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Research article

Global existence and regularity for the dynamics of viscous oriented fluids

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Abstract: We prove global existence of weak solutions to regularized versions of balance equations representing the dynamics over a torus of complex fluids, with microstructure described by a vector field taking values in the unit ball. Regularization is offered by the presence of second-neighbor microstructural interactions and our choice of filtering the balance of macroscopic momentum by inverse Helmholtz operator with unit length scale.

Keywords: **Mechanics of complex materials; spin fluids; second-neighbor interactions; hyperbolic equations**
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1. Introduction

For sufficiently differentiable maps $\tilde{u} : \mathbb{T}^2 \times [0, T] \to \widetilde{\mathbb{R}^2}$ and $\tilde{\nu} : \mathbb{T}^2 \times [0, T] \to S^2$, with $\mathbb{T}^2$ a torus and S^2 the unit sphere, we have shown in reference [12] that the system

$$
\begin{aligned}
&u_t + (u \cdot \nabla)u - \Delta u + \nabla \pi = -\nabla \cdot (\nabla \nu^\top \nabla \nu) - \nabla \nu^\top \Delta \nu_t,\\
&\nabla \cdot u = 0,\\
&\Delta \nu_t + \Delta((u \cdot \nabla)\nu) - \Delta^2 \nu = \nu_t + (u \cdot \nabla)\nu + |\nabla \nu|^2 \nu - \Delta \nu,
\end{aligned}
$$

reasonably describes the dynamics over $\mathbb{T}^2$ of oriented (i.e., polarized or spin) fluids, a representation in which we account for second-neighbor director interactions in a minimalistic way, the one giving us sufficient amount of regularity to allow existence of a certain class of weak solutions.

In the balance of microstructural actions governing the evolution of ν, an hyper-stress behaving like $\nabla^2 \nu$ accounts for second-neighbor interactions; it enters the equation through its double divergence, which generates the term $\Delta^2 \nu$. A viscous-type contribution (namely $\nabla\nu^\top \Delta\nu_t$) affects the Ericksen stress in the balance of macroscopic momentum, an equation where π is the pressure, i.e., the reactive stress associated with the volume-preserving constraint $\nabla \cdot u = 0$.

We have explicitly underlined in reference [12] the terms neglected in the previous balance equations with respect to a complete representation of second-neighbor director interactions, and their contribution to the Ericksen stress.

Also, to tackle the analysis of such balances, in reference [12] we considered transient states foreseeing $|\nu| \leq 1$ (i.e., a polarized fluid not in saturation conditions) and replaced the nonlinear term $|\nabla\nu|^2\nu$ with its approximation $\frac{1}{\varepsilon^2}(1-|\nu|^2)\nu$, ε a positive parameter. Eventually, we established just *local* existence of a certain class of weak solutions.

The description of such fluids falls within the general model-building framework of the mechanics of complex materials (a format involving manifold-valued microstructural descriptors) in references [26] and [27] (see also [28], [29]). By following that format, if we *derive* balance equations by requiring invariance of the sole external power of actions under isometric changes in observers even just for first-neighbor interactions, since the infinitesimal generator of $SO(3)$ action over S^2 is $-\nu\times$, we find the possible existence of a conservative self-action proportional to ν, i.e., something like $\lambda\nu$, with $\lambda \geq 0$.

Consequently, we consider here a relaxed version of the balances above by accounting for $|\nu| \leq 1$ and introducing the self-action $\lambda\nu$. Then, we write

$$u_t + (u\cdot\nabla)u - \Delta u + \nabla\pi = -\nabla\cdot(\nabla\nu^\top\nabla\nu) - \nabla\nu^\top\Delta\nu_t, \tag{1.1}$$

$$\nabla\cdot u = 0, \tag{1.2}$$

$$\Delta\nu_t + \Delta((u\cdot\nabla)\nu) - \Delta^2\nu = \nu_t + (u\cdot\nabla)\nu - \Delta\nu + \frac{1}{\varepsilon^2}(1-|\nu|^2)\nu - \lambda\nu, \quad |\nu| \leq 1, \tag{1.3}$$

with initial conditions

$$u|_{t=0} = u_0, \qquad \nu|_{t=0} = \nu_0. \tag{1.4}$$

We tackle its analysis by filtering the balance of macroscopic momentum by $(I-\Delta)^{-1}$. In the process, we define the regularized velocity

$$w := (I-\Delta)^{-1}u,$$

and approximate the filtered version of equation (1.1) by considering that $\nabla\cdot(I-\Delta)^{-1}(u\otimes u) \approx \nabla\cdot(w\otimes w)$. Then, we apply the inverse filter $(I-\Delta)$ (and we write once again π and ν for pressure and director field, respectively). The resulting system reads

$$w_t - \Delta w_t + (w\cdot\nabla)w - \Delta w + \nabla\pi = -\nabla\cdot(\nabla\nu^\top\nabla\nu) - \nabla\nu^\top\Delta\nu_t, \tag{1.5}$$

$$\nabla\cdot w = 0, \tag{1.6}$$

$$\Delta\nu_t + \Delta((w\cdot\nabla)\nu) - \Delta^2\nu = \nu_t + (w\cdot\nabla)\nu - \Delta\nu + \frac{1}{\varepsilon^2}(1-|\nu|^2)\nu - \lambda\nu, \quad |\nu| \leq 1. \tag{1.7}$$

For it, we prove global existence of weak solutions (defined as in reference [12]).

The obtained regularity could allow us to obtain a uniqueness result. Also, the granted global existence of weak solutions can be used for analyzing possible weak or strong attractors, which we may foresee in appropriate state spaces. All these aspects will be matter of a forthcoming work.

2. Notation and preliminaries

For $p \geq 1$, by $L^p = L^p(\mathbb{T}^2)$ we indicate the usual Lebesgue space with norm $\|\cdot\|_p$. When $p = 2$, we use the notation $\|\cdot\| := \|\cdot\|_{L^2}$ and denote by $(\cdot, \cdot)$ the related inner product. Moreover, with k a nonnegative integer and $p \geq 1$, we denote by $W^{k,p} := W^{k,p}(\mathbb{T}^2)$ the usual Sobolev space with norm $\|\cdot\|_{k,p}$ (using $\|\cdot\|_k$ when $p = 2$). We write $W^{-1,p'} := W^{-1,p'}(\mathbb{T}^2)$, $p' = p/(p-1)$, for the dual of $W^{1,p}(\mathbb{T}^2)$ with norm $\|\cdot\|_{-1,p'}$.

Let X be a real Banach space with norm $\|\cdot\|_X$. We will use the customary spaces $W^{k,p}(0,T;X)$, with norm denoted by $\|\cdot\|_{W^{k,p}(0,T;X)}$. In particular, $W^{0,p}(0,T;X) = L^p(0,T;X)$ are the standard Bochner spaces.

$(L^p)^n := L^2(\mathbb{T}^2, \mathbb{R}^n)$, $p \geq 1$, is the function space of vector-valued L^2-maps. Similarly, $(W^{k,p})^n := (W^{k,p}(\mathbb{T}^2))^n$ is the usual Sobolev space of vector-valued maps with components in $W^{k,p}$, while $(H^s)^n$ is the space of vector-valued maps with components in $H^s = W^{s,2} \cap \{w : \nabla \cdot w = 0\}$. We also define the following spaces:

$$
\begin{aligned}
&\mathcal{H} := \text{closure of } C_0^\infty(\mathbb{T}^2, \mathbb{R}^2) \cap \{w \mid \nabla \cdot w = 0\} \text{ in } (L^2)^2,\\
&\mathcal{H}^s := \text{closure of } C_0^\infty(\mathbb{T}^2, \mathbb{R}^2) \cap \{w \mid \nabla \cdot w = 0\} \text{ in } (W^{s,2})^2,\\
&\boldsymbol{H}^s := \{v \in (W^{s,2})^3\}.
\end{aligned}
$$

This last space is the usual Sobolev space of vector fields with components $W^{s,2}$-functions. Again $\boldsymbol{H} := \boldsymbol{H}^0$. By $\mathcal{H}^{-s}$ we indicate the space dual to $\mathcal{H}^s$. We denote by $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\mathcal{H}^{-1},\mathcal{H}^1}$ the duality pairing between $\mathcal{H}^{-1}$ and $\mathcal{H}^1$. We will also assume that the vector fields u and w have null average on $\mathbb{T}^2$. In particular, under such an assumption, Poincaré's inequality holds true.

