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On curvature tensors of Norden and metallic pseudo-Riemannian manifolds

Abstract: We study some properties of curvature tensors of Norden and, more generally, metallic pseudo-Riemannian manifolds. We introduce the notion of J-sectional and J-bisectional curvature of a metallic pseudo-Riemannian manifold (M, J, g) and study their properties. We prove that under certain assumptions, if the manifold is locally metallic, then the Riemann curvature tensor vanishes. Using a Norden structure (J, g) on M, we consider a family of metallic pseudo-Riemannian structures $\{J_{a,b}\}_{a,b\in\mathbb{R}}$ and show that for $a \neq 0$, the J-sectional and J-bisectional curvatures of M coincide with the $J_{a,b}$-sectional and $J_{a,b}$-bisectional curvatures, respectively. We also give examples of Norden and metallic structures on $\mathbb{R}^{2n}$.

Keywords: Norden manifolds, metallic pseudo-Riemannian structures; J-sectional curvature; J-bisectional curvature

MSC: 53C15, 53C25

1 Introduction

Let (M, g) be a pseudo-Riemannian manifold. A metallic pseudo-Riemannian structure J on M is a g-symmetric $(1, 1)$-tensor field on M such that $J^2 = pJ + qI$, for some p and q real numbers, [1], [7]. In particular, for $p = 0$ and $q = -1$, J is a Norden structure on M, [10], [2].

In this paper we study some differential geometrical properties of Norden and, more generally, metallic pseudo-Riemannian manifolds. We define nearly locally metallic and nearly Kähler-Norden manifolds, extending a result of Gray, [5], about 4-dimensional nearly Kähler manifolds, [6], (Proposition 3.2). We focus then on curvature tensors, we introduce the notion of J-sectional and J-bisectional curvature of a metallic pseudo-Riemannian manifold (M, J, g) and study their properties. We prove that under certain assumptions, if the manifold is locally metallic, then the Riemann curvature tensor vanishes (Theorem 4.8). In the case of Norden manifolds we give a formula that express the sectional curvature with respect to the J-sectional curvature and some other terms (Proposition 4.10). This formula is the analogue of Vanhecke's formula for almost hermitian manifolds, [14], however in the Kähler-Norden case it is not possible to simplify the other terms like in [14]. In this sense, we recall that Norden metrics are necessarily neutral metrics, [3], and this class arises naturally in $N = 2$ string theory, [11], [4]. There are profound differences between Riemannian and neutral geometry, Hermitian and Norden geometry and this is one of the differences. Then, using a Norden structure (J, g) on M, we consider a family of metallic pseudo-Riemannian structures $\{J_{a,b}\}_{a,b\in\mathbb{R}}$ and show that for $a \neq 0$, the J-sectional and J-bisectional curvatures of M coincide with the $J_{a,b}$-sectional and $J_{a,b}$-bisectional curvatures, respectively. We also give examples of Norden and metallic structures on $\mathbb{R}^{2n}$ and describe the geometrical meaning of the sign of $p^2 + 4q$.

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2 Preliminaries

In [1], we generalized the notion of metallic Riemannian manifold to metallic pseudo-Riemannian manifold as follows:

Definition 2.1. [1] Let (M, g) be a pseudo-Riemannian manifold and let J be a g-symmetric $(1, 1)$-tensor field on M such that $J^2 = pJ + qI$, for some p, q real numbers. Then (J, g) is called a *metallic pseudo-Riemannian structure on M* and (M, J, g) is called a *metallic pseudo-Riemannian manifold*.

Fix now a metallic structure J on M and define the associated linear connections:

Definition 2.2. [1] i) A linear connection ∇ on M is called *J-connection* if J is covariantly constant with respect to ∇, namely $\nabla J = 0$.

ii) A metallic pseudo-Riemannian manifold (M, J, g) such that the Levi-Civita connection ∇ with respect to g is a J-connection is called a *locally metallic pseudo-Riemannian manifold*.

The concept of integrability is defined in the classical manner:

Definition 2.3. A metallic structure J is called *integrable* if its Nijenhuis tensor field N_J vanishes, where $N_J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y]$, for $X, Y \in C^\infty(TM)$.

