$.

The following proposition puts in relation the algebraic convergence with the infinitesimal convergence (see [41, Thm 6.12 (i)] and the errata corrige [43, Thm 3.9]).

Proposition III.3.10. *Let $m \geq 1$ and $0 \leq q \leq \frac{m(m-1)}{2}$ be two integers.*

i) If $(\mu^{(n)}) \subset \mathcal{H}^{\mathrm{loc}}_{q,m}$ converges algebraically to $\mu^{(\infty)} \in \mathcal{H}^{\mathrm{loc}}_m$, then $(\mu^{(n)})$ converges infinitesimally to $\mu^{(\infty)}$.

ii) The converse assertion of (i) is not true.

Proof. Let us assume that $(\mu^{(n)}) \subset \mathcal{H}^{\mathrm{loc}}_{q,m}$ converges algebraically to $\mu \in \mathcal{H}^{\mathrm{loc}}_m$. From (III.3.4) we immediately get that $S^{\mu^{(n)}} \to S^{\mu^{(\infty)}}$ in the standard Euclidean topology. Then claim (i) is a consequence of (III.3.5) and (III.3.6). On the other hand, claim (ii) follows directly from [10, Ex 9.1]. □

As we will see in Chapter IV and in Chapter V, there are sequences $(\mu^{(n)}) \subset \mathcal{H}^{\mathrm{loc}}_{q,m}$ with bounded curvature, say e.g. $|\sec(\mu^{(n)})| \leq 1$, which do not converge to any element in $\mathcal{H}^{\mathrm{loc}}_m$. By Remark III.3.1, in this case it necessarily holds that $|\mu^{(n)}|_{\mathrm{st}} \to +\infty$ as $n \to +\infty$. Motivated by this fact, we recall the following

Definition III.3.11. A sequence $(\mu^{(n)}) \subset \mathcal{H}^{\mathrm{loc}}_{q,m}(1)$ is said to be *algebraically collapsed* if $|\mu^{(n)}|_{\mathrm{st}} \to +\infty$ as $n \to +\infty$, *algebraically non-collapsed* otherwise.

Notice that by Remark III.3.1 and Proposition III.3.10 we immediately get

Corollary III.3.12. *If $(\mu^{(n)}) \subset \mathcal{H}^{\mathrm{loc}}_{q,m}(1)$ is algebraically non-collapsed, then it converges infinitesimally, up to a subsequence, to an element $\mu^{(\infty)} \in \mathcal{H}^{\mathrm{loc}}_m(1)$.*

III.3.5 Proofs of Theorem III.1.1 and Theorem III.1.4

We are now ready to prove the following

Theorem III.3.13. *For each $\mu \in \mathcal{H}^{\mathrm{loc}}_m(1)$, there exists a geometric model $(\mathcal{B}_\mu, \hat{g}_\mu) = (\mathcal{B}_{\hat{g}_\mu}(o_\mu, \pi), \hat{g}_\mu)$ which is equivariantly locally isometric to $(\mathsf{G}_\mu/\mathsf{H}_\mu, g_\mu)$, and it is unique up to a global equivariant isometry.*

which coincides with Theorem III.1.1.

Proof of Theorem III.3.13. Let us fix $\mu \in \mathcal{H}^{\mathrm{loc}}_{q,m}(1)$. If μ is regular, then the claim holds true by Remark III.2.4. So, we assume that $\mu \in \mathcal{H}^{\mathrm{loc}}_{q,m} \setminus \mathcal{H}_{q,m}$. By Theorem III.3.4, we can pick a sequence $(\mu^{(n)}) \subset \mathcal{H}_{q,m}$ which converges algebraically to μ. By Proposition III.3.10 it comes that $(\mu^{(n)})$ converges infinitesimally to μ and hence we can assume that

$$|\sec(\mu^{(n)})| \leq \frac{1}{(1-\frac{\varepsilon^{(n)}}{\pi})^2} \qquad \text{for some positive } \varepsilon^{(n)} \to 0 \ .$$

By repeating the same argument as in Remark III.2.4, we can pull back the metric $g_{\mu^{(n)}}$ to the tangent ball of radius $\pi - \varepsilon^{(n)}$ at the origin of $\mathsf{G}_{\mu^{(n)}}/\mathsf{H}_{\mu^{(n)}}$ via the Riemannian exponential. By Proposition III.3.10 and Corollary III.2.9, we can pass to a subsequence in such a way that these tangent balls converges to a geometric model $(\mathcal{B}, \hat{g})$ in the pointed $\mathcal{C}^\infty$-topology. Therefore, from Theorem I.4.12, it necessary holds that $(B^m, \hat{g}^{(\infty)})$ is locally isometric to $(\mathsf{G}_\mu/\mathsf{H}_\mu, g_\mu)$, since they have the same Riemannian s-tuple model for any $s \geq \imath(m)+2$. □

Finally, we notice that the pointed $\mathcal{C}^{s+2}$-convergence of a sequence of geometric models clearly implies the s-infinitesimal convergence. Our Theorem III.1.4, which we prove below, is a kind of converse of such statement.

Proof of Theorem III.1.4. Claim (i) follows directly from the very definition of pointed $\mathcal{C}^{s+2}$-convergence and s-infinitesimal convergence. In order to prove claim (ii), let us consider a sequence $(\mu^{(n)}) \subset \mathcal{H}^{\mathrm{loc}}_m(1)$ and assume that $(\mu^{(n)})$ converges $(s+1)$-infinitesimally to an element $\mu^{(\infty)} \in \mathcal{H}^{\mathrm{loc}}_m$ for some $s \geq \imath(m)+2$. By Proposition III.3.10 we get $\mu^{(\infty)} \in \mathcal{H}^{\mathrm{loc}}_m(1)$. Fix $0 < \alpha < 1$. By Theorem I.4.12 and Corollary III.2.9, it comes that there exists a subsequence of $(\mathcal{B}_{\mu^{(n)}}, \hat{g}_{\mu^{(n)}})$ which converges to $(\mathcal{B}_{\mu^{(\infty)}}, \hat{g}_{\mu^{(\infty)}})$ in the pointed $\mathcal{C}^{s+2,\alpha}$-topology. Moreover, it holds more: any convergent subsequence of $(\mathcal{B}_{\mu^{(n)}}, \hat{g}_{\mu^{(n)}})$ in the pointed $\mathcal{C}^{s+2,\alpha}$-topology necessarily converges to $(\mathcal{B}_{\mu^{(\infty)}}, \hat{g}_{\mu^{(\infty)}})$. Let us assume then by contradiction that $(\mathcal{B}_{\mu^{(n)}}, \hat{g}_{\mu^{(n)}})$ does not converges to $(\mathcal{B}_{\mu^{(\infty)}}, \hat{g}_{\mu^{(\infty)}})$ in the pointed $\mathcal{C}^{s+2,\alpha}$-topology. Then, there exist $0 < \delta < \pi$ and a sequence $(n_j) \subset \mathbb{N}$ such that for any $j_{\mathrm{o}} \in \mathbb{N}$ and for any choice of $\mathcal{C}^{s+3,\alpha}$-embeddings $\phi^{(n_j)} : \mathcal{B}_{\hat{g}_{\mu^{(\infty)}}}(o_{\mu^{(\infty)}}, \pi - \delta) \subset \mathcal{B}_{\mu^{(\infty)}} \to \mathcal{B}_{\mu^{(n_j)}}$ with $\phi^{(n)}(o_{\mu^{(\infty)}}) = o_{\mu^{(n_j)}}$ and $j \geq j_{\mathrm{o}}$, the pulled back metrics $(\phi^{(n_j)})^* \hat{g}_{\mu^{(n_j)}}$ do not converge to $\hat{g}_{\mu^{(\infty)}}$ in $\mathcal{C}^{s+2,\alpha}$-topology. On the other hand, the sequence $(\mathcal{B}_{\mu^{(n_j)}}, \hat{g}_{\mu^{(n_j)}})$ has bounded geometry up to order $s+1$, and hence it admits a subsequence which converges to a limit geometric model in the pointed $\mathcal{C}^{s+2,\alpha}$-topology. By construction, this limit is not isometric to $(\mathcal{B}_{\mu^{(\infty)}}, \hat{g}_{\mu^{(\infty)}})$, which is a contradiction. $\square$

Chapter IV

On the infinitesimal convergence of locally homogeneous spaces

IV.1 Statement of results

In this Chapter, we investigate more deeply the notion of s-infinitesimal convergence by letting the integer s vary. In particular, we prove that keeping all the covariant derivatives of the curvature tensor bounded along a sequence of locally homogeneous spaces is a much more restrictive condition than just a bound on any finite number of them. More precisely

Theorem IV.1.1. *For any choice of $m, s \in \mathbb{N}$ such that $m \geq 3$ and $s \geq \imath(m)+2$, the notion of s-infinitesimal convergence in $\mathcal{H}_m^{\mathrm{loc}}$ is strictly weaker than the one of $(s+1)$-infinitesimal convergence.*

See Formula (I.4.8) for the definition of $\imath(m)$. In particular, we construct an explicit 2-parameter family

$$\{\mu_\star(\varepsilon,\delta) : \varepsilon, \delta \in \mathbb{R}\,,\ \varepsilon > 0\,,\ 0 \leq \delta < 1\} \subset \mathcal{H}_{0,3}(1)$$

with the following property: for any fixed integer $k \geq 0$, there are $(\varepsilon^{(n)}), (\delta^{(n)}) \subset (0,1)$ and $C > 0$ such that, letting $\mu^{(n)} := \mu_\star(\varepsilon^{(n)}, \delta^{(n)})$, it holds that

$$\begin{gathered} \big|\operatorname{Rm}^0(\mu^{(n)})\big|_{\mathrm{st}} + \big|\operatorname{Rm}^1(\mu^{(n)})\big|_{\mathrm{st}} + \ldots + \big|\operatorname{Rm}^k(\mu^{(n)})\big|_{\mathrm{st}} \leq C\ , \\ \big|\operatorname{Rm}^{k+1}(\mu^{(n)})\big|_{\mathrm{st}} \to +\infty \quad \text{as } n \to +\infty\ . \end{gathered}$$

The key idea to construct such $\mu_\star(\varepsilon,\delta)$ is to consider a slight modification of the well-known Berger spheres, which arise from the canonical variation of the round metric with respect to the Hopf fibration $S^1 \to \mathsf{SU}(2) \to \mathbb{C}P^1$ (see [3, p. 252]), which correspond to the case $\delta = 0$.

Combining Theorem III.1.4 and Theorem IV.1.1, we also obtain

Corollary IV.1.2. *For any $m, k \in \mathbb{N}$ with $m \geq 3$ and $k \geq \imath(m) + 4$, there exists a sequence of m-dimensional pointed locally homogeneous spaces converging to a limit pointed locally homogeneous space in the pointed $\mathcal{C}^{k,\alpha}$-topology for any $0 < \alpha < 1$ and which does not admit any convergent subsequence in the pointed $\mathcal{C}^{k+1}$-topology.*

To the best of our knowledge, Corollary IV.1.2 provides the first examples of locally homogeneous spaces converging in the pointed $\mathcal{C}^{k,\alpha}$-topology for any $0 < \alpha < 1$ but not in the $\mathcal{C}^{k+1}$-topology, for some fixed $k \geq 0$.

Finally, we stress that the contents of this chapter lead to the following corollary (compare with Corollary III.1.3).

Corollary IV.1.3. *If $1 \leq m < 3$, then $\mathcal{H}^{\mathrm{loc}}_m(1) = \mathcal{H}_m(1)$ is compact in the pointed $\mathcal{C}^\infty$-topology. On the contrary, $\mathcal{H}^{\mathrm{loc}}_m(1)$ is not compact in the pointed $\mathcal{C}^3$-topology for any $m \geq 3$.*

IV.2 Riemannian curvature of left invariant metrics on $\mathsf{SU}(2)$

Let us fix a left invariant metric g on the Lie group $\mathsf{SU}(2)$, which we will constantly identify with the corresponding Euclidean inner product on the Lie algebra $\mathfrak{su}(2)$. By the Milnor Theorem, it is known that there exists a g-orthogonal basis (X_0, X_1, X_2) for $\mathfrak{su}(2)$ such that

$$[X_0, X_1] = -2X_2 \ , \quad [X_0, X_2] = +2X_1 \ , \quad [X_1, X_2] = -2X_0 \ . \tag{IV.2.1}$$

Up to an automorphism of $\mathfrak{su}(2)$, we can assume that

$$X_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \ , \quad X_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \quad X_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \ . \tag{IV.2.2}$$

We also set

$$g(X_0,X_0)=\varepsilon\ ,\quad g(X_1,X_1)=\lambda_1\ ,\quad g(X_2,X_2)=\lambda_2\ . \tag{IV.2.3}$$

We aim to study the curvature of $(\mathsf{SU}(2),g)$. Firstly, by means of (IV.2.1) and (IV.2.3), we notice that the bracket $\mu\in\mathcal{H}_{0,3}$ satisfying $(\mathsf{G}_\mu,g_\mu)=(\mathsf{SU}(2),g)$ is given by

$$\mu(e_0,e_1)=-2\sqrt{\tfrac{\lambda_2}{\varepsilon\lambda_1}}\,e_2\ ,\quad \mu(e_0,e_2)=+2\sqrt{\tfrac{\lambda_1}{\varepsilon\lambda_2}}\,e_1\ ,\quad \mu(e_1,e_2)=-2\sqrt{\tfrac{\varepsilon}{\lambda_1\lambda_2}}\,e_0\ . \tag{IV.2.4}$$

Here, we indicated with (e_0,e_1,e_2) the standard basis of $\mathbb{R}^3$. From now on, we will denote by

$$\big(E_{ij}:=e^i\otimes e_j-e^j\otimes e_i\ ,\ 0\le i<j\le 2\big)$$

the standard basis of $\mathfrak{so}(3)$.

Let $S^\mu:\mathbb{R}^3\to\mathfrak{so}(3)$ be the linear operator uniquely associated to the canonical Ambrose-Singer connection of μ. By (III.3.4), one can directly check that

$$S^\mu(e_0)=+c^\mu_0E_{12}\ ,\quad S^\mu(e_1)=-c^\mu_1E_{02}\ ,\quad S^\mu(e_2)=+c^\mu_2E_{01}\ , \tag{IV.2.5}$$

where $c^\mu_0,c^\mu_1,c^\mu_2\in\mathbb{R}$ are the coefficients defined by

$$c^\mu_0:=\tfrac{-\varepsilon+\lambda_1+\lambda_2}{\sqrt{\varepsilon\lambda_1\lambda_2}}\ ,\quad c^\mu_1:=\tfrac{+\varepsilon-\lambda_1+\lambda_2}{\sqrt{\varepsilon\lambda_1\lambda_2}}\ ,\quad c^\mu_2:=\tfrac{+\varepsilon+\lambda_1-\lambda_2}{\sqrt{\varepsilon\lambda_1\lambda_2}}\ . \tag{IV.2.6}$$

In virtue of (III.3.5) and (IV.2.5), the curvature operator $\mathrm{Rm}^0(\mu):\Lambda^2\mathbb{R}^3\to\mathfrak{so}(3)$ is diagonal, i.e.

$$\mathrm{Rm}^0(\mu)(e_0\wedge e_1)=\sec(\mu)(e_0\wedge e_1)E_{01}\ ,\quad \mathrm{Rm}^0(\mu)(e_1\wedge e_2)=\sec(\mu)(e_1\wedge e_2)E_{12}\ ,$$
$$\mathrm{Rm}^0(\mu)(e_0\wedge e_2)=\sec(\mu)(e_0\wedge e_2)E_{02}\ ,$$

and the sectional curvature is given by

$$\begin{aligned}
\sec(\mu)(e_0\wedge e_1)&=-c^\mu_0c^\mu_1+c^\mu_1c^\mu_2+c^\mu_0c^\mu_2\\
&=\tfrac{1}{\lambda_1\lambda_2}\big(\varepsilon+2(\lambda_2-\lambda_1)-\varepsilon^{-1}(\lambda_2-\lambda_1)(\lambda_1+3\lambda_2)\big)\ ,\\
\sec(\mu)(e_1\wedge e_2)&=+c^\mu_0c^\mu_1-c^\mu_1c^\mu_2+c^\mu_0c^\mu_2\\
&=\tfrac{1}{\lambda_1\lambda_2}\big(-3\varepsilon+2(\lambda_1+\lambda_2)+\varepsilon^{-1}(\lambda_2-\lambda_1)^2\big)\ ,\\
\sec(\mu)(e_0\wedge e_2)&=+c^\mu_0c^\mu_1+c^\mu_1c^\mu_2-c^\mu_0c^\mu_2\\
&=\tfrac{1}{\lambda_1\lambda_2}\big(\varepsilon-2(\lambda_2-\lambda_1)+\varepsilon^{-1}(\lambda_2-\lambda_1)(\lambda_2+3\lambda_1)\big)\ .
\end{aligned} \tag{IV.2.7}$$

By applying (III.3.6), we obtain the following expressions for the non-zero components of $\mathrm{Rm}^1(\mu)$

$$\begin{aligned}
\mathrm{Rm}^1(\mu)(e_0|e_0\wedge e_1) &= 2(c^\mu_0)^2(c^\mu_1-c^\mu_2)E_{02}\\
\mathrm{Rm}^1(\mu)(e_0|e_0\wedge e_2) &= 2(c^\mu_0)^2(c^\mu_1-c^\mu_2)E_{01}\\
\mathrm{Rm}^1(\mu)(e_1|e_0\wedge e_1) &= 2(c^\mu_1)^2(c^\mu_0-c^\mu_2)E_{12}\\
\mathrm{Rm}^1(\mu)(e_1|e_1\wedge e_2) &= 2(c^\mu_1)^2(c^\mu_0-c^\mu_2)E_{01}\\
\mathrm{Rm}^1(\mu)(e_2|e_0\wedge e_2) &= 2(c^\mu_2)^2(c^\mu_0-c^\mu_1)E_{12}\\
\mathrm{Rm}^1(\mu)(e_2|e_1\wedge e_2) &= 2(c^\mu_2)^2(c^\mu_0-c^\mu_1)E_{02}
\end{aligned} \tag{IV.2.8}$$

and for the non-zero components of $\mathrm{Rm}^2(\mu)$.

$$\begin{aligned}
\mathrm{Rm}^2(\mu)(e_0,e_0|e_0\wedge e_1) &= +4(c^\mu_0)^3(c^\mu_1-c^\mu_2)E_{01}\\
\mathrm{Rm}^2(\mu)(e_0,e_0|e_0\wedge e_2) &= -4(c^\mu_0)^3(c^\mu_1-c^\mu_2)E_{02}\\
\mathrm{Rm}^2(\mu)(e_0,e_1|e_0\wedge e_2) &= -2c^\mu_0(c^\mu_1-c^\mu_2)(c^\mu_0(c^\mu_1+c^\mu_2)-c^\mu_1c^\mu_2)E_{12}\\
\mathrm{Rm}^2(\mu)(e_0,e_1|e_1\wedge e_2) &= -2c^\mu_0(c^\mu_1-c^\mu_2)(c^\mu_0(c^\mu_1+c^\mu_2)-c^\mu_1c^\mu_2)E_{02}\\
\mathrm{Rm}^2(\mu)(e_0,e_2|e_0\wedge e_1) &= -2c^\mu_0(c^\mu_1-c^\mu_2)(c^\mu_0(c^\mu_1+c^\mu_2)-c^\mu_1c^\mu_2)E_{12}\\
\mathrm{Rm}^2(\mu)(e_0,e_2|e_1\wedge e_2) &= -2c^\mu_0(c^\mu_1-c^\mu_2)(c^\mu_0(c^\mu_1+c^\mu_2)-c^\mu_1c^\mu_2)E_{01}\\
\mathrm{Rm}^2(\mu)(e_1,e_0|e_0\wedge e_2) &= -2c^\mu_1(c^\mu_0-c^\mu_2)(c^\mu_0(c^\mu_1-c^\mu_2)+c^\mu_1c^\mu_2)E_{12}\\
\mathrm{Rm}^2(\mu)(e_1,e_0|e_1\wedge e_2) &= -2c^\mu_1(c^\mu_0-c^\mu_2)(c^\mu_0(c^\mu_1-c^\mu_2)+c^\mu_1c^\mu_2)E_{02}\\
\mathrm{Rm}^2(\mu)(e_1,e_1|e_0\wedge e_1) &= +4(c^\mu_1)^3(c^\mu_0-c^\mu_2)E_{01}\\
\mathrm{Rm}^2(\mu)(e_1,e_1|e_1\wedge e_2) &= -4(c^\mu_1)^3(c^\mu_0-c^\mu_2)E_{12}\\
\mathrm{Rm}^2(\mu)(e_1,e_2|e_0\wedge e_1) &= +2c^\mu_1(c^\mu_0-c^\mu_2)(c^\mu_0(c^\mu_1-c^\mu_2)+c^\mu_1c^\mu_2)E_{02}\\
\mathrm{Rm}^2(\mu)(e_1,e_2|e_0\wedge e_2) &= +2c^\mu_1(c^\mu_0-c^\mu_2)(c^\mu_0(c^\mu_1-c^\mu_2)+c^\mu_1c^\mu_2)E_{01}\\
\mathrm{Rm}^2(\mu)(e_2,e_0|e_0\wedge e_1) &= -2c^\mu_2(c^\mu_0-c^\mu_1)(c^\mu_0(c^\mu_1-c^\mu_2)-c^\mu_1c^\mu_2)E_{12}\\
\mathrm{Rm}^2(\mu)(e_2,e_0|e_1\wedge e_2) &= -2c^\mu_2(c^\mu_0-c^\mu_1)(c^\mu_0(c^\mu_1-c^\mu_2)-c^\mu_1c^\mu_2)E_{01}\\
\mathrm{Rm}^2(\mu)(e_2,e_1|e_0\wedge e_1) &= -2c^\mu_2(c^\mu_0-c^\mu_1)(c^\mu_0(c^\mu_1-c^\mu_2)-c^\mu_1c^\mu_2)E_{02}\\
\mathrm{Rm}^2(\mu)(e_2,e_1|e_0\wedge e_2) &= -2c^\mu_2(c^\mu_0-c^\mu_1)(c^\mu_0(c^\mu_1-c^\mu_2)-c^\mu_1c^\mu_2)E_{01}\\
\mathrm{Rm}^2(\mu)(e_2,e_2|e_0\wedge e_2) &= +4(c^\mu_2)^3(c^\mu_0-c^\mu_1)E_{02}\\
\mathrm{Rm}^2(\mu)(e_2,e_2|e_1\wedge e_2) &= -4(c^\mu_2)^3(c^\mu_0-c^\mu_1)E_{12}
\end{aligned} \tag{IV.2.9}$$

We stress also that from (III.3.6), for any integer $k \geq 0$ there exists $C_k > 0$, which depends only on k, such that

$$\big|X \lrcorner \mathrm{Rm}^{k+1}(\mu)\big|_{\mathrm{st}} \leq C_k \big|S^\mu(X)\big|_{\mathrm{st}} \big|\mathrm{Rm}^k(\mu)\big|_{\mathrm{st}} . \tag{IV.2.10}$$

Finally, we prove the following

Lemma IV.2.1. *Let $k \geq 1$ be an integer and $b(k) := 1 + (k \bmod 2)$. Then, it holds that*

$$
\begin{aligned}
&\mathrm{Rm}^k(\mu)(e_0, \ldots, e_0|e_0\wedge e_1) = (-1)^{[\frac{k-1}{2}]}2^k(c_0^\mu)^{k+1}(c_1^\mu - c_2^\mu)E_{0b(k)}\\
&\mathrm{Rm}^k(\mu)(e_0, \ldots, e_0|e_0\wedge e_2) = (-1)^{[\frac{k}{2}]}2^k(c_0^\mu)^{k+1}(c_1^\mu - c_2^\mu)E_{0b(k+1)} \quad ,\\
&\mathrm{Rm}^k(\mu)(e_0, \ldots, e_0|e_1\wedge e_2) = 0
\end{aligned}
\tag{IV.2.11}
$$

where $c_0^\mu, c_1^\mu, c_2^\mu$ are defined in (IV.2.6).

Proof. We proceed by induction on $k \geq 1$. The case $k = 1$ follows directly from the computations in (IV.2.8). In order to prove that $k \Rightarrow k+1$, we first notice that

$$
[E_{12}, E_{0b(k)}] = (-1)^k E_{0b(k+1)} \ , \quad (-1)^{[\frac{k}{2}]} = (-1)^{k-1}(-1)^{[\frac{k-1}{2}]} \ . \tag{IV.2.12}
$$

So, by (IV.2.5), (III.3.6), (IV.2.12) and the inductive hypothesis we get

$$
\begin{aligned}
\mathrm{Rm}^{k+1}&(\mu)(e_0, \ldots, e_0|e_0\wedge e_1) =\\
&= -(-1)^{[\frac{k-1}{2}]}2^k(c_0^\mu)^{k+2}(c_1^\mu - c_2^\mu)[E_{12}, E_{0b(k)}]+\\
&\qquad\qquad + (-1)^{[\frac{k-1}{2}]}2^k(c_0^\mu)^{k+2}(c_1^\mu - c_2^\mu)E_{0b(k+1)}\\
&= (-1)^{k-1}(-1)^{[\frac{k-1}{2}]}2^{k+1}(c_0^\mu)^{k+2}(c_1^\mu - c_2^\mu)E_{0b(k+1)}\\
&= (-1)^{[\frac{k}{2}]}2^k(c_0^\mu)^{k+1}(c_1^\mu - c_2^\mu)E_{0b(k+1)} \ .
\end{aligned}
$$

The second formula is analogous. For the third formula, from (IV.2.5), (III.3.6) and the inductive hypothesis

$$
\begin{aligned}
\mathrm{Rm}^{k+1}&(\mu)(e_0, \ldots, e_0|e_1\wedge e_2) =\\
&= 0 + c_0^\mu \,\mathrm{Rm}^k(\mu)(e_0, \ldots, e_0|e_2 \wedge e_2) - c_0^\mu \,\mathrm{Rm}^k(\mu)(e_0, \ldots, e_0|e_1 \wedge e_1)\\
&= 0
\end{aligned}
$$

which completes the proof. $\square$

IV.3 Collapsing sequences on $\mathsf{SU}(2)$

IV.3.1 Almost-Berger sequences

We consider now a sequence $(\mu^{(n)}) \subset \mathcal{H}_{0,3}$ of brackets which corresponds to a sequence $(g_{\mu^{(n)}})$ of left-invariant metrics on $\mathsf{SU}(2)$. By (IV.2.1), we can assume that

$$\begin{gathered} g_{\mu^{(n)}}(X_0,X_0)=\varepsilon^{(n)}\ ,\quad g_{\mu^{(n)}}(X_1,X_1)=\lambda_1^{(n)}\ ,\quad g_{\mu^{(n)}}(X_2,X_2)=\lambda_2^{(n)}\ , \\ g_{\mu^{(n)}}(X_0,X_1)=g_{\mu^{(n)}}(X_0,X_2)=g_{\mu^{(n)}}(X_1,X_2)=0 \end{gathered} \tag{IV.3.1}$$

where (X_0,X_1,X_2) is the standard basis of $\mathfrak{su}(2)$ given in (IV.2.2). Let us suppose that

$$\varepsilon^{(n)}\to 0\ ,\quad \lambda_i^{(n)}\to\lambda_i^{(\infty)}\in(0,+\infty)\quad \text{as } n\to+\infty\ .$$

A direct computation based on (IV.2.7) shows that the sectional curvature $\sec(\mu^{(n)})$ is uniformly bounded if and only if

$$\big|\lambda_1^{(n)}-\lambda_2^{(n)}\big|\leq C\,\varepsilon^{(n)}\quad \text{for some } C>0\ . \tag{IV.3.2}$$

Notice that (IV.3.2) is coherent with Theorem V.3.3 in the next Chapter, which will give necessary conditions for a sequence of invariant metrics on a compact homogeneous space to diverge with bounded curvature.

