Moser-Nash kernel estimates for degenerate parabolic equations

MOSER- NASH KERNEL ESTIMATES FOR DEGENERATE PARABOLIC EQUATIONS

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ABSTRACT. In this paper we deal with the Cauchy problem associated to a class of nonlinear degenerate parabolic equations, whose prototype is the parabolic p-Laplacian $(2 < p < \infty)$. In his seminal paper Moser after stated the Harnack estimates, proved almost optimal estimates for parabolic kernel by using the so called Harnack chain method. In the linear case sharp estimates come by using Nash approach. Fabes and Stroock proved that Gaussian estimate are equivalent to a parabolic Harnack inequality. In this paper, by using the DiBenedetto-DeGiorgi approach we prove optimal kernel estimates for degenerate quasilinear parabolic equations. To get this result we need to prove the finite speed of the propagation of the support and to establish optimal estimates. Lastly we use these results to prove existence and sharp pointwise estimates from above and from below for the fundamental solutions.

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1. INTRODUCTION

In his seminal paper Moser [27] after stated the Harnack estimates, in Theorem 2 he focussed his attention on Harnack estimates at large. More specifically he proved that there exist two positive constants A and a such that, for any x and y in $\mathbb{R}^N$, for any $0 < s < t < T$ and for any nonnegative solution of

$$u_t = \sum_{,ji=1}^{N} D_i(a_{ij}(x,t)D_j u)$$

in $\mathbb{R}^N \times (0;\infty)$, we have

$$u(t,y) \geq u(s,y)\left(\frac{s}{t}\right)^a \; e^{-A\left(1+\frac{|x-y|^2}{t-s}\right)}$$

Let us remark that in the x variable we have the well known exponential behavior of the fundamental solution, whereas in the t variable we have a power like decay, which is not the optimal one.

He proved these estimates by using a technique called Harnack chain that consists in iterating the Harnack estimates. This technique produces non optimal estimates.

By using different techniques many Authors proved sharp estimates from above and from below. Among them we quote Li and Yau [23] .

They actually proved this Gaussian estimate for the heat kernel $p_t(x,y)$ on any complete Riemannian manifold M with non-negative Ricci curvature:

$$p_t(x,y) \simeq \frac{1}{V(x,\sqrt{t})} exp\left(\frac{d^2(x,y)}{ct}\right)$$

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where $d(x, y)$ is the geodesic distance between points $x, y \in M$, $V(x, r)$ is the Riemannian volume of a geodesic ball $B(x, r)$ of radius r centered at x; the sign $\simeq$ means that the ratio of the two quantities is bounded from above and below by two positive constants non depending upon x, y and t.

We recall that that the Gaussian estimate is equivalent to a parabolic Harnack inequality. For a proof see the very interesting paper by Fabes and Stroock [18] where they use the Nash's ideas [28] to prove sharp estimates on the kernel.

Sharp estimates were proved by several Authors for different linear operators and in different context. Among them we quote the importan contributions due to Coulhon, Grigor'yan and Saloff Coste (see for instance [4], [22] and [32]). In the linear context, also the Harnack chain approach was exploited for subelliptic operators (see the monograph [33] for more details).

For its flexibility the Harnack chain method can be used in the study of degenerate parabolic equations of p-laplacian or porous medium type.

The first result was proved by Auchmuty and Bao [3]. It was afterwards extended in a paper by Gianazza and Polidoro [20] and it was fully exploited in the monograph [16].

To find sharp estimates at large for the nonlinear case requires a technique more sophisticated than the Harnack chain. The first pioneering paper in such a direction is [15] where the De Giorgi techniques [12] are heavily used.

This approach was exploited in two recent papers [31, 9] where sharp pointwise upper and lower estimates for nonnegative solutions to singular parabolic p-Laplacian type equations were proved.

The aim of this paper is to establish similar estimates in the framework of degenerate parabolic equations. In this context, degenerate equations are more difficult to treat than singular ones. The extra difficulty relies on the fact that for singular equations, the speed of propagation of the support is infinite, while, for degenerate equations, it is finite. This means that solutions have a compact support at any time $t > 0$, and therefore the size of the support and the free boundaries need to be controlled. For this reason, one of the main ingredients in the proof of the pointwise estimates for degenerate parabolic equations in the present paper are precise estimates for the support of the solution; see §3.This difficulty appears also from the region where the estimates hold. Quite surprisingly it is nor the geometry induced by the Barenblatt solution neither a classical p-parabola, but a combination of these two "natural" geometries. We prove that this region is optimal. The reason is that the geometry that rules for such kind of equation is the so called "intrinsic geometry". For more details about the definition of this geometry we refer the reader to the monographs [13] and [35].

As a consequence, our pointwise estimates imply existence results and sharp estimates on the fundamental solutions. We are on the one hand able to prove that for any degenerate parabolic operator of p-Laplacian type there exists a fundamental solution. On the other hand, given any fundamental solution in this class of degenerate parabolic equations, we are able to prove estimates, from below and from above, in terms of the explicit Barenblatt solution of the prototype equation, i.e. the p-Laplacian equation. This means that – apart from a constant – the pointwise behavior of any fundamental solution is the one of the Barenbaltt solution.

1.1. **Statement of the problem.** We now introduce the precise parabolic equations and the notions of solutions we are dealing with. In the following, we consider degenerate parabolic equations of the type

$$\partial_t u - \operatorname{div} \mathbf{A}(x,t,u,Du) = 0, \quad \text{in } \mathbb{R}^N \times (0,\infty). \tag{1.1}$$

Thereby, the function $\mathbf{A} = (A_1,\dots,A_N)\colon \mathbb{R}^N \times (0,\infty) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function, i.e. we assume that the mapping $\mathbb{R}^N \times (0,\infty) \ni (x,t) \mapsto \mathbf{A}(x,t,u,\xi)$ is measurable for any $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$ and that $\mathbb{R} \times \mathbb{R}^N \ni (u,\xi) \mapsto \mathbf{A}(x,t,u,\xi)$ is continuous for almost all $(x,t) \in \mathbb{R}^N \times (0,\infty)$. Moreover, $\mathbf{A}$ is assumed to satisfy the following ellipticity and growth conditions:

$$\begin{cases} \mathbf{A}(x,t,u,\xi)\cdot\xi \geq \nu|\xi|^p, \\ |\mathbf{A}(x,t,u,\xi)| \leq L|\xi|^{p-1}, \end{cases} \tag{1.2}$$

for almost all $(x,t) \in \mathbb{R}^N \times [0,\infty)$, all $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, for some constants $0 < \nu \leq L < \infty$ and with $p > 2$. Finally, we assume that

$$\begin{cases} \big(\mathbf{A}(x,t,u,\xi_1) - \mathbf{A}(x,t,u,\xi_2)\big)\cdot(\xi_1-\xi_2) \geq 0, \\ |\mathbf{A}(x,t,u_1,\xi) - \mathbf{A}(x,t,u_2,\xi)| \leq L|u_1-u_2|\big(1+|\xi|^{p-1}\big), \end{cases} \tag{1.3}$$

holds true for almost all $(x,t) \in \mathbb{R}^N \times [0,\infty)$ and all $u, u_i \in \mathbb{R}$ and $\xi, \xi_i \in \mathbb{R}^N$, $i = 1,2$. We note that that hypotheses (1.2) and (1.3) imply a comparison principle for weak solutions of (1.1), and moreover ensure the existence of weak solutions to the associated Cauchy problem with L^1 intial data; see, for instance, [24, 16].

1.2. **Pointwise estimates for weak solutions.** The precise notion of weak solution to (1.1) we shall use in the sequel is given in the following definition.

Definition 1.1. *A function $u \in C^0((0,\infty);L^2(\mathbb{R}^N)) \cap L^p_{\mathrm{loc}}(0,\infty;W^{1,p}(\mathbb{R}^N))$ is a weak solution of (1.1) in $\mathbb{R}^N \times (0,\infty)$ if*

$$\int_{\mathbb{R}^N} u\varphi\,dx\bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \big[-u\partial_t\varphi + \mathbf{A}(x,t,u,Du)\cdot D\varphi\big]\,dxdt = 0 \tag{1.4}$$

holds for every subinterval $[t_1,t_2] \subset (0,\infty)$ and every test function $\varphi \in W^{1,2}(t_1,t_2;L^2(\mathbb{R}^N)) \cap L^p(t_1,t_2;W^{1,p}(\mathbb{R}^N))$. □

We note that in the weak formulation it is not necessary to assume that the testing function φ has compact support on the time slices $\mathbb{R}^N \times \{t\}$. For a more detailed discussion of this fact, we refer to Remark 2.1 below.

Our pointwise estimates will be expressed in terms of the Barenblatt solution. Therefore, before we state our results, we recall the precise definition of the *Barenblatt solution of mass* $M > 0$:

$$\mathcal{B}_{p,M}(x,t) := t^{-\frac{N}{\beta}}\left[C_{p,M} - \gamma_p\Big(\frac{|x|}{t^{1/\beta}}\Big)^{\frac{p}{p-1}}\right]_+^{\frac{p-1}{p-2}}, \tag{1.5}$$

where

$$\beta := N(p-2)+p \quad \text{and} \quad \gamma_p := \Big(\frac{1}{\beta}\Big)^{\frac{1}{p-1}}\Big(\frac{p-2}{p}\Big), \tag{1.6}$$

and $C_{p,M}$ is a positive constant of the form $C_{p,M} = (\frac{M}{d})^{\frac{1}{\gamma}}$, with $\gamma := \frac{(p-1)\beta}{p(p-2)}$ and a constant d depending only on N, p; see also [5], and [37, §12.3], [38]. Note that $\operatorname{spt}\mathcal{B}_{p,M_1} \subset \operatorname{spt}\mathcal{B}_{p,M_2}$ whenever $M_1 < M_2$. It is well known that $\mathcal{B}_{p,M}$ is the fundamental solution of the p-Laplacian equation with mass M in $\mathbb{R}^N \times (0,\infty)$, i.e. the explicit solution of the p-Laplacian equation with $M\delta_0$ as initial datum. We recall that the uniqueness of the fundamental solution was proved in [25]. Here, and in the following, by δ_0 we denote the delta-function at the origin. In the particular case that $M = 1$, we abbreviate $\mathcal{B}_p := \mathcal{B}_{p,1}$.

