MODULI OF FORMAL TORSORS

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Abstract. We construct the moduli stack of torsors over the formal punctured disk in characteristic $p > 0$ for a finite group isomorphic to the semidirect product of a p-group and a tame cyclic group. We prove that the stack is a limit of separated Deligne-Mumford stacks with finite and universally injective transition maps.

Introduction

The main subject of this paper is the moduli space of formal torsors, that is, G-torsors (also called principal G-bundles) over the formal punctured disk $\operatorname{Spec} k((t))$ for a given finite group (or étale finite group scheme) G and field k. More precisely, we are interested in a space over a field k whose k-points are G-torsors over $\operatorname{Spec} k((t))$. Since torsors may have non-trivial automorphisms, this space should actually be a stack in groupoids and it should not be confused with $\mathrm{B}\,G = [\operatorname{Spec} k((t))/G]$, which is a stack defined over $k((t))$.

The case where the characteristic of k and the order of G are coprime is called tame and the other case is called wild. The two cases are strikingly different: in the tame case the moduli space is expected to be zero-dimensional, while in the wild case it is expected to be infinite-dimensional.

An important work on this subject is Harbater's one [Har80]. He constructed the coarse moduli space for pointed formal torsors when k is an algebraically closed field of characteristic $p > 0$ and G is a p-group. This coarse moduli space is isomorphic to the inductive limit $\varinjlim_n \mathbb{A}^n$ of affine spaces such that the transition map $\mathbb{A}^n \to \mathbb{A}^{n+1}$ is the composition of the closed embedding $\mathbb{A}^n \hookrightarrow \mathbb{A}^{n+1}$ and the Frobenius map of $\mathbb{A}^{n+1}$. In particular it is neither a scheme nor an algebraic space, but an ind-scheme. Some of the differences between Harbater's space and our space are explained in Remark 4.26. As a consequence Harbater shows that there is a bijective correspondence between G-torsors over the affine line $\mathbb{A}^1$ and over $\operatorname{Spec} k((t))$. In this direction an important development has been given by Gabber and Katz in [Kat86]. Later Pries [Pri02] and Obus-Pries [OP10] constructed moduli/parameter spaces for groups $\mathbb{Z}/p \rtimes C$ and $\mathbb{Z}/p^m \rtimes C$ with C a tame cyclic group respectively and Fried-Mezard [FM02] constructed a parameter space of (not necessarily Galois) covers of $\operatorname{Spec} k((t))$ with given ramification data; all these works assumed k to be algebraically closed.

In recent works [Yas14, Yas17] of the second named author, an unexpected relation of this moduli space to singularities of algebraic varieties was discovered. He has formulated a conjectural generalization of the motivic McKay correspondence by Batyrev [Bat99] and Denef-Loeser [DL02] to arbitrary characteristics, which relates a motivic integral over the moduli space of formal torsors with a stringy invariant of wild quotient singularities. The motivic integral can be viewed as the motivic counterpart of mass formulas for local Galois representations, see [WY15, WY17].

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The first and largest problem for other groups is the construction of the moduli space. From the arithmetic viewpoint, the case where k is finite is the most interesting, which motivates us to remove the "algebraically closed" assumption in earlier works.

The main result of this paper is to construct the moduli stack of formal torsors and to show that it is a limit of Deligne-Mumford stacks (DM stacks for short) when k is an arbitrary field of characteristic $p > 0$ and G is an étale group scheme over k which is geometrically the semidirect product $H \rtimes C$ of a p-group H and a cyclic group C of order coprime with p. This is an important step towards the general case, because, if k is algebraically closed, then connected G-torsors over $\operatorname{Spec} k((t))$ (or equivalently Galois extensions of $k((t))$ with group G) exist only for semidirect products as before. Moreover any G'-torsor for a general G' is induced by some connected G-torsor along an embedding $G \hookrightarrow G'$.

To give the precise statement of the result, we introduce the following notation. We denote by Δ_G the category fibered in groupoids over the category of affine k-schemes such that for a k-algebra B, $\Delta_G(\operatorname{Spec} B)$ is the category of G-torsors over $\operatorname{Spec} B((t))$. The following is the precise statement of the main result:

Theorem A. *Let k be a field of positive characteristic p and G be a finite and étale group scheme over k such that $G \times_k \overline{k}$ is a semidirect product $H \rtimes C$ of a p-group H and a cyclic group C of rank coprime with p.*

1) *Then there exists a direct system $\mathcal{X}_*$ of separated DM stacks with finite and universally injective transition maps, with a direct system of finite and étale atlases (see 3.1 for the definition) $X_n \longrightarrow \mathcal{X}_n$ from affine schemes and with an isomorphism $\varinjlim_n \mathcal{X}_n \simeq \Delta_G$.*
2) *If G is a constant p-group then the stacks $\mathcal{X}_n$ can be chosen to be smooth and integral. More precisely there is a strictly increasing sequence $v\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that $X_n = \mathbb{A}^{v_n}$, the maps $\mathbb{A}^{v_n} \longrightarrow \mathcal{X}_n$ are finite and étale of degree $\sharp G$ and the transition maps $\mathbb{A}^{v_n} \longrightarrow \mathbb{A}^{v_{n+1}}$ are composition of the inclusion $\mathbb{A}^{v_n} \longrightarrow \mathbb{A}^{v_{n+1}}$ and the Frobenius $\mathbb{A}^{v_{n+1}} \longrightarrow \mathbb{A}^{v_{n+1}}$.*
3) *If G is an abelian constant group of order p^r then we also have an equivalence*
$$\left(\varinjlim_n \mathbb{A}^{v_n}\right) \times \mathrm{B}\, G \simeq \Delta_G$$
and the map from $\varinjlim_n \mathbb{A}^{v_n}$ to the sheaf of isomorphism classes of Δ_G, which is nothing but the rigidification $\Delta_G /\!\!/ G$ (see Appendix B), is an isomorphism.

As a consequence of assertion 1) of this theorem (and A.5) the fibered category Δ_G is a stack.

We now explain the outline of our construction. We first consider the case of a constant group scheme of order p^r. Following Harbater's strategy, we prove the theorem in this case by induction. We obtain the explicit description of Δ_G as in assertion 3) when $G = \mathbb{Z}/p\mathbb{Z}$ by the Artin-Schreier theory (Theorem 4.13); this is one of the two base cases. It is not difficult to generalize it to the case $G \simeq (\mathbb{Z}/p\mathbb{Z})^n$ (Lemma 4.20), which forms the initial step of induction. Since a general p-group has a central subgroup $H \subset G$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$, we have a natural map $\Delta_G \longrightarrow \Delta_{G/H}$, enabling the induction to work. We then use the fact (Proposition 4.22) that this map factors into the rigidification $\Delta_G \longrightarrow \Delta_G /\!\!/ H$ and an X_H-torsor $\Delta_G /\!\!/ H \longrightarrow \Delta_{G/H}$ with $X_H = \Delta_H /\!\!/ H$, to construct a direct system for Δ_G from one for $\Delta_{G/H}$.

Next we consider the other base case, the case of the group scheme μ_n of n-th roots of unity with n coprime to p. In this case, we have the following explicit description of Δ_G, including also the case of characteristic zero:

Theorem B. *Let k be a field and $n \in \mathbb{N}$ such that $n \in k^*$. We have an equivalence*

$$\coprod_{q=0}^{n-1} \mathrm{B}(\mu_n) \longrightarrow \Delta_{\mu_n}$$

where the map $\mathrm{B}(\mu_n) \longrightarrow \Delta_{\mu_n}$ in the index q maps the trivial μ_n-torsor to the μ_n-torsor $\frac{k((t))[Y]}{(Y^n - t^q)} \in \Delta_{\mu_n}(k)$.

When G is a constant group of the form $H \rtimes C$ and k contains all n-th roots of unity, then $C \simeq \mu_n$ and there exists a map $\Delta_G \longrightarrow \Delta_{\mu_n}$. Using Theorem A for p-groups, we show that the fiber products $\Delta_G \times_{\Delta_{\mu_n}} \operatorname{Spec} k$ with respect to n maps $\operatorname{Spec} k \to \Delta_{\mu_n}$ induced from the equivalence in Theorem B are limits of DM stacks. Finally, to conclude that Δ_G itself is a limit of DM stacks and also to reduce the problem to the case of a constant group, we need a proposition (Proposition 3.5) roughly saying that if $\mathcal{Y}$ is a G-torsor over a stack $\mathcal{X}$ for a constant group G and $\mathcal{Y}$ is a limit of DM stacks, then $\mathcal{X}$ is also a limit of DM stacks. This innocent-looking proposition turns out to be rather hard to prove and we will make full use of 2-categories.

The moduli stack of formal torsors introduced in this paper is used in [TY19] to construct a moduli space in a weaker sense for general finite étale group schemes and in [Yas] to develop the motivic integration over wild DM stacks. Moreover it is showed in [TY19] that the motivic integral in the conjecture mentioned above on the McKay correspondence makes rigorous sense and this conjecture is finally proved in [Yas]. According to discussion and observation in [TY], it appears quite meaningful to generalize the McKay correspondence further to nonreduced finite group schemes. For this reason, the moduli problem of formal torsors for such group schemes would be important as a future study.

Notice that points of Δ_G over a field L, namely G-torsors over $L((t))$, can also be seen as (not necessarily connected) Galois extensions of $L((t))$ and, taking integers, as special covers of $L[[t]]$ with an action of G. It is therefore natural to ask and indeed this has been our initial approach to the problem, if one can define a moduli space of special G-covers of $B[[t]]$ for varying B or, more precisely, give a different moduli interpretation of G-torsors of $B((t))$ in terms of covers of $B[[t]]$, in the spirit of [Ton17] and [Ton14]. We don't have a precise answer to this question, but in [TY19, Yas] we give partial answers.

The paper is organized as follows. In Section 1 we set up notation and terminology frequently used in the paper. In Section 2 we collect basic results on power series rings, finite and universally injective morphisms and torsors. In Section 3, after introducing a few notions and proving a few easy results, the rest of the section is devoted to the proof of the proposition mentioned above (Proposition 3.5). Section 4 is the main body of the paper, where we prove Theorems A and B. The proof of Theorem B is given on page 22, the one of Theorem A, 2) and 3) is given on page 26 and the one of Theorem A, 1) is given on page 30. Lastly we include two Appendices about limits of fibered categories, implicitly used in Theorem A, and rigidification, an operation introduced in [AOV08] for algebraic stacks and that we extends to more general stacks.

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1. Notation and terminology

Given a ring B we denote by $B((t))$ the ring of Laurent series $\sum_{i=r}^{\infty} b_i t^i$ with $b_i \in B$ and $r \in \mathbb{Z}$, that is, the localization $B[[t]]_t = B[[t]][t^{-1}]$ of the formal power series ring $B[[t]]$ with coefficients in B. This should not be confused with the fraction field of $B[[t]]$ (when B is a domain).

By a fibered category over a ring B we always mean a category fibered in groupoids over the category Aff/B of affine B-schemes.

Recall that a finite map between fibered categories is by definition affine and therefore represented by finite maps of algebraic spaces.

By a vector bundle on a scheme X we always mean a locally free sheaf of finite rank. A vector bundle on a ring B is a vector bundle on $\operatorname{Spec} B$ or, before sheafification, a projective B-module of finite type.

If $\mathcal{C}$ is a category, $X\colon \mathcal{C} \longrightarrow (\text{groups})$ is a functor of groups and S is a set we denote by $X^{(S)}\colon \mathcal{C} \longrightarrow (\text{groups})$ the functor so defined: if $c \in \mathcal{C}$ then $X^{(S)}(c)$ is the set of functions $u\colon S \longrightarrow X(c)$ such that $\{s \in S \mid u(s) \neq 1_{X(c)}\}$ is finite.

We recall that for a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of fibered categories over a ring B, f is faithful (resp. fully faithful, an equivalence) if and only if for every affine B-scheme U, $f_U\colon \mathcal{X}(U) \to \mathcal{Y}(U)$ is so (see [Sta17, 003Z]). A morphism of fibered categories is called a *monomorphism* if it is fully faithful. We also note that every representable (by algebraic spaces) morphism of stacks is faithful ([Sta17, 02ZY]).

A map $f\colon \mathcal{Y} \longrightarrow \mathcal{X}$ between fibered categories over Aff/k is a torsor under a sheaf of groups $\mathcal{G}$ over Aff/k if it is given a 2-Cartesian diagram

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \operatorname{Spec} k \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathrm{B}\,\mathcal{G} \end{array}$$

By a stack we mean a stack over the category Aff of affine schemes with respect to the fppf topology, unless a different site is specified.

We often abbreviate "Deligne-Mumford stack" to "DM stack".

2. Preliminaries

In this section we collect some general results that will be used later.

2.1. Some results on power series.

Lemma 2.1. *Let C be a ring, $J \subseteq C$ be an ideal and assume that C is J-adically complete. If $U \subseteq \operatorname{Spec} C$ is an open subset containing $\operatorname{Spec}(C/J)$ then $U = \operatorname{Spec} C$.*

Proof. Let $U = \operatorname{Spec} C - V(I)$, where $I \subseteq C$ is an ideal. The condition $\operatorname{Spec}(C/J) \subseteq U$ means that $I + J = C$. In particular there exists $g \in I$ and $j \in J$ such that $g = 1 + j$. Since j is nilpotent in all the rings C/J^n we see that g is invertible in all the rings C/J^n, which easily implies that g is invertible in C. Thus $I = C$. □

Lemma 2.2. *Let R be a ring, X be a quasi-affine scheme formally étale over R, C be an R-algebra and J be an ideal such that C is J-adically complete. Then the projection $C \longrightarrow C/J^n$ induces a bijection*

$$X(C) \longrightarrow X(C/J^n) \text{ for all } n \in \mathbb{N}$$

Proof. Since X is formally étale the projections $C/J^m \longrightarrow C/J^n$ for $m \geq n$ induce bijections

$$X(C/J^m) \longrightarrow X(C/J^n)$$

Thus it is enough to prove that if Y is any quasi-affine scheme over R then the natural map

$$\alpha_Y \colon Y(C) \longrightarrow \varprojlim_{n \in N} Y(C/J^n)$$

is bijective. This is clear when Y is affine. Let $B = \mathrm{H}^0(\mathcal{O}_Y)$, so that Y is a quasi-compact open subset of $U = \operatorname{Spec} B$. The fact that α_U is an isomorphism tells us that α_Y is injective. To see that it is surjective we have to show that if $B \longrightarrow C$ is a map such that all $\operatorname{Spec} C/J^n \longrightarrow \operatorname{Spec} B$ factors through Y then also $\phi\colon \operatorname{Spec} C \longrightarrow \operatorname{Spec} B$ factors through Y. But the first condition implies that $\phi^{-1}(Y)$ is an open subset of $\operatorname{Spec} C$ containing $\operatorname{Spec} C/J$. The equality $\phi^{-1}(Y) = \operatorname{Spec} C$ then follows from 2.1. □

Corollary 2.3. *Let B be a ring, $f\colon Y \longrightarrow \operatorname{Spec} B[[t]]$ an étale map, $\xi\colon \operatorname{Spec} L \longrightarrow \operatorname{Spec} B$ a geometric point and assume that the geometric point $\operatorname{Spec} L \longrightarrow \operatorname{Spec} B \longrightarrow \operatorname{Spec} B[[t]]$ is in the image on f. Then there exists an étale neighborhood $\operatorname{Spec} B' \longrightarrow \operatorname{Spec} B$ of ξ such that $\operatorname{Spec} B'[[t]] \longrightarrow \operatorname{Spec} B[[t]]$ factors through $Y \longrightarrow \operatorname{Spec} B[[t]]$.*

Proof. We can assume Y affine, say $Y = \operatorname{Spec} C$. Set $B' = C/tC$, so that the induced map $f_0\colon Y_0 = \operatorname{Spec} B' \longrightarrow \operatorname{Spec} B$ is étale. By hypothesis the geometric point $\operatorname{Spec} L \longrightarrow \operatorname{Spec} B$ is in the image of f_0 and therefore factors through f_0. Moreover the map $Y_0 \longrightarrow Y$ gives an element of $Y(B')$ which, by 2.2, lifts to an element of $Y(B'[[t]])$, that is a factorization of $\operatorname{Spec} B'[[t]] \longrightarrow \operatorname{Spec} B[[t]]$ through $Y \longrightarrow \operatorname{Spec} B[[t]]$. □

Lemma 2.4. *Let R be a ring, S be an R-algebra and consider the map*

$$\omega_{S/R}\colon R[[t]] \otimes_R S \to S[[t]]$$

The image of $\omega_{S/R}$ is the subring of $S[[t]]$ of series $\sum s_n t^n$ such that there exists a finitely generated R submodule $M \subseteq S$ with $s_n \in M$ for all $n \in \mathbb{N}$.

If any finitely generated R submodule of S is contained in a finitely presented R submodule of S then $\omega_{S/R}$ is injective.

Proof. The claim about the image of $\omega_{S/R}$ is easy.

Given an R-module M we define $M[[t]]$ as the R-module $M^{\mathbb{N}}$. Its elements are thought of as series $\sum_n m_n t^n$ and $M[[t]]$ has a natural structure of $R[[t]]$-module. This association extends to a functor $\operatorname{Mod} A \to \operatorname{Mod} A[[t]]$ which is easily seen to be exact. Moreover there is a natural map

$$\omega_{M/R}\colon R[[t]] \otimes_R M \to M[[t]]$$

Since both functors are right exact and $\omega_{M/R}$ is an isomorphism if M is a free R-module of finite rank, we can conclude that $\omega_{M/R}$ is an isomorphism if M is a finitely presented R-module. Let $\mathcal{P}$ be the set of finitely presented R submodules of S. By hypothesis this is a filtered set. Passing to the limit we see that the map

$$\omega_{S/R}\colon R[[t]] \otimes_R S \simeq \varinjlim_{M \in \mathcal{P}} (R[[t]] \otimes_R M) \longrightarrow \varinjlim_{M \in \mathcal{P}} M[[t]] = \bigcup_{M \in \mathcal{P}} M[[t]] \subseteq S[[t]]$$

is injective. □

Lemma 2.5. *Let N be a finite set and denote by $\underline{N}\colon \mathrm{Aff}/\mathbb{Z} \longrightarrow (\mathrm{Sets})$ the associated constant sheaf. Then the maps*

$$\underline{N}(B) \longrightarrow \underline{N}(B[[t]]) \longrightarrow \underline{N}(B((t)))$$

are bijective. In other words if B is a ring and $B((t)) \simeq C_1 \times \cdots \times C_l$ (resp. $B[[t]] = C_1 \times \cdots \times C_l$) then $B \simeq B_1 \times \cdots \times B_l$ and $C_j = B_j((t))$ (resp. $C_j = B_j[[t]]$).

Proof. Notice that $\underline{N}$ is an affine scheme étale over $\operatorname{Spec}\mathbb{Z}$. Since $A = B[[t]]$ is t-adically complete we obtain that $\underline{N}(B[[t]]) \longrightarrow \underline{N}(B[[t]]/tB[[t]])$ is bijective thanks to 2.2. Since $B \longrightarrow B[[t]]/tB[[t]]$ is an isomorphism we can conclude that $\underline{N}(B) \longrightarrow \underline{N}(B[[t]])$ is bijective.

Let n be the cardinality of N and C be a ring. An element of $\underline{N}(C)$ is a decomposition of $\operatorname{Spec} C$ into n-disjoint open and closed subsets. In particular if $n = 2$ then $\underline{N}(C)$ is the set of open and closed subsets of $\operatorname{Spec} C$. Taking this into account it is easy to reduce the problem to the case $n = 2$. In this case another way to describe $\underline{N}$ is $\underline{N} = \operatorname{Spec}\mathbb{Z}[x]/(x^2 - x)$, so that $\underline{N}(C)$ can be identified with the set of idempotents of C. Consider the map $\alpha_B \colon \underline{N}(B[[t]]) \longrightarrow \underline{N}(B((t)))$, which is injective since $B[[t]] \longrightarrow B((t))$ is so. If, by contradiction, α_B is not surjective, we can define $k > 0$ as the minimum positive number for which there exist a ring B and $a \in B[[t]]$ such that $a/t^k \in \underline{N}(B((t)))$ and $a/t^k \notin B[[t]]$. Let B, a as before and set $a_0 = a(0)$. It is easy to check that $a_0^2 = 0$ in B. Set $C = B/\langle a_0 \rangle$. By 2.4 we have that $B[[t]]/a_0 B[[t]] = C[[t]]$ and that $B((t))/a_0 B((t)) = C((t))$. Thus we have a commutative diagram

$$\begin{array}{ccc} \underline{N}(B[[t]]) & \xrightarrow{\alpha_B} & \underline{N}(B((t))) \\ \downarrow & & \downarrow \beta \\ \underline{N}(C[[t]]) & \xrightarrow{\alpha_C} & \underline{N}(C((t))) \end{array}$$

in which the vertical maps are bijective, since the topological space of a spectrum does not change modding out by a nilpotent. By construction $\beta(a/t^k) = a'/t^{k-1}$ where $a' = (a - a_0)/t$. By minimality of k we must have that $a'/t^{k-1} \in C[[t]]$. Since the vertical maps in the above diagram are bijective we can conclude that also $a/t^k \in B[[t]]$, a contradiction. □

Lemma 2.6. *Let M, N be vector bundles on $B((t))$. Then the functor*

$$\underline{\operatorname{Hom}}_{B((t))}(M, N) \colon \operatorname{Aff}/B \longrightarrow (\operatorname{Sets}),\ C \longmapsto \operatorname{Hom}_{C((t))}(M \otimes C((t)), N \otimes C((t)))$$

is a sheaf in the fpqc topology.

Proof. Set $H = \operatorname{Hom}_{B((t))}(M, N)$, which is a vector bundle over $B((t))$. Moreover if C is a B-algebra we have $H \otimes_{B((t))} C((t)) \simeq \underline{\operatorname{Hom}}_{B((t))}(M, N)(C)$ because M and N are vector bundles. By A.4 we have to prove descent on coverings indexed by a finite set and, by 2.5, it is enough to consider a faithfully flat map $B \longrightarrow C$. If $i_1, i_2 \colon C \longrightarrow C \otimes_B C$ are the two inclusions, descent corresponds to the exactness of the sequence

$$0 \longrightarrow H \longrightarrow C((t)) \otimes H \xrightarrow{(i_1 - i_2) \otimes \mathrm{id}_H} (C \otimes_B C)((t)) \otimes H$$

Since this sequence is obtained applying $\otimes_{B((t))} H$ to the exact sequence $0 \longrightarrow B((t)) \longrightarrow C((t)) \xrightarrow{i_1 - i_2} (C \otimes_B C)((t))$ and H is flat we get the result. □

2.2. **Finite and universally injective morphisms.**

Definition 2.7. A map $\mathcal{X} \longrightarrow \mathcal{Y}$ between algebraic stacks is *universally injective* (resp. *universally bijective*, a *universal homeomorphism*) if for all maps $\mathcal{Y}' \longrightarrow \mathcal{Y}$ from an algebraic stack the map $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \longrightarrow |\mathcal{Y}'|$ on topological spaces is injective (resp. bijective, an homeomorphism).