Since we are interested in studying sequences with bounded curvature, we assume without loss of generality that $\lambda_1^{(\infty)}=\lambda_2^{(\infty)}=1$. Notice that, if we define

$$\bar{k}:=\,\sup\Big\{k\in\mathbb{Z}\,:\,\big(\varepsilon^{(n)}\big)^{-\frac{k}{2}}\big|\lambda_1^{(n)}-\lambda_2^{(n)}\big|\to 0 \text{ as } n\to+\infty\Big\}\ , \tag{IV.3.3}$$

then condition (IV.3.2) implies that $\bar{k}\geq 1$. We introduce now the following

Definition IV.3.1. An *almost-Berger sequence* is any sequence $(\mu^{(n)})\subset\mathcal{H}_{0,3}$ which corresponds to a sequence of left-invariant metrics on $\mathsf{SU}(2)$ as in (IV.3.1) such that

i) $\varepsilon^{(n)}\to 0$ and $\lambda_i^{(n)}\to 1$ as $n\to+\infty$;

ii) $\sec(\mu^{(n)})$ is uniformly bounded, i.e. (IV.3.2) holds true.

The positive integer $\operatorname{reg}(\mu^{(n)}):=\bar{k}$ defined in (IV.3.3) is called *regularity index of* $(\mu^{(n)})$.

This nomenclature is motivated by the following fact. If $\lambda_1^{(n)} = \lambda_2^{(n)} = 1$ for any $n \in \mathbb{N}$, then $g_{\mu^{(n)}}$ comes from the canonical variation of the round metric on $S^3 = \mathsf{SU}(2)$ with respect to the Hopf fibration $S^1 \to S^3 \to S^2$ (see [3, p. 252]). The 3-sphere endowed with any such a metric is commonly named *Berger sphere*. They provide the first non trivial example of collapsing sequence with bounded curvature (see e.g. [17, 18]).

From the very definition of regularity index, the following properties hold:

- $\big(\varepsilon^{(n)}\big)^{-\frac{k}{2}}\big|\lambda_1^{(n)} - \lambda_2^{(n)}\big| \to 0$ as $n \to +\infty$ for any integer $0 \leq k \leq \mathrm{reg}(\mu^{(n)})$;
- if $\mathrm{reg}(\mu^{(n)}) = \bar{k}$ is finite, then $\big(\varepsilon^{(n)}\big)^{-\frac{\bar{k}+1}{2}}\big|\lambda_1^{(n)} - \lambda_2^{(n)}\big|$ is bounded away from zero.

Let us stress also that from (IV.2.4) it comes that any almost-Berger sequence $(\mu^{(n)})$ verifies

$$\mu^{(n)}(e_0, e_1) \to -\infty \ , \quad \mu^{(n)}(e_0, e_2) \to +\infty \ , \quad \mu^{(n)}(e_1, e_2) \to 0 \qquad \text{as } n \to +\infty \ .$$

This means that they never converge algebraically to an element of $\mathcal{H}_3^{\mathrm{loc}}$. Actually, we point out that any almost-Berger sequence is, in particular, a diverging sequence with bounded curvature as the ones in Chapter V. Hence, the fact that almost-Berger sequences cannot converge algebraically can be also derived from Proposition V.1.4.

IV.3.2 Curvature estimates

We investigate the behavior of the curvature along an almost-Berger sequence $(\mu^{(n)}) \subset \mathcal{H}_{0,3}$. Concerning the sectional curvature, from (IV.2.7) we directly get

Proposition IV.3.2. *Let* $(\mu^{(n)})$ *be an almost-Berger sequence with regularity index* $\mathrm{reg}(\mu^{(n)}) = \bar{k}$. *Then*

$$\sec(\mu^{(n)})(e_0{\wedge}e_1) \to 0 \ , \quad \sec(\mu^{(n)})(e_1{\wedge}e_2) \to 4 \ , \quad \sec(\mu^{(n)})(e_0{\wedge}e_2) \to 0$$

if and only if $\bar{k} \geq 2$.

We give now a characterization for the covariant derivatives of the curvature tensor along $(\mu^{(n)})$ in terms of the regularity index. More precisely

Proposition IV.3.3. *Let* $(\mu^{(n)})$ *be an almost Berger sequence with regularity index* $\operatorname{reg}(\mu^{(n)}) = \bar{k} \geq 2$. *Then, for any integer* $k \geq 1$ *there exists a constant* $L_k > 0$ *such that*

$$\big|\operatorname{Rm}^k(\mu^{(n)})\big|_{\mathrm{st}} \leq L_k\Big(\big(\varepsilon^{(n)}\big)^{\frac{1}{2}} + \big(\varepsilon^{(n)}\big)^{-\frac{k+2}{2}}\big|\lambda_1^{(n)} - \lambda_2^{(n)}\big|\Big) \quad \textit{for any } n \in \mathbb{N}\ . \tag{IV.3.4}$$

Proof. Let us assume, without loss of generality, that $\varepsilon^{(n)} < 1$ and $\big|\lambda_1^{(n)} - \lambda_2^{(n)}\big| < 1$ for any $n \in \mathbb{N}$. Firstly, since $\bar{k} \geq 2$ by hypothesis, from (IV.2.6) we get

$$c_1^{\mu^{(n)}}, c_2^{\mu^{(n)}} \sim \big(\varepsilon^{(n)}\big)^{\frac{1}{2}}$$

and so from (IV.2.5), (IV.2.8) and (IV.2.10) it follows that for any $k \geq 1$ there exists $L_{k,0} > 0$ such that

$$\big|\operatorname{Rm}^k(\mu^{(n)})(e_{i_1}, \ldots, e_{i_k}|e_{j_1} \wedge e_{j_2})\big|_{\mathrm{st}} \leq L_{k,0}\big(\varepsilon^{(n)}\big)^{\frac{k}{2}} \tag{IV.3.5}$$

for any $1 \leq i_1, \ldots, i_k \leq 2$, $0 \leq j_1 < j_2 \leq 2$.

Secondly, by (IV.2.6) we notice that

$$c_0^{\mu^{(n)}} \sim 2\big(\varepsilon^{(n)}\big)^{-\frac{1}{2}}$$

and so from (IV.2.11) it follows that for any $k \geq 1$ there exists $L_{k,k} > 0$ such that

$$\big|\operatorname{Rm}^k(\mu^{(n)})(e_0, \ldots, e_0|e_{j_1} \wedge e_{j_2})\big|_{\mathrm{st}} \leq L_{k,k}\big(\varepsilon^{(n)}\big)^{-\frac{k+2}{2}}\big|\lambda_1^{(n)} - \lambda_2^{(n)}\big| \tag{IV.3.6}$$

for any $0 \leq j_1 < j_2 \leq 2$. Moreover, from (IV.2.5), (IV.2.10) and (IV.3.6), it follows that for any $k, r \in \mathbb{Z}$ with $k \geq 2$ and $1 \leq r \leq k-1$, there exists $L_{k,r} > 0$ such that

$$\big|\operatorname{Rm}^k(\mu^{(n)})(e_{i_1}, \ldots, e_{i_{k-r}}, \underbrace{e_0, \ldots, e_0}_{r}|e_{j_1} \wedge e_{j_2})\big|_{\mathrm{st}} \leq L_{k,r}\big(\varepsilon^{(n)}\big)^{-\frac{r+2}{2}}\big|\lambda_1^{(n)} - \lambda_2^{(n)}\big| \tag{IV.3.7}$$

for any $1 \leq i_1, \ldots, i_{k-r} \leq 2$, $0 \leq j_1 < j_2 \leq 2$.

Thirdly, a direct computation based on (III.3.6) and the last identity in (R1) (see Section I.4.3) shows that for any $k, r \in \mathbb{Z}$ with $k \geq 0$ and $0 \leq r \leq k$, there exists $N_{k,r} > 0$ such that

$$\Big| \big| \operatorname{Rm}^{k+2}(\mu^{(n)})(e_{\alpha_1}, \dots, e_{\alpha_r}, e_{\ell_1}, e_{\ell_2}, e_{\alpha_{r+1}}, \dots, e_{\alpha_k} | e_{j_1} \wedge e_{j_2}) \big|_{\mathrm{st}} - \\ - \big| \operatorname{Rm}^{k+2}(\mu^{(n)})(e_{\alpha_1}, \dots, e_{\alpha_r}, e_{\ell_2}, e_{\ell_1}, e_{\alpha_{r+1}}, \dots, e_{\alpha_k} | e_{j_1} \wedge e_{j_2}) \big|_{\mathrm{st}} \Big| \leq \\ \leq N_{k,r} \sum_{q=0}^{r} \big| \operatorname{Rm}^q(\mu^{(n)}) \big|_{\mathrm{st}} \big| \operatorname{Rm}^{k-q}(\mu^{(n)}) \big|_{\mathrm{st}} \quad \text{(IV.3.8)}$$

for any $0 \leq \alpha_1, \dots, \alpha_k \leq 2$, $0 \leq \ell_1, \ell_2 \leq 2$, $0 \leq j_1 < j_2 \leq 2$.

Finally, we are ready to prove (IV.3.4) by induction on $k \geq 1$. For $k = 1$, it follows directly from (IV.2.6) and (IV.2.8). Let us fix now $k > 1$ and assume that (IV.3.4) holds for any $1 \leq k' \leq k$. Then

$$\begin{aligned}
&\big| \operatorname{Rm}^{k+1}(\mu^{(n)}) \big|_{\mathrm{st}} \leq \\
&\quad \leq C_1 \sum_{\substack{0 \leq \alpha_1, \dots, \alpha_{k+1} \leq 2 \\ 0 \leq j_1 < j_2 \leq 2}} \big| \operatorname{Rm}^{k+1}(\mu^{(n)})(e_{\alpha_1}, \dots, e_{\alpha_{k+1}} | e_{j_1} \wedge e_{j_2}) \big|_{\mathrm{st}} \\
&\quad \overset{\text{(IV.3.8)}}{\leq} C_2 \Bigg\{ \sum_{r=0}^{k+1} \sum_{\substack{1 \leq i_1, \dots, i_{(k+1)-r} \leq 2 \\ 0 \leq j_1 < j_2 \leq 2}} \big| \operatorname{Rm}^{k+1}(\mu^{(n)})(e_{i_1}, \dots, e_{i_{(k+1)-r}}, \underbrace{e_0, \dots, e_0}_{r} | \\
&\qquad\qquad | e_{j_1} \wedge e_{j_2}) \big|_{\mathrm{st}} + \sum_{q=0}^{k-1} \big| \operatorname{Rm}^q(\mu^{(n)}) \big|_{\mathrm{st}} \big| \operatorname{Rm}^{k-q-1}(\mu^{(n)}) \big|_{\mathrm{st}} \Bigg\} \\
&\quad \overset{\text{(IV.3.5),(IV.3.7),(IV.3.6)}}{\leq} C_3 \Bigg\{ \sum_{i=1}^{k+1} \big(\varepsilon^{(n)}\big)^{\frac{i}{2}} + \sum_{i=0}^{k+1} \big(\varepsilon^{(n)}\big)^{-\frac{i+2}{2}} \big| \lambda_1^{(n)} - \lambda_2^{(n)} \big| + \\
&\qquad\qquad + \sum_{q=0}^{k-1} \big| \operatorname{Rm}^q(\mu^{(n)}) \big|_{\mathrm{st}} \big| \operatorname{Rm}^{k-q-1}(\mu^{(n)}) \big|_{\mathrm{st}} \Bigg\} \\
&\quad \leq C_4 \Bigg\{ \big(\varepsilon^{(n)}\big)^{\frac{1}{2}} + \big(\varepsilon^{(n)}\big)^{-\frac{k+3}{2}} \big| \lambda_1^{(n)} - \lambda_2^{(n)} \big| + \\
&\qquad\qquad + \sum_{q=0}^{k-1} \big| \operatorname{Rm}^q(\mu^{(n)}) \big|_{\mathrm{st}} \big| \operatorname{Rm}^{k-q-1}(\mu^{(n)}) \big|_{\mathrm{st}} \Bigg\}
\end{aligned}$$

where C_i are some suitable positive constants. Finally, by Proposition IV.3.2 and the inductive hypothesis, we obtain

$$\sum_{q=0}^{k-1} \big|\operatorname{Rm}^q(\mu^{(n)})\big|_{\mathrm{st}} \big|\operatorname{Rm}^{k-q-1}(\mu^{(n)})\big|_{\mathrm{st}} \leq \tilde{C}\Big(\big(\varepsilon^{(n)}\big)^{\frac{1}{2}} + \big(\varepsilon^{(n)}\big)^{-\frac{k+3}{2}} \big|\lambda_1^{(n)} - \lambda_2^{(n)}\big|\Big)$$

for another suitable $\tilde{C} > 0$. Therefore, the thesis follows. □

From (IV.2.8), (IV.2.11) and (IV.3.4) we directly get

Corollary IV.3.4. *Let $(\mu^{(n)})$ be an almost Berger sequence with regularity index* $\operatorname{reg}(\mu^{(n)}) = \bar{k}$.

a) If $\bar{k} \geq 3$, then $\big|\operatorname{Rm}^k(\mu^{(n)})\big|_{\mathrm{st}} \to 0$ as $n \to +\infty$ for any $1 \leq k \leq \bar{k} - 2$.

b) If $\bar{k} \geq 2$ and it is finite, the following conditions hold true:

- *$\big|\operatorname{Rm}^{\bar{k}-1}(\mu^{(n)})\big|_{\mathrm{st}}$ does not converge to 0 as $n \to +\infty$;*
- *$\big|\operatorname{Rm}^{\bar{k}-1}(\mu^{(n)})\big|_{\mathrm{st}}$ is bounded if and only if $\big(\varepsilon^{(n)}\big)^{-\frac{\bar{k}+1}{2}} \big|\lambda_1^{(n)} - \lambda_2^{(n)}\big|$ is bounded.*

c) If $\bar{k}$ is finite, then $\big|\operatorname{Rm}^k(\mu^{(n)})\big|_{\mathrm{st}} \to +\infty$ as $n \to +\infty$ for any integer $k \geq \bar{k}$.

Proof. Claim (a) follows directly from (IV.3.4). Let us assume now that $\bar{k} \geq 2$. Then, by (IV.2.11) we get

$$\big|\operatorname{Rm}^k(\mu^{(n)})(e_0, \dots, e_0|e_0\wedge e_1)\big|_{\mathrm{st}} \sim 2^{2(k+1)}\big(\varepsilon^{(n)}\big)^{-\frac{k+2}{2}} \big|\lambda_1^{(n)} - \lambda_2^{(n)}\big| \qquad \text{(IV.3.9)}$$

for any $k \geq 1$. Therefore, claim (b) follows from (IV.3.4) and (IV.3.9), while claim (c) follows from (IV.3.9). Finally, if $\bar{k} = 1$, then claim (c) follows from (IV.2.8) and (IV.2.11). □

IV.4 Proofs of the main results

For the proofs of the main results, we begin by considering the 2-parameter family

$$\{\mu_\star = \mu_\star(\varepsilon, \delta) : \varepsilon, \delta \in \mathbb{R},\, \varepsilon > 0,\, 0 \leq \delta < 1\} \subset \mathcal{H}_{0,3}$$

defined by

$$\begin{gathered} \mu_\star(e_0, e_1) = -2\sqrt{\tfrac{1}{\varepsilon}\tfrac{2+\delta}{2-\delta}}\, e_2\ , \quad \mu_\star(e_0, e_2) = +2\sqrt{\tfrac{1}{\varepsilon}\tfrac{2-\delta}{2+\delta}}\, e_1\ , \\ \mu_\star(e_1, e_2) = -2\sqrt{\tfrac{4\varepsilon}{4-\delta^2}}\, e_0\ . \end{gathered} \qquad \text{(IV.4.1)}$$

By means of (IV.2.4), each element $\mu_\star(\varepsilon,\delta) \in \mathcal{H}_{0,3}$ corresponds to the Lie group $\mathsf{SU}(2)$ endowed with the diagonal left-invariant metric $g_{\mu_\star(\varepsilon,\delta)}$ defined by

$$g_{\mu_\star(\varepsilon,\delta)}(X_0,X_0) = \varepsilon \ , \quad g_{\mu_\star(\varepsilon,\delta)}(X_1,X_1) = 1-\tfrac{\delta}{2} \ , \quad g_{\mu_\star(\varepsilon,\delta)}(X_2,X_2) = 1+\tfrac{\delta}{2} \ , \tag{IV.4.2}$$

where (X_0,X_1,X_2) is the standard basis of $\mathfrak{su}(2)$ given in (IV.2.2).

Let us also consider the Riemannian symmetric space $(\mathbb{C}P^1 \times \mathbb{R}, g_{\mathrm{FS}}+dt^2)$, which corresponds to the element $\mu_{\mathrm{o}} \in \mathcal{H}_{1,3}$ defined by

$$\mu_{\mathrm{o}}(e_0,e_1) = -2e_2 \ , \quad \mu_{\mathrm{o}}(e_0,e_2) = +2e_1 \ , \quad \mu_{\mathrm{o}}(e_1,e_2) = -2e_0 \ , \quad \mu_{\mathrm{o}}(e_3,\,\cdot\,) = 0 \ .$$

Clearly, it holds that

$$\mathrm{Rm}^0(\mu_{\mathrm{o}}) = \begin{pmatrix} 4 & & \\ & 0 & \\ & & 0 \end{pmatrix} , \quad \mathrm{Rm}^k(\mu_{\mathrm{o}}) = 0 \quad \text{for any } k \geq 1 \ . \tag{IV.4.3}$$

Proof of Theorem IV.1.1. Let us fix $m=3$, and hence $\imath(3)=1$ (see Section I.4.3). Fix an integer $s \geq 3$ and consider the sequence $\mu^{(n)} := \mu_\star(\varepsilon^{(n)},\delta^{(n)})$ defined by

$$\varepsilon^{(n)} := \tfrac{1}{n^2} \ , \quad \delta^{(n)} := \tfrac{1}{n^{s+\frac{5}{2}}} \ .$$

Then we get

$$\big(\varepsilon^{(n)}\big)^{-\frac{k}{2}}\delta^{(n)} = n^{k-s-\frac{5}{2}} , \quad \lim_{n\to+\infty} n^{k-s-\frac{5}{2}} = \begin{cases} 0 & \text{if } 0 \leq k \leq s+2 \\ +\infty & \text{if } k > s+2 \end{cases}$$

and hence, from (IV.4.2), it comes that $(\mu^{(n)})$ is an almost Berger sequence with regularity index $\mathrm{reg}(\mu^{(n)}) = s+2$ (see Definition IV.3.1). From Proposition IV.3.2 and Corollary IV.3.4, we get

$$\mathrm{Rm}^0(\mu^{(n)}) \to \begin{pmatrix} 4 & & \\ & 0 & \\ & & 0 \end{pmatrix} , \quad \mathrm{Rm}^k(\mu^{(n)}) \to 0 \quad \text{for any } 1 \leq k \leq s \ ,$$
$$\big|\,\mathrm{Rm}^{k'}(\mu^{(n)})\big|_{\mathrm{st}} \to +\infty \quad \text{for any } k' \geq s+1 \ .$$

Therefore the thesis comes directly from (IV.4.3). For $m>3$, it is sufficient to consider the Riemannian product $(\mathsf{SU}(2)\times\mathbb{R}^{m-3}, g_{\mu^{(n)}}+g_{\mathrm{flat}})$, where $\mu^{(n)}$ is constructed as above choosing $s \geq \imath(m)+2$. □

Proof of Corollary IV.1.2. As in the proof of Theorem IV.1.1, we can reduce to the case $m = 3$. Fix an integer $k \geq \imath(3) + 4 = 5$ and consider the sequences

$$\varepsilon^{(n)} := \tfrac{1}{n^2} \ , \quad \delta^{(n)} := \tfrac{1}{n^{k+1}} \ .$$

By arguing as in the proof of Theorem IV.1.1, from Proposition IV.3.2 and Corollary IV.3.4 it comes that the sequence $\mu^{(n)} := \mu_\star(\varepsilon^{(n)}, \delta^{(n)})$ converges $(k-2)$-infinitesimally to μ_{o} and

$$\tfrac{1}{C} < \big|\operatorname{Rm}^{k-1}(\mu^{(n)})\big|_{\mathrm{st}} < C \ , \quad \big|\operatorname{Rm}^{k'}(\mu^{(n)})\big|_{\mathrm{st}} \to +\infty \quad \text{ for any } k' \geq k$$

for some $C > 1$. We can choose $R > 0$ big enough so that $|\sec(R \cdot \mu^{(n)})| \leq 1$, where the scaled bracket $R \cdot \mu^{(n)}$ is defined by (III.3.9). Letting $\tilde{\mu}^{(n)} := R \cdot \mu^{(n)}$ and $\tilde{\mu}_{\mathrm{o}} := R \cdot \mu_{\mathrm{o}}$, by Theorem III.1.4 and Corollary III.2.9, it follows that we can pass to a subsequence in such a way that $(\mathcal{B}_{\tilde{\mu}^{(n)}}, \hat{g}_{\tilde{\mu}^{(n)}})$ converges to $(\mathcal{B}_{\tilde{\mu}_{\mathrm{o}}}, \hat{g}_{\tilde{\mu}_{\mathrm{o}}})$ in the pointed $\mathcal{C}^{k,\alpha}$-topology for any $0 < \alpha < 1$. Finally, let us assume by contradiction that $(\mathcal{B}_{\tilde{\mu}^{(n)}}, \hat{g}_{\tilde{\mu}^{(n)}})$ admits a convergent subsequence in the pointed $\mathcal{C}^{k+1}$-topology. By Theorem III.1.4, this would imply that $\big|\operatorname{Rm}^{k-1}(\mu^{(n_j)})\big|_{\mathrm{st}} \to 0$ as $j \to +\infty$ for some $(n_j) \subset \mathbb{N}$, which is not possible. □

Proof of Corollary IV.1.3. Let us first prove that the moduli spaces $\mathcal{H}_1(1)$, $\mathcal{H}_2(1)$ are compact in the pointed $\mathcal{C}^\infty$-topology. For $m = 1$, $\mathcal{H}_1 = \mathcal{H}_{0,1}$ contains only the bracket $\mu = 0$ which correspond to the straight line $(\mathbb{R} = \mathbb{R}/\{0\}, dt^2)$. Therefore in this case the statement is trivially true. For $m = 2$, the moduli space $\mathcal{H}_2(1)$ decomposes as $\mathcal{H}_2(1) = \mathcal{H}_{0,2}(1) \cup \mathcal{H}_{1,2}(1)$. Then, by the well known classification of real 2-dimensional Lie algebras, we get

$$\mathcal{H}_{0,2}(1) = \{0\} \cup \{R \cdot \mu_{\mathrm{sol}} : R \in (0,1]\,\} \ ,$$

where the bracket $\mu = 0$ correspond to $(\mathbb{R}^2 = \mathbb{R}^2/\{0\}, g_{\mathrm{flat}})$, while $\mu_{\mathrm{sol}}(e_1, e_2) = e_1$ correspond to the solvmanifold presentation of the hyperbolic space $(\mathbb{R}H^2 = \mathsf{G}_{\mu_{\mathrm{sol}}}/\{0\}, g_{\mathrm{hyp}})$. On the other hand

$$\mathcal{H}_{1,2}(1) = \{\mu_0\} \cup \{R \cdot \mu_\epsilon : R \in (0,1]\,,\ \epsilon = \pm 1\} \ ,$$

where

$$\mu_0(e_0, e_1) = +e_2 \ , \quad \mu_0(e_0, e_2) = -e_2 \ , \quad \mu_0(e_1, e_2) = 0$$

corresponds to $(\mathbb{R}^2 = \mathsf{SE}(2)/\mathsf{SO}(2), g_{\text{flat}})$, while

$$\mu_\epsilon(e_0, e_1) = +e_2 \ , \quad \mu_\epsilon(e_0, e_2) = -e_2 \ , \quad \mu_\epsilon(e_1, e_2) = \epsilon\, e_0 \ , \qquad \text{with } \epsilon = \pm 1$$

correspond to $(S^2 = \mathsf{SO}(3)/\mathsf{SO}(2), g_{\text{round}})$ and $(\mathbb{R}H^2 = \mathsf{SL}(2,\mathbb{R})/\mathsf{SO}(2), g_{\text{hyp}})$, respectively. Therefore, $\mathcal{H}_2(1)$ is compact with respect to the topology of the algebraic convergence and hence the claim follows by Corollary III.1.5. Finally, to prove that $\mathcal{H}^{\text{loc}}_m(1)$ is not compact in the pointed $\mathcal{C}^3$-topology for any $m \geq 3$, one can reduce to the case $m = 3$ and applying again Proposition IV.3.2, Corollary IV.3.4 and Remark III.3.3 in order to construct a sequence $(\mu^{(n)}) \subset \mathcal{H}_3(1)$ such that $\big|\operatorname{Rm}^1(\mu^{(n)})\big|_{\text{st}} \to +\infty$ as $n \to +\infty$. □

IV.5 A note on the infinitesimal convergence

In this section, we recall the definition of infinitesimal convergence introduced by Lauret in [41] for sequences of homogeneous spaces and we prove that it is equivalent to the notion of infinitesimal convergence according to Definition III.3.9.