Now, we can state our first main result, which provides sharp pointwise estimates from below for weak solutions, starting only from the value that the solution takes in a certain point of the domain. These estimates hold in the large, and generalize the Harnack estimates proved in [14] and [15].

estimates from below

Theorem 1.2 (estimates from below)**.** *Suppose that assumptions* (1.2) *and* (1.3) *are in force. Then, there exist two constants* $\gamma = \gamma(N, p, \nu, L) > 0$ *and* $\widetilde{\gamma} = \widetilde{\gamma}(N, p, \nu, L) > 0$ *such that the following holds true: Let* u *be a nonnegative weak solution of* (1.1) *and* $P_o = (x_o, t_o) \in \mathbb{R}^N \times (0, \infty)$ *such that* $u(P_o) > 0$. *Then, there holds*

$$u(x,t) \geq \gamma\, u(P_o)\mathcal{B}_p\left(\frac{x - x_o}{t_o^{\frac{1}{p}}\, u(P_o)^{\frac{p-2}{p}}}, \frac{t}{t_o}\right),$$

for any point $(x,t) \in \mathbb{R}^N \times (t_o, \infty)$ *with* $x \in B_{r(t)}(x_o)$, *where*

$$r(t) := \widetilde{\gamma} u(P_o)^{\frac{p-2}{p}}\, t_o^{\frac{1}{p}} min\Big(\big[\frac{t - t_o}{t_o}\big]^{\frac{1}{\beta}}, \big[\frac{t - t_o}{t_o}\big]^{\frac{1}{p}}\Big).$$

Remark 1.3. The explicit solution of the parabolic p-Laplace equation easily shows that, when t is large, the estimates obtained in the previous theorem are sharp. □

The estimates from below from Theorem 1.2 imply, in a strightforward way, the estimates from above (for more details, see [9]).

estimates from above

Corollary 1.4 (estimates from above)**.** *Let* $\varepsilon > 0$ *and suppose that assumptions* (1.2) *and* (1.3) *are in force. Then, there exist two constants* $\gamma = \gamma(N, p, \nu, L) > 0$ *and* $\widetilde{\gamma} = \widetilde{\gamma}(N, p, \nu, L, \varepsilon) > 0$ *such that the following holds true: Let* u *be a nonnegative weak solution of* (1.1) *and* $P_o = (x_o, t_o) \in \mathbb{R}^N \times (0, \infty)$ *such that* $u(P_o) > 0$. *Then, for any point* $(x_1, t_1) \in \mathbb{R}^N \times (0, \frac{t_o}{1+\varepsilon})$ *there holds either*

$$|x_1 - x_o| \geq \widetilde{\gamma}\, t_1^{\frac{1}{p}} [\tfrac{t_o - t_1}{t_1}]^{\frac{1}{\beta}} u(P_1)^{\frac{p-2}{p}},$$

or

$$u(P_o) \geq \gamma u(P_1)\mathcal{B}_p\left(\frac{x_o - x_1}{t_1^{\frac{1}{p}} u(P_1)^{\frac{p-2}{p}}}, \frac{t_o}{t_1}\right).$$

Note that the estimate on $u(x_1, t_1)$ is not explicit. Nevertheless arguing as in [9] in the singular case, it is possible to prove that also this estimate is sharp.

1.3. **Results for fundamental solutions.** By our methods we can also prove existence and pointwise estimates for fundamental solutions to general degenerate parabolic equations. As already mentioned above, for the prototype equations the fundamental solution (i.e. solutions where the initial data is a Dirac mass) is explicitly known to be the Barenblatt solution. More precisely, in this case, the function $\mathcal{B}_p$ defined in (1.5) is the solution of the following initial-value problem

equationBa

$$\begin{cases} \partial_t u = \operatorname{div}(|Du|^{p-2}Du), & \text{in } \mathbb{R}^N \times (0, \infty), \\ u(\cdot, 0) = \delta_0, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.7}$$

where δ_0 denotes the delta-function at the origin. As already noted in [31], when dealing with the initial-value problem with a measure as initial data, the solution does in general not belong to the natural parabolic Sobolev space $L^p(0, T; W^{1,p}(\mathbb{R}^N))$, for $T > 0$. Instead, we have for the spatial gradient that $|\nabla u| \in \mathcal{M}^{\frac{N(p-1)}{N-1}}$, where $\mathcal{M}^q$ is the Marcinkiewicz space of order q. On the other hand, since $\frac{N(p-1)}{N-1} > p - 1$, this implies that $|\nabla u|^{p-1} \in L^1(\mathbb{R}^N \times (0, T))$. Therefore, the solution of the problem exists in the distributional sense. For more details on entropy solutions see [6, 7], while for renormalized solutions we refer to [11, 29].

Keeping this in mind and following the approach of [25, 30], we define in the following the notion of fundamental solution to parabolic equations of the type

equationfundamental

$$\begin{cases} \partial_t u = \operatorname{div} \mathbf{A}(x,t,u,Du), & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(\cdot,0) = \delta_0, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.8}$$

where the operator $\mathbf{A}$ satisfies assumptions (1.2) and (1.3).

def:fundamental-sol **Definition 1.5** (fundamental solution)**.** *A nonnegative function* $u\colon \mathbb{R}^N \times (0,\infty) \to \mathbb{R}$ *is a fundamental solution of* (1.8) *if the following conditions are satisfied:*

i) $u \in C((0,\infty); L^1(\mathbb{R}^N))$,

ii) *u is a weak solution of* $(1.8)_1$ *in* $\mathbb{R}^N \times [s,+\infty)$*, for any* $s > 0$*,*

iii) *we have* $\displaystyle\lim_{t\downarrow 0} \int_{B_\varrho} u(x,t)\,dx = 1$*, for any* $\varrho > 0$*, and*

iv) *we have* $\displaystyle\lim_{t\downarrow 0} \int_{\mathbb{R}^N \setminus B_\varrho} u(x,t)\,dx = 0$*, for any* $\varrho > 0$*.*

□

As a first result we can prove existence of fundamental solutions to general degenerate parabolic equations.

thm:existence **Theorem 1.6** (Existence of fundamental solutions)**.** *Let assumptions* (1.2) *and* (1.3) *be in force. Then, there exists at least one nonnegative fundamental solution of* (1.8) *in the sense of Definition 1.5.*

As a second result, we prove sharp pointwise estimates for fundamental solutions from below and from above. We are able to show that, up to a constant, any fundamental solution behaves like the Barenblatt fundamental solution.

thm:Barenblatt **Theorem 1.7.** *Let assumptions* (1.2) *and* (1.3) *be in force and u be a fundamental solution of* (1.8)*. Then there are two positive constants* M_1 *and* M_2 *depending only on* N, p, ν, L *such that for any* $(x,t) \in \mathbb{R}^N \times (0,\infty)$ *there holds*

$$\mathcal{B}_{p,M_1}(x,t) \le u(x,t) \le \mathcal{B}_{p,M_2}(x,t),$$

where $\mathcal{B}_{p,M}$ *denotes the Barenblatt solution of mass* M *defined in* (1.5)*.*

In the singular case, it was first proved the result for the fundamental estimates in [31]. Then, in [9], the estimates at large were proved. In the survey paper [17], it is proved that the estimates proved in [9] imply the estimates proved in [31].

1.4. **Plan of the paper.** The paper is organized as follows. In §2, we explain the notation used throughout the paper and collect some known results to be used in the proofs of our results. Susequently, in §3 we derives sharp estimates for the speed of propagation of the support. These are essential for the proofs of our results. By using the Barenblatt solution $\mathcal{B}_p$ of the p-Laplacian, the pointwise estimates from below for weak solution of 1.1 as stated in Theorem 1.2 are established in §4. The estimates from above from Corollary 1.4 are an emmediate consequence. In §5 we prove the existence of fundamental solutions as stated in Theorem 1.6. The result of Theorem 1.7, i.e. the optimal bounds from below and from above for the fundamental solutions of (1.1) are proved in §6. Finally, in §7 we exhibit some miscellaneous results. We remark that estimates obtained for the p-Laplacian can be extended to the Porous Medium equation, and by a change of variable also to the Fokker Planck equation.

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sec:prelim

2. Preliminaries

2.1. **Notation.** For $x_o \in \mathbb{R}^N$ and $\varrho > 0$ we denote by $B_\varrho(x_o)$ the Euclidean ball with center at x_o and radius ϱ. If x_o is the origin we usually omit the center point in the notation and write $B_\varrho := B_\varrho(0)$ for short. By $\omega_N := |B_1|$ we denote the volume of the unit ball. Finally, $\boldsymbol{\gamma}$ denotes a generic constant which may change from line to line.

rem:testfn

Remark 2.1. As mentioed after Definition 1.1, in the weak formulation it is not necessary to assume that the testing function φ has compact support. This can be seen as follows: for $k > 0$ we use $\zeta_k\varphi$ as a testing function, where $\zeta_k \in C_0^1(B_{2k})$ satisfies $\zeta_k \equiv 1$ in B_k and $\|D\zeta_k\|_\infty \le \frac{2}{k}$. Then, we have

$$\int_{\mathbb{R}^N} \zeta_k \varphi u \, dx \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \zeta_k \big[-u\partial_t \varphi + \mathbf{A}(x,t,u,Du) \cdot D\varphi \big] \, dxdt \\ = - \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \varphi \mathbf{A}(x,t,u,Du) \cdot D\zeta_k \, dxdt.$$

For the integral on the right-hand side, we get

$$\bigg| \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \varphi \, \mathbf{A}(x,t,u,Du) \cdot D\zeta_k \, dxdt \bigg| \le \frac{2L}{k} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |Du|^{p-1} |\varphi| \, dxdt \\ \le \frac{2L}{k} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \big(|Du|^p + |\varphi|^p \big) \, dxdt \to 0$$

in the limit $k \to \infty$. Therefore, letting $k \to \infty$, we obtain (1.4).

