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Interior gradient bounds for nonlinear elliptic systems

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Abstract. *This manuscript is dedicated to Umberto Mosco, with esteem and affection. Umberto was my mentor at the University of Rome, where I completed my four years studies in Mathematics before my PhD program in Pisa. I dedicate to him the article, which is divided in two parts. In the first section I propose some regularity theorems, precisely some interior bounds for the gradient of weak solutions to a class of nonlinear elliptic systems; the title of this manuscript takes its origin from this section. The second part of the manuscript deals with my first studies in Rome together with Umberto Mosco and with my next studies in Pisa where I met Ennio De Giorgi and where I had the good fortune of assisting to the birth of the $G-$convergence and the $\Gamma-$convergence theories, with some connections with the Mosco's convergence.*

Contents

1. Nonlinear elliptic systems in divergence form

The *nonlinear elliptic system* that we take under considerations has the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^{\alpha}\left(Du\left(x\right)\right) = 0, \qquad \alpha = 1, 2, \ldots, m, \tag{1.1}$$

and it consists of $m \geq 1$ partial differential equations in an open set $\Omega \subset \mathbb{R}^n$, for some $n \geq 2$. The map $u = u(x)$ is defined for $x = (x_i) \in \Omega \subset \mathbb{R}^n$ and takes values in $\mathbb{R}^m$. The symbol Du represents the $m \times n$ gradient-matrix of the map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$; i.e. the matrix $Du = (\frac{\partial u^{\alpha}}{\partial x_i})_{i=1,2,\ldots,n}^{\alpha=1,2,\ldots,m}$ of the partial derivatives of u.

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We assume that the vector field $A(\xi) = (a_i^\alpha(\xi))_{i=1,2,\ldots,n}^{\alpha=1,2,\ldots,m}$, $A\colon \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is of class C^1 and that it satisfies the *ellipticity condition*

$$\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{m} \frac{\partial a_i^\alpha(\xi)}{\partial \xi_j^\beta}\lambda_i^\alpha \lambda_j^\beta > 0, \qquad \forall\, \lambda, \xi \in \mathbb{R}^{m\times n} : \lambda \neq 0.$$

As often happens in the context of the Calculus of Variations, we consider the vector field $A(\xi)$ to be the *gradient* of a real function $f(\xi)$; i.e., there exists a function $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ of class $C^2(\mathbb{R}^{m\times n})$ (it is sufficient the weaker assumption $f \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^{m\times n})$) such that $A(\xi) = D_\xi f(\xi)$; in terms of components $a_i^\alpha = \dfrac{\partial f}{\partial \xi_i^\alpha} = f_{\xi_i^\alpha}$, for all $\alpha = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$. Under this variational condition, the previous ellipticity condition can be equivalently written in the form

$$\sum_{i,j,\alpha,\beta} \frac{\partial^2 f(\xi)}{\partial \xi_i^\alpha \partial \xi_j^\beta}\lambda_i^\alpha \lambda_j^\beta > 0, \qquad \forall\, \lambda, \xi \in \mathbb{R}^{m\times n} : \lambda \neq 0.$$

Thus the ellipticity condition of the system is equivalent to the positivity on $\mathbb{R}^{m\times n}$ of the quadratic form of the second derivatives $D_\xi^2 f(\xi)$, which implies the (strict) *convexity* of the function f.

In this case any *weak solution* (in a class of Sobolev maps u to be defined) to the differential elliptic system (1.1) is a *minimizer* to the *energy integral*

$$F(u) = \int_\Omega f(Du)\,dx\,. \tag{1.2}$$

That is, the map $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ satisfies the inequality

$$\int_\Omega f(Du)\;dx \le \int_\Omega f(D(u+\varphi))\;dx$$

for every test function φ with compact support in Ω; i.e., $\varphi \in C_0^1(\Omega; \mathbb{R}^m)$.

It is well known that in the *vector-valued case,* i.e. $m \geq 2$, in general we cannot expect *everywhere regularity* of the local minimizers of integrals as in (1.2), nor of the weak solutions to nonlinear differential systems as in (1.1). Examples of non-smooth solutions are originally due to De Giorgi [42], Giusti-Miranda [61], Nečas [84], and more recently to Šverák-Yan [92], De Silva-Savin [47], Mooney-Savin [79], Mooney [80].

A classical assumption finalized to the *everywhere regularity* is a modulus-dependence in the energy integrand; i.e., in terms of the function f, we require that

$$f(\xi) = g(|\xi|) \tag{1.3}$$

with a convex increasing function $g = g(t)$, $g\colon\ [0, +\infty) \to [0, +\infty)$, $g'(0) = 0$. In fact the first regularity result for weak solutions to nonlinear systems in divergence form of the type (1.1) is due to Karen Uhlenbeck obtained in her celebrated paper [97], published in 1977 and related to the energy-integral $f(\xi) = g(|\xi|) = |\xi|^p$ with

$p \geq 2$. Later Marcellini [68] in 1996 considered general energy-integrands $g(|\xi|)$ allowing *exponential growth* and Marcellini-Papi [74] in 2006 also some *slow growth*. Related regularity results, with energy-integrands $f(x,\xi) = g(x,|\xi|)$ depending on x too, are due to Mascolo-Migliorini [78], Beck-Mingione [5], Di Marco-Marcellini [48], De Filippis-Mingione [40]. See also Apushkinskaya-Bildhauer-Fuchs [1] for a local gradient bound of a-priori bounded minimizers.

The following Theorem 1.1 nowadays is the most general local Lipschitz continuity result, valid for nonlinear elliptic systems under either slow or fast growth conditions, when the integrand function $f(\xi) = g(|\xi|)$ is independent of x.

Theorem 1.1 (Marcellini-Papi [74]). *Let $t_0, H > 0$ and $\beta \in (1/n, 2/n)$. Let $g : [0,+\infty) \to [0,+\infty)$ be a convex increasing function of class $W^{2,\infty}_{\mathrm{loc}}$. We assume that for every $\alpha \in (1, n/(n-1)]$ there exist $K = K(\alpha)$ such that*

$$\frac{H}{t^{2\beta}}\left[\left(\frac{g'(t)}{t}\right)^{\frac{n-2}{n}} + \frac{g'(t)}{t}\right] \leq g''(t) \leq K\left[\frac{g'(t)}{t} + \left(\frac{g'(t)}{t}\right)^{\alpha}\right], \ \forall\, t \geq t_0 \tag{1.4}$$

(here we explicitly consider the case $n \geq 3$; if $n = 2$ the exponent $(n-2)/n$ must be replaced by any fixed real number in $(0,1)$). If u is a minimizer in $W^{1,1}_{\mathrm{loc}}(\Omega; R^m)$ of the energy integral (1.2), then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega; R^m)$. Moreover u satisfies the gradient bound: for every given $\varepsilon > 0$ and concentric balls B_ϱ,B_R compactly contained in Ω of radius respectively $\varrho < R$ there exists a constant $c = c(n,m,\alpha,\beta,H,K,\varepsilon,\varrho,R)$ such that

$$\|Du\|^2_{L^\infty(B_\varrho;\mathbb{R}^{m\times n})} \leq c\left(\int_{B_R}\{1 + f(Du)\}\,dx\right)^{1+\varepsilon}. \tag{1.5}$$

The quoted results [68],[74],[78],[5],[48],[40],[1] are sometime technical and not always easy to read. Here we give some regularity results with simpler and less technical assumptions which, at the current state of art, are valid for a large class on nonlinear variational systems and, in spite of their simplicity, are not weaker than any other Lipschitz regularity result known in the mathematical literature for autonomous energy integrals as in (1.2).