IV.5.1 Taylor expansion of real analytic Riemannian metrics

For reader's convenience, we give a detailed proof for the following known fact (see also [2, Prop E.III.7]).

Proposition IV.5.1. *For any integer $k \geq 0$, every partial derivative $\frac{\partial^{k+2} g_{ij}}{\partial^{q_1} x^1 \ldots \partial^{q_m} x^m}\big|_0$ of order $k+2$ of a real analytic Riemannian metric g in normal coordinates is expressible as a polynomial in the components of* $\operatorname{Rm}^0(g)|_0, \ldots, \operatorname{Rm}^k(g)|_0$.

To check this, consider a ball $B \subset\subset \mathbb{R}^m$ centered at the origin and denote by g a real analytic Riemannian metric on B. We also assume that the standard coordinates $(x^1, \ldots, x^m)$ of $\mathbb{R}^m$ are normal for g at 0. Then, there exists $\varepsilon > 0$ sufficiently small such that

$$g_{ij}(x) = \delta_{ij} + \sum_{k=0}^{\infty} \sum_{|q|=k+2} \frac{\partial^q g_{ij}(0)}{q!} x^q \qquad \text{for any } |x|_{\text{st}} < \varepsilon \quad ,$$

where for any multi-index $q = (q_1, \ldots, q_m)$ we denote by $q! := q_1! \ldots q_m!$, by $x^q := (x^1)^{q_1} \ldots (x^m)^{q_m}$ and by $\partial^q := \frac{\partial^{|q|}}{\partial^{q_1} x^1 \ldots \partial^{q_m} x^m}$.

Fix now $y \in B$ and consider the radial geodesic $\gamma(t) := ty$ together with the Jacobi vector field $J(t) := tw^i \frac{\partial}{\partial x^i}$ along $\gamma(t)$. Set also $f(t) := g_{\gamma(t)}(J(t), J(t))$ and, for any tensor field $A = A(t)$ along $\gamma(t)$, denote by $A^{\{k\}}$ the k-th covariant derivative $A^{\{k\}}(t) := (\nabla^g_t)^k A(t)$ along $\gamma(t)$. We recall that the Jacobi equation is

$$J^{\{2\}}(t) = R(t)\big(J^{\{0\}}(t)\big) \quad \text{with} \;\; R(t)(\,\cdot\,) := \,-\operatorname{Rm}(g)_{\gamma(t)}(\dot\gamma(t) \wedge (\,\cdot\,))\dot\gamma(t) \ .$$

By the Leibniz rule, we get

$$\begin{gathered} J^{\{0\}}(0) = 0 \ , \quad J^{\{1\}}(0) = w \ , \quad J^{\{2\}}(0) = 0 \ , \\ J^{\{k+2\}}(0) = P_k\big(R^{\{0\}}(0), \dots, R^{\{k-1\}}(0)\big)w \quad \text{for any integer } k \geq 1 \ , \end{gathered} \tag{IV.5.1}$$

where P_k are polynomials in k variables of $\deg(P_k) = \lfloor \frac{k+1}{2} \rfloor$ recursively defined by

$$\begin{gathered} P_1(a^0) = a^0 \ , \quad P_2(a^0, a^1) = 2a^1 \ , \\ P_k(a^0, \dots, a^{k-1}) = ka^{k-1} + \sum_{i=1}^{k-2} \binom{k}{i+2} a^{k-2-i} P_i(a^0, \dots, a^{i-1}) \ . \end{gathered}$$

Differentiating the function f, we get $f(0) = \dot f(0) = 0$, $\ddot f(0) = \langle w, w \rangle_{\mathrm{st}}$ and for any integer $k \geq 1$

$$\begin{aligned} f^{(2k+1)}(0) &= 2(2k+1)\big\langle J^{\{2k\}}(0), w\big\rangle_{\mathrm{st}} + \sum_{i=3}^{k} 2\binom{2k+1}{i}\big\langle J^{\{2k+1-i\}}(0), J^{\{i\}}(0)\big\rangle_{\mathrm{st}} \ , \\ f^{(2k+2)}(0) &= 2(2k+2)\big\langle J^{\{2k+1\}}(0), w\big\rangle_{\mathrm{st}} + \\ &\qquad + \sum_{i=3}^{k} 2\binom{2(k+1)}{i}\big\langle J^{\{2k+2-i\}}(0), J^{\{i\}}(0)\big\rangle_{\mathrm{st}} + \\ &\qquad\qquad + \binom{2(k+1)}{k+1}\big\langle J^{\{k+1\}}(0), J^{\{k+1\}}(0)\big\rangle_{\mathrm{st}} \ . \end{aligned} \tag{IV.5.2}$$

From (IV.5.1) and (IV.5.2) it follow that

$$f^{(k+4)}(0) = \sum_{ij} \sum_{|q|=k+2} \alpha^{[k]}_{ijq}\, y^q w^i w^j \quad \text{for any integer } k \geq 0 \ \ ,$$

where $\alpha^{[k]}_{ijq}$ are coefficients which depends polynomially only on the components of $\mathrm{Rm}^0(g)|_0, \ldots, \mathrm{Rm}^k(g)|_0$. Hence, for t sufficiently small

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k \\ &= \delta_{ij} w^i w^j t^2 + \sum_{k=0}^{\infty} \frac{f^{(k+4)}(0)}{(k+4)!} t^{k+4} \\ &= t^2 \left(\delta_{ij} + \sum_{k=0}^{\infty} \sum_{|q|=k+2} \frac{\alpha_{ijq}^{[k]}}{(k+4)!} y^q t^{k+2} \right) w^i w^j \ . \end{aligned}$$

Since $f(t) = t^2 g_{ij}(ty) w^i w^j$, we finally get

$$g_{ij}(x) = \delta_{ij} + \sum_{k=0}^{\infty} \sum_{|q|=k+2} \frac{\frac{q!}{(k+4)!} \alpha_{ijq}^{[k]}}{q!} x^q$$

and the thesis follows.

IV.5.2 The infinitesimal convergence in the sense of Lauret

A sequence $(\mu^{(n)}) \subset \mathcal{H}_m$ *converges infinitesimally to* $\mu^{(\infty)} \in \mathcal{H}_m$ *in the sense of Lauret* if there exists a sequence of smooth embeddings

$$\phi^{(n)} : \mathcal{B}_{g_{\mu^{(\infty)}}}\big(e_{\mu^{(\infty)}} \mathsf{H}_{\mu^{(\infty)}}, \varepsilon^{(n)}\big) \subset \mathsf{G}_{\mu^{(\infty)}} / \mathsf{H}_{\mu^{(\infty)}} \to \mathsf{G}_{\mu^{(n)}} / \mathsf{H}_{\mu^{(n)}} \ , \quad \text{with } \varepsilon^{(n)} \to 0^+$$

such that $\phi^{(n)}(e_{\mu^{(\infty)}} \mathsf{H}_{\mu^{(\infty)}}) = e_{\mu^{(n)}} \mathsf{H}_{\mu^{(n)}}$ and

$$\Big| \big((\nabla^{\mu})^k \big(\phi^{(n)*} g_{\mu^{(n)}} - g_{\mu^{(\infty)}} \big) \big)_{e_\mu \mathsf{H}_\mu} \Big|_{g_\mu} \to 0 \quad \text{as } n \to +\infty \ , \quad \text{for any } k \geq 0 \ . \tag{IV.5.3}$$

Fixing a system of local coordinates centered at $e_\mu \mathsf{H}_\mu$ and letting $g_{ij}^{(n)}$, $g_{ij}^{(\infty)}$ be the components of $\phi^{(n)*} g_{\mu^{(n)}}$ and $g_{\mu^{(\infty)}}$, respectively, it is easy to realize that (IV.5.3) is equivalent to require that

$$\partial^q g_{ij}^{(n)}(0) \to \partial^q g_{ij}^{(\infty)}(0) \quad \text{as } n \to +\infty \tag{IV.5.4}$$

for any multi-index q.

Proposition IV.5.2. *Let $(\mu^{(n)}) \subset \mathcal{H}_m$ be a sequence and $\mu^{(\infty)} \in \mathcal{H}_m$. Then $(\mu^{(n)})$ converges infinitesimally to $\mu^{(\infty)}$ in the sense of Lauret if and only if $(\mu^{(n)})$ converges infinitesimally to $\mu^{(\infty)}$ according to Definition III.3.9.*

Proof. Let $(B^m, \hat{g}_{\mu^{(n)}})$ and $(B^m, \hat{g}_{\mu^{(\infty)}})$ be the sequences of geometric models associated with $\mu^{(n)}$ and $\mu^{(\infty)}$, respectively. Here, we recall that $B^m := B_{\mathrm{st}}(0, \pi) \subset \mathbb{R}^m$. By IV.5.4, it comes that if $(\mu^{(n)})$ converges infinitesimally to $\mu^{(\infty)}$ in the sense of Lauret, then $\partial^q (\hat{g}_{\mu^{(n)}})_{\ell r}(0) \to \partial^q (\hat{g}_{\mu^{(\infty)}})_{ij}(0)$ as $n \to +\infty$, for any multi-index q, and so $(\mu^{(n)})$ converges infinitesimally to $\mu^{(\infty)}$ according to Definition III.3.9. On the other hand, if $(\mu^{(n)})$ converges infinitesimally to $\mu^{(\infty)}$ according to Definition III.3.9, then there exists a sequence of matrices $(a^{(n)}) \subset \mathsf{O}(m)$ such that $a^{(n)} \to I_m$ and

$$a^{(n)} \cdot \mathrm{Rm}^k(\mu^{(n)}) \to \mathrm{Rm}^k(\mu^{(\infty)}) \quad \text{as } n \to +\infty \ ,$$

for any integer $k \geq 0$, where $a^{(n)}$ acts by change of basis. Therefore, Proposition IV.5.1 implies that

$$(a^{(n)})^\ell_i (a^{(n)})^r_j \, \partial^q (\hat{g}_{\mu^{(n)}})_{\ell r}(0) \to \partial^q (\hat{g}_{\mu^{(\infty)}})_{ij}(0) \quad \text{as } n \to +\infty \ ,$$

for any multi-index q, and this completes the proof. $\square$

Chapter V

Diverging sequences of unit volume invariant metrics with bounded curvature

V.1 Statement of results

In this Chapter we study the space $\mathcal{M}_1^{\mathsf{G}}$ of G-invariant, unit volume metrics on a given compact, connected, almost-effective homogeneous space $M^m = \mathsf{G}/\mathsf{H}$. In particular, we focus on diverging sequences, i.e. which are not contained in any compact subset of $\mathcal{M}_1^{\mathsf{G}}$, and we prove some structure results for those which have bounded curvature.

In [12] the authors introduced the notion of 0-*Palais-Smale sequences*, that are sequences $(g^{(n)}) \subset \mathcal{M}_1^{\mathsf{G}}$ such that $\mathrm{scal}(g^{(n)}) \to 0$ and $\big|\operatorname{Ric}^{\mathrm{o}}(g^{(n)})\big|_{g^{(n)}} \to 0$. Here, $\operatorname{Ric}^{\mathrm{o}}(g^{(n)})$ is the traceless Ricci tensor of $g^{(n)}$ and $|\cdot|_{g^{(n)}}$ is the norm induced by $g^{(n)}$ on the tensor bundle over M. Moreover, they proved the following result: *if M admits a* 0*-Palais-Smale sequence, then there exists a closed, connected intermediate subgroup $\mathsf{H}^{\mathrm{o}} \subsetneq \mathsf{K}^{\mathrm{o}} \subset \mathsf{G}^{\mathrm{o}}$ such that the quotient $\mathsf{K}^{\mathrm{o}}/\mathsf{H}^{\mathrm{o}}$ is a torus.* (see [12, Thm 2.1]). Here, H^{o} and G^{o} denote the identity components of H and G, respectively. This last theorem is optimal if the isotropy group H is connected. In case H is disconnected, the authors conjectured that G/H is itself a homogeneous torus bundle (see [12, p. 697]).

Notice that a 0-Palais-Smale sequence $(g^{(n)})$ cannot have convergent subsequences if M is not a torus. This means that $(g^{(n)})$ goes off to infinity on the set $\mathcal{M}_1^{\mathsf{G}}$ and consequently we say that such sequences are *divergent*. Moreover, the Gap Theorem [11, Thm 4] implies that any sequence $(g^{(n)}) \subset \mathcal{M}_1^{\mathsf{G}}$ with $\mathrm{scal}(g^{(n)}) \to \delta \geq 0$ and $\big|\operatorname{Ric}^{\mathrm{o}}(g^{(n)})\big|_{g^{(n)}} \to 0$ has bounded curvature and hence, assuming that M is not a torus, 0-Palais-Smale sequences are special examples of diverging sequences with bounded curvature.

The first main result proved in this chapter is

Theorem V.1.1. *Let $M^m = \mathsf{G}/\mathsf{H}$ be a compact, connected homogenous space. If there exists a diverging sequence $(g^{(n)}) \subset \mathcal{M}_1^{\mathsf{G}}$ with bounded curvature, i.e. with $|\sec(g^{(n)})| \leq C$ for some constant $C > 0$, then there exists an intermediate closed subgroup $\mathsf{H} \subsetneq \mathsf{K} \subset \mathsf{G}$ such that the quotient K/H is a torus.*

We stress that the proof of Theorem V.1.1 is purely algebraic and constructive. In fact, we show that the sum of the eigenspaces associated to all the *shrinking eigenvalues* of any diverging sequence $(g^{(n)}) \subset \mathcal{M}_1^{\mathsf{G}}$ with bounded curvature is a reductive complement of $\mathfrak{h} = \mathrm{Lie}(\mathsf{H})$ into an intermediate $\mathrm{Ad}(\mathsf{H})$-invariant Lie subalgebra $\mathfrak{h} \subsetneq \mathfrak{l} \subsetneq \mathfrak{g} = \mathrm{Lie}(\mathsf{G})$, which uniquely detects a strictly intermediate Lie subgroup $\mathsf{H} \subsetneq \mathsf{L} \subsetneq \mathsf{G}$, possibly not closed, such that the quotient $\overline{\mathsf{L}}/\mathsf{H}$ is a torus. Clearly Theorem V.1.1 follows by setting $\mathsf{K} := \overline{\mathsf{L}}$. Actually, we know more about the structure of any such a sequence: $(g^{(n)})$ approaches asymptotically, in a precise sense, a submersion-type metric with respect to the (locally) homogeneous fibration $\mathsf{L}/\mathsf{H} \to \mathsf{G}/\mathsf{H} \to \mathsf{G}/\mathsf{L}$ whose fibers shrink as $n \to +\infty$. We refer to Theorem V.3.3 for more details.

Since 0-Palais-Smale sequences are, in particular, diverging sequences with bounded curvature and we require neither that the Lie groups H, G are connected, nor that the traceless Ricci goes to zero, Theorem V.1.1 generalizes [12, Thm 2.1]. We stress that this proves the previously mentioned conjecture in [12, p. 697]. On the other hand, we point out that [12, Thm 2.1] allows for changing the transitive group actions, while our Theorem V.1.1 does not.

Letting $\mathsf{N}_{\mathsf{G}}(\mathsf{H}^{\mathrm{o}})$ be the normalizer of H^{o} in G, from Theorem V.1.1 we immediately obtain the following

Corollary V.1.2. *If there exists no intermediate closed subgroup $\mathsf{H} \subsetneq \mathsf{K} \subset \mathsf{G}$ such that the quotient K/H is a torus, e.g. when $\mathrm{rank}(\mathsf{H}) = \mathrm{rank}(\mathsf{N}_{\mathsf{G}}(\mathsf{H}^{\mathrm{o}}))$, then any diverging 1-parameter family in $\mathcal{M}_1^{\mathsf{G}}$ has unbounded curvature. In particular,* *in such a case the scalar curvature functional satisfies the Palais-Smale condition on all of the space* $\mathcal{M}_1^{\mathsf{G}}$.

We remark that, again by means of the Gap Theorem [11, Thm 4], 0-Palais-Smale sequences get flatter and flatter as they go off to infinity. This last observation, together with the aim of providing an algebraic proof of the Palais-Smale condition for the functional scal (e.g. see [9, Sec 2] for an algebraic proof of the Bochner Theorem), brought us to study diverging sequences inside the subsets $(\mathcal{M}_1^{\mathsf{G}})_\varepsilon$, with $\varepsilon > 0$. The second main result proved in this chapter is

Theorem V.1.3. *Let* $M^m = \mathsf{G}/\mathsf{H}$ *be a compact, connected homogenous space and let* $\varepsilon > 0$. *Assume that there exists a diverging sequence* $(g^{(n)}) \subset (\mathcal{M}_1^{\mathsf{G}})_\varepsilon$ *with bounded curvature and let* K *be the intermediate closed subgroup determined by* $(g^{(n)})$ *as in Theorem V.1.1. Then, there exists a second intermediate closed subgroup* $\mathsf{K} \subsetneq \mathsf{K}' \subset \mathsf{G}$ *such that the quotient* K'/H *is not a torus.*

As above, the proof of Theorem V.1.3 is purely algebraic and constructive. In fact, we show that the sum of the eigenspaces associated to all the *generalized bounded eigenvalues* of any diverging sequence $(g^{(n)}) \subset (\mathcal{M}_1^{\mathsf{G}})_\varepsilon$ with bounded curvature is a reductive complement of $\mathfrak{h}$ into a second intermediate $\mathrm{Ad}(\mathsf{H})$-invariant Lie subalgebra $\mathfrak{h} \subsetneq \mathfrak{l} \subsetneq \mathfrak{l}' \subsetneq \mathfrak{g}$, which uniquely detects a strictly intermediate Lie subgroup $\mathsf{L} \subsetneq \mathsf{L}' \subsetneq \mathsf{G}$, possibly not closed, such that the quotient $\overline{\mathsf{L}'}/\mathsf{H}$ is not a torus. Again, Theorem V.1.3 follows by setting $\mathsf{K}' := \overline{\mathsf{L}'}$.

We also exhibit an example of a sequence of unit volume invariant metrics on the Stiefel manifold $V_3(\mathbb{R}^5) = \mathsf{SO}(5)/\mathsf{SO}(2)$ which diverges with bounded curvature and whose scalar curvature converges to a positive constant. In that case, referring to the notation above, the intermediate subgroups are $\mathsf{L} = \mathsf{K} = \mathsf{SO}(2)\times\mathsf{SO}(2)$ and $\mathsf{L}' = \mathsf{K}' = \mathsf{SO}(4)$. We stress here that, unlike the previous case, this example shows that a sequence $(g^{(n)}) \subset (\mathcal{M}_1^{\mathsf{G}})_\varepsilon$ which diverges with bounded curvature does not necessarily approach asymptotically a submersion-type metric with respect to the (locally) homogeneous fibration $\mathsf{L}'/\mathsf{H} \to \mathsf{G}/\mathsf{H} \to \mathsf{G}/\mathsf{L}'$ given by the bigger Lie subgroup L' (see Subsection V.3.2).

Finally, we relate our results on diverging sequences with bounded curvature to the algebraic convergence introduced in Chapter III. Of course algebraically collapsed sequences are necessarily divergent. Remarkably, the following weaker converse assertion follows from Theorem V.3.3.

Proposition V.1.4. *Let $M^m = \mathsf{G}/\mathsf{H}$ be a compact, connected homogenous space and suppose that $\pi_1(M)$ is finite. If $(g^{(n)}) \subset \mathcal{M}_1^{\mathsf{G}}$ is a diverging sequence with bounded curvature, then it is algebraically collapsed.*

Notice that Proposition V.1.4 is optimal. In fact, we provide an easy example of a sequence of unit volume invariant metrics on the product $S^1 \times S^2$ which diverges with bounded curvature whose associated sequence of brackets converges algebraically to $(\mathbb{R}^3, g_{\text{flat}})$.

V.2 H-subalgebras, submersion metrics and submersion directions

V.2.1 H-subalgebras

We consider a compact, connected and almost effective m-dimensional homogeneous space $M = \mathsf{G}/\mathsf{H}$, with G and H compact Lie groups, and a fixed $\mathrm{Ad}(\mathsf{G})$-invariant Euclidean inner product Q on the Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathsf{G})$. Notice that we call *Lie subgroup of* G any immersed submanifold of G which is also a subgroup. We refer to [7, 8] for what concerns H-subalgebras and submersion directions.

Since G is compact it is well known that $\mathfrak{g}$ is reductive, i.e. its radical coincides with its center $\mathfrak{z}(\mathfrak{g})$. We observe also that every Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is reductive itself. This last claim can be easily proved by noticing that restriction of Q to $\mathfrak{k}$ is an $\mathrm{Ad}(\mathsf{K}^o)$-invariant Euclidean inner product on $\mathfrak{k}$, where we indicated with K^o the connected Lie subgroup of G with Lie algebra $\mathfrak{k}$. Hence, any Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ splits as $\mathfrak{k} = [\mathfrak{k}, \mathfrak{k}] \oplus \mathfrak{z}(\mathfrak{k})$. We denote also by $\overline{\mathsf{K}^o}$ the closure of K^o in G, which is itself a Lie group, and by $\bar{\mathfrak{k}}$ its Lie algebra, which coincides with the Malcev-closure of $\mathfrak{k}$ in $\mathfrak{g}$ (see Definition I.4.3). Then, $\bar{\mathfrak{k}}$ is a compact subalgebra of $\mathfrak{g}$, possibly $\bar{\mathfrak{k}} = \mathfrak{g}$, and moreover $[\bar{\mathfrak{k}}, \bar{\mathfrak{k}}] = [\mathfrak{k}, \mathfrak{k}]$ by [58, Thm 3, p. 52].

Definition V.2.1. A H-*subalgebra of* $\mathfrak{g}$ is an $\mathrm{Ad}(\mathsf{H})$-invariant intermediate Lie subalgebra $\mathfrak{k}$ which lies properly between $\mathfrak{h} = \mathrm{Lie}(\mathsf{H})$ and $\mathfrak{g}$. An H-subalgebra $\mathfrak{k}$ is called *toral* if $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}$, *non-toral* if $[\mathfrak{k}, \mathfrak{k}] \not\subset \mathfrak{h}$.

Notice that if H is connected, then the condition of $\mathrm{Ad}(\mathsf{H})$-invariance in the definition above is redundant. However, in the general case proper intermediate subalgebras which are not $\mathrm{Ad}(\mathsf{H})$-invariant can occur.

Let us consider now an H-subalgebra $\mathfrak{k} \subset \mathfrak{g}$ and let K^{o} be the only connected Lie subgroup of G with Lie algebra $\mathrm{Lie}(\mathsf{K}^{\mathrm{o}}) = \mathfrak{k}$. Of course, if H is connected then $\mathsf{H} \subset \mathsf{K}^{\mathrm{o}}$. However, in general it only holds that the identity component of H stays in $\mathsf{H} \cap \mathsf{K}^{\mathrm{o}}$ and there is no need for the whole subgroup H to be contained in K^{o}. Anyway, we stress the following important fact.

Proposition V.2.2. *Let $\mathfrak{k}$ be an H-subalgebra of $\mathfrak{g}$ and K^{o} be the only connected Lie subgroup of G such that $\mathrm{Lie}(\mathsf{K}^{\mathrm{o}}) = \mathfrak{k}$. Then, the subgroup K generated by H and K^{o} is a Lie subgroup of G, not necessarily closed, with $\mathrm{Lie}(\mathsf{K}) = \mathfrak{k}$. Moreover, H is closed in K and the quotient K/H is connected. Finally, $\mathfrak{k}$ is toral if and only if $\overline{\mathsf{K}}/\mathsf{H}$ is a torus.*

Proof. Since $\mathfrak{k}$ is $\mathrm{Ad}(\mathsf{H})$-invariant, it follows that H normalizes K^{o}, i.e. $C(h)(\mathsf{K}^{\mathrm{o}}) \subset \mathsf{K}^{\mathrm{o}}$ for any $h \in \mathsf{H}$. Here, $C(\cdot)$ indicates the conjugation inside G. Therefore the subgroup $\mathsf{K} \subset \mathsf{G}$ generated by H and K^{o} coincides with the set $\mathsf{H}\mathsf{K}^{\mathrm{o}} = \{hk : h \in \mathsf{H}\,,\ k \in \mathsf{K}^{\mathrm{o}}\}$. Since

$$\mathsf{H}\mathsf{K}^{\mathrm{o}} \simeq (\mathsf{H}{\times}\mathsf{K}^{\mathrm{o}})/\,\mathsf{H} \cap \mathsf{K}^{\mathrm{o}}\ ,$$

where $\mathsf{H} \cap \mathsf{K}^{\mathrm{o}}$ acts freely on $\mathsf{H}{\times}\mathsf{K}^{\mathrm{o}}$ on the right by $(h,k) \cdot h' := (hh', (h')^{-1}k)$, it comes that K is a Lie subgroup of G which is closed if and only if K^{o} is closed in G. Since the identity component of H is contained in K^{o}, it follows that the identity component of K coincides with K^{o} and hence $\mathrm{Lie}(\mathsf{K}) = \mathfrak{k}$.

We notice now that K is Hausdorff and H is compact, hence H is necessarily closed in K. Moreover, by the Second Isomorphism Theorem we get $\mathsf{K}/\mathsf{K}^{\mathrm{o}} \simeq \mathsf{H}/(\mathsf{H} \cap \mathsf{K}^{\mathrm{o}})$ and hence K/H is connected.