Moreover, in the definition of weak solution we avoid using the time derivative of u, since $\partial_t u$ might not exist as a function. Indeed, one can only show that $\partial_t u \in L^{p'}(0,\infty; W^{-1,p'}(\mathbb{R}^N))$. Nevertheless, later on we shall use

weak

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^N} \big[\varphi \, \partial_t u + \mathbf{A}(x,t,u,Du) \cdot D\varphi \big] \, dxdt = 0 \tag{2.1}$$

instead of (1.4) as weak form of the equation. Thereby, the use of the time derivative has to be understood in a formal way. These computations can be made rigorous by the use of a mollification procedure with respect to time as for instance Steklov averages. □

2.2. **Auxiliary material.** We state Gagliardo-Nirenberg's inequality in the form that will be suitable for our purposes later.

lem:gag

Lemma 2.2. *Let $1 \le \sigma, p, q < \infty$ and $\vartheta \in (0,1)$ such that $-\frac{N}{q} = \vartheta(1 - \frac{N}{p}) - (1-\vartheta)\frac{N}{\sigma}$. Then there exists a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N,p,q,\sigma)$ such that for any $v \in L^\sigma(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$ there holds:*

$$\int_{\mathbb{R}^N} |v|^q \, dx \le \boldsymbol{\gamma} \bigg(\int_{\mathbb{R}^N} |v|^\sigma \, dx \bigg)^{\frac{(1-\vartheta)q}{\sigma}} \bigg(\int_{\mathbb{R}^N} |Dv|^p \, dx \bigg)^{\frac{\vartheta q}{p}}.$$

We shall use the well known DeGiorgi iteration lemma, which can for instance be found in [19, Lemma 7.1].

lem:DeGiorgi

Lemma 2.3. *Let $\alpha > 0$ and let $(\mathbf{k}_i)_{i \in \mathbb{N}}$ be a sequence of real positive numbers, satisfying the recursive inequalities*

$$\mathbf{k}_{i+1} \le C B^i \mathbf{k}_i^{1+\alpha}$$

with $C, B > 1$. If $\mathbf{k}_o \le C^{-\frac{1}{\alpha}} B^{-\frac{1}{\alpha^2}}$, we have

$$\mathbf{k}_i \le B^{-\frac{i}{\alpha}} \mathbf{k}_o$$

and hence in particular $\lim_{i \to \infty} \mathbf{k}_i = 0$.

The following result can be deduced from [21, §3] and [14, §3], adapted to our situation.

lm:Surnachev

Lemma 2.4 (Expansion of positivity). *Let the assumptions* (1.2) *be in force and* $M > 0$. *Then there exist positive constants* ε_o *and* k_o *depending only on* N, p, ν, L *and* M *such that the following holds true: let* $u \in C^0((0,\infty); L^2(\mathbb{R}^N)) \cap L^p_{\rm loc}(0,\infty; W^{1,p}(\mathbb{R}^N))$ *be a continuous nonnegative weak solution of* (1.1) *in* $\mathbb{R}^N \times [0,\infty)$ *and assume that there exist* $\delta > 0$ *and* $\varrho > 0$ *such that* $u(x,0) \geq \delta$ *for any* $x \in B_\varrho$. *Then, there holds*

$$u\big(x, k_o \delta^{2-p} \varrho^p\big) \geq \varepsilon_o \delta, \qquad \textit{for any } x \in B_{M\varrho}.$$

Let us quote the final result of [14] that we adapt to our assumptions (for instance we take into account the homogeneous structure hypotheses we assumed at the beginning).

Thm:Harnack

Theorem 2.5 (Intrinsic Harnack inequality). *Let the assumptions* (1.2) *be in force. Then there exist positive constants* c_1 *and* c_2 *depending only on* N, p, ν *and* L *such that the following holds: whenever* $u \in C^0((0,\infty), L^2(\mathbb{R}^N)) \cap L^p_{\rm loc}(0,\infty; W^{1,p}(\mathbb{R}^N))$ *is a continuous nonnegative weak solution of* (1.1) *in* $\mathbb{R}^N \times [0,\infty)$ *in the sense of Definition 1.1 and* $(x_o, t_o) \in \mathbb{R}^N \times [0,\infty)$ *and* $\varrho > 0$, *then there holds*

Harnackestimates

$$u(x_o,t_o) \leq c_1 \inf_{B_\varrho(x_o)} u(\cdot, t_o + \theta \varrho^p), \qquad \theta := \Big(\frac{c_2}{u(x_o,t_o)}\Big)^{p-2}. \tag{2.2}$$

sec:est-support

3. Estimates for the support of the solution

In this Section we consider the initial value problem

ini-equation

$$\begin{cases} \partial_t u - \operatorname{div} \mathbf{A}(x,t,u,Du) = 0, & \text{in } \mathbb{R}^N \times \mathbb{R}_{\geq 0}, \\ u(\cdot,0) = u_o, & \text{in } \mathbb{R}^N \times \mathbb{R}_{\geq 0}, \end{cases} \tag{3.1}$$

with initial datum $u_o \in L^1(\mathbb{R}^N)$ satisfying

u0

$$\operatorname{spt} u_o \subset B_{R_o} \qquad \text{for some } R_o > 0. \tag{3.2}$$

In the following, we use a method introduced in [1] and in [2] (see also a recent paper by Tedeev and Vespri [34]) to control the size of the support of the solution, in terms of R_o, the L^1-norm of u_o and the time elapsed. Our first aim is to prove that this assumption implies that for any finite time, also the solution has bounded support.

lem:bounded-support

Lemma 3.1. *Let* $R_o > 0$ *and* $u_o \in L^1(\mathbb{R}^N)$ *satisfy* (3.2) *and suppose that* $A\colon \mathbb{R}^N \times (0,\infty) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ *satisfies the assumptions* (1.2). *Then, for any weak solution* $u \in C^0((0,\infty); L^2(\mathbb{R}^N)) \cap L^p_{\rm loc}(0,\infty; W^{1,p}(\mathbb{R}^N))$ *of the initial value problem* (3.1) *we have that* $\operatorname{spt} u \cap (\mathbb{R}^N \times [0,t])$ *is bounded for any* $t > 0$.

Proof. We fix $t > 0$ and let $\eta \in C^1_0(\mathbb{R}^N)$ with $0 \leq \eta \leq 1$, $\eta \equiv 0$ in B_{R_o}. In the weak form (2.1) of equation (3.1) we choose the testing function $\varphi = \eta^p u$. For the term containing the time derivative, we compute

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^N} \partial_\tau u \varphi \, dx d\tau &= \frac{1}{2} \int_0^t \int_{\mathbb{R}^N} \eta^p \partial_\tau u^2 \, dx d\tau \\ &= \frac{1}{2} \Big[\int_{\mathbb{R}^N \times \{t\}} \eta^p u^2 \, dx - \int_{\mathbb{R}^N} \eta^p u_o^2 \, dx \Big] = \frac{1}{2} \int_{\mathbb{R}^N \times \{t\}} \eta^p u^2 \, dx, \end{aligned}$$

where in the last line we used that $\operatorname{spt} u_o \subset B_{R_o}$ and $\eta \equiv 0$ on B_{R_o}. Next, we consider the diffusion part in (2.1). Here, we have

$$\int_0^t \int_{\mathbb{R}^N} A(x,t,u,Du) \cdot D\varphi \, dx d\tau = \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} := \int_0^t \int_{\mathbb{R}^N} \eta^p \, \mathbf{A}(x,t,u,Du) \cdot Du \, dx d\tau,$$

$$\mathrm{II} := \int_0^t \int_{\mathbb{R}^N} \mathbf{A}(x,t,u,Du) \cdot D\eta^p u \, dx d\tau.$$

By $(1.2)_1$, we get

$$\mathrm{I} \geq \nu \int_0^t \int_{\mathbb{R}^N} \eta^p |Du|^p \, dx d\tau,$$

while using the growth assumption $(1.2)_2$ and Young's inequality, we find for the second term that

$$\begin{aligned} |\mathrm{II}| &\leq Lp \int_0^t \int_{\mathbb{R}^N} \eta^{p-1} |Du|^{p-1} |D\eta||u| \, dx d\tau \\ &\leq \frac{\nu}{2} \int_0^t \int_{\mathbb{R}^N} \eta^p |Du|^p \, dx d\tau + \boldsymbol{\gamma} \int_0^t \int_{\mathbb{R}^N} |D\eta|^p |u|^p \, dx d\tau, \end{aligned}$$

where $\boldsymbol{\gamma} = \boldsymbol{\gamma}(p,\nu,L)$. Combining the last two estimates yields

$$\mathrm{I} + \mathrm{II} \geq \frac{\nu}{2} \int_0^t \int_{\mathbb{R}^N} \eta^p |Du|^p \, dx d\tau - \boldsymbol{\gamma} \int_0^t \int_{\mathbb{R}^N} |D\eta|^p |u|^p \, dx d\tau.$$

Inserting the preceding estimates into (2.1), we obtain

$$\int_{\mathbb{R}^N \times \{t\}} \eta^p |u|^2 \, dx + \int_0^t \int_{\mathbb{R}^N} \eta^p |Du|^p \, dx d\tau \leq \boldsymbol{\gamma} \int_0^t \int_{\mathbb{R}^N} |D\eta|^p |u|^p \, dx d\tau,$$

with a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(p,\nu,L)$. Since $t > 0$ was arbitrary, we would obtain the same estimate for any $\tau \in (0,t)$ instead of t. Therefore, we can take the supremum over $\tau \in (0,t)$ in the first term on the left-hand side, i.e. we have

testing-1

$$\sup_{\tau \in (0,t)} \int_{\mathbb{R}^N \times \{\tau\}} \eta^p |u|^2 \, dx + \int_0^t \int_{\mathbb{R}^N} \eta^p |Du|^p \, dx d\tau \leq \boldsymbol{\gamma} \int_0^t \int_{\mathbb{R}^N} |D\eta|^p |u|^p \, dx d\tau, \tag{3.3}$$

for a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(p,\nu,L)$.

