



UNIVERSITÀ  
DEGLI STUDI  
FIRENZE

# FLORE

## Repository istituzionale dell'Università degli Studi di Firenze

### Extension and tangential CRF conditions in quaternionic analysis

Questa è la Versione finale referata (Post print/Accepted manuscript) della seguente pubblicazione:

*Original Citation:*

Extension and tangential CRF conditions in quaternionic analysis / Marco Maggesi, Donato Pertici, Giuseppe Tomassini. - In: ANNALI DI MATEMATICA PURA ED APPLICATA. - ISSN 0373-3114. - STAMPA. - 199:(2020), pp. 2263-2289. [10.1007/s10231-020-00968-5]

*Availability:*

This version is available at: 2158/1172369 since: 2024-02-20T11:07:26Z

*Published version:*

DOI: 10.1007/s10231-020-00968-5

*Terms of use:*

Open Access

La pubblicazione è resa disponibile sotto le norme e i termini della licenza di deposito, secondo quanto stabilito dalla Policy per l'accesso aperto dell'Università degli Studi di Firenze (<https://www.sba.unifi.it/upload/policy-oa-2016-1.pdf>)

*Publisher copyright claim:*

(Article begins on next page)

# EXTENSION AND TANGENTIAL CRF CONDITIONS IN QUATERNIONIC ANALYSIS

MARCO MAGGESI<sup>1</sup>, DONATO PERTICI<sup>2</sup>, AND GIUSEPPE TOMASSINI<sup>3</sup>

ABSTRACT. We prove some extension theorems for quaternionic holomorphic functions in the sense of Fueter. Starting from the existence theorem for the nonhomogeneous Cauchy-Riemann-Fueter Problem, we prove that an  $\mathbb{H}$ -valued function  $f$  on a smooth hypersurface, satisfying suitable tangential conditions, is locally a jump of two  $\mathbb{H}$ -holomorphic functions. From this, we obtain, in particular, the existence of the solution for the Dirichlet Problem with smooth data. We extend these results to the continuous case. In the final part, we discuss the octonian case.

## CONTENTS

Introduction	1
1. Generalities	2
1.1. Fueter operators and $\mathbb{H}$ -holomorphic functions	2
1.2. Several variables	4
1.3. Bochner-Martinelli Kernel	5
1.4. $\mathbb{H}$ -holomorphy and $\mathbb{H}$ -convexity	5
1.5. $\overline{\mathcal{D}}$ -problem and Hartogs Theorem	6
2. Riemann-Hilbert and Dirichlet problems for $\mathbb{H}$ -holomorphic functions.	7
2.1. The operator $\overline{\mathcal{D}}_b$ and the CRF condition	7
2.2. Solvability of the Riemann-Hilbert problem	10
3. The CRF condition in weak form	15
4. Appendix: Some generalizations to octonions	18
References	21

## INTRODUCTION

This paper aims to set forth the methods of complex analysis in the quaternionic analysis in several variables. The main objects of such a theory are the  $\mathbb{H}$ -holomorphic functions, i.e., those functions  $f = f(q_1, \dots, q_n)$ ,  $q_1, \dots, q_n \in \mathbb{H}$ , which are (left) regular in the sense of Fueter with respect to each variable. For the basic results in the quaternionic analysis in one and several variables, we refer to the articles by Sudbery [S] and [Pe1] respectively. As for a more geometric aspect of the theory, we refer to the book [IMV] and the rich bibliography quoted there.

Coming to the content of the paper, we are dealing with the boundary values and extension problems for  $\mathbb{H}$ -holomorphic functions. As it is well known, this is one of the central themes in complex analysis, which motivated the study of overdetermined systems of linear partial differential equations, the CR geometry, and the theory of extension of “holomorphic objects”.

For the sake of simplicity, we restrict ourselves to the case  $n = 2$ , even if most of the main results proved in the paper hold in any dimension.

---

*Date:* February 18, 2020.

*1991 Mathematics Subject Classification.* 30G35.

*Key words and phrases.* Quaternionic analysis and Cauchy-Riemann-Fueter operator and  $\mathbb{H}$ -holomorphic functions and Nonhomogeneous Cauchy-Riemann-Fueter system.

The authors received support from GNSAGA-INdAM and Italian MIUR..

The paper is organized into three sections.

In Section 1, after fixing the main notations, we define the differential forms  $dq_\alpha, Dq_\alpha$  that play a fundamental role, and the Cauchy-Riemann-Fueter operator  $\overline{\mathfrak{D}}$ . As an application of the Cauchy-Fueter formula in one variable [Fu1, S], we prove a result of ‘‘Carleman type’’ (Proposition 1.1). We also recall the Bochner-Martinelli formula proved in [Pe1], and we show that the Bochner-Martinelli kernel  $\mathbf{K}^{BM}(q, q_0)$  writes as a sum  $\mathbf{K}_1^{BM}(q, q_0) + \mathbf{K}_2^{BM}(q, q_0)$ , where  $\mathbf{K}_1^{BM}(q, q_0)$  and  $\mathbf{K}_2^{BM}(q, q_0)$  are complex differential forms and the latter is exact on  $\{q \neq q_0\}$ , see (12).

The Section ends with a brief overview of the main results on  $\mathbb{H}$ -holomorphy,  $\mathbb{H}$ -convexity [Pe3], and the  $\overline{\mathfrak{D}}$ -problem [ABLSS, AL, CSSS, BDS].

Sections 2 and 3 are the bulk of the paper. In the first part of Sections 2, using the differential forms  $dq_\alpha, Dq_\alpha$ , we formulate the CRF condition on a smooth hypersurface  $S$  in terms of the tangential operators  $Dq_1|_S \wedge d_{(q_1)}f, Dq_2|_S \wedge d_{(q_2)}f$  (Theorem 2.4). This allows us to give the notion of *admissible function*  $f : S \rightarrow \mathbb{H}$ , which is satisfied by the traces or, more generally, the ‘‘jumps’’ of  $\mathbb{H}$ -holomorphic functions, as done by the second author in [Pe3]. Admissibility is a second-order condition, so, unlike the complex case, the traces or, more generally, the jumps of  $\mathbb{H}$ -holomorphic functions satisfy first and second-order equations. This is not surprising since these problems are related to local solvability of the Cauchy-Riemann-Fueter Problem  $\overline{\mathfrak{D}}u = g$  and this requires a second-order differential condition for  $g$ . The main results of Section 2 are Theorem 2.9, and Theorem 2.11 reported below.

Let  $\Omega \subset \mathbb{H}^2$  be a domain. A *domain splitting*  $(S, U^+, U^-)$  of  $\Omega$  is given by a smooth (nonempty) hypersurface  $S$  closed in  $\Omega$  and two open disjoint nonempty sets  $U^+, U^-$ , such that  $\Omega \setminus S = U^+ \cup U^-$ , where both  $U^+$  and  $U^-$  have boundary  $S$  in  $\Omega$ .

We say that a continuous (smooth) function  $f : S \rightarrow \mathbb{H}$  is a *continuous (smooth) jump* relative to a domain splitting  $(S, U^+, U^-)$  of  $\Omega$ , if there exist two  $\mathbb{H}$ -holomorphic functions  $F^+, F^-$ , on  $U^+, U^-$  respectively, such that  $F^+, F^-$  are continuous (smooth) up to  $S$  and  $f = F^+|_S - F^-|_S$ .

**Theorem.** *Let  $\Omega \subset \mathbb{H}^2$  be a convex domain and  $(S, U^+, U^-)$  a domain splitting of  $\Omega$ . Let  $f : S \rightarrow \mathbb{H}$  a smooth admissible function. Then,  $f$  is a smooth jump.*

**Theorem.** *Let  $\Sigma$  be an open half-space and  $S \subset \mathbb{H}^2$  a connected closed smooth hypersurface of  $\Sigma$ . Assume that  $\Sigma \setminus S$  splits into two connected components  $D$  and  $W$ , with  $D$  bounded. Let  $f : S \cap \Sigma \rightarrow \mathbb{H}$  be a smooth admissible function. Then,  $f$  extends to  $D$  by an  $\mathbb{H}$ -holomorphic function, which is smooth up to  $S$ .*

In Section 3, we extend the previous results when the function  $f$  is admissible in a weak sense.

Finally, in the Appendix, we provide the characteristic conditions for the local solvability of the Cauchy-Riemann-Fueter Problem  $\overline{\mathfrak{D}}u = g$  in the case of  $n = 2$  octonian variables. This, allows us to generalize some of our constructions and results to the octonian case.

## 1. GENERALITIES

In this section, we summarize some of the main notions and results contained in the seminal papers [Pe1, Pe2, Pe3].

**1.1. Fueter operators and  $\mathbb{H}$ -holomorphic functions.** We fix some notations. Let  $\mathbb{H}$  be the quaternion algebra over  $\mathbb{R}$ . For a generic  $q \in \mathbb{H}$  we write

$$q = \sum_{\alpha=0}^3 x_\alpha i_\alpha, \quad \bar{q} = x_0 - \sum_{\alpha=1}^3 x_\alpha i_\alpha$$

$x_\alpha \in \mathbb{R}$ , where  $i_0 = 1, i_1 = i, i_2 = j, i_3 = k$ .

We also define the following  $\mathbb{H}$ -valued differential forms

$$(1) \quad dq = \sum_{\alpha=0}^3 i_\alpha dx_\alpha, \quad \overline{dq} = \sum_{\alpha=0}^3 \bar{i}_\alpha dx_\alpha$$

and

$$(2) \quad Dq = \sum_{\alpha=0}^3 (-1)^\alpha i_\alpha dX_{\hat{\alpha}}, \quad \overline{Dq} = \sum_{\alpha=0}^3 (-1)^\alpha \bar{i}_\alpha dX_{\hat{\alpha}},$$

where  $dX_{\hat{\alpha}} = dx_0 \wedge \cdots \wedge \widehat{dx_\alpha} \wedge \cdots \wedge dx_3$ .

Let  $F$  be a  $C^1$   $\mathbb{H}$ -valued function. Following Fueter, we define the operators

$$(3) \quad \frac{\partial F}{\partial q} = \sum_{\alpha=0}^3 \bar{i}_\alpha \frac{\partial F}{\partial x_\alpha}, \quad \frac{\partial F}{\partial \bar{q}} = \sum_{\alpha=0}^3 i_\alpha \frac{\partial F}{\partial x_\alpha}.$$

We have

$$(4) \quad \Delta F = \frac{\partial}{\partial q} \frac{\partial}{\partial \bar{q}} F = \frac{\partial}{\partial \bar{q}} \frac{\partial}{\partial q} F,$$

$$(5) \quad d(Dq \cdot F) = \frac{\partial F}{\partial \bar{q}} dx,$$

where  $dx = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$ .

The function  $F$  is said to be (*left*)  $\mathbb{H}$ -holomorphic if

$$\frac{\partial F}{\partial \bar{q}} = 0.$$

The function  $F$  is said to be (*left*)  $\mathbb{H}$ -antiholomorphic if

$$\frac{\partial F}{\partial q} = 0.$$

Right  $\mathbb{H}$ -holomorphic and  $\mathbb{H}$ -antiholomorphic functions are defined interchanging in (3)  $\partial F/\partial x_\alpha$  with  $i_\alpha$  and  $\bar{i}_\alpha$  respectively. For the corresponding derivative, we adopt the notation

$$\frac{F\partial}{\partial \bar{q}}, \quad \frac{F\partial}{\partial q}.$$

For every  $q_0 \in \mathbb{H}$ , the function

$$G(q - q_0) = \frac{\bar{q} - \bar{q}_0}{|q - q_0|^4}$$

is left and right  $\mathbb{H}$ -holomorphic.

The function  $G(q - q_0)$  is the Cauchy-Fueter kernel and is the main ingredient to prove the basic *Cauchy-Fueter formula*

$$F(q_0) = \frac{1}{2\pi^2} \int_{q \in \text{b}\Omega} G(q - q_0) Dq F(q),$$

where  $\Omega$  is a bounded domain in  $\mathbb{H}$  with  $\text{b}\Omega$  sufficiently smooth,  $q_0 \in \Omega$ , and  $F : \overline{\Omega} \rightarrow \mathbb{H}$  a  $C^1$  function which is  $\mathbb{H}$ -holomorphic in  $\Omega$  and continuous on  $\overline{\Omega}$ .

From this formula and 4, one checks immediately that left, right  $\mathbb{H}$ -holomorphic and  $\mathbb{H}$ -antiholomorphic functions are harmonic.

For other general results in one quaternionic variable we refer to [S]. Here we just want to mention the following ‘‘Carleman type’’ result:

**Proposition 1.1.** *Let  $\Omega$  be a domain in the ball  $B(r) = \{q \in \mathbb{H} : |q| < r\}$  such that  $0 \notin \overline{\Omega}$  and  $\text{b}\Omega = \Gamma \cup \Sigma$ , with  $\Gamma \subset B(r)$  and  $\Sigma \subset \text{b}B(r)$ . Let  $F$  be an  $\mathbb{H}$ -holomorphic function on a neighborhood of  $\overline{\Omega}$ . Then,  $F|_\Omega$  depends only on  $F|_\Gamma$ .*

*Proof.* Let  $q \in \Omega$ . By Cauchy-Fueter formula,

$$\begin{aligned} F(q) &= \frac{1}{2\pi^2} \int_{p \in \text{b}\Omega} G(p-q) DpF(p) \\ &= \frac{1}{2\pi^2} \int_{p \in \Gamma} G(p-q) DpF(p) + \frac{1}{2\pi^2} \int_{p \in \Sigma} G(p-q) DpF(p). \end{aligned}$$

If  $p \in \Sigma$ , then  $|q| < |p|$  and

$$G(p-q) = \sum_{m=0}^{+\infty} \sum_{\nu \in \sigma_m} P_\nu(q) G_\nu(p),$$

where  $\sigma_m = \{(m_1, m_2, m_3) \in \mathbb{N}^3 : m_1 + m_2 + m_3 = m\}$ , the  $P_\nu$  are  $\mathbb{H}$ -holomorphic polynomials, the functions  $G_\nu(p)$  are  $\mathbb{H}$ -holomorphic in  $\mathbb{H} \setminus \{0\}$ , and the series is totally convergent with respect to  $p \in \Sigma$  (see [S, Proposition 10]).