Let us suppose now that $\mathfrak{k}$ is toral. We can also assume that K^{o} is closed in G. Otherwise, one can just reply the same argument as below by replacing $\mathfrak{k}$ with its Malcev-closure $\bar{\mathfrak{k}}$ inside $\mathfrak{g}$. We notice that by the Second Isomorphism Theorem $\mathsf{K}/\mathsf{H} \simeq \mathsf{K}^{\mathrm{o}}/(\mathsf{H} \cap \mathsf{K}^{\mathrm{o}})$ and that the subgroup $\mathsf{H} \cap \mathsf{K}^{\mathrm{o}}$ is normal in K^{o}. To prove this last claim, firstly we observe that it is straightforward to show that the commutator $[\mathsf{K}^{\mathrm{o}}, \mathsf{K}^{\mathrm{o}}]$ is connected. Therefore, since $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}$ it holds that $[\mathsf{K}^{\mathrm{o}}, \mathsf{K}^{\mathrm{o}}] \subset \mathsf{H} \cap \mathsf{K}^{\mathrm{o}}$ and hence $C(k)(h) = [k,h]h \in \mathsf{H} \cap \mathsf{K}^{\mathrm{o}}$ for any $k \in \mathsf{K}^{\mathrm{o}}$, $h \in \mathsf{H} \cap \mathsf{K}^{\mathrm{o}}$. This actually proves that K/H is a compact, connected Lie group. Finally, by using the fact that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}$, the Lie algebra $\mathfrak{k}$ splits as

$$\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{a}\ , \qquad \text{with } [\mathfrak{h}, \mathfrak{a}] = [\mathfrak{a}, \mathfrak{a}] = \{0\}$$

and therefore K/H is a torus. On the other hand, it is easy to check that if $\overline{\mathsf{K}}/\mathsf{H}$ is a torus, then $[\mathfrak{k},\mathfrak{k}]=[\bar{\mathfrak{k}},\bar{\mathfrak{k}}]\subset\mathfrak{h}$ and this completes the proof. □

From now on, we will always associate to any H-subalgebra $\mathfrak{k}\subset\mathfrak{g}$ the Lie subgroup $\mathsf{K}\subset\mathsf{G}$ defined as in Proposition V.2.2. If K is closed in G, then it gives rise to the homogeneous fibration $\mathsf{K}/\mathsf{H}\to\mathsf{G}/\mathsf{H}\to\mathsf{G}/\mathsf{K}$ whose standard fiber K/H, which is not almost-effective in general, is a torus if and only if $\mathfrak{k}$ is toral. If K is not closed in G, then we get a fibration of local factor spaces $\mathsf{K}/\mathsf{H}\to\mathsf{G}/\mathsf{H}\to\mathsf{G}/\mathsf{K}$ whose standard fiber K/H is a dense submanifold of $\overline{\mathsf{K}}/\mathsf{H}$, which is a torus if and only if $\mathfrak{k}$ is toral.

Any H-subalgebra $\mathfrak{k}$ determines an $\mathrm{Ad}(\mathsf{H})$-invariant Q-orthogonal decomposition

$$\mathfrak{g}=\underbrace{\mathfrak{h}+\overbrace{\mathfrak{m}_{\mathfrak{k}}}^{}}_{\mathfrak{k}}\!\!\overbrace{\,+\mathfrak{m}_{\mathfrak{k}}^{\perp}}^{\mathfrak{m}}\ ,\qquad \text{with } [\mathfrak{k},\mathfrak{m}_{\mathfrak{k}}^{\perp}]\subset\mathfrak{m}_{\mathfrak{k}}^{\perp}\ . \tag{V.2.1}$$

Since $\mathfrak{k}$ is reductive, $\mathfrak{k}$ is toral if and only if $\mathfrak{m}_{\mathfrak{k}}$ lies in the center of $\mathfrak{k}$, i.e. $[\mathfrak{h},\mathfrak{m}_{\mathfrak{k}}]=[\mathfrak{m}_{\mathfrak{k}},\mathfrak{m}_{\mathfrak{k}}]=\{0\}$. If $\mathfrak{k}$ is not compact, i.e. if the subgroup K is not closed in G, from the equality $[\bar{\mathfrak{k}},\bar{\mathfrak{k}}]=[\mathfrak{k},\mathfrak{k}]$ we get a finer $\mathrm{Ad}(\mathsf{H})$-invariant Q-orthogonal decomposition

$$\mathfrak{g}=\underbrace{\mathfrak{k}+\overbrace{\mathfrak{a}_{\mathfrak{k}}}^{}}_{\bar{\mathfrak{k}}}\!\!\overbrace{\,+\mathfrak{m}_{\bar{\mathfrak{k}}}^{\perp}}^{\mathfrak{m}_{\mathfrak{k}}^{\perp}}=\mathfrak{h}+\mathfrak{m}_{\bar{\mathfrak{k}}}+\mathfrak{m}_{\bar{\mathfrak{k}}}^{\perp}\ ,\qquad \text{with } [\mathfrak{k},\mathfrak{a}_{\mathfrak{k}}]=[\mathfrak{a}_{\mathfrak{k}},\mathfrak{a}_{\mathfrak{k}}]=\{0\}\ . \tag{V.2.2}$$

We remark also that any submodule of $\mathfrak{m}$ is $\mathrm{Ad}(\overline{\mathsf{K}})$-invariant if and only if is $\mathrm{Ad}(\mathsf{K})$-invariant.

Finally, if we suppose that the group G is semisimple, given any not necessarily compact toral H-subalgebra $\mathfrak{k}$, the following result holds.

Lemma V.2.3. *Let $\mathfrak{k}$ be an H-subalgebra of $\mathfrak{g}$. If G is semisimple and $\mathfrak{k}$ is toral, then $\mathfrak{k}$ is faithfully represented by its adjoint action on $\mathfrak{m}_{\mathfrak{k}}^{\perp}$.*

Proof. Since G is compact and $\overline{\mathsf{K}}$ is closed in G, the quotient $\mathsf{G}/\overline{\mathsf{K}}$ is a reductive homogeneous space. Let now N be the maximal normal subgroup of G contained in $\overline{\mathsf{K}}$ and $\mathfrak{n}:=\mathrm{Lie}(\mathsf{N})$. We consider also the Q-orthogonal decomposition $\mathfrak{n}=\mathfrak{n}_1+\mathfrak{n}_2$, with $\mathfrak{n}_1:=\mathfrak{h}\cap\mathfrak{n}$. Since $\mathfrak{n}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{n}\subset\bar{\mathfrak{k}}$, it follows that $[\mathfrak{n},\mathfrak{m}_{\bar{\mathfrak{k}}}^{\perp}]=\{0\}$. Moreover, since $\mathfrak{n}_2\subset\mathfrak{m}_{\bar{\mathfrak{k}}}$ and $\mathfrak{k}$ is toral, it holds that $[\mathfrak{n}_2,\mathfrak{h}]=[\mathfrak{n}_2,\mathfrak{m}_{\bar{\mathfrak{k}}}]=\{0\}$.

But then $\mathfrak{n}_2 \subset \mathfrak{z}(\mathfrak{g}) = \{0\}$ and so $\mathfrak{n} = \mathfrak{n}_1 \subset \mathfrak{h}$. Being G/H almost-effective by assumption, it follows that $\mathfrak{n} = \{0\}$ and so $\mathsf{G}/\overline{\mathsf{K}}$ is almost-effective. Hence, its isotropy representation is faithful (see e.g. [67, Cor 6.15]). But then

$$\{X \in \mathfrak{k} : [X, \mathfrak{m}_{\mathfrak{k}}^{\perp}] = \{0\}\} \subset \{X \in \bar{\mathfrak{k}} : [X, \mathfrak{m}_{\bar{\mathfrak{k}}}^{\perp}] = \{0\}\} = \{0\}$$

and so the claim follows. □

V.2.2 Submersion metrics and submersion directions

As a standard reference for what concerns Riemannian submersion, we refer to [3, Ch 9]. We recall here the following

Definition V.2.4. Let $\mathfrak{k} \subset \mathfrak{g}$ be an H-subalgebra. An invariant metric $g \in \mathcal{M}^{\mathsf{G}}$ is called $\mathfrak{k}$-*submersion metric* if $g(\mathfrak{m}_{\mathfrak{k}}, \mathfrak{m}_{\mathfrak{k}}^{\perp}) = \{0\}$ and its restriction on $\mathfrak{m}_{\mathfrak{k}}^{\perp} \otimes \mathfrak{m}_{\mathfrak{k}}^{\perp}$ is $\mathrm{Ad}(\mathsf{K})$-invariant. The set of all $\mathfrak{k}$-submersion metrics is denoted by $\mathcal{M}^{\mathsf{G}}(\mathfrak{k})$ and the set of unit volume $\mathfrak{k}$-submersion metrics is denoted by $\mathcal{M}^{\mathsf{G}}_1(\mathfrak{k}) := \mathcal{M}^{\mathsf{G}}_1 \cap \mathcal{M}^{\mathsf{G}}(\mathfrak{k})$.

This definition is due to the fact that, given an H-subalgebra $\mathfrak{k}$, any metric $g \in \mathcal{M}^{\mathsf{G}}(\mathfrak{k})$ gives rise to a (locally) homogeneous Riemannian submersion

$$\mathsf{K}/\mathsf{H} \to (\mathsf{G}/\mathsf{H}, g) \to (\mathsf{G}/\mathsf{K}, g|_{\mathfrak{m}_{\mathfrak{k}}^{\perp} \otimes \mathfrak{m}_{\mathfrak{k}}^{\perp}}) \ . \tag{V.2.3}$$

Moreover, by means of the following lemma, the submersion (V.2.3) has totally geodesic fibers.

Lemma V.2.5. *Let* $\mathfrak{k} \subset \mathfrak{g}$ *be an* H-*subalgebra and* $g \in \mathcal{M}^{\mathsf{G}}$. *If* $g(\mathfrak{m}_{\mathfrak{k}}, \mathfrak{m}_{\mathfrak{k}}^{\perp}){=}\{0\}$ *with respect to the decomposition* (V.2.1), *then* K/H *is totally geodesic in* $(\mathsf{G}/\mathsf{H}, g)$.

Proof. Let $X_1, X_2 \in \mathfrak{m}_{\mathfrak{k}}$ and $X_3 \in \mathfrak{m}_{\mathfrak{k}}^{\perp}$. Since by hypothesis $g(\mathfrak{m}_{\mathfrak{k}}, \mathfrak{m}_{\mathfrak{k}}^{\perp}) = \{0\}$, from [3, Lemma 7.27] we directly get that

$$\begin{aligned} 2g\big(\nabla^g_{X_1^*} X_2^*, X_3^*\big) &= g([X_1^*, X_2^*], X_3^*) + g([X_1^*, X_3^*], X_2^*) + g([X_2^*, X_3^*], X_1^*) \\ &= -g([X_1, X_2]_{\mathfrak{m}}, X_3) + g([X_3, X_1]_{\mathfrak{m}}, X_2) + g([X_3, X_2]_{\mathfrak{m}}, X_1) \\ &= 0 \ , \end{aligned}$$

where we indicated with $X^*_x := \frac{d}{dt} \exp(tX) \cdot x\big|_{t=0}$ the action vector field associated to $X \in \mathfrak{g}$, with ∇^g the Levi-Civita connection of g and we used the fact that $[X, Y]^* = -[X^*, Y^*]$ for any $X, Y \in \mathfrak{g}$. This is equivalent of saying that the second fundamental form of K/H in $(\mathsf{G}/\mathsf{H}, g)$ is identically zero, and so K/H is totally geodesic. □

Let now $\mathcal{M}_1^{\mathsf{G}}$ be the space of unit volume G-invariant metrics on $M = \mathsf{G}/\mathsf{H}$ and $\Sigma \subset T_{Q_{\mathfrak{m}}}\mathcal{M}_1^{\mathsf{G}}$ the unit tangent sphere defined in (I.5.7). Fix $v \in \Sigma$ and a good decomposition φ for v. Let also

$$\hat{v}_1 < \ldots < \hat{v}_{\ell_v}$$

be the distinct eigenvalues of v ordered by size, and let $I_1^v(\varphi), \ldots, I_{\ell_v}^v(\varphi) \subset I = \{1, \ldots, \ell\}$ be the index sets defined by the condition

$$v_i = \hat{v}_s \iff i \in I_s^v(\varphi) \qquad \text{for every } s \in \{1, \ldots, \ell_v\},\ i \in I \ . \tag{V.2.4}$$

Lemma V.2.6 ([7], Lemma 4.12 and Lemma 4.13)**.** *Let $v \in \Sigma$ and let φ be a good decomposition for v. Then $\ell_v > 1$ and there exists a constant $c = c(\mathsf{G}/\mathsf{H}) > 0$, which does not depend neither on v nor φ, such that $\hat{v}_1 < -c$ and $\hat{v}_{\ell_v} > c$. Furthermore, for any $1 \leq i, j, k \leq \ell_v$, the real number $[I_i^v(\varphi) I_j^v(\varphi) I_k^v(\varphi)]_\varphi$ does not depend on the choice of the good decomposition φ.*

From (I.5.25) it follows that the scalar curvature along the geodesic $\gamma_v(t)$ is

$$\operatorname{scal}(\gamma_v(t)) = \frac{1}{2}\sum_{i \in I} d_i b_i e^{-tv_i} - \frac{1}{4}\sum_{i,j,k \in I} [ijk]_\varphi e^{t(v_i - v_j - v_k)} \ . \tag{V.2.5}$$

We recall now the following definition, firstly introduced by Böhm.

Definition V.2.7 ([7], Def 5.11)**.** Let $\mathcal{S}^\Sigma$ denote the set of all $v \in \Sigma$ with the following property: if φ is any good decomposition for v, then for all $(i, j, k) \in I^3$ it holds that

$$[ijk]_\varphi > 0 \implies v_i - v_j - v_k + \hat{v}_1 \leq 0 \ . \tag{V.2.6}$$

Any element $v \in \mathcal{S}^\Sigma$ is called *submersion direction.*

Notice that (V.2.6) does not depend on the choice of the good decomposition φ for v. Moreover, submersion directions (or *non-negative directions*, as originally named by Böhm) have the following remarkable property, which comes directly from (V.2.6).

Proposition V.2.8 ([7], Lemma 5.16)**.** *Let $v \in \mathcal{S}^\Sigma$ and let φ be a good decomposition for v. Then*

$$[I_1^v(\varphi) I_{j_1}^v(\varphi) I_{j_2}^v(\varphi)]_\varphi = 0 \qquad \textit{for any } 1 \leq j_1 < j_2 \leq \ell_v \ . \tag{V.2.7}$$

In particular, $\mathfrak{k}_1 := \mathfrak{h} + \mathfrak{m}_{I_1^v(\varphi)}$ is an H-subalgebra.

This last proposition gives rise to a partition of the set $\mathcal{S}^\Sigma$ into the sets of $\mathfrak{k}_1$-*submersion directions*, which are defined by

$$\mathcal{S}^\Sigma(\mathfrak{k}_1) := \{v \in \mathcal{S}^\Sigma : \mathfrak{m}_{I_1^v(\varphi)} = \mathfrak{m}_{\mathfrak{k}_1} \text{ for any good decomposition } \varphi \text{ for } v\} \ , \tag{V.2.8}$$

for any H-subalgebra $\mathfrak{k}_1 \subset \mathfrak{g}$. As a direct generalization of (V.2.8), we are going to introduce a descending chains of subsets of $\mathcal{S}^\Sigma$, which will play a role in the next section. First, we define *flag of* H-*subalgebras* any ordered set $\zeta := (\mathfrak{k}_1, \dots, \mathfrak{k}_p)$ of H-subalgebras of $\mathfrak{g}$ such that $\mathfrak{k}_1 \subsetneq \dots \subsetneq \mathfrak{k}_p$. The *lenght of* ζ is the cardinality $|\zeta| = p$. Notice that, by Proposition V.2.2, any flag of H-subalgebras determines univocally a finite sequence of intermediate Lie subgroups $\mathsf{H} \subsetneq \mathsf{K}_1 \subsetneq \dots \subsetneq \mathsf{K}_p \subsetneq \mathsf{G}$.

Definition V.2.9. Let $\zeta := (\mathfrak{k}_1, \dots, \mathfrak{k}_p)$ be a flag of H-subalgebras. A unit tangent vector $v \in \Sigma$ is called ζ-*submersion direction* if it satisfies the following conditions for any good decomposition φ of v:

i) $\mathfrak{k}_1 = \mathfrak{h} + \mathfrak{m}_{I_1^v(\varphi)}\ ,\ \mathfrak{k}_2 = \mathfrak{k}_1 + \mathfrak{m}_{I_2^v(\varphi)}\ , \dots,\ \mathfrak{k}_p = \mathfrak{k}_{p-1} + \mathfrak{m}_{I_p^v(\varphi)}$;

ii) for any $1 \leq q \leq p$, for any $(i,j,k) \in \{q, \dots, \ell_v\}^3$ it holds

$$[I_i^v(\varphi) I_j^v(\varphi) I_k^v(\varphi)]_\varphi > 0 \implies \hat{v}_i - \hat{v}_j - \hat{v}_k + \hat{v}_q \leq 0 \ .$$

The set of all ζ-submersion directions is denoted by $\mathcal{S}^\Sigma(\zeta)$ or $\mathcal{S}^\Sigma(\mathfrak{k}_1, \dots, \mathfrak{k}_p)$, equivalently.

Given a flag of H-subalgebras $\zeta := (\mathfrak{k}_1, \dots, \mathfrak{k}_p)$, it follows from the very definition that

$$\mathcal{S}^\Sigma(\zeta) = \mathcal{S}^\Sigma(\mathfrak{k}_1, \dots, \mathfrak{k}_p) \subset \mathcal{S}^\Sigma(\mathfrak{k}_1, \dots, \mathfrak{k}_{p-1}) \subset \dots \subset \mathcal{S}^\Sigma(\mathfrak{k}_1, \mathfrak{k}_2) \subset \mathcal{S}^\Sigma(\mathfrak{k}_1) \ .$$

Furthermore, the set $\mathcal{S}^\Sigma(\zeta)$ of ζ-submersion directions is related with the notion of submersion type metrics by the following

Proposition V.2.10. *Let $\zeta = (\mathfrak{k}_1, \dots, \mathfrak{k}_p)$ be a flag of* H-*subalgebras. Then, it holds that*

$$\mathcal{S}^\Sigma(\zeta) \subset \mathcal{S}^\Sigma \cap T_{Q_\mathfrak{m}} \mathcal{M}_1^\mathsf{G}(\mathfrak{k}_q) \quad \text{for any } 1 \leq q \leq p \ , \tag{V.2.9}$$

i.e. $\gamma_v(t) \in \mathcal{M}_1^\mathsf{G}(\mathfrak{k}_q)$ for any $v \in \mathcal{S}^\Sigma(\zeta)$, for any $t > 0$, for any $1 \leq q \leq p$.

Proof. Let $v \in \mathcal{S}^\Sigma(\zeta)$ and φ be a good decomposition for v. Fix $1 \leq q \leq p$. We have to show that the submodule $\mathfrak{m}_{I_i^v(\varphi)}$ is $\mathrm{Ad}(\mathsf{K}_q)$-invariant for any $q \leq$

$i \leq \ell_v$. Since every submodule $\mathfrak{m}_{I_i^v(\varphi)}$ is $\mathrm{Ad}(\mathsf{H})$-invariant, it follows from the very definition of K_q (see Proposition V.2.2) that it is sufficient to show that $\mathfrak{m}_{I_i^v(\varphi)}$ is $\mathrm{ad}(\mathfrak{k}_q)$-invariant for any $q \leq i \leq \ell_v$. We already know from (V.2.1) that $[\mathfrak{k}_q, \mathfrak{m}_{\mathfrak{k}_q}^{\perp}] \subset \mathfrak{m}_{\mathfrak{k}_q}^{\perp}$. From condition (ii) in Definition V.2.9, we get

$$[I_q^v(\varphi) I_{j_1}^v(\varphi) I_{j_2}^v(\varphi)] = 0 \qquad \text{for any } q \leq j_1 < j_2 \leq \ell_v \ .$$

In particular, $Q\big([\mathfrak{m}_{\mathfrak{k}_q}, \mathfrak{m}_{I_i^v(\varphi)}], \mathfrak{m}_{I_j^v(\varphi)}\big) = 0$ for any $q < i, j \leq \ell_v$, $i \neq j$. So, we can conclude that $[\mathfrak{m}_{\mathfrak{k}_q}, \mathfrak{m}_{I_i^v(\varphi)}] \subset \mathfrak{m}_{I_i^v(\varphi)}$ for any $q < i \leq \ell_v$. □

By means of Proposition V.2.10, the following geometric interpretation for the set of submersion directions arises. Given an element $v \in \mathcal{S}^{\Sigma}(\mathfrak{k}_1)$, moving along the geodesic $\gamma_v(t)$ is equivalent to shrinking the fibers of the (locally) homogeneous Riemannian submersion associated to $\mathfrak{k}_1$ as in V.2.3 and to rescaling the base space, while the volume is keeped fixed.

The set $\mathcal{S}^{\Sigma} \subset \Sigma$ of submersion directions has originally raised from the study of the scalar curvature functional $\mathrm{scal} : \mathcal{M}_1^{\mathsf{G}} \to \mathbb{R}$, aimed to get results of existence and non-existence for homogeneous Einstein metrics (see e.g. [87, 7]). It turns out that it plays a crucial role in studying the asymptotic behavior of the curvature tensor along geodesic rays γ_v. More concretely

Theorem V.2.11. *Let $v \in \Sigma$ and γ_v the corresponding geodesic ray in $\mathcal{M}_1^G$.*
a) [7, Thm 5.18] If $v \in \Sigma \setminus \mathcal{S}^{\Sigma}$, then

$$\lim_{t \to +\infty} \mathrm{scal}(\gamma_v(t)) \to -\infty \ .$$

b) If $v \in \mathcal{S}^{\Sigma}(\mathfrak{k}_1)$ for a non-toral H-subalgebra $\mathfrak{k}_1 \subset \mathfrak{g}$, then

$$\lim_{t \to +\infty} \big|\,\mathrm{Ric}(\gamma_v(t))\big|_{\gamma_v(t)} \to +\infty \ .$$

Proof. Fix $v \in \Sigma$ and a good decomposition φ for v. If $v \in \Sigma \backslash \mathcal{S}^{\Sigma}$, then there exists $\varepsilon > 0$ and a triple $(i_{\mathrm{o}}, j_{\mathrm{o}}, k_{\mathrm{o}}) \in I^3$ such that $[i_{\mathrm{o}} j_{\mathrm{o}} k_{\mathrm{o}}]_{\varphi} > \varepsilon$ and $v_{i_{\mathrm{o}}} - v_{j_{\mathrm{o}}} - v_{k_{\mathrm{o}}} + \hat{v}_1 > \varepsilon$. Since $\hat{v}_1 < 0$ by Lemma V.2.6, from (V.2.5) we get

$$\mathrm{scal}(\gamma_v(t)) < \tfrac{1}{2}\big(b_{\mathsf{G/H}} - \varepsilon e^{t\varepsilon}\big) e^{-t\hat{v}_1} \to -\infty$$

and this completes the proof of the first claim.

Let now $\mathfrak{k}_1$ be a non-toral H-subalgebra of $\mathfrak{g}$ and suppose that $v \in \mathcal{S}^{\Sigma}(\mathfrak{k}_1)$. Then, if $i \in I_1^v(\varphi)$, for any $j,k \in I$ it follows from (V.2.7) that

$$\begin{gathered}[ijk]_{\varphi}\big(1-e^{t(v_k-v_j)}\big)=0 \text{ for any } t>0\ , \\ [ijk]_{\varphi}>0 \text{ only if } j,k \in I_s^v(\varphi) \text{ for some } 1 \leq s \leq \ell_v\ .\end{gathered} \tag{V.2.10}$$

So, for any $i \in I_1^v(\varphi)$, from (I.5.21) we get

$$\begin{aligned}\mathrm{ric}_i(\gamma_v(t)) &= \frac{b_i}{2}e^{-tv_i}-\frac{1}{2d_i}\sum_{j,k\in I}[ijk]_{\varphi}e^{t(v_k-v_i-v_j)}+\frac{1}{4d_i}\sum_{j,k\in I}[ijk]_{\varphi}e^{t(v_i-v_j-v_k)} \\ &\overset{(\mathrm{I.5.13})}{=} \Big(c_i+\frac{1}{2d_i}\sum_{j,k\in I}[ijk]_{\varphi}\Big)e^{-t\hat{v}_1}-\frac{1}{2d_i}e^{-t\hat{v}_1}\sum_{j,k\in I}[ijk]_{\varphi}e^{t(v_k-v_j)}+ \\ &\qquad\qquad +\frac{1}{4d_i}e^{t\hat{v}_1}\sum_{j,k\in I}[ijk]_{\varphi}e^{-t(v_j+v_k)} \\ &\overset{(\mathrm{V.2.10})}{=} c_ie^{-t\hat{v}_1}+\frac{1}{4d_i}e^{t\hat{v}_1}\sum_{\substack{j,k\in I_s^v(\varphi)\\ 1\leq s\leq \ell_v}}[ijk]_{\varphi}e^{-2t\hat{v}_s} \\ &= \frac{1}{2d_i}\Big(2d_ic_i+\frac{1}{2}\sum_{j,k\in I_1^v(\varphi)}[ijk]_{\varphi}\Big)e^{-t\hat{v}_1}+\frac{1}{4d_i}\sum_{\substack{j,k\in I_s^v(\varphi)\\ 2\leq s\leq \ell_v}}[ijk]_{\varphi}e^{-t(2\hat{v}_s-\hat{v}_1)}\ .\end{aligned}$$

Since $\mathfrak{k}_1$ is non toral, there exists $i_{\mathrm{o}} \in I_1^v(\varphi)$ such that

$$2d_{i_{\mathrm{o}}}c_{i_{\mathrm{o}}}+\frac{1}{2}\sum_{j,k\in I_1^v(\varphi)}[i_{\mathrm{o}}jk]_{\varphi}>0$$

and so the second claim follows. □

Remark V.2.12. To prove the second claim, it is possible to argue also like this. Let $v \in \mathcal{S}^{\Sigma}(\mathfrak{k}_1)$ for a given non-toral H-subalgebra $\mathfrak{k}_1$ and $\varphi \in \mathcal{F}^{\mathsf{G}}$ be a good decomposition for v. Since $\gamma_v(t)|_{\mathsf{K}_1/\mathsf{H}} = e^{t\hat{v}_1}Q_{I_1^v(\varphi)}$ and $\hat{v}_1 < 0$, it follows that the intrinsic sectional curvature of K_1/H blows up as $t \to +\infty$. Moreover, from Lemma V.2.5 and Proposition V.2.10, we know that K_1/H is totally geodesic in $(\mathsf{G}/\mathsf{H},\gamma_v(t))$ for any $t>0$ and so also its extrinsic sectional curvature blows up. Then, claim (b) follows directly from [11, Thm 4].

