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A Fourier analytic proof of the Blaschke-Santalo Inequality

(Article begins on next page)

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A FOURIER ANALYTIC PROOF OF THE BLASCHKE-SANTALÓ INEQUALITY

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ABSTRACT. The Blaschke-Santaló Inequality is the assertion that the volume product of a centrally symmetric convex body in Euclidean space is maximized by (and only by) ellipsoids. In this paper we give a Fourier analytic proof of this fact.

1. INTRODUCTION

Let K be a *convex body* in $\mathbb{R}^N$, that is a compact convex subset of $\mathbb{R}^N$ with non-empty interior, and assume that the origin is an interior point of K. We associate to K another convex body K^*, called the *dual body* or *polar body* of K, defined by

$$K^* = \{\mathbf{y} \in \mathbb{R}^N : \mathbf{x} \cdot \mathbf{y} \leq 1 \text{ for each } \mathbf{x} \in K\},$$

where $\mathbf{x} \cdot \mathbf{y}$ is the usual scalar product. The terminology *dual body* is fitting, because the unit ball of any norm in $\mathbb{R}^N$ is a convex body and its dual body is the unit ball of the corresponding dual norm.

Assume that K is origin symmetric, i.e. $K = -K$. The product

$$P(K) = \text{vol}_N(K)\text{vol}_N(K^*),$$

where vol_N denotes N-dimensional Lebesgue measure in $\mathbb{R}^N$, is called the *volume product* of K. For a general convex body K the volume product $P(K)$ is defined as the minimum, for x in the interior of K, of $\text{vol}_N(K)\text{vol}_N\left((K-x)^*\right)$. Here $K-x$ is the translate of K by $-x$. The functional $P(K)$ is an affine invariant and thus all ellipsoids in $\mathbb{R}^N$ have the same volume product, and all parallelotopes in $\mathbb{R}^N$ have the same volume product. Furthermore, $P(K^*) = P(K)$, because $(K^*)^* = K$, and as a consequence, for instance, the volume product of the unit cube in $\mathbb{R}^3$ (the ℓ_∞ unit ball) is the same as the volume product of the octahedron (the ℓ_1 unit ball). All of these observations were made by Kurt Mahler in the 1930's, in connection to transference principles for linear forms (see Cassels [Cas92]).

A sharp upper bound for the volume product is given by the Blaschke-Santaló Inequality.

Theorem (The Blaschke-Santaló Inequality). *Let B denote the Euclidean unit ball of $\mathbb{R}^N$. For every convex body K in $\mathbb{R}^N$*

$$(1.1) \qquad\qquad P(K) \leq P(B),$$

and equality holds if and only if K is an ellipsoid.

Inequality (1.1) was first proved by Blaschke [Bla17] for $N = 2, 3$, by Santaló [San49] for any N. Petty [Pet85], completed the proof of the equality case.

These results were obtained in the context of affine differential geometry, as a consequence of results on the affine isoperimetric inequality and of its equality cases. Later proofs, which are more direct and use classical tools of convexity, are due, among others, to Saint Raymond [Sai81], for origin symmetric bodies, and to Meyer and Pajor [MP90]. See Schneider [Sch14] for a detailed account of the literature on the volume product.

It is our goal in this paper to prove the previous theorem, in the class of origin symmetric convex bodies, using a Fourier analytic approach. See Theorem 1 in Section 3 for the statement.

The minimum of the volume product for $N \geq 3$ is still unknown. It is conjectured that, for a convex body K, we have

$$(1.2) \qquad\qquad P(K) \geq \frac{(N+1)^{N+1}}{(N!)^2},$$

with equality precisely for simplices, and that, for an origin symmetric convex body K, we have

$$(1.3) \qquad\qquad P(K) \geq \frac{4^N}{N!},$$

with equality holding for affine transforms of cubes, of crosspolytopes and, more generally, for Hanner polytopes. Mahler [Mah39] was able to prove (1.2) and (1.3) when $N = 2$.

Inequality (1.3) is known as *Mahler conjecture* and it has remained open for over three quarters of a century. It has been proved in certain classes of bodies, for instance when K is a zonotope (see Reisner [Rei85, Rei86]) or when K is 1-unconditional, i.e. an affine transform of K is symmetric with respect to each coordinate hyperplane (see Saint Raymond [Sai81]). Recently it has been proved that the cube (see Petrov et al. [NPRZ10]) and every Hanner polytopes (see Kim [Kim13]) are local minimizers of the volume product (and strict local minimizers in the proper sense) in the class of origin-symmetric convex bodies. The interested reader is advised to consult Tao [Tao08] for a very nice discussion about the conjecture and some of its subtleties.