For $k > 0$ large enough, we choose a cutoff function $\zeta_k \in C^1_0(B_{2k})$, such that $0 \leq \zeta_k \leq 1$, and $\zeta_k \equiv 1$ in B_k, and $\|D\zeta_k\|_\infty \leq \frac{2}{k}$. Moreover, we let $r \geq R_o$ to be chosen later and define

$$r_i := 2r - \frac{r}{2^i} \quad \text{and} \quad s_i := \frac{r_i + r_{i+1}}{2},$$

for $i \in \mathbb{N}_0$, such that $B_{r_i} \subset B_{s_i} \subset B_{r_{i+1}}$. We let $\eta_i \in C^1(\mathbb{R}^N)$ such that $0 \leq \eta_i \leq 1$, $\eta_i \equiv 0$ in B_{r_i}, $\eta_i \equiv 1$ in $\mathbb{R}^N \setminus B_{s_i}$ and $\|D\eta_i\|_\infty \leq 2^{i+3}/r$. Choosing $\eta = \zeta_k \eta_i$ in (3.3), we find that

$$\begin{aligned} \sup_{\tau \in (0,t)} \int_{\mathbb{R}^N \setminus B_{s_i} \times \{\tau\}} \zeta_k^p |u|^2 \, dx &+ \int_0^t \int_{\mathbb{R}^N \setminus B_{s_i}} \zeta_k^p |Du|^p \, dx d\tau \\ &\leq \boldsymbol{\gamma} \Big(\frac{2^{ip}}{r^p} + \frac{2}{k^p} \Big) \int_0^t \int_{\mathbb{R}^N \setminus B_{r_i}} |u|^p \, dx d\tau. \end{aligned}$$

Letting $k \to \infty$ this yields

$$\begin{aligned} \sup_{\tau \in (0,t)} \int_{\mathbb{R}^N \setminus B_{s_i} \times \{\tau\}} |u|^2 \, dx &+ \int_0^t \int_{\mathbb{R}^N \setminus B_{s_i}} |Du|^p \, dx d\tau \\ &\leq \frac{\boldsymbol{\gamma} 2^{ip}}{r^p} \int_0^t \int_{\mathbb{R}^N \setminus B_{r_i}} |u|^p \, dx d\tau. \end{aligned}$$

Next, we choose a second cut-off function $\tilde{\eta}_i \in C^1(\mathbb{R}^N)$ satisfying $0 \leq \tilde{\eta}_i \leq 1$, $\tilde{\eta}_i \equiv 0$ in B_{s_i}, $\tilde{\eta}_i \equiv 1$ in $\mathbb{R}^N \setminus B_{r_{i+1}}$ and $\|D\tilde{\eta}_i\|_\infty \leq 2^{i+3}/r$. Letting

$$v_i := \tilde{\eta}_i |u|,$$

the preceding inequality yields

$$
\begin{aligned}
\sup_{\tau\in(0,t)}\int_{\mathbb{R}^N\times\{\tau\}} v_i^2\,dx &+ \int_0^t\int_{\mathbb{R}^N}|Dv_i|^p\,dxd\tau \\
&\le \sup_{\tau\in(0,t)}\int_{\mathbb{R}^N\times\{\tau\}}\tilde\eta_i^2|u|^2\,dx + \int_0^t\int_{\mathbb{R}^N}\big[\tilde\eta_i^p|Du|^p+|D\tilde\eta_i|^p|u|^p\big]\,dxd\tau \\
&\le \sup_{\tau\in(0,t)}\int_{\mathbb{R}^N\setminus B_{s_i}\times\{\tau\}}|u|^2\,dx + \int_0^t\int_{\mathbb{R}^N\setminus B_{s_i}}\big[|Du|^p+\tfrac{2^{ip}}{r^p}|u|^p\big]\,dxd\tau \\
&\le \frac{\boldsymbol\gamma 2^{ip}}{r^p}\int_0^t\int_{\mathbb{R}^N\setminus B_{r_i}}|u|^p\,dxd\tau.
\end{aligned}
$$

Taking into account that $v_{i-1}\equiv|u|$ on $\mathbb{R}^N\setminus B_{r_i}\times(0,t)$, this shows that

est-vi-1

$$
\sup_{\tau\in(0,t)}\int_{\mathbb{R}^N\times\{\tau\}} v_i^2\,dx + \int_0^t\int_{\mathbb{R}^N}|Dv_i|^p\,dxd\tau \le \frac{\boldsymbol\gamma 2^{ip}}{r^p}\int_0^t\int_{\mathbb{R}^N}v_{i-1}^p\,dxd\tau, \tag{3.4}
$$

where $\boldsymbol\gamma=\boldsymbol\gamma(p,\nu,L)$. Now, we define

$$
\mathbf{k}_i := \int_0^t\int_{\mathbb{R}^N}v_i^p\,dxd\tau.
$$

By Gagliardo-Nirenberg's inequality from Lemma 2.2 applied with $q=p$ and

$$
\vartheta := \frac{N(p-2)}{N(p-2)+2p}\in(0,1),
$$

and estimate (3.4), we obtain

$$
\begin{aligned}
\mathbf{k}_i &\le \boldsymbol\gamma\int_0^t\bigg(\int_{\mathbb{R}^N}|v_i|^2\,dx\bigg)^{\frac{(1-\vartheta)p}{2}}\bigg(\int_{\mathbb{R}^N}|Dv_i|^p\,dx\bigg)^{\vartheta}dt \\
&\le \boldsymbol\gamma\sup_{\tau\in(0,t)}\bigg(\int_{\mathbb{R}^N\times\{\tau\}}|v_i|^2\,dx\bigg)^{\frac{(1-\vartheta)p}{2}}\int_0^t\bigg(\int_{\mathbb{R}^N}|Dv_i|^p\,dx\bigg)^{\vartheta}dt \\
&\le \boldsymbol\gamma\, t^{1-\vartheta}\sup_{\tau\in(0,t)}\bigg(\int_{\mathbb{R}^N\times\{\tau\}}|v_i|^2\,dx\bigg)^{\frac{(1-\vartheta)p}{2}}\bigg(\int_0^t\int_{\mathbb{R}^N}|Dv_i|^p\,dxdt\bigg)^{\vartheta} \\
&\le \boldsymbol\gamma\, t^{1-\vartheta}\bigg(\frac{2^{ip}}{r^p}\,\mathbf{k}_{i-1}\bigg)^{\vartheta+\frac{(1-\vartheta)p}{2}} = \boldsymbol\gamma\, t^{1-\vartheta}\bigg(\frac{2^{ip}}{r^p}\,\mathbf{k}_{i-1}\bigg)^{1+\frac{(1-\vartheta)(p-2)}{2}},
\end{aligned}
$$

where $\boldsymbol\gamma=\boldsymbol\gamma(N,p,\nu,L)$. Applying Lemma 2.3 with

$$
\alpha := \frac{(1-\vartheta)(p-2)}{2}
$$

and

$$
B = 2^{p(1+\frac{(1-\vartheta)(p-2)}{2})} = 2^{p(1+\alpha)},\quad C=\boldsymbol\gamma\, t^{1-\vartheta}r^{-p(1+\frac{(1-\vartheta)(p-2)}{2})} = \boldsymbol\gamma\, t^{1-\vartheta}r^{-p(1+\alpha)},
$$

we find that $\lim_{i\to\infty}\mathbf{k}_i=0$, provided that

DG-cond-1

$$
\mathbf{k}_o \le \boldsymbol\gamma^{-\frac{1}{\alpha}}2^{-\frac{p(1+\alpha)}{\alpha^2}}t^{-\frac{1-\vartheta}{\alpha}}r^{\frac{p(1+\alpha)}{\alpha}} = \boldsymbol\gamma\, t^{-\frac{2}{p-2}}r^{\frac{p^2+Np-2N}{p-2}}, \tag{3.5}
$$

with a constant $\boldsymbol\gamma=\boldsymbol\gamma(N,p,\nu,L)$. Since

$$
\mathbf{k}_o \le \int_0^t\int_{\mathbb{R}^N}\tilde\eta^p|u|^p\,dxd\tau,
$$

this condition is satisfied if we choose $r\ge R_o$ such that

$$
r^{\frac{p^2+Np-2N}{p-2}} \ge \boldsymbol\gamma\, t^{\frac{2}{p-2}}\int_0^t\int_{\mathbb{R}^N}\tilde\eta^p|u|^p\,dxd\tau.
$$

As mentioned above, this ensures that (3.5) is satisfied and thus $\lim_{i\to\infty} \mathbf{k}_i = 0$. In turn, this implies that $u \equiv 0$ on $(\mathbb{R}^N \setminus B_{2r}) \times [0,t]$. This proves the assertion of the lemma. □

We remark that the initial mass is preserved. This can easily be proved by choosing the testing function 1 in the weak form (2.1) of equation $(3.1)_1$. Note that this testing function is admissible in (2.1) since the support of the solution is bounded on finite time intervals by Lemma 3.1. Therefore we have

Rm:R1

Remark 3.2. *Let $R_o > 0$ and $u_o \in L^1(\mathbb{R}^N)$ satisfy (3.2) and suppose that $\mathbf{A}$ satisfies the assumptions (1.2). Then, for any weak solution u of the initial value problem (3.1) there holds*

mass

$$\|u(t)\|_{L^1(\mathbb{R}^N)} = \|u_o\|_{L^1(\mathbb{R}^N)}, \tag{3.6}$$

for any $t > 0$.

The following result gives us sharp L^∞ estimates.

thm:Linfty

Theorem 3.3. *Let $R_o > 0$ and $u_o \in L^1(\mathbb{R}^N)$ satisfy (3.2) and suppose that $\mathbf{A}$ satisfies the assumptions (1.2). Then, there exists a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N,p,\nu,L)$ such that any weak solution u of the initial value problem (3.1) satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le \boldsymbol{\gamma}\, t^{-\frac{N}{\beta}} \|u_o\|_{L^1(\mathbb{R}^N)}^{\frac{p}{\beta}}, \qquad \text{with } \beta := N(p-2)+p,$$

for any $t > 0$.