Remark 2.8. In order to show that a map $\mathcal{X} \longrightarrow \mathcal{Y}$ is universally injective (resp. universally bijective, a universal homeomorphism) it is enough to test on maps $\mathcal{Y}' \longrightarrow \mathcal{Y}$ where $\mathcal{Y}'$ is an affine scheme. Indeed injectivity and surjectivity can be tested on the geometric fibers. Moreover if $\bigsqcup_i Y_i \longrightarrow \mathcal{Y}$ is a smooth surjective map and the Y_i are affine then $|\mathcal{X}| \longrightarrow |\mathcal{Y}|$ is open if $|\mathcal{X} \times_{\mathcal{Y}} Y_i| \longrightarrow |Y_i|$ is open for all i. In particular if $\mathcal{X} \longrightarrow \mathcal{Y}$ is representable then it is universally injective (resp. bijective, a universal homeomorphism) if and only if it is represented by map of algebraic spaces which are universally injective (resp. universally bijective, universal homeomorphisms) in the usual sense.

Proposition 2.9. *Let $f\colon \mathcal{X} \longrightarrow \mathcal{Y}$ be a map of algebraic stacks. Then f is finite and universally injective if and only if it is a composition of a finite universal homeomorphism and a closed immersion. More precisely, if $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{\mathcal{Y}} \longrightarrow f_*\mathcal{O}_{\mathcal{X}})$, then $\mathcal{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_{\mathcal{Y}}/\mathcal{I})$ is finite and a universal homeomorphism.*

Proof. The if part in the statement is clear. So assume that f is finite and universally injective and consider the factorization $\mathcal{X} \xrightarrow{g} \mathcal{Z} = \mathrm{Spec}(\mathcal{O}_{\mathcal{Y}}/\mathcal{I}) \xrightarrow{h} \mathcal{Y}$. Since f is finite, the map g is finite and surjective. Since h is a monomorphism, given a map $U \longrightarrow \mathcal{Z}$ from a scheme we have that $\mathcal{X} \times_{\mathcal{Z}} U \longrightarrow \mathcal{X} \times_{\mathcal{Y}} U$ is an isomorphism, which implies that g is also universally injective as required. □

Remark 2.10. The following properties of morphisms of schemes are stable by base change and fpqc local on the base: finite, closed immersion, universally injective, surjective and universal homeomorphism (see [Sta17, 02WE]). In particular for representable maps of algebraic stacks those properties can be checked on an atlas.

Remark 2.11. Let $f\colon \mathcal{S}' \longrightarrow \mathcal{S}$ be a map of algebraic stacks, $\mathcal{U}, \mathcal{V}$ and $\mathcal{U}', \mathcal{V}'$ algebraic stacks with a map to $\mathcal{S}$ and $\mathcal{S}'$ respectively and $u\colon \mathcal{U}' \longrightarrow \mathcal{U}$, $v\colon \mathcal{V}' \longrightarrow \mathcal{V}$ be $\mathcal{S}$-maps. If f, u, v are finite and universally injective then so is the induced map $\mathcal{U}' \times_{\mathcal{S}'} \mathcal{V}' \longrightarrow \mathcal{U} \times_{\mathcal{S}} \mathcal{V}$. The map $(\mathcal{U} \times_{\mathcal{S}} \times \mathcal{V}) \times_{\mathcal{S}} \mathcal{S}' \longrightarrow \mathcal{U} \times_{\mathcal{S}} \mathcal{V}$ is finite and universally injective. The map $\mathcal{U}' \longrightarrow \mathcal{U} \times_{\mathcal{S}} \mathcal{S}'$ is also finite and universally injective because $\mathcal{U} \times_{\mathcal{S}} \mathcal{S}' \longrightarrow \mathcal{U}$ and $\mathcal{U}' \longrightarrow \mathcal{U}$ are so (use [Sta17, 01S4] for the universal injectivity). Thus we can assume $\mathcal{S} = \mathcal{S}'$ and $f = \mathrm{id}$. In this case it is enough to use the factorization $\mathcal{U}' \times_{\mathcal{S}} \mathcal{V}' \longrightarrow \mathcal{U} \times_{\mathcal{S}} \mathcal{V}' \longrightarrow \mathcal{U} \times_{\mathcal{S}} \mathcal{V}$.

2.3. **Some results on torsors.** In what follows, actions of groups (or sheaves of groups) are supposed to be right actions. Recall that for a sheaf $\mathcal{G}$ of groups on a site $\mathcal{C}$, $\mathrm{B}\,\mathcal{G}$ denotes the category of $\mathcal{G}$-torsors over objects of $\mathcal{C}$, and that given a map $\mathcal{G} \to \mathcal{H}$ of sheaves of groups, then there exists a functor $\mathrm{B}\,\mathcal{G} \to \mathrm{B}\,\mathcal{H}$ sending a $\mathcal{G}$-torsor $\mathcal{P}$ to the $\mathcal{H}$-torsor $(\mathcal{P} \times \mathcal{H})/\mathcal{G}$.

Lemma 2.12. *Let $\mathcal{G}$ be a sheaf of groups on a site $\mathcal{C}$ and $\mathcal{H}$ be a sheaf of subgroups of the center $Z(\mathcal{G})$. Then $\mathcal{H}$ is normal in $\mathcal{G}$, the map $\mu\colon \mathcal{G} \times \mathcal{H} \longrightarrow \mathcal{G}$ restriction of the multiplication is a morphism of groups and the first diagram*

$$\begin{array}{ccc} \mathrm{B}(\mathcal{G} \times \mathcal{H}) & \longrightarrow & \mathrm{B}(\mathcal{G}) \\ \downarrow & & \downarrow \\ \mathrm{B}(\mathcal{G}) & \longrightarrow & \mathrm{B}(\mathcal{G}/\mathcal{H}) \end{array} \qquad \begin{array}{ccc} \mathcal{G} \times \mathcal{H} & \xrightarrow{\mu} & \mathcal{G} \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{G}/\mathcal{H} \end{array}$$

induced by the second one is 2-Cartesian. A quasi-inverse $\mathrm{B}(\mathcal{G}) \times_{\mathrm{B}(\mathcal{G}/\mathcal{H})} \mathrm{B}(\mathcal{G}) \longrightarrow \mathrm{B}(\mathcal{G} \times \mathcal{H}) = \mathrm{B}(\mathcal{G}) \times \mathrm{B}(\mathcal{H})$ is obtained as follows: given $(\mathcal{P}, \mathcal{Q}, \lambda) \in \mathrm{B}(\mathcal{G}) \times_{\mathrm{B}(\mathcal{G}/\mathcal{H})} \mathrm{B}(\mathcal{G})$ (so that $\lambda\colon \mathcal{P}/\mathcal{H} \longrightarrow \mathcal{Q}/\mathcal{H}$ is a $\mathcal{G}/\mathcal{H}$-equivariant isomorphism) we associate $(\mathcal{P}, \mathcal{I}_\lambda)$, where $\mathcal{I}_\lambda$ is the fiber of λ along the map $\underline{\mathrm{Iso}}^{\mathcal{G}}(\mathcal{P}, \mathcal{Q}) \longrightarrow \underline{\mathrm{Iso}}^{\mathcal{G}/\mathcal{H}}(\mathcal{P}/\mathcal{H}, \mathcal{Q}/\mathcal{H})$ and the action of $\mathcal{H}$ is given by $\mathcal{H} \longrightarrow \mathcal{G} \longrightarrow \underline{\mathrm{Aut}}(\mathcal{Q})$.

Proof. Let $(\mathcal{P},\mathcal{Q},\lambda)\in \mathrm{B}(\mathcal{G})\times_{\mathrm{B}(\mathcal{G}/\mathcal{H})}\mathrm{B}(\mathcal{G})$ over an object $c\in\mathcal{C}$. The composition $\mathcal{H}\longrightarrow\mathcal{G}\longrightarrow \underline{\mathrm{Aut}}(\mathcal{Q})$ has image in $\underline{\mathrm{Aut}}^{\mathcal{G}}(\mathcal{Q})$ because $\mathcal{H}$ is central. Moreover the map $\underline{\mathrm{Iso}}^{\mathcal{G}}(\mathcal{P},\mathcal{Q})\longrightarrow \underline{\mathrm{Iso}}^{\mathcal{G}/\mathcal{H}}(\mathcal{P}/\mathcal{H},\mathcal{Q}/\mathcal{H})$ is $\mathcal{H}$-equivariant. It is also an $\mathcal{H}$-torsor: locally when $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic to $\mathcal{G}$, the previous map become $\mathcal{G}\longrightarrow\mathcal{G}/\mathcal{H}$. Thus $\mathcal{I}_\lambda$ is an $\mathcal{H}$-torsor over $c\in\mathcal{C}$. Thus we have two well defined functors

$$\Lambda\colon \mathrm{B}(\mathcal{G})\times_{\mathrm{B}(\mathcal{G}/\mathcal{H})}\mathrm{B}(\mathcal{G})\longrightarrow \mathrm{B}(\mathcal{G})\times\mathrm{B}(\mathcal{H}) \text{ and } \Delta\colon \mathrm{B}(\mathcal{G})\times\mathrm{B}(\mathcal{H})\longrightarrow \mathrm{B}(\mathcal{G})\times_{\mathrm{B}(\mathcal{G}/\mathcal{H})}\mathrm{B}(\mathcal{G})$$

and we must show they are quasi-inverses of each other. Let's consider the composition $\Lambda\circ\Delta$ and $(\mathcal{P},\mathcal{E})\in \mathrm{B}(\mathcal{G})\times\mathrm{B}(\mathcal{H})$ over $c\in\mathcal{C}$. We have $\Delta(\mathcal{P},\mathcal{E})=(\mathcal{P},(\mathcal{P}\times\mathcal{E}\times\mathcal{G})/\mathcal{G}\times\mathcal{H},\lambda)$ where λ is the inverse of

$$[(\mathcal{P}\times\mathcal{E}\times\mathcal{G})/\mathcal{G}\times\mathcal{H}]/\mathcal{H}\longrightarrow\mathcal{P}/\mathcal{H},\ (p,e,1)\longrightarrow p$$

We have to give an $\mathcal{H}$-equivariant map $\mathcal{E}\longrightarrow\mathcal{I}_\lambda$. Given a global section $e\in\mathcal{E}$, that is a map $\mathcal{H}\longrightarrow\mathcal{E}$, we get a $\mathcal{G}\times\mathcal{H}$ equivariant morphism $\mathcal{P}\times\mathcal{H}\longrightarrow\mathcal{P}\times\mathcal{E}$ and thus a $\mathcal{G}$-equivariant morphism

$$\delta\colon \mathcal{P}\longrightarrow(\mathcal{P}\times\mathcal{H}\times\mathcal{G})/\mathcal{G}\times\mathcal{H}\longrightarrow(\mathcal{P}\times\mathcal{E}\times\mathcal{G})/\mathcal{G}\times\mathcal{H}$$

which is easily seen to induce λ. Mapping e to δ gives an $\mathcal{H}$-equivariant map $\mathcal{E}\to\mathcal{I}_\lambda$. There are several conditions that must be checked but they are all elementary and left to the reader.

Now consider $\Delta\circ\Lambda$ and $(\mathcal{P},\mathcal{Q},\lambda)\in \mathrm{B}(\mathcal{G})\times_{\mathrm{B}(\mathcal{G}/\mathcal{H})}\mathrm{B}(\mathcal{G})$ over an object $c\in\mathcal{C}$. It is easy to see that

$$(\mathcal{P}\times\mathcal{I}_\lambda\times\mathcal{G})/\mathcal{G}\times\mathcal{H}\longrightarrow\mathcal{Q},\ (p,\phi,g)\mapsto\phi(p)g$$

is a $\mathcal{G}$-equivariant morphism and it induces a morphism $\Delta\circ\Lambda(\mathcal{P},\mathcal{Q},\lambda)\longrightarrow(\mathcal{P},\mathcal{Q},\lambda)$. □

Remark 2.13. If $X\longrightarrow Y$ is integral (e.g. finite) and a universal homeomorphism of schemes and G is an étale group scheme over a field k then $\mathrm{B}\,G(Y)\longrightarrow\mathrm{B}\,G(X)$ is an equivalence. Indeed by [AGV64, Expose VIII, Theorem 1.1] the fiber product induces an equivalence between the category of schemes étale over Y and the category of schemes étale over X.

Lemma 2.14. *Let G be a finite group scheme over k of rank $\mathrm{rk}\,G$, $U\longrightarrow\mathcal{G}$ a finite, flat and finitely presented map of degree $\mathrm{rk}\,G$ and $\mathcal{G}\longrightarrow T$ be a map locally equivalent to $\mathrm{B}\,G$, where U, $\mathcal{G}$ and T are categories fibered in groupoids. If $U\longrightarrow T$ is faithful then it is an equivalence.*

Proof. By changing the base T we can assume that T is a scheme, $\mathcal{G}=\mathrm{B}\,G\times T$ and U is an algebraic space. We must prove that if $P\longrightarrow U$ is a G-torsor and $P\longrightarrow U\longrightarrow T$ is a cover of degree $\mathrm{rk}\,G$ then $f\colon U\longrightarrow T$ is an isomorphism. It follows that $f\colon U\longrightarrow T$ is flat, finitely presented and quasi-finite. Moreover $f\colon U\longrightarrow\mathrm{B}\,G\times T\longrightarrow T$ is proper. We can conclude that $f\colon U\longrightarrow T$ is finite and flat. Looking at the ranks of the involved maps we see that f must have rank 1. □

3. Direct system of Deligne-Mumford stacks

In this section we discuss some general facts about direct limits of DM stacks. For the general notion of limit see Appendix A. By a direct system in this section we always mean a direct system indexed by $\mathbb{N}$.

Definition 3.1. Let $\mathcal{X}$ be a category fibered in groupoid over $\mathbb{Z}$. A coarse ind-algebraic space for $\mathcal{X}$ is a map $\mathcal{X}\longrightarrow X$ to an ind-algebraic space X which is universal among maps from $\mathcal{X}$ to an ind-algebraic space and such that, for all algebraically closed field K, the map $\mathcal{X}(K)/\simeq\longrightarrow X(K)$ is bijective.

Lemma 3.2. *Let $\mathcal{Z}_*$ be a direct system of quasi-compact and quasi-separated algebraic stacks admitting coarse moduli spaces $\mathcal{Z}_n \longrightarrow \overline{\mathcal{Z}_n}$. Then the limit of those maps $\Delta \longrightarrow \overline{\Delta}$ is a coarse ind-algebraic space.*

Assume moreover that the transition maps of $\mathcal{Z}_*$ are finite and universally injective. Then for all $n \in \mathbb{N}$ and all reduced rings B the functors $\mathcal{Z}_n(B) \longrightarrow \mathcal{Z}_{n+1}(B) \longrightarrow \Delta(B)$ are fully faithful. In particular the maps $\overline{\mathcal{Z}_n} \longrightarrow \overline{\mathcal{Z}_{n+1}}$ are universally injective and, if all $\mathcal{Z}_m$ are DM, $\mathcal{Z}_n \longrightarrow \Delta$ preserves the geometric stabilizers.

Proof. The first claim follows easily taking into account that, since $\mathcal{Z}_n$ is quasi-compact and quasi-separated, a functor from $\mathcal{Z}_n$ to an ind-algebraic space factors through an algebraic space and therefore uniquely through $\overline{\mathcal{Z}_n}$. It is also easy to reduce the second claim to the case of some $\mathcal{Z}_n$.

Denote by $\psi\colon \mathcal{Z}_n \longrightarrow \mathcal{Z}_{n+1}$ the transition map. Let $\xi, \eta \in \mathcal{Z}_n(B)$ and $a\colon \psi(\xi) \longrightarrow \psi(\eta)$. Set $\psi(\eta) = \zeta \in \mathcal{Z}_{n+1}(B)$. If W is the base change of $\mathcal{Z}_n \longrightarrow \mathcal{Z}_{n+1}$ along $\operatorname{Spec} B \xrightarrow{\zeta} \mathcal{Z}_{n+1}$ then $\overline{\xi} = (\xi, \zeta, a), \overline{\eta} = (\eta, \zeta, \mathrm{id}) \in W(B)$. A lifting of a to an isomorphism $\xi \longrightarrow \eta$ is exactly an isomorphism $\overline{\xi} \longrightarrow \overline{\eta}$. Such an isomorphism exists and it is unique because, since $W \longrightarrow \operatorname{Spec} B$ is an homeomorphism and B is reduced it has at most one section.

Applying the above property when B is an algebraically closed field we conclude that $\overline{\mathcal{Z}_n} \longrightarrow \overline{\mathcal{Z}_{n+1}}$ is universally injective. If all $\mathcal{Z}_m$ are DM then the geometric stabilizers are constant. Since for all algebraically closed field K the functor $\mathcal{Z}_n(K) \longrightarrow \mathcal{Z}_{n+1}(K)$ is fully faithful we see that $\mathcal{Z}_n \longrightarrow \mathcal{Z}_{n+1}$ is an isomorphism on geometric stabilizers. □

Definition 3.3. Given a direct system of stacks $\mathcal{Y}_*$, a direct system of smooth (resp. étale) atlases for $\mathcal{Y}_*$ is a direct system of algebraic spaces U_* together with smooth (resp. étale) atlases $U_i \longrightarrow \mathcal{Y}_i$ and 2-Cartesian diagrams

$$\begin{array}{ccc} U_i & \longrightarrow & U_{i+1} \\ \downarrow & & \downarrow \\ \mathcal{Y}_i & \longrightarrow & \mathcal{Y}_{i+1} \end{array}$$

for all $i \in \mathbb{N}$.

Lemma 3.4. *Let $\mathcal{Y}_*$ be a direct system of stacks and $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack. Then the functor*

$$\varinjlim_n \operatorname{Hom}(\mathcal{X}, \mathcal{Y}_n) \longrightarrow \operatorname{Hom}(\mathcal{X}, \varinjlim_n \mathcal{Y})$$

is an equivalence of categories. If the transition maps of $\mathcal{Y}_*$ are faithful (resp. fully faithful) so are the transition maps in the above limit.

Proof. Denotes by $\zeta_{\mathcal{X}}$ the functor in the statement. When $\mathcal{X}$ is an affine scheme $\zeta_{\mathcal{X}}$ is an equivalence thanks to A.5. In general there is a smooth atlas $U \longrightarrow \mathcal{X}$ from an affine scheme. It is easy to see that the functor $\zeta_{\mathcal{X}}$ is faithful. If two morphisms become equal in the limit it is enough to pullback to U and get a finite index for ζ_U. By descent this index will work in general.

The next step is to look at the case when $\mathcal{X}$ is a quasi-compact scheme. Using the faithfulness just proved and taking a Zariski covering of $\mathcal{X}$ (here one uses that the intersection of two open quasi-compact subschemes of $\mathcal{X}$ is again quasi-compact) one proves that $\zeta_{\mathcal{X}}$ is an equivalence.

Finally using that ζ_U, $\zeta_{U \times_{\mathcal{X}} U}$ and $\zeta_{U \times_{\mathcal{X}} U \times_{\mathcal{X}} U}$ are equivalences and using descent one get that $\zeta_{\mathcal{X}}$ is an equivalence. The last statement can be proved directly. □

Proposition 3.5. *Consider a* 2*-Cartesian diagram*

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \operatorname{Spec} k \\ \downarrow{\scriptstyle F} & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathrm{B}\, G \end{array}$$

where G is a finite group and $\mathcal{X}$ is a stack over Aff/k*. Suppose that there exists a direct system $\mathcal{Y}_*$ of DM stacks of finite type over k with finite and universally injective transition maps, affine diagonal, with a direct system of étale atlases $Y_n \longrightarrow \mathcal{Y}_n$ from affine schemes and with an isomorphism $\varinjlim_n \mathcal{Y}_n \simeq \mathcal{Y}$. Then there exists a direct system of DM stacks $\mathcal{X}_*$ of finite type over k with finite and universally injective transition maps, affine diagonal, with a direct system of étale atlases $X_n \longrightarrow \mathcal{X}_n$ from affine schemes and with an isomorphism $\varinjlim_n \mathcal{X}_n \simeq \mathcal{X}$. If all the stacks in $\mathcal{Y}_*$ are separated then the stacks $\mathcal{X}_*$ can also be chosen separated. If $Y_n \longrightarrow \mathcal{Y}_n$ is finite and étale then $X_n \longrightarrow \mathcal{X}_n$ can also be chosen finite and étale.*

The remaining part of this section is devoted to the proof of the above Proposition. Its outline is as follows. For some data $\underline{\omega}$, we define a stack $\mathcal{X}_{\underline{\omega}}$, and for a suitable sequence $\underline{\omega_u}$, $u \in \mathbb{N}$ of such data, we will prove that the sequence

$$\mathcal{X}_{\underline{\omega_0}} \to \mathcal{X}_{\underline{\omega_1}} \to \cdots$$

has the desired property. To do this, we reduce the problem to proving a similar property for the induced sequence $\mathcal{Z}_{\underline{\omega_u}} \cong \mathcal{X}_{\underline{\omega_u}} \times_{\mathcal{X}} \mathcal{Y}$, $u \in \mathbb{N}$. Then we describe $\mathcal{Z}_{\underline{\omega}}$ using fiber products of simple stacks. Once these are done, it is straightforward to see the desired properties of $\mathcal{Z}_{\underline{\omega_u}}$, $u \in \mathbb{N}$.

We recall that stacks and more generally categories fibered in groupoids form 2-categories; 1-morphisms are base preserving functors between them and 2-morphisms are base preserving natural isomorphisms between functors. In a diagram of stacks, 1-morphisms (functors) are written as normal thin arrows and 2-morphisms as thick arrows. For instance, in the diagram of categories fibered in groupoids

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & \swarrow\!\!\swarrow{\scriptstyle \lambda} & \downarrow{\scriptstyle g} \\ C & \xrightarrow[i]{} & D \end{array}$$

A, B, C and D are categories fibered in groupoids, f, g, h and i are functors and λ is a natural isomorphism $g \circ f \to i \circ h$. We will also say that λ *makes the diagram* 2*-commutative*. For a diagram including several 2-morphisms such as

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle i}\downarrow & \swarrow\!\!\swarrow{\scriptstyle \lambda} & \downarrow & \swarrow\!\!\swarrow{\scriptstyle \lambda'} & \downarrow{\scriptstyle h} \\ D & \xrightarrow[j]{} & E & \xrightarrow[k]{} & F \end{array}$$

the *induced natural isomorphism* means that the induced natural isomorphism of the two outer paths from the upper left to the bottom right; concretely, in the above diagram, it is the natural isomorphism $h \circ g \circ f \to k \circ j \circ i$ induced by λ and λ'.