As far as we know, the Fourier analytic approach to study the volume product has been first used by F. Nazarov. He [Naz12] used it, together with Hörmander's solution to the $\bar{\partial}$ problem, to prove the Bourgain-Milman Inequality [BM87]

$$P(K) \geq c^N \frac{4^N}{N!},$$

where $c > 0$ is a constant not depending on N and $K \subset \mathbb{R}^N$ is an origin symmetric convex body. Ryabogin and Zvavitch [RZ] describes the ideas behind Nazarov's proof as well as those behind the proofs of some of the results on the Mahler conjecture mentioned above.

The main object in our investigation is the following functional.

Definition 1. Given an origin symmetric convex body $K \subset \mathbb{R}^N$ define

$$\rho(K) = \inf \int_{\mathbb{R}^N} |F(\mathbf{x})|^2 d\mathbf{x},$$

where the infimum is taken over the class of square-integrable continuous functions $F : \mathbb{R}^N \to \mathbb{C}$ that satisfy

   (1) $|F(\mathbf{0})| \geq 1$, and

(2) $\widehat{F}(\boldsymbol{\xi}) = 0$ if $\boldsymbol{\xi} \in \mathbb{R}^N \setminus K$.

As we will prove in Section 3, $\rho(K) = 1/\mathrm{vol}_N(K)$ and the only minimizers of ρ are admissible multiple of the inverse Fourier transform of the characteristic function 1_K of K. On the other hand, the Paley-Wiener Theorem (see next section) states that the analytic extension to $\mathbb{C}^N$ of every function F admissible for ρ has an asymptotic behavior at infinity which is related to the norm whose unit ball is K^*. This connection is at the hearth of this proof of the Blaschke-Santaló Inequality.

To deal with the equality cases, we will show that if K is origin symmetric and $P(K) = P(B)$, then, for each direction $\boldsymbol{\theta} \in \mathrm{S}^{N-1}$, there exists an ellipsoid E (which a priori may depend on $\boldsymbol{\theta}$) such that for each hyperplane L orthogonal to $\boldsymbol{\theta}$ the $(N-1)$-volume of the sections $K \cap L$ and $E \cap L$ coincide. This property, and a result proved by M. Meyer and S. Reisner [MR89, Lemma 3], imply that K is an ellipsoid.

In section 4 we introduce a variational quantity $\eta(K)$ associated with an origin symmetric convex body. It is essentially an L^1 version of $\rho(K)$. We state a conjecture (due to the second author and Jeffrey Vaaler) regarding the exact value of $\eta(K)$ and prove it when K is a ball or a cube.

For another problem in convex geometry where the Fourier transform of 1_K in $\mathbb{C}^n$ plays an important role see Bianchi [Bia13].

Gabriele Bianchi wishes to acknowledge that all proofs in this paper are due to Michael Kelly, except for that of the equality case in the Blaschke-Santaló Inequality, which is due to himself.

2. Background and Notation

Throughout this paper z denotes an element of the complex numbers $\mathbb{C}$, and $\overline{z}$ denotes the complex conjugate of z. The symbol $\mathcal{U} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ denotes the upper half plane of $\mathbb{C}$, where $\mathrm{Im}(z)$ is the *imaginary part* of z. We use boldface letters or symbols to denote vectors, $\mathbf{x}$ denotes a vector in $\mathbb{R}^N$, $\mathbf{z}$ a vector in $\mathbb{C}^N$, and $\mathrm{Im}(\mathbf{z})$ denotes the vector of the imaginary parts of $\mathbf{z}$. We write vol_k for k-dimensional Lebesgue measure in $\mathbb{R}^N$. By B and S^{N-1} we denote respectively the Euclidean unit ball and unit sphere in $\mathbb{R}^N$. The symbol ω_{N-1} indicates the surface area of S^{N-1}.

The *support function* of a convex body K in $\mathbb{R}^N$ is defined, for $\mathbf{x} \in \mathbb{R}^N$, by

$$h_K(\mathbf{x}) = \sup\{\mathbf{x} \cdot \mathbf{y} : \mathbf{y} \in K\}.$$

If K is an origin symmetric convex body and $\|\cdot\|_{K^*}$ denotes the norm in $\mathbb{R}^N$ whose unit ball is K^*, i.e. $\|\mathbf{x}\|_{K^*} = \inf\{\lambda > 0 : x \in \lambda K^*\}$, we have

(2.1) $$h_K(\mathbf{x}) = \|\mathbf{x}\|_{K^*}.$$

Given a convex body K in $\mathbb{R}^n$, $t \in \mathbb{R}$ and $\boldsymbol{\theta} \in \mathrm{S}^{N-1}$, we denote by $S_K(t, \boldsymbol{\theta})$ the *Radon transform* of the characteristic function 1_K of K

$$S_K(t, \boldsymbol{\theta}) = \mathrm{vol}_{N-1}\left(\{\mathbf{x} \in K : \mathbf{x} \cdot \boldsymbol{\theta} = t\}\right).$$

