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Character degrees in π-separable groups

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Abstract. If a group G is π-separable, where π is a set of primes, the set of irreducible characters $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ can be defined. In this paper, we prove variants of some classical theorems in character theory, namely the theorem of Ito–Michler and Thompson's theorem on character degrees, involving irreducible characters in the set $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$.

1 Introduction

The character theory of π-separable groups was first introduced by I. Martin Isaacs in 1984 in a series of papers, starting with [5]. In those papers, Isaacs introduced, for a π-separable group G, the family $\mathrm{I}_\pi(G)$ of *π-partial characters*, defined on π-elements of G, and a family of irreducible lifts $\mathrm{B}_\pi(G)$ for these characters. The aim of the paper was originally to generalize, for π-separable groups, the concept of Brauer characters; in fact, if the group G is p-solvable, one has that $\mathrm{I}_{p'}(G) = \mathrm{IBr}_p(G)$ (where p' is the set of all primes different from p).

In this paper, we study how the degrees of the B_π-characters (and, therefore, of the π-partial characters) influence the group structure.

In particular, we will focus on the degrees of the characters belonging to the set $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$, with π' being the complementary set of π. This set of characters is in general smaller than $\mathrm{Irr}(G)$, and it is easy to find examples of a π-separable group G and a character $\psi \in \mathrm{Irr}(G)$ such that $\psi(1) \neq \chi(1)$ for any $\chi \in \mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$. Nevertheless, in π-separable groups, the degrees of the characters in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ present some properties which are usually associated with the degrees of the characters in $\mathrm{Irr}(G)$. In this paper, we present a version for the B_π-characters of two theorems about character degrees, which are sometimes considered as dual: the theorem of Ito–Michler and Thompson's theorem on character degrees.

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The famous theorem of Ito–Michler says that a group G has a normal abelian Sylow p-subgroup, for some prime p, if and only if p does not divide the degree of any irreducible character of the group. If G is π-separable, we see that there exists a version of the theorem involving only the characters in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$.

Theorem A. *Let G be a π-separable group, and let p be any prime. Then G has a normal abelian Sylow p-subgroup if and only if p does not divide the degree of any character in* $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$.

Of course, there is an easy corollary following from this.

Corollary B. *Let G be a π-separable group, and let p be any prime. Then p divides the degree of some characters in* $\mathrm{Irr}(G)$ *if and only if it divides the degree of some characters in* $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$.

The well-known Thompson theorem on character degrees affirms that, if a prime p divides the degree of every nonlinear irreducible character of a group G, then G has a normal p-complement.

In [11], Navarro and Wolf studied a variant of the theorem involving more than one prime. Let $\mathrm{Irr}_{\pi'}(G)$ be the set of irreducible characters whose degrees is not divided by any prime in π. To ask that p divides the degree of every irreducible nonlinear character of G is equivalent to asking that $\mathrm{Irr}_{p'}(G) = \mathrm{Lin}(G)$. In [11, Corollary 3], Navarro and Wolf considered more than one prime and proved that, if G is a π-separable group and H is a Hall π-subgroup of G, then $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$ if and only if $G' \cap \mathrm{N}_G(H) = H'$.

In this paper, we first establish an equivalence between the condition on character degrees studied in the aforementioned [11, Corollary 3] and the same condition restricted to the set of characters $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$.

Theorem C. *Let G be a π-separable group. Then* $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$ *if and only if* $\mathrm{Irr}_{\pi'}(G) \cap (\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)) \subseteq \mathrm{Lin}(G)$.

Afterwards, the paper focuses on variants of Thompson's theorem considering only B_π-characters or $\mathrm{B}_{\pi'}$-characters.

Theorem D. *Let G be a π-separable group, let H be a Hall π-subgroup for G, and let $N = \mathrm{N}_G(H)$. Then*

(a) $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G) \subseteq \mathrm{Lin}(G)$ *if and only if $G' \cap H = H'$;*

(b) $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_{\pi'}(G) \subseteq \mathrm{Lin}(G)$ *if and only if $G' \cap N \leq H$.*

The reader may notice that these last results are actually strongly related with [11, Corollary 3]. In fact, since all the proofs presented here are independent from

[11, Corollary 3], Theorem C and Theorem D can provide an alternative, even if not shorter, proof of it.

2 Review of the π-theory

In this section, we are going to recall briefly some essential concepts of the character theory of π-separable group. In [5], or in the first part of [8], the reader can find a more extensive account.