As a consequence of Theorem V.2.11, the only way of *reaching the boundary of the space* $\mathcal{M}_1^{\mathsf{G}}$, moving along a geodesic γ_v while keeping the curvature bounded, is to choose $v \in \mathcal{S}^{\Sigma}(\mathfrak{k}_1)$ for some toral H-subalgebra $\mathfrak{k}_1 \subset \mathfrak{g}$. By the way, we stress the fact that this last condition is far form being sufficient.

Example V.2.13 (Berger spheres)**.** Let $M = \mathsf{G} = \mathsf{SU}(2)$. Consider the $\mathrm{Ad}(\mathsf{SU}(2))$-invariant inner product $Q(A_1, A_2) := -\frac{1}{2}\operatorname{Tr}(A_1{\cdot}A_2)$ on $\mathfrak{su}(2)$, the standard Q-orthonormal basis (X_0, X_1, X_2) such that

$$[X_0, X_1] = -2X_2 \ , \quad [X_1, X_2] = -2X_0 \ , \quad [X_2, X_0] = -2X_1$$

and set $\mathfrak{k} := \mathrm{span}(X_0)$. By means of (I.5.7) and (V.2.7), it is easy to check that $\mathcal{S}^{\Sigma}(\mathfrak{k}) = \{\bar{v}\}$, where the tangent direction $\bar{v}$ is given by

$$\bar{v} = \begin{pmatrix} -\frac{\sqrt{6}}{3} & & \\ & \frac{\sqrt{6}}{6} & \\ & & \frac{\sqrt{6}}{6} \end{pmatrix} .$$

Let us indicate now with $\big(X_0(t) := e^{\frac{\sqrt{6}}{6}t}X_0, X_1(t) := e^{-\frac{\sqrt{6}}{12}t}X_1, X_2(t) := e^{-\frac{\sqrt{6}}{12}t}X_2\big)$ the $\gamma_{\bar{v}}(t)$-orthonormal basis for $\mathfrak{su}(2)$ obtained by normalizing (X_0, X_1, X_2). Then, one can directly check that the curvature tensor

$$\mathrm{Rm}(\gamma_{\bar{v}}(t)) : \mathfrak{su}(2) \wedge \mathfrak{su}(2) \to \mathfrak{su}(2) \wedge \mathfrak{su}(2)$$

is diagonal and explicitly given by

$$\begin{aligned} \mathrm{Rm}(\gamma_{\bar{v}}(t))(X_1(t){\wedge}X_2(t)) &= e^{-\frac{2}{3}\sqrt{6}t}X_1(t){\wedge}X_2(t) \ , \\ \mathrm{Rm}(\gamma_{\bar{v}}(t))(X_1(t){\wedge}X_3(t)) &= e^{-\frac{2}{3}\sqrt{6}t}X_1(t){\wedge}X_3(t) \ , \\ \mathrm{Rm}(\gamma_{\bar{v}}(t))(X_2(t){\wedge}X_3(t)) &= \Big(4e^{-\frac{\sqrt{6}}{6}t} - 3e^{-\frac{2}{3}\sqrt{6}t}\Big)X_2(t){\wedge}X_3(t) \ . \end{aligned}$$

Hence, we conclude that $\lim_{t\to+\infty} \big|\mathrm{Rm}(\gamma_{\bar{v}}(t))\big|_{\gamma_{\bar{v}}(t)} = 0$. Notice that $(\mathsf{SU}(2), \gamma_{\bar{v}}(t))$ is a Berger sphere for any $t \geq 0$ (see Section IV.3.1).

V.3 Proofs of Theorem V.1.1 and Theorem V.1.3

V.3.1 Main results

Let us consider a sequence $(g^{(n)}) \subset \mathcal{M}_1^{\mathsf{G}}$. Then, for every $n \in \mathbb{N}$ there exist $v^{(n)} \in \Sigma$ and $t^{(n)} > 0$, univocally determined, such that $g^{(n)} = \gamma_{v^{(n)}}(t^{(n)})$. Since

Σ is compact, there exist a sequence $(n_i) \subset \mathbb{N}$ and a direction $v^{(\infty)} \in \Sigma$ such that $v^{(n_i)} \to v^{(\infty)}$. For the sake of simplicity, in this section we will assume that the whole sequence $(v^{(n)})$ converges to some $v^{(\infty)} \in \Sigma$, which we call *limit direction of* $(g^{(n)})$. We also say that $(g^{(n)})$ is *divergent* if $t^{(n)} \to +\infty$.

For any $n \in \mathbb{N}$ we choose a good decomposition $\varphi^{(n)} = (\mathfrak{m}_1^{(n)}, \dots, \mathfrak{m}_\ell^{(n)})$ of $\mathfrak{m}$ for $v^{(n)}$, so that

$$g^{(n)} = \lambda_1^{(n)} Q_{\mathfrak{m}_1^{(n)}} + \ldots + \lambda_\ell^{(n)} Q_{\mathfrak{m}_\ell^{(n)}} \ , \quad \text{with } \ \lambda_i^{(n)} := e^{t^{(n)} v_i^{(n)}} \ . \tag{V.3.1}$$

Since $v^{(n)} \to v^{(\infty)}$, we can suppose that the sequence $(\varphi^{(n)}) \subset \mathcal{F}^{\mathsf{G}}$ converges as $n \to +\infty$ to a good decomposition $\varphi^{(\infty)} = (\mathfrak{m}_1^{(\infty)}, \dots, \mathfrak{m}_\ell^{(\infty)})$ for the limit direction $v^{(\infty)}$ of $(g^{(n)})$. For simplicity of notation, since we do not need to specify the particular choice of $\varphi^{(n)}$ and $\varphi^{(\infty)}$, we will write $[ijk]^{(n)}$ and $[ijk]^{(\infty)}$ instead of $[ijk]_{\varphi^{(n)}}$ and $[ijk]_{\varphi^{(\infty)}}$, respectively. Being the map $\varphi \mapsto [ijk]_\varphi$ continuous, it holds that $[ijk]^{(n)} \to [ijk]^{(\infty)}$ as $n \to +\infty$. Furthermore, the coefficients introduced in (I.5.11) and (I.5.10) will be indicated by $b_i^{(n)}$, $c_i^{(n)}$ when they refer to the decomposition $\varphi^{(n)}$ and by $b_i^{(\infty)}$, $c_i^{(\infty)}$ when they refer to the decomposition $\varphi^{(\infty)}$, respectively. Again, it holds that $b_i^{(n)} \to b_i^{(\infty)}$ and $c_i^{(n)} \to c_i^{(\infty)}$ as $n \to +\infty$.

From now on, up to passing to a subsequence we will always assume that the decompositions $\varphi^{(n)}$ are ordered in such a way that

$$v_1^{(n)} \leq v_2^{(n)} \leq \ldots \leq v_\ell^{(n)} \quad \text{ for any } n \in \mathbb{N} \ . \tag{V.3.2}$$

For simplicity of notation, we set $I := \{1, \dots, \ell\}$, $I_s^{(\infty)} := I_s^{v^{(\infty)}}(\varphi^{(\infty)})$ for any $1 \leq s \leq \ell_{v^{(\infty)}}$ and we define the map $r : \{0, \dots, \ell_{v^{(\infty)}}\} \to \{0, \dots, \ell\}$ by imposing the conditions

$$r(0) := 0 \ , \quad I_s^{(\infty)} = \{r(s-1)+1, \dots, r(s)\} \quad \text{ for any } 1 \leq s \leq \ell_{v^{(\infty)}} \ . \tag{V.3.3}$$

Moreover, we set $I_{\geq q}^{(\infty)} := \bigcup_{s=q}^{\ell_{v^{(\infty)}}} I_s^{(\infty)}$. Let us fix for each $n \in \mathbb{N}$ a $Q_{\mathfrak{m}}$-orthonormal $\varphi^{(n)}$-adapted basis $\big(e_\alpha^{(n)}\big)$ for $\mathfrak{m}$. Since $v^{(n)} \to v^{(\infty)}$ we can suppose that there exists a $Q_{\mathfrak{m}}$-orthonormal $\varphi^{(\infty)}$-adapted basis $\big(e_\alpha^{(\infty)}\big)$ for $\mathfrak{m}$ such that $e_\alpha^{(n)} \to e_\alpha^{(\infty)}$

as $n \to +\infty$. For the sake of shortness we set

$$\sec_i\big(g^{(n)}\big) := \sum_{e^{(n)}_\alpha, e^{(n)}_{\alpha'} \in \mathfrak{m}^{(n)}_i} \sec(g^{(n)})(e^{(n)}_\alpha \wedge e^{(n)}_{\alpha'}) \quad \text{for any } i \in I \ , \tag{V.3.4}$$

$$\sec_{ij}\big(g^{(n)}\big) := \sum_{\substack{e^{(n)}_\alpha \in \mathfrak{m}^{(n)}_i \\ e^{(n)}_\beta \in \mathfrak{m}^{(n)}_j}} \sec(g^{(n)})(e^{(n)}_\alpha \wedge e^{(n)}_\beta) \quad \text{for any } i,j \in I,\ i<j \ . \tag{V.3.5}$$

From (I.5.9), (I.5.14) and (I.5.23) we obtain

$$\begin{aligned}
\sec_i\big(g^{(n)}\big) = \sum_{e^{(n)}_\alpha, e^{(n)}_{\alpha'} \in \mathfrak{m}^{(n)}_i} &\bigg\{ \big|[e^{(n)}_\alpha, e^{(n)}_{\alpha'}]_{\mathfrak{h}}\big|^2_Q + \frac{1}{4}\big|[e^{(n)}_\alpha, e^{(n)}_{\alpha'}]_{\mathfrak{m}^{(n)}_i}\big|^2_Q + \\
&+ \sum_{k \in I\setminus\{i\}} \big|[e^{(n)}_\alpha, e^{(n)}_{\alpha'}]_{\mathfrak{m}^{(n)}_k}\big|^2_Q - \frac{3}{4}\sum_{k \in I\setminus\{i\}} \big|[e^{(n)}_\alpha, e^{(n)}_{\alpha'}]_{\mathfrak{m}^{(n)}_k}\big|^2_Q \frac{\lambda^{(n)}_k}{\lambda^{(n)}_i} \bigg\} \frac{1}{\lambda^{(n)}_i} \\
&= \bigg(d_i c^{(n)}_i + \frac{1}{4}[iii]^{(n)} + \sum_{k \in I\setminus\{i\}} [iik]^{(n)} - \frac{3}{4}\sum_{k \in I\setminus\{i\}} [iik]^{(n)} \frac{\lambda^{(n)}_k}{\lambda^{(n)}_i} \bigg) \frac{1}{\lambda^{(n)}_i} \ .
\end{aligned} \tag{V.3.6}$$

Moreover, from (I.5.9) and (I.5.15) we get

$$\begin{aligned}
&\sec_{ij}\big(g^{(n)}\big) = \\
&= \sum_{\substack{e^{(n)}_\alpha \in \mathfrak{m}^{(n)}_i \\ e^{(n)}_\beta \in \mathfrak{m}^{(n)}_j}} \bigg\{ \sum_{k \in I} \big|[e^{(n)}_\alpha, e^{(n)}_\beta]_{\mathfrak{m}^{(n)}_k}\big|^2_Q \frac{\big(\lambda^{(n)}_i\big)^2 + \big(\lambda^{(n)}_j - \lambda^{(n)}_k\big)\big(-2\lambda^{(n)}_i + \lambda^{(n)}_j + 3\lambda^{(n)}_k\big)}{4\lambda^{(n)}_i\lambda^{(n)}_j\lambda^{(n)}_k} \bigg\} \\
&= \frac{1}{4}\sum_{k \in I} [ijk]^{(n)} \frac{\lambda^{(n)}_i}{\lambda^{(n)}_j\lambda^{(n)}_k} + \frac{1}{4}\sum_{k \in I}[ijk]^{(n)} \Big(\frac{\lambda^{(n)}_j}{\lambda^{(n)}_k} - 1\Big)\Big(-2\frac{\lambda^{(n)}_i}{\lambda^{(n)}_j} + 1 + 3\frac{\lambda^{(n)}_k}{\lambda^{(n)}_j}\Big)\frac{1}{\lambda^{(n)}_i} \ .
\end{aligned} \tag{V.3.7}$$

Up to passing to a subsequence we assume that each coefficient $\lambda^{(n)}_i$ is monotonic. Moreover, we introduce the following notation

$$p^{(n)}_{ij} := \frac{\lambda^{(n)}_i}{\lambda^{(n)}_j} \tag{V.3.8}$$

and, up to passing to a further subsequence, we assume that the limits $p^{(\infty)}_{ij} := \lim_n p^{(n)}_{ij} \in [0, +\infty]$ do exist. Moreover, we define

$$a^{(n)}_{ijk} := [ijk]^{(n)}\big(p^{(n)}_{jk} - 1\big)\big(-2p^{(n)}_{ij} + 1 + 3p^{(n)}_{kj}\big) \tag{V.3.9}$$

and we set $a^{(\infty)}_{ijk} := \lim_n a^{(n)}_{ijk} \in \mathbb{R} \cup \{\pm\infty\}$ whenever it exists.

The next theorem is an intermediate result, which will be crucial in the proof of Theorem V.3.3. Nonetheless, we stress that it would be enough for proving Theorem V.1.1.

Theorem V.3.1. *Let us assume that $(g^{(n)}) \subset \mathcal{M}^{\mathsf{G}}_1$ is divergent and has bounded curvature. Then, $v^{(\infty)} \in \mathcal{S}^{\Sigma}(\mathfrak{k}_1)$ for some toral H-subalgebra $\mathfrak{k}_1$. Moreover, the following necessary conditions hold.*

A) *For any $i \leq j \leq k$ such that $i \in I^{(\infty)}_1$, we have*

$$[ijk]^{(\infty)} = 0 \quad \Longrightarrow \quad \lim_{n \to +\infty} [ijk]^{(n)} p^{(n)}_{kj} = 0 \ .$$

B) *For any $j, k \in I$ we have*

$$[I^{(\infty)}_1 jk]^{(\infty)} > 0 \quad \Longrightarrow \quad p^{(\infty)}_{kj} = 1 \ .$$

Proof. From (V.2.5) it follows that

$$\begin{aligned}
\operatorname{scal}(g^{(n)}) &= \frac{1}{2} \sum_{i \in I} d_i b^{(n)}_i e^{-t^{(n)} v^{(n)}_i} - \frac{1}{4} \sum_{i,j,k \in I} [ijk]^{(n)} e^{t^{(n)}(v^{(n)}_i - v^{(n)}_j - v^{(n)}_k)} \\
&\leq \frac{1}{4} \Big(2 b_{\mathsf{G}/\mathsf{H}} - \sum_{i,j,k \in I} [ijk]^{(n)} e^{t^{(n)}(v^{(n)}_i - v^{(n)}_j - v^{(n)}_k + v^{(n)}_1)} \Big) e^{-t^{(n)} v^{(n)}_1} \ ,
\end{aligned}$$

where $b_{\mathsf{G}/\mathsf{H}}$ is defined in (I.5.12). Since by assumption $\operatorname{scal}(g^{(n)})$ is bounded from below, there exists a constant $C > 0$ such that

$$\sum_{i,j,k \in I} [ijk]^{(n)} e^{t^{(n)}(v^{(n)}_i - v^{(n)}_j - v^{(n)}_k + v^{(n)}_1)} \leq C \quad \text{for any } n \in \mathbb{N} \ . \tag{V.3.10}$$

We observe also that if $v^{(\infty)} \in \Sigma \setminus \mathcal{S}^{\Sigma}$, then (V.3.10) is never satisfied. In fact, in that case we can fix $\varepsilon > 0$ and a triple $(i_o, j_o, k_o) \in I^3$ such that $[i_o j_o k_o]^{(n)} > \varepsilon$ and $v^{(n)}_{i_o} - v^{(n)}_{j_o} - v^{(n)}_{k_o} + v^{(n)}_1 > \varepsilon$, so that

$$[i_o j_o k_o]^{(n)} e^{t^{(n)}(v^{(n)}_{i_o} - v^{(n)}_{j_o} - v^{(n)}_{k_o} + v^{(n)}_1)} > \varepsilon e^{t^{(n)} \varepsilon} \to +\infty \ .$$

Then, it holds that $v^{(\infty)} \in \mathcal{S}^{\Sigma}(\mathfrak{k}_1)$ with $\mathfrak{k}_1 := \mathfrak{h} + \mathfrak{m}_{I^{(\infty)}_1}$ (see Proposition V.2.8). Since by assumption the sectional curvature is bounded, using (V.3.6) and (V.3.7), for any $i, j \in I$ such that $i \in I_1^{(\infty)}$, $i < j$ we get

$$\lambda_i^{(n)} \sec_i(g^{(n)}) = d_i c_i^{(n)} + \frac{1}{4}[iii]^{(n)} + \sum_{k \in I \setminus \{i\}} [iik]^{(n)} - \frac{3}{4} \sum_{k \in I \setminus \{i\}} [iik]^{(n)} p_{ki}^{(n)} \longrightarrow 0 \ , \tag{V.3.11}$$

$$4\lambda_i^{(n)} \sec_{ij}(g^{(n)}) = \sum_{k \in I} \big([ijk]^{(n)} p_{ik}^{(n)} p_{ij}^{(n)} + a_{ijk}^{(n)} \big) \longrightarrow 0 \tag{V.3.12}$$

as $n \to +\infty$, where $\sec_i(g^{(n)})$, $\sec_{ij}(g^{(n)})$ were defined in (V.3.4), (V.3.5), respectively, and the coefficients $p_{ij}^{(n)}$, $a_{ijk}^{(n)}$ were introduced in (V.3.8), (V.3.9), respectively.

Step 1. We are going to apply (V.3.12) by restricting ourselves to the case $j \in I_{\geq 2}^{(\infty)}$. At first we notice that, since $i \leq r(1) < j$, for any $k \in I$ we have

$$2v_i^{(n)} - v_k^{(n)} - v_j^{(n)} \longrightarrow 2\hat{v}_1^{(\infty)} - v_k^{(\infty)} - v_j^{(\infty)} \leq \hat{v}_1^{(\infty)} - \hat{v}_2^{(\infty)} < 0 \ ,$$

where $\hat{v}_i^{(\infty)}$ are the distinct eigenvalues of $v^{(\infty)}$ ordered by size, and so

$$\lim_{n \to +\infty} [ijk]^{(n)} p_{ik}^{(n)} p_{ij}^{(n)} = 0 \quad \text{for any } i, j, k \in I \text{ such that } i \in I_1^{(\infty)},\ j \in I_{\geq 2}^{(\infty)} \ . \tag{V.3.13}$$

Therefore, from (V.3.12) and (V.3.13) we obtain for any fixed $j \in I_{\geq 2}^{(\infty)}$

$$\lim_{n \to +\infty} \Big\{ \sum_{k \in I} a_{ijk}^{(n)} \Big\} = 0 \ , \qquad \text{for any } i \in I_1^{(\infty)} \ . \tag{$\star_j$}$$

Notice that, under the assumption $i \in I_1^{(\infty)}$ and $j \in I_{\geq 2}^{(\infty)}$, it comes $p_{ij}^{(\infty)} = 0$ and so from (V.3.9) we directly get the following implications:

$$\begin{aligned}
p_{jk}^{(\infty)} = +\infty \quad &\Longrightarrow \quad a_{ijk}^{(n)} \sim [ijk]^{(n)} p_{jk}^{(n)} \geq 0 \\
p_{jk}^{(\infty)} \in [1, +\infty) \quad &\Longrightarrow \quad a_{ijk}^{(\infty)} = [ijk]^{(\infty)} \big(p_{jk}^{(\infty)} - 1\big)\big(1 + 3p_{kj}^{(\infty)}\big) \geq 0 \\
p_{jk}^{(\infty)} \in (0, 1) \quad &\Longrightarrow \quad a_{ijk}^{(\infty)} = -[ijk]^{(\infty)} \big(1 - p_{jk}^{(\infty)}\big)\big(1 + 3p_{kj}^{(\infty)}\big) \leq 0 \\
p_{jk}^{(\infty)} = 0 \quad &\Longrightarrow \quad a_{ijk}^{(n)} \sim -3[ijk]^{(n)} p_{kj}^{(n)} \leq 0
\end{aligned} \quad . \tag{V.3.14}$$

For any $q \in \{0, 1, \dots, \ell - r(1) - 1\}$, we set $j = \ell - q$ and we consider the following claim, which we denote by $P(q)$: the limit $a_{i(\ell-q)k}^{(\infty)}$ exists for any $i \in I_1^{(\infty)}$, $k \in I$ and $a_{i(\ell-q)k}^{(\infty)} = 0$.

First, we consider the case $q = 0$, i.e. $j = \ell$. From (V.3.2), we directly get that $p^{(\infty)}_{\ell k} \in [1, +\infty]$. But then, by means of (V.3.14) and $(\star_\ell)$, it follows that $P(0)$ holds.

Let us fix now $0 \leq q \leq \ell - r(1) - 2$ and assume that $P(q')$ holds for any $0 \leq q' \leq q$. In particular, this means that $a^{(\infty)}_{i(\ell-q')k} = 0$ for any $i \in I^{(\infty)}_1$, $k \in I$ and hence for any $1 \leq q' \leq q$ we have

$$\begin{cases} \lim_{n\to+\infty}[i(\ell-q')k]^{(n)}p^{(n)}_{(\ell-q')k} = 0 & \text{for any } i \in I^{(\infty)}_1\,,\ k \in I\setminus\{\ell-q'\} \\ & \qquad\qquad \text{such that } [i(\ell-q')k]^{(\infty)} = 0\ , \\ p^{(\infty)}_{(\ell-q')k} = 1 & \text{for any } k \in I \\ & \qquad\qquad \text{such that } [I^{(\infty)}_1(\ell-q')k]^{(\infty)} > 0\ . \end{cases} \tag{V.3.15}$$

Then, for any $i \in I^{(\infty)}_1$, $k \in I$ we obtain:

- if $p^{(\infty)}_{(\ell-q-1)k} \in [1, +\infty]$, then, by (V.3.14), we directly get that $a^{(n)}_{i(\ell-q-1)k}$ is definitely non negative;
- if $p^{(\infty)}_{(\ell-q-1)k} \in [0, 1)$, then, by (V.3.2), it follows that there exists $1 \leq q' \leq q$ such that $k = \ell-q'$ and so (V.3.14), (V.3.15) imply that the limit $a^{(\infty)}_{i(\ell-q-1)k}$ exists and $a^{(\infty)}_{i(\ell-q-1)k} = 0$.

By means of $(\star_{\ell-q-1})$, this actually proves that $P(q+1)$ holds. Hence, we proved by induction that $P(q)$ holds for any $0 \leq q \leq \ell - r(1) - 1$. In particular, this means that

$$a^{(\infty)}_{ijk} = 0 \quad \text{for any } i \in I^{(\infty)}_1\,,\ j \in I^{(\infty)}_{\geq 2}\,,\ k \in I$$

and hence the following two conditions must hold:

$$i \in I^{(\infty)}_1\,,\ j \in I^{(\infty)}_{\geq 2}\,,\ k \in I \ \text{ and } \ [ijk]^{(\infty)} = 0 \quad \Longrightarrow \quad \lim_{n\to+\infty}[ijk]^{(n)}p^{(n)}_{jk} = 0\ , \tag{V.3.16}$$

$$j, k \in I^{(\infty)}_{\geq 2} \ \text{ and } \ [I^{(\infty)}_1 jk]^{(\infty)} > 0 \quad \Longrightarrow \quad p^{(\infty)}_{jk} = 1\ . \tag{V.3.17}$$

Step 2. We are going to apply (V.3.12) by restricting ourselves to the case $j \in I^{(\infty)}_1$. For the sake of clarity, we set $i_1 := i$ and $i_2 := j$. At first we notice that, since $i_1 < i_2 \leq r(1)$, for any $k \in I^{(\infty)}_{\geq 2}$

$$a^{(n)}_{i_1 i_2 k} \sim -3[i_1 i_2 k]^{(n)}p^{(n)}_{k i_2} \overset{\text{(V.3.16)}}{\longrightarrow} 0\ . \tag{V.3.18}$$

Moreover, by changing indexes in (V.3.13), we get

$$\lim_{n\to+\infty}[i_1 i_2 k]^{(n)} p^{(n)}_{i_1 k} p^{(n)}_{i_1 i_2} = 0 \quad \text{for any } k \in I^{(\infty)}_{\geq 2} \; . \tag{V.3.19}$$

So, from (V.3.12), (V.3.18) and (V.3.19), we get for any fixed $i_1, i_2 \in I^{(\infty)}_1$, $i_1 < i_2$

$$\lim_{n\to+\infty}\left\{ \sum_{k\in I^{(\infty)}_1} \left([i_1 i_2 k]^{(n)} p^{(n)}_{i_1 k} p^{(n)}_{i_1 i_2} + a^{(n)}_{i_1 i_2 k} \right) \right\} = 0 \; . \tag{$\triangle_{i_1 i_2}$}$$

Let us notice that

$$\begin{aligned}
&\sum_{k\in I^{(\infty)}_1} \left([i_1 i_2 k]^{(n)} p^{(n)}_{ik}\, p^{(n)}_{i_1 i_2} + a^{(n)}_{i_1 i_2 k} \right) = \\
&\quad = \sum_{k=1}^{i_1} [i_1 i_2 k]^{(n)} \left(\frac{(p^{(n)}_{i_2 i_1} - 1)^2 (p^{(n)}_{i_1 k})^2 + 2(p^{(n)}_{i_2 i_1} + 1) p^{(n)}_{i_1 k} - 3}{p^{(n)}_{i_2 i_1} p^{(n)}_{i_1 k}} \right) + \\
&\qquad\qquad + \sum_{k=i_1+1}^{r(1)} [i_1 i_2 k]^{(n)} p^{(n)}_{i_1 k} p^{(n)}_{i_1 i_2} + \sum_{k=i_1+1}^{r(1)} a^{(n)}_{i_1 i_2 k} \; .
\end{aligned} \tag{V.3.20}$$

Furthermore, if $k \leq i_1 < i_2$, then $p^{(n)}_{i_2 i_1}, p^{(n)}_{i_1 k} \geq 1$ by (V.3.2) and hence

$$\frac{(p^{(n)}_{i_2 i_1} - 1)^2 (p^{(n)}_{i_1 k})^2 + 2(p^{(n)}_{i_2 i_1} + 1) p^{(n)}_{i_1 k} - 3}{p^{(n)}_{i_2 i_1} p^{(n)}_{i_1 k}} \geq 1 \quad \text{for any } k \leq i_1 < i_2 \; . \tag{V.3.21}$$

For any $i_1 \in \{1, \ldots, r(1){-}1\}$ and for any $q \in \{0, \ldots, r(1){-}i_1{-}1\}$, we set $i_2 = r(1){-}q$ and we consider the following claim, which we denote by $\hat{P}(i_1, q)$: the limit $a^{(\infty)}_{i_1 (r(1)-q) k}$ exists for any $k \in \{i_1{+}1, \ldots, r(1)\}$ and $a^{(\infty)}_{i(r(1)-q)k} = 0$.