For a function $F \in L^2(\mathbb{R}^N)$ the *Fourier transform* $\widehat{F}$ is defined for $\boldsymbol{\xi} \in \mathbb{R}^N$ by

$$\widehat{F}(\boldsymbol{\xi}) = \lim_{T \to \infty} \int_{[-T,T]^N} e^{-2\pi i \mathbf{x} \cdot \boldsymbol{\xi}} F(\mathbf{x}) d\mathbf{x}.$$

A function $F : \mathbb{C}^N \to \mathbb{C}$ is an *entire function* if it is holomorphic, in each coordinate separately, at each $\mathbf{z} \in \mathbb{C}^N$. If F is an entire function, the complex conjugate F^* of F, defined by $F^*(\mathbf{z}) = \overline{F(\overline{\mathbf{z}})}$, is also an entire function.

Let K be an origin symmetric convex body in $\mathbb{R}^N$. Following Stein and Weiss [SW71, §3.4] we call an entire function F *of exponential type* K^* if for every $\epsilon > 0$ there exists a constant $c_\epsilon > 0$ such that, for every $\mathbf{y} \in \mathbb{R}^N$,

$$(2.2) \qquad\qquad |F(i\mathbf{y})| \leq c_\epsilon e^{2\pi(1+\epsilon)\|\mathbf{y}\|_{K^*}}.$$

When $F \in L^2(\mathbb{R}^N)$ is such that the support of $\widehat{F}$ is contained in K then it is well known that F is the restriction to $\mathbb{R}^N$ of the entire function defined, for $\mathbf{z} \in \mathbb{C}^N$, by the formula

$$(2.3) \qquad\qquad F(\mathbf{z}) = \int_K e^{2\pi i \mathbf{z} \cdot \boldsymbol{\xi}} \widehat{F}(\boldsymbol{\xi}) d\boldsymbol{\xi}.$$

This representation and the Cauchy-Schwarz Inequality imply

$$|F(i\mathbf{y})| \leq \int_K \left| e^{-2\pi \mathbf{y} \cdot \boldsymbol{\xi}} \widehat{F}(\boldsymbol{\xi}) \right| d\boldsymbol{\xi} \leq \mathrm{vol}_N(K)^{1/2} \left(\int_K |\widehat{F}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \right)^{1/2} e^{2\pi h_K(\mathbf{y})},$$

i.e., in view of (2.1), they imply that F is of exponential type K^*. The following theorem (due to Paley and Wiener in the one dimensional case and Stein in the general case) proves that these properties are equivalent.

Theorem (Paley-Wiener-Stein [SW71])**.** *Let* $F \in L^2(\mathbb{R}^N)$ *and let* K *be an origin symmetric convex body. Then* F *is a.e. equal to the restriction to* $\mathbb{R}^N$ *of an entire function of exponential type* K^* *if and only if the support of* $\widehat{F}$ *is contained in* K.

3. Proof of the Blaschke-Santaló Inequality

Theorem 1. *For every origin symmetric convex body* K *in* $\mathbb{R}^N$

$$(3.1) \qquad\qquad P(K) \leq P(B),$$

and equality holds if and only if K *is an ellipsoid.*

Proof. Our proof of (3.1) proceeds in two parts. First we show that

$$(3.2) \qquad\qquad \rho(K) = \frac{1}{\mathrm{vol}_N(K)}$$

and then we show that

$$(3.3) \qquad\qquad \frac{\rho(B)}{\mathrm{vol}_N(B^*)} \leq \frac{\rho(K)}{\mathrm{vol}_N(K^*)}.$$

Plugging (3.2) into (3.3) and rearranging terms yields (3.1).

We say that a continuous $F \in L^2(\mathbb{R}^N)$ is *admissible for* $\rho(K)$ provided that F satisfies conditions (1) and (2) in the definition of ρ.

Let us prove (3.2). Let $F(\mathbf{x})$ be an admissible function for $\rho(K)$. Condition (1) is equivalent to $\left| \int_K \widehat{F}(\boldsymbol{\xi}) d\boldsymbol{\xi} \right| \geq 1$, due to formula (2.3). We can thus write

$$
\begin{aligned}
1 &\leq \left| \int_K \widehat{F}(\boldsymbol{\xi}) d\boldsymbol{\xi} \right|^2 \\
&\leq \mathrm{vol}_N(K) \int_K \left| \widehat{F}(\boldsymbol{\xi}) \right|^2 d\boldsymbol{\xi} \\
&= \mathrm{vol}_N(K) \int_{\mathbb{R}^N} |F(\mathbf{x})|^2 d\mathbf{x} \\
&\leq \mathrm{vol}_N(K)\rho(K).
\end{aligned}
$$

(3.4)

The inequality in the second line is a consequence of Cauchy-Schwarz Inequality, and the equality in the third line is a consequence of Parseval's Identity. Note that, by the discussion of the equality cases in Cauchy-Schwarz Inequality, F minimizes $\rho(K)$ if and only if F is an admissible multiple of the inverse Fourier transform of 1_K, i.e.

$$
(3.5) \qquad F(\mathbf{x}) = \frac{\alpha}{\mathrm{vol}_N(K)} \int_K e^{2\pi i \mathbf{x} \cdot \boldsymbol{\xi}} d\boldsymbol{\xi}
$$

for some $\alpha \in \mathbb{C}$, with $|\alpha| = 1$. This concludes the proof of (3.2).