Here and in the sequel, we denote by c (or $\bar{c}$) positive constants, which may assume different values.

We'll make use of the following well-known inequalities (see, e.g., [1, 2, 16, 19, 22, 23, 33]):

Ladyzhenskaya's,

$$\|v\|_{L^4} \leq C\|v\|^{\frac{1}{2}}\|\nabla v\|^{\frac{1}{2}}, \;\; v \in \mathcal{H}^1, \tag{2.1}$$

Agmon's,

$$\|v\|_{L^\infty} \leq C\|v\|^{\frac{1}{2}}\|\Delta v\|^{\frac{1}{2}}, \;\; v \in \mathcal{H}^2. \tag{2.2}$$

In the sequel (especially to get estimates in $\boldsymbol{H}^s$, with s non-integer) we'll also make use of commutator-type estimates as the one in the following lemma concerning the operator Λ^s, $s \in \mathbb{R}^+$ (see, e.g., [20, 21, 31], see also [32, 6]), with $\Lambda := (-\Delta)^{1/2}$.

Lemma 2.1. *For $s > 0$ and $1 < r \leq \infty$, and for smooth enough u and v, we get*

$$\|\Lambda^s(uv)\|_{L^r} \leq c(\|u\|_{L^{p_1}}\|\Lambda^s v\|_{L^{q_1}} + \|v\|_{L^{p_2}}\|\Lambda^s u\|_{L^{q_2}}), \tag{2.3}$$

where $1/r = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$ and c is a suitable positive constant.

We also recall the following result about product-laws in Sobolev spaces ([17, Theorem 2.2], see also [30])

Lemma 2.2. *Let $s_0,\ s_1,\ s_2 \in \mathbb{R}$. The product estimate*

$$\|fg\|_{H^{-s_0}} \leq c\|f\|_{H^{s_1}}\|g\|_{H^{s_2}} \tag{2.4}$$

holds, provided that

$$s_0 + s_1 + s_2 \geq \frac{n}{2}, \text{ where } n \text{ is the space dimension}, \tag{2.5}$$
$$s_0 + s_1 \geq 0, \tag{2.6}$$
$$s_0 + s_2 \geq 0, \tag{2.7}$$
$$s_1 + s_2 \geq 0, \tag{2.8}$$

If in (2.5) *the equality sign holds, inequalities* (2.6)–(2.8) *must be strict.* (2.9)

Set $\mathcal{T}_2 := 2\pi\mathbb{Z}^2/L$. $\mathbb{T}^2$ is the torus defined by $\mathbb{T}^2 := (\mathbb{R}^2/\mathcal{T}_2)$. We can expand $w \in \boldsymbol{H}^s(\mathbb{T}^2)$ in Fourier series as

$$w(x) = \sum_{k\in\mathcal{T}_2^\star} \widehat{w}_k e^{ik\cdot x},$$

with $k = (k_1, k_2) \in \mathbb{Z}^2$ the wave-number, $|k| = \sqrt{|k_1|^2 + |k_2|^2}$. The Fourier coefficients for w are defined by $\widehat{w}_k := \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} w(x)e^{-ik\cdot x}dx$. The norm in $\boldsymbol{H}^s$ is given by

$$\|w\|^2_{H^s} = \sum |k|^{2s}|\widehat{w}_k|^2,$$

and the inner product $(\,\cdot\,,\,\cdot\,)_{H^s} = (\,\Lambda^s\,\cdot\,,\,\Lambda^s\,\cdot\,)$ is characterized by

$$(w, v)_{H^s} = \sum_{|k|\geq 1} |k|^{2s}\widehat{w}_k \cdot \overline{\widehat{v}}_k,$$

where the over-bar denotes, as usual, complex conjugation. Consider the inverse Helmholtz operator

$$G := (I - \Delta)^{-1}, \tag{2.10}$$

taking values

$$Gw(x) := \int_{\mathbb{T}^2} G(x, y)w(y))\mathrm{d}y, \tag{2.11}$$

where $G(x, y)$ is the associated Green function (see, e.g. [5, 7, 8, 9, 10]). For $w \in \boldsymbol{H}^s$, take the Fourier expansion $w = \sum_{k\in\mathcal{T}_2^\star} \widehat{w}_k e^{ik\cdot x}$, so that, by inserting this expression in (2.11), we get

$$Gw := \sum_{k\in\mathcal{T}_2^\star} \frac{1}{1 + |k|^2}\widehat{w}_k e^{ik\cdot x}. \tag{2.12}$$

G is self-adjoint. It commutes with differential operators (see, e.g., [4, 5, 7]). We get also

$$(Gv, w) = (G^{1/2}v, G^{1/2}w)_{L^2} = (v, w)_{H^{-1}} \text{ and } (G^{1/2}v, G^{1/2}v)_{L^2} = \|v\|_{H^{-1}}. \tag{2.13}$$

(see also [5, 19]).

3. Existence and regularity result

We set

$$f_\varepsilon(v) := \frac{1}{\varepsilon^2}(1-|v|^2)v,\ \ \varepsilon > 0.$$

Then, we rewrite the filtered balances as

$$w_t - \Delta w_t + (w\cdot\nabla)w - \Delta w + \nabla\pi = -\nabla\cdot(\nabla v^\top \nabla v) - \nabla v^\top \Delta v_t, \tag{3.1}$$

$$\nabla\cdot w = 0, \tag{3.2}$$

$$\Delta v_t + \Delta((w\cdot\nabla)v) - \Delta^2 v = v_t + (w\cdot\nabla)v - \Delta v + f_\varepsilon(v) - \lambda v,\ \ |v|\le 1, \tag{3.3}$$

with initial conditions

$$w|_{t=0} = w_0, \qquad v|_{t=0} = v_0. \tag{3.4}$$

To keep the notation compact, here and in the sequel we omit the dependence of w and v on ε.

Definition 3.1 (Regular weak solution)**.** *For a given $T > 0$, a pair (w, v) is a regular weak solution of* (3.1)*–to–*(3.3) *if $(w, v) \in L^\infty(0, T; \mathcal{H}^{\frac{3}{2}} \times \boldsymbol{H}^{\frac{5}{2}})$, $(\partial_t w, \partial_t v) \in L^2(0, T; \mathcal{H}^1 \times \boldsymbol{H}^{\frac{3}{2}})$, and*

$$\begin{aligned}\int_0^T &\Big((w_t(s), \nu(s)) + (\nabla w_t(s),\ \nabla\nu(s)) + ((w(s)\cdot\nabla)w(s), \nu) + (\nabla w(s), \nabla\nu(s))\Big)\,\mathrm{d}s\\ &= \int_0^T \Big((\nabla v^\top \nabla v(s), \nabla\nu(s)) + (\nabla v^\top \nabla v_t(s), \nabla\nu(s))\Big)\,\mathrm{d}s,\end{aligned} \tag{3.5}$$

holds true for every $\nu \in C_0^\infty((0, T)\times\mathbb{T}^2)$, and

$$\begin{aligned}\int_0^T &\Big((\nabla v_t(s),\nabla h(s)) + (\nabla[(w(s)\cdot\nabla)v(s)], \nabla h(s)) + (\Delta v(s), \Delta h(s))\Big)\,\mathrm{d}s\\ &= \int_0^T \Big((v_t(s), h(s)) + ((w(s)\cdot\nabla))v(s), h(s))\\ &\qquad + (f_\varepsilon(v(s)), h(s)) + (\nabla v(s), \nabla h(s)) - \lambda(v(s), h(s))\Big)\,\mathrm{d}s\end{aligned} \tag{3.6}$$

for every $h(t, x) = \psi(t)\phi(x)$, with $\phi \in \boldsymbol{H}^{\frac{5}{2}}$, $\psi \in C_0^\infty(0, T)$, and $|v(x, t)| \le 1$ a.e. in $(0, T)\times\mathbb{T}^2$.