Since  $0 \notin \bar{\Omega}$ , by the Cauchy-Fueter theorem (see [Fu1, 1. Hauptsatz]) we have

$$\int_{p \in \text{b}\Omega} G_\nu(p) DpF(p) = \int_{p \in \Gamma} G_\nu(p) DpF(p) + \int_{p \in \Sigma} G_\nu(p) DpF(p) = 0,$$

for all  $\nu$ . It follows that

$$\begin{aligned} \int_{p \in \Sigma} G(p-q) DpF(p) &= \sum_{m=0}^{+\infty} \sum_{\nu \in \sigma_m} P_\nu(q) \int_{p \in \Sigma} G_\nu(p) DpF(p) \\ &= - \sum_{m=0}^{+\infty} \sum_{\nu \in \sigma_m} P_\nu(q) \int_{p \in \Gamma} G_\nu(p) DpF(p), \end{aligned}$$

whence the Carleman formula

$$F(q) = \frac{1}{2\pi^2} \int_{p \in \Gamma} G(p-q) DpF(p) - \frac{1}{2\pi^2} \sum_{m=0}^{+\infty} \sum_{\nu \in \sigma_m} P_\nu(q) \int_{p \in \Gamma} G_\nu(p) DpF(p)$$

proving the statement.  $\square$

**1.2. Several variables.** Fueter operators clearly extend to ( $\mathbb{H}$ -valued) functions of several quaternionic variables  $q_1, q_2, \dots, q_n$ .

For the sake of simplicity, from now on we assume  $n = 2$ , even if the most part of the results proved in the sequel hold for any  $n$ .

We denote  $q = (q_1, q_2)$  the generic element of  $\mathbb{H}^2$  and we set

$$q_1 = \sum_{\alpha=0}^3 x_\alpha i_\alpha, \quad q_2 = \sum_{\alpha=0}^3 y_\alpha i_\alpha.$$

The Cauchy-Riemann-Fueter operators  $\bar{\mathfrak{D}}$  and  $\mathfrak{D}$  are then defined, respectively, by

$$(6) \quad F \longmapsto (\partial F / \partial \bar{q}_1, \partial F / \partial \bar{q}_2), \quad F \longmapsto (\partial F / \partial q_1, \partial F / \partial q_2)$$

and  $F$  is said to be (*left*)  $\mathbb{H}$ -holomorphic if it is  $C^1$  and  $\bar{\mathfrak{D}}F = 0$ .

We have the identity

$$(7) \quad \begin{aligned} \frac{1}{2} (\bar{d}q_1 \wedge dq_1 \wedge dy \wedge dF + dx \wedge \bar{d}q_2 \wedge dq_2 \wedge dF) = \\ - (\bar{D}q_1 \frac{\partial F}{\partial q_1} \wedge dy + dx \wedge \bar{D}q_2 \frac{\partial F}{\partial q_2}) + \star dF, \end{aligned}$$

where  $dx = dx_0 \wedge \dots \wedge dx_3$ ,  $dy = dy_0 \wedge \dots \wedge dy_3$ , and  $\star$  is the Hodge operator.

In particular, by 7, we get that if  $F$  is  $\mathbb{H}$ -holomorphic,

$$(8) \quad \frac{1}{2} (\bar{d}q_1 \wedge dq_1 \wedge dy \wedge dF + dx \wedge \bar{d}q_2 \wedge dq_2 \wedge dF) = \star dF.$$

**Remark 1.2.** Formula (8) holds, more generally, at those points where  $\overline{\mathfrak{D}}F = 0$ .

Let  $\Delta_1$  ( $\Delta_2$ ) denote the laplacian in the coordinates  $x_\alpha$  ( $y_\alpha$ ),  $\alpha = 0, 1, 2, 3$ . Then, if  $F$  is  $\mathbb{H}$ -holomorphic,  $\Delta_1 F = \Delta_2 F = 0$ . In particular,  $F$  is harmonic.

A useful way to construct  $\mathbb{H}$ -holomorphic functions in one quaternionic variable is to start by (complex) holomorphic functions  $F = F(z) = u + iv$  and define [Fu1, 5. Satz]

$$(9) \quad F^\# = F^\#(q) := u(\operatorname{Re} q, |\operatorname{Im} q|) + \frac{\operatorname{Im} q}{|\operatorname{Im} q|} v(\operatorname{Re} q, |\operatorname{Im} q|).$$

In general,  $F^\#$  is not  $\mathbb{H}$ -holomorphic, not even harmonic, but its laplacian  $\Delta F^\#$  is.

**Example.** Let  $F(z) = z^n$ . Then,

$$F^\#(q) = (z^n)^\# = q^n.$$

In particular, for the cases  $n = 3$  and  $n = -1$ , we get

$$\Delta q^3 = -4(2q + \bar{q}),$$

$$\Delta \left( \left( \frac{1}{z} \right)^\# \right) = -4 \frac{\bar{q}}{|q|^4} = -4G(q).$$

**1.3. Bochner-Martinelli Kernel.** The *Bochner-Martinelli Kernel*  $\mathbf{K}^{BM}(q, q_0)$  was introduced in [Pe1], where a representation formula for  $\mathbb{H}$ -holomorphic functions was proved:

$$(10) \quad F(q_0) = \int_{q \in \mathfrak{b}\Omega} \mathbf{K}^{BM}(q, q_0) F(q).$$

Here  $q_0$  belongs to a bounded domain  $\Omega$  in  $\mathbb{H}^n$  with smooth boundary  $\mathfrak{b}\Omega$  and  $F$  is  $\mathbb{H}$ -holomorphic in  $\Omega$  and continuous up to  $\mathfrak{b}\Omega$ . We will use the notation  $\mathbf{K}^{BM}(q, q_0)$  instead of the original one.

Set  $q_1 = z_1 + w_1j$ ,  $q_2 = z_2 + w_2j$ ,  $z_\alpha, w_\alpha \in \mathbb{C}$ ,  $z = (z_1, z_2)$ ,  $w = (w_1, w_2)$ .

We use the notation

$$\mathbf{K}^{BM}(q, q_0) := \mathbf{K}^{BM}(z, w, z^0, w^0),$$

where  $z^0 = (z_1^0, z_2^0)$ ,  $w^0 = (w_1^0, w_2^0)$ .

The  $\mathbb{H}$ -valued differential form  $\mathbf{K}^{BM}(q, q_0)$  is a real analytic of degree 7 and

$$(11) \quad \mathbf{K}^{BM}(q, q_0) = \mathbf{K}_1^{BM}(q, q_0) + \mathbf{K}_2^{BM}(q, q_0)j,$$

where  $\mathbf{K}_1^{BM}$ ,  $\mathbf{K}_2^{BM}$  are real analytic complex-valued differential forms.

Observe that  $\mathbf{K}_1^{BM}(z, w, z_0, w_0)$  is the Bochner-Martinelli kernel for functions which are holomorphic with respect to  $z_1, z_2$  and antiholomorphic with respect to  $w_1, w_2$  and  $\mathbf{K}_2^{BM}(z, w, z_0, w_0)$  is exact on  $\mathbb{H}^2 \setminus \{(z^0, w^0)\}$ :

$$(12) \quad \mathbf{K}_2^{BM}(z, w, z_0, w_0) = d\omega_2$$

where

$$\begin{aligned} \omega_2 = (8\pi^4)^{-1} & |(z, w) - (z^0, w^0)|^{-6} \cdot \\ & (d\bar{z}_1 \wedge dw_1 \wedge d\bar{z}_2 \wedge dz_2 \wedge d\bar{w}_2 \wedge dw_2 + \\ & d\bar{z}_1 \wedge dz_1 \wedge d\bar{w}_1 \wedge dw_1 \wedge d\bar{z}_2 \wedge dw_2). \end{aligned}$$

**1.4.  $\mathbb{H}$ -holomorphy and  $\mathbb{H}$ -convexity.**  $\mathbb{H}$ -holomorphy and  $\mathbb{H}$ -convexity are defined like in the complex case [Pe3]. Kontinuitätssatz holds true [Pe3, Theorem 2], as well as the following implications [Pe3, Proposition 6, Theorem 3]

- 1) for a domain in  $\mathbb{C}^4 \simeq \mathbb{H}^2$ , holomorphy implies  $\mathbb{H}$ -holomorphy. The converse is not true in general (e.g.  $\mathbb{H} \setminus \{(0, 0)\}$  is a domain of  $\mathbb{H}$ -holomorphy, but it is not a domain of holomorphy in  $\mathbb{C}^2 \simeq \mathbb{H}$ );
- 2)  $\mathbb{H}$ -holomorphy implies  $\mathbb{H}$ -convexity;

For domains  $\Omega \subset \mathbb{H}^n$ ,  $n > 1$ , with smooth boundary  $\text{b}\Omega$ , a necessary condition for the  $\mathbb{H}$ -holomorphy can be given by the  $2^{\text{nd}}$  fundamental form  $h$  of  $\text{b}\Omega$  with respect to the orientation of  $\text{b}\Omega$  determined by the inward unit normal vector. Precisely [Pe3, Theorem 4],

3) given a point  $q_0 \in \text{b}\Omega$ , there is no right  $\mathbb{H}$ -line  $\ell$  tangent to  $\text{b}\Omega$  at  $q_0$  such that  $h(q_0)|_\ell < 0$ .

In this case, we say that  $\Omega$  (or its boundary) is *Levi  $\mathbb{H}$ -convex*. For  $n = 2$ , we say that  $\Omega$  is *strongly Levi  $\mathbb{H}$ -convex*, if for all  $q_0 \in \text{b}\Omega$ , we have  $h(q_0)|_\ell > 0$ , where  $\ell$  is the only right  $\mathbb{H}$ -line tangent to  $\text{b}\Omega$  at  $q_0$ .

In general, we say that a smooth hypersurface  $S \subset \mathbb{H}^n$  is *nondegenerate* if, there exists a right  $\mathbb{H}$ -line  $\ell$  such that the form  $h(q_0)|_\ell$  has constant sign.

Two open problems:

- i) Is a domain  $\mathbb{H}$ -convex a domain of  $\mathbb{H}$ -holomorphy?
- ii) Levi problem in  $\mathbb{H}^n$ .

1.5.  **$\bar{\mathfrak{D}}$ -problem and Hartogs Theorem.** Let  $q = (q_1, q_2) \in \mathbb{H}^2$  with

$$q_1 = \sum_{\alpha=0}^3 x_\alpha i_\alpha, \quad q_2 = \sum_{\alpha=0}^3 y_\alpha i_\alpha$$

and consider the laplacians

$$\Delta_1 = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \quad \Delta_2 = \frac{\partial^2}{\partial y_0^2} + \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2}.$$

Then, since  $\partial/\partial\bar{q}_s$  and  $\Delta_h$  commute we have

$$(13) \quad \frac{\partial}{\partial\bar{q}_s} \frac{\partial}{\partial q_h} \frac{\partial}{\partial\bar{q}_h} = \frac{\partial}{\partial\bar{q}_s} \Delta_h = \Delta_h \frac{\partial}{\partial\bar{q}_s}.$$

It follows that, if  $u$  is a smooth (local) solution of the CRF system

$$(14) \quad \bar{\mathfrak{D}}u = g, \quad g = (g_1, g_2),$$

then

$$(15) \quad \Delta_h g_s = \frac{\partial}{\partial\bar{q}_s} \frac{\partial g_h}{\partial q_h},$$

which is a nontrivial condition for  $h \neq s$ .

For every pair  $g = (g_1, g_2)$ , we set

$$(16) \quad \begin{aligned} \bar{P}_1(g) &= \frac{\partial}{\partial\bar{q}_1} \frac{\partial g_2}{\partial q_2} - \Delta_2 g_1, \\ \bar{P}_2(g) &= \frac{\partial}{\partial\bar{q}_2} \frac{\partial g_1}{\partial q_1} - \Delta_1 g_2 \end{aligned}$$

and denote  $\bar{P}$  the operator  $g = (g_1, g_2) \mapsto (\bar{P}_1(g), \bar{P}_2(g))$ . Then, if  $g = \bar{\mathfrak{D}}u$  with  $u$  smooth, we have

$$(17) \quad \bar{P}(g) = 0,$$

i.e.,

$$(18) \quad \bar{P}_1(g) = 0, \quad \bar{P}_2(g) = 0.$$

Conditions (15) for  $h, s = 1, \dots, n$  are still necessary in order to solve  $\bar{\mathfrak{D}}u = g$  for  $g = (g_1, \dots, g_n)$ . If  $g \in C_0^k$ ,  $n, k \geq 2$ , they are also sufficient and in such situation  $\bar{\mathfrak{D}}u = g$  has a  $C_0^k$  solution  $u$  (see [Pe2, Theorem 1]). In particular, this implies Hartogs Theorem. We point out that Hartogs Theorem was already proved by the second author [Pe1, Teorema 6], by solving the equation  $\bar{\mathfrak{D}}u = g$  with integral conditions on  $g$ , instead of (15). As for the system  $\bar{\mathfrak{D}}u = g$ , when  $g \in C^\infty(\Omega, \mathbb{H})$ ,  $\Omega \subset \mathbb{H}^n$ , we have the following: if  $n = 2$  and  $\Omega$  is convex, the system has a smooth solution if and only if  $\bar{P}(g) = 0$  (see [ABLSS]). If  $n > 2$ , conditions 15 are no longer sufficient in general. For

$g \in C^\infty(\Omega, \mathbb{H})$ ,  $\Omega$  convex, using the results of [AL, CSSS], necessary and sufficient conditions were proved in [BDS].