Proof. From Lemma 3.1 we know that $\operatorname{spt} u \cap (\mathbb{R}^N \times [0,t])$ is bounded for any $t > 0$. Therefore, we can apply [13, Chapter V, Theorem 4.3] to infer that there exists a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N,p,\nu,L)$ such that for any $t > 0$ there holds

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le \boldsymbol{\gamma}\, t^{-\frac{N}{\beta}} \bigg(\fint_0^t \int_{\mathbb{R}^n} u \, dx dt \bigg)^{\frac{p}{\beta}}.$$

At this point, the result follows from Remark 3.2, since the L^1-norm of $t \mapsto u(\cdot,t)$ is invariant in time. □

Now, we have the prerequisites to prove a qualitative bound for the support of $u(\cdot,t)$ at time $t > 0$. The strategy to the proof will be a refined version of the proof of Lemma 3.1. However, to get the optimal exponents in the estimate, we will use Theorem 3.3 and Remark 3.2 in the final part of the proof.

arpestimatessupport

Theorem 3.4. *Let $R_o > 0$ and $u_o \in L^1(\mathbb{R}^N)$ satisfy (3.2) and suppose that $\mathbf{A}$ satisfies the assumptions (1.2). Then, there exists a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N,p,\nu,L)$ such that any weak solution u of the initial value problem (3.1) satisfies*

supp bound

$$\operatorname{spt} u(\cdot,t) \subset B_{R(t)} \qquad \text{for any } t > 0, \tag{3.7}$$

where

R(t)

$$R(t) := 2R_o + \boldsymbol{\gamma}\, t^{\frac{1}{\beta}} \|u_o\|_{L^1(\mathbb{R}^N)}^{\frac{p-2}{\beta}}, \qquad \text{with } \beta := N(p-2)+p. \tag{3.8}$$

Proof. Let $t > 0$. From Lemma 3.1 we know that $\operatorname{spt} u \cap (\mathbb{R}^N \times [0,t])$ is bounded and therefore, we can choose $k > R_o$ so large that $\operatorname{spt} u \cap (\mathbb{R}^N \times [0,t]) \subset B_k \times [0,t]$. We let $\theta \in (0, \min\{1, \frac{p}{N}\})$ to be chosen later. In the weak form (2.1) we choose the admissible testing function $\varphi = \varphi_\varepsilon := \eta^p (u^2+\varepsilon^2)^{\frac{\theta-1}{2}} u$, with $\eta \in C^1(\mathbb{R}^N)$, $\eta \equiv 0$ in B_{R_o} and $\varepsilon \in (0,1)$. For the term containing the time derivative, we compute

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^N} \partial_\tau u \varphi_\varepsilon \, dx d\tau &= \int_0^t \int_{B_k} \partial_\tau u \varphi_\varepsilon \, dx d\tau \\
&= \frac{1}{2} \int_0^t \int_{B_k} \eta^p \partial_\tau u^2 (u^2+\varepsilon^2)^{\frac{\theta-1}{2}} \, dx d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1+\theta} \int_0^t \int_{B_k} \eta^p \partial_\tau |u^2+\varepsilon^2|^{\frac{1+\theta}{2}}\, dxd\tau \\
&= \frac{1}{1+\theta} \bigg[\int_{B_k\times\{t\}} \eta^p |u^2+\varepsilon^2|^{\frac{1+\theta}{2}}\, dx - \int_{B_k} \eta^p |u_o^2+\varepsilon^2|^{\frac{1+\theta}{2}}\, dx \bigg] \\
&= \frac{1}{1+\theta} \bigg[\int_{B_k\times\{t\}} \eta^p |u^2+\varepsilon^2|^{\frac{1+\theta}{2}}\, dx - \varepsilon^{1+\theta} \int_{B_k} \eta^p\, dx \bigg],
\end{aligned}
$$

where in the last line we used that $\operatorname{spt} u_o \subset B_{R_o}$ and $\eta \equiv 0$ in B_{R_o}. Letting $\varepsilon \downarrow 0$, we find that

$$
\lim_{\varepsilon\downarrow 0} \int_0^t \int_{\mathbb{R}^N} \partial_\tau u \varphi_\varepsilon \, dxd\tau = \frac{1}{1+\theta} \int_{B_k\times\{t\}} \eta^p |u|^{1+\theta}\, dx = \frac{1}{1+\theta} \int_{\mathbb{R}^N\times\{t\}} \eta^p |u|^{1+\theta}\, dx.
$$

Next, we consider the diffusion part in (2.1). Here, we have

$$
\int_0^t \int_{\mathbb{R}^N} \mathbf{A}(x,t,u,Du)\cdot D\varphi_\varepsilon\, dxd\tau = \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon,
$$

where

$$
\begin{aligned}
\mathrm{I}_\varepsilon &:= \int_0^t \int_{\mathbb{R}^N} \eta^p \mathbf{A}(x,t,u,Du)\cdot D\big[(u^2+\varepsilon^2)^{\frac{\theta-1}{2}} u\big]\, dxd\tau, \\
\mathrm{II}_\varepsilon &:= \int_0^t \int_{\mathbb{R}^N} \mathbf{A}(x,t,u,Du)\cdot D\eta^p \big[(u^2+\varepsilon^2)^{\frac{\theta-1}{2}} u\big]\, dxd\tau.
\end{aligned}
$$

For the term I_ε, we first compute

$$
\begin{aligned}
D\big[(u^2+\varepsilon^2)^{\frac{\theta-1}{2}} u\big] &= (u^2+\varepsilon^2)^{\frac{\theta-1}{2}} Du - (1-\theta)(u^2+\varepsilon^2)^{\frac{\theta-3}{2}} u^2 Du \\
&= (u^2+\varepsilon^2)^{\frac{\theta-3}{2}} (\theta u^2+\varepsilon^2) Du.
\end{aligned}
$$

Hence, by $(1.2)_1$, we get

$$
\begin{aligned}
\mathrm{I}_\varepsilon &\ge \nu \int_0^t \int_{\mathbb{R}^N} \eta^p (u^2+\varepsilon^2)^{\frac{\theta-3}{2}} (\theta u^2+\varepsilon^2) |Du|^p\, dxd\tau \\
&\ge \nu\theta \int_0^t \int_{\mathbb{R}^N} \eta^p (u^2+\varepsilon^2)^{\frac{\theta-1}{2}} |Du|^p\, dxd\tau \\
&\ge \frac{\nu\theta}{2} \int_0^t \int_{\mathbb{R}^N} \eta^p (|u|+\varepsilon)^{\theta-1} |Du|^p\, dxd\tau \\
&= \frac{\nu\theta}{2} \Big(\frac{p}{p+\theta-1}\Big)^p \int_0^t \int_{\mathbb{R}^N} \eta^p \big|D(|u|+\varepsilon)^{\frac{p+\theta-1}{p}}\big|^p\, dxd\tau.
\end{aligned}
$$

Using the growth assumption $(1.2)_2$ and Young's inequality, we find for the second term that

$$
\begin{aligned}
|\mathrm{II}_\varepsilon| &\le Lp \int_0^t \int_{B_k} \eta^{p-1} |D\eta| |Du|^{p-1} (u^2+\varepsilon^2)^{\frac{\theta-1}{2}} |u|\, dxd\tau \\
&\le Lp \int_0^t \int_{B_k} \eta^{p-1} |D\eta| |Du|^{p-1} (|u|+\varepsilon)^{\theta}\, dxd\tau \\
&= Lp \Big(\frac{p}{p+\theta-1}\Big)^{p-1} \int_0^t \int_{B_k} \eta^{p-1} |D\eta| \big|D(|u|+\varepsilon)^{\frac{p+\theta-1}{p}}\big|^{p-1} (|u|+\varepsilon)^{\frac{p+\theta-1}{p}}\, dxd\tau \\
&\le \frac{\nu\theta}{4} \Big(\frac{p}{p+\theta-1}\Big)^p \int_0^t \int_{B_k} \eta^p \big|D(|u|+\varepsilon)^{\frac{p+\theta-1}{p}}\big|^p\, dxd\tau \\
&\quad + \boldsymbol{\gamma} \int_0^t \int_{B_k} |D\eta|^p (|u|+\varepsilon)^{p+\theta-1}\, dxd\tau,
\end{aligned}
$$

where $\boldsymbol{\gamma} = \boldsymbol{\gamma}(p, \nu, L, \theta)$. Combining the last two estimates yields

$$\mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon \geq \frac{\nu\theta}{4}\Big(\frac{p}{p+\theta-1}\Big)^p \int_0^t \int_{\mathbb{R}^N} \eta^p \big|D(|u|+\varepsilon)^{\frac{p+\theta-1}{p}}\big|^p \, dxd\tau \\ - \boldsymbol{\gamma} \int_0^t \int_{B_k} |D\eta|^p (|u|+\varepsilon)^{p+\theta-1} \, dxd\tau.$$

Therefore, by Fatou's Lemma we find that

$$\liminf_{\varepsilon \downarrow 0} \big(\mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon\big) \geq \frac{\nu\theta}{4}\Big(\frac{p}{p+\theta-1}\Big)^p \int_0^t \int_{\mathbb{R}^N} \eta^p \big|D|u|^{\frac{p+\theta-1}{p}}\big|^p \, dxd\tau \\ - \boldsymbol{\gamma} \int_0^t \int_{\mathbb{R}^N} |D\eta|^p |u|^{p+\theta-1} \, dxd\tau.$$

Inserting the preceding estimates into (2.1), and passing to the limit $\varepsilon \downarrow 0$, we obtain

$$\int_{\mathbb{R}^N \times \{t\}} \eta^p |u|^{1+\theta} \, dx + \int_0^t \int_{\mathbb{R}^N} \eta^p \big|D|u|^{\frac{p+\theta-1}{p}}\big|^p \, dxd\tau \\ \leq \boldsymbol{\gamma} \int_0^t \int_{\mathbb{R}^N} |D\eta|^p |u|^{p+\theta-1} \, dxd\tau.$$

Since $t > 0$ was arbitrary, we would obtain the same estimate for any $\tau \in (0, t)$ instead of t. Therefore, we can take the supremum over $\tau \in (0, t)$ in the first term on the left-hand side, i.e. we have

testing

$$\sup_{\tau \in (0,t)} \int_{\mathbb{R}^N \times \{\tau\}} \eta^p |u|^{1+\theta} \, dx + \int_0^t \int_{\mathbb{R}^N} \eta^p \big|D|u|^{\frac{p+\theta-1}{p}}\big|^p \, dxd\tau \\ \leq \boldsymbol{\gamma} \int_0^t \int_{\mathbb{R}^N} |D\eta|^p |u|^{p+\theta-1} \, dxd\tau, \tag{3.9}$$

for a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(p, \nu, L, \theta)$.

