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p-length and character degrees in p-solvable groups

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Let G be a p-solvable group, where p is a prime. We prove that the p-length of G is less or equal then the number of distinct irreducible character degrees of G not divisible by p. Furthermore, we prove that the result still holds if we impose some restriction on the field of values of the characters. In particular, if $p = 2$, we can consider only rational-valued characters.

1. Introduction

Let G be a finite group and let $\mathrm{cd}_{p'}(G)$ be the set of irreducible character degrees not divisible by a prime p. In a recent paper [2], it is proved that, if $\mathrm{cd}_{p'}(G) = \{1, m\}$, then the group G is solvable and $\mathrm{O}^{pp'pp'}(G) = 1$.

If G is a finite p-solvable group; the *p-length* of G, denoted as $\ell_p(G)$, is the minimum possible number of factors that are p-groups in any normal series for G in which each factor is either a p-group or a p'-group. It is not hard to prove that it is equal to the number of p-groups in the upper p-series of G. Therefore, the result in [2] provides that, if $|\mathrm{cd}_{p'}(G)| = 2$, then G is solvable and $\ell_p(G) \leq 2$.

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The proof in the paper for the solvable case can be easily generalized to obtain that, if G is p-solvable, then $\ell_p(G) \leq |\mathrm{cd}_{p'}(G)|$.

This paper is clearly inspired by that proof; however, here a stronger result is presented for the solvable case.

Let $E \subseteq \mathbb{C}$ be a field and let G be a finite group. We call $\mathrm{Irr}_E(G)$ the set of irreducible characters which have values in E. Then, we define the set

$$\mathrm{cd}_{E,p'}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}_E(G) \text{ and } p \nmid \chi(1)\}.$$

We will see that, for some $E \subset \mathbb{C}$, $|\mathrm{cd}_{E,p'}(G)|$ can provide a bound for the p-length of a p-solvable group G.

If n is a natural number, the *n-cyclotomic extension* $\mathbb{Q}(\zeta_n)$ of the field of rational numbers is obtained by adjoining a primitive n-root of unity ζ_n to $\mathbb{Q}$. This field is sometimes denoted as $\mathbb{Q}_n$ and we adopt this notation in the paper. If π is a set of primes, $\mathbb{Q}_\pi$ denotes the extension of the field of rational numbers obtained by adjoining all complex n-th roots of unity of $\mathbb{Q}$, for all π-numbers n. Sometimes, when $\pi = \{p\}$, authors identify $\mathbb{Q}_p$ with $\mathbb{Q}_{\{p\}}$. In this paper, however, we will *not* use this notation: we write $\mathbb{Q}_p$ for the p-cyclotomic extension of $\mathbb{Q}$ and we write $\mathbb{Q}_{\{p\}}$ for the extension of $\mathbb{Q}$ obtained by adjoining all complex q-root of unity for all $q = p^a$, $a \in \mathbb{N}$. In particular, in our notation, $\mathbb{Q}_2 = \mathbb{Q}$.

It is proved in [10] that the 2-length of a solvable group G is bounded by the number of rational-valued irreducible characters of odd degree. This result was later improved in [12] and [11], where it is proved that, if G is p-solvable and $\ell = \ell_p(G)$, then G has at least 2^ℓ irreducible characters of degree coprime to p and values in $\mathbb{Q}_p$.

The results here are somewhat similar; however, we find a bound that does not depend from the number of irreducible characters but from the number of distinct irreducible character degrees.

Theorem A. *Let G be a p-solvable group and let $\ell_p(G)$ its p-length, then $\ell_p(G) \leq \left|\mathrm{cd}_{\mathbb{Q}_p,p'}(G)\right|$, with $\mathbb{Q}_p$ being a p-cyclotomic extension of $\mathbb{Q}$. In particular, if G is solvable, then $\ell_2(G) \leq \left|\mathrm{cd}_{\mathbb{Q},2'}(G)\right|$.*

Theorem A will not be proved directly. It is in fact a consequence of an analogous result, Theorem 3.1, concerning the B_p-characters, a family of characters defined in [4]. This is a bit surprising, since it is known that $|\mathrm{B}_p(G)|$ is equal to the number of conjugacy classes of p-elements (see [4, Theorem 9.3]), thus, these characters are relatively few, if compared to all the irreducible characters.