We recall that a π-separable group always has a Hall π-subgroup, and two Hall π-subgroups are always conjugate. We use standard notation, and we write $\mathrm{Hall}_\pi(G)$ to denote the set of all Hall π-subgroups of the group G.

At first, we need to define another subset of the irreducible characters, the *π-special characters*.

Definition ([2]). Let $\chi \in \mathrm{Irr}(G)$; then χ is said to be a *π-special character* if, for any $M \leq G$ subnormal in G, every irreducible constituent of χ_M has order and degree which are both π-numbers.

We write $\mathrm{X}_\pi(G)$ for the set of all π-special characters of the group G. The concept of π-special character is essential for further developments of the theory and for the definition of the B_π-characters.

Theorem 2.1 ([5]). *Let G be a π-separable group, and let $\chi \in \mathrm{Irr}(G)$. Then there exist a subgroup $W \leq G$, canonically defined up to conjugation, $\alpha \in \mathrm{X}_\pi(W)$ and $\beta \in \mathrm{X}_{\pi'}(W)$ such that $\chi = (\alpha\beta)^G$.*

In the notation of Theorem 2.1, if $\beta = 1_W$, the character χ is a B_π-character. The relation between B_π and π-special characters is actually even stronger.

Theorem 2.2 ([5, Lemma 5.4]). *Let $\chi \in \mathrm{B}_\pi(G)$; then χ is π-special if and only if $\chi(1)$ is a π-number.*

Moreover, if $\chi \in \mathrm{Irr}_{\pi'}(G)$, there exists a stronger version of Theorem 2.1, due to Isaacs and Navarro.

Theorem 2.3 ([9, Theorem 3.6]). *Let G be a π-separable group, $H \in \mathrm{Hall}_\pi(G)$, and let $\chi \in \mathrm{Irr}_{\pi'}(G)$. Then there exist a subgroup $H \leq W \leq G$, $\alpha \in \mathrm{X}_\pi(W)$ linear and $\beta \in \mathrm{X}_{\pi'}(W)$ such that $\chi = (\alpha\beta)^G$. Moreover, α and β are unique in $\mathrm{Irr}(W)$ with this property. Finally, W can be chosen as the (unique) maximal subgroup of G such that α_H extends to W.*

Theorem 2.3 may suggest to the reader a property of the π-special character extension, which we are going to state here.

Theorem 2.4. *Let G be a π-separable group, and let $H \leq G$ such that $|G : H|$ is a π'-number. Let $\psi \in \mathrm{Irr}(H)$ be a π-special character, and suppose ψ extends to G. Then ψ has an extension to G which is also a π-special character.*

Proof. This is a direct consequence of the more general result [7, Theorem F]. □

Let us recall the behavior of B_π-characters in relation to normal subgroups.

Theorem 2.5. *Let G be π-separable, let $M \trianglelefteq G$, and let $\chi \in \mathrm{B}_\pi(G)$; then every irreducible constituent of χ_M belongs to $\mathrm{B}_\pi(M)$.*

On the other hand, let $\psi \in \mathrm{B}_\pi(M)$; if $|G : M|$ is a π-number, then every irreducible constituent of ψ^G is in $\mathrm{B}_\pi(G)$ while, if $|G : M|$ is a π'-number, then there exists a unique irreducible constituent of ψ^G which belongs to $\mathrm{B}_\pi(G)$.

In particular, if $\psi \in \mathrm{B}_\pi(M)$, then there always exists at least one character $\chi \in \mathrm{Irr}(G \mid \psi)$ which belongs to $\mathrm{B}_\pi(G)$.

Proof. This is a direct consequence of [5, Theorem 6.2] and [5, Theorem 7.1] □

A basic property of B_π-characters concerns the restriction to Hall π-subgroups.

Theorem 2.6 ([5, Theorem 8.1]). *Let $\chi \in \mathrm{B}_\pi(G)$, and let $H \in \mathrm{Hall}_\pi(G)$; then there exists an irreducible constituent φ of χ_H such that $\varphi(1) = \chi(1)_\pi$. Moreover, for any irreducible constituent φ of χ_H such that $\varphi(1) = \chi(1)_\pi$, the multiplicity of φ in χ_H is 1 and φ does not appear as an irreducible constituent of the restriction to H of any other character in $\mathrm{B}_\pi(G)$.*

A character $\varphi \in \mathrm{Irr}(H)$ like the ones in Theorem 2.6 is called a *Fong character* associated with χ. It is in general quite difficult to tell if an irreducible character of the Hall π-subgroup H is a Fong character associated with some B_π-character of G; a characterization is presented in [6]. The problem is simpler, however, if one considers only the primitive characters of H.

