Periodic solutions of some autonomous Liénard equations with relativistic acceleration

Jean Mawhin
Institut de Recherche en Mathématique et Physique
Université Catholique de Louvain
chemin du Cyclotron, 2, 1348 Louvain-la-Neuve, Belgium
jean.mawhin@uclouvain.be

Gabriele Villari
Dipartamento di Matematica e informatica "Ulisse Dini"
Universitá degli Studi di Firenze
viale Morgagni, 67/A, 50137 Firenze, Italy
villari@math.unifi.it

Abstract

After giving a new proof of the existence of a stable limit cycle for the relativistic Van der Pol equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + k(x^2-1)\dot{x} + x = 0,$$

we find sufficient conditions upon f and g in order that the relativistic Liénard equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + f(x)\dot{x} + g(x) = 0$$

has a stable limit cycle. The approach and the existence conditions are distinct from those recently obtained by Pérez-González, Torregrosa and Torres [8].
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1 Introduction

In the last ten years, the study of the existence and multiplicity of periodic solutions of non-autonomous second order equations where $\ddot{x}$, with $\dot{x}$ denoting the derivative of x with respect to t, is replaced by a relativistic type acceleration

$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}$ has been considered by many authors. They either use the Leray-Schauder-type methods initiated in [1], or variational methods initiated in [2], or symplectic methods initiated in [5]. Subsequent references can be found in the surveys [6, 7].

Much less attention has been paid to the existence of limit cycles of autonomous second order equations involving a relativistic-type acceleration. A notable exception is the recent interesting paper [8] of Pérez-González, Torregrosa and Torres, where the existence and uniqueness of limit cycles of differential equations of the Liénard type

$$\frac{d}{dt}\varphi(\dot{x}) + +f(x)\psi(\dot{x}) + g(x) = 0,$$

is considered for some class of homeomorphisms φ from an open bounded interval onto $\mathbb{R}$. In particular, it is shown there that the relativistic van der Pol equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + k(x^2-1)\dot{x} + x = 0 \tag{1}$$

has a unique periodic orbit for all $k \neq 0$.

The aim is this paper is to take advantage of some techniques introduced in [3, 9, 10, 11] to obtain new results for relativistic Liénard equations of the form

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\right) + f(x)\dot{x} + g(x) = 0, \tag{2}$$

where the continuous functions f and g satisfy some conditions.

In Section 2, various reductions of equation (2) are considered for possible use in the sequel, and it is shown that the Cauchy problem for (2) is uniquely solvable when g is locally Lipschitz continuous. In Section 3, we give another proof of the existence of a limit cycle for the relativistic Van der Pol equation (1), a result already obtained in [8]. In Section 4, we obtain the existence of a stable limit cycle for the relativistic Liénard equation (2) in situations which are not covered by the results of [8], as shown by the example developed in Section 5.

2 Relativistic Duffing and Liénard equations

Let us consider the relativistic Liénard equation (2), with $g(0) = 0$, so that $(0,0)$ is an equilibrium. Solutions of equation (2) must of course be such that $|\dot{x}(t)| < 1$ for all $t \in \mathbb{R}$, so that, instead of considering the usual phase plane $\mathbb{R}^2$, one is *a priori* restricted to the strip $\mathbb{R} \times (-1, 1)$.

A way to avoid this difficulty is to make the change of variable

$$y = \frac{\dot{x}}{\sqrt{1-\dot{x}^2}}, \quad |\dot{x}| < 1,$$

which is equivalent to

$$\dot{x} = \frac{y}{\sqrt{1+y^2}}, \quad y \in \mathbb{R},$$

so that equation (2) can be written as the equivalent system

$$\dot{x} = \frac{y}{\sqrt{1+y^2}}, \quad \dot{y} = -f(x)\frac{y}{\sqrt{1+y^2}} - g(x). \tag{3}$$

Another approach, inspired by the use of the Liénard plane in the classical case, is to write equation (2) in the form

$$\frac{d}{dt}\left[\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + F(x)\right] + g(x) = 0$$

where $F(x) := \int_0^x f(s)\,ds$ ($x \in \mathbb{R}$), and make the change of variable

$$y = \frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + F(x), \quad |\dot{x}| < 1,$$

which is equivalent to

$$\dot{x} = \frac{y - F(x)}{\sqrt{1+[y-F(x)]^2}}, \quad y \in \mathbb{R}.$$

Hence, equation (2) can be written as the equivalent system

$$\dot{x} = \frac{y - F(x)}{\sqrt{1+[y-F(x)]^2}}, \quad \dot{y} = -g(x). \tag{4}$$

From this follows immediately the following regularity result.