Now let us prove (3.3). Let $F(\mathbf{x})$ be an admissible function for $\rho(K)$. Without loss of generality we may assume that $F(\mathbf{x})$ is even. This is because the even part of $F(\mathbf{x})$ is admissible for $\rho(K)$ and (by the triangle inequality) has a L^2-norm less than or equal to that of $F(\mathbf{x})$. Let us denote by F also the entire extension of F defined by (2.3).

For each $\boldsymbol{\theta} \in \mathrm{S}^{N-1}$, we define a function $G_{\boldsymbol{\theta}} : \mathbb{C} \to \mathbb{C}$ as

$$
G_{\boldsymbol{\theta}}(z) = F(z\boldsymbol{\theta}).
$$

This is an even entire function of exponential type $[-\|\boldsymbol{\theta}\|_{K^*}^{-1}, \|\boldsymbol{\theta}\|_{K^*}^{-1}]$, by (2.2). Note that, since $G_{\boldsymbol{\theta}}(z)$ is even, there exists an entire function $H_{\boldsymbol{\theta}}(z)$ such that $G_{\boldsymbol{\theta}}(z) = H_{\boldsymbol{\theta}}(z^2)$. Finally we define $R_{\boldsymbol{\theta}} : \mathbb{C}^N \to \mathbb{C}$ as the radial extension of $G_{\boldsymbol{\theta}}(z)$, i.e., as

$$
R_{\boldsymbol{\theta}}(\mathbf{z}) = H_{\boldsymbol{\theta}}\left(z_1^2 + \cdots + z_N^2 \right).
$$

By Fubini's Theorem, $\int_0^{+\infty} |F(r\boldsymbol{\theta})|^2 r^{N-1} dr$ exists finite almost for every $\boldsymbol{\theta} \in \mathrm{S}^{N-1}$, i.e. the restriction of $R_{\boldsymbol{\theta}}$ to $\mathbb{R}^N$ is square-summable almost for every $\boldsymbol{\theta} \in \mathrm{S}^{N-1}$. We clearly have

$$
\begin{aligned}
\int_{\mathbb{R}^N} |F(\mathbf{x})|^2 d\mathbf{x} &= \int_{\mathrm{S}^{N-1}} \int_0^{\infty} |F(r\boldsymbol{\theta})|^2 r^{N-1} dr \ d\sigma(\boldsymbol{\theta}) \\
&= \frac{1}{\omega_{N-1}} \int_{\mathrm{S}^{N-1}} \int_{\mathbb{R}^N} |R_{\boldsymbol{\theta}}(\mathbf{x})|^2 d\mathbf{x} \ d\sigma(\boldsymbol{\theta}),
\end{aligned}
$$

(3.6)

where $d\sigma$ is the standard surface measure on S^{N-1}.

The function $R_{\boldsymbol{\theta}}(\mathbf{z})$ satisfies $|R_{\boldsymbol{\theta}}(\mathbf{0})| \geq 1$, it is entire and the support of the Fourier transform of the restriction of $R_{\boldsymbol{\theta}}$ to $\mathbb{R}^N$ is contained in the ball $\|\boldsymbol{\theta}\|_{K^*} B$. The last claim is a consequence of the Paley-Wiener Theorem, of the fact that $R_{\boldsymbol{\theta}}(\mathbf{z})$ is of exponential type $\|\boldsymbol{\theta}\|_{K^*}^{-1} B$ and of $\|\boldsymbol{\theta}\|_{K^*}^{-1} B = (\|\boldsymbol{\theta}\|_{K^*} B)^*$. The function

$R_{\boldsymbol{\theta}}(\mathbf{x})$ is thus admissible for $\rho\left(\|\boldsymbol{\theta}\|_{K^*}B\right)$. Therefore

$$(3.7) \qquad \int_{\mathbb{R}^N} |R_{\boldsymbol{\theta}}(\mathbf{x})|^2 d\mathbf{x} \geq \rho\left(\|\boldsymbol{\theta}\|_{K^*}B\right)$$
$$= \|\boldsymbol{\theta}\|_{K^*}^{-N}\rho(B),$$

since ρ is positively homogeneous of degree $-N$.