In the following, we'll always refer to "regular weak solutions" simply as "weak solutions", for the sake of brevity,.

Theorem 3.1. *Assume $(w_0, v_0) \in \mathcal{H}^{\frac{3}{2}}\times\boldsymbol{H}^{\frac{5}{2}}$, with $|v_0(x)| \le 1$ for a.e. $x \in \mathbb{T}^2$. Then, system* (3.1)*–to–*(3.3)*, supplied with* (3.4)*, admits a weak solution (w, v) which is defined for any fixed time $T \ge 0$.*

The chosen regularity for the initial data allows the reader to compare easily the result here with what we got in reference [12], realizing our passage from local (short time) to global (large fixed time) existence. Also, by renouncing to a certain amount of solution regularity (i.e., considering a weaker class) we could accept data $(w_0, v_0) \in \mathcal{H}^1 \times \boldsymbol{H}^2$, obtaining for them once again an existence result (see Remark 4.1 below).

Remark 3.1. *For the integral* $\int_{\mathbb{T}^2} \nabla((w\cdot\nabla)v)\cdot\nabla\omega \, \mathrm{d}x$*, with* $w \in \mathcal{H}^1$*,* $\omega \in \boldsymbol{H}^1$*, and* $v \in \boldsymbol{H}^2$*, we get*

$$\begin{aligned}\int_{\mathbb{T}^2} \nabla((w\cdot\nabla)v)\cdot\nabla\omega \, \mathrm{d}x &= \int_{\mathbb{T}^2} \partial_j(w^i\partial_i v^k)\partial_j\omega^k \, \mathrm{d}x \\ &= \int_{\mathbb{T}^2} \partial_j w^i \partial_i v^k \partial_j \omega^k \, \mathrm{d}x + \int_{\mathbb{T}^2} w^i \partial_{ij} v^k \partial_j \omega^k \, \mathrm{d}x.\end{aligned}$$

The first term on the right-hand side of the above identity is such that

$$\int_{\mathbb{T}^2} \partial_j w^i \partial_i v^k \partial_j \omega^k \, \mathrm{d}x = \int_{\mathbb{T}^2} \partial_i v^k \partial_j \omega^k \partial_j w^i \, \mathrm{d}x = \int_{\mathbb{T}^2} (\nabla v^\top \nabla\omega)\cdot\nabla w \, \mathrm{d}x,$$

and for the second term we find

$$\int_{\mathbb{T}^2} w^i \partial_{ij} v^k \partial_j \omega^k \, \mathrm{d}x = \int_{\mathbb{T}^2} \partial_{ij} v^k \partial_j \omega^k w^i \, \mathrm{d}x = \int_{\mathbb{T}^2} \nabla(\nabla v)^\top \nabla\omega \cdot w \, \mathrm{d}x.$$

For the second term on the right-hand side of (3.1)*, we compute*

$$\nabla v^\top \Delta v_t = \nabla\cdot(\nabla v^\top \nabla v_t) - \nabla(\nabla v)^\top \nabla v_t. \tag{3.7}$$

4. Proofs

We introduce Galerkin's approximating functions $\{(w^n, v^n)\}$, prove a maximum principle, by which the constraint $|v^n| \leq 1$ is verified, and compute some a-priori estimates. The Aubin-Lions compactness theorem [25] allows us to get convergence of a subsequence. Actually, we apply Galerkin's procedure originally used for the standard Navier-Stokes equations, by adapting it to system (3.1)–to–(3.3). (Further details about such a scheme appear in references [24, §2], [13, Appendix A], [7] [11].)

(**Note**: In the sequel, for the sake of conciseness we'll often avoid writing explicitly the integration measure in some integrals, every time we find it appropriate.)

4.1. Approximate Galerkin solutions

Galerkin's method is applied directly only to the velocity field w (this scheme is also known as "semi-Galerkin formulation"; see, e.g, [13]).

For any positive integer i, let us denote by $(\omega_i, \pi_i) \in \mathcal{H}^2 \times W^{1,2}$ the unique solution of the following Stokes problem:

$$\begin{aligned} &\Delta\omega_i + \nabla\pi_i = -\lambda_i\omega_i, && \text{in } \mathbb{T}^2, \\ &\nabla\cdot\omega_i = 0, && \text{in } \mathbb{T}^2, \end{aligned} \tag{4.1}$$

with $\int_{\mathbb{T}^2} \pi_i dx = 0$, for $i = 1, 2, \ldots$ and $0 < \lambda_1 \leq \lambda_2 \leq \ldots \lambda_n \ldots$ with $\lambda_n \to +\infty$, as $n \to \infty$. Functions $\{\omega_i\}_{i=1}^{+\infty}$ determine an orthonormal basis in $\mathcal{H}$ made of the eigenfunctions pertaining to (4.1).

Let $\mathbf{P}_n : \mathcal{H}^{3/2} \to \mathcal{H}_n := \mathcal{H}^{3/2} \cap \operatorname{span}\{\omega_1, \omega_2, \ldots, \omega_n\}$ be the orthonormal projection of $\mathcal{H}^{3/2}$ on its finite dimensional subspace $\mathcal{H}_n$. Take $T > 0$. For every positive integer n, we look for an approximate solution $(w^n, v^n) \in C^1(0, T; \mathcal{H}_n) \times L^\infty(0, T, \boldsymbol{H}^{\frac{3}{2}}) \cap L^2(0, T, \boldsymbol{H}^{\frac{5}{2}})$ to system (3.1)–to–(3.3) with

$$w^n(t, x) = \sum_{i=1}^{n} \phi_i^n(t)\omega_i(x), \quad \phi_i^n \text{ to be determined.} \tag{4.2}$$

Consider the following problem defined a.e. in $(0,T)\times\mathbb{T}^2$:

$$\begin{aligned}&(w_t^n(t)-\Delta w_t^n(t),v^n)_{H^{\frac{1}{2}}}+\big((w^n(t)\cdot\nabla)w^n(t),v^n\big)_{H^{\frac{1}{2}}}+(\nabla w^n(t),\nabla v^n)_{H^{\frac{1}{2}}}\\&\quad=\big((\nabla(v^n)^\top\nabla v^n)(t),\nabla v^n\big)_{H^{\frac{1}{2}}}+\big((\nabla(v^n)^\top\Delta v_t)(t),v^n\big)_{H^{\frac{1}{2}}},\quad\forall v^n\in\mathcal{H}_n,\end{aligned}\tag{4.3}$$

$$(I-\Delta)[\partial_t v^n(t)-\Delta v^n(t)+((w^n(t)\cdot\nabla)v^n(t))]=-f_\varepsilon(v^n(t))+\lambda v^n,\tag{4.4}$$

$$|v^n|\le 1,\tag{4.5}$$

$$w^n(x,0)=w_0^n(x):=\mathbf{P}_n(w_0)(x),\ \ v^n(0,x)=v_0(x),\ \text{for } x\in\mathbb{T}^2,\tag{4.6}$$

where $w_0\in\mathcal{H}^{\frac{3}{2}}$ and $v_0\in\boldsymbol{H}^{\frac{5}{2}}$, with $|v_0(x)|\le 1$ a.e. in $\mathbb{T}^2$.