Lemma 1 *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $g : \mathbb{R} \to \mathbb{R}$ locally Lipschitzian, the the Cauchy problem for equation (2) or (3) or (4) is locally uniquely solvable.*

Proof. It suffices to notice that F is of class C^1, and apply standard results [4] to system (4). ∎

The second step in establishing the existence of a closed orbit for (3) is to find conditions under which the unique equilibrium $(0,0)$ is a source. This is the case if $f(0) < 0$.

The third and last step consists in proving the existence of a winding orbit around the origin, in order to apply Poincaré-Bendixson's theorem [4]. To this aim, it is of interet to study the corresponding Duffing equation, for which $f \equiv 0$,

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + g(x) = 0,$$

and the system (3) reduces to

$$\dot{x} = \frac{y}{\sqrt{1+y^2}}, \quad \dot{y} = -g(x). \tag{5}$$

System (5) has the Hamiltonian structure

$$\dot{x} = \frac{\partial H}{\partial y}(x,y), \quad \dot{y} = -\frac{\partial H}{\partial x}(x,y),$$

with

$$H(x,y) = \sqrt{1+y^2} - 1 + G(x),$$

and $G(x) = \int_0^x g(s)\,ds$. Consequently, we have the energy first integral

$$\sqrt{1+y^2} - 1 + G(x) = C.$$

We have substracted the constant 1 from $\sqrt{1+y^2}$ in order that, for $|y|$ small, the result is close to the classical expression $\frac{y^2}{2}$.

It is easy to see, like in the classical case, that the origin $(0,0)$ of our (x,y)-phase plane is a global center for the system (5) if and only if $G(x) \to +\infty$ as $|x| \to \infty$. Now, taking H as a Liapunov function for system (3), we obtain, for its time-derivative along the trajectories of (3)

$$\begin{aligned}
\dot{H}(x,y) &= \frac{\partial H}{\partial x}(x,y)\dot{x} + \frac{\partial H}{\partial y}(x,y)\dot{y} \\
&= g(x)\frac{y}{\sqrt{1+y^2}} - \frac{y}{\sqrt{1+y^2}}\left[f(x)\frac{y}{\sqrt{1+y^2}} + g(x)\right] \\
&= -f(x)\frac{y^2}{1+y^2}.
\end{aligned}$$

Therefore, at points where $f(x)$ is positive, the trajectories of system (3) enter trajectories of system (5), while, at points where $f(x)$ is negative, the trajectories of system (3) exit trajectories of system (5).

Moreover, the slope of the trajectories of system (3) is given by the following expression, where y' denotes the derivative of y with respect to x,

$$y'(x) = \frac{\dot{y}}{\dot{x}} = -f(x) - g(x)\frac{\sqrt{1+y^2}}{y}, \tag{6}$$

and the 0-isocline, namely the curve in which $\dot{y} = 0$, is given by

$$\frac{y}{\sqrt{1+y^2}} = -\frac{g(x)}{f(x)}.$$

3 The relativistic Van der Pol equation revisited

At first we discuss the Van der Pol equation (1) where $k \neq 0$, although interesting results, and in particular the existence of limit cycles, can be proved in a similar way for equation (2). Notice that the case where $k < 0$ is reduced to the case where $k > 0$ by changing t into $-t$, so that we can assume without loss of generality that $k > 0$.