Now, we let $r \geq R_o$ to be chosen later and define

$$r_i := 2r - \frac{r}{2^i} \quad \text{and} \quad s_i := \frac{r_i + r_{i+1}}{2}$$

for $i \in \mathbb{N}_0$ and choose $\eta_i \in C^1(\mathbb{R}^N)$ such that $0 \leq \eta_i \leq 1$, $\eta_i \equiv 0$ in B_{r_i}, $\eta_i \equiv 1$ in $\mathbb{R}^N \setminus B_{s_i}$ and $\|D\eta_i\|_\infty \leq 2^{i+3}/r$. Choosing $\eta = \eta_i$ in (3.9) we find that

est-3

$$\sup_{\tau \in (0,t)} \int_{\mathbb{R}^N \setminus B_{s_i} \times \{\tau\}} |u|^{1+\theta} \, dx + \int_0^t \int_{\mathbb{R}^N \setminus B_{s_i}} \big|D|u|^{\frac{p+\theta-1}{p}}\big|^p \, dxd\tau \\ \leq \frac{\boldsymbol{\gamma} 2^{ip}}{r^p} \int_0^t \int_{\mathbb{R}^N \setminus B_{r_i}} |u|^{p+\theta-1} \, dxd\tau. \tag{3.10}$$

Next, we choose a second cut-off function $\tilde\eta_i \in C^1(\mathbb{R}^N)$ satisfying $0 \leq \tilde\eta_i \leq 1$, $\tilde\eta_i \equiv 0$ in B_{s_i}, $\tilde\eta_i \equiv 1$ in $\mathbb{R}^N \setminus B_{r_{i+1}}$ and $\|D\tilde\eta_i\|_\infty \leq 2^{i+3}/r$. Setting

$$v_i := \tilde\eta_i |u|^{\frac{p+\theta-1}{p}} \quad \text{and} \quad \sigma := \frac{p(1+\theta)}{p+\theta-1} \in (1, p),$$

and by using (3.10) we get

$$\sup_{\tau \in (0,t)} \int_{\mathbb{R}^N \times \{\tau\}} v_i^\sigma \, dx + \int_0^t \int_{\mathbb{R}^N} |Dv_i|^p \, dxd\tau \\ \leq \sup_{\tau \in (0,t)} \int_{\mathbb{R}^N \times \{\tau\}} \tilde\eta_i^\sigma |u|^{1+\theta} \, dx \\ + \int_0^t \int_{\mathbb{R}^N} \Big[\tilde\eta_i^p \big|D|u|^{\frac{p+\theta-1}{p}}\big|^p + |D\tilde\eta_i|^p |u|^{p+\theta-1}\Big] \, dxd\tau$$

$$
\begin{aligned}
&\le \sup_{\tau\in(0,t)} \int_{\mathbb{R}^N\setminus B_{s_i}\times\{\tau\}} |u|^{1+\theta}\, dx \\
&\qquad + \int_0^t \int_{\mathbb{R}^N\setminus B_{s_i}} \Big[\big|D|u|^{\frac{p+\theta-1}{p}}\big|^p + \tfrac{2^{ip}}{r^p}|u|^{p+\theta-1} \Big]\, dxd\tau \\
&\le \frac{\gamma 2^{ip}}{r^p} \int_0^t \int_{\mathbb{R}^N\setminus B_{r_i}} |u|^{p+\theta-1}\, dxd\tau,
\end{aligned}
$$

where in the last inequality we used (3.10). Taking into account that $v_{i-1} \equiv |u|^{\frac{p+\theta-1}{p}}$ on $\mathbb{R}^N \setminus B_{r_i} \times (0,t)$, this shows that

est-vi

$$
\sup_{\tau\in(0,t)} \int_{\mathbb{R}^N\times\{\tau\}} v_i^\sigma\, dx + \int_0^t \int_{\mathbb{R}^N} |Dv_i|^p\, dxd\tau \le \frac{\gamma 2^{ip}}{r^p} \int_0^t \int_{\mathbb{R}^N} v_{i-1}^p\, dxd\tau, \tag{3.11}
$$

where $\gamma = \gamma(p,\nu,L,\theta)$. Now, we define

$$
\mathbf{k}_i := \int_0^t \int_{\mathbb{R}^N} v_i^p\, dxd\tau.
$$

By Gagliardo-Nirenberg's inequality from Lemma 2.2 applied with $q = p$ and

$$
\vartheta := \frac{N(p-2)}{N(p-2)+p(1+\theta)} \in (0,1)
$$

and estimate (3.11), we obtain

$$
\begin{aligned}
\mathbf{k}_i &\le \gamma \int_0^t \bigg(\int_{\mathbb{R}^N} |v_i|^\sigma\, dx \bigg)^{\frac{(1-\vartheta)p}{\sigma}} \bigg(\int_{\mathbb{R}^N} |Dv_i|^p\, dx \bigg)^{\vartheta} dt \\
&\le \gamma \sup_{\tau\in(0,t)} \bigg(\int_{\mathbb{R}^N\times\{\tau\}} |v_i|^\sigma\, dx \bigg)^{\frac{(1-\vartheta)p}{\sigma}} \int_0^t \bigg(\int_{\mathbb{R}^N} |Dv_i|^p\, dx \bigg)^{\vartheta} dt \\
&\le \gamma\, t^{1-\vartheta} \sup_{\tau\in(0,t)} \bigg(\int_{\mathbb{R}^N\times\{\tau\}} |v_i|^\sigma\, dx \bigg)^{\frac{(1-\vartheta)p}{\sigma}} \bigg(\int_0^t \int_{\mathbb{R}^N} |Dv_i|^p\, dxdt \bigg)^{\vartheta} \\
&\le \gamma\, t^{1-\vartheta} \bigg(\frac{2^{ip}}{r^p}\, \mathbf{k}_{i-1} \bigg)^{\vartheta + \frac{(1-\vartheta)p}{\sigma}} = \gamma\, t^{1-\vartheta} \bigg(\frac{2^{ip}}{r^p}\, \mathbf{k}_{i-1} \bigg)^{1 + \frac{(1-\vartheta)(p-\sigma)}{\sigma}},
\end{aligned}
$$

where $\gamma = \gamma(N,p,\nu,L,\theta)$. Applying Lemma 2.3 with

$$
\alpha := \frac{(1-\vartheta)(p-\sigma)}{\sigma}
$$

and

$$
B = 2^{p(1+\frac{(1-\vartheta)(p-\sigma)}{\sigma})} = 2^{p(1+\alpha)},
$$

and

$$
C = \gamma\, t^{1-\vartheta} r^{-p(1+\frac{(1-\vartheta)(p-\sigma)}{\sigma})} = \gamma\, t^{1-\vartheta} r^{-p(1+\alpha)},
$$

we find that $\lim_{i\to\infty} \mathbf{k}_i = 0$, provided that

DG-cond

$$
\mathbf{k}_o \le \gamma^{-\frac{1}{\alpha}} 2^{-\frac{p(1+\alpha)}{\alpha^2}} t^{-\frac{1-\vartheta}{\alpha}} r^{\frac{p(1+\alpha)}{\alpha}} =: \gamma_1 t^{-\frac{1-\vartheta}{\alpha}} r^{\frac{p(1+\alpha)}{\alpha}}, \tag{3.12}
$$

with a constant $\gamma_1 = \gamma_1(N,p,\nu,L,\theta)$. On the other hand, by Remark 3.2 and Theorem 3.3 we have

$$
\begin{aligned}
\mathbf{k}_o &\le \int_0^t \int_{\mathbb{R}^N} |u|^{p+\theta-1}\, dxd\tau \\
&\le \int_0^t \|u(\cdot,\tau)\|_{L^\infty(\mathbb{R}^N)}^{p+\theta-2} \int_{\mathbb{R}^N} |u(\cdot,\tau)|\, dxd\tau \\
&= \|u_o\|_{L^1(\mathbb{R}^N)} \int_0^t \|u(\cdot,\tau)\|_{L^\infty(\mathbb{R}^N)}^{p+\theta-2}\, d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le \gamma \|u_o\|_{L^1(\mathbb{R}^N)}^{1+\frac{p(p+\theta-2)}{\beta}} \int_0^t \tau^{-\frac{N(p+\theta-2)}{\beta}}\, d\tau \\
&\le \gamma_2 \|u_o\|_{L^1(\mathbb{R}^N)}^{1+\frac{p(p+\theta-2)}{\beta}} t^{1-\frac{N(p+\theta-2)}{\beta}}.
\end{aligned}
$$

Note that $\frac{N(p+\theta-2)}{\beta} < 1$, since $\theta < \frac{p}{N}$. Moreover γ_2 is a constant depending only on N, p, ν, L. Therefore, condition (3.12) is implied by

$$
\gamma_2 \|u_o\|_{L^1(\mathbb{R}^N)}^{1+\frac{p(p+\theta-2)}{\beta}} t^{1-\frac{N(p+\theta-2)}{\beta}} \le \gamma_1 t^{-\frac{1-\vartheta}{\alpha}} r^{\frac{p(1+\alpha)}{\alpha}},
$$

which is the same as

$$
r^{\frac{p(1+\alpha)}{\alpha}} \ge \frac{\gamma_2}{\gamma_1} \|u_o\|_{L^1(\mathbb{R}^N)}^{1+\frac{p(p+\theta-2)}{\beta}} t^{1+\frac{1-\vartheta}{\alpha}-\frac{N(p+\theta-2)}{\beta}}.
$$

At this point we compute that

$$
\frac{\alpha}{p(1+\alpha)} \left[1 + \frac{p(p+\theta-2)}{\beta}\right] = \frac{p-2}{\beta}
$$

and

$$
\frac{\alpha}{p(1+\alpha)} \left[1 + \frac{1-\vartheta}{\alpha} - \frac{N(p+\theta-2)}{\beta}\right] = \frac{1}{\beta}.
$$

Therefore, the last inequality turns into

r

$$
r \ge \gamma\, t^{\frac{1}{\beta}} \|u_o\|_{L^1(\mathbb{R}^N)}^{\frac{p-2}{\beta}}, \tag{3.13}
$$

for a constant $\gamma = (\frac{\gamma_2}{\gamma_1})^{\frac{\alpha}{p(1+\alpha)}}$ depending only on N, p, ν, L, θ. We now fix $\theta \in (0, \min\{1, \frac{N}{p}\})$; for instance, we can take $\theta = \min\{\frac{1}{2}, \frac{N}{2p}\}$. This fixes γ in dependence on N, p, ν, L. Since $r \ge R_o$ was arbitrary, we now choose

$$
r = R_o + \gamma\, t^{\frac{1}{\beta}} \|u_o\|_{L^1(\mathbb{R}^N)}^{\frac{p-2}{\beta}}.
$$

By the preceding arguments, this ensures that (3.12) is satisfied. Therefore, we have that $\lim_{i\to\infty} \mathbf{k}_i = 0$. In turn, this implies that $u \equiv 0$ on $(\mathbb{R}^N \setminus B_{2r}) \times [0, t]$. This finishes the proof of the theorem. □

Remark 3.5. The explicit solution of the p-Laplace equation shows that the estimates obtained in this section are sharp. □

sec:est-below

4. Estimates from below

In this section we derive estimates from below as stated in Theorem 1.2 to the solution of (1.1) under the assumptions (1.2) and (1.3), by using the Barenblatt solution $\mathcal{B}_p$ of the p-Laplacian.