First, we are going to prove that $\hat{P}(i_1, 0)$ holds for any $1 \leq i_1 \leq r(1){-}1$. By the very definition (V.3.9), it follows that each $a^{(n)}_{i_1 r(1) k}$, with $i_1{+}1 \leq k \leq r(1)$ is definitely non negative. Hence, by applying $(\triangle_{i_1 r(1)})$ and (V.3.20), we get the claim.

Let us fix now $1 \leq i_1 \leq r(1){-}1$ and $0 \leq q \leq r(1){-}i{-}2$ and assume that $\hat{P}(i_1, q')$ holds for any $0 \leq q' \leq q$. By means of $(\triangle_{i_1 (r(1)-q')})$ and (V.3.20), we get $a^{(\infty)}_{i_1 (r(1)-q') k} = 0$ for any $i_1{+}1 \leq k \leq r(1)$. Again, for any $i_1{+}1 \leq k \leq r(1)$, we have:

- if $p^{(\infty)}_{(r(1)-q-1)k} \in [1,+\infty]$, then, by the very definition (V.3.9), we directly get that $a^{(n)}_{i_1(r(1)-q-1)k}$ is definitely non negative;
- if $p^{(\infty)}_{(r(1)-q-1)k} \in [0,1)$, then, by (V.3.2), it follows that there exists $1 \le q' \le q$ such that $k = r(1)-q'$ and so the limit $a^{(\infty)}_{i_1(r(1)-q-1)k}$ exists and $a^{(\infty)}_{i_1(r(1)-q-1)k} = 0$.

By means of $(\triangle_{i(\ell-q-1)})$, this actually proves that $\hat{P}(i_1, q{+}1)$ holds. Hence, we proved by induction that $\hat{P}(i,q)$ holds for any $1 \le i_1 \le r(1)-1$, $0 \le q \le r(1){-}i_1{-}1$. In particular, by (V.3.20) we obtain

$$
\begin{aligned}
&(\triangle_{i_1 i_2}) \iff \\
&\iff \begin{cases}
\lim\limits_{n\to+\infty} [i_i i_2 k]^{(n)} \left(\dfrac{(p^{(n)}_{i_2 i_1}-1)^2 (p^{(n)}_{i_1 k})^2 + 2(p^{(n)}_{i_2 i_1}+1)p^{(n)}_{i_1 k} - 3}{p^{(n)}_{i_2 i_1} p^{(n)}_{i_1 k}} \right) = 0 & 1 \le k < i_1 \\
\lim\limits_{n\to+\infty} [i_1 i_1 i_2]^{(n)} p^{(n)}_{i_2 i_1} = 0 & \\
\lim\limits_{n\to+\infty} [i_1 i_2 k]^{(n)} p^{(n)}_{i_1 k} p^{(n)}_{i_1 i_2} = 0 & i_1{+}1 \le k \le r(1) \\
a^{(\infty)}_{i_1 i_2 k} = 0 & i_1{+}1 \le k \le r(1)
\end{cases} \\
&\implies \begin{cases}
\lim\limits_{n\to+\infty} [i_1 i_2 k]^{(n)} = 0 & 1 \le k \le i_1 \\
\lim\limits_{n\to+\infty} [i_1 i_1 i_2]^{(n)} p^{(n)}_{i_2 i_1} = 0 & \\
\lim\limits_{n\to+\infty} [i_1 i_2 k]^{(n)} \big(p^{(n)}_{i_2 k} - 1\big) = 0 & i_1{+}1 \le k \le r(1)
\end{cases} .
\end{aligned}
$$

Therefore, we get

$$
\lim_{n\to+\infty} [i_1 i_2 i_3]^{(n)} p^{(n)}_{i_3 i_2} = 0 \quad \text{for any } i_1, i_2, i_3 \in I^{(\infty)}_1 ,\ i_1 \le i_2 < i_3 . \tag{V.3.22}
$$

Step 3. We are going to apply (V.3.11). Notice that, by changing indexes in (V.3.16), it holds

$$
\lim_{n\to+\infty} [iik]^{(n)} p^{(n)}_{ki} = 0 \quad \text{for any } i \in I^{(\infty)}_1,\ k \in I^{(\infty)}_{\ge 2} . \tag{V.3.23}
$$

Therefore from (V.3.11) and (V.3.23) we directly get

$$
\lim_{n\to+\infty} \left\{ \sum_{k \in I^{(\infty)}_1 \setminus \{i\}} [iik]^{(n)} \left(p^{(n)}_{ki} - \frac{4}{3} \right) \right\} = \frac{4}{3} d_i c^{(\infty)}_i + \frac{1}{3} [iii]^{(\infty)} , \quad i \in I^{(\infty)}_1 . \tag{∇_i}
$$

By applying (V.3.22) it follows that for any $i \in I^{(\infty)}_1$ all the summands inside the curly brackets in the left-hand side of (∇_i) are infinitesimal or definitely non positive, while all the summands in the right-hand side are non negative. Hence, it holds necessarily

$$c_{i_1}^{(\infty)} = 0 \;, \quad [i_1 i_1 i_2]^{(\infty)} = 0 \quad \text{for any } i_1, i_2 \in I_1^{(\infty)} \;. \tag{V.3.24}$$

The thesis follows now from (V.3.16), (V.3.17), (V.3.22) and (V.3.24). □

Next, we aim to extend Theorem V.3.1 by considering not only the most shrinking direction, but all the shrinking directions of $(g^{(n)})$. First, we need the following

Proposition V.3.2 ([7], Lemma 5.55). *Assume that there exists a flag of* H-*subalgebras* $\zeta = (\mathfrak{k}_1, \ldots, \mathfrak{k}_p)$ *such that* $v^{(\infty)} \in \mathcal{W}^{\Sigma}(\zeta)$. *If* $\mathfrak{k}_q$ *is toral for some* $1 \leq q \leq p$, *then*

$$\operatorname{scal}\big(g^{(n)}\big) \leq \frac{1}{2} \sum_{i > r(q)} d_i b_i^{(n)} e^{-t^{(n)} v_i^{(n)}} - \frac{1}{4} \sum_{i,j,k > r(q)} [ijk]^{(n)} e^{t^{(n)}(v_i^{(n)} - v_j^{(n)} - v_k^{(n)})} \;, \tag{V.3.25}$$

where the map $r : s \mapsto r(s)$ *is defined in* (V.3.3).

Since the estimate (V.3.25) plays a fundamental role in the proof of our main results, we present a proof of Proposition V.3.2 in the next section.

Let us consider $p \in \{1, \ldots, \ell_{v^{(\infty)}} - 1\}$ in such a way that $\lambda^{(n)}_{r(p-1)+1}$ is bounded and $\lambda^{(n)}_{r(p)+1} \to +\infty$. We set $I^{\mathrm{gb}} := \cup_{q=1}^{p} I_q^{(\infty)} = \{1, \ldots, r(p)\}$ and we call it *index set of the generalized bounded eigenvalues of* $(g^{(n)})$. This name is due to the fact that for any $i \in I$, if $\lambda_i^{(n)}$ is bounded then $i \in I^{\mathrm{gb}}$. Notice that it can happen that $\lambda_i^{(n)} \to +\infty$ for some $i \in I^{\mathrm{gb}}$.

Let also $I^{\mathrm{sh}} := \{1, \ldots, \widetilde{r}\} \subsetneq I$ be the *index set of the shrinking eigenvalues of* $(g^{(n)})$, i.e. $\lambda_{\widetilde{r}}^{(n)} \to 0$ and $\lambda_{\widetilde{r}+1}^{(n)}$ is bounded away from zero. We define then

$$\begin{gathered}
\mathfrak{k}_1 := \mathfrak{h} + \mathfrak{m}_{I_1^{(\infty)}}^{(\infty)} \;, \quad \mathfrak{k}_2 := \mathfrak{k}_1 + \mathfrak{m}_{I_2^{(\infty)}}^{(\infty)} \;, \quad \ldots \quad , \quad \mathfrak{k}_{p-1} := \mathfrak{k}_{p-2} + \mathfrak{m}_{I_{p-1}^{(\infty)}}^{(\infty)} \;, \\
\mathfrak{l}' := \mathfrak{k}_p := \mathfrak{k}_{p-1} + \mathfrak{m}_{I_p^{(\infty)}}^{(\infty)} = \mathfrak{h} + \sum_{i \in I^{\mathrm{gb}}} \mathfrak{m}_i^{(\infty)}
\end{gathered} \tag{V.3.26}$$

and also

$$\mathfrak{l} := \mathfrak{h} + \sum_{i \in I^{\mathrm{sh}}} \mathfrak{m}_i^{(\infty)} \;. \tag{V.3.27}$$

Notice that it necessary holds that $r(p-1) \leq \widetilde{r} \leq r(p)$, and hence $\mathfrak{k}_{p-1} \subset \mathfrak{l} \subset \mathfrak{l}'$.

We are ready to prove our main result. Notice that both Theorem V.1.1 and Theorem V.1.3 are consequences of the following

Theorem V.3.3. *The set* $\zeta := (\mathfrak{k}_1, \ldots, \mathfrak{k}_{p-1}, \mathfrak{l}')$ *defined in* (V.3.26) *is a flag of* H*-subalgebras of* $\mathfrak{g}$ *and* $v^{(\infty)} \in \mathcal{S}^{\Sigma}(\zeta)$*. Moreover, the subspace* $\mathfrak{l}$ *defined in* (V.3.27) *is a toral* H*-subalgebra of* $\mathfrak{g}$ *and the following conditions hold true.*
A) *For any* $i \leq j \leq k$ *such that* $i \in I^{\mathrm{sh}}$*, we have*

$$[ijk]^{(\infty)} = 0 \quad \Longrightarrow \quad \lim_{n \to +\infty} [ijk]^{(n)} p^{(n)}_{kj} = 0\ .$$

B) *For any* $j, k \in I$ *we have*

$$[I^{\mathrm{sh}} jk]^{(\infty)} > 0 \quad \Longrightarrow \quad \lim_{n \to +\infty} p^{(n)}_{kj} = 1\ .$$

Finally, if $\mathfrak{l}'$ *is toral, e.g. if* $\mathfrak{l} = \mathfrak{l}'$*, then* $\lim_{n \to +\infty} \mathrm{scal}\big(g^{(n)}\big) \leq 0$.

Proof. If $p = 1$, i.e. if $\lambda^{(n)}_{r(1)+1} \to +\infty$, then the first part of the theorem coincide with the statement of Theorem V.3.1. Let us suppose then that $p > 1$. If $p = 2$, one can skip the next part of the proof.

We suppose now that $p > 2$. For any $q \in \{1, \ldots, p-1\}$, we consider the following claim, which we denote by $\tilde{P}(q)$: $\mathfrak{k}_q$ is a toral H-subalgebra, $v^{(\infty)} \in \mathcal{W}^{\Sigma}(\mathfrak{k}_1, \ldots, \mathfrak{k}_q)$ and both (A), (B) hold after having replaced the index set I^{sh} with $I^{(\infty)}_q$.

Notice that $\tilde{P}(1)$ follows directly from Theorem V.3.1. Let us fix now $1 \leq q \leq p-2$ and assume that $\tilde{P}(q')$ holds for any $1 \leq q' \leq q$. From (V.3.25) we get

$$\begin{aligned}
\mathrm{scal}\big(g^{(n)}\big) &\leq \frac{1}{2} \sum_{i > r(q)} d_i b^{(n)}_i e^{-t^{(n)} v^{(n)}_i} - \frac{1}{4} \sum_{i,j,k > r(q)} [ijk]^{(n)} e^{t^{(n)} (v^{(n)}_i - v^{(n)}_j - v^{(n)}_k)} \\
&\leq \frac{1}{4} \bigg(2 \sum_{i > r(q)} d_i b^{(n)}_i - \sum_{i,j,k > r(q)} [ijk]^{(n)} e^{t^{(n)} \left(v^{(n)}_i - v^{(n)}_j - v^{(n)}_k + v^{(n)}_{r(q)+1}\right)} \bigg) \frac{1}{\lambda^{(n)}_{r(q)+1}}
\end{aligned}$$

and so, since by assumption $\mathrm{scal}\big(g^{(n)}\big)$ is bounded from below, there exists necessarily $C > 0$ such that

$$\sum_{i,j,k > r(q)} [ijk]^{(n)} e^{t^{(n)} \left(v^{(n)}_i - v^{(n)}_j - v^{(n)}_k + v^{(n)}_{r(q)+1}\right)} \leq C \quad \text{for any } n \in \mathbb{N}\ .$$

Then, by arguing as at the beginning of the proof of Theorem V.3.1, we directly get

$$i,j,k > r(q)\,,\ [ijk]^{(\infty)} > 0 \quad \Longrightarrow \quad v_i^{(\infty)} - v_j^{(\infty)} - v_k^{(\infty)} + \hat{v}_{q+1}^{(\infty)} \leq 0\ . \quad \text{(V.3.28)}$$

As a consequence $\mathfrak{k}_{q+1}$ is an H-subalgebra of $\mathfrak{g}$ and $v^{(\infty)} \in \mathcal{W}^{\Sigma}(\mathfrak{k}_1, \ldots, \mathfrak{k}_{q+1})$. Since $\lambda_{r(q+1)}^{(n)} \to 0$ as $n \to +\infty$, for any $i,j \in I$ such that $i \in I_{q+1}^{(\infty)}$, $i < j$ it follows that

$$\sec_i(g^{(n)}) \cdot \lambda_i^{(n)} = d_i c_i^{(n)} + \frac{1}{4}[iii]^{(n)} + \sum_{k \in I \backslash \{i\}} [iik]^{(n)} - \frac{3}{4} \sum_{k \in I \backslash \{i\}} [iik]^{(n)} p_{ki}^{(n)} \longrightarrow 0\ , \quad \text{(V.3.29)}$$

$$\sec_{ij}(g^{(n)}) \cdot 4\lambda_i^{(n)} = \sum_{k \in I} \big([ijk]^{(n)} p_{ik}^{(n)} p_{ij}^{(n)} + a_{ijk}^{(n)}\big) \longrightarrow 0\ , \quad \text{(V.3.30)}$$

where $\sec_i(g^{(n)})$ and $\sec_{ij}(g^{(n)})$ are defined in (V.3.4) and (V.3.5), respectively, and the coefficients $a_{ijk}^{(n)}$ were introduced in (V.3.9). So, one can apply, mutatis mutandis, Step 1, Step 2 and Step 3 already seen in the proof of Theorem V.3.1 to conclude that $\tilde{P}(q+1)$ holds true. Hence, it follows by induction that $\tilde{P}(q)$ holds for any $1 \leq q \leq p-1$.

From now on, it does not matter if $p = 2$ or $p > 2$. Since $\mathfrak{k}_{p-1}$ is toral and $\lambda_{r(p-1)+1}^{(n)}$ is bounded, from (V.3.25) it follows that $\mathfrak{l}'$ is an H-subalgebra of $\mathfrak{g}$ and $v^{(\infty)} \in \mathcal{S}^{\Sigma}(\mathfrak{k}_1, \ldots, \mathfrak{l}')$. Moreover, by repeating once again Step 1, Step 2 and Step 3 letting the index i run from 1 to $\widetilde{r}$, one can prove that $\mathfrak{l}$ is a toral subalgebra and that both conditions (A), (B) hold true.

Finally, for the proof of the last claim, we do not assume anymore that $p > 1$, i.e. we allow p to be 1. Let us suppose by contradiction that $\mathfrak{l}'$ is toral and $\mathrm{scal}(g^{(n)}) > \delta$ definitely, for some $\delta > 0$. By (V.3.25) it holds that for any n large enough

$$\frac{1}{2} \sum_{i > r(p)} d_i b_i^{(n)} e^{-t^{(n)} v_i^{(n)}} - \frac{1}{4} \sum_{i,j,k > r(p)} [ijk]^{(n)} e^{t^{(n)}(v_i^{(n)} - v_j^{(n)} - v_k^{(n)})} > \delta\ .$$

Hence, there exists a constant $C' > 0$ such that

$$4\delta \lambda_{r(p)+1}^{(n)} + \sum_{i,j,k > r(p)} [ijk]^{(n)} e^{t^{(n)}(v_i^{(n)} - v_j^{(n)} - v_k^{(n)} + v_{r(p)+1}^{(n)})} < C' \quad \text{for any } n \in \mathbb{N} \quad \text{(V.3.31)}$$

which is clearly not possible, since all the terms in (V.3.31) are non negative and $\lambda_{r(p)+1}^{(n)}$ is unbounded. □

V.3.2 An explicit example on $V_3(\mathbb{R}^5)$

We exhibit an example of a sequence of $\mathsf{SO}(5)$-invariant metrics on the Stiefel manifold $V_3(\mathbb{R}^5)$, i.e. the space of orthonormal 3-frames in $\mathbb{R}^5$, which diverges with bounded curvature.

Let $M = V_3(\mathbb{R}^5) = \mathsf{SO}(5)/\mathsf{SO}(2)$ and consider the $\mathrm{Ad}(\mathsf{SO}(5))$-invariant inner product $Q(A_1, A_2) := -\frac{1}{2}\,\mathrm{Tr}(A_1{\cdot}A_2)$ on $\mathfrak{so}(5)$. We choose the Q-orthonormal basis for the Lie algebra $\mathfrak{so}(5)$ given by

$$
\begin{gathered}
E := e^4{\otimes}e_5 - e^5{\otimes}e_4 \ , \quad X_1 := e^2{\otimes}e_3 - e^3{\otimes}e_2 \ , \\
X_2 := e^3{\otimes}e_4 - e^4{\otimes}e_3 \ , \quad X_3 := e^3{\otimes}e_5 - e^5{\otimes}e_3 \ , \\
X_4 := e^2{\otimes}e_4 - e^4{\otimes}e_2 \ , \quad X_5 := e^2{\otimes}e_5 - e^5{\otimes}e_2 \ , \quad X_6 := e^1{\otimes}e_4 - e^4{\otimes}e_1 \ , \\
X_7 := e^1{\otimes}e_5 - e^5{\otimes}e_1 \ , \quad X_8 := e^1{\otimes}e_3 - e^3{\otimes}e_1 \ , \quad X_9 := e^1{\otimes}e_2 - e^2{\otimes}e_1 \ ,
\end{gathered}
$$

where we denoted by $(e_1, \ldots, e_5)$ the standard basis of $\mathbb{R}^5$ and by $(e^1, \ldots, e^5)$ its dual frame. Then, the isotropy algebra is $\mathfrak{so}(2) = \mathrm{span}(E)$ and its Q-orthogonal reductive complement $\mathfrak{m}$ decomposes into six $\mathrm{Ad}(\mathsf{SO}(2))$-irreducible submodules:

$$
\begin{gathered}
\mathfrak{m}_1 = \mathrm{span}(X_1) \ , \quad \mathfrak{m}_2 = \mathrm{span}(X_2, X_3) \ , \quad \mathfrak{m}_3 = \mathrm{span}(X_4, X_5) \ , \\
\mathfrak{m}_4 = \mathrm{span}(X_6, X_7) \ , \quad \mathfrak{m}_5 = \mathrm{span}(X_8) \ , \quad \mathfrak{m}_6 = \mathrm{span}(X_9) \ .
\end{gathered}
$$

Notice that $\mathfrak{m}_2 \simeq \mathfrak{m}_3 \simeq \mathfrak{m}_4$ are equivalent to the standard representation of $\mathsf{SO}(2)$, while $\mathfrak{m}_1 \simeq \mathfrak{m}_5 \simeq \mathfrak{m}_6$ are trivial. One can directly check that the coefficients related to this decomposition are

$$
\begin{gathered}
c_1 = 0 \ , \quad c_2 = c_3 = c_4 = 1 \ , \quad c_5 = c_6 = 0 \ , \\
b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = 6 \ , \\
[123] = 2 \ , \quad [156] = 1 \ , \quad [245] = 2 \ , \quad [346] = 2 \ .
\end{gathered}
\tag{V.3.32}
$$

We define also

$$
\mathfrak{k}_1 := \mathfrak{h} + \mathfrak{m}_1 \simeq \mathfrak{so}(2) \oplus \mathfrak{so}(2) \ , \quad \mathfrak{k}_2 := \mathfrak{k}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 \simeq \mathfrak{so}(4) \ ,
$$

which are $\mathsf{SO}(2)$-subalgebras of $\mathfrak{so}(5)$. Notice that $\mathfrak{k}_1$ is toral, while $\mathfrak{k}_2$ is non-toral.

Let us consider the sequence $(g^{(n)}) \subset \mathcal{M}_1^{\mathsf{SO}(5)}$ defined by

$$
g^{(n)} := \tfrac{1}{4n^4} Q_{\mathfrak{m}_1} + Q_{\mathfrak{m}_2} + Q_{\mathfrak{m}_3} + nQ_{\mathfrak{m}_4} + 2nQ_{\mathfrak{m}_5} + 2nQ_{\mathfrak{m}_6} \ . \tag{V.3.33}
$$

Notice that the eigenvalues of the tangent direction $v^{(n)}$ are

$$v_1^{(n)} = -\frac{2+4\log_2 n}{\sqrt{20(\log_2 n)^2+20\log_2 n+6}} \ , \quad v_2^{(n)} = v_3^{(n)} = 0 \ ,$$
$$v_4^{(n)} = \frac{\log_2 n}{\sqrt{20(\log_2 n)^2+20\log_2 n+6}} \ , \quad v_5^{(n)} = v_6^{(n)} = \frac{1+\log_2 n}{\sqrt{20(\log_2 n)^2+20\log_2 n+6}}$$

and so $v^{(n)} \in \mathcal{S}^{\Sigma}(\mathfrak{k}_1)$, but $v^{(n)} \notin \mathcal{S}^{\Sigma}(\mathfrak{k}_1, \mathfrak{k}_2)$. From (V.2.9) it follows that $(g^{(n)})$ lies in the space $\mathcal{M}_1^{\mathsf{G}}(\mathfrak{k}_1)$ of unit volume $\mathfrak{k}_1$-submersion metrics. One can directly check that the Ricci operator of $g^{(n)}$ is diagonal, with eigenvalues

$$\mathrm{ric}_1\big(g^{(n)}\big) = \tfrac{8n^2+1}{32n^6} \ , \quad \mathrm{ric}_2\big(g^{(n)}\big) = \mathrm{ric}_3\big(g^{(n)}\big) = \tfrac{14n^4+2n^2-1}{8n^4} \ ,$$
$$\mathrm{ric}_4\big(g^{(n)}\big) = -\tfrac{3n^2-6n+1}{2n^2} \ , \quad \mathrm{ric}_5\big(g^{(n)}\big) = \mathrm{ric}_6\big(g^{(n)}\big) = \tfrac{48n^6+48n^5-16n^4-1}{32n^6} \ .$$

By [11, Thm 4] it follows that $(g^{(n)})$ has bounded curvature. For the sake of thoroughness, we provide the explicit expression of all the components of the curvature operator $\mathrm{Rm}(g^{(n)})$. Let us consider the $g^{(n)}$-orthonormal frame

$$X_1^{(n)} := 2n^2 X_1 \ , \quad X_2^{(n)} := X_2 \ , \quad X_3^{(n)} := X_3 \ , \quad X_4^{(n)} := X_4 \ , \quad X_5^{(n)} := X_5 \ ,$$
$$X_6^{(n)} := \tfrac{1}{\sqrt{n}} X_6 \ , \quad X_7^{(n)} := \tfrac{1}{\sqrt{n}} X_7 \ , \quad X_8^{(n)} := \tfrac{1}{\sqrt{2n}} X_8 \ , \quad X_9^{(n)} := \tfrac{1}{\sqrt{2n}} X_9 \ .$$

Then, the curvature operator $\mathrm{Rm}(g^{(n)}) : \Lambda^2\mathfrak{m} \to \Lambda^2\mathfrak{m}$ takes the following form.