Theorem 2.7 ([6, Corollary 6.1] or [8, Theorem 5.13]). *Let H be a Hall π-subgroup of a π-separable group G, and let $\varphi \in \mathrm{Irr}(H)$. If φ is primitive, then it is a Fong character associated with some character in $\mathrm{B}_\pi(G)$. If φ_1 is another primitive irreducible character of H, then φ and φ_1 are associated with the same character in $\mathrm{B}_\pi(G)$ if and only if they are $\mathrm{N}_G(H)$-conjugated.*

Considering the nature of the results presented in this paper, a natural question a reader may ask is whether the set $\mathrm{B}_\pi(G) \cup \mathrm{B}_\pi(G)$ is actually strictly smaller than $\mathrm{Irr}(G)$. As we anticipated in the introduction, it happens quite often. In fact, one of the properties of B_π-characters (see [5, Theorem 9.3]) is that $|\mathrm{B}_\pi(G)|$ is equal to the number of conjugacy classes of π-elements of G. Therefore, we have $\mathrm{B}_\pi(G) \cup \mathrm{B}_\pi(G) = \mathrm{Irr}(G)$ if and only if each element of the π-separable group G is either a π-element or a π'-element. This is proved in [3, Lemma 4.2] to happen if and only if G is a Frobenius or a 2-Frobenius group and each Frobenius complement and Frobenius kernel is either a π-group or a π'-group.

Let us denote by $\mathrm{cd}(G)$ the set of irreducible character degrees of G, and let us write $\mathrm{cd}_{\mathrm{B}_\pi}(G)$ and $\mathrm{cd}_{\mathrm{B}_{\pi'}}(G)$ for the sets of character degrees of, respectively, B_π-characters and $\mathrm{B}_{\pi'}$-characters of G. Even when $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ is strictly smaller than $\mathrm{Irr}(G)$, it may happen that $\mathrm{cd}(G) = \mathrm{cd}_{\mathrm{B}_\pi}(G) \cup \mathrm{cd}_{\mathrm{B}_{\pi'}}(G)$. This happens, for example, if we consider the group $\mathrm{SL}(2,3) \ltimes (\mathbb{Z}_3)^2$, with $\pi = \{2\}$, or the group $(\mathrm{C}_3 \ltimes \mathrm{C}_7) \wr \mathrm{C}_2$, with $\pi = \{7\}$.

However, for a π-separable group G, in general, $\mathrm{cd}(G) \neq \mathrm{cd}_{\mathrm{B}_\pi}(G) \cup \mathrm{cd}_{\mathrm{B}_{\pi'}}(G)$. A first, obvious example of this fact is when $G = H \times K$, with H a π-group and K a π'-group, both nonabelian. In this case, in fact, we have that $\mathrm{B}_\pi(G) = \mathrm{Irr}(H)$ and $\mathrm{B}_{\pi'}(G) = \mathrm{Irr}(K)$.

Let us also see a less trivial example of this fact.

Example 2.8. Let $G = H \ltimes M$, with $H = \mathrm{SL}(2,3)$ acting canonically on the vector space $M = (\mathbb{Z}_3)^2$. Computing the character table of G, we can see that $\mathrm{cd}(G) = \{1,2,3,8\}$ and, with a little more work, it is not hard to prove that $\mathrm{cd}_{\mathrm{B}_3}(G) = \{1,8\}$ and $\mathrm{cd}_{\mathrm{B}_2}(G) = \{1,2,3\}$.

Now, let $\Gamma = G \wr \mathrm{C}_2$, let $\theta \in \mathrm{B}_3(G)$ of degree 8, and let $\eta \in \mathrm{B}_2(G)$ of degree 3. The character $\theta \times \eta \in \mathrm{Irr}(G \times G)$ induces irreducibly to Γ and $\chi(1) = 48$.