The set K^* can be represented in polar coordinates as

$$K^* = \{\rho\boldsymbol{\theta} : \boldsymbol{\theta} \in \mathrm{S}^{N-1}, 0 \leq \rho \leq \|\boldsymbol{\theta}\|_{K^*}^{-1}\},$$

and therefore

$$(3.8) \qquad \frac{\mathrm{vol}_N(K^*)}{\mathrm{vol}_N(B^*)} = \frac{1}{\omega_{N-1}} \int_{\mathrm{S}^{N-1}} \|\boldsymbol{\theta}\|_{K^*}^{-N} d\sigma(\boldsymbol{\theta}).$$

Using (3.6), (3.7) and (3.8) we obtain

$$(3.9) \qquad \int_{\mathbb{R}^N} |F(\mathbf{x})|^2 d\mathbf{x} \geq \rho(B)\frac{\mathrm{vol}_N(K^*)}{\mathrm{vol}_N(B^*)}.$$

The inequality (3.3) then follows upon taking the infimum over all admissible functions $F(\mathbf{x})$.

Let us now prove that we have equality in (3.1) **only when** K **is an ellipsoid.** Let $F(x)$ be as in (3.5), with $\alpha = 1$. This function is admissible for $\rho(K)$, is even and $\int_{\mathbb{R}^N} |F(x)|^2 d\mathbf{x} = \rho(K)$. Therefore equality holds in (3.1) if and only if equality holds in (3.9). The proof of (3.9) reveals that this happens if and only if $R_{\boldsymbol{\theta}}(\mathbf{x})$ minimizes $\rho(\|\boldsymbol{\theta}\|_{K^*}B)$ almost for every $\boldsymbol{\theta} \in \mathrm{S}^{N-1}$. In view of the discussion at the end of the proof of (3.2) and of the definition of $R_{\boldsymbol{\theta}}$, this is equivalent to saying that almost for every $\boldsymbol{\theta} \in \mathrm{S}^{N-1}$ and for every $r \geq 0$, $F(r\boldsymbol{\theta})$ coincides with an admissible multiple of the restriction to the ray $\{r\theta : r \geq 0\}$ of the inverse Fourier transform of $1_{\|\boldsymbol{\theta}\|_{K^*}B}$. This is equivalent to saying that there exists $\alpha(\theta) \in \mathbb{C}$ with $|\alpha(\theta)| = 1$ such that for each $r \in \mathbb{R}$

$$(3.10) \qquad \frac{1}{\mathrm{vol}_N(K)} \int_K e^{2\pi i r\boldsymbol{\theta}\cdot\boldsymbol{\xi}} d\boldsymbol{\xi} = \frac{\alpha(\boldsymbol{\theta})}{\mathrm{vol}_N(\|\boldsymbol{\theta}\|_{K^*}B)} \int_{\|\boldsymbol{\theta}\|_{K^*}B} e^{2\pi i r\boldsymbol{\theta}\cdot\boldsymbol{\xi}} d\boldsymbol{\xi}.$$

Since, by Fubini's Theorem, the n-dimensional inverse Fourier transform of 1_K is the 1-dimensional inverse Fourier transform of the Radon transform S_K of K, (3.10) can be rewritten as

$$\frac{1}{\mathrm{vol}_N(K)} \int_{\mathbb{R}} e^{2\pi i r t} S_K(t, \boldsymbol{\theta}) \, dt = \frac{\alpha(\boldsymbol{\theta})}{\mathrm{vol}_N(\|\boldsymbol{\theta}\|_{K^*}B)} \int_{\mathbb{R}} e^{2\pi i r t} S_{\|\boldsymbol{\theta}\|_{K^*}B}(t, \boldsymbol{\theta}) \, dt.$$

This identity implies that for each $t \in \mathbb{R}$ and almost for every $\boldsymbol{\theta} \in \mathrm{S}^{N-1}$

$$(3.11) \qquad \frac{1}{\mathrm{vol}_N(K)} S_K(t, \boldsymbol{\theta}) = \frac{\alpha(\boldsymbol{\theta})}{\mathrm{vol}_N(\|\boldsymbol{\theta}\|_{K^*}B)} S_{\|\boldsymbol{\theta}\|_{K^*}B}(t, \boldsymbol{\theta}).$$

By continuity the previous identity holds for each $\boldsymbol{\theta} \in \mathrm{S}^{N-1}$. Moreover, since $|\alpha(\boldsymbol{\theta})| = 1$ and each other term in (3.11) is non-negative, we have $\alpha(\boldsymbol{\theta}) = 1$.