**Remark 1.3.** *The same is true if  $g$  is replaced by a distribution. This is a consequence of the “division of distributions” [Eh, Ma, Pa, AN, N]. We will use this generalization in Section 3.*

As far as we know, nothing is known about the existence of the equation  $\overline{\mathcal{D}}u = g$  in more general domains.

## 2. RIEMANN-HILBERT AND DIRICHLET PROBLEMS FOR $\mathbb{H}$ -HOLOMORPHIC FUNCTIONS.

**2.1. The operator  $\overline{\mathcal{D}}_b$  and the CRF condition.** Let  $\Omega \subset \mathbb{H}^2$  be a domain. A *domain splitting*  $(S, U^+, U^-)$  of  $\Omega$  is given by a smooth (nonempty) hypersurface  $S$  closed in  $\Omega$  and two open disjoint nonempty sets  $U^+, U^-$ , such that  $\Omega \setminus S = U^+ \cup U^-$ , where both  $U^+$  and  $U^-$  have boundary  $S$  in  $\Omega$ .

We say that a continuous (resp. smooth<sup>1</sup>) function  $f : S \rightarrow \mathbb{H}$  is a *continuous (resp. smooth) jump* relative to a domain splitting  $(S, U^+, U^-)$  of  $\Omega$ , if there exist two  $\mathbb{H}$ -holomorphic functions  $F^+, F^-$ , on  $U^+, U^-$  respectively, such that  $F^+, F^-$  are continuous (resp. smooth) up to  $S$  and  $f = F^+|_S - F^-|_S$ .

A function  $f : S \rightarrow \mathbb{H}$  (continuous or smooth) is *locally a jump* if, for every  $q_0 \in S$ , there exists a neighborhood  $U$  of  $q_0$  such that  $f|_{U \cap S}$  is a jump in  $U$ .

Observe that the functions  $F^+, F^-$  are determined up an  $\mathbb{H}$ -holomorphic function in  $U$ . In particular, if  $S$  is the boundary of a bounded domain in  $\mathbb{H}^2$ , Dirichlet problem reduces to Riemann-Hilbert problem via the Hartogs theorem.

Both these problems require conditions on the given function  $f : S \rightarrow \mathbb{H}$  that we call CRF conditions.

Let  $S$  be defined by  $\rho = 0$ . We say that a smooth function  $f : S \rightarrow \mathbb{H}$  is a (*left*) *CRF function* if, there is a smooth extension  $F$  of  $f$  on a neighborhood of  $S$ , such that we have

$$(19) \quad \overline{\mathcal{D}}F = \rho \cdot A + \overline{\mathcal{D}}\rho \cdot B,$$

with  $A$  and  $B$  smooth. The CRF condition is independent of the extension  $F$ , as well as of the equation of  $S$ .

The CRF condition can be given in a more intrinsic way, as shown in Theorem 2.4 below.

**Remark 2.1.** *Observe that,  $f$  is a CRF function if and only if there exists a smooth extension  $F_1$  of  $f$  with  $\overline{\mathcal{D}}F_1 = 0$  on  $S$ . (It is enough to take  $F_1 = F - \rho \cdot B$ , where  $F$  satisfies 19.)*

Clearly, if  $F$  is an  $\mathbb{H}$ -holomorphic function on one sided neighborhood of  $S$ , then  $F|_S$  is a CRF function, in particular, every local jump  $f$  on  $S$  is a CRF function.

We will see below that, unlike the complex case, trace conditions on  $f$  involve both first-order and second-order differential equations (Remark 2.3).

This is not surprising, due to the fact that Riemann-Hilbert problem is related to local solvability of  $\overline{\mathcal{D}}u = g$  and this requires a second-order differential condition for  $g$ .

If  $F = U + Vj$  is an extension of  $f$ ,  $q_1 = z_1 + w_1j$ ,  $q_2 = z_2 + w_2j$ , where  $U, V, z_1, w_1, z_2, w_2$  are complex, then the CRF condition writes

$$(20) \quad \text{rank} \begin{pmatrix} U_{\bar{z}_1} - \overline{V} \overline{w}_1 & \rho_{\bar{z}_1} & -\rho \overline{w}_1 \\ \overline{V} z_1 + U_{w_1} & \rho_{w_1} & \rho z_1 \\ U_{\bar{z}_2} - \overline{V} \overline{w}_2 & \rho_{\bar{z}_2} & -\rho \overline{w}_2 \\ \overline{V} z_2 + U_{w_2} & \rho_{w_2} & \rho z_2 \end{pmatrix} < 3.$$

<sup>1</sup>For convenience of exposition, since our work reposes in an essential way to the theory of Ehrenpreis and its applications [Eh, CSSS], we restrict ourselves to the class of  $C^\infty$  functions, even if some definitions and constructions can be given in a more general setting.



2.1.1. *CRF condition and extendability.* Suppose  $S$  oriented. Denote  $\omega$  the volume form of  $S$  and  $\nu = (\nu_1, \nu_2)$ ,  $\nu_1, \nu_2 \in \mathbb{H}$ , the unit normal vector which gives the orientation of  $S$ .

Let  $\langle \cdot, \cdot \rangle : \mathbb{H}^2 \times \mathbb{H}^2 \rightarrow \mathbb{H}$  be the scalar product

$$\langle (q_1, q_2), (p_1, p_2) \rangle = \bar{q}_1 p_1 + \bar{q}_2 p_2.$$

By direct computation, one verify that

$$(21) \quad (\overline{Dq_1} \wedge dy)|_S = -\bar{\nu}_1 \omega, \quad (dx \wedge \overline{Dq_2})|_S = -\bar{\nu}_2 \omega.$$

Let  $f : S \rightarrow \mathbb{H}$  be smooth and  $F$  a smooth extension of  $f$  on a neighborhood of  $S$ . Then, by restriction to  $S$ , from (7) we get

$$(22) \quad -\frac{1}{2} \left( \overline{dq_1} \wedge dq_1 \wedge dy + dx \wedge \overline{dq_2} \wedge dq_2 \right) \Big|_S \wedge df = \left( -\langle \nu, \overline{\mathfrak{D}F}|_S \rangle + \frac{\partial F}{\partial \nu} \right) \omega,$$

where  $\overline{\mathfrak{D}F} = \left( \frac{\partial F}{\partial \bar{q}_1}, \frac{\partial F}{\partial \bar{q}_2} \right)$ .

Let  $f^\perp : S \rightarrow \mathbb{H}$  be the smooth function defined by

$$(23) \quad -\frac{1}{2} \left( \overline{dq_1} \wedge dq_1 \wedge dy + dx \wedge \overline{dq_2} \wedge dq_2 \right) \Big|_S \wedge df = f^\perp \cdot \omega$$

and set

$$(24) \quad \frac{\partial}{\partial x_\alpha} \Big|_S = \tau_{x_\alpha} + \left( \frac{\partial}{\partial x_\alpha}, \nu \right) \nu, \quad \frac{\partial}{\partial y_\alpha} \Big|_S = \tau_{y_\alpha} + \left( \frac{\partial}{\partial y_\alpha}, \nu \right) \nu$$

$\alpha = 0, 1, 2, 3$ , where  $(\cdot, \cdot)$  denotes the euclidean scalar product of  $\mathbb{R}^8$  and  $\tau_{x_\alpha}, \tau_{y_\alpha}$  are the tangential components of  $\frac{\partial}{\partial x_\alpha} \Big|_S, \frac{\partial}{\partial y_\alpha} \Big|_S$  respectively.

We set

$$(25) \quad \begin{aligned} f_{(x_\alpha)} &= \tau_{x_\alpha}(f) + \left( \frac{\partial}{\partial x_\alpha}, \nu \right) f^\perp, \\ f_{(y_\alpha)} &= \tau_{y_\alpha}(f) + \left( \frac{\partial}{\partial y_\alpha}, \nu \right) f^\perp, \\ f_{(\bar{q}_1)} &= f_{(x_0)} + \mathbf{i}f_{(x_1)} + \mathbf{j}f_{(x_2)} + \mathbf{k}f_{(x_3)}, \\ f_{(\bar{q}_2)} &= f_{(y_0)} + \mathbf{i}f_{(y_1)} + \mathbf{j}f_{(y_2)} + \mathbf{k}f_{(y_3)}; \end{aligned}$$

they are smooth functions on  $S$ .

**Proposition 2.2.** *Let  $f : S \rightarrow \mathbb{H}$  be a smooth CRF function and  $F$  a smooth local extension of  $f$  such that  $\overline{\mathfrak{D}F} = 0$  on  $S$ . Then,*

$$(26) \quad \frac{\partial F}{\partial \nu} = f^\perp, \quad \frac{\partial F}{\partial x_\alpha} \Big|_S = f_{(x_\alpha)}, \quad \frac{\partial F}{\partial y_\alpha} \Big|_S = f_{(y_\alpha)},$$

for  $\alpha = 0, 1, 2, 3$ .

*Proof.* Since  $\overline{\mathfrak{D}F} = 0$  on  $S$  and  $F|_S = f$ , by Remark 22

$$-\frac{1}{2} \left( \overline{dq_1} \wedge dq_1 \wedge dy \wedge df + dx \wedge \overline{dq_2} \wedge dq_2 \wedge df \right) \Big|_S = \frac{\partial F}{\partial \nu} \omega$$

and comparing with (23) we then have  $\frac{\partial F}{\partial \nu} = f^\perp$ . Formulas (24) now imply  $\frac{\partial F}{\partial x_\alpha} \Big|_S = f_{(x_\alpha)}$ ,  $\frac{\partial F}{\partial y_\alpha} \Big|_S = f_{(y_\alpha)}$ ,  $\alpha = 0, 1, 2, 3$ .  $\square$

**Remark 2.3.** *If  $f$  is the boundary value of an  $\mathbb{H}$ -holomorphic function  $F$ , then, by Proposition 2.2, we get*

$$\begin{aligned} \frac{\partial F}{\partial x_\alpha} \Big|_S &= f_{(x_\alpha)} \\ \frac{\partial F}{\partial y_\alpha} \Big|_S &= f_{(y_\alpha)} \end{aligned} \quad \text{for } \alpha = 0, 1, 2, 3.$$

Since the operators  $\overline{\mathfrak{D}}$ ,  $\partial/\partial x_\alpha$ ,  $\partial/\partial y_\alpha$  commute,  $f_{(x_\alpha)}$  and  $f_{(y_\alpha)}$  are restrictions of the  $\mathbb{H}$ -holomorphic functions  $\frac{\partial F}{\partial x_\alpha}$  and  $\frac{\partial F}{\partial y_\alpha}$  respectively, hence  $f_{(x_\alpha)}$ ,  $f_{(y_\alpha)}$  are CRF functions too.

A smooth CRF function  $f : S \rightarrow \mathbb{H}$  is said to be *admissible* if  $f_{(x_\alpha)}$ ,  $f_{(y_\alpha)}$ ,  $\alpha = 0, 1, 2, 3$ , are CRF functions too. Unlike the complex case, a CRF function is not admissible in general. Here is a counterexample:

**Example.** Let  $S = \{y_3 = 0\}$ ,  $f = -x_1 y_0 \mathbf{j} + x_0 y_0 \mathbf{k}$ . Since  $\partial f / \partial \bar{q}_1 = 0$ ,  $f$  is CRF. Moreover,  $f^\perp = f_{(y_3)} = -x_0 + x_1 \mathbf{i}$ . In particular, if  $f_{(y_3)}$  were CRF we should have  $\partial f_{(y_3)} / \partial \bar{q}_1 = 0$ , whereas  $\partial f_{(y_3)} / \partial \bar{q}_1 = -2$ .

2.1.2. *The tangential operator  $\overline{\mathfrak{D}}_b$ .* The CRF condition determines a differential operator on  $S$  that will be denoted by  $\overline{\mathfrak{D}}_b$ . We want to write explicitly the operator  $\overline{\mathfrak{D}}_b$ .

Consider on  $S$  the following  $\mathbb{H}$ -valued differential forms

$$(27) \quad \begin{aligned} d_{(q_1)} f &= f_{(x_0)} dx_0|_S + f_{(x_1)} dx_1|_S + f_{(x_2)} dx_2|_S + f_{(x_3)} dx_3|_S \\ d_{(q_2)} f &= f_{(y_0)} dy_0|_S + f_{(y_1)} dy_1|_S + f_{(y_2)} dy_2|_S + f_{(y_3)} dy_3|_S. \end{aligned}$$

The following equalities hold

$$(28) \quad \begin{aligned} Dq_1|_S \wedge d_{(q_1)} f &= -f_{(\bar{q}_1)} dx|_S \\ Dq_2|_S \wedge d_{(q_2)} f &= -f_{(\bar{q}_2)} dy|_S. \end{aligned}$$

We have the following

**Theorem 2.4.** *For a given smooth function  $f$  on  $S$  the following conditions are equivalent:*

- a)  $f$  is a CRF function;
- b)  $f_{(\bar{q}_1)} \equiv f_{(\bar{q}_2)} \equiv 0$ ;
- c)  $Dq_1|_S \wedge d_{(q_1)} f \equiv Dq_2|_S \wedge d_{(q_2)} f \equiv 0$ .