For this particular equation, system (3) becomes

$$\dot{x} = \frac{y}{\sqrt{1+y^2}}, \quad \dot{y} = -k(x^2-1)\frac{y}{\sqrt{1+y^2}} - x, \tag{7}$$

and the 0-isocline is given by

$$\frac{y}{\sqrt{1+y^2}} = -\frac{x}{k(x^2-1)}. \tag{8}$$

Observe first that, for $f(x) = k(x^2-1)$, $f(0) = -1 < 0$ and hence the origin of the phase plane is a source.

The 0-isocline in the classical Van der Pol equation is given by

$$y = -\frac{x}{k(x^2-1)}. \tag{9}$$

Of course, points of (8) only correspond to those x for which $-\frac{x}{k(x^2-1)} \in (-1,1)$, i.e., as easily shown, to the x belonging to the set

$$(-\infty, -x_2) \cup (-x_1, x_1) \cup (x_2, +\infty),$$

where

$$x_1 = -\frac{1}{2k} + \sqrt{\frac{1}{4k^2} + 1} \in (0,1), \; x_2 = \frac{1}{2k} + \sqrt{\frac{1}{4k^2} + 1} \in (1, +\infty).$$

Hence (8) can be seen as 'stretching' the restriction of (9) to $\mathbb{R} \times (-1,1)$ to $\mathbb{R}^2$ (see Fig. 1 and Fig. 2).

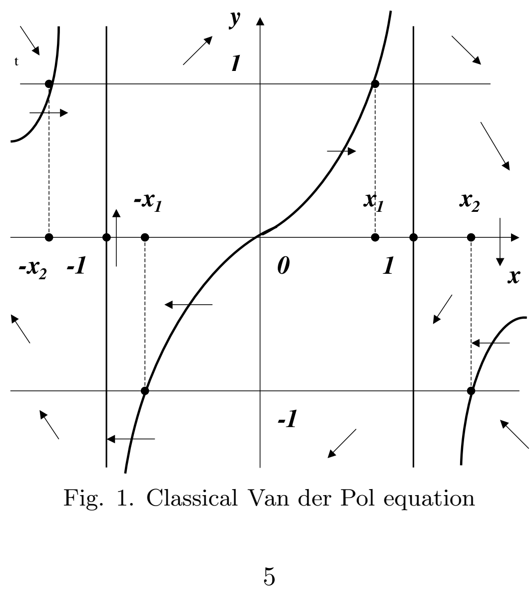

Fig. 1. Classical Van der Pol equation

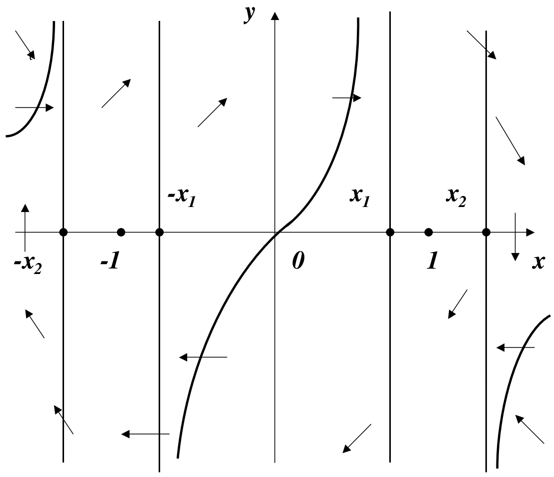

Fig. 2. Relativistic Van der Pol equation

At this point, arguing in the same way as in the classical case considered in [9], we are able to produce a winding trajectory. As the origin is a source, we can apply the Poincaré-Bendixson theorem [4] and get the existence of at least one limit cycle for (7).