For $\boldsymbol{\theta} \in S^{N-1}$ and $t \in \mathbb{R}$ let

$$D_K(t, \boldsymbol{\theta}) = \mathrm{vol}_N\left(\{\mathbf{x} \in K : \mathbf{x} \cdot \boldsymbol{\theta} \geq t \|\boldsymbol{\theta}\|_{K^*}\}\right).$$

Meyer and Reisner [MR89, Lemma 3] proves that if $D_K(t,\boldsymbol{\theta})$ does not depend on $\boldsymbol{\theta}$ for each $t \in [0,1]$ then K is an ellipsoid. We prove that this is the case. We write

$$(3.12) \qquad D_K(t,\boldsymbol{\theta}) = \int_{t\|\boldsymbol{\theta}\|_{K^*}}^{\|\boldsymbol{\theta}\|_{K^*}} S_K(r,\boldsymbol{\theta})dr.$$

Formula (3.11) implies

$$\begin{aligned}
D_K(t,\boldsymbol{\theta}) &= \frac{\mathrm{vol}_N(K)}{h_K(\theta)^N \mathrm{vol}_N(B)} \int_{t\|\boldsymbol{\theta}\|_{K^*}}^{\|\boldsymbol{\theta}\|_{K^*}} S_{\|\boldsymbol{\theta}\|_{K^*}B}(r,\boldsymbol{\theta})dr \\
&= \frac{\omega_{N-2}\mathrm{vol}_N(K)}{h_K(\theta)^N \mathrm{vol}_N(B)} \int_{t\|\boldsymbol{\theta}\|_{K^*}}^{\|\boldsymbol{\theta}\|_{K^*}} \left(\|\boldsymbol{\theta}\|_{K^*}^2 - r^2\right)^{\frac{N-1}{2}} dr \\
&= \frac{\omega_{N-2}\mathrm{vol}_N(K)}{\mathrm{vol}_N(B)} \int_t^1 \left(1 - s^2\right)^{\frac{N-1}{2}} ds.
\end{aligned}$$

This concludes the proof. $\qquad \square$

Remark 1. The validity of (3.11), with $\alpha(\boldsymbol{\theta}) = 1$, for a given $\boldsymbol{\theta}$ and for each $t \in \mathbb{R}$ is equivalent to the existence of an ellipsoid $E(\boldsymbol{\theta})$ such that $S_K(t,\boldsymbol{\theta}) = S_{E(\boldsymbol{\theta})}(t,\boldsymbol{\theta})$ for each $t \in \mathbb{R}$.

4. A RELATED VARIATIONAL QUANTITY

Another extremal quantity related to $\rho(K)$ is the "L^1−version" $\eta(K)$.

Definition 2. Given a convex body K define

$$\eta(K) = \inf \int_{\mathbb{R}^N} F(\mathbf{x})d\mathbf{x}$$

where the infimum is taken over the class of non-zero continuous functions $F(\mathbf{x})$ that satisfy

    (1) $F(\mathbf{x}) \geq 0$ for every $\mathbf{x} \in \mathbb{R}^N$,
    (2) $F(\mathbf{0}) \geq 1$, and
    (3) $\widehat{F}(\boldsymbol{\xi}) = 0$ if $\boldsymbol{\xi} \in \mathbb{R}^N \setminus K$.

When K is a cube, the infimum is achieved by the Fejér kernel. An extremal function for a generic origin symmetric convex body K can then be thought of as a "Fejér kernel associated with K." On another level, the determination of $\eta(K)$ is perhaps the simplest form of the so-called *Beurling-Selberg extremal problem* in several variables. The difficulty in determining $\eta(K)$ is the non-negativity, which is awkward from the Fourier analytic point of view. In the single variable case the function $F(x)$ can be factored as $F(x) = |U(x)|^2$ where $U(x)$ is admissible for $\rho(K/2)$. In several variables such a factorization is not generally available, and is known to be false for trigonometric polynomials of two or more variables. However, Jeff Vaaler and the second author conjecture that there are extremal functions for $\eta(K)$ that do admit such a factorization.

Conjecture. *For any origin symmetric convex body $K \subset \mathbb{R}^N$, we have*

$$\eta(K) = \frac{2^N}{\mathrm{vol}_N(K)}.$$

Remark 2. From (3.2) it follows that $\eta(K) \leq \rho(K/2) = 2^N \rho(K) = 2^N \text{vol}_N(K)^{-1}$. The above conjecture asserts that there is equality in this inequality for *every* origin symmetric convex body K.

Our main goal in this section is to prove that this conjecture holds when K is a ball and when K is a cube.

Theorem 2. *Let $B \subset \mathbb{R}^N$ be the Euclidean unit ball and $Q \subset \mathbb{R}^N$ be the Euclidean unit cube. Then*

$$(4.1) \qquad \eta(B) = \frac{2^N}{\text{vol}_N(B)}$$

and

$$(4.2) \qquad \eta(Q) = \frac{2^N}{\text{vol}_N(Q)}.$$

The result (4.1) is implicit in the work of Holt and Vaaler [HV96]. Since the proof of this result does not require the full force of the Holt-Vaaler machinery we will provide a self contained proof here.