We proved in [1] that if (M, J, g) is a locally metallic pseudo-Riemannian manifold, then J is integrable. Necessary and sufficient conditions for the integrability of a polynomial structure J whose characteristic polynomial has only simple roots were given by Vanzura in [13] who proved that if there exists a symmetric linear J-connection ∇, then the structure J is integrable.

Remark that any pseudo-Riemannian manifold admits locally metallic pseudo-Riemannian structures, namely $J = \mu I$, where $\mu = \frac{p \pm \sqrt{p^2 + 4q}}{2}$ with $p^2 + 4q \geq 0$ and we call them *trivial metallic structures*.

Definition 2.4. [1] A metallic pseudo-Riemannian manifold (M, J, g) such that the Levi-Civita connection ∇ with respect to g satisfies the condition

$$(\nabla_X J)Y + (\nabla_Y J)X = 0,$$

for any $X, Y \in C^\infty(TM)$, is called a *nearly locally metallic pseudo-Riemannian manifold*.

Proposition 2.5. *[1] A nearly locally metallic pseudo-Riemannian manifold (M, J, g) such that $J^2 = pJ + qI$ with $p^2 + 4q > 0$ is a locally metallic pseudo-Riemannian manifold if and only if J is integrable.*

3 Nearly Kähler-Norden manifolds

We recall that a Norden manifold is a smooth manifold M with a g-symmetric almost complex structure J, in particular (M, J, g) is a metallic manifold with $p = 0$ and $q = -1$.

Definition 3.1. If (M, J, g) is a Norden manifold which is nearly locally metallic we call M a *nearly Kähler-Norden* manifold.

For nearly Kähler-Norden manifolds, the following result extends the analogous for nearly Kähler manifolds, [5].

Proposition 3.2. *A 4-dimensional nearly Kähler-Norden manifold is a Kähler-Norden manifold.*

Proof. Let X, Y be local orthonormal vector fields: $g(X, X) = g(Y, Y) = 1$, $g(X, Y) = 0$. Then JX, JY satisfy $g(JX, JX) = g(JY, JY) = -1$, $g(JX, JY) = 0$ and $\{X, Y, JX, JY\}$ is a local frame for TM. We now prove that $(\nabla_X J)Y$ is g-orthogonal to X, Y, JX, JY and this will imply $\nabla J = 0$.

Let us compute:

$$g((\nabla_X J)Y, X) = g(\nabla_X JY, X) - g(J\nabla_X Y, X) = Xg(JY, X) - g(JY, \nabla_X X) - g(\nabla_X Y, JX)$$
$$= -g(JY, \nabla_X X) + g(Y, \nabla_X JX)$$
$$= g(Y, (\nabla_X J)X) = 0.$$

From the hypothesis $(\nabla_X J)Y + (\nabla_Y J)X = 0$, we get:

$$g((\nabla_X J)Y, Y) = -g((\nabla_Y J)X, Y) = 0.$$

Moreover:

$$g((\nabla_X J)Y, JX) = g(J(\nabla_X J)Y, X) = g(\nabla_X Y, X) + g(J\nabla_X JY, X) = -g(Y, \nabla_X X) - g(Y, \nabla_X JX) = -g(Y, J(\nabla_X J)X) = 0$$

and

$$g((\nabla_X J)Y, JY) = -g((\nabla_Y J)X, JY) = 0.$$

Then $(\nabla_X J)Y = 0$.
Moreover we have $(\nabla_X J)Y = J(\nabla_X J)JY$, then we get $(\nabla_X J)JY = 0$, $(\nabla_{JY} J)X = 0$, $(\nabla_{JY} J)JX = 0$ and $(\nabla_{JX} J)JY = 0$. Thus $\nabla J = 0$ and the proof is complete. □

4 Curvature tensors of a metallic pseudo-Riemannian manifold

In this section, we give some properties of the Riemann curvature tensor of a metallic pseudo-Riemannian manifold (M, J, g) and introduce the notion of J-sectional and J-bisectional curvature in this setting.