$$\begin{aligned}
\mathrm{Rm}(g^{(n)})(X_1^{(n)}\wedge X_2^{(n)}) &= \tfrac{1}{16n^4} X_1^{(n)}\wedge X_2^{(n)} + \tfrac{3n-1}{16\sqrt{2}n^4} X_6^{(n)}\wedge X_9^{(n)} \\
\mathrm{Rm}(g^{(n)})(X_1^{(n)}\wedge X_3^{(n)}) &= \tfrac{1}{16n^4} X_1^{(n)}\wedge X_3^{(n)} + \tfrac{3n-1}{16\sqrt{2}n^4} X_7^{(n)}\wedge X_9^{(n)} \\
\mathrm{Rm}(g^{(n)})(X_1^{(n)}\wedge X_4^{(n)}) &= \tfrac{1}{16n^4} X_1^{(n)}\wedge X_4^{(n)} - \tfrac{3n-1}{16\sqrt{2}n^4} X_6^{(n)}\wedge X_8^{(n)} \\
\mathrm{Rm}(g^{(n)})(X_1^{(n)}\wedge X_5^{(n)}) &= \tfrac{1}{16n^4} X_1^{(n)}\wedge X_5^{(n)} - \tfrac{3n-1}{16\sqrt{2}n^4} X_7^{(n)}\wedge X_8^{(n)} \\
\mathrm{Rm}(g^{(n)})(X_1^{(n)}\wedge X_6^{(n)}) &= \tfrac{2n^2+n-1}{16\sqrt{2}n^4} X_2^{(n)}\wedge X_9^{(n)} - \tfrac{2n^2+n-1}{16\sqrt{2}n^4} X_4^{(n)}\wedge X_8^{(n)} \\
\mathrm{Rm}(g^{(n)})(X_1^{(n)}\wedge X_7^{(n)}) &= \tfrac{2n^2+n-1}{16\sqrt{2}n^4} X_3^{(n)}\wedge X_9^{(n)} - \tfrac{2n^2+n-1}{16\sqrt{2}n^4} X_5^{(n)}\wedge X_8^{(n)} \\
\mathrm{Rm}(g^{(n)})(X_1^{(n)}\wedge X_8^{(n)}) &= \tfrac{1}{64n^6} X_1^{(n)}\wedge X_8^{(n)} - \tfrac{n-1}{8\sqrt{2}n^3} X_4^{(n)}\wedge X_6^{(n)} - \tfrac{n-1}{8\sqrt{2}n^3} X_5^{(n)}\wedge X_7^{(n)} \\
\mathrm{Rm}(g^{(n)})(X_1^{(n)}\wedge X_9^{(n)}) &= \tfrac{1}{64n^6} X_1^{(n)}\wedge X_9^{(n)} + \tfrac{n-1}{8\sqrt{2}n^3} X_2^{(n)}\wedge X_6^{(n)} + \tfrac{n-1}{8\sqrt{2}n^3} X_3^{(n)}\wedge X_7^{(n)} \\
\mathrm{Rm}(g^{(n)})(X_2^{(n)}\wedge X_3^{(n)}) &= X_2^{(n)}\wedge X_3^{(n)} + \tfrac{16n^4-1}{16n^4} X_4^{(n)}\wedge X_5^{(n)} - \tfrac{n^2-6n+1}{8n^2} X_6^{(n)}\wedge X_7^{(n)} \\
\mathrm{Rm}(g^{(n)})(X_2^{(n)}\wedge X_4^{(n)}) &= \tfrac{16n^4-3}{16n^4} X_2^{(n)}\wedge X_4^{(n)} + \tfrac{8n^4-1}{8n^4} X_3^{(n)}\wedge X_5^{(n)} - \\
&\qquad - \tfrac{2n^5-12n^4+2n^3+1}{16n^5} X_8^{(n)}\wedge X_9^{(n)} \\
\mathrm{Rm}(g^{(n)})(X_2^{(n)}\wedge X_5^{(n)}) &= -\tfrac{1}{16n^4} X_3^{(n)}\wedge X_4^{(n)} \\
\mathrm{Rm}(g^{(n)})(X_2^{(n)}\wedge X_6^{(n)}) &= \tfrac{n-1}{8\sqrt{2}n^3} X_1^{(n)}\wedge X_9^{(n)} - \tfrac{7n^2-2n-1}{8n^2} X_2^{(n)}\wedge X_6^{(n)} - \tfrac{n-1}{2n} X_3^{(n)}\wedge X_7^{(n)} \\
\mathrm{Rm}(g^{(n)})(X_2^{(n)}\wedge X_7^{(n)}) &= -\tfrac{(n+1)(3n-1)}{8n^2} X_3^{(n)}\wedge X_6^{(n)} \\
\mathrm{Rm}(g^{(n)})(X_2^{(n)}\wedge X_8^{(n)}) &= \tfrac{5n^2-2n+1}{8n^2} X_2^{(n)}\wedge X_8^{(n)} + \tfrac{8n^5+8n^4-1}{32n^5} X_4^{(n)}\wedge X_9^{(n)} \\
\mathrm{Rm}(g^{(n)})(X_2^{(n)}\wedge X_9^{(n)}) &= \tfrac{(n+1)(2n-1)}{16\sqrt{2}n^4} X_1^{(n)}\wedge X_6^{(n)} + \tfrac{12n^5-16n^4+43+1}{32n^5} X_4^{(n)}\wedge X_8^{(n)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Rm}(g^{(n)})(X_3^{(n)}\wedge X_4^{(n)}) &= -\tfrac{1}{16n^4}X_2^{(n)}\wedge X_5^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_3^{(n)}\wedge X_5^{(n)}) &= -\tfrac{8n^4-1}{8n^4}X_2^{(n)}\wedge X_4^{(n)} + \tfrac{16n^4-3}{16n^4}X_3^{(n)}\wedge X_5^{(n)} -\\
&\qquad - \tfrac{2n^5+2n^3-12n^4+1}{16n^5}X_8^{(n)}\wedge X_9^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_3^{(n)}\wedge X_6^{(n)}) &= -\tfrac{(n+1)(3n-1)}{8n^2}X_2^{(n)}\wedge X_7^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_3^{(n)}\wedge X_7^{(n)}) &= \tfrac{n+1}{8\sqrt{2}n^3}X_1^{(n)}\wedge X_9^{(n)} - \tfrac{n-1}{2n}X_2^{(n)}\wedge X_6^{(n)} - \tfrac{7n^2-2n-1}{8n^2}X_3^{(n)}\wedge X_7^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_3^{(n)}\wedge X_8^{(n)}) &= \tfrac{5n^2-2n+1}{8n^2}X_3^{(n)}\wedge X_8^{(n)} + \tfrac{8n^5+8n^4-1}{32n^5}X_5^{(n)}\wedge X_9^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_3^{(n)}\wedge X_9^{(n)}) &= \tfrac{(n+1)(2n-1)}{16\sqrt{2}n^4}X_1^{(n)}\wedge X_7^{(n)} + \tfrac{12n^5-16n^4+4n^3+1}{32n^5}X_5^{(n)}\wedge X_8^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_4^{(n)}\wedge X_5^{(n)}) &= \tfrac{16n^4-1}{16n^4}X_2^{(n)}\wedge X_3^{(n)} + X_4^{(n)}\wedge X_5^{(n)} - \tfrac{n^2-6n+1}{8n^2}X_6^{(n)}\wedge X_7^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_4^{(n)}\wedge X_6^{(n)}) &= -\tfrac{n-1}{8\sqrt{2}n^3}X_1^{(n)}\wedge X_8^{(n)} - \tfrac{7n^2-2n-1}{8n^2}X_4^{(n)}\wedge X_6^{(n)} - \tfrac{n-1}{2n}X_5^{(n)}\wedge X_7^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_4^{(n)}\wedge X_7^{(n)}) &= -\tfrac{(n+1)(3n-1)}{8n^2}X_5^{(n)}\wedge X_6^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_4^{(n)}\wedge X_8^{(n)}) &= -\tfrac{(n+1)(2n-1)}{16\sqrt{2}n^4}X_1^{(n)}\wedge X_6^{(n)} + \tfrac{12n^5-16n^4+4n^3+1}{32n^5}X_2^{(n)}\wedge X_9^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_4^{(n)}\wedge X_9^{(n)}) &= -\tfrac{8n^5+8n^4-1}{32n^5}X_2^{(n)}\wedge X_8^{(n)} + \tfrac{5n^2-2n+1}{8n^2}X_4^{(n)}\wedge X_9^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_5^{(n)}\wedge X_6^{(n)}) &= -\tfrac{(n+1)(3n-1)}{8n^2}X_4^{(n)}\wedge X_7^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_5^{(n)}\wedge X_7^{(n)}) &= -\tfrac{n-1}{8\sqrt{2}n^3}X_1^{(n)}\wedge X_8^{(n)} - \tfrac{n-1}{2n}X_4^{(n)}\wedge X_6^{(n)} - \tfrac{7n^2-2n-1}{8n^2}X_5^{(n)}\wedge X_7^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_5^{(n)}\wedge X_8^{(n)}) &= -\tfrac{(n+1)(2n-1)}{16\sqrt{2}n^4}X_1^{(n)}\wedge X_7^{(n)} + \tfrac{12n^5-16n^4+4n^3+1}{32n^5}X_3^{(n)}\wedge X_9^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_5^{(n)}\wedge X_9^{(n)}) &= \tfrac{8n^5+8n^4-1}{32n^5}X_3^{(n)}\wedge X_8^{(n)} + \tfrac{5n^2-2n+1}{8n^2}X_5^{(n)}\wedge X_9^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_6^{(n)}\wedge X_7^{(n)}) &= -\tfrac{n^2-6n+1}{8n^2}X_2^{(n)}\wedge X_3^{(n)} - \tfrac{n^2-6n+1}{8n^2}X_4^{(n)}\wedge X_5^{(n)} - \tfrac{1}{n}X_6^{(n)}\wedge X_7^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_6^{(n)}\wedge X_8^{(n)}) &= -\tfrac{3n-1}{16\sqrt{2}n^4}X_1^{(n)}\wedge X_4^{(n)} + \tfrac{n^2+6n-3}{8n^2}X_6^{(n)}\wedge X_8^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_6^{(n)}\wedge X_9^{(n)}) &= \tfrac{3n-1}{16\sqrt{2}n^4}X_1^{(n)}\wedge X_2^{(n)} + \tfrac{n^2+6n-3}{8n^2}X_6^{(n)}\wedge X_9^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_7^{(n)}\wedge X_8^{(n)}) &= -\tfrac{3n-1}{16\sqrt{2}n^4}X_1^{(n)}\wedge X_5^{(n)} + \tfrac{n^2+6n-3}{8n^2}X_7^{(n)}\wedge X_8^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_7^{(n)}\wedge X_9^{(n)}) &= \tfrac{3n-1}{16\sqrt{2}n^4}X_1^{(n)}\wedge X_3^{(n)} + \tfrac{n^2+6n-3}{8n^2}X_7^{(n)}\wedge X_9^{(n)}\\
\mathrm{Rm}(g^{(n)})(X_8^{(n)}\wedge X_9^{(n)}) &= -\tfrac{2n^5-12n^4+2n^3+1}{16n^5}X_2^{(n)}\wedge X_4^{(n)} - \tfrac{2n^5-12n^4+2n^3+1}{16n^5}X_3^{(n)}\wedge X_5^{(n)} +\\
&\qquad + \tfrac{32n^5-3}{64n^6}X_8^{(n)}\wedge X_9^{(n)}
\end{aligned}$$

This example shows that, in some sense, Theorem V.3.3 is optimal. Indeed, we have

$$\begin{gathered}
p = 2\ , \quad I_1^{(\infty)} = I_{p-1}^{(\infty)} = I^{\mathrm{sh}} = \{1\}\ , \quad I_2^{(\infty)} = I_p^{(\infty)} = \{2,3\}\ ,\\
I^{\mathrm{gb}} = \{1,2,3\}\ , \quad I_3^{(\infty)} = \{4,5,6\}
\end{gathered} \tag{V.3.34}$$

and so $\mathfrak{l} = \mathfrak{k}_1$, $\mathfrak{l}' = \mathfrak{k}_2$. Moreover

$$[245] > 0\ , \quad \frac{\lambda_5^{(n)}}{\lambda_4^{(n)}} = 2 \neq 1\ . \tag{V.3.35}$$

So, even though $v^{(\infty)} \in \mathcal{S}^{\Sigma}(\mathfrak{l},\mathfrak{l}')$ because

$$v_1^{(\infty)} = -\tfrac{4}{\sqrt{20}}\ , \quad v_2^{(\infty)} = v_3^{(\infty)} = 0\ , \quad v_4^{(\infty)} = v_5^{(\infty)} = v_6^{(\infty)} = \tfrac{1}{\sqrt{20}}\ ,$$

from (V.3.35) it follows that claim (B) is not true anymore if one replaces the index set I^{sh} with I^{gb}. This means that $(g^{(n)})$ does not approach asymptotically a $\mathfrak{l}'$-submersion metric.

Moreover

$$\mathrm{scal}\big(g^{(n)}\big) = \tfrac{224n^6+288n^5-32n^4-8n^2-1}{32n^6} \to 7 > 0$$

and this shows that: *it is possible for a sequence of invariant metrics to diverge with bounded curvature and positive scalar curvature bounded away from zero.*

Finally, along the geodesic $\gamma_{v^{(n)}}(t)$ we have

$$\begin{aligned}\mathrm{scal}(\gamma_{v^{(n)}}(t)) = 12 - 2e^{t(v_5^{(n)}-v_4^{(n)})} - e^{tv_1^{(n)}} - 6e^{-tv_4^{(n)}} - 6e^{-tv_5^{(n)}} - \\ - \tfrac12 e^{-t(2v_5^{(n)}-v_1^{(n)})} - 2e^{-t(v_4^{(n)}+v_5^{(n)})} - 2e^{-t(v_5^{(n)}-v_4^{(n)})}\end{aligned}$$

and so $\lim_{t\to+\infty}\mathrm{scal}(\gamma_{v^{(n)}}(t)) = -\infty$ for any $n\in\mathbb{N}$. On the other hand, one can directly check that along the limit geodesic $\gamma_{v^{(\infty)}}(t)$, the Ricci operator is diagonal with eigenvalues

$$\begin{gathered}\mathrm{ric}_1(\gamma_{v^{(\infty)}}(t)) = e^{tv_1^{(\infty)}} + \tfrac12 e^{-t(2v_4^{(\infty)}-v_1^{(\infty)})}\ , \\ \mathrm{ric}_2(\gamma_{v^{(\infty)}}(t)) = \mathrm{ric}_3(\gamma_{v^{(\infty)}}(t)) = 2 - \tfrac12 e^{tv_1^{(\infty)}} + \tfrac12 e^{-2tv_4^{(\infty)}} \\ \mathrm{ric}_4(\gamma_{v^{(\infty)}}(t)) = 3e^{-tv_4^{(\infty)}} - e^{-2tv_4^{(\infty)}}\ , \\ \mathrm{ric}_5(\gamma_{v^{(\infty)}}(t)) = \mathrm{ric}_6(\gamma_{v^{(\infty)}}(t)) = 3e^{-tv_4^{(\infty)}} - e^{-2tv_4^{(\infty)}} - \tfrac12 e^{-t(2v_4^{(\infty)}-v_1^{(\infty)})}\end{gathered}$$

and so, by applying again [11, Thm 4], $\big|\mathrm{Rm}(\gamma_{v^{(\infty)}}(t))\big|_{\gamma_{v^{(\infty)}}(t)}$ is bounded. The limit values of the Ricci eigenvalues along the original sequence $(g^{(n)})$ are

$$\begin{gathered}\lim_{n\to+\infty}\mathrm{ric}_1\big(g^{(n)}\big) = 0\ , \quad \lim_{n\to+\infty}\mathrm{ric}_2\big(g^{(n)}\big) = \lim_{n\to+\infty}\mathrm{ric}_3\big(g^{(n)}\big) = \tfrac74\ , \\ \lim_{n\to+\infty}\mathrm{ric}_4\big(g^{(n)}\big) = -\tfrac32\ , \quad \lim_{n\to+\infty}\mathrm{ric}_5\big(g^{(n)}\big) = \lim_{n\to+\infty}\mathrm{ric}_6\big(g^{(n)}\big) = \tfrac32\ ,\end{gathered}$$

while along the limit geodesic $\gamma_{v^{(\infty)}}(t)$

$$\begin{gathered}\lim_{t\to+\infty}\mathrm{ric}_1(\gamma_{v^{(\infty)}}(t)) = 0\ , \quad \lim_{t\to+\infty}\mathrm{ric}_2(\gamma_{v^{(\infty)}}(t)) = \lim_{t\to+\infty}\mathrm{ric}_3(\gamma_{v^{(\infty)}}(t)) = 2\ , \\ \lim_{t\to+\infty}\mathrm{ric}_4(\gamma_{v^{(\infty)}}(t)) = \lim_{t\to+\infty}\mathrm{ric}_5(\gamma_{v^{(\infty)}}(t)) = \lim_{t\to+\infty}\mathrm{ric}_6(\gamma_{v^{(\infty)}}(t)) = 0\ .\end{gathered}$$

This actually shows that a diverging sequence $(g^{(n)}) \subset \mathcal{M}_1^{\mathsf{G}}$ with bounded curvature and limit direction $v^{(\infty)}$ can develop a different asymptotic behavior with respect to to the limit geodesic $\gamma_{v^{(\infty)}}(t)$.

Finally, let us mention that in our previous example $\widetilde{r} = r(p-1)$. It is also easy to exhibit examples where $\widetilde{r} = r(p)$, e.g. by considering again Berger spheres as in Example V.2.13. However, it is not clear whether it is actually possible to construct a sequence of invariant metrics which diverges with bounded curvature with $r(p-1) < \widetilde{r} < r(p)$. We stress that, for this to be the case, it is necessary that the limit direction $v^{(\infty)}$ admits the eigenvalue $\hat{v}_p^{(\infty)} = 0$ and the module $\mathfrak{m}_{I_p^{(\infty)}}$ needs to be $\mathrm{Ad}(\mathsf{K}_{p-1})$-reducible.

V.4 Proof of Proposition V.3.2

For the convenience of the reader, we provide here a proof of Proposition V.3.2 following Böhm's original approach. First, we need the following estimate.

Proposition V.4.1. *Let* G *be a compact* N*-dimensional Lie group with a fixed* $\mathrm{Ad}(\mathsf{G})$*-invariant Euclidean inner product* Q *on* $\mathfrak{g} := \mathrm{Lie}(\mathsf{G})$, $\mathfrak{a} \subset \mathfrak{g}$ *an abelian Lie subalgebra and* $(e_1, \dots, e_N)$ *a* Q*-orthonormal basis for* $\mathfrak{g}$ *such that* $\mathfrak{a} = \mathrm{span}(e_1, \dots, e_{q+1})$ *for some* $0 \leq q \leq N-1$. *Let also* $(e_1^{(n)}, \dots, e_N^{(n)})$ *be a sequence of* Q*-orthonormal bases for* $\mathfrak{g}$ *such that* $e_i^{(n)} \to e_i$ *as* $n \to +\infty$ *for any* $1 \leq i \leq N$. *Then, there exist* $\bar{n} \in \mathbb{N}$ *and* $C > 0$ *such that*

$$\sum_{i,j \leq q+1} Q\big([e_1^{(n)}, e_i^{(n)}], e_j^{(n)}\big)^2 \leq C \sum_{\substack{i \leq q+1 \\ k > q+1}} Q\big([e_1^{(n)}, e_i^{(n)}], e_k^{(n)}\big)^2 \quad \textit{for any } n \geq \bar{n} \ . \tag{V.4.1}$$

Proof. Of course (V.4.1) holds true if $\mathfrak{g}$ is abelian or $q = 0, 1$. Hence, we assume that $1 < q < N-1$ and that $\mathfrak{g}$ is not abelian. Let $I := \{1, \dots, N\}$, $I_1 := \{2, \dots, q+1\}$ and $I_2 := \{q+2, \dots, N\}$. Notice that we will pass whenever convenient to a subsequence without mentioning it explicitly. Moreover, for any subspace $\mathfrak{p} \subset \mathfrak{g}$, we denote by $\mathfrak{p}^\perp$ its Q-orthogonal complement inside $\mathfrak{g}$.

Let us suppose by contradiction that

$$\sum_{i,j \in I_1} Q\big([e_1^{(n)}, e_i^{(n)}], e_j^{(n)}\big)^2 > c^{(n)} \sum_{\substack{i \in I_1 \\ k \in I_2}} Q\big([e_1^{(n)}, e_i^{(n)}], e_k^{(n)}\big)^2 \quad \text{for any } n \in \mathbb{N} \ , \tag{V.4.2}$$

for some sequence $c^{(n)} \to +\infty$.

Let also $\mathfrak{t} \subset \mathfrak{g}$ be a maximal abelian Lie subalgebra of $\mathfrak{g}$ such that $e_1 \in \mathfrak{t}$. We claim that it is possible to assume that $e_1^{(n)} \in \mathfrak{t}$ for any $n \in \mathbb{N}$. In fact, we can choose a sequence $(\mathfrak{t}^{(n)})$ of maximal abelian subalgebras of $\mathfrak{g}$ such that $e_1^{(n)} \in \mathfrak{t}^{(n)}$ and $\mathfrak{t}^{(n)} \to \mathfrak{t}$ as $n \to +\infty$. But then, there exists a sequence $(x^{(n)}) \subset \mathsf{G}$ such that $\mathrm{Ad}(x^{(n)})(\mathfrak{t}^{(n)}) = \mathfrak{t}$ and $x^{(n)} \to 1_{\mathsf{G}}$. Therefore, by setting $e_i'^{(n)} := \mathrm{Ad}(x^{(n)})(e_i^{(n)})$ for any $i \in I$, we obtain a new Q-orthonormal basis $(e_1'^{(n)}, \dots, e_N'^{(n)})$ which converges to $(e_1, \dots, e_N)$.

For any $i \in I_1$ we write

$$\mathfrak{t}^{\perp} \ni [e_1^{(n)}, e_i^{(n)}] = \sum_{j \in I_1 \setminus \{i\}} a_{ij}^{(n)} e_j^{(n)} + z_i^{(n)} \ , \qquad \text{with } z_i^{(n)} \in \mathrm{span}(e_{q+2}^{(n)}, \dots, e_N^{(n)}) \tag{V.4.3}$$

and we choose $j(i) \in I_1 \setminus \{i\}$ such that $|a_{ij(i)}^{(n)}| \geq |a_{ij}^{(n)}|$ for any $j \in I_1 \setminus \{i\}$, for any $n \in \mathbb{N}$. Moreover, up to reorder the index set I_1, we may assume that $|a_{23}^{(n)}| \geq |a_{ij(i)}^{(n)}|$. So, by means of (V.4.2) and (V.4.3), we get

$$\big|a_{23}^{(n)}\big|^2 \geq \frac{1}{q} \sum_{i \in I_1} \big|a_{ij(i)}^{(n)}\big|^2 > \frac{c^{(n)}}{q^2} \sum_{i \in I_1} |z_i^{(n)}|_Q^2 \quad \text{for any } n \in \mathbb{N} \ . \tag{V.4.4}$$

We claim now that it is possible to assume that for any $i \in I_1$

$$\lim_{n \to +\infty} \frac{\big|a_{ij(i)}^{(n)}\big|}{\big|a_{23}^{(n)}\big|} > 0 \ . \tag{V.4.5}$$

In fact, let $I_1' := \{i \in I_1 : \ i \text{ satisfies (V.4.5)}\}$ and $I_1'' := I_1 \setminus I_1'$. Of course $\{2,3\} \subset I_1'$. Then, by (V.4.4)

$$\begin{aligned}
(1 + |I_1''|)\big|a_{23}^{(n)}\big|^2 &= \big|a_{23}^{(n)}\big|^2 + \sum_{i \in I_1''} \frac{\big|a_{23}^{(n)}\big|^2}{\big|a_{ij(i)}^{(n)}\big|^2} \big|a_{ij(i)}^{(n)}\big|^2 \\
&> \frac{c^{(n)}}{q^2} \sum_{i \in I_1} |z_i^{(n)}|_Q^2 + \frac{1}{q} \sum_{\substack{i \in I_1'' \\ j \in I_1 \setminus \{i\}}} \frac{\big|a_{23}^{(n)}\big|^2}{\big|a_{ij(i)}^{(n)}\big|^2} \big|a_{ij}^{(n)}\big|^2 \ , \\
&\geq \tilde{c}^{(n)} \sum_{\substack{i \in I_1' \\ k \in I_1'' \cup I_2}} Q\big([e_1^{(n)}, e_i^{(n)}], e_k^{(n)}\big)^2
\end{aligned} \tag{V.4.6}$$

where

$$\tilde{c}^{(n)} := \min \left\{ \frac{c^{(n)}}{q^2}, \frac{1}{q} \min_{i \in I_1''} \left\{ \frac{\big|a_{23}^{(n)}\big|^2}{\big|a_{ij(i)}^{(n)}\big|^2} \right\} \right\} \to +\infty \ .$$

On the other hand

$$\sum_{i\in I_1'} \big|a^{(n)}_{ij(i)}\big|^2 \sim C'\,\big|a^{(n)}_{23}\big|^2 \quad \text{for some } C'>0 \tag{V.4.7}$$

and so by (V.4.6) and (V.4.7) we directly get that

$$\sum_{i,j\in I_1'} Q\big([e_1^{(n)},e_i^{(n)}],e_j^{(n)}\big)^2 > \hat c^{(n)} \sum_{\substack{i\in I_1'\\ k\in I_1''\cup I_2}} Q\big([e_1^{(n)},e_i^{(n)}],e_k^{(n)}\big)^2 \quad \text{for any } n\in\mathbb{N}$$

for some sequence $\hat c^{(n)}\to+\infty$.

So, from now on, we assume $I_1=I_1'$ and hence $\big|a^{(n)}_{ij(i)}\big|>0$ for any $n\in\mathbb{N}$, $i\in I_1$. Let also $d:=\dim(\mathfrak{t})$ be the rank of $\mathfrak{g}$.

We are going to prove by induction that there exists a Q-orthonormal basis $(e_{1,1},e_{1,2},\dots,e_{1,d})$ for $\mathfrak{t}$ and a set of vectors $E_i^{(\infty)}\in\mathfrak{a}\setminus\{0\}$, $i\in I_1$, such that for any $s\in\{1,\dots,d\}$ the following claim, which we denote by $\bar P(s)$, holds: there exist a sequence $(e^{(n)}_{1,s})\subset \operatorname{span}(e_{1,s},\dots,e_{1,d})\subset\mathfrak{t}$, with $e^{(n)}_{1,s}\to e_{1,s}$ and, for any $i\in I_1$, a sequence of real numbers $\hat a^{(n)}_{i,s}>0$, with $\hat a^{(n)}_{i,s}\to 0$, such that, if we set

$$e^{(n)}_{i,s} := \begin{cases} e_i^{(n)} & \text{if } s=1\\ \operatorname{pr}_{\mathfrak{c}_{\mathfrak{g}}(e_{1,1})\cap\ldots\cap\mathfrak{c}_{\mathfrak{g}}(e_{1,s-1})}(e_i^{(n)}) & \text{if } s>1\end{cases} ,$$

then

$$\frac{1}{\hat a^{(n)}_{i,s}}[e^{(n)}_{1,s},e^{(n)}_{i,s}] \to E_i^{(\infty)} , \quad e^{(n)}_{i,s}\to e_i \quad \text{as } n\to+\infty \quad , \quad \text{for any } i\in I_1 \quad . \tag{V.4.8}$$

First, we consider the case $s=1$ and we set

$$e_{1,1}:=e_1 , \quad e^{(n)}_{1,1}:=e^{(n)}_1 , \quad \hat a^{(n)}_{i,1}:=a^{(n)}_{ij(i)} \quad \text{for any } i\in I_1 \quad .$$

Next, we define

$$E^{(n)}_{i,1} := \frac{1}{\hat a^{(n)}_{i,1}} \sum_{j\in I_1\setminus\{i\}} a^{(n)}_{ij}\, e^{(n)}_{j,1} , \quad Z^{(n)}_{i,1} := \frac{1}{\hat a^{(n)}_{i,1}}\, z^{(n)}_i$$

in such a way that

$$\frac{1}{\hat a^{(n)}_{i,1}}[e^{(n)}_{1,1},e^{(n)}_{i,1}] = E^{(n)}_{i,1}+Z^{(n)}_{i,1} \quad \text{for any } i\in I_1 \quad . \tag{V.4.9}$$

By (V.4.4) and (V.4.5), it follows that

$$\sum_{i\in I_1} \big|Z^{(n)}_{i,1}\big|^2_Q \leq \varepsilon^{(n)} \quad \text{for some } \varepsilon^{(n)} \to 0 \ ,$$

while, by construction, $E^{(\infty)}_i := \lim_{n\to+\infty} E^{(n)}_{i,1} \neq 0$ and $E^{(\infty)}_i \in \mathfrak{a}\cap\mathfrak{t}^{\perp}$. Hence, it follows that $\bar{P}(1)$ holds. Let us fix now $1 \leq s \leq d-1$ and assume that $\bar{P}(s')$ holds true for any $1 \leq s' \leq s$. Notice that, by the inductive hypothesis, we get $[e_{1,s'}, e_i] = 0$ for any $1 \leq s' \leq s$, $i \in I_1$ and then $\mathfrak{a} \subset \mathfrak{c}_{\mathfrak{g}}(e_{1,1}) \cap \ldots \cap \mathfrak{c}_{\mathfrak{g}}(e_{1,s})$. Here, we denoted by $\mathfrak{c}_{\mathfrak{g}}(X)$ the centralizer of $X \in \mathfrak{g}$ in $\mathfrak{g}$.