Instead of exploiting test functions in L^2, we take directly the formulation in $H^{1/2}$, for it provides the needed regularity, The pertinent analysis develops in two steps:

–Step A: Let $\overline{w}^n\in C^1(0,T;\mathcal{H}_n)$ be a given velocity field of the form $\overline{w}^n(t,x)=\sum_{i=1}^n\overline{\phi}_i^n(t)\omega_i(x)$, with $\overline{\phi}_i^n$ assigned. For

$$(I-\Delta)[v_t^n(t)-\Delta v^n(t)+(\overline{w}^n(t)\cdot\nabla)v^n(t)]=-f_\varepsilon(v^n(t))+\lambda v^n(t),\quad\text{a.e. in }(0,T)\times\mathbb{T}^2,$$

with $v^n(0,x)=v_0(x),$ for $x\in\mathbb{T}^2$, we actually look for a vector field

$$v^n\in L^\infty(0,T;\boldsymbol{H}^{\frac{3}{2}})\cap L^2(0,T;\boldsymbol{H}^{\frac{5}{2}}),\quad v_t\in L^2(0,T;\boldsymbol{H}^1)$$

solving a.e. on $(0,T)\times\mathbb{T}^2$ the following system:

$$v_t^n(t)-\Delta v^n(t)+(\overline{w}^n(t)\cdot\nabla)v^n(t)=-G(f_\varepsilon(v^n(t)))+\lambda G(v^n(t)),\tag{4.7}$$

$$v^n(0,x)=v_0(x),\ \text{for } x\in\mathbb{T}^2,\tag{4.8}$$

where G is once again the inverse Helmholtz operator $G=(I-\Delta)^{-1}$ introduced in (4.1). Since G has Fourier symbol corresponding to the inverse of two spatial derivatives, the right-hand side part of (4.7) results to be regularized (i.e., the terms $-Gf_\varepsilon(v^n)$ gains two additional spatial derivatives with respect to $f_\varepsilon(v^n)$; the same occurs for $Gv^n(t)$). Thus, this new expression can be rewritten equivalently as a semilinear parabolic equation in the unknown v^n. The existence of such v^n is guaranteed by the classical theory of parabolic equations (see, e.g., [18]), which also provides higher regularity results (see [18, Theorem 6, Ch. 7.1]). They allow us to use the regularity of initial data $v_0\in\boldsymbol{H}^{\frac{5}{2}}$ to get $v^n\in L^\infty(0,T;\boldsymbol{H}^{\frac{3}{2}})\cap L^2(0,T;\boldsymbol{H}^{\frac{5}{2}})$ and $v_t^n\in L^2(0,T;\boldsymbol{H}^1)$ (by interpolation we also have that $v_t\in C(0,T;\boldsymbol{H}^1)$). The following lemma (see [12, Lemma 4.1] and also [15, Lemma 2.1]) guarantees the constraint $|v|\le 1$.

Lemma 4.1 (Weak maximum principle)**.** *Let $v_0\in\boldsymbol{H}^{\frac{5}{2}}$ be such that $|v_0(x)|\le 1$ for a.e. $x\in\mathbb{T}^2$. Take $\bar{w}^n\in C(0,T;\mathcal{H}_n)$. Then, there exists a weak solution $v^n\in L^\infty(0,T;\boldsymbol{H}^{\frac{3}{2}})\cap L^2(0,T;\boldsymbol{H}^{\frac{5}{2}})$ to the problem* (4.7)–(4.8)*. Moreover, fixed $\epsilon>0$ large enough in the definition of f_ϵ, every such weak solution verifies $|v^n(x,t)|\le 1$ a.e. on $\mathbb{T}^2\times[0,T]$.*

In performing the next calculations, we could relax hypotheses by assuming that $v_0\in\boldsymbol{H}^1$ and is such that $|v_0(x)|\le 1$ a.e. $x\in\mathbb{T}^2$, with $\bar{w}^n\in C(0,T;\boldsymbol{H}_n)$. Then, there would exist a weak solution $v\in L^\infty(0,T;\boldsymbol{H}^1)\times L^2(0,T;\boldsymbol{H}^2)$, with $|v(x,t)|\le 1$ a.e. in $\mathbb{T}^2\times(0,T)$. However, for the sake of simplicity, we still use the same regularity assumptions previously introduced and we denote by v and w the quantities v^n and $\bar{w}^n$, respectively for the sake of conciseness.

Proof. Existence of the solution $v^n \in L^\infty(0,T;\boldsymbol{H}^{\frac{3}{2}}) \cap L^2(0,T;\boldsymbol{H}^{\frac{5}{2}})$ to (4.7)–(4.8) has been already mentioned above.

Define $\varphi(x,t) = (|v(x,t)|^2 - 1)_+$, where $z_+ = \max\{z,0\}$ for each $z \in \mathbb{R}$. Assume there exists a measurable subset $B \subset \mathbb{T}^2$ with positive measure $|B| > 0$ such that $|v(x,t)| > 1$ a.e. in $B \times (t_1,t_2]$, $0 \le t_1 < t_2 \le T$, and $|v(x,t)| = 1$ a.e. in $\partial B \times (t_1,t_2]$. By taking φv as a test function against (4.7), we get

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{T}^2} \partial_t(|v|^2)\varphi + \int_{\mathbb{T}^2}(w\cdot\nabla)|v|^2\varphi + \int_{\mathbb{T}^2}\nabla v\cdot(\varphi v)\\ -\frac{1}{\epsilon^2}\int_{\mathbb{T}^2} G(|v|^2-1)v\cdot\varphi v + \lambda\int_B (Gv)\cdot(\varphi v) = 0,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\frac{1}{2}\int_B \partial_t(|v|^2)\varphi + \int_B(w\cdot\nabla)|v|^2\varphi + \int_B\nabla v\cdot\nabla(\varphi v)\\ -\frac{1}{\epsilon^2}\int_B G^{1/2}(\varphi v)\cdot G^{1/2}(\varphi v) + \lambda\int_B (G^{1/2}v)\cdot(G^{1/2}v) = 0.\end{aligned} \tag{4.9}$$

With $\|\cdot\|$ indicating $\|\cdot\|_{L^2(B)}$, we can also write

$$\frac{1}{2}\int_B \partial_t(|v|^2)\varphi = \frac{1}{2}\int_B \partial_t(|v|^2-1)\varphi = \frac{1}{4}\frac{d}{dt}\|\varphi\|^2,$$

$$\int_B(w\cdot\nabla)|v|^2\varphi = \int_B (w\cdot\nabla)(|v|^2-1)\varphi = \int_B(w\cdot\nabla)\varphi\cdot\varphi = 0,$$

$$\begin{aligned}\int_B \nabla v\cdot\nabla(\varphi v) &= \frac{1}{2}\int_B \nabla(|v|^2)\cdot\nabla\varphi + \int_B|\nabla v|^2\varphi\\ &= \frac{1}{2}\int_B\nabla(|v|^2-1)\nabla\varphi + \int_B|\nabla v|^2\varphi\\ &= \frac{1}{2}\|\nabla\varphi\|^2 + \int_B|\nabla v|^2\varphi \ge \frac{1}{2}\|\nabla\varphi\|^2 \ge 0.\end{aligned}$$

Then, equation (4.9) becomes

$$\begin{aligned}\frac{d}{dt}\|\varphi\|^2 + 2\|\nabla\varphi\|^2 + 4\int_B|\nabla v|^2\varphi - \frac{4}{\epsilon^2}\int_B G^{1/2}(\varphi v)\cdot G^{1/2}(\varphi v)\\ + 4\lambda\int_B (G^{1/2}v)\cdot G^{1/2}(\varphi v) = 0.\end{aligned} \tag{4.10}$$

Since $\varphi(t_2) \ge \varphi(t_1)$ (here, $\varphi(t_1) = 0$), by integrating in time over $(t_1,t_2]$, we get

$$\begin{aligned}2\int_{t_1}^{t_2}\|\nabla\varphi\|^2 + 4\int_{t_1}^{t_2}\Big(\int_B|\nabla v|^2\varphi - \frac{4}{\epsilon^2}\int_B (G^{1/2}v)\cdot G^{1/2}(\varphi v)\\ + 4\lambda\int_B (G^{1/2}v)\cdot G^{1/2}(\varphi v)\Big) \le 0.\end{aligned}$$

In principle, B may have more than one connected component with positive measure. However, these components are finite in number for $\overline{B}$ is compact. Thus, previous inequality can be rewritten as

$$\sum_i \Big(2\int_{t_1}^{t_2}\int_{B_i}|\nabla\varphi|^2 + 4\int_{t_1}^{t^2}\Big(\int_{B_i}|\nabla v|^2\varphi - \frac{4}{\epsilon^2}\int_{B_i}G^{1/2}(\varphi v)\cdot G^{1/2}(\varphi v) \\ +4\lambda\int_{B_i}(G^{1/2}v)\cdot G^{1/2}(\varphi v)\Big)\Big) \le 0.$$