Let Δ_1 be the component of the curve defined by (8) for $x \in (-\infty, -x_2)$, i.e. the graph of the increasing positive function

$$u_1(x) = -\frac{x}{\sqrt{k^2(x^2-1)^2 - x^2}}, \tag{10}$$

and let Δ_2 be the component of the same curve for $x \in (x_2, +\infty)$, i.e. the graph of the increasing negative function given by (10) for $x \in (x_2, +\infty)$, that we denote by $u_2(x)$. We have

$$\lim_{x \to -x_2-} u_1(x) = +\infty, \qquad \lim_{x \to x_2+} u_2(x) = -\infty,$$
$$\lim_{x \to -\infty} u_1(x) = 0, \qquad \lim_{x \to +\infty} u_2(x) = 0.$$

Let $\alpha \in \Delta_1$ with abscissa $x_\alpha < -x_2$. Now let

$$G(x,y) = -k(x^2-1)\frac{y}{\sqrt{1+y^2}} - x.$$

Since the function $\frac{y}{\sqrt{1+y^2}}$ is an increasing function of y and $k(x^2-1) > 0$ for $x \notin [-1,1]$, we see that, for each fixed $x \notin [-1,1]$, $G(x,y)$ is a decreasing function of y. The trajectory which passes through the point α comes from 'infinity' without intersecting the x-axis before reaching the point $(x_\alpha, u_1(x_\alpha)) \in \Delta_1$. From

$$y'(x) = \frac{\dot{y}}{\dot{x}} = -k(x^2-1) - x\frac{\sqrt{1+y^2}}{y}, \tag{11}$$

it follows that the trajectory does not have vertical asymptotes and, being bounded away from the x-axis, it must cross the y-axis. By an analogous argument, we can claim that the trajectory, after entering the $x > 0$ half-plane, either will cross the x-axis on the $0 < x \leq x_2$ segment, or will cross the line $x = x_2$. In the latter case, $y(x)$ will decrease after $x = x_2$. As

$$|x| + \frac{k(x^2 - 1)}{\sqrt{1 + y^2}} > |x| > x_2 > 0 \ \text{ if } \ |x| > x_2, \tag{12}$$

there cannot be an horizontal asymptote for the trajectory, which must eventually cross the x-axis at some $x > x_2$. The trajectory is now in the $y < 0$ half-plane. As a consequence of (11) again, the trajectory must meet the y-axis at some level $y < 0$.

Afterwards, as a consequence of (12) again, the trajectory cuts the x-axis either on the interval $(-x_2, 0)$, or at some $x \leq -x_2$. In this case, the trajectory must remain below the graph Δ_1, and hence is bounded in the future. The ω-limit set is compact and non-empty. Since the only critical point, at the origin, is repulsive, we can conclude that the limit set must be a cycle. Hence we have proved the following result.

Theorem 1 *For each $k \neq 0$, equation (1) has a least one nontrivial periodic solution.*

This theorem was already proved in [8] but our argument is quite different from the one given there.

4 The relativistic Liénard equation

We return to system (3) and first compare the slope of the relativistic Liénard system (6) with the slope of the classical Liénard system, namely

$$y'(x) = -f(x) - \frac{g(x)}{y}.$$

A direct comparison of the slopes at the same point (x, y) shows that if $xy > 0$, the trajectories of system (3) enter the trajectories of the classical Liénard system

$$\dot{x} = y, \ \ \dot{y} = -f(x)y - g(x), \tag{13}$$

while if $xy < 0$, the trajectories of system (3) exit the trajectories of system (13). So, when $xy > 0$, the trajectories of (3) are guided by those of (13). The question is then the intersection of a positive semitrajectory with the x-axis, because in this way one can prove that trajectories are clockwise and then apply the Poincaré-Bendixson theorem.

When $F(x)$ is bounded from below for x positive large enough and bounded from above for x negative large enough, Villari [10] has proved that the condition

$$\limsup_{x \to +\infty} [G(x) + F(x)] = +\infty \tag{14}$$

is necessary and sufficient in order that a positive semitrajectory starting with a nonnegative y intersects the x-axis, and that the condition

$$\limsup_{x\to-\infty}[G(x)-F(x)]=+\infty$$

is necessary and sufficient in order that a positive semitrajectory starting with a nonpositive y intersects the x-axis. The results are proved in the Liénard plane but hold as well in the phase plane.