We consider now the following Q-orthogonal decompositions:

$$\begin{aligned} e^{(n)}_{1,s} &:= \alpha^{(n)}_s e_{1,s} + \tilde{e}^{(n)}_{1,s+1} \ , \\ e^{(n)}_{i,s} &:= T^{(n)}_i + V^{(n)}_{i,s+1} + W^{(n)}_{i,s+1} \ , \quad i \in I_1 \ , \end{aligned}$$

with $\tilde{e}^{(n)}_{1,s+1} \in \mathfrak{t}$ and $T^{(n)}_i \in \mathfrak{t}$, $V^{(n)}_{i,s+1} \in \mathfrak{c}_{\mathfrak{g}}(e_{1,1}) \cap \ldots \cap \mathfrak{c}_{\mathfrak{g}}(e_{1,s}) \cap \mathfrak{t}^{\perp}$, $W^{(n)}_{i,s+1} \in \big(\mathfrak{c}_{\mathfrak{g}}(e_{1,1}) \cap \ldots \cap \mathfrak{c}_{\mathfrak{g}}(e_{1,s})\big)^{\perp}$. Then

$$[e^{(n)}_{1,s}, e^{(n)}_{i,s}] = [\tilde{e}^{(n)}_{1,s+1}, V^{(n)}_{i,s+1}] + [e^{(n)}_{1,s}, W^{(n)}_{i,s+1}] \ ,$$

with $[\tilde{e}^{(n)}_{1,s+1}, V^{(n)}_{i,s+1}] \in \mathfrak{c}_{\mathfrak{g}}(e_{1,1}) \cap \ldots \cap \mathfrak{c}_{\mathfrak{g}}(e_{1,s}) \cap \mathfrak{t}^{\perp}$ and $[e^{(n)}_1, W^{(n)}_{i,s+1}] \in \big(\mathfrak{c}_{\mathfrak{g}}(e_{1,1}) \cap \ldots \cap \mathfrak{c}_{\mathfrak{g}}(e_{1,s})\big)^{\perp}$. If we set

$$\widetilde{E}^{(n)}_{i,s} := \frac{1}{\hat{a}^{(n)}_{i,s}} [e^{(n)}_{1,s}, e^{(n)}_{i,s}] \ ,$$

we get

$$[\tilde{e}^{(n)}_{1,s+1}, V^{(n)}_{i,s+1}] = \hat{a}^{(n)}_{i,s} \operatorname{pr}_{\mathfrak{c}_{\mathfrak{g}}(e_{1,1})\cap\ldots\mathfrak{c}_{\mathfrak{g}}(e_{1,s})}(\widetilde{E}^{(n)}_{i,s}) \tag{V.4.10}$$

and hence, since $\operatorname{pr}_{\mathfrak{c}_{\mathfrak{g}}(e_{1,1})\cap\ldots\mathfrak{c}_{\mathfrak{g}}(e_{1,s})}(\widetilde{E}^{(n)}_{i,s}) \to E^{(\infty)}_i \neq 0$ as $n \to +\infty$, we deduce that $\tilde{e}^{(n)}_{1,s+1} \neq 0$. Next, we set

$$e^{(n)}_{1,s+1} := \frac{\tilde{e}^{(n)}_{1,s+1}}{|\tilde{e}^{(n)}_{1,s+1}|_Q} \ , \quad e_{1,s+1} := \lim_{n\to+\infty} e^{(n)}_{1,s+1} \ , \quad \hat{a}^{(n)}_{i,s+1} := \frac{\hat{a}^{(n)}_{i,s}}{|\tilde{e}^{(n)}_{1,s+1}|_Q} \ .$$

Since $e^{(n)}_{i,s+1} = T^{(n)}_i + V^{(n)}_{i,s+1}$, it follows that

$$\frac{1}{\hat{a}^{(n)}_{i,s+1}} [e^{(n)}_{1,s+1}, e^{(n)}_{i,s+1}] = \operatorname{pr}_{\mathfrak{c}_{\mathfrak{g}}(e_{1,1})\cap\ldots\mathfrak{c}_{\mathfrak{g}}(e_{1,s})}(\widetilde{E}^{(n)}_{i,s}) = E^{(n)}_{i,s+1} + Z^{(n)}_{i,s+1} \ ,$$

where

$$E^{(n)}_{i,s+1} := \mathrm{pr}_{\mathrm{span}(e^{(n)}_{2,s+1},\dots,e^{(n)}_{q+1,s+1})}\big(\mathrm{pr}_{\mathfrak{c}_{\mathfrak{g}}(e_{1,1})\cap\dots\mathfrak{c}_{\mathfrak{g}}(e_{1,s})}(\widetilde{E}^{(n)}_{i,s})\big) \ ,$$
$$Z^{(n)}_{i,s+1} := \mathrm{pr}_{(\mathrm{span}(e^{(n)}_{2,s+1},\dots,e^{(n)}_{q+1,s+1}))^{\perp}}\big(\mathrm{pr}_{\mathfrak{c}_{\mathfrak{g}}(e_{1,1})\cap\dots\mathfrak{c}_{\mathfrak{g}}(e_{1,s})}(\widetilde{E}^{(n)}_{i,s})\big) \ .$$

Since by inductive hypothesis $\mathfrak{a} \subset \mathfrak{c}_{\mathfrak{g}}(e_{1,1})\cap\ldots\cap\mathfrak{c}_{\mathfrak{g}}(e_{1,s})$, it follows that $e^{(n)}_{i,s+1} \to e_i$ for any $i \in I_1$ and hence

$$E^{(n)}_{i,s+1} \to E^{(\infty)}_i \ , \quad Z^{(n)}_{i,s+1} \to 0 \quad \text{as } n \to +\infty \ .$$

Since $[e_{1,s+1}, e_i] = \hat{a}^{(\infty)}_{i,s+1}E^{(\infty)}_i$, with $\hat{a}^{(\infty)}_{i,s+1} := \lim_{n\to+\infty} \hat{a}^{(n)}_{i,s+1}$, and $e_i, E^{(\infty)}_i \in \mathfrak{a}$, it follows that $\hat{a}^{(\infty)}_{i,s+1} = 0$. This proves that $\bar{P}(s{+}1)$ holds and hence, by induction that $\bar{P}(s)$ holds for any $1 \leq s \leq d$.

By (V.4.8), it follows that

$$[e_{1,s}, e_i] = 0 \ , \quad E^{(\infty)}_i \in \mathfrak{a}\cap\mathfrak{t}^{\perp} \quad \text{for any } i \in I_1,\ 1 \leq s \leq d \ ,$$

and hence $[\mathfrak{t},\mathfrak{a}] = \{0\}$, $\mathfrak{a}\cap\mathfrak{t}^{\perp} \neq \{0\}$. Therefore, $\mathfrak{t}+\mathfrak{a}$ is an abelian Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{t} \subsetneq \mathfrak{t}+\mathfrak{a}$, which is not possible since $\mathfrak{t}$ is maximal by assumption. □

Proof of Proposition V.3.2. From now until the end of the proof, we adopt the notation introduced at the beginning of Section V.3. Assume that $v^{(\infty)} \in \mathcal{S}^{\Sigma}(\mathfrak{k}_1,\dots,\mathfrak{k}_p)$ and that $\mathfrak{k}_q$ is toral for some $1 \leq q \leq p$. From (I.5.25) it follows directly that

$$\begin{aligned}
&\mathrm{scal}\big(g^{(n)}\big) = \\
&\quad = \frac{1}{2}\sum_{i\in I} d_i b^{(n)}_i e^{-t^{(n)}v^{(n)}_i} - \frac{1}{4}\sum_{i,j,k\in I}[ijk]^{(n)}e^{t^{(n)}(v^{(n)}_i-v^{(n)}_j-v^{(n)}_k)} \\
&\quad = \frac{1}{2}\sum_{i\leq r(q)} e^{-t^{(n)}v^{(n)}_i}\Bigg\{\sum_{j,k\leq r(q)}[ijk]^{(n)}\Big(1-\frac{1}{2}e^{t^{(n)}(v^{(n)}_j-v^{(n)}_k)}\Big)+ \\
&\qquad\qquad + \sum_{\substack{j\leq r(q)\\ k>r(q)}}[ijk]^{(n)}\Big(2-\frac{1}{2}e^{t^{(n)}(v^{(n)}_k-v^{(n)}_j)}\Big)- \\
&\qquad\qquad\qquad - \sum_{j,k>r(q)}[ijk]^{(n)}\Big(\frac{1}{2}e^{t^{(n)}(v^{(n)}_j-v^{(n)}_k)}+\frac{1}{2}e^{t^{(n)}(v^{(n)}_k-v^{(n)}_j)}-1\Big)-
\end{aligned}$$

$$- \sum_{\substack{j \leq r(q) \\ k > r(q)}} [ijk]^{(n)} e^{t^{(n)}(v_j^{(n)} - v_k^{(n)})} - \frac{1}{2} \sum_{j,k > r(q)} [ijk]^{(n)} e^{t^{(n)}(2v_i^{(n)} - v_j^{(n)} - v_k^{(n)})} \Bigg\} +$$

$$+ \frac{1}{2} \sum_{i > r(q)} d_i b_i^{(n)} e^{-t^{(n)} v_i^{(n)}} - \frac{1}{4} \sum_{i,j,k > r(q)} [ijk]^{(n)} e^{t^{(n)}(v_i^{(n)} - v_j^{(n)} - v_k^{(n)})} \ .$$

Since $\mathfrak{k}_q$ is toral, it splits as $\mathfrak{k}_q = \mathfrak{h} + \mathfrak{a}$, with $[\mathfrak{h}, \mathfrak{a}] = [\mathfrak{a}, \mathfrak{a}] = \{0\}$ and $\mathfrak{a} \neq \{0\}$. Hence, from (V.4.1), it follows that there exist $\bar{n} \in \mathbb{N}$ and a constant $C > 0$ such that

$$\sum_{j,k \leq r(q)} [ijk]^{(n)} \leq C \sum_{\substack{j \leq r(q) \\ k > r(q)}} [ijk]^{(n)} \quad \text{for any } n \geq \bar{n} \ , \quad 1 \leq i \leq r(q) \ . \tag{V.4.11}$$

We can also assume that there exists $\varepsilon > 0$ such that $v_k^{(n)} - v_j^{(n)} > \varepsilon$ for any $j \leq r(q)$, $k > r(q)$ and $n \geq \bar{n}$. Then

$$\begin{aligned} \sum_{j,k \leq r(q)} [ijk]^{(n)} \Big(1 - \frac{1}{2} e^{t^{(n)}(v_j^{(n)} - v_k^{(n)})}\Big) + \sum_{\substack{j \leq r(q) \\ k > r(q)}} [ijk]^{(n)} \Big(2 - \frac{1}{2} e^{t^{(n)}(v_k^{(n)} - v_j^{(n)})}\Big) \leq \\ \leq \sum_{j,s \leq r(q)} [ijk]^{(n)} + \sum_{\substack{j \leq r(q) \\ k > r(q)}} [ijk]^{(n)} \Big(2 - \frac{1}{2} e^{t^{(n)}\varepsilon}\Big) \\ \leq -\frac{1}{2} \sum_{\substack{j \leq r(q) \\ k > r(q)}} [ijk]^{(n)} \Big(e^{t^{(n)}\varepsilon} - \tilde{C}\Big) \end{aligned}$$

with $\tilde{C} := 2C + 4$. Since

$$\tfrac{1}{2} e^{t^{(n)}(v_j^{(n)} - v_k^{(n)})} + \tfrac{1}{2} e^{t^{(n)}(v_k^{(n)} - v_j^{(n)})} = \cosh t^{(n)}(v_j^{(n)} - v_k^{(n)}) \geq 1 \ ,$$

the claim follows. □

V.5 Algebraically collapsed sequences of G-invariant metrics

In this last section, we are going to apply Theorem V.3.3 to give a characterization of algebraically collapsed sequences of invariant metrics on a given compact homogeneous manifold.

Let $M = \mathsf{G}/\mathsf{H}$ be again a fixed compact, connected and almost effective m-dimensional homogeneous space, with G and H compact Lie groups, and fix an $\mathrm{Ad}(\mathsf{G})$-invariant product Q on $\mathfrak{g}$. If $q = \dim(\mathsf{H})$, then one can embed the space $\mathcal{M}^{\mathsf{G}}$ into $\mathcal{H}_{q,m}$ by associating to any $g \in \mathcal{M}^{\mathsf{G}}$ the unique $\mu \in \mathcal{H}_{q,m}$ such that $(\mathsf{G}_\mu/\mathsf{H}_\mu, g_\mu) = (\mathsf{G}/\mathsf{H}, g)$. We aim to prove the following

Proposition V.5.1. *Let* $(g^{(n)}) \subset \mathcal{M}^{\mathsf{G}}_1$ *be a diverging sequence with bounded curvature and* $(\mu^{(n)})$ *the corresponding sequence of brackets. If* $\pi_1(M)$ *is finite, then* $(\mu^{(n)})$ *is algebraically collapsed.*

which coincides with Proposition V.1.4.

To this purpose, let $g \in \mathcal{M}^{\mathsf{G}}$, $\varphi \in \mathcal{F}^{\mathsf{G}}$ a good decomposition for g, i.e. it takes the form (I.5.5), and μ the bracket corresponding to g. We consider also the Q-orthogonal splitting

$$\mu = (\mu|_{\mathfrak{h}\wedge\mathfrak{g}}) + \mu_{\mathfrak{h}} + \mu_{\mathfrak{m}} \ , \qquad \text{with} \qquad \mu_{\mathfrak{h}} : \mathfrak{m}\wedge\mathfrak{m} \to \mathfrak{h} \ , \quad \mu_{\mathfrak{m}} : \mathfrak{m}\wedge\mathfrak{m} \to \mathfrak{m} \ . \tag{V.5.1}$$

Let (e_α) be a φ-adapted $Q_{\mathfrak{m}}$-orthonormal basis for $\mathfrak{m}$ and (z_γ) be a $Q_{\mathfrak{h}}$-orthonormal basis for $\mathfrak{h}$. Then, we obtain

$$|\mu|^2_{\mathrm{st}} = \big|(\mu|_{\mathfrak{h}\wedge\mathfrak{g}})\big|^2_{\mathrm{st}} + |\mu_{\mathfrak{h}}|^2_{\mathrm{st}} + |\mu_{\mathfrak{m}}|^2_{\mathrm{st}} \tag{V.5.2}$$

with

$$\begin{aligned}
\big|(\mu|_{\mathfrak{h}\wedge\mathfrak{g}})\big|^2_{\mathrm{st}} &= \big|(\mu|_{\mathfrak{h}\wedge\mathfrak{h}})\big|^2_{\mathrm{st}} + \sum_{i\in I}\sum_{\substack{e_\alpha\in\mathfrak{m}_i\\ z_\gamma\in\mathfrak{h}}} \Big|\Big[z_\gamma, \tfrac{e_\alpha}{\sqrt{\lambda_i}}\Big]_{\mathfrak{m}_i}\Big|^2_g \\
&= \big|(\mu|_{\mathfrak{h}\wedge\mathfrak{h}})\big|^2_{\mathrm{st}} + \sum_{i\in I}\sum_{\substack{e_\alpha\in\mathfrak{m}_i\\ z_\gamma\in\mathfrak{h}}} \big|[z_\gamma, e_\alpha]\big|^2_Q \\
&= \big|(\mu|_{\mathfrak{h}\wedge\mathfrak{h}})\big|^2_{\mathrm{st}} + \sum_{i\in I} d_i c_i
\end{aligned} \tag{V.5.3}$$

and

$$\begin{aligned}
|\mu_{\mathfrak{h}}|^2_{\mathrm{st}} &= \sum_{i,j\in I}\sum_{\substack{e_\alpha\in\mathfrak{m}_i\\ e_\beta\in\mathfrak{m}_j}} \Big|\Big[\tfrac{e_\alpha}{\sqrt{\lambda_i}}, \tfrac{e_\beta}{\sqrt{\lambda_j}}\Big]_{\mathfrak{h}}\Big|^2_Q = \sum_{i\in I}\frac{1}{\lambda_i}\sum_{e_\alpha,e_{\alpha'}\in\mathfrak{m}_i} \big|[e_\alpha, e_{\alpha'}]_{\mathfrak{h}}\big|^2_Q = \sum_{i\in I}\frac{d_i c_i}{\lambda_i} \ , \\
|\mu_{\mathfrak{m}}|^2_{\mathrm{st}} &= \sum_{i,j,k\in I}\sum_{\substack{e_\alpha\in\mathfrak{m}_i\\ e_\beta\in\mathfrak{m}_j}} \Big|\Big[\tfrac{e_\alpha}{\sqrt{\lambda_i}}, \tfrac{e_\beta}{\sqrt{\lambda_j}}\Big]_{\mathfrak{m}_k}\Big|^2_g = \sum_{i,j,k\in I} [ijk]_\varphi \frac{\lambda_k}{\lambda_i\lambda_j} \ .
\end{aligned} \tag{V.5.4}$$

Proof of Proposition V.5.1. Since M is connected and the fundamental group $\pi_1(M)$ is finite, up to enlarging the space $\mathcal{M}^{\mathsf{G}}$ of invariant metrics, we can assume that the group G is connected and semisimple. Let us fix a sequence $(g^{(n)}) \subset \mathcal{M}^{\mathsf{G}}_1$ which diverges with bounded curvature. From now until the end of the proof, we adopt the notation introduced in Section V.3. By Lemma V.2.3 and Theorem V.3.3, we can choose $i_{\mathrm{o}} \in I^{\mathrm{sh}}$ and $j_{\mathrm{o}}, s_{\mathrm{o}} \in I \setminus I^{\mathrm{sh}}$ such that $[i_{\mathrm{o}} j_{\mathrm{o}} s_{\mathrm{o}}]^{(\infty)} > 0$. Then, by Theorem V.3.3 and (V.5.4) we directly get

$$\big|\mu^{(n)}_{\mathfrak{m}}\big|^2_{\mathrm{st}} \geq [i_{\mathrm{o}} j_{\mathrm{o}} s_{\mathrm{o}}]^{(n)} \frac{\lambda^{(n)}_{s_{\mathrm{o}}}}{\lambda^{(n)}_{i_{\mathrm{o}}} \lambda^{(n)}_{j_{\mathrm{o}}}} \sim [i_{\mathrm{o}} j_{\mathrm{o}} s_{\mathrm{o}}]^{(\infty)} \frac{1}{\lambda^{(n)}_{i_{\mathrm{o}}}} \to +\infty$$

and so the claim follows. □

The next easy example shows that the finiteness hypothesis on the fundamental group $\pi_1(M)$ cannot be removed.

Example V.5.2. Let $M^3 = S^1 {\times} S^2 = \mathsf{G}/\mathsf{H}$, with $\mathsf{G} := \mathsf{U}(1){\times}\mathsf{SU}(2)$ and $\mathsf{H} := \{1\}{\times}\mathsf{U}(1) \subset \mathsf{G}$. Let us fix an $\mathrm{Ad}(\mathsf{G})$-invariant inner product Q on $\mathfrak{g} = \mathrm{Lie}(\mathsf{G})$ and a Q-orthonormal basis (X_0, X_1, X_2, X_3) for $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}_1 + \mathfrak{m}_2 \ , \quad \mathfrak{h} = \mathrm{span}(X_1)\, , \ \mathfrak{m}_1 = \mathrm{span}(X_0)\, , \ \mathfrak{m}_2 = \mathrm{span}(X_2, X_3) \ ,$$
$$[X_0, X_i] = 0 \ , \quad [X_1, X_2] = -2X_3 \ , \quad [X_2, X_3] = -2X_1 \ , \quad [X_3, X_1] = -2X_2 \ .$$

We consider now the sequence of metrics $g^{(n)} := \frac{1}{n^2} Q_{\mathfrak{m}_1} + n Q_{\mathfrak{m}_2}$, together with the $g^{(n)}$-normalized frame

$$X^{(n)}_0 := n X_0 \ , \quad X^{(n)}_2 := \tfrac{1}{\sqrt{n}} X_2 \ , \quad X^{(n)}_3 := \tfrac{1}{\sqrt{n}} X_3 \ .$$

Then, one can directly check that the curvature operator $\mathrm{Rm}(g^{(n)}) : \Lambda^2\mathfrak{m} \to \Lambda^2\mathfrak{m}$ is diagonal and explicitly given by

$$\mathrm{Rm}(g^{(n)})(X^{(n)}_0 {\wedge} X^{(n)}_2) = \mathrm{Rm}(g^{(n)})(X^{(n)}_0 {\wedge} X^{(n)}_3) = 0 \ ,$$
$$\mathrm{Rm}(g^{(n)})(X^{(n)}_2 {\wedge} X^{(n)}_3) = \tfrac{4}{n} X^{(n)}_2 {\wedge} X^{(n)}_3 \ ,$$

while

$$[X^{(n)}_0, X^{(n)}_2] = [X^{(n)}_0, X^{(n)}_3] = 0 \ , \quad [X^{(n)}_2, X^{(n)}_3] = -\tfrac{2}{n} X_1 \ .$$

So, the sequence $(g^{(n)})$ diverges with bounded curvature and it is algebraically non-collapsed.

Finally, let us consider a sequence $(g^{(n)}) \subset \mathcal{M}^{\mathsf{G}}$ and, up to a normalization, for any $n \in \mathbb{N}$ fix the scale of the most shrinking direction to be 1. This is equivalent of saying that, with respect to a diagonal decomposition as (V.3.1) in the previous section, $\min\{\lambda_1^{(n)}, \dots, \lambda_\ell^{(n)}\} = 1$ for any $n \in \mathbb{N}$. In this case, we say that $(g^{(n)})$ *is normalized with respect to the most shrinking direction.* Notice that any such a sequence is divergent if and only if $\mathrm{vol}(g^{(n)}) \to +\infty$.

Proposition V.5.3. *If* $(g^{(n)}) \subset \mathcal{M}^{\mathsf{G}}$ *is normalized with respect to the most shrinking direction and has bounded curvature, then it is algebraically non-collapsed.*

Proof. Let $(g^{(n)})$ be a divergent sequence of G-invariant metrics with bounded curvature and suppose that it is normalized with respect to the most shrinking direction. As in the proof of Proposition V.1.4, from now on we adopt the notation introduced at the beginning of Section V.3. By (I.5.21), the diagonal terms of the Ricci tensor along the sequence are given by

$$\mathrm{ric}_i(g^{(n)}) = \frac{b_i^{(n)}}{2\lambda_i^{(n)}} - \frac{1}{2d_i}\sum_{j,k\in I}[ijk]^{(n)}\frac{\lambda_k^{(n)}}{\lambda_i^{(n)}\lambda_j^{(n)}} + \frac{1}{4d_i}\sum_{j,k\in I}[ijk]^{(n)}\frac{\lambda_i^{(n)}}{\lambda_j^{(n)}\lambda_k^{(n)}}\ . \qquad \text{(V.5.5)}$$

Suppose by contradiction that $(g^{(n)})$ is algebraically collapsed. Since from our normalization $\lambda_i^{(n)} \geq 1$ for any $n \in \mathbb{N}$, $1 \leq i \leq \ell$, from (V.5.4) we get necessarily that $|\mu_{\mathfrak{m}}|_{g^{(n)}} \to +\infty$. So, again by (V.5.4) there exists a triple $(i_1, i_2, i_3) \in I^3$ such that $[i_1i_2i_3]^{(n)}\frac{\lambda_{i_1}^{(n)}}{\lambda_{i_2}^{(n)}\lambda_{i_3}^{(n)}} \to +\infty$. Since $\mathrm{ric}_{i_1}(g^{(n)})$ is bounded, by (V.5.5) there exist $i_4, i_5 \in I$ such that $[i_1i_4i_5]^{(n)}\frac{\lambda_{i_4}^{(n)}}{\lambda_{i_1}^{(n)}\lambda_{i_5}^{(n)}} \to +\infty$. By the way, $\mathrm{ric}_{i_4}(g^{(n)})$ is bounded too and then there exist $i_6, i_7 \in I$ such that $[i_4i_6i_7]^{(n)}\frac{\lambda_{i_6}^{(n)}}{\lambda_{i_4}^{(n)}\lambda_{i_7}^{(n)}} \to +\infty$. Iterating this procedure, we obtain two sequences $(i_s), (j_s) \subset I$ such that $[i_sj_sj_{s+1}]^{(n)}\frac{\lambda_{j_{s+1}}^{(n)}}{\lambda_{i_s}^{(n)}\lambda_{j_s}^{(n)}} \to +\infty$. Since $I = \{1, \dots, \ell\}$ is finite and the relation defined on the set $\{\lambda_1^{(n)}, \dots, \lambda_\ell^{(n)}\}$ by

$$a^{(n)} \prec b^{(n)} \iff \tfrac{b^{(n)}}{a^{(n)}} \to +\infty$$

is asymmetric and transitive, the sequences (i_s) and (j_s) are necessarily finite too, i.e. they are of the form $(i_1, \dots, i_{s_o})$ and $(j_1, \dots, j_{s_o})$, respectively. So, it follows that $\mathrm{ric}_{j_{s_o}}(g^{(n)}) \to +\infty$ and this is a contradiction. $\square$

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