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A fixed-point approach for decaying solutions of difference equations

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A boundary value problem associated with the difference equation with advanced argument

$$\Delta\big(a_n\Phi(\Delta x_n)\big)+b_n\Phi(x_{n+p})=0,\quad n\geq 1 \qquad (*)$$

is presented, where $\Phi(u)=|u|^{\alpha}\mathrm{sgn}\ u$, $\alpha>0, p$ is a positive integer and the sequences a,b, are positive. We deal with a particular type of decaying solution of $(*)$, that is the so-called intermediate solution (see below for the definition). In particular, we prove the existence of this type of solution for $(*)$ by reducing it to a suitable boundary value problem associated with a difference equation without deviating argument. Our approach is based on a fixed-point result for difference equations, which originates from existing ones stated in the continuous case. Some examples and suggestions for future research complete the paper.

This article is part of the theme issue 'Topological degree and fixed point theories in differential and difference equations'.

1. Introduction

Consider the equation

$$\Delta\big(a_n\Phi(\Delta x_n)\big)+b_n\Phi(x_{n+p})=0, \qquad \text{(P)}$$

where Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$, Φ is the operator $\Phi(u)=|u|^{\alpha}\mathrm{sgn}\,u$, $\alpha>0$, p is a positive integer and the sequences a,b are positive for $n\geq 1$,

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and satisfy

$$\sum_{i=1}^{\infty} \Phi^*\left(\frac{1}{a_i}\right) < \infty, \quad \sum_{i=1}^{\infty} b_i = \infty, \tag{1.1}$$

where Φ^* is the inverse of the map Φ, that is $\Phi^*(u) = |u|^{1/\alpha}\operatorname{sgn} u$.

Equation (P) appears in the discretization process for searching spherically symmetric solutions of certain nonlinear elliptic equations with weighted φ-Laplacian, see e.g. [1]. A special case of (P) is the discrete half-linear equation

$$\Delta\big(a_n\Phi(\Delta y_n)\big) + b_n\Phi(y_{n+1}) = 0, \tag{H}$$

which has been studied extensively from various points of view, especially with regard to the oscillation and the qualitative behaviour of non-oscillatory solutions, see [2, ch. 3] and references therein.

As usual, a non-trivial solution x of (P) is said to be *non-oscillatory* if x_n is either positive or negative for any large n and *oscillatory* otherwise. By virtue of the Sturm separation criterion, see e.g. [2, Section 8.2.1.], all the solutions of (H) have the same behaviour with respect to the oscillation. In other words, either all non-trivial solutions of (H) are non-oscillatory or all the solutions of (H) are oscillatory. Thus, equivalently, we say that (H) is non-oscillatory if (H) has a non-oscillatory solution. Nevertheless, for equation (P) with $p \neq 1$, the situation is different, since in this case oscillatory solutions and non-oscillatory solutions may coexist.

In this paper, we deal with the existence of particular types of non-oscillatory solutions, that is solutions x of (P) such that $x_n > 0$, $\Delta x_n < 0$ for large n and

$$\lim_n x_n = 0, \quad \lim_n x_n^{[1]} = a_n\Phi(\Delta x_n) = -\infty, \tag{1.2}$$

where $x^{[1]}$ is called the quasi-difference of x. Solutions satisfying (1.2) are called *intermediate solutions*. This terminology originates from the corresponding continuous case and is due to the fact that, when (1.1) is satisfied, any non-oscillatory solution x of (P) satisfies either $\lim_n x_n = \ell_x \neq 0$, or (1.2), or $\lim_n x_n = 0$, $\lim_n x_n^{[1]} = -\ell_x$, $0 < \ell_x < \infty$, see [3,4]. The investigation of intermediate solutions is a hard problem, due to difficulties in finding suitable sharp upper and lower bounds for these solutions, see e.g. [5, p. 241] and [6, p. 3], in which these facts are pointed out for the continuous case.

In the half-linear case the problem of the existence of intermediate solutions has been completely solved by the following.

Theorem 1.1. *Assume* (*1.1*).

(i_1) *Equation* (H) *does not have intermediate solutions if*

$$\sum_{n=1}^{\infty} b_n\Phi\left(\sum_{k=n+1}^{\infty}\Phi^*\left(\frac{1}{a_k}\right)\right) + \sum_{n=1}^{\infty}\Phi^*\left(\frac{1}{a_{n+1}}\sum_{k=1}^{n} b_k\right) < \infty.$$

(i_2) *Equation* (H) *has intermediate solutions if* (H) *is non-oscillatory and*

$$\sum_{n=1}^{\infty} b_n\Phi\left(\sum_{k=n+1}^{\infty}\Phi^*\left(\frac{1}{a_k}\right)\right) + \sum_{n=1}^{\infty}\Phi^*\left(\frac{1}{a_{n+1}}\sum_{k=1}^{n} b_k\right) = \infty.$$

Proof. The assertion follows from theorem 3.1 (b), (c) and theorem 3.2. in [7], with minor changes. ∎

Here, we present a comparison result which allows us to solve the boundary value problem (BVP)

$$\begin{cases} \Delta(a_n\Phi(\Delta x_n)) + b_n\Phi(x_{n+p}) = 0, \quad p > 1, \\ \lim_n x_n = 0, \ \lim_n x_n^{[1]} = -\infty \end{cases} \tag{1.3}$$

by reducing it to the existence of intermediate solutions in the half-linear case. The main result is the following.