To this aim we consider separately the *linear*, the *superlinear* and the *sublinear growth*. Precisely, we consider the energy integral (1.2) with integrand $f(\xi) = g(|\xi|)$ as in (1.3) and g: $[0,+\infty) \to [0,+\infty)$ convex increasing function of class $W^{2,\infty}_{\mathrm{loc}}$; i.e. the second derivative g'' of g is locally bounded in $(0,+\infty)$. In this case, with the modulus dependence $f(\xi) = g(|\xi|)$, we have

$$a_i^\alpha(\xi) = \frac{\partial f(\xi)}{\partial \xi_i^\alpha} = \frac{\partial g(|\xi|)}{\partial \xi_i^\alpha} \quad \text{and} \quad \frac{\partial |\xi|}{\partial \xi_i^\alpha} = \frac{\xi_i^\alpha}{|\xi|}.$$

Therefore $a_i^\alpha(\xi) = g'(|\xi|)\dfrac{\xi_i^\alpha}{|\xi|}$ and the differential system (1.1) assumes the form

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{g'(|Du|)}{|Du|}u^\alpha_{x_i}\right) = 0, \qquad \alpha = 1,2,\ldots,m. \tag{1.6}$$

We can see which kind of *ellipticity conditions* are satisfied by the system (1.6). This aspect turns out to be equivalent to test which kind of *uniform convexity conditions* are satisfied by the energy integrand $f(\xi)$. With a computation, for instance as in [74, formula (3.3)], we find

$$\begin{aligned} \min\left\{\frac{g'(|\xi|)}{|\xi|}, g''(|\xi|)\right\}|\lambda|^2 &\leq \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(\xi)\,\lambda_i^\alpha \lambda_j^\beta \\ &\leq \max\left\{\frac{g'(|\xi|)}{|\xi|}, g''(|\xi|)\right\}|\lambda|^2 \end{aligned} \tag{1.7}$$

for every $\lambda,\ \xi \in \mathbb{R}^{m\times n}$. Therefore the real functions $t \to \frac{g'(t)}{t}$ and $t \to g''(t)$ play a relevant role to establish the convexity condition of the energy integrand and ellipticity conditions of the nonlinear differential system. As clearly described in section 2 of [72], a first crucial step for the *everywhere regularity* is to establish the local Lipschitz continuity of minimizers and respectively of weak solutions; in this context the role of the functions $\frac{g'(t)}{t}$ and $g''(t)$ is relevant when $t \to +\infty$; i.e., for $t \geq t_0$ for some given $t_0 > 0$.

We consider separately three cases. We say that the system (1.1), or equivalently the system (1.6), has either *linear* or *superlinear or sublinear growth* respectively if

$$\exists\, L_1, L_2 \in (0,+\infty):\qquad L_1 \leq \frac{g'(t)}{t} \leq L_2\,,\quad \forall\, t \geq t_0\,, \tag{1.8}$$

$$\lim_{t\to+\infty} \frac{g'(t)}{t} = +\infty\,, \tag{1.9}$$

$$\lim_{t\to+\infty} \frac{g'(t)}{t} = 0\,. \tag{1.10}$$

For instance, for the $p-$Laplacian we have $g(t) = t^p$ and $\frac{g'(t)}{t} = pt^{p-2}$; therefore (1.8),(1.9),(1.10) respectively correspond to $p = 2$, $p > 2$ and to $p < 2$. In the three cases the *$p-$Laplace equation* or the *$p-$Laplace system*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left\{|Du|^{p-2}\, u^\alpha_{x_i}\right\} = 0\,,\qquad \alpha = 1,2,\ldots,m, \tag{1.11}$$

corresponds to the *Laplace system* $\Delta u^\alpha = 0$, $\alpha = 1,2,\ldots,m$ when $p = 2$, which of course is a *linear* system (of m Laplace equations, each one independent from the others); it is a superlinear system if $p > 2$ and it is sublinear when $p \in (1,2)$.

Of course from the above scheme (1.8),(1.9),(1.10) it is excluded the case

$$\liminf_{t\to+\infty} \frac{g'(t)}{t} = 0 \quad\text{and}\quad \limsup_{t\to+\infty} \frac{g'(t)}{t} > 0 \tag{1.12}$$

or the case

$$\liminf_{t\to+\infty} \frac{g'(t)}{t} < +\infty \quad\text{and}\quad \limsup_{t\to+\infty} \frac{g'(t)}{t} = +\infty, \tag{1.13}$$

when the system (1.1),(1.6) has an *indefinite oscillating growth* as $t = |\xi| \to +\infty$. However *some* of these cases enter in the regularity Theorem A in Marcellini-Papi [74] and in [68].

1.1. Elliptic systems with linear growth

In this section we consider the system (1.1), or equivalently (1.6), with *linear growth* as in (1.8). Let $t_0 > 0$ and $\frac{1}{n} < \beta < \frac{2}{n}$ be fixed. We assume that there exists positive constants $m = m(\beta)$ and M such that

$$\frac{m(\beta)}{t^{2\beta}} \le g''(t) \le M, \quad \forall\, t \ge t_0\,. \tag{1.14}$$

Note that the condition (1.14) is satisfied if for instance the second derivative of g is bounded away from zero for large values of t; i.e., if there exist positive constants t_0, m, M such that

$$m \le g''(t) \le M, \quad \forall\, t \ge t_0\,. \tag{1.15}$$

Under these conditions the following regularity result holds.

Theorem 1.2 (gradient bound under linear growth)**.** *Let $g : [0, +\infty) \to [0, +\infty)$ be a convex increasing function of class $W^{2,\infty}_{\mathrm{loc}}$ satisfying (1.8) and (1.14); i.e., there exist positive constants $L_1, L_2, m(\beta), M$ and $\beta \in \left(\frac{1}{n}, \frac{2}{n}\right)$, $t_0 \in (0, +\infty)$ such that*

$$L_1 \le \frac{g'(t)}{t} \le L_2\,, \quad \frac{m(\beta)}{t^{2\beta}} \le g''(t) \le M, \quad \forall\, t \ge t_0\,. \tag{1.16}$$

Then every weak solution u to the nonlinear elliptic system (1.1) is locally Lipschitz continuous in Ω. Moreover the following gradient bound holds: there exists an exponent $\omega > 1$ and, for every ϱ, R, $0 < \varrho < R$, there exists a positive constant C such that

$$\|Du\|^2_{L^\infty(B_\varrho;\mathbb{R}^{m\times n})} \le c \left(\int_{B_R} \{1 + f(Du)\}\, dx \right)^{\omega}. \tag{1.17}$$

The exponent ω depends on β, n, while the constant C depends on $\varrho, R, n, \alpha, \beta, t_0$ and $\sup\{g''(t) : \ t \in (0, t_0)\}$. Here B_ϱ and B_R are concentric balls compactly contained in Ω of radius respectively ϱ and R.