Let us denote the functor $\mathcal{X} \longrightarrow \mathrm{B}\, G$ in 3.5 by Q. An object of $\mathcal{Y}$ over $T \in \mathrm{Aff}/k$ is identified with a pair (ξ, c) such that $\xi \in \mathcal{X}(T)$ and c is a section of the G-torsor $Q(\xi) \longrightarrow T$, in other words $c \in Q(\xi)(T)$. For each $g \in G$, we define an automorphism $\iota_g \colon \mathcal{Y} \to \mathcal{Y}$ sending (x, c) to (x, cg). By construction we have $\iota_1 = \mathrm{id}$

and $\iota_g \circ \iota_h = \iota_{hg}$ and we interpret those maps as a map $\iota\colon \mathcal{Y} \times G \longrightarrow \mathcal{Y}$. For all $\xi \in \mathcal{X}$ the map ι induces the action of G on $Q(\xi)$.

Definition 3.6. We define a category fibered in groupoids $\widetilde{\mathcal{X}}$ as follows. An object of $\widetilde{\mathcal{X}}$ over a scheme T is a tuple (P, η, μ) where P is a G-torsor over $T \in \mathrm{Aff}/k$ with a G-action $m_P\colon P \times G \to P$, $\eta\colon P \to \mathcal{Y}$ is a morphism and μ is a natural isomorphism

(3.1)
$$\begin{array}{ccc} P \times G & \xrightarrow{m_P} & P \\ {\scriptstyle \eta \times \mathrm{id}_G}\downarrow & \overset{\mu}{\swarrow} & \downarrow{\scriptstyle \eta} \\ \mathcal{Y} \times G & \xrightarrow[\iota]{} & \mathcal{Y} \end{array}$$

such that if μ_g denotes the natural isomorphism induced from μ by composing $P \simeq P \times \{g\} \hookrightarrow P \times G$ and $m_g\colon P \to P$ denotes the action of g, then the diagram

(3.2)
$$\begin{array}{ccccc} P & \xrightarrow{m_h} & P & \xrightarrow{m_g} & P \\ {\scriptstyle \eta}\downarrow & \overset{\mu_h}{\swarrow} & {\scriptstyle \eta}\downarrow & \overset{\mu_g}{\swarrow} & \downarrow{\scriptstyle \eta} \\ \mathcal{Y} & \xrightarrow[\iota_h]{} & \mathcal{Y} & \xrightarrow[\iota_g]{} & \mathcal{Y} \end{array}$$

induces μ_{hg}. A morphism $(P, \eta, \mu) \to (P', \eta', \mu')$ over T is a pair (α, β) where $\alpha\colon P \to P'$ is a G-equivariant isomorphism over T and β is a natural isomorphism

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & P' \\ & {\scriptstyle \eta}\searrow \quad \overset{\beta}{\swarrow} \quad \swarrow{\scriptstyle \eta'} & \\ & \mathcal{Y} & \end{array}$$

such that the diagram

(3.3)
$$\begin{array}{ccccc} P \times G & & & & P \\ & \searrow^{\alpha \times \mathrm{id}} \quad \overset{\mathrm{id}}{\swarrow} & & \swarrow^{\alpha} & \\ {\scriptstyle \eta \times \mathrm{id}}\downarrow & \overset{\beta^{-1} \times \mathrm{id}}{\Leftarrow} & P' \times G & \overset{\beta}{\Rightarrow} & \downarrow{\scriptstyle \eta} \\ & \swarrow_{\eta' \times \mathrm{id}} \quad \overset{\mu'}{\searrow} & & \searrow_{\eta'} & \\ \mathcal{Y} \times G & & \xrightarrow[\iota]{} & & \mathcal{Y} \end{array}$$

induces μ.

There is a functor $\mathcal{X} \longrightarrow \widetilde{\mathcal{X}}$: given $\xi \in \mathcal{X}(T)$ one gets a G-torsor $Q(\xi) \to T$ and a morphism $\eta\colon Q(\xi) \to \mathcal{Y}$ and using the Cartesian diagram relating $\mathcal{X}$ and $\mathcal{Y}$ we also get a natural transformation as above. The following is a generalization of [Rom05, Theorem 4.1] for stacks without geometric properties.

Lemma 3.7. *The functor $\mathcal{X} \longrightarrow \widetilde{\mathcal{X}}$ is an equivalence.*

Proof. The forgetful functor $\widetilde{\mathcal{X}} \longrightarrow \mathrm{B}\, G$ composed with the functor in the statement is $Q\colon \mathcal{X} \longrightarrow \mathrm{B}\, G$. Since $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ are stacks, it is enough to show that the functor $\Xi\colon \mathcal{Y} \longrightarrow \widetilde{\mathcal{Y}} = \widetilde{\mathcal{X}} \times_{\mathrm{B}\, G} \operatorname{Spec} k$ is an equivalence.

An object of the stack $\widetilde{\mathcal{Y}}$ can be regarded as a pair (η, μ) such that $\eta\colon T \times G \to \mathcal{Y}$ is a morphism and μ is a natural isomorphism as in (3.1) with $m_P\colon P \times G \to P$ replaced with $\mathrm{id}_T \times m_G\colon T \times G \times G \to T \times G$, where m_G is the multiplication of G. A morphism $(\eta, \mu) \to (\eta', \mu')$ in $\widetilde{\mathcal{Y}}(T)$ is a natural isomorphism $\beta\colon \eta \to \eta'$ satisfying the same compatibility as in (3.3) where P and P' are replaced with $T \times G$ and α is replaced with $\mathrm{id}_{T \times G}$. The functor Ξ sends an object $\rho \in \mathcal{Y}(T)$

to $(\widetilde{\rho}\colon T\times G\to\mathcal{Y},\mu)$ such that $\widetilde{\rho}|_{T\times\{g\}}=\iota_g\circ\rho$ and μ is the canonical natural isomorphism, and a morphism $\gamma\colon\rho\to\rho'$ to $(\mathrm{id}_{T\times G},\widetilde{\gamma})$ where $\widetilde{\gamma}|_{T\times\{g\}}=\iota_g(\gamma)$. One also gets a functor $\Lambda\colon\widetilde{\mathcal{Y}}\longrightarrow\mathcal{Y}$ by composing with the identity of G and it is easy to see that $\Lambda\circ\Xi=\mathrm{id}$. The compatibilities defining the objects of $\widetilde{\mathcal{Y}}$ also allow to define an isomorphism $\Xi\circ\Lambda\simeq\mathrm{id}$. □

Set $\delta_u\colon\mathcal{Y}_u\longrightarrow\mathcal{Y}$ for the structure maps and $\delta_{u,v}\colon\mathcal{Y}_u\longrightarrow\mathcal{Y}_v$ for the transition maps for all $u\leq v\in\mathbb{N}$. Given $w\geq v\geq u\in\mathbb{N}$ we denote by $\mathcal{R}(u,v,w)$ the collection of tuples $(\omega,\omega',\theta,\theta')$ forming 2-commutative diagrams:

(3.4)
$$\begin{array}{ccc}\mathcal{Y}_u\times G & \xrightarrow{\omega} & \mathcal{Y}_v\\ {\scriptstyle\delta_u\times\mathrm{id}}\downarrow & \swarrow\!\!\swarrow\ {\scriptstyle\theta} & \downarrow{\scriptstyle\delta_v}\\ \mathcal{Y}\times G & \xrightarrow[\iota]{} & \mathcal{Y}\end{array}$$

(3.5)
$$\begin{array}{ccc}\mathcal{Y}_v\times G & \xrightarrow{\omega'} & \mathcal{Y}_w\\ {\scriptstyle\delta_v\times\mathrm{id}}\downarrow & \swarrow\!\!\swarrow\ {\scriptstyle\theta'} & \downarrow{\scriptstyle\delta_w}\\ \mathcal{Y}\times G & \xrightarrow[\iota]{} & \mathcal{Y}\end{array}$$

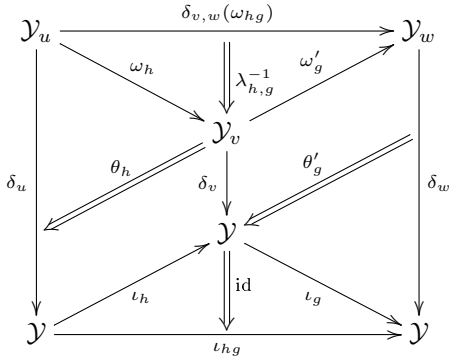
We also require the existence of a natural isomorphism

(3.6)
$$\begin{array}{ccc}\mathcal{Y}_u\times G & \xrightarrow{\omega} & \mathcal{Y}_v\\ {\scriptstyle\delta_{u,v}\times\mathrm{id}}\downarrow & \swarrow\!\!\swarrow\ {\scriptstyle\zeta} & \downarrow{\scriptstyle\delta_{v,w}}\\ \mathcal{Y}_v\times G & \xrightarrow[\omega']{} & \mathcal{Y}_w\end{array}$$

compatible with θ and θ' and, for $g,h\in G$, the existence of a natural isomorphism $\lambda_{h,g}$

(3.7)
$$\begin{array}{ccc}\mathcal{Y}_u & \xrightarrow{\omega_h} & \mathcal{Y}_v\\ {\scriptstyle\omega_{hg}}\downarrow & \swarrow\!\!\swarrow\ {\scriptstyle\lambda_{h,g}} & \downarrow{\scriptstyle\omega'_g}\\ \mathcal{Y}_v & \xrightarrow[\delta_{v,w}]{} & \mathcal{Y}_w\end{array}$$

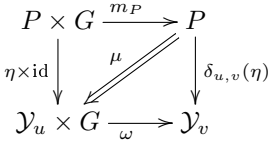
such that the natural isomorphism induced by

(3.8)


coincides with the one induced by θ_{hg} and $\delta_{v,w}$. Since the transition maps of $\mathcal{Y}_*$ are faithful, the functor $\mathrm{Hom}(\mathcal{Y}_u,\mathcal{Y}_w)\to\mathrm{Hom}(\mathcal{Y}_u,\mathcal{Y})$ is faithful as well, which means that natural transformations ζ and $\lambda_{g,h}$ are uniquely determined.

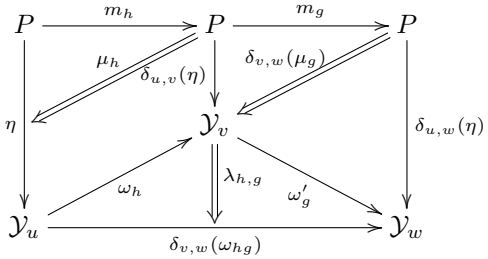
Definition 3.8. For $\underline{\omega} = (\omega, \omega', \theta, \theta') \in \mathcal{R}(u, v, w)$, we define the following category fibered in groupoids $\mathcal{X}_{\underline{\omega}}$ as follows.

An object of $\mathcal{X}_{\underline{\omega}}$ over T is a triple (P, η, μ) of a G-torsor P over T, $\eta\colon P \longrightarrow \mathcal{Y}_u$ and a natural isomorphism μ making the diagram



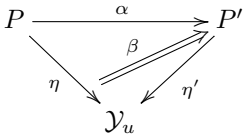
2-commutative such that the diagram

(3.9)



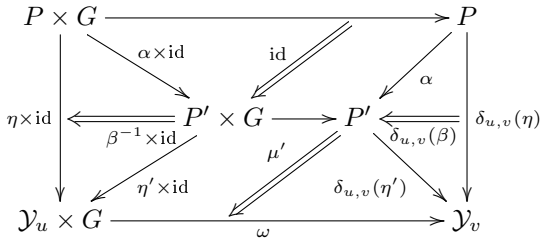
induces $\delta_{v,w}(\mu_{hg})$.

A morphism $(P, \eta, \mu) \longrightarrow (P', \eta', \mu')$ in $\mathcal{X}_{\underline{\omega}}(T)$ is a pair (α, β) where $\alpha\colon P \longrightarrow P'$ is a G-equivariant isomorphism over T and β is a natural isomorphism making the following diagram 2-commutative



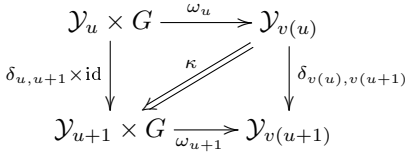
such that the diagram

(3.10)



induces the natural isomorphism μ.

By 3.4 for all algebraic stacks $\mathcal{X}$ the functor $\varinjlim_w \mathrm{Hom}(\mathcal{X}, \mathcal{Y}_w) \to \mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ is an equivalence. This allows us to choose increasing functions $v, w\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that $u \leq v(u) \leq w(u)$ and $\underline{\omega_u} = (\omega_u, \omega'_u, \theta_u, \theta'_u) \in \mathcal{R}(u, v(u), w(u))$, so that $\mathcal{X}_{\underline{\omega_u}}$ is defined, for all $u \in \mathbb{N}$. Moreover we can assume there exist natural isomorphisms κ and κ'



$$\begin{array}{ccc} \mathcal{Y}_{v(u)} \times G & \xrightarrow{\omega'_u} & \mathcal{Y}_{w(u)} \\ {\scriptstyle \delta_{v(u),v(u+1)}\times \mathrm{id}}\downarrow & \swarrow_{\kappa'} & \downarrow{\scriptstyle \delta_{w(u),w(u+1)}} \\ \mathcal{Y}_{v(u+1)} \times G & \xrightarrow[\omega'_{u+1}]{} & \mathcal{Y}_{w(u+1)} \end{array}$$

such that $\delta_{v(u+1)}(\kappa)$ is induced from θ_u and θ_{u+1} and $\delta_{w(u+1)}(\kappa')$ is induced from θ'_u and θ'_{u+1}. Again, since the transition maps of $\mathcal{Y}_*$ are faithful, natural transformations κ and κ' are uniquely determined.

For each $u \in \mathbb{N}$, there exist canonical functors $\mathcal{X}_{\underline{\omega_u}} \to \mathcal{X}_{\underline{\omega_{u+1}}}$ and $\mathcal{X}_{\underline{\omega_u}} \to \widetilde{\mathcal{X}}$, which lead to a functor

$$\varinjlim_u \mathcal{X}_{\underline{\omega_u}} \longrightarrow \widetilde{\mathcal{X}}.$$

Proposition 3.9. *The functor* $\varinjlim_u \mathcal{X}_{\underline{\omega_u}} \longrightarrow \widetilde{\mathcal{X}}$ *is an equivalence.*

Proof. By definition, every object and every morphism of $\widetilde{\mathcal{X}}$ come from ones of $\mathcal{X}_{\underline{\omega_u}}$ for some u. Namely the above functor is essentially surjective and full. To see the faithfulness, we take objects (P, η, μ), (P', η', μ') of $\mathcal{X}_{\underline{\omega_u}}(T)$ and their images $(P, \eta_\infty, \mu_\infty)$, $(P, \eta'_\infty, \mu'_\infty)$ in $\widetilde{\mathcal{X}}(T)$. The map

$$\mathrm{Hom}_{\mathcal{X}_{\underline{\omega_u}}(T)}((P, \eta, \mu), (P', \eta', \mu')) \to \mathrm{Hom}_{\widetilde{\mathcal{X}}(T)}((P, \eta_\infty, \mu_\infty), (P', \eta'_\infty, \mu'_\infty))$$

is compatible to projections to the set $\mathrm{Iso}_T^G(P, P')$ of G-equivariant isomorphisms over T. The fibers over $\alpha \in \mathrm{Iso}_T^G(P, P')$ are respectively identified with subsets of $\mathrm{Hom}_{\mathcal{Y}_u(P)}(\eta, \eta' \circ \alpha)$ and of $\mathrm{Hom}_{\mathcal{Y}(P)}(\eta_\infty, \eta'_\infty \circ \alpha)$. Since $\mathcal{Y}(P)$ is the limit of the categories $\mathcal{Y}_u(P)$ by 3.4 we get the faithfulness. □

Definition 3.10. For $\underline{\omega} = (\omega, \omega', \theta, \theta') \in \mathcal{R}(u, v, w)$, we define $\mathcal{Z}_{\underline{\omega}}$ as the stack of pairs (η, μ) where $\eta\colon T \times G \longrightarrow \mathcal{Y}_u$ is a morphism and μ is a natural isomorphism making the diagram

$$\begin{array}{ccc} T \times G \times G & \xrightarrow{\mathrm{id}\times m_G} & T \times G \\ {\scriptstyle \eta\times\mathrm{id}}\downarrow & \swarrow_{\mu} & \downarrow{\scriptstyle \delta_{u,v}(\eta)} \\ \mathcal{Y}_u \times G & \xrightarrow[\omega]{} & \mathcal{Y}_v \end{array} \tag{3.11}$$

2-commutative and such that

$$\begin{array}{ccccc} T \times G & \xrightarrow{\mathrm{id}\times h} & T \times G & \xrightarrow{\mathrm{id}\times g} & T \times G \\ & {\scriptstyle \mu_h}\swarrow\ {\scriptstyle \delta_{u,v}(\eta)} & \downarrow & {\scriptstyle \delta_{v,w}(\mu_g)}\swarrow & \\ {\scriptstyle \eta}\downarrow & & \mathcal{Y}_v & & \downarrow{\scriptstyle \delta_{u,w}(\eta)} \\ & {\scriptstyle \omega_h}\nearrow & \Downarrow{\scriptstyle \lambda_{h,g}} & \searrow{\scriptstyle \omega'_g} & \\ \mathcal{Y}_u & & \xrightarrow[\delta_{v,w}(\omega_{hg})]{} & & \mathcal{Y}_w \end{array} \tag{3.12}$$

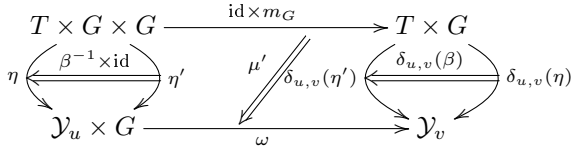
induces $\delta_{v,w}(\tau_{hg})$.

A morphism $(\eta, \mu) \to (\eta', \mu')$ in $\mathcal{Z}_{\underline{\omega}}$ is a natural isomorphism $\beta\colon \eta \to \eta'$,

$$\begin{array}{ccc} & T \times G & \\ {\scriptstyle \eta}\swarrow & \Downarrow{\scriptstyle \beta} & \searrow{\scriptstyle \eta'} \\ & \mathcal{Y}_u & \end{array}$$

such that the diagram

(3.13)



induces the natural isomorphism μ.

Lemma 3.11. *There exists a natural equivalence $\mathcal{Z}_{\underline{\omega}} \simeq \mathcal{X}_{\underline{\omega}} \times_{\mathcal{X}} \mathcal{Y}$. Here the morphism $\mathcal{X}_{\underline{\omega}} \longrightarrow \mathcal{X}$ implicit in the fiber product is the composite of the morphism $\mathcal{X}_{\underline{\omega}} \longrightarrow \widetilde{\mathcal{X}}$ and a quasi-inverse $\widetilde{\mathcal{X}} \xrightarrow{\sim} \mathcal{X}$ of the equivalence in 3.7.*

Proof. Set $\widetilde{\mathcal{Z}}_{\underline{\omega}} = \mathcal{X}_{\underline{\omega}} \times_{\mathcal{X}} \mathcal{Y}$. We may identify an object of $\widetilde{\mathcal{Z}}_{\underline{\omega}}(T)$ with a tuple (P, η, μ, s) such that (P, η, μ) is an object of $\mathcal{X}_{\underline{\omega}}(T)$ and s is a section of $P \to T$. A morphism $(P, \eta, \mu, s) \to (P', \eta', \mu', s')$ is identified with a morphism $(\alpha, \beta)\colon (P, \eta, \mu) \to (P', \eta', \mu')$ in $\mathcal{X}_{\underline{\omega}}$ satisfying $\alpha \circ s = s'$. The section s induces a G-equivariant isomorphism $T \times G \longrightarrow P$. Identifying P with $T \times G$ through this isomorphism, we see that $\widetilde{\mathcal{Z}}_{\underline{\omega}}$ is equivalent to $\mathcal{Z}_{\underline{\omega}}$. □

Notice that if $\mathcal{U} \longrightarrow \mathcal{V}$ is a G-torsor then $\mathcal{V}$ has affine diagonal (resp. is separated) if and only if $\mathcal{U}$ has the same property. The "only if" part is clear. The "if" part follows because $\mathrm{B}\,G$ is separated, descent and the 2-Cartesian diagrams

$$\begin{array}{ccccc} \mathcal{U} & \longrightarrow & \mathcal{U} \times \mathcal{U} & \longrightarrow & \operatorname{Spec} k \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{V} & \longrightarrow & \mathcal{V} \times_{\mathrm{B}\,G} \mathcal{V} & \longrightarrow & \mathrm{B}\,G \end{array}$$

From this remark and from 3.7, 3.9 and 3.11, the proof of 3.5 reduces to:

Lemma 3.12. *The stacks $\mathcal{Z}_{\underline{\omega_*}}$ form a direct system of DM stacks of finite type over k with affine diagonal, with finite and universally injective transition maps, and with a direct system of étale atlases $Z_* \longrightarrow \mathcal{Z}_{\underline{\omega_*}}$ from affine schemes. Moreover if all $\mathcal{Y}_*$ are separated so are the $\mathcal{Z}_{\underline{\omega_*}}$ and if $Y_n \longrightarrow \mathcal{Y}_n$ is finite and étale then $Z_* \longrightarrow \mathcal{Z}_{\underline{\omega_*}}$ can be chosen to be finite and étale.*

To prove this lemma, we will describe $\mathcal{Z}_{\underline{\omega}}$ by using fiber products of simpler stacks. In what follows, for a stack $\mathcal{K}$ and a finite set I, we denote by $\mathcal{K}^I$ the product $\prod_{i \in I} \mathcal{K}$ and identify its objects over a scheme T with the morphisms $T \times I = \sqcup_i T \to \mathcal{K}$.

Let $\mathcal{W}_{\underline{\omega}}$ be the stack of pairs (η, μ) where $\eta\colon T \times G \longrightarrow \mathcal{Y}_u$ and μ is a natural isomorphism as in (3.11), but not necessarily satisfying the compatibility imposed on objects of $\mathcal{Z}_{\underline{\omega}}$.