Theorem 1.2. *Assume* (1.1) *and*

$$\limsup_n b_n < \infty. \tag{1.4}$$

Then the BVP (1.3) *is solvable, i.e. equation* (P) *with* $p > 1$ *has intermediate solutions, if and only if the half-linear equation*

$$\Delta\big(a_{n+p-1}\Phi(\Delta y_n)\big) + b_n\Phi(y_{n+1}) = 0, \tag{H1}$$

has intermediate solutions.

Theorem 1.2 will be proved by means of a fixed-point result for discrete operators acting in Fréchet spaces, see theorem 2.2.

Combining theorem 1.1 and the comparison theorem 1.2, we get necessary and sufficient conditions for the existence of intermediate solutions of difference equation with advanced argument.

Corollary 1.3. *Assume* (1.1) *and* (1.4).

(i_1) *Equation* (P) *with* $p > 1$ *has intermediate solutions if* (H1) *is non-oscillatory and*

$$\sum_{n=1}^{\infty} b_n\Phi\left(\sum_{k=n+1}^{\infty}\Phi^*\left(\frac{1}{a_{k+p-1}}\right)\right) + \sum_{n=1}^{\infty}\Phi^*\left(\frac{1}{a_{n+p}}\sum_{k=1}^{n} b_k\right) = \infty. \tag{1.5}$$

(i_2) *Equation* (P) *with* $p > 1$ *does not have intermediate solutions if*

$$\sum_{n=1}^{\infty} b_n\Phi\left(\sum_{k=n+1}^{\infty}\Phi^*\left(\frac{1}{a_{k+p-1}}\right)\right) + \sum_{n=1}^{\infty}\Phi^*\left(\frac{1}{a_{n+p}}\sum_{k=1}^{n} b_k\right) < \infty.$$

2. Fixed-point approaches

Boundary value problems for difference equations in $\mathbb{R}^n$ are often solved by reducing the problem to a fixed-point equation for a possibly nonlinear operator in a suitable function space. Thus, the existence of a solution is obtained by applying a fixed-point theorem, for instance the Tychonoff theorem, the Schauder theorem, the Leray–Schauder continuation principle or Krasnoselkii-type fixed point theorems on cones. For a survey on this topic, we refer to the papers [8–10] and the monographies [11,12]. In particular, in [12, ch. 2], see also [11, ch. 5], certain BVPs are studied by means of a nonlinear Leray–Schauder alternative. This approach is based on a very general method given in [13]. In particular, in [13] the authors present a Leray–Schauder continuation principle in locally convex topological vector spaces, which unifies the Leray–Schauder alternative theorem and the Tichonov fixed-point theorem. More precisely, let $\mathbb{E}$ be a Hausdorff locally convex topological vector space with a family of seminorms generating the topology. The following holds.

Theorem 2.1 ([13, [Theorem 1.1]]). *Let Q be a convex closed subset of $\mathbb{E}$ and let $T : Q \times [0,1] \to \mathbb{E}$ be a continuous map with relatively compact image. Assume that:*

(i_1) $T(x, 0) \in Q$ *for any* $x \in Q$;

(i_2) for any $(x,\lambda) \in \partial Q \times [0,1)$ with $T(x,\lambda) = x$ there exists open neighbourhoods U_x of x in $\mathbb{E}$ and I_λ of λ in $[0,1)$ such that

$$T\Big((U_x \cap \partial Q) \times I_\lambda\Big) \subset Q.$$

Then the equation

$$x = T(x,1)$$

has a solution.

Some of the above quoted results have a discrete counterpart. For instance, a method for solving BVPs associated with difference systems is given in [10, theorem 2.1]. Due to the peculiarities of the discrete case, it may be applied to functional difference equations, including equations with deviating arguments or sum difference equations.

Now, we present an existence result which generalizes, in the particular case of scalar difference equations, [10, theorem 2.1].