4.1 Riemannian curvature

Let (M, J, g) be a locally metallic pseudo-Riemannian manifold with $J^2 = pJ + qI$, p, q real numbers. Denoting by R the Riemann curvature tensor of g:

$$R(X, Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

and

$$R(X, Y, Z, W) := g(R(X, Y)Z, W),$$

for $X, Y, Z, W \in C^\infty(TM)$, since:

$$R(X, Y)JZ = J(R(X, Y)Z) + \nabla_X(\nabla_Y J)Z - \nabla_Y(\nabla_X J)Z - (\nabla_{[X,Y]} J)Z + (\nabla_X J)\nabla_Y Z - (\nabla_Y J)\nabla_X Z,$$

we obtain:

$$R(X, Y)JZ = J(R(X, Y)Z). \tag{1}$$

From a direct computation, we successively get:

$$R(JX, Y, Z, W) = g(R(JX, Y)Z, W) = g(R(Z, W)JX, Y) = g(J(R(Z, W)X), Y) \tag{2}$$

$$= g(R(Z, W)X, JY) = R(X, JY, Z, W),$$

$$R(X, Y, JZ, W) = R(X, Y, Z, JW), \tag{3}$$

$$R(JX, JY, Z, W) = g(R(JX, JY)Z, W) = g(R(Z, W)JX, JY) = g(J(R(Z, W)X), JY) \tag{4}$$
$$= pg(R(Z, W)X, JY) + qg(R(Z, W)X, Y) = pg(R(X, JY)Z, W) + qg(R(X, Y)Z, W)$$

and also:

$$R(X, Y, JZ, JW) = pg(R(X, Y)Z, JW) + qg(R(X, Y)Z, W). \tag{5}$$

From (2) and (3) we obtain:

$$R(X, JY, JZ, W) = R(JX, Y, JZ, W) = R(JX, Y, Z, JW) = R(X, JY, Z, JW) \tag{6}$$

and we can state:

Proposition 4.1. *If (M, J, g) is a locally metallic pseudo-Riemannian manifold, then the Riemann curvature tensor R of g defined by $R(X, Y, Z, W) := g(R(X, Y)Z, W)$ satisfies the identities:*
1. *$R(JX, Y, Z, W) = R(X, JY, Z, W)$;*
2. *$R(X, Y, JZ, W) = R(X, Y, Z, JW)$;*
3. *$R(JX, JY, Z, W) = pR(X, JY, Z, W) + qR(X, Y, Z, W)$;*
4. *$R(X, Y, JZ, JW) = pR(X, Y, Z, JW) + qR(X, Y, Z, W)$;*
5. *$R(X, JY, JZ, W) = R(JX, Y, JZ, W) = R(JX, Y, Z, JW) = R(X, JY, Z, JW)$,*

for any $X, Y, Z, W \in C^\infty(TM)$.

4.2 *J*-sectional curvature

An analogous of the holomorphic sectional curvature will be defined on a metallic pseudo-Riemannian manifold (M, J, g) for non degenerate plane sections as follows.

Definition 4.2. Let X be a non zero tangent vector field and let $\pi_{X, JX}$ be the plane generated by X and JX. If $g(X, X)g(JX, JX) - [g(X, JX)]^2 \neq 0$, then $\pi_{X, JX}$ is called *non degenerate* and we define the *J-sectional curvature* as:

$$K^J(X) := \frac{g(R(X, JX)X, JX)}{g(X, X)g(JX, JX) - [g(X, JX)]^2}. \tag{7}$$

If K^J is a constant, we say that *M has constant J-sectional curvature.*

Denote by:

$$k^J(X) := R(X, JX, X, JX),$$

for any $X \in C^\infty(TM)$.

Proposition 4.3. *If (M, J, g) is a locally metallic pseudo-Riemannian manifold, then $k^J(X) = 0$, for any $X \in C^\infty(TM)$ and the J-sectional curvature of non degenerate plane sections vanishes.*

Proof. We have:

$$k^J(X) = R(X, JX, X, JX) = R(JX, X, X, JX) = -R(X, JX, X, JX) = -k^J(X),$$

for any $X \in C^\infty(TM)$. Then the statement.   □

4.3 *J*-bisectional curvature

Define now on a metallic pseudo-Riemannian manifold (M, J, g) the *J-bisectional curvature* for non degenerate plane sections as follows.

Definition 4.4. Let X and Y be two linearly independent tangent vector fields and let $\pi_{X,JX}$ and $\pi_{Y,JY}$ be the planes generated by X and JX, and Y and JY, respectively. If $\pi_{X,JX}$ and $\pi_{Y,JY}$ are non degenerate, we define the *J-bisectional curvature* as:

$$K^J(X, Y) := \tag{8}$$

$$= \frac{g(R(X, JX)Y, JY)\sqrt{|g(X, X)g(JX, JX) - [g(X, JX)]^2|}\sqrt{|g(Y, Y)g(JY, JY) - [g(Y, JY)]^2|}}{(g(X, X)g(JX, JX) - [g(X, JX)]^2)(g(Y, Y)g(JY, JY) - [g(Y, JY)]^2)}.$$

If K^J is a constant, we say that *M has constant J-bisectional curvature*.