Anyway, now we can state as a corollary of Theorem A what we have anticipated earlier in the introduction.

Corollary B. *Let G be a p-solvable group and let $\ell_p(G)$ its p-length, then $\ell_p(G) \leq |\mathrm{cd}_{p'}(G)|$.*

We conclude this section by mentioning that, as noticed by the reviewer, since the p'-degree characters of a group are the height zero characters in p-blocks of maximal defect, it is likely that the results of this paper can be generalized to the height zero characters of the p-blocks of non-maximal defect. In particular, Fong-Reynoldt reduction is also relevant to this problem.

This may be the object of future investigations, however, we are not going to treat the problem here.

2. Review of the π-theory

In this section, we will briefly summarize the theory of characters of π-separable groups, which is extensively used in the proof of the main results of the paper.

Unfortunately, many of the most interesting consequences of the theory will be left out. An interested reader should consult [4], or the first part of [8], for a complete exposition of the theory.

Let π be a set of primes and denote as π' its complementary set. A finite group is said to be a π-group if its order is a π-number, which means that all its prime divisors lie in π.

A finite group G is said to be *π-separable* if every quotient in a composition series of the group is either a π-group or a π'-group. It is said to be *π-solvable* if every quotient is either a π'-group or a group of prime order. The reader will notice that, if π consists of the sole prime p, then the two concepts coincide and in this case we simply talk about *p-solvable groups*.

If G is a finite group, a character $\chi \in \mathrm{Irr}(G)$ is said to be *π-special* if its degree and its order are π-numbers and, for any $M \lhd \lhd G$ and any irreducible constituent φ of χ_M, $o(\varphi)$ is a π-number. There is no need for the group to be π-separable in order to define π-special characters. However, if G is a π-separable group, α is a π-special character and β is a π'-special character, it has been proved in [1] that $\alpha\beta$ is irreducible, too.

This step is crucial in the definition of B_π-characters. Let G be π-separable and $\chi \in \mathrm{Irr}(G)$, in [4] Isaacs described an algorithm which associates to χ a set $\mathrm{nuc}(\chi)$ of (conjugated) *inducing pairs* (W, μ) such that $W \leq G$, $\mu = \alpha\beta \in \mathrm{Irr}(W)$ is a product of a π-special and a π'-special character of W and $\mu^G = \chi$. Note that $\mathrm{nuc}(\chi)$ is not characterized by this three properties, so it is necessary to refer to Isaacs' algorithm in order to define it. Unfortunately, the algorithm is too complex to explain it here. We refer again to [4] for details. For what concerns the aim of this paper, the reader may accept that, given $\chi \in \mathrm{Irr}(G)$, there is a canonical way to obtain, up to conjugation, an inducing pair $(W, \mu) \in \mathrm{nuc}(\chi)$.

Definition 2.1. Let $\chi \in \mathrm{Irr}(G)$, where G is a π-separable group, and let $(W, \mu) \in \mathrm{nuc}(\chi)$. If μ is π-special, then χ is a B_π-character. We denote as $\mathrm{B}_\pi(G)$ the set of B_π-characters of the group G.

The original aim of the theory behind these characters was (likely) to find a subset of the irreducible characters which *behave well* when restricted to the π-elements of the group. In fact, by restricting the characters in $\mathrm{B}_\pi(G)$ to the π-elements, we obtain a basis for the class functions on these elements. In particular, if $\pi = p'$, then $\mathrm{B}_{p'}(G)$ is a family of lifts for the irreducible Brauer characters of G.