Denote by $\mathbb{N}_n$ and $\mathbb{N}_{m,n}$, the sets

$$\mathbb{N}_n = \{i \in \mathbb{N} : i \geq n \in \mathbb{N}\}$$

and

$$\mathbb{N}_{m,n} = \{i \in \mathbb{N}_m : i < n,\ m, n \in \mathbb{N}, m < n\}$$

and let $\mathbb{X}$ be the space of all real sequences defined on $\mathbb{N}_m$. Hence $\mathbb{X}$ is a Frechét space with the topology of pointwise convergence on $\mathbb{N}_m$. From the discrete Arzelà–Ascoli theorem (e.g. [11, theorem 5.3.1]), any bounded set in $\mathbb{X}$ is relatively compact. We recall that a set $\Omega \subset \mathbb{X}$ is bounded if and only if it consists of sequences which are equibounded on $\mathbb{N}_{m,n}$ for any $n > m$. Clearly, if $\Omega \subset \mathbb{X}$ is bounded, then $\Omega^\Delta = \{\Delta u,\ u \in \Omega\}$ is bounded, too.

Using, with minor changes, a discrete counterpart of a compactness and continuity result stated in [14, theorem 1.3] for the continuous case, we have the following.

Theorem 2.2. *Consider the BVP*

$$\begin{cases} \Delta(a_n \Phi(\Delta x_n)) = g(n,x), & n \in \mathbb{N}_m \\ x \in S, \end{cases} \tag{2.1}$$

where $g : \mathbb{N}_m \times \mathbb{X} \to \mathbb{R}$ is a continuous map, and S is a subset of $\mathbb{X}$.
Let $G : \mathbb{N}_m \times \mathbb{X}^2 \to \mathbb{R}$ be a continuous map such that $G(k,u,u) = g(k,u)$ for all $(k,u) \in \mathbb{N}_m \times \mathbb{X}$. If there exist a non-empty, closed, convex set $\Omega \subset \mathbb{X}$, and a bounded, closed subset $S_C \subset S \cap \Omega$ such that the problem

$$\begin{cases} \Delta(a_n \Phi(\Delta x_n)) = G(n,x,q), & n \in \mathbb{N}_m \\ x \in S_C, \end{cases} \tag{2.2}$$

has a unique solution for any $q \in \Omega$ fixed, then (2.1) has at least a solution.

Proof. Let T be the operator $T : \Omega \to S_C$ which maps every $q \in \Omega$ into the unique solution $x = T(q)$ of (2.2). Let us show that the operator T is continuous with relatively compact image. The relatively compactness of $T(\Omega)$ follows immediately since S_C is bounded. To prove the continuity of T in Ω, let $\{q^j\}$ be a sequence in Ω, $q^j \to q^\infty \in \Omega$, and let $v^j = T(q^j)$. Since $T(\Omega)$ is relatively compact, $\{v^j\}$ admits a subsequence (still indicated with $\{v^j\}$) which converges to $v^\infty \in \mathbb{X}$. As $v^j \in S_C$ and S_C is closed, then $v^\infty \in S_C$. Taking into account the continuity of G, we obtain

$$\Delta(a_n \Phi(\Delta v_k^\infty)) = \lim_j \Delta(a_n \Phi(\Delta v_k^j)) = \lim_j G(k, q^j, v^j) = G(k, q^\infty, v^\infty).$$

The uniqueness of the solution of (2.2) yields $v^\infty = T(q^\infty)$, and therefore T is continuous on Ω. By the Tychonoff fixed-point theorem, T has at least one fixed point in Ω, which is a solution of (2.1), as it can be easily checked, taking into account that $S_C \subset S$. ■

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As follows from the proof of theorem 2.2, no explicit form of the fixed-point operator is needed for the solvability of (2.1). A key point for the unique solvability of (2.2) in theorem 2.2 is the choice of the map G. To this aim, in our opinion, the best cases are the following two, namely

$$(i_1)\ \ G(n,q,x)=\tilde{g}(n,q), \qquad (i_2)\ \ G(n,q,x)=\tilde{g}(n,q)\Phi(x_{n+1}).$$

In the case (i_2), the equation in (2.2) is a half-linear equation and in this situation a very large variety of results is known, see e.g. [2, ch. VIII]. An application in this direction is in [10, Section 4].

In the case (i_1), that is when the function G does not depend on x and the equation in (2.2) is affine, theorem 2.2 can be particularly useful to solve BVPs associated to difference equations with deviating arguments. Indeed, in this case, it can lead to a BVP associated with a second-order difference equation without deviating argument. An application of this fact is in the following section.

3. Proof of theorem 1.2

For proving theorem 1.2, the following auxiliary result is needed.

Lemma 3.1. *Assume (1.1) and (1.4). Let x be an eventually positive decreasing solution of* (P) *such that* $\lim_n x_n=0$. *Then the series*

$$\sum_{i=2}^{\infty}\Phi^*\left(\frac{1}{a_{i+p-1}}\sum_{k=1}^{i-1}b_k\Phi(x_{k+p})\right)$$

converges.