Denote by:

$$k^J(X, Y) := R(X, JX, Y, JY),$$

for any $X, Y \in C^\infty(TM)$.

Proposition 4.5. *If (M, J, g) is a locally metallic pseudo-Riemannian manifold, then $k^J(X, Y) = 0$, for any X, $Y \in C^\infty(TM)$ and the J-bisectional curvature of non degenerate plane sections vanishes.*

Proof. We have:

$$k^J(X, Y) = R(X, JX, Y, JY) = R(X, JX, JY, Y) = -R(X, JX, Y, JY) = -k^J(X, Y),$$

for any $X, Y \in C^\infty(TM)$. Then the statement. □

4.4 *RM*-manifolds

Definition 4.6. We say that the metallic pseudo-Riemannian manifold (M, J, g) is an *RM-manifold* if M has *J*-invariant Riemannian curvature i.e.

$$R(JX, JY, JZ, JW) = R(X, Y, Z, W),$$

for any $X, Y, Z, W \in C^\infty(TM)$.

Proposition 4.7. *Let (M, J, g) be a metallic pseudo-Riemannian manifold. If M is an RM-manifold and $q^2 \neq 1$, then $k^J(X) = 0$ and $k^J(X, Y) = 0$, for any $X, Y \in C^\infty(TM)$. In particular, for an RM-manifold with $q^2 \neq 1$, the J-sectional curvature and the J-bisectional curvature of non degenerate plane sections vanish.*

Proof. We have:

$$k^J(X) = R(X, JX, X, JX) = R(JX, J^2X, JX, J^2X) = q^2R(X, JX, X, JX) = q^2k^J(X),$$

for any $X \in C^\infty(TM)$ and

$$k^J(X, JY) = R(X, JX, Y, JY) = R(JX, J^2X, JY, J^2Y) =$$

$$= q^2R(JX, X, JY, Y) = q^2R(X, JX, Y, JY) = q^2k^J(X, JY),$$

for any $X, Y \in C^\infty(TM)$. Then the statement. □

We have the following:

Theorem 4.8. *Let (M, J, g) be a locally metallic pseudo-Riemannian manifold with $J^2 = pJ + qI$ which is an RM-manifold. If $q[p^2 - (q - 1)^2] \neq 0$ and $p \neq 0$, $q \neq -1$, then its Riemann curvature tensor vanishes.*

Proof. For a locally metallic Riemannian manifold the following identities hold:

$$R(JX, Y, Z, W) = R(X, JY, Z, W)$$

$$qR(X, Y, Z, W) = R(JX, JY, Z, W) - pR(X, JY, Z, W)$$

and by using Bianchi's identity:

$$0 = R(X, JX, Y, JY) = -R(JX, Y, X, JY) - R(Y, X, JX, JY) =$$

$$= -R(X, JY, X, JY) + R(X, Y, X, J^2Y) =$$

$$= -R(X, JY, X, JY) + pR(X, Y, X, JY) + qR(X, Y, X, Y).$$

Thus

$$R(X, JY, X, JY) = qR(X, Y, X, Y) + pR(X, Y, X, JY),$$

for any $X, Y \in C^\infty(TM)$.

Since $q \neq 0$, J is invertible and $J^{-1} = \frac{1}{q}J - \frac{p}{q}I$, therefore, by using the *RM*-hypothesis we get

$$R(X, JY, X, JY) = R(J^{-1}X, Y, J^{-1}X, Y) =$$

$$= \frac{1}{q^2}R(JX, Y, JX, Y) - \frac{2p}{q^2}R(X, Y, JX, Y) + \frac{p^2}{q^2}R(X, Y, X, Y) =$$

$$= \frac{1}{q^2}R(X, JY, X, JY) - \frac{2p}{q^2}R(X, Y, JX, Y) + \frac{p^2}{q^2}R(X, Y, X, Y)$$

or

$$(1 - \frac{1}{q^2})R(X, JY, X, JY) = -\frac{2p}{q^2}R(X, Y, JX, Y) + \frac{p^2}{q^2}R(X, Y, X, Y).$$