There are other properties of these character we are going to need, however. One of them concerns their behaviour with normal subgroups.

Proposition 2.2. *Let G be π-separable and let $M \trianglelefteq G$. If $\chi \in \mathrm{B}_\pi(G)$, then every irreducible constituent of χ_M belongs to $\mathrm{B}_\pi(M)$. On the other hand, if $\psi \in \mathrm{B}_\pi(M)$ and G/M is a π-group, then every character in $\mathrm{Irr}(G \mid \psi)$ belongs to $\mathrm{B}_\pi(G)$ while, if G/M is a π'-group, then there exists a unique character in $\mathrm{Irr}(G \mid \psi)$ which belongs to $\mathrm{B}_\pi(G)$.*

Proof. It is a direct consequence of [4, Theorem 6.2] and [4, Theorem 7.1] □

What makes B_π-characters extremely useful for the purposes of this paper, however, is their behaviour when restricted to a Hall π-subgroup. A *Hall π-subgroup H* of a group G is a subgroup such that $|H| = |G|_\pi$, where $|G|_\pi$ is the π-part of the order of G. Such subgroup is not always guaranteed to exist; however, if the group G is π-separable, a Hall π-subgroup always exists and any two of them are conjugated in G.

Theorem 2.3 *([4, Theorem 8.1]). Let $\chi \in \mathrm{B}_\pi(G)$, with G π-separable, and let $H \in \mathrm{Hall}_\pi(G)$. The degree of every irreducible constituent of χ_H is at least $\chi(1)_\pi$. Moreover, there exist some irreducible constituents $\varphi \in \mathrm{Irr}(H)$ of χ_H such that $\varphi(1) = \chi(1)_\pi$. Finally, if φ is one of such constituents, and $\psi \in \mathrm{B}_\pi(G)$, then $[\psi_H, \varphi]$ is equal to 1 if $\psi = \chi$ and to 0 otherwise.*

Characters $\varphi \in \mathrm{Irr}(H)$ like the ones in Theorem 2.3 are called *Fong characters* associated with χ.

As a first consequence of this theorem, we have informations about the field of values of the characters in $\mathrm{B}_\pi(G)$.

Corollary 2.4 *([4, Corollary 12.1]). If $\chi \in \mathrm{B}_\pi(G)$, then it has values in $\mathbb{Q}_\pi$.*

If H is an Hall π-subgroup of a π-separable group G, it is in general hard to determine whether a character $\varphi \in \mathrm{Irr}(H)$ is a Fong character associated with some $\chi \in \mathrm{B}_\pi(G)$. The task, however, become easier under some extra assumptions.

Theorem 2.5 *([5, Corollary 6.1] or [8, Theorem 5.13]). Let H be a Hall π-subgroup of a π-separable group G and let $\varphi \in \mathrm{Irr}(H)$. If φ is primitive, then it is a Fong character associated with some character $\chi \in \mathrm{B}_\pi(G)$. Moreover, $\eta \in \mathrm{Irr}(H)$ is a Fong character associated with χ if and only if φ and η are $\mathrm{N}_G(H)$-conjugated.*

Furthermore, if a character in $\mathrm{Irr}(H)$ is not only primitive but linear, we can rely to an even stronger result, which allows us to determine the associated B_π-character without the use of Isaacs' algorithm.

Theorem 2.6. *Let G be a π-solvable group and let H be a Hall π-subgroup of G. Let $\varphi \in \mathrm{Irr}(H)$. Then, there exists a unique maximal subgroup W of G such that φ extends to W, and φ has a unique extension $\hat{\varphi}$ to W which is π-special.*

Moreover, if φ is linear, then $\chi = (\hat{\varphi})^G$ is an irreducible character in $\mathrm{B}_\pi(G)$ and φ is a Fong character associated with χ.