Proof. By (1.16)

$$\frac{m(\beta)}{t^{2\beta}} \frac{g'(t)}{L_2 t} \le \frac{m(\beta)}{t^{2\beta}} \le g''(t) \le M \le M \frac{g'(t)}{L_1 t}, \quad \forall\, t \ge t_0\,. \tag{1.18}$$

We are in the conditions required in the assumption (1.4) of Theorem 1.1. In fact from (1.18) we can also deduce that

$$\frac{m(\beta)}{2t^{2\beta}} \left[\left(\frac{g'(t)}{L_2 t} \right)^{\frac{n-2}{n}} + \frac{g'(t)}{L_2 t} \right] \le g''(t) \le \frac{M}{L_1} \frac{g'(t)}{t} \tag{1.19}$$

and by the regularity Theorem 1.1 we get the conclusion of Theorem 1.2. □

1.2. Elliptic systems with superlinear growth

In this section we consider the system (1.1), or equivalently (1.6), with *superlinear growth* as in (1.9). We assume that for some $t_0 > 0$ and for every $\alpha > 1$ there exists positive constants $m, M = M(\alpha)$ such that

$$m\frac{g'(t)}{t} \le g''(t) \le M\left(\frac{g'(t)}{t}\right)^{\alpha}, \quad \forall\, t \ge t_0\,. \tag{1.20}$$

Theorem 1.3 (gradient bound under superlinear growth)**.** *Let* $g : [0, +\infty) \to [0, +\infty)$ *be a convex increasing function of class* $W^{2,\infty}_{\mathrm{loc}}$ *satisfying (1.9) and (1.20); i.e., there exist positive constants* m, M *and* $t_0 \in (0, +\infty)$ *such that*

$$\lim_{t\to+\infty}\frac{g'(t)}{t} = +\infty\,, \quad m\frac{g'(t)}{t} \le g''(t) \le M\left(\frac{g'(t)}{t}\right)^{\alpha}, \quad \forall\, t \ge t_0\,. \tag{1.21}$$

Then every weak solution u *to the nonlinear elliptic system (1.1) is locally Lipschitz continuous in* Ω*. Moreover* u *satisfies the gradient estimate in (1.17).*

Proof. Since by (1.9) $\lim_{t\to+\infty}\frac{g'(t)}{t} = +\infty$, then also $\lim_{t\to+\infty}\left(\frac{g'(t)}{t}\right)^{\frac{2}{n}} = +\infty$. Therefore there exists $t_1 > 0$ such that $\left(\frac{g'(t)}{t}\right)^{\frac{2}{n}} \ge 1$ for all $t \ge t_1$. We also have

$$\left(\frac{g'(t)}{t}\right)^{\frac{n-2}{n}} = \frac{\frac{g'(t)}{t}}{\left(\frac{g'(t)}{t}\right)^{\frac{2}{n}}} \le \frac{g'(t)}{t}, \quad \forall\, t \ge t_1\,. \tag{1.22}$$

By (1.20),(1.22) we get

$$\frac{m}{2}\left[\left(\frac{g'(t)}{t}\right)^{\frac{n-2}{n}} + \frac{g'(t)}{t}\right] \le g''(t)\,.$$

We are under the conditions of the assumption (1.4) in Theorem 1.1, in the form

$$\frac{m}{2}\left[\left(\frac{g'(t)}{t}\right)^{\frac{n-2}{n}} + \frac{g'(t)}{t}\right] \le g''(t) \le M\left(\frac{g'(t)}{t}\right)^{\alpha}, \quad \forall\, t \ge \max\{t_0, t_1\}$$

and the conclusion of Theorem 1.3 follows from Theorem 1.1. □

1.3. Elliptic systems with sublinear growth

In the case with *sublinear growth* as in (1.10) we assume that for some $t_0 > 0$ and for some β such that $\frac{1}{n} < \beta < \frac{2}{n}$ there exists positive constants $m = m(\beta), M$ with the properties

$$\frac{m(\beta)}{t^{2\beta}}\left(\frac{g'(t)}{t}\right)^{\frac{n-2}{n}} \le g''(t) \le M\frac{g'(t)}{t}, \quad \forall\, t \ge t_0\,. \tag{1.23}$$

Theorem 1.4 (gradient bound under sublinear growth). *Let $g : [0,+\infty) \to [0,+\infty)$ be a convex increasing function of class $W^{2,\infty}_{\mathrm{loc}}$ satisfying (1.10) and (1.23); i.e., there exist positive constants $\beta \in \left(\frac{1}{n}, \frac{2}{n}\right)$, $m(\beta)$, M and $t_0 \in (0,+\infty)$ such that*

$$\lim_{t\to+\infty} \frac{g'(t)}{t} = 0\,, \quad \frac{m(\beta)}{t^{2\beta}} \left(\frac{g'(t)}{t}\right)^{\frac{n-2}{n}} \le g''(t) \le M\frac{g'(t)}{t}, \quad \forall\, t \ge t_0\,. \tag{1.24}$$

Then every weak solution u to the nonlinear elliptic system (1.1) is locally Lipschitz continuous in Ω. Moreover u satisfies the gradient estimate in (1.17).

Proof. Since by (1.10) $\lim_{t\to+\infty} \frac{g'(t)}{t} = 0$, then also $\lim_{t\to+\infty} \left(\frac{g'(t)}{t}\right)^{\frac{2}{n}} = 0$. Therefore there exists $t_1 > 0$ such that $\left(\frac{g'(t)}{t}\right)^{\frac{2}{n}} \le 1$ for all $t \ge t_1$. We also have

$$\frac{g'(t)}{t} = \left(\frac{g'(t)}{t}\right)^{\frac{2}{n}} \left(\frac{g'(t)}{t}\right)^{\frac{n-2}{n}} \le \left(\frac{g'(t)}{t}\right)^{\frac{n-2}{n}}, \quad \forall\, t \ge t_1\,. \tag{1.25}$$

By (1.23),(1.25) we get

$$\frac{m(\beta)}{2t^{2\beta}} \left[\left(\frac{g'(t)}{t}\right)^{\frac{n-2}{n}} + \frac{g'(t)}{t}\right] \le g''(t)\,.$$

Thus the condition (1.4) holds in the form

$$\frac{m(\beta)}{2t^{2\beta}} \left[\left(\frac{g'(t)}{t}\right)^{\frac{n-2}{n}} + \frac{g'(t)}{t}\right] \le g''(t) \le M\frac{g'(t)}{t}, \quad \forall\, t \ge \max\{t_0, t_1\}$$

and the conclusion of Theorem 1.4 follows from Theorem 1.1. □

1.4. Some references on related regularity results under general growth

The first gradient bound for weak solutions to a class of nonlinear *elliptic systems with general growth* has been given in [68]. Other classes of elliptic systems with general growth have been considered in [74] under *slow growth*, and in [78, 5, 48, 40] under $x-$dependence too.