*Proof.* Let  $f$  be CRF. Then, there exists a smooth extension  $F$  of  $f$  with the property  $\overline{\mathfrak{D}}F = 0$  on  $S$  (Remark 2.1). From (26), we get

$$\partial F / \partial \nu = f^\perp, \quad \frac{\partial F}{\partial x_\alpha} \Big|_S = f_{(x_\alpha)}, \quad \frac{\partial F}{\partial y_\alpha} \Big|_S = f_{(y_\alpha)},$$

$\alpha = 0, 1, 2, 3$ . Consequently

$$(29) \quad \frac{\partial F}{\partial \bar{q}_1} \Big|_S = f_{(\bar{q}_1)}, \quad \frac{\partial F}{\partial \bar{q}_2} \Big|_S = f_{(\bar{q}_2)}.$$

By hypothesis,  $\overline{\mathfrak{D}}F = 0$  on  $S$  hence  $f_{(\bar{q}_1)} = f_{(\bar{q}_2)} = 0$  and therefore, by (28)

$$(30) \quad Dq_1|_S \wedge d_{(q_1)} f = Dq_2|_S \wedge d_{(q_2)} f = 0,$$

i.e., b) and c).

Assume that  $f$  satisfies c). Then, by (28), we have  $f_{(\bar{q}_1)} dx|_S \equiv 0$ ,  $f_{(\bar{q}_2)} dy|_S \equiv 0$  and, if  $dx|_S(p) \neq 0$ ,  $dy|_S(p) \neq 0$ ,  $p \in S$ , then  $f_{(\bar{q}_1)}(p) = f_{(\bar{q}_2)}(p) = 0$ . Suppose, for instance, that  $dx|_S(p) = 0$ . Then, the second of (21) implies  $\nu_2(p) = 0$ , i.e.,  $\nu(p) = (\nu_1(p), 0)$ , where  $\nu_1(p) \neq 0$ . Thus,  $dy|_S(p) \neq 0$ , otherwise (again by (21)), we should have  $\nu_1(p) = 0$ , consequently,  $f_{(\bar{q}_2)}(p) = 0$ .

Let us show that necessarily  $f_{(\bar{q}_1)}(p) = 0$ . By standard argument of differential topology, it is easy to construct a smooth extension  $F$  of  $f$  such that  $\partial F / \partial \nu = f^\perp$ . Identity (22) and definition of  $f^\perp$  then imply that  $\langle \nu(p), \overline{\mathfrak{D}}F(p) \rangle = 0$ , i.e.,  $\bar{\nu}_1(p) \partial F / \partial \bar{q}_1(p) = 0$ , whence  $\partial F / \partial \bar{q}_1(p) = 0$ . Arguing as in the first part of the proof, we get  $\partial F / \partial \bar{q}_1|_S = f_{(\bar{q}_1)}$ ,  $\partial F / \partial \bar{q}_2|_S = f_{(\bar{q}_2)}$ , in particular, also  $f_{(\bar{q}_1)}(p) = 0$  for every  $p \in S$ , and c) imply b). Furthermore,  $\overline{\mathfrak{D}}F = 0$  on  $S$ , hence c) implies a) too.  $\square$

We denote  $\overline{\mathfrak{D}}_b$  the operator

$$(31) \quad \overline{\mathfrak{D}}_b : f \mapsto \left( Dq_1|_S \wedge d_{(q_1)} f, Dq_2|_S \wedge d_{(q_2)} f \right).$$

**2.2. Solvability of the Riemann-Hilbert problem.** We want to prove that for smooth admissible functions the local Riemann-Hilbert problem is always solvable.

We consider an orientable smooth hypersurface  $S$ , given as the zero set of a smooth function  $\rho$  such that  $\nabla\rho \neq 0$  around  $S$ .

**Proposition 2.5.** *Let  $f : S \rightarrow \mathbb{H}$  be a smooth function. The following properties are equivalent*

- i)  $f$  is admissible;
- ii) there exists a smooth extension  $F$  of  $f$  such that around  $S$  one has  $\overline{\mathfrak{D}}F = \rho^2 u$ , with  $u$  a smooth  $\mathbb{H}^2$ -valued map.

*Proof.* Let  $F$  as in ii). Clearly  $f$  is CRF. By Proposition 2.2, we have  $\partial F/\partial x_\alpha|_S = f_{(x_\alpha)}$ ,  $\partial F/\partial y_\alpha|_S = f_{(y_\alpha)}$ ,  $\alpha = 0, 1, 2, 3$ . Moreover,

$$(32) \quad \begin{aligned} \overline{\mathfrak{D}}\left(\frac{\partial F}{\partial x_\alpha}\right) &= \frac{\partial}{\partial x_\alpha}\left(\overline{\mathfrak{D}}F\right) = \rho\left(2\frac{\partial\rho}{\partial x_\alpha}u + \rho\frac{\partial u}{\partial x_\alpha}\right) \\ \overline{\mathfrak{D}}\left(\frac{\partial F}{\partial y_\alpha}\right) &= \frac{\partial}{\partial y_\alpha}\left(\overline{\mathfrak{D}}F\right) = \rho\left(2\frac{\partial\rho}{\partial y_\alpha}u + \rho\frac{\partial u}{\partial y_\alpha}\right), \end{aligned} \quad \text{for } \alpha = 0, 1, 2, 3.$$

Therefore,  $\partial F/\partial x_\alpha$  ( $\partial F/\partial y_\alpha$ ),  $\alpha = 0, 1, 2, 3$ , is a smooth extension of  $f_{(x_\alpha)}$  ( $f_{(y_\alpha)}$ ), whose  $\overline{\mathfrak{D}}$  is vanishing on  $S$ . It follows that  $f$  is admissible.

Assume now that  $f$  is admissible, in particular CRF. Therefore, there is a smooth extension  $G$  of  $f$  and a smooth  $\mathbb{H}^2$ -valued map  $\psi$  such that  $\overline{\mathfrak{D}}G = \rho\psi$ . Again, by Proposition 2.2, one has  $\partial G/\partial x_\alpha|_S = f_{(x_\alpha)}$ ,  $\partial G/\partial y_\alpha|_S = f_{(y_\alpha)}$ ,  $\alpha = 0, 1, 2, 3$ . Since also  $f_{(x_\alpha)}$  is CRF, there is a smooth extension  $F^{(x_\alpha)}$  of  $f_{(x_\alpha)}$  such that  $\overline{\mathfrak{D}}F^{(x_\alpha)} = \rho\eta^{(x_\alpha)}$  with  $\eta^{(x_\alpha)}$  smooth, whence

$$(33) \quad \partial G/\partial x_\alpha - F^{(x_\alpha)} = \rho\psi^{(x_\alpha)}$$

with  $\psi^{(x_\alpha)}$  smooth,  $\alpha = 0, 1, 2, 3$ .

Applying  $\overline{\mathfrak{D}}$  to (33), and taking into account that  $\overline{\mathfrak{D}} \circ (\partial/\partial x_\alpha) = (\partial/\partial x_\alpha) \circ \overline{\mathfrak{D}}$ , we obtain

$$(34) \quad \frac{\partial(\overline{\mathfrak{D}}G)}{\partial x_\alpha} = \rho H^{(x_\alpha)} + \overline{\mathfrak{D}}\rho \cdot \psi^{(x_\alpha)}$$

with  $H^{(x_\alpha)}$  smooth,  $\alpha = 0, 1, 2, 3$ .

In the same way,

$$(35) \quad \frac{\partial(\overline{\mathfrak{D}}G)}{\partial y_\alpha} = \rho H^{(y_\alpha)} + \overline{\mathfrak{D}}\rho \cdot \psi^{(y_\alpha)}$$

with  $H^{(y_\alpha)}$  smooth,  $\alpha = 0, 1, 2, 3$ .

Let

$$\nu = \nabla\rho/|\nabla\rho| = |\nabla\rho|^{-1} \sum_{\alpha=0}^3 \left( \frac{\partial\rho}{\partial x_\alpha} \frac{\partial}{\partial x_\alpha} + \frac{\partial\rho}{\partial y_\alpha} \frac{\partial}{\partial y_\alpha} \right).$$

By hypothesis,  $\overline{\mathfrak{D}}G = \rho\psi$ , so

$$(36) \quad \frac{\partial(\overline{\mathfrak{D}}G)}{\partial\nu} = \frac{\partial\rho}{\partial\nu}\psi + \rho\frac{\partial\psi}{\partial\nu} = |\nabla\rho|\psi + \rho\frac{\partial\psi}{\partial\nu}$$

On the other hand, from (34), (35), we derive

$$(37) \quad \frac{\partial(\overline{\mathfrak{D}}G)}{\partial\nu} = |\nabla\rho|^{-1} \left\{ \rho \sum_{\alpha=0}^3 A_\alpha + \overline{\mathfrak{D}}\rho \cdot \sum_{\alpha=0}^3 B_\alpha \right\}.$$

Equalizing (36) and (37), we get

$$(38) \quad \psi = \rho\Phi + 2\overline{\mathfrak{D}}\rho \cdot \Theta$$

and consequently

$$\overline{\mathfrak{D}}G = \rho\psi = \rho^2\Phi + \overline{\mathfrak{D}}\rho^2 \cdot \Theta.$$

Then,  $F := G - \rho^2\Theta$  is the desired extension of  $f$ .  $\square$

**Lemma 2.6.** *Let  $U$  be a domain in  $\mathbb{H}^2$ ,  $S = \{\rho = 0\}$  where  $\rho : U \rightarrow \mathbb{H}$  is smooth and  $\nabla\rho \neq 0$  on  $S$ . Let  $\{h_k\}_k \in \mathbb{N}$  be a sequence of smooth functions  $S \rightarrow \mathbb{H}$ . Then, there exists a smooth function  $E : U \rightarrow \mathbb{H}$  with the following properties*

- 1)  $E|_S = h_0$ ;
- 2)  $\frac{\partial^k E}{\partial \rho^k}|_S = h_k, \forall k \geq 1$ .

This lemma is a straightforward generalization of [AnH, Proposition 2.2].

**Proposition 2.7.** *Let  $U$  be a domain in  $\mathbb{H}^2$ ,  $S = \{\rho = 0\}$  where  $\rho : U \rightarrow \mathbb{R}$  is smooth and  $\nabla\rho \neq 0$  on  $S$ . Let  $f : S \rightarrow \mathbb{H}$  be a smooth and admissible function. Then there are a smooth function  $F : U \rightarrow \mathbb{H}$  and two sequences  $\{\alpha_k\}_{k \geq 2}, \{\beta_k\}_{k \geq 2}$  of smooth functions  $U \rightarrow \mathbb{H}$  and  $U \rightarrow \mathbb{H}^2$ , respectively, satisfying the following conditions:*

- 1)  $F|_S = f$ ;
- 2)  $(\partial^k \alpha_m / \partial \rho^k)|_S = 0, \forall k \geq 1, m \geq 2$ ;
- 3)  $\bar{\mathcal{D}}(F - \sum_{k=2}^m (\rho^k/k)\alpha_k) = \rho^m \beta_m, \forall m \geq 2$ .

*Proof.* Since  $f$  is admissible, by Proposition 2.5 there is a smooth extension  $F : U \rightarrow \mathbb{H}$  of  $f$  such that  $\bar{\mathcal{D}}F = \rho^2\sigma$ . We construct the sequences by recurrence assuming  $\alpha_2 = 0, \beta_2 = \sigma$  in such a way second and third conditions of the proposition are satisfied for  $m = 2$ .

Suppose that  $\alpha_2, \dots, \alpha_m, \beta_2, \dots, \beta_m$  are already constructed in such a way that the above conditions 2) and 3) are satisfied for all integers  $s \leq m, k \geq 1$ , in order to define  $\alpha_{m+1}$  and  $\beta_{m+1}$ .

Set

$$G = F - \sum_{k=2}^m (\rho^k/k)\alpha_k, \quad \beta_m = (\zeta_1, \zeta_2) \in \mathbb{H}^2.$$

By definition,  $\partial G / \partial \bar{q}_h = \rho^m \zeta_h, h = 1, 2$ , hence  $\bar{\mathcal{D}}G$  satisfies the condition (15), that is

$$\frac{\partial}{\partial \bar{q}_s} \frac{\partial}{\partial q_s} \frac{\partial G}{\partial \bar{q}_h} = \frac{\partial}{\partial \bar{q}_h} \frac{\partial}{\partial q_s} \frac{\partial G}{\partial \bar{q}_s},$$

which gives

$$\frac{\partial}{\partial \bar{q}_s} \left( m\rho^{m-1} \frac{\partial \rho}{\partial q_s} \zeta_h + \rho^m \frac{\partial \zeta_h}{\partial q_s} \right) = \frac{\partial}{\partial \bar{q}_h} \left( m\rho^{m-1} \frac{\partial \rho}{\partial q_s} \zeta_s + \rho^m \frac{\partial \zeta_s}{\partial q_s} \right).$$

Taking into account that  $\rho$  is real and  $m \geq 2$  we get

$$(39) \quad \begin{aligned} m(m-1)\rho^{m-2} \frac{\partial \rho}{\partial \bar{q}_s} \frac{\partial \rho}{\partial q_s} \zeta_h + m\rho^{m-1} \frac{\partial}{\partial \bar{q}_s} \left( \frac{\partial \rho}{\partial q_s} \zeta_h \right) + m\rho^{m-1} \frac{\partial \rho}{\partial \bar{q}_s} \frac{\partial \zeta_h}{\partial q_s} + \rho^m \frac{\partial}{\partial \bar{q}_s} \left( \frac{\partial \zeta_h}{\partial q_s} \right) = \\ m(m-1)\rho^{m-2} \frac{\partial \rho}{\partial \bar{q}_h} \frac{\partial \rho}{\partial q_s} \zeta_s + m\rho^{m-1} \frac{\partial}{\partial \bar{q}_h} \left( \frac{\partial \rho}{\partial q_s} \zeta_s \right) + m\rho^{m-1} \frac{\partial \rho}{\partial \bar{q}_h} \frac{\partial \zeta_s}{\partial q_s} + \rho^m \frac{\partial}{\partial \bar{q}_h} \left( \frac{\partial \zeta_s}{\partial q_s} \right). \end{aligned}$$

Summing with respect to  $s$  the above equalities and dividing by  $m(m-1)$ , for fixed  $h = 1, 2$  we get

$$(40) \quad \rho^{m-2} |\nabla \rho|^2 \zeta_h = \rho^{m-2} \frac{\partial \rho}{\partial \bar{q}_h} \left( \sum_{s=1}^2 \frac{\partial \rho}{\partial q_s} \zeta_s \right) + \rho^{m-1} l_h$$

with  $l_h \in C^\infty(U)$   $h = 1, 2$  whence

$$(41) \quad |\nabla \rho|^2 \zeta_h = \frac{\partial \rho}{\partial \bar{q}_h} \left( \sum_{s=1}^2 \frac{\partial \rho}{\partial q_s} \zeta_s \right) + \rho l_h$$

$h = 1, 2$ . Since  $\nabla \rho \neq 0$  on  $S$ , on an open neighborhood  $V \subset U$  of  $S$  we have

$$\zeta_h = \frac{\partial \rho}{\partial \bar{q}_h} g + \rho \gamma_h$$

$h = 1, 2$ , with  $g, \gamma_h \in C^\infty(V)$ .