Suppose that there exists $\psi \in \mathrm{B}_2(\Gamma) \cup \mathrm{B}_3(\Gamma)$ such that $\psi(1) = \chi(1)$, and let $\lambda_1 \times \lambda_2$ be an irreducible constituent of $\psi_{G\times G}$. Then λ_1 and λ_2 are either both in $\mathrm{B}_2(G)$ or they are both in $\mathrm{B}_3(G)$. Moreover, since $\psi(1) = 48$, then we have $\lambda_1(1)\lambda_2(1) \in \{24, 48\}$. However, neither 24 nor 48 can be written as a product of two numbers in $\mathrm{cd}_{\mathrm{B}_2}(G)$ or as a product of two numbers in $\mathrm{cd}_{\mathrm{B}_3}(G)$. It follows that $48 \in \mathrm{cd}(\Gamma)$ but $48 \notin \mathrm{cd}_{\mathrm{B}_2}(\Gamma) \cup \mathrm{cd}_{\mathrm{B}_3}(\Gamma)$.

3 Character degrees and normal subgroups

In this section, we are going to see the proof of Theorem A. We also give a different proof of a result which appears in [8].

The technique we use to prove the results in this section mirrors the one used in [1]. In particular, the key result for the section is the following lemma, borrowed from [1].

Lemma 3.1. *Let G be a group, let N be a normal minimal π'-subgroup, and let $M \trianglelefteq G$ such that M/N is an abelian π-group. Furthermore, suppose that*

$O_\pi(M) = 1$. *Then there exists a character* $\chi \in B_{\pi'}(G)$ *such that* $\chi(1)$ *is divided by* $|M : N|$.

Proof. Let A be a complement for N in M, which exists by the Schur–Zassenhaus theorem. Since $C_A(N) \leq Z(M)$ and A is a π-group,

$$C_A(N) \leq O_\pi(M) \leq O_\pi(G) = 1.$$

Thus, A acts faithfully on N. By [1, Lemma 2.8], there exists some character $\tau \in \mathrm{Irr}(N)$ such that $\eta = \tau^M \in \mathrm{Irr}(M)$. In particular, $|M : N|$ divides $\eta(1)$. Since $\tau \in B_{\pi'}(N) = \mathrm{Irr}(N)$ and N is normal in M, by Theorem 2.5, the character η is in $B_{\pi'}(M)$, too. It follows that $B_{\pi'}(G \mid \eta)$ is nonempty and $|M : N|$ divides the degree of every character in $B_{\pi'}(G \mid \eta)$. □

Now, as anticipated, we are going to present a different proof of [8, Theorem 3.17] using Lemma 3.1.

Theorem 3.2 ([8, Theorem 3.17]). *Let* G *be* π*-separable; then* $B_\pi(G) = X_\pi(G)$ *if and only if* G *has a normal* π*-complement.*

Proof. Note at first that, if G has a normal π-complement H, then it follows that $B_\pi(G) = X_\pi(G) = \mathrm{Irr}(G/H)$. Thus, there is only one implication to be proved.

Let us assume that $B_\pi(G) = X_\pi(G)$ and prove the thesis by induction on $|G|$. At first, let us assume that $O_{\pi'}(G) = 1$ since, otherwise, the thesis would follow by induction.

Let N be a normal minimal subgroup of G, and suppose it to be a π-group. Since the hypotheses are preserved by factor groups, if H is a Hall π'-subgroup of G, then, by induction on $|G|$, we have that HN is normal in G. In particular, it follows that there exists $K \lhd G$ such that K/N is a π'-chief factor of G. Since $|N|$ and $|K/N|$ are coprime, at least one of them is odd and, thus, since an odd group is solvable and both N and K/N are normal minimal in G, at least one of them is abelian.

Suppose K/N is abelian. Since we have assumed $O_{\pi'}(G) = 1$, it follows by Lemma 3.1 that there exists a character in $B_\pi(G)$ whose degree is divided by the π'-number $|K : N|$, contradicting the hypothesis.

Suppose that K/N is not abelian, so N has to be, and let $\lambda \in \mathrm{Irr}(N) = B_\pi(N)$. If λ is not K-invariant, then the degree of some $\theta \in B_\pi(K \mid \lambda)$ is divided by some primes in π' and, therefore, so is the degree of some character $\chi \in B_\pi(G \mid \theta)$, in contradiction with the hypothesis. It follows that K fixes every character of N and, thus, it also centralizes N since N is abelian. If B is a complement of N in K, then it is normal in K. In particular, $1 < B = O_{\pi'}(K) \leq O_{\pi'}(G)$.