We also quote some recent related regularity results for elliptic and parabolic equations and systems under general growth conditions. We start from the general *$p, q-$growth* case, whose regularity theory, based on the local gradient bounds, takes its origins from some articles published by the author in the years 1989-1993. For more recent results we refer to Eleuteri-Marcellini-Mascolo [52]-[55], Cupini-Marcellini-Mascolo [28]-[34], De Filippis [38], Duzgun-Marcellini-Vespri [50, 51], Carozza-Giannetti-Leonetti-Passarelli [18], Carozza-Kristensen-Passarelli [19]. In particular in [69, 70, 72, 73] it is possible to find an updated list of references. We quote the recent studies of the so-called *double face case* by Colombo-Mingione

[26, 27], Baroni-Colombo-Mingione [4]; see also Eleuteri-Marcellini-Mascolo [55], De Filippis-Ho [41], Nguyen-Tran [85], Fiscella-Pinamonti [58].

The case of the $p(x)$ −exponent, i.e. variable exponents by Eleuteri-Marcellini-Mascolo [53], Cencelja-Rădulescu-Repovš [20], Papageorgiou-Rădulescu-Repovš-Dušan [86], Chlebicka [21], Chlebicka-De Filippis [22, 23], Ding-Zhang-Zhou [49], Chlebicka-De Filippis-Koch [23]. General growth conditions even for the one-dimensional case $n = 1$ have been studied in [15, 60]. For the general case $n > 1$ and $m > 1$ under *quasiconvexity conditions* see [67] and the *integral convexity condition* [8] by Bögelein-Dacorogna-Duzaar-Marcellini-Scheven; see also [9]-[14].

2. The origins of the $\Gamma-$convergence in Pisa and the links with the Mosco's convergence

2.1. A story that began in 1968

Once upon a time ... a young student at the University of Roma, i.e. at the *Sapienza Università di Roma*, if more precisely we use the today's name. At that time only one State University, named *Università degli Studi di Roma,* existed in Rome. It was the year 1968. Yes, the protests year, protest movements not only in the States but also in many European Universities, also in Rome. In 1968 I knew Umberto Mosco as a teacher to students in Mathematics at their third year of university studies. At that time I was younger than Umberto, only few years younger, and till now I am younger than Umberto!

Umberto was fascinating as a teacher. Immediately I was strongly interested in his classes. We must not think that - at that moment - other good mathematicians did not teach *Analysis* at the University of Rome; the opposite! There was for instance Guido Stampacchia, a great mathematician as well as strong personality, and Beppe Da Prato, a special mathematician always very deep and precise, strong expert in *functional analysis* and its applications to *partial differential equations*. At the Mathematical Institute in Rome (at that time Departments were not jet born in Italy) Gaetano Fichera, a strong mathematician, was also teaching there; I had a discussion with Fichera that probably pushed me to choose different directions of research. I was fascinated by the classes of Guido Stampacchia too. The subjects of his teaching were the fundamental tools in Analysis, such as Lebesgue integrals, L^p and Sobolev spaces, in order to arrive soon to the theory of second order linear elliptic equations with measurable coefficients; and then to the *variational inequalities*, which were one of his main mathematical interests and that received strong consideration in the mathematical literature of that years, in particular by the French and the Italian schools.

As well as some other students in these years, for instance Lucio Boccardo, Italo Capuzzo Dolcetta, Michele Matzeu, Maria Agostina Vivaldi, I was mostly attracted by the warm classes by Umberto Mosco about *convex analysis*; but let me say - as a student - I was more attracted by the *weak topology* and *weak convergence* in Banach spaces or in locally convex topological vectorial spaces. At that time, for me as a young student, it was very exciting to discover the relative compactness in the weak topology of bounded sequences in infinite dimensional Hilbert or Banach spaces. It was also a very interesting tool to discover that this property does not come for free in the infinite dimensional case, but it needs the *reflexivity* of the space, or at least it needs the *weak** *topology*, especially in the way explained by Umberto Mosco.

Today I meet some students who have attended the first course of Analysis in my classes and who have now reached the end of their studies in Mathematics; sometimes I ask them: *among your studies, which subject was more relevant and more useful for your subsequent studies in Analysis?* Some students answered *Taylor's Formula* for its applications to numerical analysis; some others, knowing my research interest in the *Calculus of Variations*, answered the *Weierstrass Theorem* about the existence of the maximum and the minimum values of a continuous function on a closed and bounded interval of the real line. I mention to them my opinion: it is the relative compactness of the bounded sequences in $\mathbb{R}$, the theorem that in Italy we call *Bolzano-Weierstrass Theorem.* This theorem gave origin to a chain of relevant compactness theorems in Functional Analysis, such as for instance the *Ascoli-Arzelà Compactness Theorem* for equicontinuous and equibounded sequences of functions; the *Rellich Theorem*, also named *Rellich–Kondrachov* theorem, on the compact embedding concerning *Sobolev Spaces.* The weak convergence enters too: the relative weak compactness of bounded sequences in an infinite dimensional Hilbert space is a main - and relatively simple - example of application of the Bolzano-Weierstrass Theorem.

Thus I was attracted by the fascinating classes by Umberto Mosco and I decided to prepare my bachelor thesis in Rome under his supervision. I discussed my thesis on November 1970, on *Bochner integrals* about multivalued applications. I never studied anymore Bochner integrals for multivalued functions, however I continued to be strongly interested in the scientific researches that Umberto Mosco was carried out in these years. In fact, also in the period that I spent to prepare my thesis, I continued to study convex analysis and in particular the Mosco's convergence of convex sets and of convex functions.

Immediately after the conclusion of my thesis in Rome, I applied for a PhD position at the Scuola Normale in Pisa. At that time, and also now, the PhD Program at the Scuola Normale in Pisa is named *Corso di Perfezionamento.* I applied for a PhD position and I had to pass a colloquium. Edoardo Vesentini was the president of the committee of my colloquium; later he was also Director of the Scuola Normale. I passed the colloquium. Edoardo Vesentini liked me as a student! I say this because, although Geometer, he remained scientifically in touch with me also when I started my research studies in Analysis; after some years we also published together, as editors, the Lecture Notes [87].