Then, there exists at least one connected component B_j, with $|B_j|>0$, on which

$$2\int_{t_1}^{t_2}\int_{B_j}|\nabla\varphi|^2 + 4\int_{t_1}^{t_2}\left(\int_{B_j}|\nabla v|^2\varphi - \frac{4}{\epsilon^2}\int_{B_j}G^{1/2}(\varphi v)\cdot G^{1/2}(\varphi v) \right.\\ \left. +4\lambda\int_{B_j}G^{1/2}(v)\cdot G^{1/2}(\varphi v)\right) \le 0,$$

and hence, by(2.13), we have

$$\begin{aligned} 2\int_{t_1}^{t_2}\|\nabla\varphi\|^2_{L^2(B_j)} + 4\int_{t_1}^{t^2}\int_{B_j}|\nabla v|^2\varphi &\le \frac{4}{\epsilon^2}\int_{t_1}^{t^2}\int_{B_j}G^{1/2}(\varphi v)\cdot G^{1/2}(\varphi v) \\ &\quad - 4\lambda\int_{t_1}^{t^2}\int_{B_j}G^{1/2}(v)\cdot G^{1/2}(\varphi v)\Big) \\ &\le \frac{c}{\epsilon^2}\int_{t_1}^{t^2}\|\varphi v\|^2_{H^{-1}(B_j)} \\ &\quad + 4\lambda\left|\int_{t_1}^{t^2}\int_{B_j}G^{1/2}(v)\cdot G^{1/2}(\varphi v)\right|. \end{aligned} \tag{4.11}$$

Since $|v(x,t)|=1$ a.e. on $\partial B\times(t_1,t_2)$, we get $\varphi(x,t)=0$ a.e. on $\partial B\times(t_1,t_2)$ and, in particular, $\varphi(x,t)=0$ a.e. on $\partial B_j\times(t_1,t_2)$. Assume that B_j is the closure of an open set. By using the Poincaré inequality on left-hand side first term of (4.11), along with the control (2.4) (see also [30]), we obtain

$$\|\varphi v\|^2_{H^{-1}} \le \|\varphi\|^2_{H^1(B_j)}\|v\|^2_{L^2(B_j)} \le c\|\varphi\|^2_{H^1(B_j)}\|v\|^2_{L^\infty(0,T;L^2(\mathbb{T}^2))} = \tilde{c}\|\varphi\|^2_{H^1(B_j)},$$

and

$$\begin{aligned} 4\lambda\left|\int_B G^{1/2}(v)\cdot G^{1/2}(\varphi v)\right| &\le 4\lambda\|v\|\|\varphi v\|_{H^{-1}} \\ &\le 4\lambda\|v\|^2\|\varphi\|_{H^1} \\ &\le c\lambda\Big(\int_{B_j}(\|v\|^2-1)\,\mathrm{d}x+1\Big)\|\varphi\|_{H^1} \\ &\le c\lambda\left(\int_{B_j}(\|v\|^2-1)^2\,\mathrm{d}x\right)^{\frac{1}{2}}\|\varphi\|_{H^1} + c\lambda\|\varphi\|_{H^1} \\ &\le c\lambda\|\varphi\|\|\varphi\|_{H^1} + \bar{c}\lambda\|\varphi\|^2_{H^1} \\ &\le \hat{c}\lambda\|\varphi\|^2_{H^1}. \end{aligned} \tag{4.12}$$

Hence, the inequality

$$C\int_{t_1}^{t_2}\|\varphi\|^2_{L^2(B_j)}+\int_{t_1}^{t_2}\|\nabla\varphi\|^2_{L^2(B_j)}+4\int_{t_1}^{t^2}\int_{B_j}|\nabla v|^2\varphi\leq\Big(\frac{\tilde{c}}{\epsilon^2}+\hat{c}\lambda\Big)\int_{t_1}^{t_2}\|\varphi\|^2_{H^1(B_j)},$$

where C is the constant involved in the Poincaré inequality, holds true. Then, we find

$$c\int_{t_1}^{t_2}\|\varphi\|^2_{H^1(B_j)}+4\int_{t_1}^{t^2}\int_{B_j}|\nabla v|^2\varphi\leq\Big(\frac{\tilde{c}}{\epsilon^2}+\hat{c}\lambda\Big)\int_{t_1}^{t_2}\|\varphi\|^2_{H^1(B_j)}, \tag{4.13}$$

which gives an absurd by assuming that ϵ is sufficiently large as λ is small.

The general case, when B_j is not the closure of an open set, follows the same line of the argument in reference [12]. □

–Step B: Let $v^n\in L^\infty(0,T;\boldsymbol{H}^{\frac{3}{2}})\cap L^2(0,T;\boldsymbol{H}^{\frac{5}{2}})$ be the vector field just determined in the previous step. We search the approximating velocity field $w^n\in C^1(0,T;\mathcal{H}_n)$ satisfying the equation

$$\begin{aligned}(w^n_t(t),v^n)_{H^{\frac{1}{2}}}+&(\nabla w^n_t(t),\nabla v^n)_{H^{\frac{1}{2}}}+(\nabla w^n(t),\nabla v^n)_{H^{\frac{1}{2}}}+\big((\overline{w}^n(t)\cdot\nabla)w^n(t),v^n\big)_{H^{\frac{1}{2}}},\\ &=\big((\nabla(v^n)^\top\nabla v^n),\nabla v^n\big)_{H^{\frac{1}{2}}}+\big(\nabla(v^n)^\top\Delta v_t,v^n\big)_{H^{\frac{1}{2}}},\quad\forall v^n\in\mathcal{H}_n,\end{aligned}$$

with

$$w^n(x,0)=w^n_0(x)=\mathbf{P}_n(w_0)(x),\ \text{for } x\in\mathbb{T}^2,$$

where both v^n and $\overline{w}^n$ are given. Thanks to the Cauchy-Lipschitz theorem, we can prove existence of a unique maximal solution w^n of the above problem.

4.2. Global existence

In the sequel, for the sake of compactness, we use the same symbol $\|\cdot\|_{L^p(0,T;L^k)}$ for the norm in $L^p(0,T;L^k)$ and $L^p(0,T;(L^k)^n)$. We employ the same convention also for $L^p(0,T;W^{s,k})$ and $L^p(0,T;(W^{s,k})^n)$ (also $L^p(0,T;\boldsymbol{H}^s)$ and $L^p(0,T;(\boldsymbol{H}^s)^n)$).

Proof of Theorem 3.1. First, we deduce a priori estimates. Then, we apply a compactness criterion proving that the limiting pair $(\hat{w},\hat{v})$ is actually a weak solution to (3.1)–(3.3), supplemented by (1.4). Only for the sake of conciseness we use (w,v) instead of (w^n,v^n).

–Step 1: *Energy a priori estimates.* Consider equation (3.3), to which we apply the operator $G=(I-\Delta)^{-1}$, and take the L^2-product with test v, obtaining

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2+\|\nabla v\|^2\leq&\frac{1}{\varepsilon^2}\int_{\mathbb{T}^2}|G((1-|v|^2)v)||v|\,\mathrm{d}x+\lambda\int_{\mathbb{T}^2}|G^{\frac{1}{2}}v|^2\,\mathrm{d}x\\ \leq&\frac{c}{\varepsilon^2}\|(1-|v|^2)v\|_{H^{-2}}\|v\|+c\lambda\|v\|^2_{H^{-1}}\\ \leq&\frac{c}{\varepsilon^2}\|(1-|v|^2)v\|\|v\|+c\lambda\|v\|^2\\ \leq&\Big(\frac{c}{\varepsilon^2}+c\lambda\Big)\|v\|^2,\end{aligned} \tag{4.14}$$

where the constraint $|v|^2 \leq 1$ plays a role. Given $T > 0$, the Gronwall Lemma implies $v \in L^\infty(0,T;L^2) \cap L^2(0,T;\boldsymbol{H}^1)$.