Remark 3.13. Let $F, G\colon \mathcal{W}_1 \longrightarrow \mathcal{W}_0$ be two maps of stacks and denote by $\mathcal{W}_2$ the stack of pairs (w, ζ) were $w \in \mathcal{W}_1(T)$ and $\zeta\colon G(w) \longrightarrow F(w)$ is an isomorphism in $\mathcal{W}_0(T)$. Then there is a 2-Cartesian diagram

$$\begin{array}{ccc} \mathcal{W}_2 & \xrightarrow{p} & \mathcal{W}_1 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \Gamma_G} \\ \mathcal{W}_1 & \xrightarrow{\Gamma_F} & \mathcal{W}_1 \times \mathcal{W}_0 \end{array}$$

where Γ_* denotes the graph and p the projection. Notice that the sheaf of isomorphisms of an object of a fiber product can be expressed as fiber products of the sheaves of isomorphisms of its factors. In particular, if $\mathcal{W}_0, \mathcal{W}_1$ have affine

diagonals, then $\mathcal{W}_2$ has affine diagonal. If $\mathcal{W}_1$ has affine diagonal and F is affine then p is also affine. This is because the graph Γ_F can be factors as the diagonal $\mathcal{W}_1 \longrightarrow \mathcal{W}_1 \times \mathcal{W}_1$ followed by $\mathrm{id} \times F \colon \mathcal{W}_1 \times \mathcal{W}_1 \longrightarrow \mathcal{W}_1 \times \mathcal{W}_0$.

This remark particularly gives:

Lemma 3.14. *Let $\Phi \colon \mathcal{Y}_u^G \to \mathcal{Y}_v^{G\times G}$ be the morphism sending $\eta \colon T \times G \to \mathcal{Y}_u$ to the composition*

$$\Phi(\eta) \colon T \times G \times G \xrightarrow{\mathrm{id}\times m_G} T \times G \xrightarrow{\eta} \mathcal{Y}_u \xrightarrow{\delta_{u,v}} \mathcal{Y}_v$$

and let $\Psi \colon \mathcal{Y}_u^G \to \mathcal{Y}_v^{G\times G}$ be the morphism sending $\eta \colon T\times G \to \mathcal{Y}_u$ to the composition

$$\Psi(\eta) \colon T \times G \times G \xrightarrow{\eta\times\mathrm{id}} \mathcal{Y}_u \times G \xrightarrow{\omega} \mathcal{Y}_v.$$

Let $\Gamma_\Phi, \Gamma_\Psi \colon \mathcal{Y}_u^G \to \mathcal{Y}_u^G \times \mathcal{Y}_v^{G\times G}$ be their respective graph morphisms. Then

$$\mathcal{W}_{\underline{\omega}} \simeq \mathcal{Y}_u^G \times_{\Gamma_\Phi, \mathcal{Y}_u^G \times \mathcal{Y}_v^{G\times G}, \Gamma_\Psi} \mathcal{Y}_u^G.$$

Let $(\eta, \mu) \in \mathcal{W}_{\underline{\omega}}(T)$. In the two diagrams,

(3.14)
$$\begin{array}{ccccc}
T\times G\times G\times G & \xrightarrow{\mathrm{id}_T\times m_G\times \mathrm{id}_G} & T\times G\times G & \xrightarrow{\mathrm{id}_T\times m_G} & T\times G \\
\downarrow{\scriptstyle \eta} & \overset{\mu\times\mathrm{id}}{\swarrow} \quad \overset{\delta_{u,v}(\eta)}{\downarrow} & & \overset{\delta_{v,w}(\mu)}{\swarrow} \quad \downarrow{\scriptstyle \delta_{u,w}(\eta)} & \\
\mathcal{Y}_u\times G\times G & \xrightarrow[\omega\times\mathrm{id}_G]{} & \mathcal{Y}_v\times G & \xrightarrow[\omega']{} & \mathcal{Y}_w
\end{array}$$

and

(3.15)
$$\begin{array}{ccccc}
T\times G\times G\times G & \xrightarrow{\mathrm{id}_{T\times G}\times m_G} & T\times G\times G & \xrightarrow{\mathrm{id}_T\times m_G} & T\times G \\
\downarrow{\scriptstyle \eta} & \overset{\text{canonical}}{\swarrow} \quad \overset{\delta_{u,v}(\eta)}{\downarrow} & & \overset{\delta_{v,w}(\mu)}{\swarrow} \quad \downarrow{\scriptstyle \delta_{u,w}(\eta)} & \\
\mathcal{Y}_u\times G\times G & \xrightarrow[\delta_{u,v}\times m_G]{} & \mathcal{Y}_v\times G & \xrightarrow[\omega']{} & \mathcal{Y}_w
\end{array}$$

the paths from $T\times G\times G\times G$ to $\mathcal{Y}_w$ through the upper right corner are identical; we denote this morphism $T \times G \times G \times G \to \mathcal{Y}_w$ by $r(\eta)$. As for the paths through the left bottom corner, there is a natural isomorphism between them given by ζ (3.6) and $\lambda_{h,g}$ (3.7). We identify the two lower paths through this natural isomorphism and denote it by $s(\eta)$. We denote the natural isomorphism $r(\eta) \to s(\eta)$ induced from the former diagram by $\alpha(\mu)$ and the one from the latter diagram by $\beta(\eta)$. The compatibility (3.12) is nothing but $\alpha(\mu) = \beta(\mu)$.

For a stack $\mathcal{K}$, we denote by $I(\mathcal{K})$ its inertia stack. An object of $I(\mathcal{K})$ is a pair (x, α) where x is an object of $\mathcal{K}$ and α is an automorphism of x. There is an equivalence $I(\mathcal{K}) \simeq \mathcal{K} \times_{\Delta,\mathcal{K}\times\mathcal{K},\Delta} \mathcal{K}$. We have the forgetting morphism $I(\mathcal{K}) \to \mathcal{K}$, which has the section $\mathcal{K} \to I(\mathcal{K})$, $x \mapsto (x, \mathrm{id})$. If $\mathcal{K}$ is a DM stack of finite type with finite diagonal, then $I(\mathcal{K}) \to \mathcal{K}$ is finite and unramified and $\mathcal{K} \to I(\mathcal{K})$ is a closed immersion.

Lemma 3.15. *Let $\Theta, \Lambda \colon \mathcal{W}_{\underline{\omega}} \to I(\mathcal{Y}_w^{G\times G\times G})$ be the functors sending an object (η, μ) of $\mathcal{W}_{\underline{\omega}}$ to $(r(\eta), \beta(\mu)^{-1} \circ \alpha(\mu))$ and to $(r(\eta), \mathrm{id})$ respectively. Consider also the functor $\mathcal{Z}_{\underline{\omega}} \to \mathcal{Y}_w^{G\times G\times G}$ sending (η, μ) to $r(\eta)$. Then*

$$\mathcal{Z}_{\underline{\omega}} \times_{\mathcal{Y}_w^{G\times G\times G}} I(\mathcal{Y}_w^{G\times G\times G}) \simeq \mathcal{W}_{\underline{\omega}} \times_{\Gamma_\Theta, \mathcal{W}_{\underline{\omega}} \times I(\mathcal{Y}_w^{G\times G\times G}), \Gamma_\Lambda} \mathcal{W}_{\underline{\omega}}.$$

Proof. From 3.13, the right hand side is regarded as the stack of pairs $((\eta, \mu), \epsilon)$ such that (η, μ) is an object of $\mathcal{W}_{\underline{\omega}}$ and ϵ is a natural isomorphism $\Theta((\eta, \mu)) \to \Lambda((\eta, \mu))$.

Thus ϵ is an isomorphism $r(\eta) \to r(\eta)$ making the diagram

$$\begin{array}{ccc} r(\eta) & \xrightarrow{\epsilon} & r(\eta) \\ {\scriptstyle \beta(\mu)^{-1}\circ\alpha(\mu)}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ r(\eta) & \xrightarrow[\epsilon]{} & r(\eta) \end{array}$$

commutative. Therefore $\beta(\mu)^{-1} \circ \alpha(\mu) = \mathrm{id}$, equivalently, the compatibility (3.12) holds, and ϵ can be an arbitrary automorphism of $r(\eta)$. This shows the equivalence of the lemma. □

Lemma 3.16. *The stack $\mathcal{Z}_{\underline{\omega}}$ is a DM stack of finite type with affine diagonal and it is separated if all the $\mathcal{Y}_*$ are separated. Moreover, for functions $v, w\colon \mathbb{N} \to \mathbb{N}$ and $\underline{\omega_u} \in \mathcal{R}(u, v(u), w(u))$, $u \in \mathbb{N}$, the morphism $\mathcal{Z}_{\underline{\omega_u}} \to \mathcal{Z}_{\underline{\omega_{u+1}}}$ is finite and universally injective.*

Proof. If $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ are DM stacks of finite type with affine (resp. finite) diagonals, then so is $\mathcal{U} \times_{\mathcal{W}} \mathcal{V}$. Indeed, $\mathcal{U} \times \mathcal{V}$ is a DM stack of finite type with affine (resp. finite) diagonal and $\mathcal{U} \times_{\mathcal{W}} \mathcal{V} \to \mathcal{U} \times \mathcal{V}$ is an affine (resp. finite) morphism since it is a base change of the diagonal $\mathcal{W} \to \mathcal{W} \times \mathcal{W}$. Hence $\mathcal{U} \times_{\mathcal{W}} \mathcal{V}$ is a DM stack of finite type with affine (resp. finite) diagonal.

From 3.14 and 3.15, $\mathcal{Z}_{\underline{\omega}} \times_{\mathcal{Y}_w^{G\times G\times G}} I(\mathcal{Y}_w^{G\times G\times G})$ is a DM stack of finite type with affine (resp. finite, provided that all $\mathcal{Y}_*$ are separated) diagonal. Since the section $\mathcal{Y}_w^{G\times G\times G} \to I(\mathcal{Y}_w^{G\times G\times G})$ is a closed immersion, the same conclusion holds for $\mathcal{Z}_{\underline{\omega}}$.

From 2.11, we can conclude that the morphism $I(\mathcal{Y}_{w(u)}^{G\times G\times G}) \to I(\mathcal{Y}_{w(u+1)}^{G\times G\times G})$ is finite and universally injective. From 2.11, 3.14 and 3.15,

$$\mathcal{Z}_{\underline{\omega_u}} \times_{\mathcal{Y}_{w(u)}^{G\times G\times G}} I(\mathcal{Y}_{w(u)}^{G\times G\times G}) \to \mathcal{Z}_{\underline{\omega_{u+1}}} \times_{\mathcal{Y}_{w(u+1)}^{G\times G\times G}} I(\mathcal{Y}_{w(u+1)}^{G\times G\times G})$$

is finite and universally injective, and so is the composition

$$\mathcal{Z}_{\underline{\omega_u}} \to \mathcal{Z}_{\underline{\omega_u}} \times_{\mathcal{Y}_{w(u)}^{G\times G\times G}} I(\mathcal{Y}_{w(u)}^{G\times G\times G}) \to \mathcal{Z}_{\underline{\omega_{u+1}}} \times_{\mathcal{Y}_{w(u+1)}^{G\times G\times G}} I(\mathcal{Y}_{w(u+1)}^{G\times G\times G}).$$

The morphism $\mathcal{Z}_{\underline{\omega_u}} \to \mathcal{Z}_{\underline{\omega_{u+1}}}$ factorizes this and hence is finite and universally injective. □

If $\mathcal{U} \to \mathcal{V}$ is a map of stacks and $\mathcal{V}$ has affine diagonal then $\mathcal{U} \times_{\mathcal{V}} \mathcal{U} \to \mathcal{U} \times \mathcal{U}$ is affine, because it is the base change of $\Delta\colon \mathcal{V} \to \mathcal{V} \times \mathcal{V}$ along $\mathcal{U} \times \mathcal{U} \to \mathcal{V} \times \mathcal{V}$. Therefore the morphism

$$\mathcal{Z}_{\underline{\omega_u}} \to \mathcal{W}_{\underline{\omega_u}} \times \mathcal{W}_{\underline{\omega_u}} \to (\mathcal{Y}_u^G \times \mathcal{Y}_u^G) \times (\mathcal{Y}_u^G \times \mathcal{Y}_u^G)$$

induced from equivalences in 3.14 and 3.15 is affine. Pulling back a direct system of étale atlases for $(\mathcal{Y}_u^G \times \mathcal{Y}_u^G) \times (\mathcal{Y}_u^G \times \mathcal{Y}_u^G)$ to $\mathcal{Z}_{\underline{\omega_u}}$, we obtain a system of atlases as in 3.12, which completes the proof of 3.12 and the one of 3.5.

4. The stack of formal G-torsors

We fix a field k and an étale group scheme G over k. In this section we will introduce and study the stack of formal G-torsors.

Definition 4.1. We denote by Δ_G the category fibered in groupoids over Aff/k whose objects over B are $\Delta_G(B) = \mathrm{B}\, G(B((t)))$.

Remark 4.2. By construction we have that if k'/k is a field extension then $\Delta_G \times_k k' \simeq \Delta_{G\times_k k'}$.

Corollary 4.3. *The fiber category Δ_G is a pre-stack in the fpqc topology.*

Proof. Let $D_1, D_2 \in \Delta_G(B) = \mathrm{B}\, G(B((t)))$. We must show that

$$I = \underline{\mathrm{Iso}}_{\Delta_G}(D_1, D_2)\colon \mathrm{Aff}/B \longrightarrow (\mathrm{Sets})$$

is an fpqc sheaf. By 2.6, $\underline{\mathrm{Hom}}_{B((t))}(D_1, D_2)$ is a sheaf, so that in particular I is separated. Thus we must show that if $B \longrightarrow C$ is an fpqc covering, $\phi\colon D_1 \longrightarrow D_2$ is a map and $\phi \otimes C((t))$ is a G-equivariant morphisms of $C((t))$-algebras, then ϕ is also G-equivariant. But this is obvious since D_i is a subset of $D_i \otimes C((t))$. □

Definition 4.4. If $\mathcal{X}$ is a category fibered in groupoids over $\mathbb{F}_p$ then its Frobenius $F_{\mathcal{X}}\colon \mathcal{X} \longrightarrow \mathcal{X}$ is the functor mapping $\xi \in \mathcal{X}(B)$ to $F_B^*(\xi) \in \mathcal{X}(B)$, where $F_B\colon B \longrightarrow B$ is the absolute Frobenius. The Frobenius is $\mathbb{F}_p$-linear, natural in $\mathcal{X}$ and coincides with the usual Frobenius if $\mathcal{X}$ is a scheme.

A category fibered in groupoid $\mathcal{X}$ over $\mathbb{F}_p$ is called perfect if the Frobenius $F_{\mathcal{X}}\colon \mathcal{X} \longrightarrow \mathcal{X}$ is an equivalence.

Example 4.5. As a consequence of 2.13 DM stacks étale over a perfect field are perfect.

We have the following basic property of Δ_G, although we will not use it later.

Proposition 4.6. *If k is perfect the fiber category Δ_G is perfect.*

Proof. By 2.13 the functor $\mathrm{B}\, G(B((t))) \longrightarrow \mathrm{B}\, G(B((t)))$ induced by the Frobenius $F_B\colon B \longrightarrow B$ is an equivalence: the p-th powers of elements in $B((t))$ are in the image of $F_B\colon B((t)) \longrightarrow B((t))$ and therefore the spectrum of this map is integral and a universal homeomorphism. □

Another example of a perfect object that will be used later is the following:

Definition 4.7. If X is a functor $\mathrm{Aff}/\mathbb{F}_p \longrightarrow (\mathrm{Sets})$ we denote by X^∞ the direct limit of the direct system of Frobenius morphisms $X \xrightarrow{F} X \xrightarrow{F} \cdots$.

Notice that if X is a k-pre-sheaf then X^∞ does not necessarily have a k-structure unless k is perfect.

Proposition 4.8. *Let H be a central subgroup of G. Then the equivalence $\mathrm{B}\, G \times \mathrm{B}\, H \longrightarrow \mathrm{B}\, G \times_{\mathrm{B}(G/H)} \mathrm{B}\, G$ of 2.12 induces an equivalence $\Delta_G \times \Delta_H \longrightarrow \Delta_G \times_{\Delta_{G/H}} \Delta_G$. If $\mathcal{X}$ is a fibered category with a map $\mathcal{X} \longrightarrow \Delta_G$ then we have a 2-Cartesian diagram*

$$\begin{array}{ccc} \mathcal{X} \times \Delta_H & \xrightarrow{\alpha} & \Delta_G \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow \\ \mathcal{X} & \longrightarrow & \Delta_{G/H} \end{array}$$

where α is given by $\mathcal{X} \times \Delta_H \longrightarrow \Delta_G \times \Delta_H = \Delta_{G\times H} \longrightarrow \Delta_G$ and the last map is induced by the multiplication $G \times H \longrightarrow G$.

Proof. The first claim is clear. For the other we have the following Cartesian diagrams

$$\begin{array}{ccccc} \mathcal{X} \times \Delta_H & \longrightarrow & \Delta_G \times \Delta_H & \longrightarrow & \Delta_G \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow \\ \mathcal{X} & \longrightarrow & \Delta_G & \longrightarrow & \Delta_{G/H} \end{array}$$

□

4.1. **The group $G = \mathbb{Z}/p\mathbb{Z}$ in characteristic p.** In this section we consider $G = \mathbb{Z}/p\mathbb{Z}$ over $k = \mathbb{F}_p$.

Let C be an $\mathbb{F}_p$-algebra. By Artin-Schreier a $\mathbb{Z}/p\mathbb{Z}$-torsor over C is of the form $C[X]/(X^p - X - c)$, where $c \in C$ and the action is induced by $X \longmapsto X + f$ for $f \in \mathbb{F}_p$.

Lemma 4.9. *Let $c, d \in C$. Then*

$$\begin{aligned} \{u \in C \mid u^p - u + c = d\} &\longrightarrow \mathrm{Iso}_C^{\mathbb{Z}/p\mathbb{Z}}\left(\tfrac{C[X]}{(X^p - X - c)}, \tfrac{C[X]}{(X^p - X - d)}\right) \\ u &\longmapsto (X \longmapsto X - u) \end{aligned}$$

is bijective.

Proof. The map in the statement is well defined and it induces a morphism

$$\mathrm{Spec}(C[X]/(X^p - X - (d-c))) \longrightarrow \underline{\mathrm{Iso}}^{\mathbb{Z}/p\mathbb{Z}}(C[X]/(X^p - X - c), C[X]/(X^p - X - d)) = I$$

The group $\mathbb{Z}/p\mathbb{Z}$ acts on both sides and the map is equivariant. Since both sides are $\mathbb{Z}/p\mathbb{Z}$-torsors it follows that the above map is an isomorphism. □

Notation 4.10. If C is an $\mathbb{F}_p$-algebra, according to 4.9, we identify $(\mathrm{B}\,\mathbb{Z}/p\mathbb{Z})(C)$ with the category whose objects are elements of C and a morphism $c \xrightarrow{u} d$ is an element $u \in C$ such that $u^p - u + c = d$. Composition is given by the sum, identities correspond to $0 \in C$ and the inverse of $u \in C$ is $-u$.

In particular we see that if $c \in (\mathrm{B}\,\mathbb{Z}/p\mathbb{Z})(C)$ then $c \simeq c^p$.

Lemma 4.11. *Any element $b \in tB[[t]]$ is of the form $u^p - u$ for a unique element $u \in tB[[t]]$.*

Proof. Let $b, u \in tB[[t]]$ and b_s, u_s for $s \in \mathbb{N}$ their coefficients, so that $b_0 = u_0 = 0$. We extend the symbol b_s, u_s for $s \in \mathbb{Q}$ by setting $b_s = u_s = 0$ if $s \notin \mathbb{N}$. The equation $u^p - u = b$ translates in $b_s = u^p_{s/p} - u_s$ for all $s \in \mathbb{N}$. A simple computation shows that, given b, the only solution of the system is

$$u_s = -\sum_{n \in \mathbb{N}} b^{p^n}_{s/p^n}$$

□

Notation 4.12. In what follows we set

$$S = \{n \geq 1 \mid p \nmid n\}$$

and $\mathbb{A}^{(S)}\colon \mathrm{Aff}/\mathbb{F}_p \longrightarrow (\mathrm{Sets})$ where $\mathbb{A}^{(S)}(B)$ is the set of maps $b\colon S \longrightarrow B$ such that $\{s \in S \mid b_s \neq 0\}$ is finite.

Given $k \in \mathbb{N}$ we set

$$\phi_k\colon \mathbb{A}^{(S)} \longrightarrow \Delta_{\mathbb{Z}/p\mathbb{Z}},\ \phi_k(b) = \sum_{s \in S} b_s t^{-sp^k} \in B((t)) = \Delta_{\mathbb{Z}/p\mathbb{Z}}(B)$$

and $\psi_k\colon \mathbb{A}^{(S)} \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z}) \longrightarrow \Delta_{\mathbb{Z}/p\mathbb{Z}}$, $\psi_k(b, b_0) = \phi_k(b) + b_0$. Let $F_{\mathbb{A}^{(S)}}$ be the Frobenius morphism of $\mathbb{A}^{(S)}$ defined in 4.4 and let $(\mathbb{A}^{(S)})^\infty$ be the limit defined as in 4.7. For all $b \in \mathbb{A}^{(S)}(B)$ and $b_0 \in B$ there is a natural map

$$\psi_{k+1} \circ (F_{\mathbb{A}^{(S)}} \times \mathrm{id}_{\mathrm{B}(\mathbb{Z}/p\mathbb{Z})})(b, b_0) \xrightarrow{-\phi_k(b)} \psi_k(b, b_0)$$

which therefore induces a functor $(\mathbb{A}^{(S)})^\infty \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z}) \longrightarrow \Delta_{\mathbb{Z}/p\mathbb{Z}}$.

Theorem 4.13. *The functor $(\mathbb{A}^{(S)})^\infty \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z}) \longrightarrow \Delta_{\mathbb{Z}/p\mathbb{Z}}$ is an equivalence of fibered categories.*

Proof. Essential surjectivity. Let $b(t) = \sum_j b_j t^j \in \Delta_{\mathbb{Z}/p\mathbb{Z}}(B)$. By 4.11 and the definition of the map in the statement we can assume that $b_j = 0$ for $j > 0$. Let $k \in \mathbb{N}$ be a sufficiently large positive integer such that every $j < 0$ with $b_j \neq 0$ is written as $-j = p^{k-m(j)}s(j)$ for some $m(j) \geq 0$ and $s(j) \in S$. Then $b_j t^j \simeq (b_j t^j)^{p^{m(j)}} = b_j^{p^{m(j)}} t^{-p^k s(j)}$ if $b_j \neq 0$. We see therefore that, up to change b with an isomorphic element, b can be written as $\psi_k(c)$ for some $c \in (\mathbb{A}^{(S)} \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z}))(B)$.

Faithfulness. Let $([b,k],b_0),([c,k],c_0) \in (\mathbb{A}^{(S)})^\infty(B)\times\mathrm{B}(\mathbb{Z}/p\mathbb{Z})(B)$ and $u,v\colon (b,b_0) \longrightarrow (c,c_0)$ two morphisms in $\mathbb{A}^{(S)} \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z})$, that is $b = c$ and $u^p - u = v^p - v = c_0 - b_0$ with $u, v \in B$. If $\psi_k(u) = \psi_k(v)$ then $u = v$ by definition of ψ_k as desired.