Proof. Suppose $F(\mathbf{z})$ is an admissible function for $\eta(B)$. By averaging over $SO(N)$ we find that

$$\int_{\mathbb{R}^N} F(x)d\mathbf{x} = \int_{\mathbb{R}^N} \int_{SO(N)} F(\mathbf{g}\mathbf{x})d\mu(\mathbf{g})d\mathbf{x}$$

where μ is the normalized Haar measure on $SO(N)$, and that the function

$$\mathbf{x} \mapsto \int_{SO(N)} F(\mathbf{g}\mathbf{x})d\mu(\mathbf{g})$$

is admissible. In view of this observation we can safely limit our search to extremal functions that are *radial*. We will see momentarily that the extremal function we find can be factored as $F(\mathbf{z}) = U(\mathbf{z})U^*(\mathbf{z})$ where $U(\mathbf{z})$ is square integrable and radial on $\mathbb{R}^N$ and $\widehat{U}(\boldsymbol{\xi})$ is supported in $1/2B$. This allows us to recast the extremal problem as a minimization problem in a Hilbert space of the form

$$\mathbf{H}_\delta = C(\mathbb{R}^N) \cap \left\{ U(\mathbf{x}) \in L^2(\mathbb{R}^N) \ : \ \widehat{U}(\boldsymbol{\xi}) = 0 \text{ whenever } \xi \notin \delta B \right\},$$

specifically when $\delta = 1/2$. The space $\mathbf{H}_\delta$ is a Hilbert space with respect to the $L^2(\mathbb{R}^N)$-inner product $\langle \cdot, \cdot \rangle$ with the property that for every $\mathbf{z} \in \mathbb{C}^N$ and $f \in \mathbf{H}_\delta$

$$(4.3) \qquad f(\mathbf{z}) = \langle f, K(\mathbf{z}, \cdot) \rangle$$

where

$$(4.4) \qquad K(\boldsymbol{\omega}, \mathbf{z}) = \int_{\delta B} e^{-2\pi i (\mathbf{z} - \overline{\boldsymbol{\omega}}) \cdot \boldsymbol{\xi}} d\boldsymbol{\xi}.$$

We identify the elements of $\mathbf{H}_\delta$ with their entire extensions to $\mathbb{C}^N$. Let H_δ be the 1-dimensional case of $\mathbf{H}_\delta$, that is $H_\delta = \mathbf{H}_\delta$ when $N = 1$. Functions in H_δ which are real-valued and non-negative on the real axis enjoy a factorization akin to that for non-negative trigonometric polynomials given by the Fejér-Riesz theorem. The following proposition[1] is of central importance in the establishment of (4.1), because it allows us to take an awkward L^1-minimization problem and reformulate it as a minimization problem in Hilbert space.

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[1]This proposition, due to Ahiezer [Ahi48, Boa54, dB68], is essentially the original Fejér-Riesz theorem [RSN55].

Proposition 1. *Suppose $F(z) \in H_\delta$ is real valued and non-negative on the real axis and that $F(z)$ is not identically zero. Then there exists an entire function $U(z) \in H_{\delta/2}$ such that $U(z)$ is zero-free in $\mathcal{U}$ and $F(z) = U(z)U^*(z)$. If $F(z)$ is also even, then $F(z)$ admits the factorization*

$$F(z) = z^{2k}Q(z)V(z)V^*(z)$$

where k is the multiplicity of the possible zero at $z = 0$, $Q(z)$ has only purely imaginary zeros, and $V(z)$ is even.

Proof. Let $\{\omega_n : n = 1, 2...\}$ be the zeros of $F(z)$, listed with appropriate multiplicity, in the upper half plane and let

$$B_N(z) = \prod_{n=1}^{N} \frac{1 - z/\overline{\omega_n}}{1 - z/\omega_n}.$$

We define a sequence of entire functions $F_N(z)$ by $F_N(z) = B_N(z)F(z)$. Each of the functions $F_N(z)$ is in H_δ by the Paley-Wiener theorem. Since $\|F\| = \|F_N\|$ for each N, it follows that a subsequence of F_N converges weakly to some $G(z)$ in the Hilbert space. By (4.3) it follows that $F_N(z) \to G(z)$ pointwise for a subsequence. Since $|B_N(z)| \geq 1$ if $z \in \mathcal{U}$ with equality when z is real, it follows that $G(z)$ is zero free in $\mathcal{U}$ and that $|G(t)| = |F(t)|$ for real t. This shows that $F(z)^2 = F(z)F^*(z) = G(z)G^*(z)$. In particular the non-real zeros of $G(z)$ occur with even multiplicity.

Since $F(z)$ is real valued and non-negative on $\mathbb{R}$, the zeros of $G(z)$ occur with even multiplicity and so there is an entire function $U(z)$ for which $G(z) = U(z)^2$. Then $F(z)^2 = \{U(z)U^*(z)\}^2$ and since $F(z)$ is real valued and non-negative on $\mathbb{R}$ it follows that $F(z) = U(z)U^*(z)$.