More general situations have been considered by Villari and Zanolin in [11], that we shall adapt to the present situation. Like in [11], given $f:\mathbb{R}\to\mathbb{R}$ continuous, $F(x):=\int_0^x f(s)\,ds$, $g:\mathbb{R}\to\mathbb{R}$ continuous, we define $\Gamma_+:\mathbb{R}\to\mathbb{R}$ by

$$\Gamma_+(x):=\int_0^x (1+F_+(s))^{-1}g(s)\,ds,$$

where $F_+(x):=\max\{0,F(x)\}$. We also define $G(x):=\int_0^x g(s)\,ds$.

Theorem 2 *Assume that the following conditions hold.*

1. $f:\mathbb{R}\to\mathbb{R}$ *is continuous,* $g:\mathbb{R}\to\mathbb{R}$ *is locally Lipschitzian,* $xg(x)>0$ *for* $x\neq 0$*, and* $f(0)<0$.
2. *there exists* $a>0$ *such that* $f(x)>0$ *when* $x>a$*,* $\lim_{x\to+\infty}G(x)=K<+\infty$*,* $\lim_{x\to+\infty}F(x)=+\infty$.
3. *There exists* $0<\alpha<4$ *such that*

$$\lim_{x\to-\infty}[\alpha\Gamma_+(x)-F(x)]=+\infty.$$

Then equation (2) has at least a stable limit cycle.

Proof. Notice that Assumtion 2 rules the behavior of f and g for $x>0$ and Assumption 3 for $x<0$.

We first consider the behavior of a trajectory when $x>0$. Let $K>0$ be such that $G(x)<K$ for all $x\in\mathbb{R}$, according to the second condition in Assumption 2. We define $H:\mathbb{R}^2\to\mathbb{R}$ by

$$H(x,y):=\sqrt{1+y^2}-1+G(x)$$

and consider the corresponding curve of equation

$$K=\sqrt{1+y^2}-1+G(x). \tag{15}$$

It intersects the y-axis at the point $(0,-\sqrt{K^2+2K})$. On the other hand, as $G(x)<K$ for all $x\in\mathbb{R}$ the curve with equation (15) does not intersect the x-axis. For $a>0$ given in Assumption 2, the curve with equation (15) intersects the line $x=a$ at the point of ordinate

$$y=-\beta:=-\sqrt{K^2+2K-2G(a)(K+1)+G^2(a)}.$$

When $G(x) \to K$, this expression tends to 0, as expected. Following an argument that appeared in [10] and [3] and a slope comparison, we observe that the negative semi-trajectory $\gamma^-(P)$ with $P = (a, -\beta)$ does not intersect the x-axis. On the other hand, as its slope is bounded, the semi-trajectory $\gamma^+(P)$ intersects the y-axis, say at point $Q = (0, \overline{y})$ with $\overline{y} < 0$.

We now consider the behavior of a trajectory when $x < 0$. For the classical Liénard system

$$\dot{x} = y - F(x), \ \dot{y} = -g(x),$$

we know from [11] that if Assumption 3 holds, then the positive semi-trajectory $\widehat{\gamma}^+(Q)$ starting from some point $Q = (\alpha, -\beta)$ with $\alpha \in (0, 4)$ given in Assumption 3 and $\beta > 0$ intersects the vertical isocline, and therefore the x-axis at some point $R = (\widehat{x}, 0)$. The interesting case is the one where $f(x)$ is eventually negative, which corresponds to the last condition in Assumption 2. Hence, by definition of Γ_+, $G(x)$ must dominate $F(x)$. Using a comparison argument, the positive semi-trajectory $\gamma^+(Q)$ of (3) must intersect the x-axis at some point $S = (x, 0)$, with $\widehat{x} < x < 0$. Now, as its slope is bounded, the semi-trajectory $\gamma^*(S)$ must intersect the y-axis at some point $(0, y)$ with $y > 0$ and, in virtue of (14), eventually intersects the x-axis at some point $(x, 0)$ with $x > 0$.

Therefore $\gamma(P)$ is winding. The origin being a source because of the last condition in Assumption 1, we apply the Poincaré-Bendixson theorem [4] and obtain the existence of a stable limit cycle. ■

Like in [11], a 'dual' result holds if the conditions for $x > 0$ and $x < 0$ are interverted, whose statement is left to the reader.