Fullness. Let $([b,k],b_0),([c,k'],c_0) \in (\mathbb{A}^{(S)})^\infty(B)\times\mathrm{B}(\mathbb{Z}/p\mathbb{Z})(B)$ and let $u\colon \psi_k(b,b_0) \longrightarrow \psi_{k'}(c,c_0)$ be a map in $\Delta_{\mathbb{Z}/p\mathbb{Z}}$. We want to lift this morphism to $(\mathbb{A}^{(S)})^\infty \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z})$. We can assume $k = k'$. The element $u = \sum_q u_q t^q \in B((t))$ can be written as $u = u_- + u_+$, where $u_- = \sum_{q<0} u_q t^q$ and $u_+ = \sum_{q\geq 0} u_q t^q$. In particular we obtain that $u_-^p - u_- = \phi_k(c) - \phi_k(b)$ and $u_+^p - u_+ = c_0 - b_0$. By 4.11 it follows that $u_+ \in B$. It suffices to show that $u_- = 0$. To see this, we first show that $c - b$ is nilpotent. We have $\phi_k(c) \simeq \phi_k(b)$ and, applying F_B to both side we get $\phi_0(b) \simeq \phi_0(b)^{p^k} = F_B^k(\phi_k(b)) \simeq F_B^k(\phi_k(c)) = \phi_0(c)^{p^k} \simeq \phi_0(c)$. Thus there exists $v = \sum_{q<0} v_q t^q \in B((t))$ such that

$$\phi_0(c) - \phi_0(b) = \sum_{s\in S}(c_s - b_s)t^{-s} = v^p - v = \sum_{q<0}(v_{q/p}^p - v_q)t^q$$

where we set $v_l = 0$ if $l \in \mathbb{Q} - \mathbb{Z}_{<0}$. In particular for $s \in S$ and $l \in \mathbb{N}$ we obtain $b_s - c_s = v_{-s}$ and $v_{-sp^l} = v_{-s}^{p^l}$. Since $v_{-sp^l} = 0$ for $l \gg 0$ we see that $b_s - c_s$ is nilpotent. This means that there exists $j > 0$ such that $F^j_{\mathbb{A}^{(S)}}(b) = F^j_{\mathbb{A}^{(S)}}(c)$. Thus, up to replace k by $k+j$, we can assume $b = c$, so that $u_-^p = u_-$. If $u_- = \sum_{q<0} u_q t^q$ and we put $u_q = 0$ for $q \notin \mathbb{Z}$ then we have $u_q = (u_{q/p^l})^{p^l}$ for every $q \in \mathbb{Q}$ with $q < 0$ and $l \in \mathbb{N}$. For each q, taking a sufficiently large l with $q/p^l \notin \mathbb{Z}$, we see $u_q = 0$ as desired. □

Remark 4.14. The addition $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$ induces maps $\mathrm{B}(\mathbb{Z}/p\mathbb{Z}) \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{B}(\mathbb{Z}/p\mathbb{Z})$ and $\Delta_{\mathbb{Z}/p\mathbb{Z}} \times \Delta_{\mathbb{Z}/p\mathbb{Z}} \longrightarrow \Delta_{\mathbb{Z}/p\mathbb{Z}}$. The ind-scheme $(\mathbb{A}^{(S)})^\infty$ also has a natural group structure by addition. Notice that the functor in the last theorem preserves the induced "group structure" on both sides. This is because the maps ψ_k preserve the sum and the Frobenius of $\mathbb{A}^{(S)}$ is a group homomorphism. In particular the induced map from $(\mathbb{A}^{(S)})^\infty$ to the coarse ind-algebraic space of $\Delta_{\mathbb{Z}/p\mathbb{Z}}$ is an isomorphism of sheaves of groups.

Remark 4.15. If B is an $\mathbb{F}_p$-algebra, G is any constant p-group and H is a central subgroup consisting of elements of order at most p then any map $\operatorname{Spec} B \longrightarrow \Delta_{G/H}$ lifts to a map $\operatorname{Spec} B \longrightarrow \Delta_G$. More generally any G/H-torsor over B extends to a G-torsor. This follows from the fact that there is an exact sequence of sets $\mathrm{H}^1(B,G) \longrightarrow \mathrm{H}^1(B,G/H) \longrightarrow \mathrm{H}^2(B,H) = 0$. The last vanishing follows because $H \simeq (\mathbb{Z}/p\mathbb{Z})^r$ for some r and using the Artin-Schreier sequence.

Corollary 4.16. *If G is an étale p-group scheme over a field k then Δ_G is a stack in the fpqc topology.*

Proof. If B is a k-algebra and A/k is a finite k-algebra then $(B \otimes_k A)((t)) \simeq B((t)) \otimes_k A$ by 2.4. Therefore Δ_G satisfies descent along coverings of the form $B \longrightarrow B \otimes_k A$. This implies that it is enough to show that $\Delta_G \times_k L \simeq \Delta_{G\times_k L}$ is a stack, where L/k is a finite field extension such that $G \times_k L$ is constant. Again using base change, we can assume $k = \mathbb{F}_p$ and G a constant p-group. If $\sharp G = p^l$ we proceed by induction on l. If $l = 1$ then $\Delta_{\mathbb{Z}/p\mathbb{Z}} \simeq (\mathbb{A}^{(S)})^\infty \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z})$ which is a product of

stacks. For a general G let H a non-trivial central subgroup. By induction $\Delta_{G/H}$ is a stack and it is enough to show that all base change of $\Delta_G \longrightarrow \Delta_{G/H}$ along a map $\operatorname{Spec} B \longrightarrow \Delta_{G/H}$ is a stack. This fiber product is $\operatorname{Spec} B \times \Delta_H$ thanks to 4.8 and 4.15, which is a stack by inductive hypothesis. □

4.2. **Tame cyclic case.** Let k be a field and $n \in \mathbb{N}$ such that $n \in k^*$. The aim of this section is to prove Theorem B.

Set $G = \mu_n$, the group of n-th roots of unity, which is a finite and étale group scheme over k. In particular $\Delta_G(B)$ can be seen as the category of pairs $(\mathcal{L}, \sigma)$ where $\mathcal{L}$ is an invertible sheaf over $B((t))$ and $\sigma\colon \mathcal{L}^{\otimes n} \longrightarrow B((t))$ is an isomorphism. When $\mathcal{L} = B((t))$ the isomorphism σ will often be thought of as an element $\sigma \in B((t))^*$.

Lemma 4.17. *An invertible sheaf $\mathcal{L}$ over $B((t))$ with $\mathcal{L}^{\otimes n} \simeq B((t))$ is the pullback of an invertible sheaf over $\operatorname{Spec} B$. More precisely , the n-torsion part of $\operatorname{Pic}(B((t)))$ is naturally isomorphic to the one of $\operatorname{Pic}(B)$.*

Proof. Gabber's formula [BČ19, (1.2.2)] says

$$\operatorname{Pic}(B((t))) \simeq \operatorname{Pic}(B[t^{-1}]) \oplus \mathrm{H}^1_{\text{ét}}(B, \mathbb{Z}).$$

This is proved in a slightly more general form in Theorem 3.1.7 of the cited paper by Bouthier-Česnavičius. Let $N\operatorname{Pic}(B)$ denote the kernel of $\operatorname{Pic}(B[t]) \to \operatorname{Pic}(B)$ so that

$$\operatorname{Pic}(B[t^{-1}]) \simeq \operatorname{Pic}(B[t]) = \operatorname{Pic}(B) \oplus N\operatorname{Pic}(B).$$

According to [Swa80, Th. 6.1], $N\operatorname{Pic}(B)$ has no n-torsion if and only if B_{red} is n-seminormal. The n-seminormality is defined as follows. For a reduced ring A, there exists an extension $A \subset \overline{A}$ called the seminormalization (see [Swa80, Section 4]). For our purpose, we only need to know its existence. We say that A is n-seminormal if every element $x \in \overline{A}$ with $x^2, x^3, nx \in A$ belongs to A. In our situation, since n is invertible in k, every k-algebra is n-seminormal. Thus B_{red} is n-seminormal and $N\operatorname{Pic}(B)$ has no n-torsion.

It remains to show that $\mathrm{H}^1_{\text{ét}}(B, \mathbb{Z})$ has no n-torsion. To do so, we consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

of constant étale sheaves on the small étale site of $\operatorname{Spec} B$. Taking cohomology groups, we get the following exact sequence:

$$\mathrm{H}^0(B, \mathbb{Z}) \longrightarrow \mathrm{H}^0(B, \mathbb{Z}/n\mathbb{Z}) \longrightarrow (\text{the } n\text{-torsion part of } \mathrm{H}^1_{\text{ét}}(B, \mathbb{Z})) \longrightarrow 0$$

The left map is surjective, since every locally constant function $\operatorname{Spec} B \longrightarrow \mathbb{Z}/n\mathbb{Z}$ lifts to a locally constant function $\operatorname{Spec} B \longrightarrow \mathbb{Z}$. It follows that $\mathrm{H}^1_{\text{ét}}(B, \mathbb{Z})$ has no n-torsion. □

Lemma 4.18. *For a k-algebra B, we have*

$$\mu_n(B) = \{b \in B^* \mid b^n = 1\} = \{b \in B((t))^* \mid b^n = 1\} = \mu_n(B((t))).$$

Proof. Let L and R denote the left and right sides respectively. Obviously $L \subset R$. It is also easy to see $R \cap B[[t]] = L$. Thus it suffices to show that $R \subset B[[t]]$. Conversely, we suppose that it was not the case. We define the naive order $\operatorname{ord}_{naive}(a)$ of $a = \sum_{i\in\mathbb{Z}} a_i t^i \in B[[t]]$ as $\min\{i \mid a_i \neq 0\}$. Elements outside $B[[t]]$ have negative naive orders and choose an element $c = \sum_{i\in\mathbb{Z}} c_i t^i \in R \setminus B[[t]]$ such that $\operatorname{ord}_{naive}(c)$ attains the maximum, say $i_0 < 0$. Taking derivatives of $c^n = 1$, we get $ncc' = 0$ with c' the derivative of c. Since nc is invertible, $c' = 0$. If $\operatorname{char} k = 0$ it immediately follows that $c \in B$. So assume $\operatorname{char} k = p > 0$. In this case $c_i = 0$ for all i with $p \nmid i$. This means that c is in the image of the injective B-algebra

homomorphism $f\colon B[[t]] \to B[[t]]$, $t \mapsto t^p$. Let d be the unique preimage of c under f, which is explicitly given by $d = \sum_{i\in\mathbb{Z}} c_{pi} t^i$. In particular,

$$\operatorname{ord}_{naive}(d) = \operatorname{ord}_{naive}(c)/p > \operatorname{ord}_{naive}(c).$$

Since $f(d^n) = c^n = 1$ and f is injective, we have $d^n = 1$ and $d \in R \setminus B[[t]]$. This contradicts the way of choosing c. We have proved the lemma. □

Proof of Theorem B. We first define the functor $\psi\colon \bigsqcup_{q=0}^{n-1} \mathrm{B}(G) \longrightarrow \Delta_G$. An object of $\bigsqcup_{q=0}^{n-1} \mathrm{B}(G)$ over a k-algebra A is a factorization $A = \prod_q A_q$ plus a tuple $(L_q, \xi_q)_q$ where L_q is an A_q-module and $\xi_q\colon L_q^{\otimes n} \longrightarrow A_q$ is an isomorphism. A morphism $(A = \prod_q A_q, L_q, \xi_q) \longrightarrow (A = \prod_q A'_q, L'_q, \xi'_q)$ exists if and only if $A_q \simeq A_{q'}$ as A-algebras (so that such isomorphism is unique) and in this case is a collection of isomorphisms $L_q \longrightarrow L'_q$ compatible with the maps ξ_q and ξ'_q. To such an object we associate the invertible $A((t)) = \prod_q A_q((t))$-module $L = \prod_q (L_q \otimes_{A_q} A_q((t)))$ together with the map

$$L^{\otimes n} \simeq \prod_q (L_q^{\otimes n} \otimes_{A_q} A_q((t))) \longrightarrow \prod_q A_q((t)) = A((t)),\ (x_q \otimes 1)_q \mapsto (\xi_q(x_q) t^q)_q$$

It is easy to see that the functor ψ on $\mathrm{B}\,G$ in the index q is the one in the statement. We are going to show that ψ is an equivalence. Since Δ_G is a prestack by 4.3, it will be enough to show that ψ is an epimorphism and that it is fully faithful. Indeed this would imply that Δ_G is also a stack for the following reason. Given a descent datum for Δ_G, since Δ_G is a prestack, in order to show that it is effective we can refine this datum, that is refine the covering over which is defined. If ψ is fully faithful and an epimorphism, it follows that we can always assume that the descent datum for Δ_G comes from a descent datum for $\bigsqcup_{q=0}^{n-1} \mathrm{B}(G)$, which is therefore effective.

ψ epimorphism. Let $\chi \in \Delta_G(B)$. From 4.17, we can assume that the associated invertible sheaf is trivial and $\chi = (B((t)), b)$. We have $(B((t)), b) \simeq (B((t)), b')$ if and only if there exists $u \in B((t))^*$ such that $u^n b = b'$. For $c \in B((t))^*$ we define $\operatorname{ord} c\colon \operatorname{Spec} B \to \mathbb{Z}$ as follows: if $x \in \operatorname{Spec} B$ is a point with the residue field κ and $c_x \in \kappa((t))$ is the induced power series, then $(\operatorname{ord} c)(x) := \operatorname{ord} c_x$. This function is upper semicontinuous. From the additivity of orders, $\operatorname{ord} b + \operatorname{ord}(b^{-1})$ is constant zero. Since $\operatorname{ord} b$ and $\operatorname{ord}(b^{-1})$ are both upper semicontinuous, they are in fact locally constant. Thus we may suppose that $\operatorname{ord} b$ is constant, equivalently that if b_j are coefficients of b, then for some i, b_i is a unit and b_j are nilpotents for $j < i$.

Thus we can write $b = b_- + t^i b_+$ with $b_t \in B[[t]]^*$ and $b_- \in B((t))$ nilpotent. Set $\omega = b/(t^i b_+) \in B((t))$, $A = B((t))[Y]/(Y^n - \omega)$ and $C = B((t))/(b_-)$. We have that $\omega = 1$ in C and therefore that $A \otimes_{B((t))} C$ has a section. Since $A/B((t))$ is étale and $B((t)) \longrightarrow C$ is surjective with nilpotent kernel the section extends, that is ω is an n-th power. Thus we can assume $b_- = 0$. Since $B[[t]][Y]/(Y^n - b_+)$ is étale over $B[[t]]$, by 2.3 we can assume there exists $\hat{b} \in B[[t]]^*$ such that $\hat{b}^n = b_+$. In conclusion we reduce to the case $b = t^i$ and, multiplying by a power of t^n, we can finally assume $0 \le i < n$.

ψ fully faithful. If $(\mathcal{L}, \sigma) \in \Delta_G(B)$ then, by Lemma 4.18, its automorphisms are canonically isomorphic to $\mu_n(B((t))) = \mu_n(B)$. This easily implies that the restriction of ψ on each component is fully faithful. Given two objects $\alpha, \beta \in \bigsqcup_{q=0}^{n-1} \mathrm{B}(G)$ and an isomorphism $\psi(\alpha) \longrightarrow \psi(\beta)$ of their images the problem of finding an isomorphism $\alpha \longrightarrow \beta$ inducing the given one is local and easily reducible to the following claim: if $(B((t)), t^q) \simeq (B((t)), t^{q'})$ then $q \equiv q' \mod n$. But the first condition means that there exists $u \in L((t))^*$ such that $u^n t^q = t^{q'}$ and, applying ord, we get the result. □

4.3. **General p-groups.** In this section we consider the case of a constant p-group G over a field k of characteristic p and the aim is to prove Theorem A in this case. We setup the following notation for this section. All groups considered in this section are constant.

Definition 4.19. We set

$$S = \{n \geq 1 \mid p \nmid n\}$$

and, given a finite dimensional $\mathbb{F}_p$-vector space H (regarded as an abelian p-group), we define a sheaf of abelian groups

$$X_H = (\mathbb{A}^{(S)})^\infty \otimes_{\mathbb{F}_p} H \colon \mathrm{Aff}/\mathbb{F}_p \longrightarrow (\text{Abelian groups})$$

(that is $X_H = [(\mathbb{A}^{(S)})^\infty]^n$ if $\dim_{\mathbb{F}_p} H = n$ after the choice of a basis of H). We also define sheaves of abelian groups $X_{H,m} = \mathbb{A}^{(S_m)} \otimes_{\mathbb{F}_p} H$ for $m \in \mathbb{N}$ with $S_m = \{n \in S \mid n \leq m\}$. We finally define $\Delta_{H,m} = X_{H,m} \times \mathrm{B}(H)$.

Lemma 4.20. *1) We have an isomorphism $X_H \times \mathrm{B}\, H \longrightarrow \Delta_H$.*

2) We have $X_H = \varinjlim_m X_{H,m}$ as sheaves of abelian groups, where the transition map $X_{H,m} \longrightarrow X_{H,m+1}$ is the composition of the inclusion $\mathbb{A}^{(S_m)} \otimes_{\mathbb{F}_p} H \longrightarrow \mathbb{A}^{(S_{m+1})} \otimes_{\mathbb{F}_p} H$ and the Frobenius of $\mathbb{A}^{(S_{m+1})} \otimes_{\mathbb{F}_p} H$.

3) We have an equivalence $\varinjlim_m (\Delta_{H,m}) \simeq \Delta_H$.

Proof. The last two assertions are obvious. We prove the first one. If H is the cyclic group of order p, then this is just 4.13. Otherwise we take a subgroup $1 \neq I \subsetneq H$. Since the quotient map $H \to H/I$ has a section, the morphism $\Delta_H \to \Delta_{H/I}$ also has a section. From 4.8, we have $\Delta_H \simeq \Delta_{H/I} \times \Delta_I$. The assertion follows from induction on the order of H. □

In the following proposition, we use rigidification, an operation introduced in [AOV08] for algebraic stacks. Roughly speaking, it kills some subgroups of stabilizers. Generalization to non-algebraic stacks will be treated in Appendix B. Note that from 4.16, Δ_G is a stack for a p-group G.

Lemma 4.21. *Let H be a finite dimensional $\mathbb{F}_p$-vector space. We have $\Delta_H /\!\!/ H \simeq X_H$.*

Proof. By 4.20 we have $\Delta_H \simeq X_H \times \mathrm{B}\, H$. Since H is abelian the result follows from B.4. □

Proposition 4.22. *Let G be a p-group and H be a central subgroup which is an $\mathbb{F}_p$-vector space. Then H is naturally a subgroup sheaf of the inertia stack of Δ_G (see Appendix B for the inertia stack as a group sheaf) and the quotient map $\Delta_G \longrightarrow \Delta_{G/H}$ is the composition of the rigidification $\Delta_G \longrightarrow \Delta_G /\!\!/ H$ and an X_H-torsor $\Delta_G /\!\!/ H \longrightarrow \Delta_{G/H}$, where the action of X_H on $\Delta_G /\!\!/ H$ is induced by $\Delta_G \times \Delta_H \longrightarrow \Delta_G$ and rigidification.*

Proof. The subgroup H acts on any G-torsor because its central. Moreover the functor $\Delta_G \longrightarrow \Delta_{G/H}$ sends isomorphisms coming from H to the identity and therefore factors through the rigidification $\Delta_G /\!\!/ H$ by B.3, 2). Rigidifying both sides of $\Delta_G \times \Delta_H \longrightarrow \Delta_G$ we get a map $(\Delta_G /\!\!/ H) \times X_H \longrightarrow \Delta_G /\!\!/ H$ over $\Delta_{G/H}$. Using 4.8 and B.3, 3) we can deduce that $\Delta_G /\!\!/ H \longrightarrow \Delta_{G/H}$ is an X_H-torsor. □

Lemma 4.23. *Let G be a p-group and H be a central subgroup which is an $\mathbb{F}_p$-vector space. Let also $\mathcal{Y}_*$ be a direct system of quasi-separated stacks over $\mathbb{N}$ with a direct system of smooth (étale) atlases U_* made of quasi-compact schemes, $\varinjlim_n \mathcal{Y}_n \longrightarrow \Delta_{G/H}$ a map and $\varinjlim_n U_n \longrightarrow \Delta_G$ a lifting. Then there exists a strictly increasing map $q \colon \mathbb{N} \longrightarrow \mathbb{N}$, a direct system of quasi-separated stacks $\mathcal{Z}_*$ with a direct system of smooth (étale) atlases $U_* \times X_{H,q_*}$ (where the transition morphisms*

$U_i \times X_{H,q_i} \longrightarrow U_{i+1} \times X_{H,q_{i+1}}$ is the product of the given map $U_i \longrightarrow U_{i+1}$ and the map $X_{H,q_i} \longrightarrow X_{H,q_{i+1}}$ of 4.20), compatible maps $\mathcal{Z}_i \longrightarrow \mathcal{Y}_i$ induced by the projection $U_i \times X_{H,q_i} \longrightarrow U_i$ and which are a composition of an H-gerbe $\mathcal{Z}_i \longrightarrow \mathcal{Z}_i /\!\!/ H$ and a X_{H,q_i}-torsor $\mathcal{Z}_i /\!\!/ H \longrightarrow \mathcal{Y}_i$ and an equivalence

$$\varinjlim_n \mathcal{Z}_n \simeq (\varinjlim_n \mathcal{Y}_n) \times_{\Delta_{G/H}} \Delta_G$$

Moreover there is a factorization $U_i \times X_{H,q_i} \longrightarrow U_i \times_{\mathcal{Y}_i} \mathcal{Z}_i \longrightarrow \mathcal{Z}_i$ where the first arrow is an H-torsor.