If $F(z)$ is even, write $U(z) = z^k p(z)R(z)R^*(-z)$ where: $R(z)$ contains the zeros of $U(z)$ which have strictly positive real part, $p(z)$ contains only purely imaginary zeros, and k is the multiplicity of the zero at 0. Let $V(z) = R(z)R(-z)$ and $Q(z) = p(z)p^*(z)$. $\qquad \square$

We now introduce a notation for restrictions and extensions for radial functions. If the restriction of $G(\mathbf{z})$ to $\mathbb{R}^N$ is radial, we let $g(z)$ denote its restriction to one of the coordinate axes. Similarly if $g(z)$ is an even entire function, we may extend $g(z)$ to a radial function $G(\mathbf{z})$ on $\mathbb{C}^N$ by

$$G(\mathbf{z}) = \sum_{\ell=0}^{\infty} \frac{g^{(2\ell)}(0)}{(2\ell)!} \{z_1^2 + \cdots + z_N^2\}^\ell.$$

Let $F(\mathbf{z})$ be an admissible function for our problem and assume that $F(\mathbf{z})$ is radial. Then the corresponding restriction $f(z)$ is an even function in H_1 that is real-valued and non-negative on the real axis. Therefore $f(z)$ admits the representation

$$f(z) = q(z)v(z)v^*(z)$$

where $q(z)$ and $v(z)$ are even entire functions and $q(z)$ has only purely imaginary zeros. We choose the functions in such a way that $|v(0)|^2 = q(0) = 1$. Seeing that $q(z)$ and $v(z)$ are even, we extend them to $\mathbb{C}^N$ to obtain the following factorization for $F(\mathbf{z})$

$$F(\mathbf{z}) = Q(\mathbf{z})V(\mathbf{z})V^*(\mathbf{z}).$$

The integral of $F(\mathbf{x})$ now has the form

$$\int_{\mathbb{R}^N} F(\mathbf{x})d\mathbf{x} = \int_{\mathbb{R}^N} Q(\mathbf{x})|V(\mathbf{x})|^2 d\mathbf{x}$$

But if $F(\mathbf{x})$ is extremal, then $q(z)$ is zero free. Suppose, by way of contradiction, that $q(z)$ has a zero at say iy for $y > 0$. Then

$$q(z) = \left(1 + \frac{z^2}{y^2}\right) \tilde{q}(z)$$

for some even entire function $\tilde{q}(z)$ such that $\tilde{q}(0) = 1$, and $\tilde{q}(x) \geq 0$ for real x. In particular, $\tilde{q}(x) < q(x)$ for all non-zero real numbers x. This plainly shows that the admissible function $\tilde{F}(\mathbf{z}) = \tilde{Q}(\mathbf{z})V(\mathbf{z})V^*(\mathbf{z})$ has smaller L^1-norm than $F(\mathbf{z})$. Therefore we may assume

$$F(\mathbf{z}) = V(\mathbf{z})V^*(\mathbf{z})$$

where $V(\mathbf{x}) \in \mathbf{H}_{1/2}$. But by the Cauchy-Schwarz Inequality and (4.3)

$$1 \leq F(\mathbf{0}) = |V(\mathbf{0})|^2 \leq K(\mathbf{0},\mathbf{0})\|V\|_2^2 = \mathrm{vol}_N(^1\!/_2 B)\|V\|_2^2.$$

where equality occurs if and only if $F(\mathbf{0}) = 1$ and $V(z)$ is a scalar multiple of $K(\mathbf{0},\mathbf{z})$. But

$$\|V\|_2^2 = \int_{\mathbb{R}^N} F(\mathbf{x})d\mathbf{x}.$$

Therefore

$$\eta(B) = \frac{2^N}{\mathrm{vol}_N(B)}.$$

Now we will show (4.2), but for $Q = [-1,1]^N$.

Suppose that $F(\mathbf{x})$ is an admissible function for $\eta(Q)$. Then by the Poisson summation formula (see, for instance, [SW71])

$$1 \leq \sum_{\mathbf{n}\in\mathbb{Z}^N} F(\mathbf{n}) = \sum_{\mathbf{m}\in\mathbb{Z}^N} \widehat{F}(\mathbf{m}) = \widehat{F}(0) = \int_{\mathbb{R}^N} F(\mathbf{x})d\mathbf{x}.$$

We note that both expressions in the Poisson summation formula converge absolutely by a classical result of Polyá and Plancherel [PP37]. By taking the infimum over all admissible functions $F(\mathbf{x})$, we find that $\eta(Q) \geq 1$. But the function

$$F(\mathbf{x}) = \prod_{n=1}^{N} \left\{ \frac{\sin \pi x_n}{\pi x_n} \right\}^2.$$

is admissible for $\eta(Q)$ and integrating $F(\mathbf{x})$ one variable at a time, we find that its integral is equal to 1. This shows $\eta(Q) = 2^N \mathrm{vol}_N(Q)^{-1} = 1$.

$\square$

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