Substituting $R(X, JY, X, JY)$ we get

$$(1 - \frac{1}{q^2})qR(X, Y, X, Y) + (1 - \frac{1}{q^2})pR(X, Y, X, JY) =$$

$$= -\frac{2p}{q^2}R(X, Y, X, JY) + \frac{p^2}{q^2}R(X, Y, X, Y)$$

or

$$(q^3 - q - p^2)R(X, Y, X, Y) = -p(q^2 + 1)R(X, Y, X, JY).$$

In particular, we get

$$R(X, JY, X, JY) = \frac{p^2 + 2q}{q^2 + 1}R(X, Y, X, Y)$$

and replacing X by JX:

$$R(JX, JY, JX, JY) = \frac{p^2 + 2q}{q^2 + 1}R(JX, Y, JX, Y)$$

or

$$R(X, Y, X, Y) = (\frac{p^2 + 2q}{q^2 + 1})^2 R(X, Y, X, Y).$$

If $(\frac{p^2+2q}{q^2+1})^2 \neq 1$, that is if $p^2 \neq (q-1)^2$ and $p \neq 0$, $q \neq -1$, R must be zero. □

Remark 4.9. If J is the trivial metallic Riemannian structure on (M, g), then M is an *RM*-manifold if and only if $\mu^4 = 1$, that is, if and only if $\mu^2 = 1$. Now $(\frac{p\pm\sqrt{p^2+4q}}{2})^2 = 1$ if and only if $p^2 = (q-1)^2$. In this case, there are no restrictions on the curvature. Moreover, we remark that the case $p = 0$, $q = -1$, is the case of Norden structures. Kähler-Norden manifolds are *RM*-manifolds, but not necessarily flat, [8], [9], [12].

4.5 Curvature tensor of Norden manifolds

Let (M, J, g) be a Norden manifold and let R be the Riemann curvature tensor of g. Denote by:

$$\lambda(X, Y) := R(X, Y, X, Y) - R(X, Y, JX, JY),$$

for any $X, Y \in C^\infty(TM)$.

Proposition 4.10. *The following formula holds:*

$$R(X, Y, X, Y) + R(JX, JY, JX, JY) =$$

$$= \frac{1}{16}[3k^J(X + JY) + 3k^J(X - JY) - k^J(X + Y) - k^J(X - Y) - 4k^J(X) - 4k^J(Y)] +$$

$$+ \frac{5}{8}[\lambda(X, Y) + \lambda(JX, JY)] + \frac{1}{8}[\lambda(X, JY) + \lambda(JX, Y)],$$

for any $X, Y \in C^\infty(TM)$.

Proof. A direct computation gives:

$$k^J(X + JY) = k^J(X) + k^J(Y) - 2R(X, JX, X, Y) + 2R(X, JX, JY, JX) +$$

$$+ 2R(X, Y, JY, Y) - 2R(X, JX, JY, Y) - 2R(JY, JX, X, Y) - 2R(JY, JX, JY, Y) +$$

$$+ R(X, Y, X, Y) + R(JX, JY, JX, JY).$$

Thus:

$$k^J(X + JY) + k^J(X - JY) = 2k^J(X) + 2k^J(Y) - 4R(X, JX, JY, Y) - 4R(JY, JX, X, Y) +$$

$$+ 2R(X, Y, X, Y) + 2R(JX, JY, JX, JY)$$

and

$$k^J(X + Y) + k^J(X - Y) = 2k^J(X) + 2k^J(Y) - 4R(X, JX, JY, Y) + 2R(X, JY, X, JY) +$$

$$+ 2R(JX, Y, JX, Y) + 4R(Y, JX, X, JY).$$

Then, by using first Bianchi's identity and substituting it, we get the statement. $\square$

Corollary 4.11. *If (M, J, g) is an RM-manifold, then we obtain the following formula:*

$$R(X, Y, X, Y) = \frac{1}{32}[3k^J(X + JY) + 3k^J(X - JY) - k^J(X + Y) +$$

$$- k^J(X - Y) - 4k^J(X) - 4k^J(Y)] + \frac{5}{8}\lambda(X, Y) + \frac{1}{8}\lambda(X, JY),$$

for any $X, Y \in C^\infty(TM)$.