Proof. Consider one index $i \in \mathbb{N}$ and the Cartesian diagrams

$$\begin{array}{ccccc} P'_{U,i} & \longrightarrow & P'_i & \longrightarrow & \Delta_G \\ \downarrow & & \downarrow & & \downarrow \\ P_{U,i} & \longrightarrow & P_i & \longrightarrow & \Delta_G /\!\!/ H \\ \downarrow & & \downarrow & & \downarrow \\ U_i & \longrightarrow & \mathcal{Y}_i & \longrightarrow & \Delta_{G/H} \end{array}$$

Set also $R_i = U_i \times_{\mathcal{Y}_i} U_i$, which is a quasi-compact algebraic space. By 4.22 $P_i \longrightarrow \mathcal{Y}_i$ is a X_H-torsor, $P'_i \longrightarrow P_i$ an H-gerbe. Moreover the lifting $U_i \longrightarrow \Delta_G$ gives an isomorphism $P'_{U,i} \simeq U_i \times \Delta_H$ and $P_{U,i} \simeq U_i \times X_H$ by 4.8. Thus the X_H-torsor $P_i \longrightarrow \mathcal{Y}_i$, by descent along $U_i \longrightarrow \mathcal{Y}_i$ is completely determined by the identification $R_i \times X_H \simeq_{R_i} R_i \times X_H$, which consists of an element $\omega_i \in X_H(R_i)$ satisfying the cocycle condition on $U_i \times_{\mathcal{Y}_i} U_i \times_{\mathcal{Y}_i} U_i$. The given equivalence $P_i \simeq (P_{i+1})_{|\mathcal{Y}_i}$ of X_H-torsors over $\mathcal{Y}_i$ is completely determined by its pullback on U_i, which is given by multiplication by $\gamma_i \in X_H(U_i)$. The compatibility this element has to satisfy is expressed by

$$(\omega_{i+1})_{|R_i}(s_i^* \gamma_i) = (t_i^* \gamma_i)\omega_i \text{ in } X_H(R_i)$$

where $s_i, t_i \colon R_i \longrightarrow U_i$ are the two projections. Since all R_i are quasi-compact and $X_H \simeq \varinjlim_j X_{H,j}$ we can find an increasing sequence of natural numbers $q \colon \mathbb{N} \longrightarrow \mathbb{N}$ and elements $e_{q_i} \in X_{H,q_i}(R_i)$, $f_{q_i} \in X_{H,q_{i+1}}(U_i)$ such that:

1) the element e_{q_i} is mapped to ω_i under the map $X_{H,q_i}(R_i) \longrightarrow X_H(R_i)$ and it satisfies the cocycle condition in $X_{H,q_i}(U_i \times_{\mathcal{Y}_i} U_i \times_{\mathcal{Y}_i} U_i)$;
2) the element f_{q_i} is mapped to γ_i under the map $X_{H,q_{i+1}}(U_i) \longrightarrow X_H(U_i)$ and, if $\overline{e_{q_i}}$ is the image of e_{q_i} under the map $X_{H,q_i}(R_i) \longrightarrow X_{H,q_{i+1}}(R_i)$, it satisfies

$$(e_{q_{i+1}})_{|R_i}(s_i^* f_{q_i}) = (t_i^* f_{q_i})\overline{e_{q_i}} \text{ in } X_{H,q_{i+1}}(R_i)$$

The data of 1) determine X_{H,q_i}-torsors $Q_i \longrightarrow \mathcal{Y}_i$ with a map $U_i \longrightarrow Q_i$ over $\mathcal{Y}_i$ and together with an X_{H,q_i}-equivariant map $Q_i \longrightarrow P_i$ such that $U_i \longrightarrow Q_i \longrightarrow P_i$ is the given map. The data of 2) determine an X_{H,q_i}-equivariant map $Q_i \longrightarrow (Q_{i+1})_{|\mathcal{Y}_i}$inducing the equivalence $P_i \simeq (P_{i+1})_{|\mathcal{Y}_i}$.

Consider also the H-gerbe $Q'_i \longrightarrow Q_i$ pullback of $P'_i \longrightarrow P_i$ along $Q_i \longrightarrow P_i$. We set $\mathcal{Z}_i = Q'_i$. We have Cartesian diagrams

$$\begin{array}{ccccc} Q'_i & \longrightarrow & P'_i & \longrightarrow & P'_{i+1} \\ \downarrow & & \downarrow & & \downarrow \\ Q_i & \longrightarrow & P_i & \longrightarrow & P_{i+1} \end{array} \qquad \begin{array}{ccccc} Q'_i & \longrightarrow & Q'_{i+1} & \longrightarrow & P'_{i+1} \\ \downarrow & & \downarrow & & \downarrow \\ Q_i & \longrightarrow & Q_{i+1} & \longrightarrow & P_{i+1} \end{array}$$

Notice that, if $\mathcal{M}$ is a stack over $\mathcal{Y}_i$, then $\mathcal{M} \times_{\mathcal{Y}_{i+1}} U_{i+1} \simeq \mathcal{M} \times_{\mathcal{Y}_i} U_i$ because U_* is a direct system of atlases. Pulling back along $U_{i+1} \longrightarrow \mathcal{Y}_{i+1}$ the above diagrams,

we obtain the bottom rows of the following diagrams.

$$\begin{array}{ccccc} X_{H,q_i}\times U_i & \longrightarrow & X_H\times U_i & \longrightarrow & X_H\times U_{i+1}\\ \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_{i+1}}\\ Q'_i\times_{\mathcal{Y}_{i+1}} U_{i+1} & \longrightarrow & \Delta_H\times U_i & \longrightarrow & \Delta_H\times U_{i+1}\\ \downarrow & & \downarrow & & \downarrow\\ X_{H,q_i}\times U_i & \longrightarrow & X_H\times U_i & \longrightarrow & X_H\times U_{i+1} \end{array}$$

$$\begin{array}{ccccc} X_{H,q_i}\times U_i & \longrightarrow & X_{H,q_{i+1}}\times U_{i+1} & \longrightarrow & X_H\times U_{i+1}\\ \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \beta_{i+1}} & & \downarrow{\scriptstyle \alpha_{i+1}}\\ Q'_i\times_{\mathcal{Y}_{i+1}} U_{i+1} & \longrightarrow & Q'_{i+1}\times_{\mathcal{Y}_{i+1}} U_{i+1} & \longrightarrow & \Delta_H\times U_{i+1}\\ \downarrow & & \downarrow & & \downarrow\\ X_{H,q_i}\times U_i & \longrightarrow & X_{H,q_{i+1}}\times U_{i+1} & \longrightarrow & X_H\times U_{i+1} \end{array}$$

The top rows of the above diagrams is instead obtained using 4.8, where the map α_i are induced by the map $X_H \longrightarrow X_H \times \mathrm{B}\, H = \Delta_H$ and the β_i are induced by the α_i. Since $Q'_i \times_{\mathcal{Y}_{i+1}} U_{i+1} \simeq Q'_i \times_{\mathcal{Y}_i} U_i$ we see that the atlases $U_i \times X_{H,q_i} \xrightarrow{\beta_i} Q'_i \times_{\mathcal{Y}_i} U_i \longrightarrow Q'_i = \mathcal{Z}_i$ define a direct system of smooth (resp. étale) atlases satisfying the requests of the statement.

Let us show the last equivalence in the statement. By A.3 the map

$$\varinjlim_n (U_n \times_{\mathcal{Y}_n} P_n) = \varinjlim_n (U_n \times X_H) \longrightarrow \varinjlim_n P_n$$

is a smooth atlas. The map $\varinjlim_n Q_n \longrightarrow \varinjlim_n P_n$ is therefore an equivalence because its base change along the above atlas is $\varinjlim_n (U_n \times X_{H,q_n}) \longrightarrow \varinjlim_n (U_n \times X_H)$, which is an isomorphism. Here we have used A.2. Using again this we see that the map

$$\varinjlim_n \mathcal{Z}_n = \varinjlim_n (Q_n \times_{P_n} P'_n) \longrightarrow \varinjlim_n P'_n = \varinjlim_n (\mathcal{Y}_n \times_{\Delta_{G/H}} \Delta_G)$$

is an equivalence as well. □

Lemma 4.24. *Let G be a p-group, H be a central subgroup which is an $\mathbb{F}_p$-vector space and $X \longrightarrow Y$ be a finite, finitely presented and universally injective map of affine schemes. Then a 2-commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & \Delta_G\\ \downarrow & \nearrow & \downarrow\\ Y & \longrightarrow & \Delta_{G/H} \end{array}$$

always admits a dashed map.

Proof. Set $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} C$ and consider the induced map $C \longrightarrow B$. Since $\mathrm{H}^2(B((t)), H) = \mathrm{H}^2(C((t)), H) = 0$ by the Artin-Schreier sequence, we have a commutative diagram

$$\begin{array}{ccccccc} \mathrm{H}^1(C((t)), H) & \longrightarrow & \mathrm{H}^1(C((t)), G) & \longrightarrow & \mathrm{H}^1(C((t)), G/H) & \longrightarrow & 0\\ \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow & & \\ \mathrm{H}^1(B((t)), H) & \longrightarrow & \mathrm{H}^1(B((t)), G) & \longrightarrow & \mathrm{H}^1(B((t)), G/H) & \longrightarrow & 0 \end{array}$$

with exact rows. By hypothesis there are $u \in \mathrm{H}^1(B((t)), G)$ and $v \in \mathrm{H}^1(C((t)), G/H)$ which agree in $\mathrm{H}^1(B((t)), G/H)$. We can find a common lifting in $\mathrm{H}^1(C((t)), G)$ by proving that the map α is surjective. By 2.4 we have that $\operatorname{Spec} B((t)) \longrightarrow \operatorname{Spec} C((t))$ is the base change of $\operatorname{Spec} B \longrightarrow \operatorname{Spec} C$ and therefore is finite and universally injective. Let D be the image of $C((t)) \longrightarrow B((t))$. The map $\mathrm{H}^1(C((t)), H) \longrightarrow \mathrm{H}^1(D, H)$ is surjective because $H \simeq (\mathbb{Z}/p\mathbb{Z})^l$ for some l and using the description of $\mathbb{Z}/p\mathbb{Z}$-torsors in 4.10. By 2.9 the map $\operatorname{Spec} B((t)) \longrightarrow \operatorname{Spec} D$ is a finite universal homeomorphism. Thus $\mathrm{H}^1(D, H) \longrightarrow \mathrm{H}^1(B((t)), H)$ is bijective by 2.13. □

Proof of Theorem A, 2) and 3). Since p-groups have non trivial center we can find a sequence of quotients

$$G = G_l \longrightarrow G_{l-1} \longrightarrow G_{l-2} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow G_{-1} = 0$$

where $\operatorname{Ker}(G_u \longrightarrow G_{u-1})$ is central in G_u and an $\mathbb{F}_p$-vector space. We proceed by induction on l. In the base case $l = 0$, so that $G = G_0$ is an $\mathbb{F}_p$-vector space, following 4.20 it is enough to set $\mathcal{X}_n = X_{G,n} \times \mathrm{B}\, G$. Consider now the inductive step and set $H = \operatorname{Ker}(G = G_l \longrightarrow G_{l-1})$. Of course we can assume $H \neq 0$, so that we can use the inductive hypothesis on G/H, obtaining a direct system $\mathcal{Y}_*$ and a map $\overline{v}\colon \mathbb{N} \longrightarrow \mathbb{N}$ with a direct system of atlases $\mathbb{A}^{\overline{v}_*}$. The first result follows applying 4.23 with $U_n = \mathbb{A}^{\overline{v}_n}$. We just have to prove the existence of a lifting $\varinjlim_n U_n \longrightarrow \Delta_G$. Since the schemes U_n are affine one always find a lifting $U_n \longrightarrow \Delta_G$ thanks to 4.15. Thanks to 4.24 any lifting $U_n \longrightarrow \Delta_G$ always extends to a lifting $U_{n+1} \longrightarrow \Delta_G$.

Assume now G abelian and set $X_G = X_{G/H} \times X_H$. By induction we can assume $\Delta_{G/H} = X_{G/H} \times \mathrm{B}(G/H)$. By 4.15 and 4.24 there is a lifting $X_{G/H} \longrightarrow \Delta_G$ of the given map $X_{G/H} \longrightarrow \Delta_{G/H}$. In particular, using 4.8, we obtain a map $X_G = X_{G/H} \times X_H \longrightarrow X_{G/H} \times \Delta_H \longrightarrow \Delta_G$, which is finite and étale of degree $\sharp G$. Since G is abelian and using 2.5 we have $(\underline{\mathrm{Aut}}_{\Delta_G} P)(B) = G(B((t))) = G(B)$ for all $P \in \Delta_G(B)$. By B.5, it follows that the rigidification $F = \Delta_G /\!\!/ G$ is the sheaf of isomorphism classes of Δ_G and that $\Delta_G \longrightarrow F$ is a gerbe locally $\mathrm{B}\, G$. Since $X_G \longrightarrow \Delta_G$ and, thanks to A.3, $\mathbb{A}^v = \varinjlim_n \mathbb{A}^{v_n} \longrightarrow \Delta_G$ are finite and étale of degree $\sharp G$, by 2.14 we can conclude that $X_G \longrightarrow F$ and $\mathbb{A}^v \longrightarrow F$ are isomorphisms. Since a gerbe having a section is trivial we get our result. □

With notation and hypothesis from Theorem A set $\mathbb{A}^v = \varinjlim \mathbb{A}^{v_*}$, $\overline{\Delta_G}$ for the coarse ind-algebraic space of Δ_G and consider the induced map $\mathbb{A}^v \longrightarrow \overline{\Delta_G}$. We want to show that when G is non-abelian this map is not an isomorphism in general. The key point is the following Lemma.

Lemma 4.25. *If K is an algebraically closed field and $P \in \Delta_G(K)$ then $H = \underline{\mathrm{Aut}}_{\Delta_G}(P)$ is (non canonically) a subgroup of G and the fiber of $\mathbb{A}^v(K) \longrightarrow \overline{\Delta_G}(K) \simeq \Delta_G(K)/\simeq$ over P has cardinality $\sharp G/\sharp H$.*

Proof. There exist $n \in \mathbb{N}$ and $P_n \in \mathcal{X}_n(K)$ inducing $P \in \Delta_G(K)$. By 3.2 we have $\underline{\mathrm{Aut}}_{\mathcal{X}_n}(P_n) = H$ and, since $\mathcal{X}_n$ is a quasi-separated DM stack, it follows that H is a finite and constant group scheme. Moreover the map

$$H(K) = \mathrm{Aut}^G_{K((t))}(P) \longrightarrow \mathrm{Aut}^G_{\overline{K((t))}}(P \times \overline{K((t))}) \simeq G$$

is injective (the last isomorphism depends on the choice of a section in $P(\overline{K((t))})$). Thanks to A.3 we have 2-Cartesian diagrams

$$\begin{array}{ccccccccc} Y & \longrightarrow & W & \longrightarrow & V & \longrightarrow & \mathbb{A}^{v_n} & \longrightarrow & \mathbb{A}^{v} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & \mathrm{B}\, H & \longrightarrow & U & \longrightarrow & \mathcal{X}_n & \longrightarrow & \Delta_G \\ & & & & \downarrow & & \downarrow & & \\ & & & & \operatorname{Spec} K & \longrightarrow & \overline{\mathcal{X}}_n & & \end{array}$$

If $F \subseteq \mathbb{A}^v(K)$ is the fiber we are looking for then we get an induced map $V(K) \longrightarrow F$ which is easily seen to be surjective. From 3.2 it follows that $\mathbb{A}^{v_n}(K) \longrightarrow \mathbb{A}^v(K)$ is injective, which implies that $V(K) \longrightarrow F$ is bijective. From [Sta17, 06ML] one get that $\mathrm{B}\, H$ is the reduction of U. Thus $W(K) = V(K)$. Notice that, since $\overline{\mathcal{X}_n}$ has schematically representable diagonal, V, W and Y are all schemes. Since the vertical maps in the top row are finite and étale of degree $\sharp G$ and $Y \longrightarrow W$ is an H-torsor we conclude that $\sharp Y = G$ and $\sharp W = \sharp Y/\sharp H$ as required. □

We see that if G is not abelian and P is a Galois extension of $\overline{K((t))}$ with group G, where K is an algebraically closed field, then the map $\mathbb{A}^v \longrightarrow \overline{\Delta_G}$ is not injective. Indeed the fiber of $\mathbb{A}^v(K) \longrightarrow \overline{\Delta_G}(K)$ over P has cardinality $\sharp G/\sharp Z(G)$ because $\operatorname{Aut}^G_{K((t))}(P) = Z(G)$.

Remark 4.26. The moduli functor F' described in [Har80, Proof of 2.1] is very similar to the sheaf of isomorphism classes of Δ_G but with some differences. Firstly F' maps pointed connected affine schemes to sets, while we look at the category of all (non-pointed) affine schemes, which is standard in modern moduli theory. Secondly, for a connected and pointed affine scheme $\operatorname{Spec} B$, he defines $F'(\operatorname{Spec} B)$ as the set of equivalence classes of pointed G-torsors on $B \otimes_k k((t))$ rather than $B((t))$ as in our case. Two covers are defined to be equivalent if they agree after a finite étale pullback of $B \otimes_k k[[t]]$. This equivalence relation plays the role of "killing terms of positive degrees", while the same role is played by 4.11 in our setting. This is better understood in the case $G = \mathbb{Z}/p\mathbb{Z}$ where one can show that F' is exactly the sheaf $\overline{\Delta}$ of isomorphism classes of $\Delta_{\mathbb{Z}/p\mathbb{Z}}$ (one can ignore base points here because $\mathbb{Z}/p\mathbb{Z}$ is abelian).

A map $\alpha\colon F' \longrightarrow \overline{\Delta}$ is well defined because if two torsors P, Q over $B \otimes k((t))$ become isomorphic after an étale cover of $B \otimes k[[t]]$, by 2.3 $P \times B((t)), Q \times B((t))$ become isomorphic over $C((t))$, where C/B is an étale covering. The surjectivity of α is easy: from the description of $\Delta_{\mathbb{Z}/p\mathbb{Z}}$ a torsor in $\overline{\Delta}(B)$ is given by an element $b \in B((t))$ with zero positive part and, therefore, belonging to $B \otimes k((t)) \subseteq B((t))$ (see also 2.4). For the injectivity take $e \in B \otimes k((t))$ defining a torsor over $B \otimes k((t))$ which become trivial in $\overline{\Delta}(B)$. Write $e = e_- + e_+$ as usual. Since $e_+ \in B \otimes k[[t]]$ (which means that its associated torsor extends to $B \otimes k[[t]]$) one has $e = e_-$ in $F'(B)$. Using the same notation and strategy of 4.13, in particular of the essential surjectivity, one can assume $e = \phi_k(\underline{b})$ for $\underline{b} \in \mathbb{A}^{(S)}$. Since $e = 0$ in $\overline{\Delta} = (\mathbb{A}^{(S)})^\infty$ it follows that the coefficients of b and e are nilpotent. Since $e^p = e$ in $F'(B)$ it follows that $e = 0$ in $F'(B)$.

4.4. **Semidirect products.** The aim of this section is to complete the proof of Theorem A. So let k be a field of positive characteristic p and G be a finite and étale group scheme over k such that $G \times_k \overline{k}$ is a semidirect product of a p-group and cyclic group of rank coprime with p.

Extending the base field by a Galois extension and using 4.2 and 3.5 we can assume that G is constant, say $G = H \rtimes C_n$ where H is a p-group and C_n is a cyclic

group of order n coprime with p. We can moreover assume that the base field k has all n-roots of unity, so that $C_n \simeq \mu_n$ as group schemes. More precisely we assume that $C_n = \mu_n(k) \subseteq k^*$ is the group of n-th roots of unity of k.

Consider $\operatorname{Spec} k \longrightarrow \Delta_{C_n}$ given by $(k((t)), t^q)$ as in Theorem B and denote by $\mathcal{Z}_{G,q}$ the fiber product $\operatorname{Spec} k \times_{\Delta_{C_n}} \Delta_G$, which is the fibered category of pairs (P, δ) where $P \in \Delta_G(B)$ and $\delta\colon P \longrightarrow \operatorname{Spec}(B((t))[X]/(X^n - t^q))$ is a G-equivariant map.

Proposition 4.27. *Let $d = (n, q)$ and $G_d = H \rtimes C_{n/d} < G$. Then the functor $\mathcal{Z}_{G_d,q/d} \longrightarrow \mathcal{Z}_{G,q}$ induced by $\Delta_{G_d} \longrightarrow \Delta_G$ and $\Delta_{C_{n/d}} \longrightarrow \Delta_{C_n}$ is an equivalence.*

Proof. Set $Q_{n,q} = \operatorname{Spec}(B((t))[X]/(X^n - t^q))$. The map $X \longmapsto X$ induces a map $Q_{n/d,q/d} \longrightarrow Q_{n,q}$ which is $C_{n/d}$-equivariant, that is $Q_{n,q}$ is the C_n-torsor induced by the $C_{n/d}$-torsor $Q_{n/d,q/d}$. We obtain a quasi-inverse $\mathcal{Z}_{G,q} \longrightarrow \mathcal{Z}_{G_d,q/d}$ mapping $P \xrightarrow{\phi} Q_{n,q}$ to the fiber product $P \times_{Q_{n,q}} Q_{n/d,q/d} \longrightarrow Q_{n/d,q/d}$. □

Remark 4.28. If $(n, q) = 1$ we have an isomorphism of B-algebras

$$B((s)) \longrightarrow B((t))[X](X^n - t^q)$$

such that the C_n-action induced on the left is $s \longmapsto \xi^\beta s$ for $\xi \in C_n$, where $\beta \in \mathbb{Z}/n\mathbb{Z}$ is the inverse of q. Indeed write $\beta q = 1 + \alpha n$ for some $\alpha, \beta \in \mathbb{N}$. We have $(X^\beta/t^\alpha)^n = t$ in $B((t))[X]/(X^n - t^q)$ and isomorphisms

$$\begin{array}{ccccc} s & \longmapsto & X & \longmapsto & X^\beta/t^\alpha \\ B((s)) & \xrightarrow{\simeq} & B((t))[X]/(X^n - t) & \xrightarrow{\simeq} & B((t))[X]/(X^n - t^q) \end{array}$$

Definition 4.29. Given a p-group H and an autoequivalence $\phi\colon \Delta_H \longrightarrow \Delta_H$ we define $\mathcal{Z}_\phi$ as the stack of pairs (P, u) where $P \in \Delta_H$ and $u\colon P \longrightarrow \phi(P)$ is an isomorphism in Δ_H.

There are two natural autoequivalences of Δ_H: $\phi_\psi\colon \Delta_H \longrightarrow \Delta_H$ obtained composing by an isomorphism $\psi\colon H \longrightarrow H$; $\phi_\xi\colon \Delta_H \longrightarrow \Delta_H$ induced by a n-th root of unity ξ using the Cartesian diagram

$$\begin{array}{ccc} \phi_\xi(P) & \longrightarrow & P \\ \downarrow & & \downarrow \\ \operatorname{Spec} B((t)) & \longrightarrow & \operatorname{Spec} B((t)) \\ \xi t & \longmapsto & t \end{array}$$

Proposition 4.30. *Assume $(q, n) = 1$ and let $\zeta \in C_n$ be a primitive n-th root of unity and $\psi\colon H \longrightarrow H$ be the automorphism image of ζ under $C_n \longrightarrow \operatorname{Aut}(H)$. Set $\xi = \zeta^\beta$ where $\beta \in \mathbb{Z}/n\mathbb{Z}$ is the inverse of q and $\phi = \phi_\psi \circ \phi_\xi\colon \Delta_H \longrightarrow \Delta_H$. Then $\mathcal{Z}_{G,q}$ is an open and closed substack of $\mathcal{Z}_\phi$.*

Proof. Let $P \in \Delta_H$. We have $\phi(P) = \phi_\xi(P)$ as schemes. Thus an isomorphism $u\colon P \longrightarrow \phi(P)$, via $\phi(P) = \phi_\xi(P) \longrightarrow P$, corresponds to an isomorphism $v\colon P \longrightarrow P$. The morphism u is over $\operatorname{Spec} B((t))$ if and only if the following diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{v} & P \\ \downarrow & & \downarrow \\ \operatorname{Spec} B((t)) & \longrightarrow & \operatorname{Spec} B((t)) \\ \xi t & \longmapsto & t \end{array}$$

Finally, by going through the definitions, we see that u is H-equivariant if and only if $\psi(h) = vhv^{-1}$ in $\mathrm{Aut}(P)$ for all $h \in H$. We identify $\mathcal{Z}_\phi$ with the stack of pairs (P, v) as above.