Proof. The first part is a consequence of [6, Theorem A] and of [7, Theorem F], while the second part follows from [5, Theorem B] and [9, Theorem 3.6]. □

3. Results

We will prove Theorem A as a consequence of the following theorem, concerning the B_p-characters.

Theorem 3.1. *Let G be a p-solvable group, let $\ell_p(G)$ its p-length and let*

$$\mathrm{cd}^{\mathrm{B}_p}_{\mathbb{Q}_p,p'}(G) = \{\chi(1) \mid \chi \in \mathrm{B}_p(G) \cap \mathrm{Irr}_{\mathbb{Q}_p}(G) \text{ and } p \nmid \chi(1)\},$$

with $\mathbb{Q}_p$ being a p-cyclotomic extension of $\mathbb{Q}$. Then, $\ell_p(G) \leq \left|\mathrm{cd}^{\mathrm{B}_p}_{\mathbb{Q}_p,p'}(G)\right|$. In particular, if G is solvable, then $\ell_2(G) \leq \left|\mathrm{cd}^{\mathrm{B}_2}_{\mathbb{Q},2'}(G)\right|$.

To prove Theorem 3.1, few preliminary results are needed. At first, we need some results in order to work with rational-valued character.

Proposition 3.2. *Let G be a π-separable group and H a Hall π-subgroup of G. Let $\chi \in \mathrm{B}_\pi(G)$ and let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|})$. If $\varphi \in \mathrm{Irr}(H)$ is a Fong character associated with χ and $\varphi^\sigma = \varphi$, then also $\chi^\sigma = \chi$.*

In particular, if φ is rational-valued, then so is χ.

Moreover, if $\pi = \{p\}$ and $o(\sigma)$ is a power of p, then σ fixes χ if and only if it fixes some of the Fong characters associated with χ.

Proof. Suppose there exists a Fong character $\varphi \in \mathrm{Irr}(H)$ associated with χ such that $\varphi^\sigma = \varphi$. Since χ^σ is again a B_π-character (see [4]) and it lies over φ, it follows from the uniqueness part of Theorem 2.3 that $\chi^\sigma = \chi$.

In particular, since a character is rational-valued if and only if it is fixed by every $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|})$, we have that χ is rational valued if φ is.

Assume now that $\pi = \{p\}$, $H = P$ is a Sylow p-subgroup of G and $o(\sigma)$ is a power of p. If $\chi^\sigma = \chi$, then σ permutes the Fong characters associated with χ. Suppose none of these Fong characters is fixed by σ. If $C_1, ..., C_t$ are the orbits of this action, then $p \mid |C_i|$ for each $i = 1, ..., t$.

Now, let $\chi(1)_p = p^a$, so that p^a is the maximal power of p dividing $\chi(1)$, and notice that, by Theorem 2.3, we can write

$$\chi_P = \sum_{i=1}^{t} \sum_{\varphi \in C_i} \varphi + \Delta,$$

where either Δ is zero or the degree of each irreducible constituent of Δ is divided by p^{a+1}. Moreover, by definition, $\varphi(1) = p^a$ for each Fong character φ associated with χ.

Let φ_i be a representative for C_i for each i, then $\chi(1) = \sum_{i=1}^{t} |C_i| \, \varphi_i(1) + \Delta(1)$. Since p^{a+1} divides $\Delta(1)$ and p divides $|C_i|$ for every i, it follows that p^{a+1} divides $\chi(1)$, in contradiction with the maximality of p^a.

It follows that $|C_i| = 1$ for at least one index i and, thus, σ fixes at least one Fong character of χ. □

Corollary 3.3. *Let G be a solvable group and P a Sylow p-subgroup of G. Let $\chi \in \mathrm{B}_p(G) \cap \mathrm{Irr}_{p'}(G)$ and let $\lambda \in \mathrm{Lin}(P)$ be a Fong character associated with χ. Then, χ has values in $\mathbb{Q}_p$ if and only if $o(\lambda) = p$.*

Proof. We know from Corollary 2.4 that a B_p-character has values in $\mathbb{Q}_{\{p\}}$. As a consequence, a B_p-character has values in $\mathbb{Q}_p$ if and only if it is fixed by every morphism in $\mathrm{Aut}(\mathbb{Q}_{\{p\}}/\mathbb{Q}_p)$ or, equivalently, by every $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|})$ of order a power of p.