Setting  $\gamma = (\gamma_1, \gamma_2)$ , and recalling that  $\beta_m = (\zeta_1, \zeta_2)$ , on  $V$  we have  $\beta_m = (\overline{\mathfrak{D}}\rho)g + \rho\gamma$ , so, by the beginning assumption, we derive

$$(42) \quad \begin{aligned} \overline{\mathfrak{D}}\left(F - \sum_{k=2}^m (\rho^k/k)\alpha_k\right) &= \rho^m \beta_m = \rho^m \overline{\mathfrak{D}}\rho \cdot g + \rho^{m+1}\gamma \\ &= \overline{\mathfrak{D}}(\rho^{m+1}g/(m+1)) - \frac{\rho^{m+1}}{m+1}\overline{\mathfrak{D}}g + \rho^{m+1}\gamma. \end{aligned}$$

With

$$\theta = \gamma - \overline{\mathfrak{D}}g/(m+1)$$

equation (42) rewrites

$$(43) \quad \overline{\mathfrak{D}}\left(F - \sum_{k=2}^m (\rho^k/k)\alpha_k - \rho^{m+1}g/(m+1)\right) = \rho^{m+1}\theta.$$

Observe that  $g$  and  $\theta$  can be chosen in such a way that an equality like (43) holds on  $U$ . (It is enough to consider a closed neighborhood  $V' \subset V$  of  $S$ , a smooth extension of  $g|_{V'}$  to  $U$  and take  $\theta$  according to (43)).

By Lemma 2.6, there exists a smooth function  $\alpha_{m+1} : U \rightarrow \mathbb{H}$ , such that  $\alpha_{m+1}|_S = g|_S$ ,  $\partial^k \alpha_{m+1}/\partial \rho^k|_S = 0$  for every  $k \geq 1$ . Then,  $\alpha_{m+1} - g = \rho\varepsilon$  with  $\varepsilon : U \rightarrow \mathbb{H}$  smooth and, consequently,

$$\begin{aligned} -\overline{\mathfrak{D}}\left(\frac{\rho^{m+1}g}{m+1}\right) &= \overline{\mathfrak{D}}\left(\frac{\rho^{m+2}\varepsilon}{m+1}\right) - \overline{\mathfrak{D}}\left(\frac{\rho^{m+1}}{m+1}\alpha_{m+1}\right) = \\ &= -\overline{\mathfrak{D}}\left(\frac{\rho^{m+1}}{m+1}\alpha_{m+1}\right) + \frac{\rho^{m+1}}{m+1}((m+2)\overline{\mathfrak{D}}\rho \cdot \varepsilon + \rho\overline{\mathfrak{D}}\varepsilon). \end{aligned}$$

If we define

$$\zeta_{m+1} = \theta - \frac{1}{m+1}((m+2)\overline{\mathfrak{D}}\rho \cdot \varepsilon + \rho\overline{\mathfrak{D}}\varepsilon)$$

$\alpha_{m+1}$  and  $\zeta_{m+1}$  satisfy conditions 2) and 3) of the proposition for  $m+1$ .  $\square$

Let  $U, \rho, S$  be as in Proposition 2.7 and let  $G : U \rightarrow \mathbb{H}^r$  be a smooth map. We say that  $G$  *vanishes of infinite order* on  $S$  or that  $G$  is *flat* on  $S$  if, for any integer  $k$ ,

$$\lim_{\rho \rightarrow 0} G/\rho^k = 0$$

uniformly on the compact sets of  $U$ .

**Proposition 2.8.** *With  $U, \rho, S$  as above, let  $f : S \rightarrow \mathbb{H}$  be a smooth admissible function. Then, there exists a smooth extension  $G$  of  $f$  to  $U$  such that  $\overline{\mathfrak{D}}G$  is flat on  $S$ .*

*Proof.* By Proposition 2.7, there exist smooth functions  $F : U \rightarrow \mathbb{H}$ ,  $\alpha_j : U \rightarrow \mathbb{H}$ ,  $\beta_j : U \rightarrow \mathbb{H}^2$ , ( $j \geq 2$ ) such that

- $F|_S = f$ ;
- $(\partial^k \alpha_j / \partial \rho^k)|_S = 0, \forall k \geq 1, j \geq 2$ ;
- $\overline{\mathfrak{D}}(F - \sum_{k=2}^m (\rho^k/k)\alpha_k) = \rho^m \beta_m, \forall m \geq 2$ .

Moreover, by Lemma 2.6, there exists a smooth function  $E : U \rightarrow \mathbb{H}$  such that  $E|_S = 0$  and

$$\frac{\partial^k E}{\partial \rho^k} \Big|_S = \frac{k!}{k+1} \alpha_{k+1} \Big|_S$$

for all  $k \geq 1$ .

Let  $T = \rho E$ . Then, since  $T|_S = 0$  and

$$\frac{\partial^k T}{\partial \rho^k} = k \frac{\partial^{k-1} E}{\partial \rho^{k-1}} + \rho \frac{\partial^k E}{\partial \rho^k}$$

for  $k \geq 1$  in a neighborhood of  $S$ , we get

$$\alpha_k|_S = \frac{k}{(k-1)!} \frac{\partial^{k-1} E}{\partial \rho^{k-1}} \Big|_S = \frac{1}{(k-1)!} \frac{\partial^k T}{\partial \rho^k} \Big|_S$$

for all  $k \geq 2$  and  $\frac{\partial T}{\partial \rho}|_S = E|_S = 0$ .

Now, fix a point  $p$  of  $S$  and let  $W_p$  be a neighborhood of  $p$  where  $\rho$  is one of the real coordinates, say the first, and denote  $\xi_1, \dots, \xi_7$  the remaining. Let  $\pi : W_p \rightarrow W_p \cap S$  denote the projection  $(\rho, \xi_1, \dots, \xi_7) \rightarrow (0, \xi_1, \dots, \xi_7)$ . By what is preceding, we deduce that in  $W_p$ , for all  $m \geq 2$ , the following holds true

$$T - \sum_{k=2}^m \frac{\rho^k}{k} (\alpha_k \circ \pi) = T - \sum_{k=0}^m \frac{\rho^k}{k!} \left( \frac{\partial^k T}{\partial \rho^k} \circ \pi \right) = \rho^{m+1} \zeta$$

with  $\zeta : W_p \rightarrow \mathbb{H}$  smooth. Consequently,

$$(44) \quad \bar{\mathcal{D}} \left( T - \sum_{k=2}^m \frac{\rho^k}{k} (\alpha_k \circ \pi) \right) = \rho^m v$$

with  $v : W_p \rightarrow \mathbb{H}^2$  smooth. Moreover, since  $(\partial^k \alpha_j / \partial \rho^k)|_S = 0, \forall k \geq 1, j \geq 2$ , we get

$$\sum_{k=2}^m \frac{\rho^k}{k} (\alpha_k - \alpha_k \circ \pi) = \rho^{m+1} \theta,$$

$\theta : W_p \rightarrow \mathbb{H}$  smooth. It follows that

$$(45) \quad \bar{\mathcal{D}} \left( \sum_{k=2}^m (\rho^k/k) \alpha_k - \sum_{k=2}^m (\rho^k/k) (\alpha_k \circ \pi) \right) = \rho^m u,$$

where  $u : W_p \rightarrow \mathbb{H}^2$  is smooth.

Finally, we define  $G = F - T$ . Clearly  $G|_S = f$  and by (44), (45) we get

$$\begin{aligned} \bar{\mathcal{D}} G &= \bar{\mathcal{D}} \left( F - \sum_{k=2}^m (\rho^k/k) \alpha_k \right) + \bar{\mathcal{D}} \left( \sum_{k=2}^m (\rho^k/k) \alpha_k - \sum_{k=2}^m (\rho^k/k) (\alpha_k \circ \pi) \right) \\ &\quad - \bar{\mathcal{D}} \left( T - \sum_{k=2}^m (\rho^k/k) (\alpha_k \circ \pi) \right) \\ &= \rho^m (\beta_m + u - v) = \rho^m w_p. \end{aligned}$$

Here  $w_p : W_p \rightarrow \mathbb{H}^2$  is smooth and uniquely determined by the condition  $\bar{\mathcal{D}} G = \rho^m w_p$ . If  $p \notin S$ , we take  $W_p$  such that  $W_p \cap S = \emptyset$  and  $w_p = \bar{\mathcal{D}} G / \rho^m$ . Therefore, the family of the local maps  $w_p$  defines a smooth map  $w_m : U \rightarrow \mathbb{H}^2$  such that  $\bar{\mathcal{D}} G = \rho^m w_m$  for every integer  $m \geq 2$ , i.e.  $G$  is a smooth extension of  $f$  to  $U$  such that  $\bar{\mathcal{D}} G$  is flat on  $S$ .

This proves Proposition 2.8.  $\square$

We apply Proposition 2.8 in order to prove that the local Riemann-Hilbert problem is always solvable. This will follow from the following

**Theorem 2.9.** *Let  $\Omega \subset \mathbb{H}^2$  be a convex domain and  $(S, U^+, U^-)$  a domain splitting of  $\Omega$ . Let  $f : S \rightarrow \mathbb{H}$  a smooth admissible function. Then,  $f$  is a smooth jump.*

*Proof.* Observe that  $S$  is orientable, so  $S$  is defined by  $\rho = 0$ , where  $\rho \in C^\infty(\Omega)$ . Let  $G : \Omega \rightarrow \mathbb{H}$  a smooth extension of  $f$ , with  $\bar{\mathcal{D}} G$  flat on  $S$  (Proposition 2.8). Define  $\eta : \Omega \rightarrow \mathbb{H}^2$  by

$$\eta = \begin{cases} -\bar{\mathcal{D}} G & \text{on } U^+ \\ 0 & \text{on } S \\ \bar{\mathcal{D}} G & \text{on } U^-. \end{cases}$$

$\eta$  is smooth in  $\Omega$ , since  $\overline{\mathcal{D}G}$  is flat on  $S$ . Set  $\eta = (\eta_1, \eta_2)$ . Then, the conditions

$$\Delta_1 \eta_2 = \frac{\partial}{\partial \bar{q}_2} \frac{\partial \eta_1}{\partial q_1}, \quad \Delta_2 \eta_1 = \frac{\partial}{\partial \bar{q}_1} \frac{\partial \eta_2}{\partial q_2}$$

are satisfied on  $U^+ \cup U^-$  (see (17)) whence on  $\Omega$ . Since  $\Omega$  is convex, there exists  $\psi : \Omega \rightarrow \mathbb{H}$  smooth such that  $\overline{\mathcal{D}\psi} = \eta$  [ABLSS]. Defining  $F^+ = (\psi + G)/2$ ,  $F^- = (\psi - G)/2$ , we have the following:  $F^+$  and  $F^-$  are smooth up to  $S$ ,  $\overline{\mathcal{D}F^+} = 0$  ( $\overline{\mathcal{D}F^-} = 0$ ) in  $U^+$  ( $U^-$ ) and  $F^+|_S - F^-|_S = f$ .

This ends the proof of Theorem 2.9.  $\square$

### 2.2.1. Two applications.

**Theorem 2.10.** *Let  $\Omega$  be a bounded domain with connected smooth boundary  $\text{b}\Omega$ . Then, every smooth admissible function  $f : \text{b}\Omega \rightarrow \mathbb{H}$  extends to  $\Omega$  by an  $\mathbb{H}$ -holomorphic function, smooth up to  $\text{b}\Omega$ .*

*Proof.* In our hypothesis,  $\mathbb{H}^2 \setminus \overline{\Omega}$  is connected with boundary  $\text{b}\Omega$ . Since  $(\text{b}\Omega, \Omega, \mathbb{H}^2 \setminus \overline{\Omega})$  is a domain splitting of  $\mathbb{H}^2$ , by Theorem 2.9,  $f = F^+|_S - F^-|_S$ , where  $F^+, F^-$  are  $\mathbb{H}$ -holomorphic. By Hartogs' Theorem  $F^-$  extends to all of  $\mathbb{H}^2$  by an  $\mathbb{H}$ -holomorphic function  $\widehat{F}^-$ . And this implies that  $f$  is the boundary value of  $F^+ - \widehat{F}^-$ .  $\square$

**Theorem 2.11.** *Let  $\Sigma$  be an open half-space and  $S \subset \mathbb{H}^2$  a connected closed smooth hypersurface of  $\Sigma$ . Assume that  $\Sigma \setminus S$  splits into two connected components  $D$  and  $W$  with  $D$  bounded. Let  $f : S \rightarrow \mathbb{H}$  be a smooth admissible function. Then,  $f$  extends to  $D$  by an  $\mathbb{H}$ -holomorphic function  $F$  which is smooth up to  $S$ .*

*Proof.* Without loss of generality, we can assume that  $\Sigma$  be the half space  $\{y_3 > 0\}$ . Let  $B$  be an open ball centered at origin such that  $S$  divides  $B \cap \Sigma$  into two connected components  $U^+$  and  $U^- = D$  and  $D$  is relatively compact in  $B$ . By Theorem 2.9, there are  $\mathbb{H}$ -holomorphic functions  $F^+ : U^+ \rightarrow \mathbb{H}$ ,  $F^- : D \rightarrow \mathbb{H}$ , smooth up to  $S$ , such that  $f = F^+|_S - F^-|_S$ . It is enough to show that  $F^+$  extends  $\mathbb{H}$ -holomorphically to  $B \cap \{y_3 > 0\}$ . We may assume that  $F^+$  is defined on an neighborhood of  $\text{b}B \cap \Sigma$  in  $\Sigma$ .