Set $S_B = \operatorname{Spec} B((s))$ with the C_n-action given by $s \mapsto \lambda^\beta s$ for $\lambda \in C_n$. By 4.28 S_B is isomorphic to $\operatorname{Spec} B((t))[X]/(X^n - t^q)$ and therefore, by construction, an object $(P, \delta) \in \mathcal{Z}_{G,q}(B)$ is a G-torsor over $B((t))$ with a G-equivariant map $\delta\colon P \longrightarrow S_B$. In particular $P \xrightarrow{\delta} S_B$ is an H-torsor. Set $g_0 = (1, \zeta) \in H \rtimes C_n = G$, so that $\psi(h) = g_0 h g_0^{-1}$ in G. Since δ is G-equivariant it follows that $(P, g_0) \in \mathcal{Z}_\phi$. We therefore get a map $\mathcal{Z}_{G,q} \longrightarrow \mathcal{Z}_\phi$. This map is fully faithful. Indeed if $(P, \delta), (P', \delta') \in \mathcal{Z}_{G,q}(B)$ the map on isomorphisms is

$$\begin{array}{c} \mathrm{Iso}_{\mathcal{Z}_{G,q}}((P,\delta),(P',\delta')) = \mathrm{Iso}^G_{S_B}(P,P') \\ \big\downarrow \\ \mathrm{Iso}_{\mathcal{Z}_\phi}((P,g_0),(P',g_0)) = \{\omega \in \mathrm{Iso}^H_{S_B}(P,P') \mid g_0\omega = \omega g_0\} \end{array}$$

We are going to show that the substack $\overline{\mathcal{Z}}_\phi$ of pairs (P, v) where $v^n = \mathrm{id}$ is the essential image of $\mathcal{Z}_{G,q} \longrightarrow \mathcal{Z}_\phi$. Since $g_0^n = 1$ we have the inclusion $\subseteq$. Now let $(P, v) \in \overline{\mathcal{Z}}_\phi$. Since $v^n = 1$ we get the map $G = H \rtimes C_n \longrightarrow \mathrm{Aut}(P)$ sending (h, ζ^m) to hv^m, defining a G-action on P. By construction the map $\delta\colon P \longrightarrow S_B$ is G-equivariant. Since $P/H \simeq S_B$ it remains to show that G acts freely on P. If $p(hv^l) = p$ for some $p \in P$, then $\delta(ph)\zeta^l = \delta(p)\zeta^l = \delta(p)$ and therefore $n \mid l$ and $v^l = 1$. Finally $ph = p$ implies $h = 1$ in $H \subseteq \mathrm{Aut}(P)$ because P is an H-torsor. Thus G acts on P freely.

We now show that $\overline{\mathcal{Z}}_\phi$ is open and closed in $\mathcal{Z}_\phi$. If $(P, v) \in \mathcal{Z}_\phi(B)$ we have that $v^n \in \mathrm{Aut}^H_{B((t))}(P)$ because $\psi(h) = vhv^{-1}$ and ψ has order n. The group scheme $\underline{\mathrm{Aut}}^H_{B((t))}(P) \longrightarrow \operatorname{Spec} B((t))$ is finite and étale, thus the locus W in $\operatorname{Spec} B((t))$ where $v^n = \mathrm{id}$ is open and closed in $\operatorname{Spec} B((t))$. By 2.5 there is an open and closed subset $\widetilde{W}$ in $\operatorname{Spec} B$ inducing W. By construction the base change of $\overline{\mathcal{Z}}_\phi \longrightarrow \mathcal{Z}_\phi$ along $(P, v)\colon \operatorname{Spec} B \longrightarrow \mathcal{Z}_\phi$ is $\widetilde{W}$, which ends the proof. □

Proposition 4.31. *If $\phi\colon \Delta_H \longrightarrow \Delta_H$ is an equivalence then there exists a direct system of separated DM stacks $\mathcal{Z}_*$ with finite and universally injective transition maps, with a direct system of finite and étale atlases $Z_n \longrightarrow \mathcal{Z}_n$ of degree $(\sharp H)^2$ from affine schemes and an equivalence $\varinjlim_n \mathcal{Z}_n \simeq \mathcal{Z}_\phi$.*

Proof. Consider a direct system of DM stacks $\mathcal{Y}_*$ as in Theorem A for the p-group H. Denote by $\Gamma_\phi\colon \Delta_H \longrightarrow \Delta_H \times \Delta_H$ be the graph of ϕ and by $\gamma_{u,v}\colon \mathcal{Y}_u \longrightarrow \mathcal{Y}_v$ the transition maps. By 3.13 $\mathcal{Z}_\phi$ is the fiber product of Γ_ϕ and the diagonal of Δ_H. There exist an increasing function $\delta\colon \mathbb{N} \longrightarrow \mathbb{N}$ and 2-commutative diagrams

$$\begin{array}{ccccc} \mathcal{Y}_n & \longrightarrow & \mathcal{Y}_{n+1} & \longrightarrow & \Delta_H \\ \big\downarrow{\scriptstyle \phi_n} & & \big\downarrow{\scriptstyle \phi_{n+1}} & & \big\downarrow{\scriptstyle \phi} \\ \mathcal{Y}_{\delta_n} & \longrightarrow & \mathcal{Y}_{\delta_{n+1}} & \longrightarrow & \Delta_H \end{array}$$

Similarly $\mathcal{Y}_n \longrightarrow \mathcal{Y}_n \times \mathcal{Y}_n \xrightarrow{\gamma_{n,\delta_n} \times \phi_n} \mathcal{Y}_{\delta_n} \times \mathcal{Y}_{\delta_n}$ and $\mathcal{Y}_n \longrightarrow \mathcal{Y}_n \times \mathcal{Y}_n \xrightarrow{\gamma_{n,\delta_n} \times \gamma_{n,\delta_n}} \mathcal{Y}_{\delta_n} \times \mathcal{Y}_{\delta_n}$ approximate Γ_ϕ and the diagonal of Δ_H respectively. By A.2 it follows that the fiber product $\mathcal{Z}_n = \mathcal{Y}_n \times_{\mathcal{Y}_{\delta_n} \times \mathcal{Y}_{\delta_n}} \mathcal{Y}_n$ of the two maps form a direct system of separated DM stacks whose limit is $\mathcal{Z}_\phi$. By 2.11 the transition maps are finite and universally injective. Let $Y_n \longrightarrow \mathcal{Y}_n$ be the finite and étale atlases of degree $\sharp H$ given in Theorem A. The induced map $Z_n = Y_n \times_{\mathcal{Y}_{\delta_n} \times \mathcal{Y}_{\delta_n}} Y_n \longrightarrow \mathcal{Z}_n$ is finite and étale of degree $(\sharp H)^2$. Since $\mathcal{Y}_{\delta_n} \times \mathcal{Y}_{\delta_n}$ has affine diagonal it follows that Z_n is affine. Finally, using the usual properties of fiber products and the fact that $Y_n \longrightarrow \mathcal{Y}_n$

is a direct system of atlases, we see that the maps $Z_n \longrightarrow Z_{n+1} \times_{\mathcal{Z}_{n+1}} \mathcal{Z}_n$ are isomorphisms. □

Proof of Theorem A, 1). Recall that we have reduced the problem to the case of a constant group $G = H \rtimes C_n$ at the beginning of this subsection 27. Consider $\pi\colon \Delta_G \longrightarrow \Delta_{C_n}$ and the decomposition $\Delta_{C_n} = \bigsqcup_{q=1}^{n} \mathrm{B}\, G_q$, where $G_q = C_n$, of Theorem B. The map

$$\bigsqcup_{q=1}^{n} (\mathrm{B}\, G_q \times_{\Delta_{C_n}} \Delta_G) \longrightarrow \Delta_G$$

is well defined and an equivalence because given k-algebras A_1 and A_2 the map $\Delta_G(A_1 \times A_2) \longrightarrow \Delta_G(A_1) \times \Delta_G(A_2)$ is an equivalence. Thus in the statement of the theorem Δ_G can be replaced by $\widetilde{\mathcal{Z}}_{G,q} = (\mathrm{B}\, G_q \times_{\Delta_{C_n}} \Delta_G)$. We must show that $\widetilde{\mathcal{Z}}_{G,q}$ is a stack in the fpqc topology if $\mathcal{Z}_{G,q}$ is so. Let B be a ring, $\mathcal{U} = \{B \longrightarrow B_i\}_{i\in I}$ a covering and $\xi \in \widetilde{\mathcal{Z}}_{G,q}(\mathcal{U})$ be a descent datum. Given a B-scheme Y we denote by $\mathcal{U}_Y = \mathcal{U} \times_B Y$ and by $\xi_Y \in \widetilde{\mathcal{Z}}_{G,q}(\mathcal{U}_Y)$ the pullback. Denote by $r\colon \widetilde{\mathcal{Z}}_{G,q} \longrightarrow \mathrm{B}\, C_n$ the structure map. The descent datum $r(\xi)$ yields a C_n-torsor $F \longrightarrow \operatorname{Spec} B$. Let $Y \longrightarrow \operatorname{Spec} B$ a B-scheme with a factorization $Y \longrightarrow F$. This factorization is a trivialization of the C_n-torsor over Y and therefore it induces a descent datum of $(\widetilde{\mathcal{Z}}_{G,q} \times_{\mathrm{B}\, C_n} \operatorname{Spec} k)(\mathcal{U}_Y) = \mathcal{Z}_{G,q}(\mathcal{U}_Y)$ which is therefore effective, yielding $\eta_Y \in \mathcal{Z}_{G,q}(Y)$ and $\widetilde{\eta}_Y \in \widetilde{\mathcal{Z}}_{G,q}(Y)$. By construction $\widetilde{\eta}_Y \in \widetilde{\mathcal{Z}}_{G,q}(Y)$ induces the descent datum $\xi_Y \in \widetilde{\mathcal{Z}}_{G,q}(\mathcal{U}_Y)$. In particular we get $\widetilde{\eta}_F \in \widetilde{\mathcal{Z}}_{G,q}(F)$. Since $\widetilde{\mathcal{Z}}_{G,q}$ is a prestack, the objects $\widetilde{\eta}_{F\times_B F}$ obtained using the two projections $F \times_B F \longrightarrow F$ are isomorphic via a given isomorphism: they both induce $\xi_{F\times_B F} \in \widetilde{\mathcal{Z}}_{G,q}(\mathcal{U}_{F\times_B F})$ which does not depend on the projections being a pullback of $\xi \in \widetilde{\mathcal{Z}}_{G,q}(\mathcal{U})$. In conclusion $\widetilde{\eta}_F$ gives a descent datum for $\widetilde{\mathcal{Z}}_{G,q}$ over the covering $F \longrightarrow \operatorname{Spec} B$. In order to get a global object in $\widetilde{\mathcal{Z}}_{G,q}(B)$ inducing the given descent datum ξ it is enough to notice that, by 2.4, Δ_G satisfies descent along coverings $U \longrightarrow \operatorname{Spec} B$ which are finite, flat and finitely presented.

Thanks to 3.5, it is enough to show that $\mathcal{Z}_{G,q}$ is a limit as in the statement. Using 4.27 we can further assume n and q coprime and, using 4.30, we can replace $\mathcal{Z}_{G,q}$ by $\mathcal{Z}_\phi$, where $\phi = \phi_\psi \circ \phi_\xi$ as in 4.30. The conclusion now follows from 4.31. □

Appendix A. Limit of fibered categories

In this appendix we discuss the notion of inductive limit of stacks. To simplify the exposition and since general colimits were not needed in this paper we will only talk about limit over the natural numbers $\mathbb{N}$. General results can be found in [TZ19, Appendix A].

A *direct system* of categories $\mathcal{C}_*$ (indexed by $\mathbb{N}$) is a collection of categories $\mathcal{C}_n$ for $n \in \mathbb{N}$ and functors $\psi_n\colon \mathcal{C}_n \longrightarrow \mathcal{C}_{n+1}$. Given indexes $n < m$ we also set

$$\psi_{n,m}\colon C_n \xrightarrow{\psi_n} \mathcal{C}_{n+1} \longrightarrow \cdots \longrightarrow \mathcal{C}_{m-1} \xrightarrow{\psi_{m-1}} \mathcal{C}_m$$

and $\psi_{n,n} = \mathrm{id}_{\mathcal{C}_n}$. The limit $\varinjlim_{n\in\mathbb{N}} C_n$ or $\mathcal{C}_\infty$ is the category defined as follows. Its objects are pairs (n, x) with $n \in \mathbb{N}$ and $x \in \mathcal{C}_n$. Given pairs (n, x) and (m, y) we set

$$\operatorname{Hom}_{\mathcal{C}_\infty}((n, x), (m, x)) = \varinjlim_{q>n+m} \operatorname{Hom}_{\mathcal{C}_q}(\psi_{n,q}(x), \psi_{m,q}(y))$$

Composition is defined in the obvious way. There are obvious functors $\mathcal{C}_n \longrightarrow \mathcal{C}_\infty$.

Given a category $\mathcal{D}$ we denote by $\operatorname{Hom}(\mathcal{C}_*, \mathcal{D})$ the category whose objects are collections (F_n, α_n) where $F_n\colon \mathcal{C}_n \longrightarrow \mathcal{D}$ are functors and $\alpha_n\colon F_{n+1} \circ \psi_n \longrightarrow F_n$

are natural isomorphisms of functors $\mathcal{C}_n \longrightarrow \mathcal{D}$. There is an obvious functor $\mathrm{Hom}(\mathcal{C}_\infty, \mathcal{D}) \longrightarrow \mathrm{Hom}(\mathcal{C}_*, \mathcal{D})$ and we have:

Proposition A.1. [TZ19, Remark A.3] *The functor* $\mathrm{Hom}(\mathcal{C}_\infty, \mathcal{D}) \longrightarrow \mathrm{Hom}(\mathcal{C}_*, \mathcal{D})$ *is an equivalence.*

This justifies calling $\mathcal{C}_\infty$ the limit of the direct system $\mathcal{C}_*$.

Let $\mathcal{S}$ be a category with fiber products. A direct system of fibered categories $\mathcal{X}_*$ over $\mathcal{S}$ (indexed by $\mathbb{N}$) is a direct system of categories $\mathcal{X}_*$ together with maps $\mathcal{X}_n \longrightarrow \mathcal{S}$ making $\mathcal{X}_n$ into a fibered category over $\mathcal{S}$ and such that the transition maps $\mathcal{X}_n \longrightarrow \mathcal{X}_{n+1}$ are maps of fibered categories. Result [TZ19, Proposition A.4] translates into what follows.

The induced functor $\mathcal{X}_\infty \longrightarrow \mathcal{S}$ makes $\mathcal{X}_\infty$ into a fibered category over $\mathcal{S}$ and the maps $\mathcal{X}_n \longrightarrow \mathcal{X}_\infty$ are map of fibered categories. Given an object $s \in \mathcal{S}$ there is an induced direct system of categories $\mathcal{X}_*(s)$ and there is a natural equivalence

$$\varinjlim_{n\in\mathbb{N}} \mathcal{X}_n(s) \xrightarrow{\simeq} \mathcal{X}_\infty(s)$$

In particular if all the $\mathcal{X}_n$ are fibered in sets (resp. groupoids) so is $\mathcal{X}_\infty$.

If $\mathcal{Y}$ is another fibered category over $\mathcal{S}$ denotes by $\mathrm{Hom}_{\mathcal{S}}(\mathcal{X}_*, \mathcal{Y})$ the subcategory of $\mathrm{Hom}(\mathcal{X}_*, \mathcal{Y})$ of objects (F_n, α_n) where F_n are base preserving functors and α_n are base preserving natural transformations. Also the arrows in the category $\mathrm{Hom}_{\mathcal{S}}(\mathcal{X}_*, \mathcal{Y})$ are required to be base preserving natural transformations. There is an induced functor $\mathrm{Hom}_{\mathcal{S}}(\mathcal{X}_\infty, \mathcal{Y}) \longrightarrow \mathrm{Hom}_{\mathcal{S}}(\mathcal{X}_*, \mathcal{Y})$ which is an equivalence of categories.

A direct check using the definition of fiber product yields the following.

Proposition A.2. *Let* $\mathcal{X}_*, \mathcal{Y}_*$ *and* $\mathcal{Z}_*$ *be direct system of categories fibered in groupoids over* $\mathcal{S}$ *and assume they are given* 2*-commutative diagrams*

$$\begin{array}{ccc} \mathcal{X}_n & \longrightarrow & \mathcal{X}_{n+1} \\ \downarrow{\scriptstyle a_n} & & \downarrow{\scriptstyle a_{n+1}} \\ \mathcal{Y}_n & \longrightarrow & \mathcal{Y}_{n+1} \end{array} \qquad \begin{array}{ccc} \mathcal{Z}_n & \longrightarrow & \mathcal{Z}_{n+1} \\ \downarrow{\scriptstyle b_n} & & \downarrow{\scriptstyle b_{n+1}} \\ \mathcal{Y}_n & \longrightarrow & \mathcal{Y}_{n+1} \end{array}$$

Then the canonical map

$$\varinjlim_{n\in\mathbb{N}} (\mathcal{X}_n \times_{\mathcal{Y}_n} \mathcal{Z}_n) \longrightarrow \varinjlim_{n\in\mathbb{N}} \mathcal{X}_n \times_{\varinjlim_{n\in\mathbb{N}} \mathcal{Y}_n} \varinjlim_{n\in\mathbb{N}} \mathcal{Z}_n$$

is an equivalence.

Corollary A.3. *Let* $\mathcal{X}_*$ *and* $\mathcal{Y}_*$ *be direct systems of categories fibered in groupoids over* $\mathcal{S}$ *and assume to have* 2*-Cartesian diagrams*

$$\begin{array}{ccc} \mathcal{Y}_n & \longrightarrow & \mathcal{Y}_{n+1} \\ \downarrow & & \downarrow \\ \mathcal{X}_n & \longrightarrow & \mathcal{X}_{n+1} \end{array}$$

Then the following diagrams are also 2*-Cartesian for all* $n \in \mathbb{N}$*:*

$$\begin{array}{ccc} \mathcal{Y}_n & \longrightarrow & \varinjlim \mathcal{Y}_* \\ \downarrow & & \downarrow \\ \mathcal{X}_n & \longrightarrow & \varinjlim \mathcal{X}_* \end{array}$$

Proof. We have maps $\mathcal{Y}_n \xrightarrow{\simeq} \mathcal{X}_n \times_{\mathcal{X}_m} \mathcal{Y}_m \longrightarrow \mathcal{X}_n \times_{\mathcal{X}_\infty} \mathcal{Y}_\infty$ for all $m > n$. Passing to the limit on m we get the result. □

Lemma A.4. *Assume that $\mathcal{S}$ has a Grothendieck topology such that for all coverings $\mathcal{V} = \{U_i \longrightarrow U\}_{i\in I}$ there exists a finite subset $J \subseteq I$ for which $\mathcal{V}_J = \{U_j \longrightarrow U\}_{j\in J}$ is also a covering. Let also $\mathcal{Y}$ be a fibered category. Then $\mathcal{Y}$ is a stack (resp. pre-stack) if and only if given a finite covering $\mathcal{U}$ of $U \in \mathcal{S}$ the functor $\mathcal{Y}(U) \longrightarrow \mathcal{Y}(\mathcal{U})$ is an equivalence (resp. fully faithful), where $\mathcal{Y}(\mathcal{U})$ is the category of descent data of $\mathcal{Y}$ over $\mathcal{U}$.*

Proof. The "only if" part is trivial. We show the "if" part. Let $\mathcal{V} = \{U_i \longrightarrow U\}_{i\in I}$ be a general covering and consider a finite subset $J \subseteq I$ for which $\mathcal{V}_J = \{U_i \longrightarrow U\}_{i\in J}$ is also a covering. Thus the composition $\mathcal{Y}(U) \longrightarrow \mathcal{Y}(\mathcal{V}) \longrightarrow \mathcal{Y}(\mathcal{V}_J)$ is an equivalence (resp. fully faithful) and it is enough to show that $\mathcal{Y}(\mathcal{V}) \longrightarrow \mathcal{Y}(\mathcal{V}_J)$ is faithful. This follows because there is a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{Y}(\mathcal{V}) & \xrightarrow{\quad a \quad} & \prod_{i\in I} \mathcal{Y}(U_i) \\ \downarrow & & \downarrow b \\ \mathcal{Y}(\mathcal{V}_J) & \longrightarrow & \prod_{i\in I}(\prod_{j\in J} \mathcal{Y}(U_i \times_U U_j)) \end{array}$$

where the functors a and b are faithful. □

Proposition A.5. *In the hypothesis of A.4, if $\mathcal{X}_*$ is a direct system of stacks (resp. pre-stacks) over $\mathcal{S}$ then $\mathcal{X}_\infty$ is also a stack (resp. pre-stack) over $\mathcal{S}$.*

Proof. It is easy to prove descent (resp. descent on morphisms) and its uniqueness along coverings indexed by finite sets. By A.4 this is enough. □

Clearly the site $\mathcal{S}$ we have in mind in the above proposition is a category fibered in groupoids over the category of affine schemes Aff with any of the usual topologies, for instance Aff/X, the category of affine schemes together with a map to a given scheme X.

Appendix B. Rigidification revisited

Rigidification is an operation that allows us to "kill" automorphisms of a given stack by modding out stabilizers by a given subgroup of the inertia. This operation is described in [AOV08, Appendix A] in the context of algebraic stacks, but one can easily see that this is a very general construction. In this appendix we discuss it in its general form so that we can apply it to non-algebraic stacks like Δ_G.