Proof. Without loss of generality, suppose $0<x_n\le 1$, $\Delta x_n<0$ for $n\ge m_0\ge 1$. We claim that for any $k,j\ge m_0$ we have

$$\left|x_j^{[1]}-x_k^{[1]}\right|\le B|j-k|, \tag{3.1}$$

where $B=\sup_{i\ge 1}b_i$. For simplicity, suppose $k\ge j$. Summing (P) we have

$$x_k^{[1]}-x_j^{[1]}=-\sum_{i=j}^{k-1}b_i\Phi(x_{i+p}), \tag{3.2}$$

where, as usual, $\sum_{i=k_1}^{k}\gamma_i=0$ if $k<k_1$. Since $|x_i|\le 1$ and $b_i\le B$, the inequality (3.1) follows. Further, from (3.2),

$$\begin{aligned}\sum_{i=m_0}^{\infty}\Phi^*\left(\frac{1}{a_{i+p-1}}\sum_{k=m_0}^{i-1}b_k\Phi(x_{k+p})\right)&=\sum_{i=m_0}^{\infty}\Phi^*\left(\frac{1}{a_{i+p-1}}\left(x_{m_0}^{[1]}-x_i^{[1]}\right)\right)\\&=\sum_{i=m_0}^{\infty}\Phi^*\left(\frac{1}{a_{i+p-1}}\left(x_{m_0}^{[1]}-x_i^{[1]}-x_{i+p-1}^{[1]}+x_{i+p-1}^{[1]}\right)\right).\end{aligned}$$

Since

$$\left|x_i^{[1]}-x_{i+p-1}^{[1]}+x_{i+p-1}^{[1]}-x_{m_0}^{[1]}\right|\le\left|x_i^{[1]}-x_{i+p-1}^{[1]}\right|+\left|x_{i+p-1}^{[1]}\right|+\left|x_{m_0}^{[1]}\right|,$$

in view of (3.1) we obtain

$$\left|\sum_{i=m_0}^{\infty}\Phi^*\left(\frac{1}{a_{i+p-1}}\left(x_i^{[1]}-x_{m_0}^{[1]}\right)\right)\right|\le\sum_{i=m_0}^{\infty}\Phi^*\left(\frac{B(p-1)+\left|x_{m_0}^{[1]}\right|}{a_{i+p-1}}+\frac{\left|x_{i+p-1}^{[1]}\right|}{a_{i+p-1}}\right).$$

In virtue of (1.1), the series

$$\sum_{i=m_0}^{\infty}\Phi^*\left(\frac{B(p-1)+\left|x_{m_0}^{[1]}\right|}{a_{i+p-1}}\right)$$

converges. Since

$$\sum_{i=m_0}^{\infty} \Phi^*\left(\frac{\left|x_{i+p-1}^{[1]}\right|}{a_{i+p-1}}\right) = -\sum_{i=m_0}^{\infty} \Delta x_{i+p-1} = x_{m_0+p-1},$$

using (1.1) and the inequality

$$\Phi^*(X+Y) \le \sigma_\alpha(\Phi^*(X)+\Phi^*(Y)),$$

where

$$\sigma_\alpha = \begin{cases} 1 & \text{if } \alpha \ge 1 \\ 2^{(1-\alpha)/\alpha} & \text{if } \alpha < 1 \end{cases},$$

we obtain the assertion. ■

Proof of theorem 1.2. First, we prove that if (P) has intermediate solutions, then (H1) has intermediate solutions.

Let x be an intermediate solution of (P) and, without loss of generality, assume for $n \ge n_0 \ge 1$

$$0 < x_n < 1, \quad \Delta x_n < 0. \tag{3.3}$$

In view of (1.4), there exists $L > 0$ such that for any $n \ge n_0$

$$\sum_{i=n}^{n+p-2} b_i \le L. \tag{3.4}$$

Moreover, let M be a positive constant, $M < 1$, such that

$$\Phi(M) \le \frac{\left|x_{n_0}^{[1]}\right|}{L+\left|x_{n_0}^{[1]}\right|}. \tag{3.5}$$

Let $\mathbb{X}$ be the Fréchet space of real sequences defined for $n \ge n_0$, endowed with the topology of convergence on $\mathbb{N}_{n_0}$, and consider the subset $\Omega \subset \mathbb{X}$ defined by

$$\Omega = \left\{u \in \mathbb{X} : Mx_{n+p-1} \le u_n \le x_{n+p-1}\right\}.$$

For any $u \in \Omega$ consider the BVP

$$\begin{cases} \Delta\left(a_{n+p-1}\Phi(\Delta z_n)\right) + b_n\Phi(u_{n+1}) = 0, & n \ge n_0 \\ z_{n_0}^{[1]} = x_{n_0}^{[1]}, & \lim_n z_n = 0, \end{cases} \tag{3.6}$$

where $z^{[1]}$ denotes the quasi-difference of z, that is

$$z_n^{[1]} = a_{n+p-1}\Phi(\Delta z_n). \tag{3.7}$$

For any $u \in \Omega$ we have

$$\sum_{k=n_0}^{n} b_k\Phi(u_{k+1}) \le \sum_{k=n_0}^{n} b_k\Phi(x_{k+p}).$$

Hence, using lemma 3.1, we have

$$\lim_n \sum_{i=n}^{\infty} \Phi^*\left(\frac{1}{a_{i+p-1}}\left(\sum_{k=n_0}^{i-1} b_k\Phi(u_{k+1})\right)\right) = 0.$$

Thus, a standard calculation shows that for any $u \in \Omega$ the BVP (3.6) has the unique solution z. Let $\mathcal{T}$ be the operator which associates to any $u \in \Omega$ the unique solution z of (3.6).