Fix  $\varepsilon > 0$  sufficiently small. For every  $c > 0$ , let  $S_c$  be the sphere centered at  $(0, -ck)$  and passing through  $\text{b}B \cap \{y_3 = \varepsilon\}$ . Consider the set  $\mathcal{C}$  of  $c \in \mathbb{R}$  such that  $F^+|_{S_c \cap \overline{B}}$  extends to a neighborhood of  $\overline{S_c} \cap \overline{B}$  in  $\overline{B}$ . We have  $\mathcal{C} \neq \emptyset$ . Let  $c_0 = \sup \mathcal{C}$ , and assume by contradiction that  $c_0$  is finite. Observe that  $F^+$  is defined in a neighborhood of  $\text{b}B \cap \{y_3 = \varepsilon\}$  in  $\overline{B}$ . Consider  $B_{c_0}$ , the open ball having  $S_{c_0}$  as its boundary and let  $U = B \setminus \overline{B}_{c_0}$ . Then, the second fundamental form of  $S_{c_0} \cap \Sigma$  (as part of the boundary of  $U$ ) is negative definite. Hence, as in the proof of Theorem 4 of [Pe3], we get that, for every  $q_0 \in S_{c_0} \cap \overline{B}$ , there exists a domain  $\Delta_{q_0} \subset U$  such that every  $\mathbb{H}$ -holomorphic function in  $\Delta_{q_0}$  extends to a bigger domain  $\widehat{\Delta}_{q_0}$  containing  $q_0$ . It follows that  $F^+$  extends  $\mathbb{H}$ -holomorphically to a neighborhood of  $S_{c_0} \cap \overline{B}$  in  $\overline{B}$ : contradiction. This means that  $c_0 = +\infty$ , thus  $F^+$  extends to  $B \cap \{y_3 > \varepsilon\}$ , for every  $\varepsilon$  near  $0^+$ . By analytic continuation (see Theorem 2.13 below), this completes the proof.  $\square$

**Remark 2.12.** *With the same notations of the above Theorem, let  $F$  be the  $\mathbb{H}$ -holomorphic extension of  $f$ . If  $|f|$  is bounded on  $S$ , then for every  $q \in D$*

$$|F(q)| \leq \sup_S |f|.$$

We mention that, an extension theorem of different type, has been recently found by Baracco, Fassina and Pinton [BFP].

Let  $S$  be a connected smooth hypersurface in  $\mathbb{H}^2$ . We say that the *analytic continuation principle* holds for smooth admissible functions on  $S$  when the following is true: if  $f : S \rightarrow \mathbb{H}$  is a smooth admissible function which vanishes on a nonempty open set of  $S$ , then  $f \equiv 0$ .

**Theorem 2.13.** *Let  $S$  be a connected smooth hypersurface in  $\mathbb{H}^2$ . Then, the analytic continuation principle for smooth admissible functions holds on  $S$  in the following two cases:*



- i)  $S$  is the boundary of a domain  $\Omega \Subset \mathbb{H}^2$  satisfying the hypothesis of Theorem 2.10;
- ii)  $S$  is nondegenerate.

*Proof.* i) Consider a smooth admissible function  $f$  on  $S$ ,  $F$  its  $\mathbb{H}$ -holomorphic extension on  $\Omega$ , and let  $Z = \{f = 0\}$ . Let  $q_0 \in \overset{\circ}{Z}$  and  $U$  be a neighborhood of  $q_0$  relatively compact in  $\overset{\circ}{Z}$ . Then there exists a domain  $\Omega_1$  with smooth boundary, satisfying the hypothesis of Theorem 2.10, such that  $\Omega \subset \Omega_1$ ,  $\text{b}\Omega_1 \setminus \text{b}\Omega \subset \mathbb{H}^2 \setminus \overline{\Omega}$ , and  $\text{b}\Omega \setminus U = \text{b}\Omega_1 \cap \text{b}\Omega$ . The function  $f_1$  on  $\text{b}\Omega_1$  that coincides with  $f$  on  $\text{b}\Omega_1 \cap \text{b}\Omega$  and is zero elsewhere, is smooth admissible and, by Theorem 2.10, extends to an  $\mathbb{H}$ -holomorphic function  $F_1$  on  $\Omega_1$ , smooth up to the boundary. By the Bochner-Martinelli formula, it follows immediately that  $F_1$  is an extension of  $F$ . By construction,  $F_1$  vanishes on the boundary of  $\Omega_1 \setminus \Omega$ , and then,  $F_1$  vanishes on  $\Omega_1 \setminus \Omega$ . By analytic continuation,  $F_1 \equiv 0$  and therefore  $F \equiv 0$  and  $f \equiv 0$  too.

ii) Let  $f$  be a smooth admissible function on  $S = \{\rho = 0\}$  and let  $Z = \{f = 0\}$ . Assume  $f$  is not identically zero. By Theorem 6 [Pe3], there exists a neighborhood  $U$  of  $S$  in, say,  $\{\rho \leq 0\}$ , such that the function  $f$  extends by an  $\mathbb{H}$ -holomorphic function  $F$ . Take a point  $p \in S$ , there exists a domain  $\Omega \subset \{\rho < 0\}$ , whose boundary contains  $p$ , such that the set  $\text{b}\Omega \cap Z$  has interior points in  $S$ , and the hypothesis of i) holds for  $\Omega$ . Using i),  $F|_{\text{b}\Omega} = 0$ , in particular,  $f(p) = 0$ . This concludes the proof,  $p$  being a generic point of  $S$ .  $\square$

**Remark 2.14.** *The analytic continuation principle does not hold for an arbitrary smooth hypersurface  $S$ . For instance, all smooth functions  $f = f(y_0, y_1, y_2)$  are admissible on  $S = \{y_3 = 0\}$ .*

### 3. THE CRF CONDITION IN WEAK FORM

In order to treat the Riemann-Hilbert problem (in particular the boundary problem) for  $\mathbb{H}$ -holomorphic functions with continuous boundary data we need to give the CRF conditions in a weak form. We need some preliminaries.

Let  $(S, U^+, U^-)$  be a domain splitting of a domain  $\Omega$  in  $\mathbb{R}^n$  and  $\varrho \in C^\infty(\Omega)$  such that

$$S = \{\varrho = 0\}, \quad U^+ = \{\varrho > 0\}, \quad U^- = \{\varrho < 0\}, \quad \nabla\varrho|_S \neq 0$$

and consider on  $S$  the orientation determined on the boundary of  $U^+$  by the inward normal vector.

By the existence of tubular neighborhoods we may assume that for  $-\varepsilon_0 < \varepsilon < \varepsilon_0$ ,  $\varepsilon_0 > 0$ , the hypersurface  $S_\varepsilon = \{\varrho = \varepsilon\}$  is diffeomorphic to  $S$  by a diffeomorphism  $\pi_\varepsilon : S_\varepsilon \rightarrow S$ .

Let  $T$  be a distribution on  $S$ . We say that  $T$  is the *trace* or the *boundary value* (in the sense of distributions) of a function  $u \in L^1_{\text{loc}}(U^+)$  following  $\{S_\varepsilon\}_{0 < \varepsilon < \varepsilon_0}$  if

$$\lim_{\varepsilon \rightarrow 0^+} \int_{S_\varepsilon} \pi_\varepsilon^*(\phi) u = (T, \phi)$$

for every real-valued test form  $\phi$  of class  $C_0^\infty$  on  $S$  of degree  $n - 1$ . In such a situation we set  $\gamma^+(u) = T$ .

In the same manner we give the notion of trace  $\gamma^-(u)$  if  $u \in L^1_{\text{loc}}(U^-)$ .

The following result was proved in [LT, Corollary I. 2. 6]. Let  $P(D)$  be a linear elliptic operator on  $U^+$  with smooth coefficients and  $u \in C^\infty(U^+)$  a solution of the equation  $P(D)u = 0$ . Then  $u$  has a boundary value  $\gamma^+(u)$  if and only if  $u$  extends as distribution through  $S$ .

Now we are in position to state the CRF condition in a weak form.

**Proposition 3.1.** *Let  $f : S \rightarrow \mathbb{H}$  be continuous function and  $T_1 = T_{1,f}$ ,  $T_2 = T_{2,f}$  the distributions on  $\Omega$  supported by  $S$  defined by*

$$(46) \quad \begin{cases} \phi \mapsto \int_S Dq_1 \wedge f\phi \, dy \\ \phi \mapsto \int_S Dq_2 \wedge f\phi \, dx, \end{cases}$$



(where  $\phi \in C_0^\infty(\Omega)$  is a real valued test function). If  $f$  is locally a jump of  $\mathbb{H}$ -holomorphic functions continuous up  $S$ , then the system

$$(47) \quad \begin{cases} \frac{\partial v}{\partial \bar{q}_1} = T_1 \\ \frac{\partial v}{\partial \bar{q}_2} = T_2 \end{cases}$$

is locally solvable along  $S$ .

*Proof.* Let  $q_0 \in S$  and  $F^\pm$   $\mathbb{H}$ -holomorphic functions in  $U^\pm$ , smooth up to  $S$  such that  $f|_{U \cap S} = F^+|_S - F^-|_S$ . Let

$$F = \begin{cases} -F^+ & \text{in } U^+ \\ -F^- & \text{in } U^- \end{cases}$$

and denote by the same letter  $F$  the distribution

$$\phi \mapsto \int_{\Omega} \phi F \, dx \wedge dy :$$

here  $\phi$  is a real valued test function. Then

$$(48) \quad \begin{aligned} \frac{\partial F}{\partial \bar{q}_1}(\phi) &= - \int_{\Omega} \phi_{\bar{q}_1} F \, dx \wedge dy \\ &= \int_{U^+} \phi_{\bar{q}_1} F^+ \, dx \wedge dy + \int_{U^-} \phi_{\bar{q}_1} F^- \, dx \wedge dy. \end{aligned}$$

Denote  $d_x$  ( $d_y$ ) the differential with respect to the  $x$  ( $y$ )-variables. Then, since  $Dq_1$  is closed, we have

$$\begin{aligned} \int_{U^+} \phi_{\bar{q}_1} F^+ \, dx \wedge dy &= \int_{U^+} d_x(Dq_1 \phi) F^+ \wedge dy \\ &= \int_{U^+} d_x(Dq_1 \cdot F^+ \cdot \phi) \wedge dy - \int_{U^+} d_x(Dq_1 \cdot F^+) \phi \wedge dy \\ &= \int_{U^+} d(Dq_1 \cdot F^+ \phi \wedge dy) - \int_{U^+} d_x(Dq_1 \cdot F^+) \phi \wedge dy \\ &= \int_S Dq_1 \cdot F^+ \phi \wedge dy - \int_{U^+} \frac{\partial F^+}{\partial \bar{q}_1} \phi \, dx \wedge dy \\ &= \int_S Dq_1 \cdot F^+ \phi \wedge dy \end{aligned}$$

by (5) and the  $\mathbb{H}$ -holomorphy of  $F^+$ .

In the same manner,

$$\int_{U^-} \phi_{\bar{q}_1} F^- \, dx \wedge dy = - \int_S Dq_1 \cdot F^- \phi \wedge dy$$

( $bU^- \cap S = -bU^+ \cap S$ ), whence

$$(49) \quad \begin{aligned} \frac{\partial F}{\partial \bar{q}_1}(\phi) &= \int_S Dq_1 \cdot F^+ \phi \wedge dy - \int_S Dq_1 \cdot F^- \phi \wedge dy \\ &= \int_S Dq_1 \cdot f \phi \wedge dy. \end{aligned}$$

Analogously, we get

$$(50) \quad \frac{\partial F}{\partial \bar{q}_2}(\phi) = \int_S Dq_2 \cdot f \phi \wedge dx.$$

Equations (49) and (50) show that the distribution  $F$  is a solution of (47).