I arrived at the Scuola Normale Superiore in Pisa as a PhD Student at the end of 1970, beginning of 1971. I started to follow the courses of Guido Stampacchia, who in the meantime moved form the University of Rome to the Scuola Normale in Pisa; I also followed courses by Sergio Spagnolo, Antonio Marino, some classes by Giovanni Prodi, Sergio Campanato, Franco Conti, Aldo Andreotti, some seminars by Enrico Bombieri and others, of course. In these years it was easy to meet in

Pisa, coming from Firenze, Enrico Giusti, Mariano Giaquinta, Giorgio Talenti. Last but not least, I followed some courses by Ennio De Giorgi, one in Logic, one about Evolution Problems, one in Calculus of Variations. At the beginning it was difficult for me to well understand his classes. The reasons? For sure I had to study a lot before to understand better; however on one side the level of his courses was very high, on another side for Ennio De Giorgi all the mathematical subjects were easy although sometimes difficult for others. I was strongly attracted by his mathematical charisma.

As already said above, at that time (1971-1973) Ennio De Giorgi was interested in Logic too, a lot of his time was dedicated to the fundaments of mathematics; but of course to partial differential equations and calculus of variations too. I had the pleasure in these years to meet him, not only in his office at the third floor of the Scuola Normale, facing Piazza dei Cavalieri in Pisa, but also for lunch and dinner, sometime with his colleagues mathematicians and with my colleagues PhD students, but also sometime - more often on Sundays - with my wife Manuela; we were used to go for the Sunday lunch to the *Salustri Restaurant* in San Giuliano Terme, near Pisa, where Ennio was well known to the family who ran the restaurant. When in his office, De Giorgi used to let me write on his blackboard while he seated in front of me on a large brown armchair, reading in French *Le Monde*. Sometime he was saying "*Sì, sì!* Yes, go on!" while continuing to read *Le Monde*! You would have a wrong impression to think that he was not aware: at the end always he was posing questions, remarks and advice. Maybe you may think that he already knew what I was describing!

In this context I had the opportunity to describe to Ennio De Giorgi the *Mosco's convergence*, and the relations with the *variational convergences* that at that time we started to understand; it was the time of the starting of the $G-$*convergence* and of the $\Gamma-$*convergence* in Pisa. The $G-$convergence by Ennio De Giorgi and Sergio Spagnolo started in the articles [91, 77, 46] (see the next section for more precise details). The first manuscript [45] by Ennio De Giorgi about $\Gamma-$*convergence* was published in 1975 in collaboration with Tullio Franzoni. Since then, the definition the $\Gamma-$*convergence* was simplified; the modern approach is described below in the Definition 2.1. At that time, also inspired by the previous approach that Umberto Mosco gave to the convergence that now takes his name (see the Definition 2.2 below) in a joint paper with Lucio Boccardo [7] we gave exactly the definition adopted nowadays, with $\liminf_{k\to+\infty} f_k(x_k) \geq f(x)$, and so on, as in the Definition 2.1 below, although in the specific context of the $\Gamma-$*convergence* of convex energy integrals of the calculus of variations. See also the Definition 3 in [64].

For completeness it is correct to mention that the theory of the $G-$*convergence* and of the $\Gamma-$*convergence* took origin not only in Pisa, but for instance also from the researches by Babuska and by the French school, in particular by Jacques Louis Lions, Luc Tartar, François Murat, Gilles Francfort, Doina Cioranescu and many others, whose results arrived in Italy only later the first approaches in Pisa. For some reference books of the French school, who mainly used the name *homogenization*, we quote Hédy Attouch [2], Alain Bensoussan, Jacques Louis Lions and

George Papanicolaou [6], Doina Cioranescu and Patrizia Donato [25], Luc Tartar [96] and the references therein.

In Italy at the beginning the $\Gamma-$convergence was named $G-$convergence.

2.2. The De Giorgi-Spagnolo $G-$convergence

The *energy integral* of the calculus of variations that we consider is, for instance, of the type

$$F(u) = \int_{\Omega} f(x, Du(x))\, dx\,, \tag{2.1}$$

where $u = u(x)$ is a real function defined for x in a bounded open set $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, $Du = Du(x)$ is its gradient in Ω, and $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, i.e., $f = f(x, \xi)$ is *convex* with respect to the gradient variable $\xi \in \mathbb{R}^n$ and measurable with respect to $x \in \Omega$. The $x-$dependence of the integrand f corresponds to an energy which is not homogeneous with respect to the body Ω. This is the *inhomogeneous*, or the *non-homogeneous case*. One of the aspects who gave origin to the $G-$convergence and to the $\Gamma-$convergence is to consider the inhomogeneity spread in Ω in a random way, like for two or more materials mixed together. A mathematical approach is to consider a *periodic* distribution of the two (or more) materials with a small $n-$dimensional period, say depending on a positive (small) parameter ε, or equivalently on an integer number $k \to +\infty$, with $\varepsilon = \frac{1}{k}$; any corresponding minimizer u_ε of the energy integral of the type in (2.1) with integrand $f = f_\varepsilon(x, \xi) = f\left(\frac{x}{\varepsilon}, \xi\right)$ describes the microscopic behavior of the physical system. Then a natural approach is to go from *microscopic* to *macroscopic*. The mathematical process corresponds to consider a minimizer u_ε of the energy integral as in (2.1), related to the integrand $f_\varepsilon(x, \xi)$, and let ε go to zero, describing the *weak* limit $u_\varepsilon \rightharpoonup u$, where u is a minimizer of an *homogeneous* energy of the form $\int_{\Omega} f_0(Du(x))\, dx$. This is the method of *homogenization*, also well known under the names $\Gamma-$*convergence*, $G-$*convergence*, $H-$*convergence*. In the next section we give some details, as well as we describe some connections between these notions and the *Mosco-convergence*.

We consider the *energy integral* $F(u)$ as in (2.1). For instance, the *energy* $F(u)$ may have the expression

$$F(u) = \int_{\Omega} \sum_{i=1}^{n} a_i(x) \left(\frac{\partial u}{\partial x_i}\right)^2 dx \tag{2.2}$$

for some *bounded measurable* coefficients $a_1(x), a_2(x), \ldots, a_n(x)$ greater than some positive constant. The $x-$dependence corresponds to a not homogeneous body Ω. This is a *non-homogeneous case* (for a discussion related to this aspect see also [71]). If we fix the potential at the boundary $\partial\Omega$, say $u(x) = u_0(x)$ for every $x \in \partial\Omega$, then an *equilibrium potential* u satisfies the Dirichlet problem

$$\begin{cases} \sum_{i=1}^{n} \dfrac{\partial}{\partial x_i}\left(a_i(x)\dfrac{\partial u}{\partial x_i}\right) = 0, & x \in \Omega, \\ u(x) = u_0(x), & x \in \partial\Omega. \end{cases} \tag{2.3}$$

A similar, but more general energy in this *inhomogeneous* context is

$$F(u) = \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, dx, \tag{2.4}$$

where $A(x)$ is the square $n \times n$ matrix

$$\begin{pmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{pmatrix},$$

while in the previous case (2.2) the vector $(a_1(x), \ldots, a_n(x))$ deserves to be represented as the diagonal $n \times n$ matrix

$$\begin{pmatrix} a_1(x) & 0 & \cdots & 0 \\ 0 & a_2(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n(x) \end{pmatrix}.$$