Summing the equation in (3.6) and using (3.2) we get

$$z_n^{[1]} = x_{n_0}^{[1]} - \sum_{k=n_0}^{n-1} b_k \Phi(u_{k+1}) \geq x_{n_0}^{[1]} - \sum_{k=n_0}^{n-1} b_k \Phi(x_{k+p}) = x_n^{[1]}.$$

Since $x^{[1]}$ is decreasing for $n \geq n_0$ and $p > 1$, we obtain for $n \geq n_0$

$$z_n^{[1]} \geq x_{n+p-1}^{[1]} \tag{3.8}$$

i.e. in view of (3.7),

$$a_{n+p-1}\Phi(\Delta z_n) \geq a_{n+p-1}\Phi(\Delta x_{n+p-1}),$$

that is,

$$\Delta z_n \geq \Delta x_{n+p-1}.$$

Since $\lim_i z_i = \lim_i x_i = 0$, we obtain for $n \geq n_0$

$$z_n \leq x_{n+p-1}. \tag{3.9}$$

Now, let us prove that for $n \geq n_0$

$$z_n \geq M x_{n+p-1}. \tag{3.10}$$

Summing the equation in (3.6) and using (3.7) we get

$$z_n^{[1]} = x_{n_0}^{[1]} - \sum_{k=n_0}^{n-1} b_k \Phi(u_{k+1}) \leq x_{n_0}^{[1]} - \Phi(M) \sum_{k=n_0}^{n-1} b_k \Phi(x_{k+p}),$$

or, using (3.2),

$$\begin{aligned} z_n^{[1]} &\leq x_{n_0}^{[1]} + \Phi(M)\left(x_n^{[1]} - x_{n_0}^{[1]}\right) \\ &= \Phi(M) x_{n+p-1}^{[1]} + \Phi(M)\left(x_n^{[1]} - x_{n+p-1}^{[1]}\right) + (1 - \Phi(M)) x_{n_0}^{[1]}. \end{aligned} \tag{3.11}$$

From (3.2)–(3.4), we also have

$$x_n^{[1]} - x_{n+p-1}^{[1]} = \sum_{i=n}^{n+p-2} b_i \Phi(x_{i+p}) \leq \sum_{i=n}^{n+p-2} b_i \leq L.$$

Thus, from (3.11) we obtain

$$z_n^{[1]} \leq \Phi(M) x_{n+p-1}^{[1]} + \left(L + \left|x_{n_0}^{[1]}\right|\right)\Phi(M) + x_{n_0}^{[1]}. \tag{3.12}$$

In the view of (3.5) we have

$$\left(L + \left|x_{n_0}^{[1]}\right|\right)\Phi(M) + x_{n_0}^{[1]} \leq 0.$$

Hence, from (3.12) we get for $n \geq n_0$

$$z_n^{[1]} \leq \Phi(M) x_{n+p-1}^{[1]} \tag{3.13}$$

or, in view of (3.7), $\Delta z_n \leq M\,\Delta x_{n+p-1}$, and (3.10) follows, since $\lim_i z_i = \lim_i x_i = 0$. Thus, in virtue of (3.9) and (3.10), the operator $\mathcal{T}$ maps Ω into itself, that is

$$\mathcal{T}(\Omega) \subset \Omega.$$

Denote by S the boundary conditions in (3.6), i.e.

$$S = \left\{v \in \mathbb{X} : a_{n_0+p-1}\Phi(\Delta v_{n_0}) = x_{n_0}^{[1]},\ \lim_n v_n = 0\right\}$$

For any $z \in \mathcal{T}(\Omega)$ we have $z \in S$. Since $\mathcal{T}(\Omega) \subset \Omega$, we get $z \in \Omega \cap S$.