Therefore, $O_{\pi'}(G) \neq 1$, and the thesis follows by induction on $|G|$. □

We now prove Theorem A. We mention that the proof was simplified after some suggestions from an anonymous reviewer, who we thank.

Proof of Theorem A. It can be observed that there is little to prove in one direction, it being a consequence of the Ito–Michler theorem. Thus, we assume that p does not divide the degree of any character in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$, and we first prove that G has a normal Sylow p-subgroup. We argue by induction on $|G|$.

Let N be a minimal normal subgroup of G. Without loss of generality, we can assume N to be a π-group. Suppose $p \in \pi$. By induction, let K/N be a normal Sylow p-subgroup of G/N; then K is a normal π-subgroup of G which contains a Sylow p-subgroup $P \in \mathrm{Syl}_p(G)$. If P is normal in K, then it is also normal in G. Otherwise, there exists $\theta \in \mathrm{Irr}(K) = \mathrm{B}_\pi(K)$ such that $p \mid \theta(1)$ and, by Theorem 2.5, there exists $\chi \in \mathrm{B}_\pi(G)$ lying over θ. As a consequence, $p \mid \chi(1)$, in contradiction with the hypothesis.

Therefore, we can assume $p \notin \pi$ and, in particular, p does not divide $|N|$. Since N is arbitrarily chosen, we can assume that $\mathrm{O}_p(G) = 1$. As in the previous paragraph, let K/N be a normal Sylow p-subgroup of G/N, which is nontrivial because p divides $|G : N|$, and let $C/N = \mathrm{Z}(K/N)$. Note that $N < C \trianglelefteq G$ and C/N is an abelian π'-group. Then, by Lemma 3.1, there exists a character χ in $\mathrm{B}_\pi(G)$ such that $|C : N|$ divides $\chi(1)$. However, since $|C : N|$ is a power of p, this would contradict the hypothesis.

Finally, if P is a normal Sylow p-subgroup of G and $\gamma \in \mathrm{Irr}(P)$, then, by Theorem 2.5, there exists $\chi \in \mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ lying over γ and, thus, $\gamma(1) \mid \chi(1)$. Since $p \nmid \chi(1)$, then γ is linear. It follows that $\mathrm{Irr}(P) = \mathrm{Lin}(P)$ and, thus, P is abelian. □

4 Variants on Thompson's theorem for B_π-characters

In this section, we prove Theorem C and Theorem D, concerning some variations of Thompson's theorem for B_π-characters and for more than one prime.

For the section, we need a variant of the McKay conjecture for π-separable groups, due to T. Wolf.

Theorem 4.1 ([12, Theorem 1.15]). *Let π and ω be two sets of primes, and let G be both π-separable and ω-separable. Let H be a Hall ω-subgroup of G, and let $N = \mathrm{N}_G(H)$. Then*

$$|\{\chi \in \mathrm{B}_\pi(G) \mid \chi(1) \text{ is an } \omega'\text{-number}\}| = |\{\psi \in \mathrm{B}_\pi(N) \mid \psi(1) \text{ is an } \omega'\text{-number}\}|.$$

In particular, we need its obvious corollary.

Corollary 4.2. *Let G be a π-separable group, let H be a Hall π-subgroup of G, and let $N = \mathrm{N}_G(H)$. Then*

$$|\{\chi \in \mathrm{B}_\pi(G) \mid \chi(1) \text{ is a } \pi'\text{-number}\}| = |\{\psi \in \mathrm{B}_\pi(N) \mid \psi(1) \text{ is a } \pi'\text{-number}\}|,$$
$$|\mathrm{X}_{\pi'}(G)| = |\mathrm{X}_{\pi'}(N)| = |\mathrm{Irr}(N/H)|.$$

At first, an easy lemma is needed, which uses the properties of the Fong characters associated with a B_π-character.

Lemma 4.3. *Let G be a π-separable group, and let H be a Hall π-subgroup of G; then $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G) \subseteq \mathrm{Lin}(G)$ if and only if every linear character in H extends to G.*

Proof. Let λ be a linear character in H. By Theorem 2.7 and Theorem 2.6, λ is the Fong character associated with some character $\chi \in \mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G)$. It follows that, if χ is linear, then it extends λ while, on the other hand, if λ extends to G, then, by Theorem 2.4, it has a linear π-special extension, which coincides with χ by Theorem 2.6. □

Now, we can already prove Theorem C, which relates the families of characters $\mathrm{Irr}(G)$ and $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ for what concerns the hypothesis of Thompson's theorem.