From this relevant example the *G−convergence, Γ-convergence, homogenization* theories took their origin, in the '70, with the contribution of De Giorgi, Marino, Spagnolo *et al.* The first references on this subject is the article [46] published by De Giorgi-Spagnolo in 1973, a paper which followed the first pioneering attempts by Spagnolo [91] and Marino-Spagnolo [77] in 1968-69, all of them being related to second order linear elliptic (and parabolic) operators of the form $\sum_{ij=1}^{n} D_i(a_{ij}(x)D_j)$, whose *energy functional* is expressed in (2.4). The celebrated Marino-Spagnolo [77] result *in loose form* states that solutions of Dirichlet problems associated to the general energy integral in (2.4) can be approximated, in the strong L^2−topology, as well as in the weak topology of $W^{1,2}(\Omega)$, by the solutions of Dirichlet problems related to simpler energy integrals as in (2.2). A drastic difference with respect to the convergence of the coefficients: *the non-diagonal zero coefficients may become nonzero in the G−limit*!

This is nowadays a well known fact, which gave origin to the *G−convergence* and to the *Γ−convergence* theories, and we will not go further here. A reader can see the well known books by Dal Maso [35], Braides [16]. Reference books for *homogenization* are due to Bensoussan-Lions-Papanicolaou [6] and to Cioranescu-Donato [25]. Relevant are the Lecture Notes book by Luc Tartar [96] and the article [59] by Francfort-Murat-Tartar, who gave also several other relevant contributions to the *G−convergence, H−convergence, homogenization*. See also Attouch [2], Attouch-Buttazzo-Michaille [3], Buttazzo [17] and Cioranescu-Damlamian-Griso [24]. The first homogenization formula in the nonlinear case was proposed in [65]. Nowadays some hundreds papers deal with *G−convergence, H−convergence, homogenization*, and it is impossible to describe all the points of view; just to refer to some more recent papers, neither from the French school nor from the Italian school, we mention for instance [56, 57].

Here we observe that the described phenomenon related to matrices and to *linear* Dirichlet problems was discovered, at the very beginning of the $G-$convergence and $\Gamma-$convergence theories, also when lower order terms are considered; see for instance Carlo Sbordone [88, 89, 90, 76] and Luc Tartar [93, 94, 95]. For the linear case with lower order terms see also the not usual approach in [66].

The first attempts of $\Gamma-$*convergence* are due to Ennio De Giorgi [45] in 1975 in collaboration with Tullio Franzoni. With this paper the $G-$*convergence*, that at the beginning was related to the convergence of the weak solutions to elliptic and parabolic pde's, with the new name of $\Gamma-$*convergence*, became a tool to treat also general energy functional and their minimizers. In the same year De Giorgi published in the *Rendiconti di Matematica* the paper [43] which, in the generalized form that Carlo Sbordone gave to it in [90], was and still is a fundamental step for the construction of the $\Gamma-$convergence theory, as nowadays is presented in the book [35] by Gianni Dal Maso.

In the years 1973-1976 the first nonlinear attempts for the $\Gamma-$convergence in the nonlinear context of *convex energy functionals* can be also found in [63, 64], in connection with the *Mosco convergence* too, and some aspects are described in the next section. The definition of $\Gamma-$convergence was proposed and discussed in [7] in connection with the convergence of minimizers, the convergence of eigenvalues and eigenfunctions, and the convergence of solution to variational inequalities. In [75] the authors characterize the Mosco and the $\Gamma-$convergence of various classes of energy integrals, with respect to the L^p and $W^{1,p}$ weak and strong topologies, in terms of the integrands and their conjugates; moreover some $\Gamma-$compactness results are proved without coerciveness assumptions.

2.3. Connections between the Γ and the Mosco convergences

Relations between $\Gamma-$*convergence* and the *Mosco convergence* are well known, although few details seem to be less known and maybe can be pointed out. We refer to the approaches started in the '70^{th}s, in the period of time when in Pisa took origin the $\Gamma-$convergence theory, mainly by the fundamental work of Ennio De Giorgi, as described in the previous section. We emphasize here some of the Mosco's results where his contribution is more related to the $\Gamma-$convergence; in particular we refer to the Mosco's papers [81, 82]; in this context we also refer to the Mosco's paper [83] about convergence of Dirichlet forms. See also Dal Maso-Mosco [36, 37] and Lancia-Mosco-Vivaldi [62].

We recall the *sequential* definition of $\Gamma-$convergence. Details can be found in the reference books by Dal Maso [35] and Braides [16]. See also Attouch [2], Attouch-Buttazzo-Michaille [3], Buttazzo [17], Cioranescu-Donato [25] Tartar [96].

Definition 2.1 (of $\Gamma-$convergence)**.** Let (X,τ) be a topological space and $(f_k)_{k\in\mathbb{N}}$ be a sequence of functions defined in X with values in $\mathbb{R}\cup\{+\infty\}$. We say that f_k $\Gamma-$converges to $f: X\to\mathbb{R}\cup\{+\infty\}$ in the (sequential) topology τ if the following two conditions hold:

(i) (equi-lower semicontinuity) for every $x\in X$ and every sequence $(x_k)_{k\in\mathbb{N}}$ converging to x in the topology τ

$$\liminf_{k\to+\infty} f_k\left(x_k\right)\geq f\left(x\right)\ ;$$

(ii) (optimality) for every $x\in X$ there exist a sequence $(x_k)_{k\in\mathbb{N}}$ converging to x in the topology τ such that the equality holds; i.e.,

$$\lim_{k\to+\infty} f_k\left(x_k\right)=f\left(x\right)\ .$$

Definition 2.2 (of the Mosco convergence)**.** Let X be a real Banach space and $(f_k)_{k\in\mathbb{N}}$ be a sequence of functions defined in X with values in $\mathbb{R}\cup\{+\infty\}$. We say that f_k $M-$converges (Mosco-converges) to $f: X\to\mathbb{R}\cup\{+\infty\}$ in X if f_k $\Gamma-$converges to f in the strong topology as well as in the weak topology of X.

Of course many properties satisfied by sequences $\Gamma-$converging either in the weak or in the strong topology of a real Banach space are satisfied also by Mosco-converging sequences and vice-versa. Clearly this fact is not always true; for instance the compactness properties of weakly $\Gamma-$converging sequences, which are typical and which characterize the $\Gamma-$convergence. However the Mosco-convergence, other than useful for instance in the convergence of solutions to variational inequalities, which was one of the main Mosco's motivations (see [81]), is very elegant because it allows to treat variational problems in a symmetric general way, for instance in the original Mosco's description of the continuity of the *Young-Fenchel transform* (see the definition below in (2.5) and details in the Mosco' paper [82]).