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Denote by S_C the subset of $\mathbb{X}$ given by

$$S_C = S \cap \Omega.$$

Since $\lim_n x_n = 0$, it holds

$$S_C = \left\{ v \in \mathbb{X} : a_{n_0+p-1}\Phi(\Delta v_{n_0}) = x_{n_0}^{[1]},\ Mx_{n+p-1} \le v_n \le x_{n+p-1} \right\}.$$

Thus S_C is a bounded and closed subset of $\mathbb{X}$. Applying theorem 2.2 we obtain that the operator $\mathcal{T}$ has a fixed point $\bar{z} \in S_C$. Clearly the sequence $\bar{z}$ is a solution of (H1) and $\lim_n \bar{z}_n = 0$. Since $\bar{z} \in \mathcal{T}(\Omega)$, from (3.8) and (3.13), we get

$$x_{n+p-1}^{[1]} \le \bar{z}_n^{[1]} \le \Phi(M) x_{n+p-1}^{[1]}$$

and so $\lim_n \bar{z}_n^{[1]} = -\infty$. Hence $\bar{z}$ is an intermediate solution of (H1).

Now, we prove the opposite, that is if (H1) has intermediate solutions, then (P) has intermediate solutions.

The argument is similar to the one given above, with minor changes. Let y be an intermediate solution of (H1) such that for $n \ge n_0 \ge 1$

$$0 < y_n < 1, \quad \Delta y_n < 0,$$

and define

$$n_1 = n_0 + p.$$

In view of (1.4), there exists $\Lambda > 0$ such that for any $n \ge n_1$

$$\sum_{i=n-p+1}^{n-1} b_i \le \Lambda. \tag{3.14}$$

Without loss of generality, we can suppose

$$\Lambda < \left| y_{n_1}^{[1]} \right|, \tag{3.15}$$

where $y_{n_1}^{[1]} = a_{n_1+p-1}\Phi(\Delta y_{n_1})$. Moreover, let $H > 1$ be a positive constant such that

$$\Phi(H) \ge \frac{\left| y_{n_1}^{[1]} \right|}{\left| y_{n_1}^{[1]} \right| - \Lambda}. \tag{3.16}$$

Let $\mathbb{X}_1$ be the Fréchet space of the real sequences defined for $n \ge n_1$, endowed with the topology of convergence on $\mathbb{N}_{n_1}$. Define the subset Ω_1 of $\mathbb{X}_1$

$$\Omega_1 = \left\{ u \in \mathbb{X}_1 : y_{n-p+1} \le u_n \le H y_{n-p+1} \right\}$$

and for any $u \in \Omega_1$ consider the BVP

$$\begin{cases} \Delta\big(a_n \Phi(\Delta w_n)\big) + b_n \Phi(u_{n+p}) = 0, & n \ge n_1 \\ w_{n_1}^{[1]} = y_{n_1}^{[1]},\ \lim_n w_n = 0, \end{cases} \tag{3.17}$$

where $w^{[1]}$ and $y^{[1]}$ denote the quasi-differences of w and y, respectively, that is the sequences

$$w_n^{[1]} = a_n \Phi(\Delta w_n), \quad y_n^{[1]} = a_{n+p-1}\Phi(\Delta y_n). \tag{3.18}$$

As before, for any $u \in \Omega_1$ the BVP (3.17) has a unique solution w. Thus, let $\mathcal{T}$ be the operator which associates with any $u \in \Omega_1$ the unique solution w of (3.17). Summing the equation in (3.17) and using (3.18) we get

$$w_n^{[1]} = y_{n_1}^{[1]} - \sum_{k=n_1}^{n-1} b_k \Phi(u_{k+p}) \le y_{n_1}^{[1]} - \sum_{k=n_1}^{n-1} b_k \Phi(y_{k+1}) = y_n^{[1]}. \tag{3.19}$$

Since $y^{[1]}$ is decreasing for $n \ge n_0$ and $p > 1$, we have $y_n^{[1]} \le y_{n-p+1}^{[1]}$ for $n \ge n_1$, and from (3.19) we obtain

$$w_n^{[1]} \le y_{n-p+1}^{[1]} \tag{3.20}$$

i.e.

$$\Delta w_n \le \Delta y_{n-p+1},$$

which implies

$$w_n \ge y_{n-p+1}, \tag{3.21}$$

since $\lim_i w_i = \lim_i y_i = 0$.

Now, let us prove that

$$w_n \le H\, y_{n-p+1}. \tag{3.22}$$

Summing the equation in (3.17) we get

$$w_n^{[1]} = y_{n_1}^{[1]} - \sum_{k=n_1}^{n-1} b_k \Phi(u_{k+p}) \ge y_{n_1}^{[1]} - \Phi(H) \sum_{k=n_1}^{n-1} b_k \Phi(y_{k+1}),$$

that is

$$\begin{aligned} w_n^{[1]} &\ge y_{n_1}^{[1]} + \Phi(H)\left(y_n^{[1]} - y_{n_1}^{[1]}\right) \\ &= \Phi(H) y_{n-p+1}^{[1]} + \left(y_n^{[1]} - y_{n-p+1}^{[1]}\right)\Phi(H) + (1 - \Phi(H)) y_{n_1}^{[1]}. \end{aligned} \tag{3.23}$$