If  $f$  is only continuous, we approximate  $S$  by hypersurfaces  $S_\varepsilon$ , with  $0 < |\varepsilon| < \varepsilon_0$ , and we apply the previous argument to  $F^+$  on  $\{\rho \geq \varepsilon\}$  when  $\varepsilon > 0$ , and  $F^-$  on  $\{\rho \leq \varepsilon\}$  when  $\varepsilon < 0$ . Hence, by taking the limits, identities (49) and (50) holds in the continuous case too.  $\square$

From the above Proposition and (15), it follows

**Corollary 3.2.** *If  $f$  is a jump of  $\mathbb{H}$ -holomorphic functions continuous up to  $S$ , then in  $\Omega$  we have*

$$(51) \quad \begin{aligned} \frac{\partial}{\partial \bar{q}_1} \frac{\partial T_2}{\partial q_2} - \Delta_2 T_1 &= 0 \\ \frac{\partial}{\partial \bar{q}_2} \frac{\partial T_1}{\partial q_1} - \Delta_1 T_2 &= 0 \end{aligned}$$

in the distribution sense, i.e.,

$$(52) \quad \begin{aligned} \int_S \left[ \left( \frac{\partial}{\partial \bar{q}_1} \frac{\partial}{\partial q_2} \phi \right) dx \wedge Dq_2 - (\Delta_2 \phi) dy \wedge Dq_1 \right] f &= 0 \\ \int_S \left[ \left( \frac{\partial}{\partial \bar{q}_2} \frac{\partial}{\partial q_1} \phi \right) dy \wedge Dq_1 - (\Delta_1 \phi) dx \wedge Dq_2 \right] f &= 0 \end{aligned}$$

for every  $\phi \in C_0^\infty(\Omega)$ .

We say that a continuous function  $f : S \rightarrow \mathbb{H}$  is a *weakly admissible function* if it satisfies (52).

We have the

**Theorem 3.3.** *A continuous function  $f : S \rightarrow \mathbb{H}$  is locally a jump of  $\mathbb{H}$ -holomorphic functions, continuous up to  $S$  if and only if it is a weakly admissible function. In particular, assume that  $S$  is the connected boundary of a bounded domain  $\Omega$ . Then, every continuous admissible function  $f : b\Omega \rightarrow \mathbb{H}$  extends to  $\Omega$  by an  $\mathbb{H}$ -holomorphic function which is continuous up to  $b\Omega$ .*

*Proof.* We only have to prove that if  $f$  is weakly admissible then it is locally a jump of  $\mathbb{H}$ -holomorphic functions continuous up to  $S$ .

Let  $q_0 \in S$ , and  $(S \cap \Omega, U^+, U^-)$  be a splitting domain of a convex domain  $\Omega$  containing  $q_0$ . Since  $f$  is weakly admissible and  $\Omega$  is convex, by (51) and Remark 1.3 there exists a distribution  $F$  in  $\Omega$  such that

$$(53) \quad \begin{cases} \frac{\partial F}{\partial \bar{q}_1} = T_1 \\ \frac{\partial F}{\partial \bar{q}_2} = T_2. \end{cases}$$

Since  $\frac{\partial F}{\partial \bar{q}_1}|_{\Omega \setminus S} = \frac{\partial F}{\partial \bar{q}_2}|_{\Omega \setminus S} = 0$ ,  $F$  is  $\mathbb{H}$ -holomorphic on  $\Omega \setminus S$ .

Let  $F^\pm = F|_{U^\pm}$ . Since  $F^\pm$  are pluriharmonic, the results of [LT] apply. In particular,  $F^\pm$  have traces  $\gamma(F^\pm)$  on  $S$  in the sense of distributions and  $\gamma(F^+) - \gamma(F^-) = f$  [LT, Corollaire I.2.6 and Théorème II.1.3].

Let  $V^\pm$  be domains with smooth boundary such that  $V^\pm \Subset \Omega$ ,  $V^\pm \subset U^\pm$  and  $(bV^+ \cap S) = (bV^- \cap S)$  is a relative neighborhood  $S_0$  of  $q_0$  in  $S$ . Again, by [LT, Corollaire I.2.6],  $F^\pm$  have traces  $\theta^\pm$  on  $bV^\pm$  in the sense of distributions, and, by the Bochner-Martinelli formula for  $\mathbb{H}$ -holomorphic functions, we have

$$F^\pm(q) = \langle \theta^\pm, \mathbf{K}^{BM}(\cdot, q) \rangle$$

for  $q \in V^\pm$ .

Let  $\psi \in C_0^\infty(S_0)$  such that  $\psi = 1$  on a neighborhood of  $q_0$ . Then,

$$\begin{aligned} \langle \psi \theta^+, \mathbf{K}^{BM}(\cdot, q) \rangle + \langle (1 - \psi) \theta^+, \mathbf{K}^{BM}(\cdot, q) \rangle &= \begin{cases} F^+(q) & q \in V^+ \\ 0 & q \notin \bar{V}^+ \end{cases} \\ \langle \psi \theta^-, \mathbf{K}^{BM}(\cdot, q) \rangle + \langle (1 - \psi) \theta^-, \mathbf{K}^{BM}(\cdot, q) \rangle &= \begin{cases} F^-(q) & q \in V^- \\ 0 & q \notin \bar{V}^- \end{cases} \end{aligned}$$

The functions  $\langle (1-\psi)\theta^\pm, \mathbf{K}^{BM}(\cdot, q) \rangle, \langle \psi\theta^\pm, \mathbf{K}^{BM}(\cdot, q) \rangle$  are smooth near  $q_0$  and, since  $\theta^+ - \theta^- = f$ , we have

$$\begin{aligned} F^+(q) &= \langle \psi f, \mathbf{K}^{BM}(\cdot, q) \rangle + \langle \psi\theta^-, \mathbf{K}^{BM}(\cdot, q) \rangle + \langle (1-\psi)\theta^+, \mathbf{K}^{BM}(\cdot, q) \rangle \\ F^-(q) &= -\langle \psi f, \mathbf{K}^{BM}(\cdot, q) \rangle + \langle \psi\theta^+, \mathbf{K}^{BM}(\cdot, q) \rangle + \langle (1-\psi)\theta^-, \mathbf{K}^{BM}(\cdot, q) \rangle \end{aligned}$$

and consequently

$$\begin{aligned} F^+(q) &= \int_{\text{bV}^+} \psi f \mathbf{K}^{BM}(\cdot, q) + u(q) \\ F^-(q) &= - \int_{\text{bV}^-} \psi f \mathbf{K}^{BM}(\cdot, q) + v(q) \end{aligned}$$

with  $u = u(q), v = v(q)$  smooth near  $q_0$ . Now, as a consequence of the classical potential theory [Mi],  $F^\pm$  are continuous up to the boundary and this concludes the proof of the general case. In the particular case when  $S$  is the boundary of  $\Omega$ , the proof runs as in the smooth case of Theorem 2.10.  $\square$

**Remark 3.4.** *Theorem 2.11 also generalizes.*

**Remark 3.5.** *A smooth function  $f : S \rightarrow \mathbb{H}$  is weakly admissible if and only if it is admissible.*

*Proof.* First assume that  $f$  is  $C^\infty$  and weakly admissible. By Remark 2.3 the functions  $F^\pm$  of the previous Proposition are smooth up to the boundary  $S$ , hence  $f$  is admissible.

Next, assume  $f$  is  $C^\infty$  and admissible, then  $f = F^+ - F^-$ . Since  $F^\pm$  are smooth up to  $S$ , then  $f$  is weakly admissible.  $\square$

Let  $S$  be a connected smooth hypersurface in  $\mathbb{H}^2$ . We say that the *analytic continuation principle* holds for weakly admissible functions on  $S$  when the following is true: if  $f : S \rightarrow \mathbb{H}$  is a continuous weakly admissible function which vanishes on a nonempty open set of  $S$ , then  $f \equiv 0$ .

**Theorem 3.6.** *Let  $S$  be a connected smooth hypersurface in  $\mathbb{H}^2$ . Then, analytic continuation principle holds for weakly admissible functions on  $S$  in the following two cases:*

- i)  $S$  is the boundary of a domain  $\Omega \Subset \mathbb{H}^2$  satisfying the hypothesis of Theorem 2.10;
- ii)  $S$  is nondegenerate.

*Proof.* The proof is analogous to the one of Theorem 2.13 using Theorem 3.3 instead of Theorem 2.10.  $\square$

**Proposition 3.7.** *Let  $S = \{\rho = 0\}$  be a smooth hypersurface in  $\mathbb{H}^2$ ,  $\nabla\rho \neq 0$  on  $S$ , and  $\Omega^- = \{\rho < 0\}$ . Assume that  $\Omega^-$  is strongly Levi  $\mathbb{H}$ -convex along  $S$ . Then, every weakly admissible function  $f : S \rightarrow \mathbb{H}$  extends to a neighborhood  $U$  of  $S$  in  $S \cup \Omega^-$  by an  $\mathbb{H}$ -holomorphic function in  $U$ , continuous up to  $S$ .*

*Proof.* By Theorem 3.3, using Kontinuitätssatz as in the proof of Theorem 4 of [Pe3], for every point of  $p \in S$  there exists a ball  $B(p)$  such that  $f|_{B(p) \cap \{\rho=0\}}$  extends  $\mathbb{H}$ -holomorphically on  $B(p) \cap \Omega^-$  by a function  $F_p$ . This implies that there exists an open covering  $B(p_j)$  of  $S$  and  $\mathbb{H}$ -holomorphic functions  $F_j : B(p_j) \cap \Omega^- \rightarrow \mathbb{H}$ , continuous up to  $S$ , such that  $F_j$  and  $f$  agree on  $B(p_j) \cap S$ . By construction,  $F_j = F_k$  on the intersection  $B(p_j) \cap B(p_k) \cap S$ , hence, by the analytic continuation principle,  $F_j = F_k$  on  $B(p_j) \cap B(p_k) \cap \Omega^-$ . Thus the functions  $\{F_k\}$  defines the required extension of  $f$ .  $\square$

#### 4. APPENDIX: SOME GENERALIZATIONS TO OCTONIONS

We sketch some generalizations of our results to octonian regular functions. We denote by  $i_0 = 1$  the real unit and by  $i_1, \dots, i_7$  the imaginary units of the division algebra of the octonions  $\mathbb{O}$ . Thus, every element  $p$  of  $\mathbb{O}$  can be written in the form

$$p = \sum_{\alpha=0}^7 x_\alpha i_\alpha \quad \text{with } x_\alpha \in \mathbb{R}.$$

As usual, we set  $\operatorname{Re}(p) = x_0$ ,  $\operatorname{Im}(p) = \sum_{\alpha=1}^7 x_\alpha i_\alpha$  and  $\bar{p} = \operatorname{Re}(p) - \operatorname{Im}(p)$ . We recall that the product of octonions is noncommutative and nonassociative.

Let  $U$  be an open set in  $\mathbb{O}$  and  $u : U \rightarrow \mathbb{O}$  a smooth function. The Cauchy-Riemann-Fueter operator  $\partial_{\bar{p}}$  acts on  $u$  in the following way:

$$\partial_{\bar{p}}u = \sum_{\alpha=0}^7 i_\alpha \frac{\partial u}{\partial x_\alpha} = \left( \sum_{\alpha=0}^7 i_\alpha \partial_{x_\alpha} \right) u.$$

We say that  $u$  is (*left*)  $\mathbb{O}$ -holomorphic in  $U$  if  $\partial_{\bar{p}}u = 0$  on  $U$ . We also consider the conjugate operator

$$\partial_p u = \overline{\partial_{\bar{p}}u} = \sum_{\alpha=0}^7 \bar{i}_\alpha \frac{\partial u}{\partial x_\alpha} = \left( \sum_{\alpha=0}^7 \bar{i}_\alpha \partial_{x_\alpha} \right) u.$$

In the case of several octonion variables  $p_1, \dots, p_n$ , we set

$$(54) \quad p_h = \sum_{\alpha=0}^7 x_{h,\alpha} i_\alpha, \quad \text{with } x_{h,\alpha} \in \mathbb{R},$$

and given an open subset  $U$  of  $\mathbb{O}^n$ , we consider the set  $\mathcal{E}^r(U)$  of the smooth maps  $U \rightarrow \mathbb{O}^r$ .

Let us consider a function  $u \in \mathcal{E}^1(U)$ ,  $u = u(p_1, \dots, p_n)$ . We define the operator

$$(55) \quad \overline{\mathfrak{D}}u = (\partial_{\bar{p}_1}u, \dots, \partial_{\bar{p}_n}u).$$

We have  $\overline{\mathfrak{D}}u \in \mathcal{E}^n(U)$ . The kernel of the operator  $\overline{\mathfrak{D}}$  consists of the (*left*)  $\mathbb{O}$ -holomorphic functions in the sense of Fueter.

For some of the basic results in octonion analysis, we refer to [DS, LP, WR].

Let  $f = (f_1, \dots, f_n) \in \mathcal{E}^n(U)$ . The *non-homogeneous Cauchy-Riemann-Fueter problem* asks for the existence of a solution of

$$(56) \quad \overline{\mathfrak{D}}u = f$$

that is

$$(57) \quad \partial_{\bar{p}_h}u = f_h$$

for  $h = 1, \dots, n$ .

In this Appendix, we aim to study conditions on  $U \subseteq \mathbb{O}^2$  and  $f = (f_1, f_2)$  which guarantee the existence of a solution  $u \in \mathcal{E}^1(U)$  of (56). In other words, to characterize the image of the operator  $\overline{\mathfrak{D}}$  for  $n = 2$ .

We start by looking at the necessary conditions on the datum  $f$  for arbitrary  $n$ . Let us recall that  $\partial_{p_m} \partial_{\bar{p}_m} = \partial_{\bar{p}_m} \partial_{p_m} = \Delta_{p_m}$  is the laplacian with respect to the real coordinates of the octonion variable  $p_m$ . If the system of Cauchy-Riemann-Fueter (56) has a solution, the datum  $f$  must satisfy the equations

$$(58) \quad \Delta_{p_m} f_l = \partial_{\bar{p}_l}(\partial_{p_m} f_m)$$

for  $l, m = 1, \dots, n$ . Indeed, if  $u$  is a solution of (56), i.e.  $\partial_{\bar{p}_i}u = f_i$ ,  $i = 1, \dots, n$ . Then

$$\Delta_{p_m} f_l = \Delta_{p_m}(\partial_{\bar{p}_l}u) = \partial_{\bar{p}_l}(\Delta_{p_m}u) = \partial_{\bar{p}_l}(\partial_{p_m}(\partial_{\bar{p}_m}u)) = \partial_{\bar{p}_l}(\partial_{p_m} f_m).$$

Wang and Ren proved in [WR] that such conditions are actually sufficient when the data  $f_1, \dots, f_n$  have a compact support. For the sufficiency in the general case we follow the method of Ehrenpreis [Eh].