5 An example

Consider system (3), i.e. equation (2), with $f(0) < 0$ and $f(x) = \frac{1}{x \log x}$ for x positive large enough, and

$$g(x) = \begin{cases} x & \text{for} \quad 0 \leq x < 1, \\ \frac{1}{x^2} & \text{for} \quad x > 1. \end{cases}$$

Clearly, $G(x)$ is bounded for x positive, and hence system (5) has not a global center, and in the half phase plane with x positive, there are trajectories which back in time do not intersect the x-axis. Because $f(x) > 0$ and coming back to $\dot{H}(x, y)$, we know that the trajectories of system (3) enter trajectories of system (5). But this means that back in time, such trajectories also do not intersect the x-axis. Now, system (3) has no vertical asymptotes, and hence its trajectories intersect the y-axis in $y < 0$. Now, defining $f(x)$ and $g(x)$ for x negative in such a way that condition

$$\limsup_{x \to -\infty} [G(x) - F(x)] = +\infty$$

holds, we can see that such trajectories intersect the x-axis for x negative. As the slope is bounded, such trajectories intersect the y-axis for y positive. Using the same argument, we can see that such trajectories intersect the x-axis for x positive because $\int_2^x \frac{1}{s\log s}\,ds \to +\infty$ when $x \to +\infty$, and therefore condition

$$\limsup_{x\to+\infty}[G(x)+F(x)] = +\infty$$

holds. Hence, we have produced a winding trajectory. Adding the standard assumption that $f(0) < 0$, we can conclude the existence of at least one stable limit cycle for system (3).

Notice that the Assumption (E_2) of [8] does not hold because

$$\lim_{x\to\infty} x[g(x)+f(x)] = 0.$$

In the case where $G(x) \to +\infty$, the previous argument does not work, but one can still get results using the stretching trick applied to the Van der Pol case. For instance, $f(x) = \frac{2}{x\log x}$ and $g(x) = \frac{1}{x\log x}$ give the desired result. Again, Assumption (E_2) of [8] does not hold.

Interesting cases can also obtained using the results of [11]

References

[1] C. Bereanu and J. Mawhin, Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian, J. Differential Equations 243 (2007), 536–557.

[2] H. Brezis and J. Mawhin, Periodic solutions of the forced relativistic pendulum, Differential Integral Equations 23 (2010), 801–810.

[3] M. Cioni and Gab. Villari, An extension of Dragilev's theorem for the existence of periodic solutions of the Liénard equation, Nonlinear Analysis TMA 127 (2015), 55?-70.

[4] E.A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[5] A. Fonda and R. Toader, Periodic solutions of pendulum-like Hamiltonian systems in the plane, Advanced Nonlinear Studies 12 (2012), 395–408.

[6] J. Mawhin, Resonance problems for some nonautonomous differential equations, in Stability and Bifurcation Theory for Non-Autonomous Differential Equations (CIME, Cetraro, 2011), R. Johnson and M.P. Pera ed., Lecture Notes in Math. No. 2065, p. 103–184, Springer, Berlin, 2013.

[7] J. Mawhin, Multiplicity of Solutions of relativistic-type systems with periodic nonlinearities : a survey, in Tenth MSU Conference on Differential Equations and Computational Simulations, Electronic J. Differential Equations, Conference 23 (2016), 77–86.

[8] S. Pérez-González, J. Torregrosa and P.J. Torres, Existence and uniqueness of limit cycles for generalized φ-Laplacian Liénard equations, J. Math. Anal. Appl. 439 (2016), 745–765.

[9] Gab. Villari, Periodic solutions of Liénard's equation, J. Math. Anal. Appl. 86 (1982), 379–386.

[10] Gab. Villari, On the qualitative behavior of solutions of Liénard equations, J. Differential Equations 67 (1987), 269–277.

[11] Gab. Villari and F. Zanolin, On a dynamical system in the Liénard plane. Necessary and sufficient conditions for the intersection with vertical isocline et applications, Funkcialaj Ekvacioj 33 (1990), 19–38.