It follows from Proposition 3.2 that χ has values in $\mathbb{Q}_p$ if and only if so does some Fong character $\rho \in \mathrm{Lin}(P)$ associated with χ. However, by Corollary 2.5, $\lambda = \rho^n$ for some $n \in \mathrm{N}_G(P)$; thus, λ has values in $\mathbb{Q}_p$.

Finally, since λ is a linear character, it has values in $\mathbb{Q}_p$ if and only if $o(\lambda) = p$. □

If a linear character of a subgroup extends to the whole group G, it is not always true that there exists an extension which preserves the order. However, it happens to be possible under special conditions.

Lemma 3.4. *Let $M \lhd G$ and suppose M is complemented in G, i.e., there exists $H \leq G$ such that $G = HM$ and $H \cap M = 1$. If $\lambda \in \mathrm{Lin}(M)$ is invariant in G, then there exists $\varphi \in \mathrm{Lin}(G)$ such that θ extends λ and $o(\theta) = o(\lambda)$.*

It is likely that this result is already known. However, not having being able to find a reference, we prove it here.

Proof. Since M is complemented in G and λ is linear, it follows from [3, Exercise 6.18] that λ extends to G. Let $\nu \in \mathrm{Lin}(G)$ be any extension of λ and notice that, in order to have $o(\nu) = o(\lambda)$, it is sufficient that $\nu_H = 1_H$. In fact, since M is in the kernel of $\nu^{o(\lambda)}$ and $G = HM$, we have that $o(\nu^{o(\lambda)}) = o(\nu_H^{o(\lambda)})$. Thus, if $\nu_H = 1_H$, then $\nu^{o(\lambda)} = 1_G$.

Let $\xi \in \mathrm{Lin}(G/M)$ such that $\xi_H = \nu_H$, which exists because $H \cong G/M$, and take $\varphi = \nu\bar{\xi}$, then φ extends λ and $\varphi_H = 1_H$, thus, $o(\varphi) = o(\lambda)$ for the first paragraph and we have the thesis. □

Proposition 2.2 tells us how B_π-characters behave in relation with normal subgroups. However, a refinement is needed, in order to have more precise informations on character degrees.

Proposition 3.5. *Let G be a π-separable group and let $M \trianglelefteq G$. Let $\lambda \in \mathrm{Lin}(M)$ such that $o(\lambda)$ is a π-number and assume also that λ extends to $\nu \in \mathrm{Lin}(HM)$, where H is a Hall π-subgroup of G. Then, there exists a character $\chi \in \mathrm{B}_\pi(G)$ such that χ lies over λ and $\chi(1)_\pi = 1$. Moreover, ν_H is a Fong character associated with χ.*

Proof. Let $\nu \in \mathrm{Lin}(HM)$ be an extension of λ to HM and let $\varphi = \nu_H$, then φ is a linear character of the Hall π-subgroup H and, by Theorem 2.5, there exists $\chi \in \mathrm{B}_\pi(G)$ which lies over φ and $\chi(1)_\pi = \varphi(1) = 1$. Thus, φ is a Fong character associated with χ.

Moreover, if $K = H \cap M$, then $\xi = \varphi_K$ is a Fong character associated with λ and it also lies under χ. Let θ be an irreducible constituent of χ_M which lies over ξ, then θ is a B_π-character, because of Proposition 2.2, and it follows that $\theta = \lambda$ for the uniqueness part of Theorem 2.3. Thus, χ lies over λ and the proof is concluded. □

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. In order to simplify the notation, since there is no ambiguity, we will write $\mathrm{Bcd}_{p'}(G)$ to refer to $\mathrm{cd}^{\mathrm{B}_p}_{\mathbb{Q}_p,p'}(G)$.