Remark 4.12. The above formula is the formula in Theorem 1 of [14], however here it is not possible to express the sectional curvature with the respect to the J-sectional curvature. Namely, in Kähler-Norden case, λ doesn't vanish. Actually, in Kähler-Norden case, k^J is zero and the formula is just an identity.

5 Metallic pseudo-Riemannian structures and Norden structures

In [1], we showed that any metallic pseudo-Riemannian structure (J, g) on M satisfying $J^2 = pJ + qI$, where p and q are real numbers with $p^2 + 4q < 0$, defines two Norden structures:

$$J_\pm := \pm\left(\frac{2}{\sqrt{-p^2 - 4q}}J - \frac{p}{\sqrt{-p^2 - 4q}}I\right)$$

and conversely, any Norden structure (J, g) on M defines a family of metallic pseudo-Riemannian structures:

$$J_{a,b} := aJ + bI,$$

where a, b are real numbers, and we proved the following:

Proposition 5.1. *[1] If $a \neq 0$, then:*
1. *$J_{a,b}$ is integrable if and only if J is integrable;*
2. *$J_{a,b}$ is locally metallic if and only if J is Kähler-Norden;*
3. *$J_{a,b}$ is nearly locally metallic if and only if J is nearly Kähler-Norden.*

Assume now that $a \neq 0$, let X be a non zero tangent vector field and let $\pi_{X,JX}$ be the plane generated by X and JX.

Proposition 5.2. *If $\pi_{X,JX}$ is non degenerate, then $\pi_{X,J_{a,b}X}$ is non degenerate, too, and*

$$K^{J_{a,b}}(X) = K^J(X),$$

for any real numbers a, b with $a \neq 0$.

Proof. We have:

$$g(X, X)g(J_{a,b}X, J_{a,b}X) - [g(X, J_{a,b}X)]^2 = a^2\{g(X, X)g(JX, JX) - [g(X, JX)]^2\}$$

and

$$g(R(X, J_{a,b}X)X, J_{a,b}X) = a^2 g(R(X, JX)X, JX).$$

Then the statement. □

Analogously, we get the following:

Proposition 5.3. *If $\pi_{X,JX}$ and $\pi_{Y,JY}$ are non degenerate, then $\pi_{X,J_{a,b}X}$ and $\pi_{Y,J_{a,b}Y}$ are non degenerate, too, and*

$$K^{J_{a,b}}(X, Y) = K^J(X, Y),$$

for any real numbers a, b with $a \neq 0$.

Proof. We have:

$$g(R(X, J_{a,b}X)Y, J_{a,b}Y) = a^2 g(R(X, JX)Y, JY).$$

Then the statement. □

6 Examples of Norden and metallic structures

In this section we construct examples of Kähler-Norden, locally metallic pseudo-Riemannian and locally metallic Riemannian structures on $\mathbb{R}^{2n}$.

6.1 Examples of Kähler-Norden structures on $\mathbb{R}^{2n}$

Let us denote by $\mathbb{R}(n)$ the set of real square matrices of order n, let $M \in \mathbb{R}(2)$ be the Minkowski matrix of order 2:

$$M := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and let g be the neutral metric on $\mathbb{R}^2$ defined by M.

$J \in \mathbb{R}(2)$ defines a Norden structure on $(\mathbb{R}^2, g)$ if and only if $J^2 = -I$ and ${}^t JM = MJ$. A direct computation gives the expression of J:

$$J = J_{a_+} := \begin{pmatrix} a & \sqrt{a^2 + 1} \\ -\sqrt{a^2 + 1} & -a \end{pmatrix}$$

or

$$J = J_{a_-} := \begin{pmatrix} a & -\sqrt{a^2 + 1} \\ \sqrt{a^2 + 1} & -a \end{pmatrix},$$

for some real number a.