From (H 1) and (3.14) we have

$$y_n^{[1]} - y_{n-p+1}^{[1]} = -\sum_{i=n-p+1}^{n-1} b_i \Phi(y_{i+1}) \ge -\Lambda.$$

Thus, from (3.23) we obtain

$$w_n^{[1]} \ge \Phi(H) y_{n-p+1}^{[1]} - \Lambda\Phi(H) + y_{n_1}^{[1]} - \Phi(H) y_{n1}^{[1]} = \Phi(H) y_{n-p+1}^{[1]} + \left(\left|y_{n_1}^{[1]}\right| - \Lambda\right)\Phi(H) + y_{n_1}^{[1]},$$

or, in view of (3.15) and (3.16),

$$w_n^{[1]} \ge \Phi(H) y_{n-p+1}^{[1]} \tag{3.24}$$

i.e.

$$\Delta w_n \ge H\, \Delta y_{n-p+1}.$$

Since $\lim_i w_i = \lim_i y_i = 0$, from here we get (3.22). Hence, in virtue of (3.21) and (3.22), the operator $\mathcal{T}$ maps Ω_1 into itself, i.e.

$$\mathcal{T}(\Omega_1) \subset \Omega_1.$$

Using the same argument to the one given in the sufficiency part, denote by S_1 the boundary conditions in (3.17). Applying theorem 2.2 with $S_C = S_1 \cap \Omega_1$, we get that the operator $\mathcal{T}$ has a fixed point $\overline{w} \in S_1 \cap \Omega_1$. Clearly the sequence $\overline{w}$ is a solution of (P) and $\lim_n \overline{w}_n = 0$. Since $\overline{w} \in \mathcal{T}(\Omega_1)$, from (3.20) and (3.24), we get

$$H y_{n-p+1}^{[1]} \le \overline{w}_n^{[1]} \le y_{n-p+1}^{[1]}$$

and so $\lim_n \overline{w}_n^{[1]} = -\infty$.

Hence $\overline{w}$ is an intermediate solution of (P) and the proof is complete. ■

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4. Suggestions and examples

The following example illustrates theorem 1.2 and corollary 1.3.

Example 4.1. Consider the difference equation with advanced argument

$$\Delta\big((n-p+1)^{1+\alpha}\Phi(\Delta x_n)\big)+\gamma\Phi(x_{n+p})=0,\quad n\geq p\geq 2, \tag{4.1}$$

where γ is a positive constant. Using theorem 1.2 and corollary 1.3, it is easy to show that (4.1) has intermediate solutions if and only if

$$0<\gamma\leq\left(\frac{1}{1+\alpha}\right)^{\alpha+1}. \tag{4.2}$$

Indeed, consider the half-linear equation

$$\Delta\big(n^{1+\alpha}\Phi(\Delta x_n)\big)+\gamma\Phi(x_{n+1})=0. \tag{4.3}$$

A standard calculation shows that (1.1) is satisfied. Moreover, using the change of variable

$$y_n=n^{1+\alpha}\Phi(\Delta x_n) \tag{4.4}$$

the equation (4.3) is transformed into the generalized discrete Euler equation

$$\Delta\big(\Phi^*(\Delta y_n)\big)+\gamma^{1/\alpha}\left(\frac{1}{n+1}\right)^{(1+\alpha)/\alpha}\Phi^*(y_{n+1})=0,$$

which is non-oscillatory if γ satisfies (4.2) and oscillatory if

$$\gamma>\left(\frac{1}{1+\alpha}\right)^{\alpha+1}, \tag{4.5}$$

see e.g. [15]. Since the transformation (4.4) maintains the oscillatory behaviour, see e.g. [16], we get that (4.3) is non-oscillatory if and only if (4.2) is satisfied. Moreover, as $a_n=n^{1+\alpha}$, $b_n=\gamma$, we have

$$\sum_{n=p}^{\infty}\Phi^*\left(\frac{1}{a_{n+p}}\sum_{k=p}^{n}b_k\right)=\gamma^{1/\alpha}\sum_{n=p}^{\infty}\left(\frac{n}{n+p}\right)^{1/\alpha}\frac{1}{n+p}=\infty$$

and so the condition (1.5) is satisfied. Hence, from corollary 1.3–i_1) equation (4.1) has intermediate solutions if (4.2) is satisfied. When (4.5) holds, as claimed, the half-linear equation (4.3) is oscillatory and it does not admit intermediate solutions. Thus, from theorem 1.2, equation (4.1) does not have intermediate solutions, either.

The existence of intermediate solutions for (P) does not depend on condition (1.4), as the following example shows.

Example 4.2. Consider the difference equation with advanced argument

$$\Delta\big((n+1)!\,\Delta x_n\big)+(n+p)!\,x_{n+p}=0,\quad p\geq 2. \tag{4.6}$$

A direct computation shows that $x=\{1/n!\}$ is an intermediate solution of (4.6). Nevertheless, for (4.6), assumption (1.4) is not verified and theorem 1.2 cannot be applied. Hence, it is an open problem if theorem 1.2 continues to hold when condition (1.4) failed.