Proposition 4.4 (Theorem C). *Let G be a π-separable group. Then we have that $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$ if and only if $\mathrm{Irr}_{\pi'}(G) \cap \big(\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)\big) \subseteq \mathrm{Lin}(G)$.*

Proof. One direction is obviously true. Thus, let one assume

$$\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G) \subseteq \mathrm{Lin}(G),$$

and suppose there exists a nonlinear character $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)$ is a π'-number. By Theorem 2.3, there exist $W \leq G$, $\alpha \in \mathrm{X}_\pi(W)$ linear and $\beta \in \mathrm{X}_{\pi'}(W)$ such that $\chi = (\alpha\beta)^G$, W contains a Hall π-subgroup H of G and it is the maximal subgroup of G such that α_H extends to W. However, by Lemma 4.3, α_H extends to G; thus $W = G$. It follows that β is a nonlinear π'-special character of G, negating the fact that every character in $\mathrm{X}_{\pi'}(G) = \mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_{\pi'}(G)$ is linear, in contradiction with the hypothesis. □

At this point, we can already prove a related result, concerning a sub-case of Theorem D and of [11, Corollary 3].

Corollary 4.5. *Let G be a π-separable group, and let H be a Hall π-subgroup of G. Then the following statements hold.*

(i) $\mathrm{Irr}_{\pi'}(G) = \{1_G\}$ *if and only if* $\mathrm{Irr}_{\pi'}(G) \cap (\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)) = \{1_G\}$.

(ii) $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G) = \{1_G\}$ *if and only if* $H = H'$.

(iii) $\mathrm{X}_{\pi'}(G) = \{1_G\}$ *if and only if* H *is self-normalizing.*

(iv) $\mathrm{Irr}_{\pi'}(G) = \{1_G\}$ *if and only if* $H = H'$ *and* H *is self-normalizing.*

Proof. For point (i), only one direction is needed. Suppose, thus, that

$$\mathrm{Irr}_{\pi'}(G) \cap (\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)) = \{1_G\} \subseteq \mathrm{Lin}(G).$$

Then, by Proposition 4.4, $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$. It follows that every character in $\mathrm{Irr}_{\pi'}(G)$ can be factorized as a product $\alpha\beta$, with $\alpha \in \mathrm{Irr}_{\pi'} \cap \mathrm{X}_\pi(G)$, $\beta \in \mathrm{X}_{\pi'}(G)$; however, $\mathrm{Irr}_{\pi'} \cap \mathrm{X}_\pi(G)$ is a subset of $\mathrm{Irr}_{\pi'} \cap \mathrm{B}_\pi(G)$, while

$$\mathrm{X}_{\pi'}(G) = \mathrm{Irr}_{\pi'} \cap \mathrm{B}_{\pi'}(G),$$

and the two sets of characters both coincide with $\{1_G\}$ by hypothesis. Therefore, it follows that $\alpha = \beta = 1_G$ and $\mathrm{Irr}_{\pi'}(G) = \{1_G\}$.

Point (ii) follows directly from Lemma 4.3. In fact, if

$$\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G) = \{1_G\} \subseteq \mathrm{Lin}(G),$$

then every character in $\mathrm{Lin}(H)$ extends to G and, by Theorem 2.4, it has an extension in $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G)$, and thus $\mathrm{Lin}(H) = \{1_H\}$. On the other hand, if $\mathrm{Lin}(H) = \{1_H\}$, then there are no nonprincipal linear Fong characters of H in G, and it follows that $|\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G)| = 1$, and the thesis follows.

Finally, point (iii) is a direct consequence of Corollary 4.2, and point (iv) follows from points (i), (ii) and (iii). □

Let us now proceed by proving Theorem D. In particular, point (a) of Theorem D can be seen as a consequence of a result which does not depend on Isaacs π-theory.

Proposition 4.6. *Let* G *be a finite group, and let* H *be a Hall* π*-subgroup for* G*. Then* $G' \cap H = H'$ *if and only if every character in* $\mathrm{Irr}(H/H')$ *extends to* G.