Let X be a *reflexive* real Banach space and let $f: X\to\mathbb{R}\cup\{+\infty\}$, not identically $+\infty$. The *conjugate*, or the Young-Fenchel *transform* $f^*: X^*\to\mathbb{R}\cup\{+\infty\}$, is the function defined in the dual Banach space X^* of X by

$$f^*\left(x^*\right)=\sup\left\{\langle x^*,x\rangle-f\left(x\right):\ x\in X\right\}. \tag{2.5}$$

The following Theorem 2.5 is a modification of a similar result given by Mosco [81, 82] in the version proposed in [63, 64]; see in particular Lemma 1 in [64]. In the original terminology by Mosco Theorem 2.5 below is one of the main steps for the so-called continuity of the Young-Fenchel transform with respect to the Mosco-convergence. Despite of this fact, we give below essentially part of the original results by Umberto Mosco about the convergence of convex sets and of convex functions, which he studied in [81, 82].

Lemma 2.3 and Lemma 2.4 below, and their consequence Theorem 2.5, are the main results of this section. As already said, they are a modification of similar results given by Mosco [81],[82] in the version proposed in [63],[64]; see in particular Lemma 1 in [64]. Although originally stated under convexity assumptions, the following Lemma 2.3, Lemma 2.4 and Theorem 2.5 hold independently of the convexity of the functions f_k, $f\colon X\to\mathbb{R}\cup\{+\infty\}$.

Lemma 2.3. *Let f_k, $f\colon X\to\mathbb{R}\cup\{+\infty\}$, not identically $+\infty$. Let us assume that the optimality condition (ii) holds for f_k, f, in the weak (respectively strong) topology of X. Then the equi-lower semicontinuity condition (i) for f_k^*, f^* holds in the strong (respectively weak) topology of X^*.*

Proof. We follow Lemma 1b in [64]. In order to obtain the equi-lower semicontinuity of f_k^*, for every $x\in X$ with $f(x)$ finite by (ii) in the Definition 2.1 we can consider $(x_k)_{k\in\mathbb{N}}$ converging to x in the weak (respectively strong) topology of X such that $\lim_{k\to+\infty} f_k(x_k)=f(x)$. If x_k^* converges to x^* in the strong (respectively weak) topology of X^* then

$$f_k^*(x_k^*)=\sup_{x\in X}\{\langle x_k^*,x\rangle - f(x)\}\geq \langle x_k^*,x_k\rangle - f(x_k)\,.$$

As $k\to+\infty$ we obtain

$$\liminf_{k\to+\infty} f_k^*(x_k^*)\geq \langle x^*,x\rangle - f(x)\,. \tag{2.6}$$

Since (2.6) trivially holds if $f(x)=+\infty$, then it is satisfied for all $x\in X$ and

$$\liminf_{k\to+\infty} f_k^*(x_k^*)\geq \sup_{x\in X}\{\langle x^*,x\rangle - f(x)\} = f^*(x^*)\,. \tag{2.7}$$

□

In the next Lemma we make two assumptions. One is stated below in (2.9) and it is a *coercivity condition*. An other technical condition is necessary to avoid trivial cases such as $f_k(x)=g(x)+k$ for some given function $g(x)$, that in the limit as $k\to+\infty$ would give f identically equal to $+\infty$, in contrast with the fact that we allow functions $f\colon X\to\mathbb{R}\cup\{+\infty\}$ not identically $+\infty$. More precisely, we require that there exists a sequence y_k in X converging in the weak topology of X such that

$$\limsup_{k\to+\infty} f_k(y_k)<+\infty\,. \tag{2.8}$$

Lemma 2.4. *We assume (2.8) and that there exist constants $m>0$ and $p>1$ such that*

$$f_k(x)\geq m\,\|x\|_X^p\ ,\quad \forall\, x\in X,\ \ \forall\, k\in\mathbb{N}. \tag{2.9}$$

If the equi-lower semicontinuity (i) holds for f_k, f in the weak topology of X, then for every $x^*\in X^*$ we have

$$\limsup_{k\to+\infty} f_k^*(x^*)\leq f^*(x^*)\ . \tag{2.10}$$

Proof. We consider the sequence y_k in X converging in the weak topology of X such that (2.8) holds; then for every $x^* \in X^*$

$$\inf_{x\in X} \{f_k(x) - \langle x^*, x\rangle\} \leq [f_k(x) - \langle x^*, x\rangle]_{x=y_k} = f_k(y_k) - \langle x^*, y_k\rangle . \qquad (2.11)$$

Let us first consider the case $\limsup_{k\to+\infty} f_k^*(x^*) < +\infty$. Then $f_k^*(x^*)$ is finite for every large value of k and there exists a sequence $x_k \in X$ such that

$$\inf_{x\in X} \{f_k(x) - \langle x^*, x\rangle\} + \frac{1}{k} > f_k(x_k) - \langle x^*, x_k\rangle . \qquad (2.12)$$

From (2.11),(2.12) we get

$$f_k(y_k) - \langle x^*, y_k\rangle > f_k(x_k) - \langle x^*, x_k\rangle - \frac{1}{k} .$$

By the coercivity condition (2.9) we obtain

$$f_k(y_k) + \|x^*\|_{X^*} \|y_k\|_X \geq f_k(y_k) - \langle x^*, y_k\rangle > f_k(x_k) - \langle x^*, x_k\rangle - \frac{1}{k} \qquad (2.13)$$

$$\geq m\|x_k\|_X^p - \|x^*\|_{X^*}\|x_k\|_X - \frac{1}{k} \geq \|x_k\|_X \left(m\|x_k\|_X^{p-1} - \|x^*\|_{X^*}\right) - 1 .$$

By (2.8) the real sequence $f_k(y_k)$ is bounded from above. Since y_k converges in the weak topology of X , the left hand side of (2.13) is bounded from above. Therefore also the right hand side remains bounded and, being $p > 1$, also $\|x_k\|_X$ is bounded. In fact, if by contradiction $\|x_k\|_X \to +\infty$ for a subsequence of $k \to +\infty$, then the quantity $\left(m\|x_k\|_X^{p-1} - \|x^*\|_{X^*}\right)$ would be positive for large values of k and thus all the left hand side of (2.13) would go to $+\infty$.

Up to a subsequence, which we continue to denote with the same symbol, as $k \to +\infty$ the sequence x_k weakly converges to some $x_0 \in X$. We rewrite (2.12) by using the definition of the Young-Fenchel transform (2.5) $f_k^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f_k(x)\}$; we obtain

$$-f_k^*(x^*) + \frac{1}{k} > f_k(x_k) - \langle x^*, x_k\rangle$$

and, passing to the lim inf as $k \to +\infty$,

$$-\limsup_{k\to+\infty} f_k^*(x^*) \geq \liminf_{k\to+\infty} [f_k(x_k) - \langle x^*, x_k\rangle] .$$

We use the equi-lower semicontinuity (i) in the Definition 2.1; we get

$$\limsup_{k\to+\infty} f_k^*(x^*) \leq \langle x^*, x_0\rangle - \liminf_{k\to+\infty} f_k(x_k) \leq \langle x^*, x_0\rangle - f(x_0) .$$

Since $f^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f(x)\}$ we finally get

$$\limsup_{k\to+\infty} f_k^*(x^*) \leq \langle x^*, x_0\rangle - f(x_0) \leq f^*(x^*) , \qquad (2.14)$$

which is the conclusion (2.10) for the case $\limsup_{k\to+\infty} f_k^*(x^*) < +\infty$.