Proof of Theorem 1.2. Throughout this proof we denote by c_1 and c_2 the corresponding constants from Theorem 2.5 depending only on N, p, ν, L. Let $P_o = (x_o, t_o) \in \mathbb{R}^N \times (0, \infty)$ such that $u(P_o) > 0$. By a change of variable we define the function $w \colon \mathbb{R}^N \times (-1, \infty) \to \mathbb{R}$ by

$$
w(x,t) := \frac{u\big(x_o + t_o^{\frac{1}{p}} u(P_o)^{\frac{p-2}{p}} x, t_o(t+1)\big)}{u(P_o)}.
$$

Straightforward computations show that w is nonnegative, $w(0,0) = 1$, $w \in C^0((-1,\infty); L^2(\mathbb{R}^N)) \cap L^p_{\rm loc}(-1,\infty; W^{1,p}(\mathbb{R}^N))$ and that w is a solution of the following parabolic equation

$$
\partial_t w = \operatorname{div} \tilde{\mathbf{A}}(x,t,w,Dw), \quad \text{in } \mathbb{R}^N \times [-1, \infty),
$$

where $\tilde{\mathbf{A}}$ is defined by

$$\tilde{\mathbf{A}}(x,t,w,\xi) := \Big(\frac{t_o}{u(P_o)^2}\Big)^{\frac{p-1}{p}} \mathbf{A}\Big(x_o + t_o^{\frac{1}{p}} u(P_o)^{\frac{p-2}{p}} x, t_o(t+1), u(P_o)w, \Big(\frac{u(P_o)^2}{t_o}\Big)^{\frac{1}{p}} \xi\Big)$$

for $(x,t) \in \mathbb{R}^N \times (-1,\infty)$, $w \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$. It can be easily verified that $\tilde{\mathbf{A}}$ satisfies assumptions (1.2) with the same parameters ν and L, while (1.3) is satisfied with $Lu(P_o)\max\{(t_o/u(P_o)^2)^{(p-1)/p}, 1\}$ instead of L.

Applying the intrinsic Harnack inequality from Theorem 2.5 to the function w in the point $(0,0)$ and taking into account $w(0,0) = 1$ we infer that

Harnack-1

$$\inf_{B_\varrho} w\big(\cdot, c_2^{p-2}\varrho^p\big) \ge \tfrac{1}{c_1}, \qquad \text{for any } \varrho > 0. \tag{4.1}$$

In particular, for $R_o = R_o(N,p,\nu,L) := (\frac{1}{2} c_2^{2-p})^{\frac{1}{p}}$ this implies

$$w(x, \tfrac{1}{2}) > \tfrac{1}{c_1}, \qquad \text{for any } x \in B_{R_o}.$$

We now let

$$v_o := \begin{cases} \frac{1}{c_1} & \text{in } B_{R_o}, \\ 0 & \text{in } \mathbb{R}^N \setminus B_{R_o}. \end{cases}$$

and consider the initial value problem

equation-v

$$\begin{cases} \partial_t v = \operatorname{div} \tilde{\mathbf{A}}(x,t,v,Dv), & \text{in } \mathbb{R}^N \times [\frac{1}{2}, \infty), \\ v(\cdot, \frac{1}{2}) = v_o & \text{in } \mathbb{R}^N. \end{cases} \tag{4.2}$$

Due to assumptions (1.2) and (1.3) there exists a nonnegative solution $v \in C^0([\frac{1}{2},\infty); L^2(\mathbb{R}^N)) \cap L^p_{\rm loc}(\frac{1}{2},\infty; W^{1,p}(\mathbb{R}^N))$ to (4.2) with $v \le w$ in $\mathbb{R}^N \times [\frac{1}{2},\infty)$. By virtue of Remark 3.2 the initial mass is preserved, i.e. for any $t \ge \frac{1}{2}$ we have that

$$\|v(t)\|_{L^1(\mathbb{R}^N)} = \|v_o\|_{L^1(\mathbb{R}^N)} = \tfrac{\omega_N R_o^N}{c_1}.$$

Moreover, from Theorem 3.4 we know that for any $t \ge \frac{1}{2}$ there holds

$$\operatorname{spt} v(\cdot,t) \subset B_{R(t)} \quad \text{with } R(t) := 2R_o + \boldsymbol{\gamma}_1 (t - \tfrac{1}{2})^{\frac{1}{\beta}} \big(\tfrac{\omega_N R_o^N}{c_1}\big)^{\frac{p-2}{\beta}},$$

for a constant $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_1(N,p,\nu,L)$ and $\beta := N(p-2)+p$. Therefore, for any $t \ge \frac{1}{2}$ there exists at least one point $x_t \in B_{R(t)}$, such that

$$v(x_t,t) \ge \frac{1}{c_1}\Big(\frac{R_o}{R(t)}\Big)^N \ge \boldsymbol{\gamma} t^{-\frac{N}{\beta}},$$

with a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N,p,\nu,L) > 0$. Since $w \ge v$ we also have that

$$w(x_t,t) \ge \boldsymbol{\gamma} t^{-\frac{N}{\beta}}, \qquad \text{for any } t \ge \tfrac{1}{2}.$$

For $t \ge \frac{1}{2}$ we now apply the Harnack inequality from Theorem 2.5 to the function w in the point (x_t,t) to get

low-point

$$\boldsymbol{\gamma} t^{-\frac{N}{\beta}} \le w(x_t,t) \le c_1 \inf_{B_\varrho(x_t)} w\Big(\cdot, t + \big(\tfrac{c_2}{w(x_t,t)}\big)^{p-2}\varrho^p\Big), \quad \text{for any } \varrho > 0. \tag{4.3}$$

In particular, for $\tilde{\varrho}(t) := [\frac{1}{2} t (\frac{w(x_t,t)}{c_2})^{p-2}]^{\frac{1}{p}}$ this implies

$$\boldsymbol{\gamma} t^{-\frac{N}{\beta}} \le c_1 \inf_{B_{\tilde{\varrho}(t)}(x_t)} w(\cdot, \tfrac{3}{2}t).$$

In virtue of (4.3) we have $\tilde{\varrho}(t) \ge \varrho(t) := \boldsymbol{\gamma}_2 t^{\frac{1}{\beta}}$, for a constant $\boldsymbol{\gamma}_2$ depending only on N,p,ν,L. Thus, the last inequality implies

$$w(x, \tfrac{3}{2}t) \ge \boldsymbol{\gamma}_3 t^{-\frac{N}{\beta}}, \qquad \text{for any } t \ge \tfrac{1}{2} \text{ and } x \in B_{\varrho(t)}(x_t),$$

with a constant $\boldsymbol{\gamma}_3 = \boldsymbol{\gamma}_3(N, p, \nu, L) > 0$. Letting

$$M := \frac{8R_o + 2\boldsymbol{\gamma}_1\big(\frac{\omega_N R_o^N}{c_1}\big)^{\frac{p-2}{\beta}}}{\boldsymbol{\gamma}_2},$$

we observe from the definitions of $R(t)$ and $\varrho(t)$ that $M\varrho(t) \geq 2R(t)$ for any $t \geq \frac{1}{2}$, so that $B_{\varrho(t)}(x_t) \supset B_{R(t)}$. Applying Lemma 2.4 to the function $(x, \tau) \mapsto w(x - x_t, \frac{3}{2}t + \tau)$ with (ϱ, δ, M) replaced by $(\varrho(t), \boldsymbol{\gamma}_3 t^{-\frac{N}{\beta}}, M)$, we infer that

$$w\big(x, \tfrac{3}{2}t + k_o \boldsymbol{\gamma}_2^p \boldsymbol{\gamma}_3^{2-p} t\big) \geq \varepsilon_o \boldsymbol{\gamma}_3 t^{-\frac{N}{\beta}}, \qquad \text{for any } t \geq \tfrac{1}{2} \text{ and } x \in B_{R(t)},$$

where k_o and ε_o denote the constants from Lemma 2.4 depending only on N, p, ν, L. After a change of variable we can thus deduce the following result: there exist constants $\boldsymbol{\gamma}_4, \boldsymbol{\gamma}_5 > 0$ and $\boldsymbol{\gamma}_6 \geq \frac{3}{4}$ depending only on N, p, ν, L such that

$$w(x, t) \geq \boldsymbol{\gamma}_4 t^{-\frac{N}{\beta}}, \qquad \text{for any } t \geq \boldsymbol{\gamma}_6 \text{ and } x \in B_{\widetilde{R}(t)},$$

where $\widetilde{R}(t) := \boldsymbol{\gamma}_5 t^{\frac{1}{\beta}}$. This proves the lower estimate for w we were looking for, for any $t \geq \boldsymbol{\gamma}_6$.