Let G be a counterexample of minimal order to the theorem and let K be a minimal normal subgroup of G, then $\ell_p(G) > |\mathrm{Bcd}_{p'}(G)|$ and $\ell_p(G/K) \leq |\mathrm{Bcd}_{p'}(G/K)|$ because of the minimality in the choice of G. Moreover, since K is normal minimal, $\ell_p(G/K) \leq \ell_p(G) \leq \ell_p(G/K) + 1$, while $|\mathrm{Bcd}_{p'}(G/K)| \leq |\mathrm{Bcd}_{p'}(G)|$ because $\mathrm{Bcd}_{p'}(G/K) \subseteq \mathrm{Bcd}_{p'}(G)$. It follows that

$$\ell_p(G/K) \leq |\mathrm{Bcd}_{p'}(G/K)| \leq |\mathrm{Bcd}_{p'}(G)| < \ell_p(G) \leq \ell_p(G/K) + 1.$$

As a consequence, $\ell_p(G) = \ell_p(G/K) + 1$ and $\mathrm{Bcd}_{p'}(G/K) = \mathrm{Bcd}_{p'}(G)$.

Therefore, we have at first that $K = \mathrm{O}_p(G)$, since otherwise $\ell_p(G/K) = \ell_p(G)$. Since K is arbitrarily chosen, it follows that $\mathrm{O}_{p'}(G) = 1$. Moreover, if $N = \mathrm{O}_{p',p}(G)$, we have that $\mathrm{Bcd}_{p'}(G/N) = \mathrm{Bcd}_{p'}(G/K) = \mathrm{Bcd}_{p'}(G)$.

Let Y be a complement for K in N. By the Frattini argument, we have that $G = \mathrm{N}_G(Y)N = \mathrm{N}_G(Y)K$. Moreover,

$$\mathrm{N}_G(Y) \cap K = \mathrm{C}_K(Y) = \mathrm{O}_p(\mathrm{Z}(N)) \lhd G.$$

Since, however, K is a normal minimal subgroup of G and $K \not\leq \mathrm{Z}(N)$ (otherwise, $Y \leq \mathrm{O}_{p'}(G) = 1$), we have $\mathrm{N}_G(Y) \cap K = 1$. Hence, K is complemented in G and, by

Lemma 3.4, every $\lambda \in \mathrm{Irr}(K)$ has an extension to its inertia subgroup G_λ of the same order.

Since we have proved that $\mathrm{C}_K(Y) = 1$, the only element in K to be centralized by Y is 1. Since K is abelian, it follows that no nonprincipal character in $\mathrm{Irr}(K)$ is N-invariant. Thus, for any $\lambda \in \mathrm{Irr}(K)$, $N \nleq G_\lambda$. On the other hand, let P be a Sylow p-subgroup of the group G, then $\mathrm{Z}(P) \cap K \neq 1$. Therefore, there exists $\lambda \in \mathrm{Irr}(K) \setminus \{1_K\}$ which is P-invariant.

Let $\lambda \in \mathrm{Irr}(K)$ be a nonprincipal P-invariant character and let $T = G_\lambda$, so that $P \leq T$ and λ has an extension to T of order p (notice that $o(\lambda) = p$ because K is elementary abelian). It follows from Proposition 3.5 that there exist some characters in $\mathrm{B}_p(G) \cap \mathrm{Irr}_{p'}(G)$ lying over λ with an associated Fong character of order p and, by Corollary 3.3, they have values in $\mathbb{Q}_p$. Among these characters, let χ be the one of maximal degree. By the third paragraph of the proof, $\mathrm{Bcd}_{p'}(G/N) = \mathrm{Bcd}_{p'}(G)$, thus, there exists $\psi \in \mathrm{B}_p(G/N)$ having values in $\mathbb{Q}_p$ such that $\psi(1) = \chi(1)$. Let $\gamma \in \mathrm{Lin}(PN/N)$ be a Fong character associated with ψ. If we consider ψ and γ as characters of, respectively, G and PN, we have that $\varepsilon = \gamma_P$ is a Fong character associated with ψ (as a character in $\mathrm{B}_p(G)$), since it is a constituent of ψ_P and $\varepsilon(1) = \psi(1)_p = 1$. Notice that $o(\varepsilon) = p$ because of Corollary 3.3.