Once written the system (56) in the form  $\overline{D}u = f$ , where  $\overline{D}$  is the real matrix of differential operator  $\overline{\mathfrak{D}}$ , the problem reduces to find the generators of the module of relations of the rows of  $\overline{D}$ .

In real coordinates, given the function

$$u(p) = \sum_{\alpha=0}^7 u_{\alpha}(p) i_{\alpha} = (u_0(p), \dots, u_7(p)),$$

the Cauchy-Riemann-Fueter operator (in one variable) takes the form

$$(59) \quad \overline{\mathfrak{D}}u = \overline{D} \begin{bmatrix} u_0 \\ \vdots \\ u_7 \end{bmatrix} = \begin{bmatrix} \partial_{x_0} & -\partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} & -\partial_{x_4} & -\partial_{x_5} & -\partial_{x_6} & -\partial_{x_7} \\ \partial_{x_1} & \partial_{x_0} & -\partial_{x_3} & \partial_{x_2} & -\partial_{x_5} & \partial_{x_4} & \partial_{x_7} & -\partial_{x_6} \\ \partial_{x_2} & \partial_{x_3} & \partial_{x_0} & -\partial_{x_1} & -\partial_{x_6} & -\partial_{x_7} & \partial_{x_4} & \partial_{x_5} \\ \partial_{x_3} & -\partial_{x_2} & \partial_{x_1} & \partial_{x_0} & -\partial_{x_7} & \partial_{x_6} & -\partial_{x_5} & \partial_{x_4} \\ \partial_{x_4} & \partial_{x_5} & \partial_{x_6} & \partial_{x_7} & \partial_{x_0} & -\partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} \\ \partial_{x_5} & -\partial_{x_4} & \partial_{x_7} & -\partial_{x_6} & \partial_{x_1} & \partial_{x_0} & \partial_{x_3} & -\partial_{x_2} \\ \partial_{x_6} & -\partial_{x_7} & -\partial_{x_4} & \partial_{x_5} & \partial_{x_2} & -\partial_{x_3} & \partial_{x_0} & \partial_{x_1} \\ \partial_{x_7} & \partial_{x_6} & -\partial_{x_5} & -\partial_{x_4} & \partial_{x_3} & \partial_{x_2} & -\partial_{x_1} & \partial_{x_0} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix}$$

(see also [LP]). Analogously, in the multivariate case, the Cauchy-Riemann-Fueter operator can be written in real components as

$$\overline{\mathfrak{D}}u = \overline{D} \begin{bmatrix} u_0 \\ \vdots \\ u_7 \end{bmatrix},$$

where

$$(60) \quad \overline{D} = \begin{bmatrix} \overline{D}_{p_1} \\ \vdots \\ \overline{D}_{p_n} \end{bmatrix}$$

is a  $8n \times 8$  matrix with entries in the polynomial ring with  $8n$  indeterminates

$$R_n = \mathbb{R}[\partial_{x_{1,0}}, \dots, \partial_{x_{1,7}}, \dots, \partial_{x_{n,0}}, \dots, \partial_{x_{n,7}}],$$

and  $\overline{D}_{p_i}$  denotes the matrix  $\overline{D}$  relative to the variable  $p_i$ .

We denote by  $\text{Syz}$  the module of syzygies of the rows of the matrix  $\overline{D}$ , which is a graded module with grading inherited by the polynomial ring  $R_n$ . By taking the real components, we get eight such real syzygies from each one of the octonian conditions (58).

**Proposition 4.1.** *The  $n(n-1)$  conditions  $\Delta_{p_m} f_l = \partial_{\overline{p}_l}(\partial_{p_m} f_m)$ , of (58) for  $l, m = 1, \dots, n$ ,  $l \neq m$ , give  $8n(n-1)$  real quadratic relations. These relations corresponds to linearly independent elements over  $\mathbb{R}$  in  $\text{Syz}$ .*

*Proof.* We want to prove that the operators  $z_{l,m}(f) = \Delta_{p_m} f_l - \partial_{\overline{p}_l}(\partial_{p_m} f_m)$  for  $l, m = 1, \dots, n$ ,  $l \neq m$  are linear independent on  $\mathbb{R}$ . Given  $a, b = 1, \dots, n$ ,  $a \neq b$ , we will prove that  $z_{a,b}$  is not a linear combination of the other  $z_{l,m}$ . Indeed, consider the test data  $g = (g_1, \dots, g_n)$  where

$$(61) \quad g_k(p_1, \dots, p_n) = \begin{cases} x_{b,0}^2 & \text{for } k = a, \\ 0 & \text{otherwise} \end{cases}$$

with notations as in (54). Then,  $z_{l,m}(g)$  is nonzero if and only if  $(l, m) = (a, b)$ .  $\square$

Now we focus on the case of  $n = 2$  octonian variables  $p_1, p_2$ . Conditions (58) become

$$\begin{aligned} \Delta_{p_2} f_1 &= \partial_{\overline{p}_1}(\partial_{p_2} f_2), \\ \Delta_{p_1} f_2 &= \partial_{\overline{p}_2}(\partial_{p_1} f_1). \end{aligned}$$

Using a computer program that calculates the generators and the Betti numbers of a graded module, one checks directly that the module  $\text{Syz}$  is generated in degree 2, and  $\text{Syz}_2$ , its component of degree 2, has real dimension 16.<sup>2</sup> From this we get

**Proposition 4.2.** *For  $n = 2$  octonian variables, the conditions (58) correspond to 16 real relations that form a basis of the module of syzygies  $\text{Syz}$  as a real vector space.*

*Proof.* It follows immediately from the above computer verification and Proposition 4.1.  $\square$

Proposition 4.2 and Ehrenpreis' Theorem [Eh, Theorem 6.2, p. 176] now imply

**Theorem 4.3.** *Let  $U \subset \mathbb{O}^2$  be a convex domain and  $f \in \mathcal{E}^2(U)$ . Then, the Cauchy-Riemann-Fueter Problem  $\overline{\mathcal{D}}u = f$  has a solution  $u \in \mathcal{E}^1(U)$  if and only if  $f$  satisfies conditions (58).*

**Remark 4.4.** *We stress that conditions (58) do not generate the module of syzygies  $\text{Syz}$  for  $n > 2$ . For  $n = 3$ , this can be directly checked (employing a computer algebra system), hence conditions (58) are not sufficient to guarantee the existence of a solution to (56).*

As in the quaternionic case, we can introduce the notion of admissible octonian function. Thus, in view of Theorem 4.3, we can run through the proofs of Theorems 2.9 and 2.10 and we get:

**Theorem 4.5.** *Let  $\Omega \subset \mathbb{O}^2$  be a domain.*

- (1) *If  $\Omega$  is convex and  $(S, U^+, U^-)$  is a domain splitting of  $\Omega$ , then every smooth admissible function  $f : S \rightarrow \mathbb{O}$  is a smooth jump.*
- (2) *If  $\Omega$  is bounded with connected smooth boundary  $b\Omega$ , then every smooth admissible function  $f : b\Omega \rightarrow \mathbb{O}$  extends to  $\Omega$  by an  $\mathbb{O}$ -holomorphic function, smooth up to  $b\Omega$ .*

## REFERENCES

- [AL] W.W. Adams, P. Loustaunau *Analysis of the module determining the properties of regular functions of several quaternionic variables*, Pacific J. 196 (2001) 1–15. (Cited on p. 2, 7.)
- [ABLSS] W.W. Adams, C.A. Berenstein, P. Loustaunau, I. Sabadini, D. Struppa *Regular Functions of Several Quaternionic Variables and the Cauchy-Fueter Complex*, J. of Geometric Analysis 9, (1), 1999) 1–15. (Cited on p. 2, 6, 14.)
- [AnH] A. Andreotti, D. Hill *E. E. Levi convexity and Hans Lewy Problem. Part I: reduction to vanishing theorems*, Ann. Sc. Norm. Sup. (3), **26** (1972) 325–363. . (Cited on p. 11.)
- [AN] A. Andreotti, M. Nacinovich *Complex of partial differential operators*, Ann. Sc. Norm. Sup. (4), **26** (1976) 553–621. . (Cited on p. 7.)
- [BFP] L. Baracco, M. Fassina, S. Pinton *An extension theorem for regular functions of two quaternionic variables*, ArXiv: 1810.0895v2 [math CV] 24 May 2019. (Cited on p. 14.)
- [BDS] J. Bureš, A. Damiano, I. Sabatini *Explicit resolution for the complex of several Fueter operators*, Journal of Geometry and Physics 57, (2007) 765–775. (Cited on p. 2, 7.)
- [CSSS] F. Colombo, I. Sabadini, F. Sommen, D.C. Struppa *Analysis of Dirac Systems and Computational Algebra*. Progress in Mathematical Physics Series, vol. 39. Birkhäuser, Boston (2004) (Cited on p. 2, 7.)
- [DS] P. Dentoni, M. Sce, *Funzioni regolari nell'algebra di Cayley*, Rend. Sem. Mat. Univ. Padova 50 (1973), 251–267 (Cited on p. 19.)
- [Eh] L. Ehrenpreis, *Fourier analysis in several complex variables*, John Wiley & Sons, Inters. Pub., New York 1970. (Cited on p. 7, 19, 21.)
- [Fu1] R. Fueter, *Die Funktionentheorie der Differentialgleichungen  $\Delta u = 0$  und  $\Delta \Delta u = 0$  mit vier reellen Variablen*, Comment. Math. Helv. 7, (1935), 307–330. (Cited on p. 2, 4, 5.)
- [IMV] S. Ivanov, I. Minchev, D. Vassilev, *Quaternionic Contact Einstein Structures and the Quaternionic Contact Yamabe Problem*, MEMOIRS of the Am. Math. Soc. **231**, 1086, Sept. 2014. (Cited on p. 1.)
- [LT] S. Lojasiewicz, G. Tomassini, *Valeurs au bord des formes holomorphes*, Several Complex Variables, Proceedings of International Conferences, Cortona, Italy, Scuola Normale Sup. Pisa, (1976–1977), 222–246. (Cited on p. 15, 17.)
- [LP] Xingmin Li and Lizhong Peng, *The Cauchy integral formulas on the octonions*. *Bulletin of the Belgian Mathematical Society, Simon Stevin*, 9, 01 2002. doi:10.36045/bbms/1102715140. (Cited on p. 19, 20.)

<sup>2</sup>We performed the mentioned computation, as well as the one of Remark 4.4, using the commands `syz` and `betti` of the computer algebra system Macaulay 2 [M2].

- [M2] D.R. Grayson, M.E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at <http://www.math.uiuc.edu/Macaulay2/> (Cited on p. 21.)
- [Ma] B. Malgrange, *Systèmes différentiels à coefficients constants*, Séminaire Bourbaki 1962/1963, N. 246. (Cited on p. 7.)
- [Mi] C. Miranda, *Partial Differential Equations of Elliptic Type*, 2nd ed., Berlin-Heidelberg-New York: Springer-Verlag 1970 (Cited on p. 18.)
- [N] M. Nacinovich, *On boundary Hilbert differential complexes*, *Annales Polonici Mathematici* **XLVI**, (1985), 213–235. (Cited on p. 7.)
- [Pa] V. P. Palamodov, *Linear differential operators with constant coefficients*, Springer, Berlin 1970. (Cited on p. 7.)
- [Pe1] D. Pertici, *Funzioni regolari di più variabili quaternioniche*, *Ann. Mat. Pur. Appl. Serie IV-Tomo CLI*, (1985), 213–65. (Cited on p. 1, 2, 5, 6.)
- [Pe2] D. Pertici, *Trace Theorems for Regular Functions of Several Quaternionic Variables*, *Forum Math.* **3**, (1991), 461–478. (Cited on p. 2, 6.)
- [Pe3] D. Pertici, *Quaternion Regular Functions and Domains of Regularity*, *Boll. UMI (7) 7-B*, (1993), 973–978. (Cited on p. 2, 5, 6, 14, 15, 18.)
- [S] A. Sudbery, *Quaternionic Analysis*, *Math. Proc. Camb. Phil. Soc.*, (1979), 199–225. (Cited on p. 1, 2, 3, 4.)
- [WR] Haiyan Wang and Guangbin Ren, Octonion analysis of several variables. *Communications in Mathematics and Statistics*, 2(2):163–185, Jun 2014. URL: <https://doi.org/10.1007/s40304-014-0034-x>, doi:10.1007/s40304-014-0034-x. (Cited on p. 19.)

<sup>(1)</sup> DIPARTIMENTO DI MATEMATICA E INFORMATICA 'ULISSE DINI' VIALE MORGAGNI, 67/A 50134 FIRENZE  
Email address: [marco.maggese@unifi.it](mailto:marco.maggese@unifi.it)

<sup>(2)</sup> DIPARTIMENTO DI MATEMATICA E INFORMATICA 'ULISSE DINI' VIALE MORGAGNI, 67/A 50134 FIRENZE  
Email address: [donato.pertici@unifi.it](mailto:donato.pertici@unifi.it)

<sup>(3)</sup> SCUOLA NORMALE SUPERIORE, PIAZZA DEI CAVALIERI, 7 - I-56126 PISA, ITALY  
Email address: [giuseppe.tomassini@sns.it](mailto:giuseppe.tomassini@sns.it)