Proof. Suppose that, for every $\lambda \in \mathrm{Irr}(H/H')$, there exists $\chi \in \mathrm{Lin}(G)$ such that $\chi_H = \lambda$. It follows that

$$H' \leq G' \cap H \leq \bigcap_{\chi \in \mathrm{Lin}(G)} \ker(\chi_H) = \bigcap_{\lambda \in \mathrm{Lin}(H)} \ker(\lambda) = H'$$

and, therefore, $G' \cap H = H'$.

On the other hand, suppose $G' \cap H = H'$. Then one can write

$$\frac{G}{G'} = \frac{HG'}{G'} \times \frac{K}{G'} \cong \frac{H}{H'} \times \frac{K}{G'}$$

for some $K/G' \in \mathrm{Hall}_{\pi'}(G/G')$ and, if we identify $\mathrm{Irr}(H/H')$ with $\mathrm{Irr}(HG'/G')$, we have that every $\lambda \in \mathrm{Irr}(H/H')$ extends to $\lambda \times 1_{K/G'} \in \mathrm{Irr}(G/G')$. □

Corollary 4.7 (Theorem D (a)). *Let G be a π-separable group, and let H be a Hall π-subgroup for G. Then $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G) \subseteq \mathrm{Lin}(G)$ if and only if $G' \cap H = H'$.*

Proof. From Lemma 4.3, the property that every character in $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G)$ is linear is equivalent to the fact that every character in H/H' extends to G. By Proposition 4.6, we deduce that it happens if and only if $G' \cap H = H'$. □

Proposition 4.8 (Theorem D (b)). *Let G be a π-separable group, let H be a Hall π-subgroup for G, and let $N = \mathrm{N}_G(H)$. Then $\mathrm{X}_{\pi'}(G) \subseteq \mathrm{Lin}(G)$ if and only if $G' \cap N \leq H$.*

Proof. Assume at first that $\mathrm{X}_{\pi'}(G) \subseteq \mathrm{Lin}(G)$; therefore, if $\chi \in \mathrm{X}_{\pi'}(G)$, then χ_N is linear. Suppose that, for some $\chi, \psi \in \mathrm{X}_{\pi'}(G)$, we have $\chi_N = \psi_N$; then

$$N \leq \ker(\chi\bar{\psi}) \lhd G.$$

It follows that $\ker(\chi\bar{\psi}) = G$, by the Frattini argument, and thus $\chi = \psi$. Therefore, the restriction realizes an injection from $\mathrm{X}_{\pi'}(G)$ to $\mathrm{X}_{\pi'}(N)$ and, since we have $|\mathrm{X}_\pi(G)| = |\mathrm{X}_\pi(N)|$, by Corollary 4.2, it is actually a bijection. It follows that every character in $\mathrm{Irr}(N/H)$ is the restriction of a linear character of G; thus we have that

$$G' \cap N \leq \bigcap_{\chi \in \mathrm{Lin}(G)} \ker(\chi_N) \leq \bigcap_{\lambda \in \mathrm{Irr}(N/H)} \ker(\lambda) = H.$$

On the other hand, suppose that $G' \cap N \leq H$. Let X be a complement for H in N, and notice that X is abelian. Moreover, notice that NG' is normal in G and it contains N; thus $G = NG'$ for the Frattini argument. It follows that

$$\frac{G}{G'} \cong \frac{N}{G' \cap N} = X \times \frac{H}{G' \cap H}$$

and, thus, there is a bijection between characters in $\mathrm{Irr}(X) = \mathrm{Irr}(N/H)$ and characters in $\mathrm{X}_{\pi'}(G/G')$. However, by Corollary 4.2, we have that

$$|\mathrm{X}_{\pi'}(G/G')| = |\mathrm{Irr}(N/H)| = |\mathrm{X}_{\pi'}(G)|$$

and, thus, it follows that every π'-special character in G is linear. □

As anticipated in the introduction, it can be easily seen that [11, Corollary 3] can also be obtained as a corollary of these last results.

Corollary 4.9 ([11, Corollary 3]). *Let G be a π-separable group, and let H be a Hall π-subgroup for G and $N = \mathrm{N}_G(H)$. Then $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$ if and only if $G' \cap N = H'$.*

Proof. If N is the normalizer in G of a Hall π-subgroup H, Proposition 4.4, Corollary 4.7 and Proposition 4.8 provide that $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$ if and only if both $G' \cap N \leq H$ and $G' \cap H = H'$, and the thesis follows directly from this. □

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