By taking the L^2-product of (3.1) with w, we compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|w\|^2 + \|\nabla w\|^2) + \|\nabla w\|^2 &= \int_{\mathbb{T}^2} (\nabla v^\top \nabla v) \cdot \nabla w \, \mathrm{d}x - \int_{\mathbb{T}^2} \nabla v^\top \Delta v_t \cdot w \, \mathrm{d}x \\ &= \int_{\mathbb{T}^2} (\nabla v^\top \nabla v) \cdot \nabla w \, \mathrm{d}x + \int_{\mathbb{T}^2} (\nabla v^\top \nabla v_t) \cdot \nabla w \, \mathrm{d}x \\ &\quad + \int_{\mathbb{T}^2} \nabla(\nabla v)^\top \nabla v_t \cdot w \, \mathrm{d}x,\end{aligned} \tag{4.15}$$

where, to get the second equality, we have used relation (3.7), integrating by parts the first term obtained.

By multiplying (3.3) by v_t and integrating over $\mathbb{T}^2$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\nabla v\|^2 + \|\Delta v\|^2) + \|v_t\|^2 + \|\nabla v_t\|^2 &= -\int_{\mathbb{T}^2} \nabla((w \cdot \nabla)v) \cdot \nabla v_t \, \mathrm{d}x \\ &\quad - \int_{\mathbb{T}^2} (w \cdot \nabla)v) \cdot v_t \, \mathrm{d}x \\ &\quad - \int_{\mathbb{T}^2} f_\varepsilon(v) \cdot v_t \, \mathrm{d}x + \lambda \int_{\mathbb{T}^2} v \cdot v_t \, \mathrm{d}x.\end{aligned} \tag{4.16}$$

Remark 3.1 implies that the first term in the right-hand side of (4.16) can be rewritten as

$$\int_{\mathbb{T}^2} \nabla((w \cdot \nabla)v) \cdot \nabla v_t \, \mathrm{d}x = \int_{\mathbb{T}^2} (\nabla v^\top \nabla v_t) \cdot \nabla w \, \mathrm{d}x + \int_{\mathbb{T}^2} \nabla(\nabla v)^\top \nabla v_t \cdot w \, \mathrm{d}x.$$

Then, by summing up (4.15) and (4.16), we infer

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|w\|^2 + \|\nabla w\|^2 + \|\nabla v\|^2 + \|\Delta v\|^2\Big) + \|\nabla w\|^2 + \|v_t\|^2 + \|\nabla v_t\|^2 \\ &\quad \leq \int_{\mathbb{T}^2} |\nabla w||\nabla v|^2 \, \mathrm{d}x + \int_{\mathbb{T}^2} |w||\nabla v||v_t| \, \mathrm{d}x \\ &\quad\quad + \int_{\mathbb{T}^2} |f_\epsilon(v)||v_t| \, \mathrm{d}x + \lambda \int_{\mathbb{T}^2} |v||v_t| \, \mathrm{d}x =: \sum_{i=1}^4 I_i.\end{aligned} \tag{4.17}$$

For the terms I_i, $i = 1, 2, 3$, we have the following bounds

$$\begin{aligned}I_1 &\leq \|\nabla w\| \|\nabla v\|_{L^4}^2 \\ &\leq c\|\nabla w\| \|\nabla v\| \|\Delta v\| \\ &\leq c\varepsilon \|\nabla w\|^2 \|\nabla v\|^2 + C_\varepsilon \|\Delta v\|^2,\end{aligned} \tag{4.18}$$

$$\begin{aligned}I_2 &\leq \|w\|_{L^4} \|\nabla v\|_{L^4} \|v_t\| \\ &\leq c\|w\|^{\frac{1}{2}} |\nabla w\|^{\frac{1}{2}} \|\nabla v\|^{\frac{1}{2}} \|\Delta v\|^{\frac{1}{2}} \|v_t\| \\ &\leq \frac{c}{\delta} \|w\| \|\nabla w\| \|\nabla v\| \|\Delta v\| + \delta \|v_t\|^2 \\ &\leq \frac{\bar{c}}{\delta} \|\nabla w\| \|\nabla v\| \|\Delta v\| + \delta \|v_t\|^2 \\ &\leq \frac{\bar{c}\epsilon}{\delta^2} \|\nabla w\|^2 \|\nabla v\|^2 + c_\epsilon \|\Delta v\|^2 + \delta \|v_t\|^2,\end{aligned} \tag{4.19}$$

$$
\begin{aligned}
I_3 &\leq \frac{2}{\varepsilon^2}\|v\|\|v_t\| \\
&\leq \frac{c}{\varepsilon^4\delta}\|v\|^2 + \delta\|v_t\|^2,
\end{aligned}
\tag{4.20}
$$

and

$$
\begin{aligned}
I_4 &\leq \lambda\|v\|\|v_t\| \\
&\leq \frac{c\lambda}{\delta}\|v\|^2 + \lambda\delta\|v_t\|^2.
\end{aligned}
\tag{4.21}
$$

Estimates above, together with inequalities (4.14) and (4.17), allow us to write

$$
\begin{aligned}
\frac{d}{dt}(\|w\|^2 + \|\nabla w\|^2 + \|\nabla v\|^2 + \|\Delta v\|^2) + \|\nabla w\|^2 + (1 - (\lambda + 2)\delta)\|v_t\|^2 + \|\nabla v_t\|^2 \\
\leq c_{\varepsilon,\delta,\lambda}\|v\|^2 + c_\varepsilon\|\Delta v\|^2 + c_{\varepsilon,\delta}\|\nabla w\|^2\|\nabla v\|^2.
\end{aligned}
\tag{4.22}
$$

In the present case the penalisation parameter $\varepsilon > 0$ is constant, so we omit such a term along with δ and λ in the next calculations. From (4.22) we obtain

$$
\begin{aligned}
\frac{d}{dt}(\|w\|^2 + \|\nabla w\|^2 + \|\nabla v\|^2 + \|\Delta v\|^2) &\leq c(\|v\|^2 + \|\Delta v\|^2 + \|\nabla w\|^2\|\nabla v\|^2) \\
&\leq c\|v\|^2 + c(\|\nabla w\|^2 + \|\Delta v\|^2)(1 + \|\nabla v\|^2).
\end{aligned}
$$

Set $y = (\|w\|^2 + \|\nabla w\|^2 + \|\nabla v\|^2 + \|\Delta v\|^2)$. The differential inequality

$$
y' \leq c\|v\|^2 + y(1 + \|\nabla v\|^2)
$$

implies

$$
\begin{aligned}
&(\|w\|^2 + \|\nabla w\|^2 + \|\nabla v\|^2 + \|\Delta v\|^2)(t) \\
&\quad\leq (\|w_0, \nabla w_0, \nabla v_0, \Delta v_0\|^2)e^{\int_0^t(1+\|\nabla v(s)\|^2)ds} \\
&\quad\quad + \|v\|_{L^\infty(0,T;L^2)}\int_0^t e^{\int_s^t(1+\|\nabla v(\ell)\|^2)d\ell}\,\mathrm{d}s.
\end{aligned}
\tag{4.23}
$$

Since (w, v) stands for (w^n, v^n), as a consequence of the above estimates, for any fixed $T > 0$, it follows that $\|w^n, \nabla w^n\|^2_{L^\infty(0,T;\mathcal{H})} + \|\nabla v^n, \Delta v^n\|^2_{L^\infty(0,T;\boldsymbol{H})}$ is uniformly bounded with respect to $n \in \mathbb{N}$. The control (4.22) implies $v_t \in L^2(0, T; \boldsymbol{H}^1)$.