In the case $\limsup_{k\to+\infty} f_k^*(x^*) = +\infty$ then up to a subsequence, which we continue to denote with the same symbol, $\lim_{k\to+\infty} f_k^*(x^*) = +\infty$. For every $M \geq 0$ there exists $k_M \in \mathbb{N}$ such that $f_k^*(x^*) > M$ for all $k > k_M$. Then, being $f_k^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f_k(x)\} > M$, for every $k > k_M$ there exists $x_k \in X$ such that

$$\langle x^*, x_k\rangle - f_k(x_k) > M\,. \tag{2.15}$$

By the coercivity assumption (2.9)

$$-M > f_k(x_k) - \langle x^*, x_k\rangle \geq \|x_k\|_X \left(m\, \|x_k\|_X^{p-1} - \|x^*\|_{X^*}\right)$$

which, being $M \geq 0$, implies $\|x_k\|_X^{p-1} \leq \frac{1}{m}\|x^*\|_{X^*}$. Up to a subsequence, which we continue to denote with the same symbol, as $k \to +\infty$ the sequence x_k weakly converges to some $x_0 \in X$. Passing to the limit as $k \to +\infty$ in (2.15), by the equi-lower semicontinuity (i) in the Definition 2.1 we get

$$\begin{aligned} M &\leq \limsup_{k\to+\infty} \left[\langle x^*, x_k\rangle - f_k(x_k)\right] = \langle x^*, x_0\rangle - \liminf_{k\to+\infty} f_k(x_k) \\ &\leq \langle x^*, x_0\rangle - f(x_0) \leq f^*(x^*)\,. \end{aligned}$$

The arbitrarily of M gives $f^*(x^*) = +\infty$ which proves

$$\limsup_{k\to+\infty} f_k^*(x^*) = +\infty = f^*(x^*)\,. \tag{2.16}$$

This gives the conclusion (2.10) also in the case $\limsup_{k\to+\infty} f_k^*(x^*) = +\infty$. □

The following Theorem 2.5 is direct consequence of Lemma 2.3 and Lemma 2.4. See also the version in [73]. We have only to verify that the technical condition (2.8) is satisfied. This is a consequence of the assumption that f_k is a sequence of functions which $\Gamma-$converges to f in the weak topology of X: since $f : X \to \mathbb{R}\cup\{+\infty\}$ is not identically $+\infty$, then there exists $y_0 \in X$ with $f(y_0) \in \mathbb{R}$ and $\lim_{k\to+\infty} f_k(y_k) = f(y_0) < +\infty$.

Theorem 2.5. *Let X be a reflexive real Banach space. Under the coercivity assumption (2.9), let f_k, f functions defined in X with values in $\mathbb{R}\cup\{+\infty,\}$, not identically $+\infty$. If f_k $\Gamma-$converges to f in the weak topology of X then f_k^* $\Gamma-$converges to f^* in the strong topology of X^*. Moreover*

$$\lim_{k\to+\infty} f_k^*(x^*) = f^*(x^*)\,. \tag{2.17}$$

As shown in the proof above, the result of Theorem 2.5 is *independent of the convexity* of the functions f_k; it also gives the convergence of the minimum values when the sequence f_k $\Gamma-$converges. In fact, if each function $f_k : X \to \mathbb{R}\cup\{+\infty\}$

is also weakly lower semicontinuous on X and it is not identically equal to $+\infty$, then the coercivity assumption (2.9) ensures the existence of the minimum value

$$\min\left\{ f_k(x) - \langle x^*, x\rangle :\; x \in X\right\} \tag{2.18}$$

for any linear perturbation $x^* \in X^*$ and the minimum value holds $-f_k^*(x^*)$ by the same definition of the *Young-Fenchel transform* f^* in (2.5). The limit condition (2.17) states that, as $k \to +\infty$, the minimum value in (2.18) converges to the corresponding minimum value for f. Under the coercivity condition (2.9) we can also obtain the weak convergence (up to a subsequence) of the minimizers.

For instance we refer to the well known context in pdes when $X = W^{1,2}(\Omega)$, Ω bounded open set in $\mathbb{R}^n$ for some $n \geq 1$, and the notation $f_k : X \to \mathbb{R} \cup \{+\infty\}$ is modified into $F_k \colon W^{1,2}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ and it is given, for example, by the *Dirichlet integral*

$$F_k(u) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}^k(x)\, u_{x_i} u_{x_j}\, dx\,, \quad u \in W^{1,2}(\Omega)\,, \tag{2.19}$$

where u is defined in Ω, in the Sobolev class $W^{1,2}(\Omega)$, and $Du = (u_{x_i})_{i=1.2,\dots,n}$ is its gradient. For every $k \in \mathbb{N}$ the $n \times n$ symmetric matrix $\left(a_{ij}^k\right)$ is positive definite and bounded, in the sense that $m\,|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}^k(x)\,\xi_i\xi_j \leq M\,|\xi|^2$ for some positive constants m, M, for all $\xi \in \mathbb{R}^n$, a.e. $x \in \Omega$ and for every $k \in \mathbb{N}$. With the notation of this section, x^* is an element of the dual space of $W_0^{1,2}(\Omega)$ of functions $u \in W^{1,2}(\Omega)$ with zero boundary value on $\partial\Omega$; for instance x^* is a generic function $h \in L^2(\Omega)$. The minimization problem corresponding to (2.18) is

$$\min\left\{ \int_\Omega \left[\sum_{i,j=1}^{n} a_{ij}^k(x)\, u_{x_i} u_{x_j} - h(x)\, u(x) \right] dx :\; u \in W_0^{1,2}(\Omega) \right\}. \tag{2.20}$$

As well known, in this context the $\Gamma-$convergence of the sequence of Dirichlet integrals $F_k \colon W_0^{1,2}(\Omega) \to \mathbb{R}$ in (2.19) implies the convergence of the minimum values $F_k(u_k)$ and also the convergence of the minimizers u_k in the weak topology of $W_0^{1,2}(\Omega)$. In fact in this context the $\Gamma-$convergence of the Dirichlet energy integrals F_k as in (2.19) is equivalent to the convergence, for every $h \in L^2(\Omega)$, of the minimum values in $W_0^{1,2}(\Omega)$ of

$$F_k(u) - \int_\Omega hu\, dx$$

and also to the weak convergence in $W_0^{1,2}(\Omega)$ of the corresponding minimizers u_k of the Dirichlet problem (2.20).

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