If we denote by $J_{a_\pm, n}$ the matrices of order $2n$ with n diagonal blocks equal to $J_{a_\pm}$:

$$J_{a_\pm, n} := \begin{pmatrix} J_{a_\pm} & O & \dots & \dots & O \\ O & J_{a_\pm} & \dots & \dots & O \\ \vdots & \dots & \ddots & \dots & \vdots \\ O & \dots & \dots & J_{a_\pm} & O \\ O & \dots & \dots & O & J_{a_\pm} \end{pmatrix},$$

then $J_{a_\pm, n}$ define flat Kähler-Norden structures on $\mathbb{R}^{2n}$ with respect to the neutral metric g_n defined by:

$$g_n := \begin{pmatrix} M & O & \dots & \dots & O \\ O & M & \dots & \dots & O \\ \vdots & \dots & \ddots & \dots & \vdots \\ O & \dots & \dots & M & O \\ O & \dots & \dots & O & M \end{pmatrix}.$$

6.2 Examples of locally metallic pseudo-Riemannian structures on $(\mathbb{R}^{2n}, g_n)$

Let $J_{a_\pm, n}$ be the flat Kähler-Norden structures on $\mathbb{R}^{2n}$ defined above, with respect to the neutral metric g_n, and let α, β be real numbers. Then

$$J_{\alpha, \beta, n} := \alpha J_{a_\pm, n} + \beta I$$

is a family of flat locally metallic pseudo-Riemannian structures on $(\mathbb{R}^{2n}, g_n)$. In block matrix form, we have the following expression:

$$J_{\alpha, \beta, n} = \begin{pmatrix} J_{\alpha, \beta} & O & \dots & \dots & O \\ O & J_{\alpha, \beta} & \dots & \dots & O \\ \vdots & \dots & \ddots & \dots & \vdots \\ O & \dots & \dots & J_{\alpha, \beta} & O \\ O & \dots & \dots & O & J_{\alpha, \beta} \end{pmatrix},$$

where:

$$J_{\alpha, \beta} := \alpha J_{a_\pm} + \beta I = \begin{pmatrix} \alpha a + \beta & \pm\sqrt{a^2 + 1} \\ \mp\sqrt{a^2 + 1} & -\alpha a + \beta \end{pmatrix}$$

and we have that $J_{\alpha, \beta, n}^2 = p J_{\alpha, \beta, n} + q I$ with $p = 2\beta$ and $q = -\alpha^2 - \beta^2$. In particular, $p^2 + 4q \le 0$.

6.3 Examples of locally metallic Riemannian structures on $(\mathbb{R}^{2n}, < , >)$

Definition 6.1. $A \in \mathbb{R}(n)$ is called a *metallic matrix* if there exist two real numbers p and q such that $A^2 = pA + qI$, where I is the identity matrix.

Let α, p, q be real numbers such that $\alpha^2 - p\alpha - q \leq 0$. Then, for any such α,

$$J_{\alpha_\pm} := \begin{pmatrix} \alpha & \pm\sqrt{q + p\alpha - \alpha^2} \\ \pm\sqrt{q + p\alpha - \alpha^2} & p - \alpha \end{pmatrix}$$

satisfy $J_{\alpha_\pm}^2 = pJ_{\alpha_\pm} + qI$, hence $J_{\alpha_\pm}$ are metallic matrices.

If we denote by $J_{\alpha_\pm, n}$ the matrices of order $2n$ with n diagonal blocks equal to $J_{\alpha_\pm}$:

$$J_{\alpha_\pm, n} := \begin{pmatrix} J_{\alpha_\pm} & O & \ldots & \ldots & O \\ O & J_{\alpha_\pm} & \ldots & \ldots & O \\ \vdots & \ldots & \ddots & \ldots & \vdots \\ O & \ldots & \ldots & J_{\alpha_\pm} & O \\ O & \ldots & \ldots & O & J_{\alpha_\pm} \end{pmatrix},$$

then $J_{\alpha_\pm, n}$ are symmetric metallic matrices, thus they define metallic Riemannian structures on $\mathbb{R}^{2n}$, with respect to the standard euclidean metric $<,>$, which are locally metallic.

We remark that the condition $\alpha^2 - p\alpha - q \leq 0$ implies $p^2 + 4q \geq 0$ and $\frac{p - \sqrt{p^2+4q}}{2} \leq \alpha \leq \frac{p + \sqrt{p^2+4q}}{2}$.

Remark 6.2. The sign of $p^2 + 4q$ characterizes the geometry of a metallic structure (M, J, g) with $J^2 = pJ + qI$. Namely, $p^2 + 4q < 0$ is a necessary and sufficient condition in order that J defines a Norden structure on (M, g). In particular, in this case, M is even dimensional and g must be a neutral metric.

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