–Step 2: *Further a priori estimates.* We take the $\boldsymbol{H}^{1/2}$-inner product of (3.1) and (3.2) with w and v_t, respectively, as in the case of equations (4.15) and (4.16). After integration by parts, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|w\|^2_{H^{\frac{1}{2}}} + \|\nabla w\|^2_{H^{\frac{1}{2}}}) + \|\nabla w\|^2_{H^{\frac{1}{2}}} \\
&= -((w \cdot \nabla)w), w)_{H^{\frac{1}{2}}} + ((\nabla v^\top \nabla v), \nabla w)_{H^{\frac{1}{2}}} \\
&\quad + ((\nabla v^\top \nabla v_t), \nabla w)_{H^{\frac{1}{2}}} + (\nabla(\nabla v)^\top \nabla v_t, w)_{H^{\frac{1}{2}}},
\end{aligned}
\tag{4.24}
$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla v\|^2_{H^{\frac{1}{2}}}+\|\Delta v\|^2_{H^{\frac{1}{2}}})+\|v_t\|^2_{H^{\frac{1}{2}}}+\|\nabla v_t\|^2_{H^{\frac{1}{2}}}\\
&\qquad=-\big((\nabla v^{\top}\nabla v_t),\nabla w\big)_{H^{\frac{1}{2}}}-\big(\nabla(\nabla v)^{\top}\nabla v_t,w\big)_{H^{\frac{1}{2}}}\\
&\qquad\quad-\big((w\cdot\nabla)v),v_t\big)_{H^{\frac{1}{2}}}-\big(f_\varepsilon(v),v_t\big)_{H^{\frac{1}{2}}}+\lambda(v,v_t)_{H^{\frac{1}{2}}}.
\end{aligned}\tag{4.25}$$

From equations (4.24) and (4.25), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|w\|^2_{H^{\frac{1}{2}}}+\|\nabla w\|^2_{H^{\frac{1}{2}}}+\|\nabla v\|^2_{H^{\frac{1}{2}}}+\|\Delta v\|^2_{H^{\frac{1}{2}}})+\|\nabla w\|^2_{H^{\frac{1}{2}}}+\|v_t\|^2_{H^{\frac{1}{2}}}+\|\nabla v_t\|^2_{H^{\frac{1}{2}}}\\
&\qquad\le\left|\big((w\cdot\nabla)w),w\big)_{H^{\frac{1}{2}}}\right|+\left|\big((\nabla v^{\top}\nabla v),\nabla w\big)_{H^{\frac{1}{2}}}\right|\\
&\qquad\quad+\left|\big((w\cdot\nabla)v),v_t\big)_{H^{\frac{1}{2}}}\right|+\left|\big(f_\varepsilon(v),v_t\big)_{H^{\frac{1}{2}}}\right|\\
&\qquad\quad+\lambda\left|(v,v_t)_{H^{\frac{1}{2}}}\right|=:\sum_{i=1}^{5}L_i.
\end{aligned}\tag{4.26}$$

For the terms L_i, $i=1,\dots,5$ we actually use the norm induced by $(\,\cdot\,,\,\cdot\,)_{\dot{H}^{\frac{1}{2}}}=(\Lambda^{\frac{1}{2}}(\cdot),\Lambda^{\frac{1}{2}}(\cdot))$ instead of the full norm $H^{\frac{1}{2}}$, although we still keep the same norm notation $\|\cdot\|_{H^{\frac{1}{2}}}$. Previous evaluation of the lower-order terms in the steps already described motivates our notational choice. Also, for the velocity vector filed w, the norm $\|w\|_{\dot{H}^{\frac{1}{2}}}$ is equivalent to the full norm $\|w\|_{H^{\frac{1}{2}}}$.

Consider L_1. Since $\int_{\mathbb{T}^2}(w\cdot\nabla)\Lambda^{\frac{1}{2}}w\cdot\Lambda^{\frac{1}{2}}w\,ds=0$, we get

$$\begin{aligned}
L_1&\le\int_{\mathbb{T}^2}|(\Lambda^{\frac{1}{2}}w\cdot\nabla)w|\,|\Lambda^{\frac{1}{2}}w|\,ds\\
&\le\|\Lambda^{\frac{1}{2}}w\|^2_{L^4}\|\nabla w\|\\
&\le\|\Lambda^{\frac{1}{2}}w\|\|\Lambda^{\frac{1}{2}}\nabla w\|\|\nabla w\|\le c\|w\|_{H^{\frac{1}{2}}}\|\nabla w\|\|\nabla w\|_{H^{\frac{1}{2}}}\\
&\le\frac{c}{\epsilon}\|w\|^2_{H^{\frac{1}{2}}}\|\nabla w\|^2+\epsilon\|\nabla w\|^2_{H^{\frac{1}{2}}}.
\end{aligned}\tag{4.27}$$

Then, by exploiting (2.3), with $s=1/2$, $r=2$ and $p_1=p_2=q_1=q_2=4$, we find

$$\begin{aligned}
L_2\le\|\nabla v^{\top}\nabla v\|_{H^{\frac{1}{2}}}\|\nabla w\|_{H^{\frac{1}{2}}}&\le c\|\Lambda^{\frac{1}{2}}\nabla v\|_{L^4}\|\nabla v\|_{L^4}\|\nabla w\|_{H^{\frac{1}{2}}}\\
&\le c\big(\|\Lambda^{\frac{1}{2}}\nabla v\|^{\frac{1}{2}}\|\Lambda^{\frac{1}{2}}\Delta v\|^{\frac{1}{2}}\|\nabla v\|^{\frac{1}{2}}\|\Delta v\|^{\frac{1}{2}}\big)\|\nabla w\|_{H^{\frac{1}{2}}}\\
&\le\frac{c}{\epsilon}(\|\nabla v\|^2_{H^{\frac{1}{2}}}\|\nabla v\|^2+\|\Delta v\|^2_{H^{\frac{1}{2}}}\|\Delta v\|^2)+\epsilon\|\nabla w\|^2_{H^{\frac{1}{2}}},
\end{aligned}\tag{4.28}$$

$$\begin{aligned}
L_3\le\|(w\cdot\nabla)v\|_{H^{\frac{1}{2}}}\|v_t\|_{H^{\frac{1}{2}}}&\le(\|\Lambda^{\frac{1}{2}}w\|_{L^4}\|\nabla v\|_{L^4}+\|w\|_{L^4}\|\Lambda^{\frac{1}{2}}\nabla v\|_{L^4})\|v_t\|_{H^{\frac{1}{2}}}\\
&\le\frac{c}{\epsilon}(\|\nabla w\|^2\|\nabla v\|^2_{H^{\frac{1}{2}}}+\|w\|^2_{H^{\frac{1}{2}}}\|\Delta v\|^2)+\epsilon\|v_t\|^2_{H^{\frac{1}{2}}},
\end{aligned}\tag{4.29}$$

$$\begin{aligned}
L_4=\left|(\Lambda^{\frac{1}{2}}f_\varepsilon(v),\Lambda^{\frac{1}{2}}v_t)\right|&\le c\|f_\varepsilon(v)\|_{H^1}\|v_t\|_{H^{\frac{1}{2}}}\\
&\le\frac{c}{\epsilon\varepsilon^4}(\|v\|^2+\|\nabla v\|^2)+\epsilon\|v_t\|^2_{H^{\frac{1}{2}}},
\end{aligned}\tag{4.30}$$

and

$$L_5 = \lambda \left|(\Lambda^{\frac{1}{2}} v, \Lambda^{\frac{1}{2}} v_t)\right| \leq c\lambda \|v\|_{H^{\frac{1}{2}}} \|v_t\|_{H^{\frac{1}{2}}} \\ \leq \frac{c\lambda}{\epsilon} \|v\|^2_{H^{\frac{1}{2}}} + \epsilon \|v_t\|^2_{H^{\frac{1}{2}}}, \tag{4.31}$$

after using Hölder's, Ladyzhenskaya's, and Young's inequalities as well as the continuous embedding $W^{1/2,2}(\mathbb{T}^2) \subset L^4(\mathbb{T}^2)$.