Let $\mathcal{S}$ be a site, $\mathcal{X}$ be a stack in groupoids over $\mathcal{S}$ and denote by $I(\mathcal{X}) \longrightarrow \mathcal{X}$ the inertia stack. The inertia stack can be also thought as the sheaf $\mathcal{X}^{\mathrm{op}} \longrightarrow$ (Groups) mapping $\xi \in \mathcal{X}(U)$ to $\mathrm{Aut}_{\mathcal{X}(U)}(\xi)$. By a subgroup sheaf of the inertia stack we mean a subgroup sheaf of the previous functor. Notice that given a sheaf $F\colon \mathcal{X}^{\mathrm{op}} \longrightarrow$ (Sets) and an object $\xi \in \mathcal{X}(U)$ one get a sheaf F_ξ on U by composing $(\mathcal{S}/U)^{\mathrm{op}} \xrightarrow{\xi} \mathcal{X}^{\mathrm{op}} \longrightarrow$ (Sets), where the first arrow comes from the 2-Yoneda lemma. Concretely one has $F_\xi(V \xrightarrow{g} U) = F(g^*\xi)$. If $f\colon V \longrightarrow U$ is any map in $\mathcal{S}$ there is a canonical isomorphism $F_\xi \times_U V \simeq F_{f^*\xi}$.

Notice moreover that a subgroup sheaf $\mathcal{H}$ of $I(\mathcal{X})$ is automatically normal: if $\xi \in \mathcal{X}(U)$ and $\omega \in I(\mathcal{X})(\xi) = \mathrm{Aut}_{\mathcal{X}(U)}(\xi)$ then ω induces the conjugation $I(\mathcal{X})(\xi) \longrightarrow I(\mathcal{X})(\xi)$ and, since $\mathcal{H}$ is a subsheaf, the subgroup $\mathcal{H}(\xi)$ is preserved by the conjugation.

We now describe how to rigidify $\mathcal{X}$ by any subgroup sheaf $\mathcal{H}$ of the inertia. We define the category $\widetilde{\mathcal{X}/\!\!/\mathcal{H}}$ as follows. The objects are the same as the ones of $\mathcal{X}$. Given $\xi \in \mathcal{X}(U)$ and $\eta \in \mathcal{X}(V)$ an arrow $\xi \longrightarrow \eta$ in $\widetilde{\mathcal{X}/\!\!/\mathcal{H}}$ is a pair (f, ϕ)

where $f\colon U \longrightarrow V$ and $\phi \in (\underline{\mathrm{Iso}}_U(f^*\eta, \xi)/\mathcal{H}_\xi)(U)$. Given $\zeta \in \mathcal{X}(W)$ and arrows $\xi \xrightarrow{(f,\phi)} \eta \xrightarrow{(g,\psi)} \zeta$ we have that $(\underline{\mathrm{Iso}}_U(f^*g^*\zeta, f^*\eta)/\mathcal{H}_{f^*\eta}) \simeq (\underline{\mathrm{Iso}}_V(g^*\zeta, \eta)/\mathcal{H}_\eta) \times_V U$, because the action of $\mathcal{H}_*$ is free. Moreover composition induces a map

$$(\underline{\mathrm{Iso}}_U(f^*g^*\zeta, f^*\eta)/\mathcal{H}_{f^*\eta}) \times (\underline{\mathrm{Iso}}_U(f^*\eta, \xi)/\mathcal{H}_\xi) \longrightarrow (\underline{\mathrm{Iso}}_U(f^*g^*\zeta, \xi)/\mathcal{H}_\xi)$$

One set $(g, \psi) \circ (f, \phi) = (gf, \omega)$ where ω is the image of $(f^*\psi, \phi)$ under the above map. It is elementary to show that this defines a category $\widetilde{\mathcal{X}/\!\!/\mathcal{H}}$ together with a map $\widetilde{\mathcal{X}/\!\!/\mathcal{H}} \longrightarrow \mathcal{S}$ making it into a category fibered in groupoids. The map $\mathcal{X} \longrightarrow \widetilde{\mathcal{X}/\!\!/\mathcal{H}}$ is also a map of fibered categories.

Definition B.1. The rigidification $\mathcal{X}/\!\!/\mathcal{H}$ of $\mathcal{X}$ by $\mathcal{H}$ over the site $\mathcal{S}$ is a stackification of the category fibered in groupoids $\widetilde{\mathcal{X}/\!\!/\mathcal{H}}$ constructed above.

Depending on the chosen foundation and the notion of category used, a stackification does not necessarily exists. The usual workaround is to talk about universes but in our case one can directly construct a stackification $\mathcal{X}/\!\!/\mathcal{H}$. We denote by $\mathcal{Z}$ the category constructed as follows. Its objects are pairs $(\mathcal{G} \longrightarrow U, F)$ where $U \in \mathcal{S}$, $\mathcal{G} \longrightarrow U$ is a gerbe and $F\colon \mathcal{G} \longrightarrow \mathcal{X}$ is a map of fibered categories satisfying the following condition: for all $y \in \mathcal{G}$ lying over $V \in \mathcal{S}$ the map $\mathrm{Aut}_{\mathcal{G}(V)}(y) \longrightarrow \mathrm{Aut}_{\mathcal{X}(V)}(F(y))$ is an isomorphism onto $\mathcal{H}(F(y))$. An arrow $(\mathcal{G}' \longrightarrow U', F') \longrightarrow (\mathcal{G} \longrightarrow U, F)$ is a triple (f, ω, δ) where

$$\begin{array}{ccc} \mathcal{G}' & \xrightarrow{\omega} & \mathcal{G} \\ \downarrow & & \downarrow \\ U' & \xrightarrow{f} & U \end{array}$$

is a 2-Cartesian diagram and $\delta\colon F \circ \omega \longrightarrow F'$ is a base preserving natural isomorphism. The class of arrows between two given objects is in a natural way a category rather than a set. On the other hand, since the maps from the gerbes to $\mathcal{X}$ are faithful by definition, this category is equivalent to a set: between two 1-arrows there exist at most one 2-arrow. In particular $\mathcal{Z}$ is a 1-category. It is not difficult to show that $\mathcal{Z}$ is fibered in groupoids over $\mathcal{S}$ and that it satisfies descent, i.e., it is a stack in groupoids over $\mathcal{S}$.

There is a functor $\Delta\colon \mathcal{X} \longrightarrow \mathcal{Z}$ mapping $\xi \in \mathcal{X}(U)$ to $F_\xi\colon \mathrm{B}\,\mathcal{H}_\xi \longrightarrow \mathcal{X} \times U \longrightarrow \mathcal{X}$. If $\psi\colon \xi' \longrightarrow \xi$ is an isomorphism in $\mathcal{X}(U)$, then $\Delta(\psi) = (\mathrm{B}(c_\psi), \lambda_\psi)$ where $c_\psi\colon \mathcal{H}_{\xi'} \longrightarrow \mathcal{H}_\xi$ is the conjugation by ψ and λ_ψ is the unique natural transformation $F_{\xi'} \longrightarrow F_\xi \circ \mathrm{B}(c_\psi)$ that evaluated in $\mathcal{H}_{\xi'}$ yields $\xi' \xrightarrow{\psi} \xi$. For the existence and uniqueness of λ_ψ recall that, by descent, a natural transformation of functors Q, Q' from a stack of torsors to a stack, is the same datum of an isomorphism between the values of Q and Q' on the trivial torsor which is functorial with respect to the automorphisms of the trivial torsor. In our case a natural transformation $F_{\xi'} \longrightarrow F_\xi \circ \mathrm{B}(c_\psi)$ is an isomorphism $\omega\colon \xi' \longrightarrow \xi$ (the values of the functors on the trivial torsor $\mathcal{H}_{\xi'}$) such that $c_\psi(u) = \omega u \omega^{-1}$ for all $\xi' \xrightarrow{u} \xi' \in \mathcal{H}_{\xi'}$ (that is for all automorphisms of the trivial torsor).

Given an object $z = (\mathcal{G}, F) \in \mathcal{Z}(U)$ there is a natural isomorphism making the following diagram 2-Cartesian:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{F} & \mathcal{X} \\ \downarrow & & \downarrow \Delta \\ U & \xrightarrow{z} & \mathcal{Z} \end{array}$$

For this reason we call $\Delta\colon \mathcal{X} \longrightarrow \mathcal{Z}$ the universal gerbe. The key point in proving this is that if we have a gerbe $\mathcal{G}$ over U and a section $x \in \mathcal{G}(U)$ then the functor

$$\mathcal{G} \longrightarrow \mathrm{B}(\underline{\mathrm{Aut}}_{\mathcal{G}}(x)),\ y \longmapsto \underline{\mathrm{Iso}}_{\mathcal{G}}(x,y)$$

is well defined and an equivalence. In particular if $(\mathcal{G},F) \in \mathcal{Z}(U)$ then $\underline{\mathrm{Aut}}_{\mathcal{G}}(x) \simeq \mathcal{H}_{F(x)}$ via F.

Proposition B.2. *The functor $\Delta\colon \mathcal{X} \longrightarrow \mathcal{Z}$ induces a fully faithful epimorphism $\widetilde{\mathcal{X}/\!\!/\mathcal{H}} \longrightarrow \mathcal{Z}$. In particular $\mathcal{Z}$ is a rigidification $\mathcal{X}/\!\!/\mathcal{H}$ of $\mathcal{X}$ by $\mathcal{H}$.*

Proof. Given a functor $T \longrightarrow \mathcal{Z}$ induced by $(\mathcal{G} \longrightarrow T, F) \in \mathcal{Z}(T)$ then $T \times_{\mathcal{Z}} \mathcal{X}$ is the stack of triples (f,ξ,ω) where $f\colon S \longrightarrow T$, $\xi \in \mathcal{X}(S)$ and ω is an isomorphism $(\mathrm{B}\,\mathcal{H}_\xi, F_\xi) \simeq (\mathcal{G},F)$. Denote by $\Delta\colon \mathcal{X} \longrightarrow \mathcal{Z}$ the functor and let $\xi,\xi' \in \mathcal{X}(U)$. Since $\mathcal{X} \longrightarrow \mathcal{Z}$ is clearly an epimorphism, we have to prove that

$$\underline{\mathrm{Iso}}_U(\xi',\xi) \longrightarrow \underline{\mathrm{Iso}}_U(\Delta(\xi'),\Delta(\xi))$$

is invariant by the action of $\mathcal{H}_\xi$ and an $\mathcal{H}_\xi$-torsor. Notice that a functor of the form $\mathrm{B}\,\mathcal{H}_{\xi'} \longrightarrow \mathrm{B}\,\mathcal{H}_\xi$ is locally induced by a group homomorphism $\mathcal{H}_{\xi'} \longrightarrow \mathcal{H}_\xi$. Thus it is enough to prove that if $c\colon \mathcal{H}_{\xi'} \longrightarrow \mathcal{H}_\xi$ is an isomorphism of groups and $\lambda\colon F_{\xi'} \longrightarrow F_\xi \circ \mathrm{B}\,c$ is an isomorphism then the set J of $\phi\colon \xi' \longrightarrow \xi$ inducing $(\mathrm{B}(c),\lambda)\colon \Delta(\xi') \longrightarrow \Delta(\xi)$ is non empty and $\mathcal{H}_\xi(U)$ acts transitively of this set.

The natural transformation λ evaluated on the trivial torsor $\mathcal{H}_{\xi'}$ yields an isomorphism $\phi\colon \xi' \longrightarrow \xi$. The fact that λ is a natural transformation implies that $c = c_\phi$ and $\lambda = \lambda_\phi$, that is $\phi \in J$. Now let $\xi' \xrightarrow{\psi} \xi$ be an isomorphism. A natural isomorphism $\mathrm{B}(c_\psi) \longrightarrow \mathrm{B}(c_\phi)$ is given by $h \in \mathcal{H}_\xi(U)$ (more precisely the multiplication $\mathcal{H}_\xi \longrightarrow \mathcal{H}_\xi$ by h) such that $hc_\psi(\omega) = c_\phi(\omega)h$ for all $\omega \in \mathcal{H}_{\xi'}(U)$. Such an h induces a morphism $\Delta(\psi) \longrightarrow \Delta(\phi)$ if and only if $h\psi = \phi$. Since this condition implies the previous one we see that $J = \mathcal{H}_\xi(U)\phi$. □

We denote by $\mathrm{B}_{\mathcal{X}}\,\mathcal{H}$ the stack of $\mathcal{H}$-torsors over $\mathcal{X}$ (thought of as a site). An object of $\mathrm{B}_{\mathcal{X}}\,\mathcal{H}$ is by definition an object $\xi \in \mathcal{X}(U)$ together with an $\mathcal{H}_{|\mathcal{X}/\xi}$-torsor over $\mathcal{X}/\xi$. Since $\mathcal{X}$ is fibered in groupoids the forgetful functor $\mathcal{X}/\xi \longrightarrow \mathcal{S}/U$ is an equivalence. Thus an object of $\mathrm{B}_{\mathcal{X}}\,\mathcal{H}$ is an object $\xi \in \mathcal{X}(U)$ together with a $\mathcal{H}_\xi$-torsor over U.

Proposition B.3. *We have:*

1) *given $\xi,\eta \in \mathcal{X}(U)$ we have $\underline{\mathrm{Iso}}_U(\Delta(\xi),\Delta(\eta)) \simeq \underline{\mathrm{Iso}}(\xi,\eta)/\mathcal{H}_\eta$;*
2) *the functor $\Delta\colon \mathcal{X} \longrightarrow \mathcal{X}/\!\!/\mathcal{H}$ is universal among maps of stacks $F\colon \mathcal{X} \longrightarrow \mathcal{Y}$ such that, for all $\xi \in \mathcal{X}(U)$, $\mathcal{H}_\xi$ lies in the kernel of $\underline{\mathrm{Aut}}_U(\xi) \longrightarrow \underline{\mathrm{Aut}}_U(F(\xi))$;*
3) *if*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{b} & \mathcal{X} \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \Delta} \\ \mathcal{R} & \longrightarrow & \mathcal{X}/\!\!/\mathcal{H} \end{array}$$

is a 2-Cartesian diagram of stacks then for all $\eta \in \mathcal{Y}(U)$ the map

$$\mathrm{Ker}(\underline{\mathrm{Aut}}_U(\eta) \longrightarrow \underline{\mathrm{Aut}}_U(a(\eta))) \longrightarrow \underline{\mathrm{Aut}}_U(b(\eta))$$

is an isomorphism onto $\mathcal{H}_{b(\eta)}$, so that $b^*\mathcal{H}$ is naturally a subgroup sheaf of $I(\mathcal{Y})$, and the induced map $\mathcal{Y}/\!\!/b^*\mathcal{H} \longrightarrow \mathcal{R}$ is an equivalence;

4) *there is an isomorphism $\mathcal{X} \times_{\mathcal{X}/\!\!/\mathcal{H}} \mathcal{X} \simeq \mathrm{B}_{\mathcal{X}}(\mathcal{H})$;*
5) *the map $\mathcal{X} \to \mathcal{X}/\!\!/\mathcal{H}$ is a relative gerbe (see* [Sta17, Tag 06P1]*).*

Proof. Point 1) follows from B.2, while point 2) is a direct consequence of the definition of rigidification. Now consider point 3). The kernel in the statement corresponds to the group of automorphisms of the object η in the fiber product $U \times_{\mathcal{R}} \mathcal{Y}$. Using that $\mathcal{X} \longrightarrow \mathcal{X}/\!\!/\mathcal{H}$ is the universal gerbe we get the the isomorphism in the statement. In particular there is an epimorphism $\mathcal{Y}/\!\!/b^*\mathcal{H} \longrightarrow \mathcal{R}$. This is fully faithful because, given $\eta, \eta' \in \mathcal{Y}(U)$, by definition of fiber product one get a Cartesian diagram

$$\begin{array}{ccc} \underline{\mathrm{Iso}}_U(\eta', \eta) & \longrightarrow & \underline{\mathrm{Iso}}_U(b(\eta'), b(\eta)) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Iso}}_U(a(\eta'), a(\eta)) & \longrightarrow & \underline{\mathrm{Iso}}_U(b(\eta'), b(\eta))/\mathcal{H}_{b(\eta)} \end{array}$$

For point 4), denotes by $\mathcal{Y}$ the fiber product in the statement. It is the stack of triples (ξ, ξ', ϕ) where $\xi, \xi' \in \mathcal{X}(U)$ and $\phi \in \underline{\mathrm{Iso}}_U(\xi', \xi)/\mathcal{H}_\xi$. The functor $\mathcal{Y} \longrightarrow \mathrm{B}_{\mathcal{X}}\mathcal{H}$ which maps (ξ, ξ', ϕ) to (ξ, P_ϕ), where P_ϕ is defined by the Cartesian diagram

$$\begin{array}{ccc} P_\phi & \longrightarrow & \underline{\mathrm{Iso}}_U(\xi', \xi) \\ \downarrow & & \downarrow \\ U & \xrightarrow{\phi} & \underline{\mathrm{Iso}}_U(\xi', \xi)/\mathcal{H}_\xi \end{array}$$

is an equivalence. This is because the functor $\mathcal{X} \longrightarrow \mathrm{B}_{\mathcal{X}}\mathcal{H}$ sending ξ to ξ with the trivial torsor is an epimorphism and the base change $\mathcal{Y} \times_{\mathrm{B}_{\mathcal{X}}\mathcal{H}} \mathcal{X} \longrightarrow \mathcal{X}$ is an equivalence since $\mathcal{Y} \times_{\mathrm{B}_{\mathcal{X}}\mathcal{H}} \mathcal{X}$ is the stack of triples (ξ, ξ', ψ) where $\xi, \xi' \in \mathcal{X}(U)$ and $\psi\colon \xi' \longrightarrow \xi$ is an isomorphism in $\mathcal{X}(U)$.

For point 5), since $\Delta\colon \mathcal{X} \to \mathcal{X}/\!\!/\mathcal{H}$ is an epimorphism, it has local sections. Moreover given two objects $\xi, \eta \in \mathcal{X}(U)$ and an isomorphism $\Delta(\xi) \to \Delta(\eta)$, by point 1), this isomorphism locally comes from an isomorphism $\xi \to \eta$ in $\mathcal{X}$, as required. □

Proposition B.4. *Let $\mathcal{X}$ be a stack in groupoid over $\mathcal{S}$ and $\mathcal{G}\colon \mathcal{S}^{\mathrm{op}} \to (\mathrm{Ab})$ be a sheaf of abelian groups. Then the map*

$$(\mathcal{X} \times \mathrm{B}_{\mathcal{S}}\mathcal{G})/\!\!/\mathcal{G} \to \mathcal{X}$$

is an equivalence.

Proof. Let $U \in \mathcal{S}$ be an object and $P \in \mathrm{B}_{\mathcal{S}}\mathcal{G}(U)$ a $\mathcal{G}$-torsor. Since $\mathcal{G}$ is commutative the action of $\mathcal{G}$ on P is $\mathcal{G}$-equivariant and therefore the map

$$\mathcal{G} \times U \to \underline{\mathrm{Aut}}_{\mathrm{B}_{\mathcal{S}}\mathcal{G}}(P)$$

is well defined and, checking locally, an isomorphism. This implies that $\mathcal{G}$, more precisely the restriction $(\mathcal{X} \times \mathrm{B}_{\mathcal{S}}\mathcal{G})^{\mathrm{op}} \to \mathcal{S}^{\mathrm{op}} \to (\mathrm{Ab})$, is a subgroup of the inertia stack of $(\mathcal{X} \times \mathrm{B}_{\mathcal{S}}\mathcal{G})$, thought of as a sheaf of groups. Since the functor $\mathcal{X} \times \mathrm{B}_{\mathcal{S}}\mathcal{G} \to \mathcal{X}$ kills the automorphisms in $\mathcal{G}$ we obtain the map in the statement thanks to B.3, 2). By B.3, 3) we can assume $\mathcal{X} = \mathcal{S}$. In order to show that $\mathrm{B}_{\mathcal{S}}\mathcal{G}/\!\!/\mathcal{G} \to \mathcal{S}$ is an equivalence, it is enough to show that the functor $\widetilde{\mathrm{B}_{\mathcal{S}}\mathcal{G}/\!\!/\mathcal{G}} \to \mathcal{S}$ is fully faithful. By construction, the objects of the first category are torsors $P, Q \in \mathrm{B}_{\mathcal{S}}\mathcal{G}(U)$ and the morphisms are $\underline{\mathrm{Iso}}^{\mathcal{G}}(P, Q)/\mathcal{G}$. But this is a sheaf which is locally trivial and therefore it is trivial. It follows that $\mathrm{Iso}_{\widetilde{\mathrm{B}_{\mathcal{S}}\mathcal{G}/\!\!/\mathcal{G}}}(P, Q)$ consists of just one element. □

Proposition B.5. *Let $\mathcal{X}$ be a stack in groupoid over $\mathcal{S}$ and $\mathcal{G}\colon \mathcal{S}^{\mathrm{op}} \to (\mathit{Groups})$ be a sheaf of groups. Assume there is an isomorphism between the restriction $\mathcal{G}_{\mathcal{X}}\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{S}^{\mathrm{op}} \to (\mathit{Groups})$ and the inertia $I(\mathcal{X})\colon \mathcal{X}^{\mathrm{op}} \to (\mathit{Groups})$. Then $\mathcal{G}_{\mathcal{X}}$ is a sheaf of abelian groups, $\mathcal{X}/\!\!/\mathcal{G}$ is the sheaf of isomorphism classes of $\mathcal{X}$ and*

$\mathcal{X} \to \mathcal{X}/\!\!/\mathcal{G}$ is a relative gerbe. Moreover for any object $U \in \mathcal{S}$ and map $U \to \mathcal{X}/\!\!/\mathcal{G}$ the fiber $\mathcal{X} \times_{\mathcal{X}/\!\!/\mathcal{G}} U \to U$ is locally of the form $\mathrm{B}_U \mathcal{G}_{|U} \to U$.

Proof. Let $h \in \mathcal{G}_{\mathcal{X}}(\xi)$ for $\xi \in \mathcal{X}(U)$. The naturality of the isomorphism $\mathcal{G}_{\mathcal{X}}(\xi) \to I(\mathcal{X})(\xi) = \mathrm{Aut}_{\mathcal{X}(U)}(\xi)$ on the morphism $h\colon \xi \to \xi$ exactly implies that the conjugation by h on $I(\mathcal{X})(\xi)$ is the identity. Thus $\mathcal{G}_{\mathcal{X}} \simeq I(\mathcal{X})$ is abelian. By B.3, 2) any map $\mathcal{X} \to \mathcal{F}$ to a sheaf factors through $\mathcal{X}/\!\!/\mathcal{G}$ and by B.3,1) the stack $\mathcal{X}/\!\!/\mathcal{G}$ is a actually a sheaf. This implies that $\mathcal{X}/\!\!/\mathcal{G}$ is the sheaf of isomorphism classes of $\mathcal{X}$. It is a gerbe thanks to B.3, 5). For the local form of $\mathcal{X} \to \mathcal{X}/\!\!/\mathcal{G}$ any map $U \to \mathcal{X}/\!\!/\mathcal{G}$ locally factors through $\mathcal{X}$ itself. In this case the fiber is exactly $\mathrm{B}_U \mathcal{G}_{|U} \to U$ thanks to B.3, 4). □

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