Example 4.1 suggests the following two comparison results.

royalsocietypublishing.org/journal/rsta *Phil. Trans. R. Soc. A* **379**: 20190374

Corollary 4.3. *Assume* (1.1) *and* (1.4). *Suppose that*

$$a_n \geq n^{1+\alpha}, \quad \text{and} \quad \sum_{n=N}^{\infty} \Phi^*\left(\frac{n}{a_{n+p}}\right) = \infty,$$

where $N \geq p$. *Set* $L = \sup_{n \geq N} b_n$. *If* b *is bounded away from zero and*

$$L < \left(\frac{1}{1+\alpha}\right)^{1+\alpha},$$

then the equation

$$\Delta\big((n-p+1)^{1+\alpha}\Phi(\Delta x_n)\big) + b_n \Phi(x_{n+p}) = 0, \quad n \geq p \geq 2, \tag{4.7}$$

has intermediate solutions.

Proof. Consider the equation

$$\Delta\big(n^{1+\alpha}\Phi(\Delta x_n)\big) + L\Phi(x_{n+1}) = 0. \tag{4.8}$$

Reasoning as in example 4.1, we get that (4.8) is non-oscillatory. Hence, in virtue of the Sturm comparison theorem, see e.g. [2, ch. 8.2], the half-linear equation (H) is non-oscillatory. Moreover, since b is bounded away from zero, there exists $\varepsilon > 0$ such that $b_n \geq \varepsilon$ for any $n \geq 1$. Hence

$$\sum_{n=N}^{\infty} \Phi^*\left(\frac{1}{a_{n+1}} \sum_{k=1}^{n} b_k\right) \geq \Phi^*(\varepsilon) \sum_{n=N}^{\infty} \Phi^*\left(\frac{n}{a_{n+1}}\right) = \infty,$$

which implies (1.5). Hence, applying corollary 1.3–i_1) to the equation (4.7), we get the assertion. ■

Corollary 4.4. *Assume* (1.1) *and* (1.4). *Moreover, suppose that for* $n \geq N \geq p$,

$$a_n \leq n^{1+\alpha}.$$

Set $\ell = \inf_{n \geq N} b_n$. *If*

$$\ell > \left(\frac{1}{1+\alpha}\right)^{1+\alpha},$$

then the equation (4.7) *does not have intermediate solutions.*

Proof. The argument is similar to the one given in corollary 4.3. Consider the half-linear equation

$$\Delta\big(n^{1+\alpha}\Phi(\Delta x_n)\big) + \ell\, \Phi(x_{n+1}) = 0. \tag{4.9}$$

Reasoning as before, we get that (4.9) is oscillatory. Hence, in virtue of the Sturm comparison theorem, also (H) is oscillatory, and so (H) does not have intermediate solutions. Thus, applying theorem 1.2 to the equation (H), we obtain the assertion. ■

Some suggestions for future research are in order.

(1) As claimed, theorem 2.1 from [13] gives a very general fixed point result, which is based on a continuation principle in a Hausdorff locally convex space. Further, in [13] the solvability to certain BVPs in the continuous case is also given. It should be interesting to establish corresponding discrete versions of these existence results, especially for [13, theorem 2.2], which deals with a scalar equation. It could be useful for studying discrete BVPs when it is hard to find an appropriate bounded closed set Ω which is mapped into itself, as, for instance occurs for intermediate solutions to Emden–Fowler superlinear discrete equation

$$\Delta(a_n|\Delta x_n|^{\alpha} \operatorname{sgn} \Delta x_n) + b_n|x_{n+1}|^{\beta} \operatorname{sgn} x_{n+1} = 0, \quad \alpha < \beta,$$

see, e.g. [7, Section V.].

(2) The proof of theorem 1.2 does not work if $p \leq 0$. Indeed, in this case the half-linear equation (H 1) is not defined, due to the shift in the weight coefficient a of the discrete operator

$$(Dx)_n = \Delta(a_n \Phi(\Delta x_n)).$$

Further, when $p < 0$, the sequence $\{u_{n+p}\}$ in BVP (3.17) has to be defined not only for $n \geq n_1$, but also for any $i \geq n_1 + p$. Consequently, when (P) is an equation with delay, the solvability of (1.3) requires a different approach, which will be presented in a forthcoming paper [17].

(3) Theorem 1.2 establishes a comparison between the asymptotic decay of intermediate solutions of (P) with the one of an associated half-linear equation. Recently, in some particular cases, a precise asymptotic analysis of intermediate solutions for discrete half-linear equations has been made in the framework of regular variation, see [18]. It should be interesting to apply this approach for obtaining a precise description of the asymptotic behaviour also for intermediate solutions of the equations with advanced argument.

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