Let W be the unique maximal subgroup of G such that ε extends to W and notice that $N \leq W$. Moreover, let $\hat{\lambda}$ be an extension of λ to T of order p, let $\upsilon = \hat{\lambda}_P$ and let V be the unique maximal subgroup of G such that the linear character $\rho = \varepsilon\upsilon$ extends to V. Notice that both W and V exist by Theorem 2.6 and, for the last part of that theorem, $\psi(1) = |G : W|$.

Now, let $\theta \in \mathrm{Irr}(V)$ be the unique p-special extension of ρ to V, then $\theta_K = \rho_K = \lambda$, as $K \leq \ker(\varepsilon)$. Therefore, λ extends to V and it follows that $V \leq T$. Moreover, if $\xi = \hat{\lambda}_V$, then $(\theta\bar{\xi})_P = \rho\bar{\upsilon} = \varepsilon$; since W is the unique maximal extension subgroup for ε, it follows that $W \geq V$. However, for the second part of Theorem 2.6, we have that $\theta^G \in \mathrm{B}_p(G)$ and, by Corollary 3.3, θ^G have values in $\mathbb{Q}_p$, because $o(\rho) = p$. Since θ lies over λ, the choice of χ to be of maximal degree leads to

$$|G : V| = \theta^G(1) \leq \chi(1) = \psi(1) = |G : W| .$$

Therefore, $V = W$ and W is contained in T.

However, we showed earlier in the proof that $N \nleq T$. On the other hand, we also have that $N \leq W \leq T$ and this leads to a contradiction. □

We can now proof Theorem A.

Proof of Theorem A. Since $\mathrm{cd}_{\mathbb{Q}_p,p'}(G) \supseteq \mathrm{cd}^{\mathrm{B}_p}_{\mathbb{Q}_p,p'}(G)$, the conclusion is true for Theorem 3.1. □

To conclude, let us see some example to show that, in general, for a p-solvable group G, $|\mathrm{cd}_{p'}(G)| \neq \left|\mathrm{cd}_{\mathbb{Q}_p,p'}(G)\right| \neq \left|\mathrm{cd}^{\mathrm{B}_p}_{\mathbb{Q}_p,p'}(G)\right|$.

Example 3.6. Let $G = \mathrm{A}_5 \times \mathrm{C}_7$ and $p = 7$. We can see that $\mathrm{cd}_{7'}(G) = \{1, 3, 4, 5\}$, $\mathrm{cd}_{\mathbb{Q}_7,7'}(G) = \{1, 4, 5\}$ and $\mathrm{cd}^{\mathrm{B}_7}_{\mathbb{Q}_7,7'}(G) = \{1\}$. Clearly, $\ell_7(G) = 1$.

Example 3.7. Let G be the semidirect product of $\mathrm{SL}(2, 3)$ acting naturally on $(\mathbb{Z}_3)^2$ and let $p = 3$, an easy computation can show that $\mathrm{cd}_{3'}(G) = \mathrm{cd}_{\mathbb{Q}_3,3'}(G) = \{1, 2, 8\}$ and $\mathrm{cd}^{\mathrm{B}_3}_{\mathbb{Q}_3,3'}(G) = \{1, 2\}$. Notice that, in this case, $\ell_3(G) = 2$.

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