By using the estimates (4.27)–to–(4.30) along with (4.26), and absorbing the parameter ε^{-4} in a generic constant c, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|w\|^2_{H^{\frac{1}{2}}} + \|\nabla w\|^2_{H^{\frac{1}{2}}} + \|\nabla v\|^2_{H^{\frac{1}{2}}} + \|\Delta v\|^2_{H^{\frac{1}{2}}}) + (1 - c\epsilon)\|\nabla w\|^2_{H^{\frac{1}{2}}} + \|v_t\|^2_{H^{\frac{1}{2}}} + \|\nabla v_t\|^2_{H^{\frac{1}{2}}} \\ &\leq c\|w\|^2_{H^{\frac{1}{2}}}(1 + \|\nabla w\|^2 + \|\nabla v\| + \|\Delta v\|^2) + c\|\nabla v\|^2_{H^{\frac{1}{2}}}(1 + \|w\|^2 + \|\nabla v\|^2 + \|\Delta v\|^2) \\ &\quad + c\|\Delta v\|^2_{H^{\frac{1}{2}}}\|\Delta v\|^2 \end{aligned}$$

with $\epsilon > 0$ small enough in a way that the coefficient $\bar{c} := (1 - c\epsilon)$ is positive. Fix $T > 0$. By Grönwall's lemma, we get

$$\begin{aligned} \|w(t)\|^2_{H^{\frac{1}{2}}} &+ \|\nabla w(t)\|^2_{H^{\frac{1}{2}}} + \|\nabla v(t)\|^2_{H^{\frac{1}{2}}} + \|\Delta v(t)\|^2_{H^{\frac{1}{2}}} \\ &\quad + \bar{c}\int_0^t (\|\nabla w\|^2_{H^{\frac{1}{2}}} + \|v_t\|^2_{H^{\frac{1}{2}}} + \|\nabla v_t\|^2_{H^{\frac{1}{2}}})\,\mathrm{d}s \\ &\leq \beta \exp\left\{c\int_0^t [(1 + \|w\|^2 + \|\nabla w\|^2 + \|\nabla v\|^2 + \|\Delta v\|^2)]\,\mathrm{d}s\right\} \end{aligned}$$

for any $0 < t \leq T$, with

$$\beta = c(\|w_0\|^2_{H^{\frac{1}{2}}} + \|\nabla w_0\|^2_{H^{\frac{1}{2}}} + \nabla v_0\|^2_{H^{\frac{1}{2}}} + \|\Delta v_0\|^2_{H^{\frac{1}{2}}}),$$

and the quantity on the right-hand side of the above inequality is bounded, for $0 < t \leq T$, thanks to equation (4.23) and the hypotheses on initial data.

Until here, we mainly used the notation (w, v) in place of (w^n, v^n) but, in view of extracting a convergent subsequence, in the last part of the proof we'll employ the (w^n, v^n) notation.

–Step 3: *Estimate for* w^n_t. In order to extract a convergent subsequence of $\{(u^n, v^n)\}$, we exploit the classical Aubin-Lions lemma; to this end we have first to provide a suitable control on w^n_t. The next calculations also fixes a minor issue present in the analogous control in reference [12], where we estimate acceleration in $L^1(0, T; \mathcal{H}^{-1})$.

Consider equation (3.1). For $\varphi \in \mathcal{H}^1$, $\int_{\mathbb{T}^2} \varphi \, \mathrm{d}x = 0$, with $\|\nabla\varphi\| = 1$. Then, we get

$$\begin{aligned}
\langle w_t^n - \Delta w_t^n, \varphi\rangle_{\mathcal{H}^{-1},\mathcal{H}^1} \leq& |((w^n \cdot \nabla)\varphi, w^n)| + |(\nabla w^n, \nabla\varphi)| \\
&+ \int_{\mathbb{T}^2} |\nabla v^n|^2 |\nabla\varphi| \, \mathrm{d}x + c\int_{\mathbb{T}^2} |\nabla v^n||\nabla v_t^n||\nabla\varphi| \, \mathrm{d}x \\
&+ c\left|(\nabla(\nabla v^n)^\top \nabla v_t^n, \varphi)\right| \\
\leq& \|w^n\|_{L^4}^2 \|\nabla\varphi\| + \|\nabla w^n\| \|\nabla\varphi\| \\
&+ \|\nabla v^n\|_{L^4}^2 \|\nabla\varphi\| + c\|\nabla v^n\|_{L^4} \|\nabla v_t^n\|_{L^4} \|\nabla\varphi\| \\
&+ c\|\nabla(\nabla v^n)^\top \nabla v_t^n\|_{H^{-1}} \|\nabla\varphi\| \\
\leq& c\big(\|w^n\|\|\nabla w^n\| + c\|\nabla w^n\| + \|\nabla v^n\|\|\Delta v^n\| \\
&+ \|\nabla v^n\|_{H^{\frac12}} \|\nabla v_t^n\|_{H^{\frac12}} + \|\Delta v^n\|_{H^{\frac12}} \|\nabla v_t^n\|_{H^{\frac12}}\big),
\end{aligned} \tag{4.32}$$

after using the estimates performed in previous steps along with Hölder's, Ladyzhenskaya's, and Poincaré's inequalities. In the last inequality above, we have also exploited the continuous embedding $W^{1/2,2}(\mathbb{T}^2) \subset L^4(\mathbb{T}^2)$ and the Sobolev product laws (see, e.g., [3, 14, 30]) to get the estimate

$$\begin{aligned}
\|\nabla(\nabla v^n)^\top \nabla v_t^n\|_{H^{-1}} &\leq \|\Delta v^n\| \|\nabla v_t^n\|_{H^{\frac12}} \\
&\leq \|\Delta v^n\|_{H^{\frac12}} \|\nabla v_t^n\|_{H^{\frac12}}.
\end{aligned}$$

Hence, we find

$$\begin{aligned}
\int_0^T \|w_t^n\|_{\mathcal{H}^1}^2 \, \mathrm{d}s \leq& c\Big[(1 + \|w^n\|_{L^\infty(0,T;L^2)}^2)\|\nabla w^n\|_{L^2(0,T;L^2)}^2 + \|\Delta v^n\|_{L^2(0,T;L^2)}^2 \\
&+ (\|\nabla v^n\|_{L^\infty(0,T;H^{\frac12})}^2 + \|\Delta v^n\|_{L^\infty(0,T;H^{\frac12})}^2)\|\nabla v_t^n\|_{L^2(0,T;H^{\frac12})}^2\Big].
\end{aligned}$$

As a final step in our argument, to extract a convergent subsequence from $\{(w^n, v^n)\}$, we can use the Aubin-Lions lemma following the same line as in the proof of [12, Theorem 3.1–**Step 3**]. Also, passage to the limit in weak formulation follows the same path exploited in reference [12]. So, we can conclude stating existence. □

Remark 4.1. *By assuming initial data $(w_0, v_0) \in \mathcal{H}^1 \times \boldsymbol{H}^2$, we can still reproduce the same calculations of* **Step 1***, while* **Step 2** *would require higher-order estimates, which are not available in the present setting. However, by using an approach similar to the one in* **Step 3***, we could obtain a weaker control on w_t by using equation* (3.1) *and providing a uniform estimate on*

$$\|\Delta w_t^n\|_{\mathcal{H}^{-2}} = \sup_{\|\varphi\|_{\dot{H}^2}=1} |\langle \Delta w_t^n, \varphi\rangle_{\mathcal{H}^{-2},\mathcal{H}^2}|$$

Indeed, also in this case, the worst term to be controlled is $\left|(\nabla(\nabla v^n)^\top \nabla v_t^n, \varphi)\right|$. For it, we get

$$\begin{aligned}
\int_0^T \left|(\nabla(\nabla v^n)^\top \nabla v_t^n, \varphi)\right|^2 \mathrm{d}s &\leq \int_0^T \|(\nabla(\nabla v^n)^\top \nabla v_t^n\|_{\mathcal{H}^{-2}}^2 \|\varphi\|_{\mathcal{H}^2}^2 \, \mathrm{d}s \\
&\leq \int_0^T \|\Delta v^n\|^2 \|\nabla v_t^n\|^2 \, \mathrm{d}s \\
&\leq \|\Delta v^n\|_{L^\infty(0,T;L^2)}^2 \int_0^T \|\nabla v_t^n\|^2 \, \mathrm{d}s,
\end{aligned}$$

on the basis of inequalities (4.22)*,* (4.23)*, and the product law* (2.4)*. Then, to conclude about the existence of weak solutions, we can use again the same idea behind limiting and convergence procedures in [12, Theorem 3.1–**Step 3**].*

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Conflict of Interest

The authors declare no conflicts of interest in this paper.

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