Our next aim is to obtain a similar estimate for times $t < \boldsymbol{\gamma}_6$ which are not too close to 0. First, we observe that (4.1) can be rewritten in the form

$$w(x, t) \geq \tfrac{1}{c_1}, \qquad \text{for any } t > 0 \text{ and } x \in B_{(c_2^{2-p} t)^{1/p}}.$$

Therefore, for fixed $\varepsilon > 0$, we get

$$w(x, t) \geq \tfrac{1}{c_1}(t+1)^{-\frac{N}{\beta}}, \qquad \text{for any } t \in [\varepsilon, \boldsymbol{\gamma}_6) \text{ and } x \in B_{\hat{R}(t)}.$$

where $\hat{R}(t) := c_2^{(2-p)/p} \varepsilon^{\frac{1}{p} - \frac{1}{\beta}} t^{\frac{1}{\beta}}$.

Together, we have proved that for any $\varepsilon > 0$ there exist constants $\boldsymbol{\gamma}_7 = \boldsymbol{\gamma}_7(N, p, \nu, L) > 0$ and $\boldsymbol{\gamma}_8 = \boldsymbol{\gamma}_8(N, p, \nu, L, \varepsilon) > 0$ such that

$$w(x, t) \geq \boldsymbol{\gamma}_7 \mathcal{B}_p(x, t+1), \qquad \text{for any } t \geq \varepsilon \text{ and } x \in B_{\tilde{r}(t)},$$

where $\tilde{r}(t) = \boldsymbol{\gamma}_8 t^{\frac{1}{\beta}}$ and $\mathcal{B}_p(x, t)$ is the Barenblatt solution of the p-Laplacian defined in (1.5). Rescaling back from w to u, we find that

$$u(x, t) = u(P_o) w\left(\frac{x - x_o}{t_o^{\frac{1}{p}} u(P_o)^{\frac{p-2}{p}}}, \frac{t - t_o}{t_o}\right) \geq \boldsymbol{\gamma}_7 u(P_o) \mathcal{B}_p\left(\frac{x - x_o}{t_o^{\frac{1}{p}} u(P_o)^{\frac{p-2}{p}}}, \frac{t}{t_o}\right)$$

holds true for any $t \geq t_o(1 + \varepsilon)$ and $x \in B_{r(t)}(x_o)$ with $r(t) := \boldsymbol{\gamma}_8 u(P_o)^{\frac{p-2}{p}} t_o^{\frac{1}{p}} [\frac{t - t_o}{t_o}]^{\frac{1}{\beta}}$. This finishes the proof of Theorem 1.2. □

sec:existence

5. Existence of fundamental solutions

In this section we will use the L^∞-estimates and the estimates on the support of the solution in order to prove the existence of fundamental solutions claimed in Theorem 1.6. However, the uniquesness of such a fundamental solution is an extremely more delicate question and we will not go into this issue here (see, for instance [31] for a deeper discussion on this topic).

Proof of Theorem 1.6. For $k \in \mathbb{N}$ we define the function $\psi_k \colon \mathbb{R}^N \to \mathbb{R}$ by

$$\psi_k(x) := \begin{cases} \frac{k^N}{\omega_N}, & \text{if } |x| \leq \frac{1}{k}, \\ 0, & \text{if } |x| > \frac{1}{k}, \end{cases}$$

and note that $\|\psi_k\|_{L^1(\mathbb{R}^N)} = 1$ and $\psi_k \to \delta_0$ as $k \to \infty$ in the sense of measures. By $u_k \in C^0([0,\infty); L^2(\mathbb{R}^N)) \cap L^p_{\rm loc}(0,\infty; W^{1,p}(\mathbb{R}^N))$ we denote the solution of the initial value problem

equationapp-
$$\begin{cases} \partial_t u_k = \operatorname{div} \mathbf{A}\big(x,t,u_k,Du_k\big), & \text{in } \mathbb{R}^N \times (0,\infty), \\ u_k(\cdot,0) = \psi_k, & \text{in } \mathbb{R}^N. \end{cases} \tag{5.1}$$

Note that the existence of u_k is ensured by assumptions (1.2) and (1.3) and that u_k is non-negative. By the conservation of mass from Remark 3.2, we know that $\|u_k(t)\|_{L^1(\mathbb{R}^N)} = \|\psi_k\|_{L^1(\mathbb{R}^N)} = 1$ for any $t > 0$. Moreover, Theorem 3.3 ensures that

exist-bound
$$\|u_k(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le \boldsymbol{\gamma} t^{-\frac{N}{\beta}}, \quad \text{for any } t > 0, \tag{5.2}$$

with $\beta = N(p-2)+p$ and a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N,p,\nu,L)$. Furthermore, from Theorem 3.4 we obtain the following bound for the support of u_k:

$$\operatorname{spt} u_k(t) \subset B_{R_k(t)}, \quad \text{with } R_k(t) := \tfrac{1}{k} + \boldsymbol{\gamma}_1 t^{\frac{1}{\beta}},$$

for any $t > 0$ and a constant $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_1(N,p,\nu,L)$. Using u_k as testing function in the weak form (2.1) of $(5.1)_1$ we infer the following energy estimate for u_k:

energy
$$\frac{1}{2}\int_{\mathbb{R}^N\times\{t_2\}} |u_k|^2\,dx + \nu \int_{t_1}^{t_2}\int_{\mathbb{R}^N} |Du_k|^p\,dxdt \le \frac{1}{2}\int_{\mathbb{R}^N\times\{t_1\}} |u_k|^2\,dx, \tag{5.3}$$

for any $0 < t_1 < t_2 < \infty$.

We now let $0 < \varepsilon < T < \infty$ and consider the sequence $\{u_k\}_{k\ge k_o}^\infty$, where $k_o \in \mathbb{N}$ is chosen such that $k_o > \frac{1}{\varepsilon}$. By (5.2), the sequence $\{u_k\}_{k\ge k_o}^\infty$ is uniformly bounded on $\mathbb{R}^N \times [\varepsilon, T]$. Moreover, classical regularity results, cf. [13, §III.1, Theorem 1.1] ensure that the sequence $\{u_k\}_{k\ge k_0}^\infty$ is also equi-Hölder continuous on $\mathbb{R}^N \times [\varepsilon,T]$. By Ascoli-Arzelá's theorem there exists a subsequence $\{u_{k_i}\}_{k_i\ge k_0}^\infty$ converging uniformly to a Hölder-continuous function $u\colon \mathbb{R}^N \times [\varepsilon,T] \to \mathbb{R}$ with $\|u(t)\|_{L^1(\mathbb{R}^N)} = 1$ and $\operatorname{spt} u(t) \subset B_{R(t)}$ for any $t \in [\varepsilon,T]$, where $R(t) := \boldsymbol{\gamma}_1 t^{\frac{1}{\beta}}$. By (5.3), the sequence $\{u_k\}_{k\ge k_o}^\infty$ is bounded in $L^p(\varepsilon,T;W^{1,p}(\mathbb{R}^N))$ and therefore we also have $u \in L^p(\varepsilon,T;W^{1,p}(\mathbb{R}^N))$ and $Du_k \rightharpoonup Du$ weakly in $L^p(\mathbb{R}^N\times(\varepsilon,T))$. Since ε and T were arbitrary, we conclude the existence of a Hölder-continuous function $u\colon \mathbb{R}^N\times(0,\infty)\to\mathbb{R}$ with $u \in L^p_{\rm loc}(0,\infty;W^{1,p}(\mathbb{R}^N))$, $\|u(t)\|_{L^1(\mathbb{R}^N)} = 1$ and $\operatorname{spt} u(t) \subset B_{R(t)}$ for any $t \in (0,\infty)$. This ensures that u satisfies properties i), iii) and iv) of Definition 1.5. It remains to prove that u satisfies propertiy ii). Therefore, we have to pass to the limit $k\to\infty$ in the weak form of equation $(5.1)_1$. Since the mapping $(u,\xi) \mapsto \mathbf{A}(x,t,u,\xi)$ is continuous for a.e. $(x,t) \in \mathbb{R}^N \times (0,\infty)$ and the vector field $\mathbf{A}$ satisfies assumptions (1.3), this can be achieved with the help of Minty's Lemma [26] (see also [8, Lemma 2.7] for a parabolic version). This proves that u is the desired fundamental solution to (1.8) in the sense of Definition 1.5. □

sec:pot-est

6. Potential estimates

The aim of this section is to prove the estimates from above and from below for solutions of (1.1) with the Dirac mass in $\mathbb{R}^N$ as initial condition as claimed in Theorem 1.7, in the same spirit as the estimates proved in [31] for singular parabolic equations.

Proof of Theorem 1.7. Let u be a fundamental solution of (1.8). The proof of the upper and lower estimates is now divided into three steps.

Step 1. Estimates on the support of the fundamental solution. For $\varrho > 0$ and $n \in \mathbb{N}$ we define the function $\psi_{n,\varrho}\colon \mathbb{R}^N \to \mathbb{R}$ by

$$\psi_{n,\varrho}(x) := \begin{cases} u(x,\frac{1}{n}), & \text{if } |x| \le \varrho, \\ 0, & \text{if } |x| > \varrho. \end{cases}$$

By $u_{n,\varrho} \in C^0([\frac{1}{n},\infty); L^2(\mathbb{R}^N)) \cap L^p_{\mathrm{loc}}(\frac{1}{n},\infty; W^{1,p}(\mathbb{R}^N))$ we denote the solution of the initial value problem

equationapp

$$\begin{cases} \partial_t u_{n,\varrho} = \operatorname{div} \mathbf{A}\big(x,t,u_{n,\varrho},Du_{n,\varrho}\big), & \text{in } \mathbb{R}^N \times (\frac{1}{n},\infty), \\ u_{n,\varrho}(\cdot,\frac{1}{n}) = \psi_{n,\varrho}, & \text{in } \mathbb{R}^N. \end{cases} \tag{6.1}$$

Note that the existence of the solution $u_{n,\varrho}$ is ensured by assumptions (1.2) and (1.3). Moreover, $u_{n,\varrho}$ is nonnegative, and $u_{n,\varrho} \le u$ in $\mathbb{R}^N \times (\frac{1}{n},\infty)$ by the maximum principle. By the conservation of the mass from Remark 3.2 we know that for any $t \ge \frac{1}{n}$ there holds $\|u_{n,\varrho}(t)\|_{L^1(\mathbb{R}^N)} = \|\psi_{n,\varrho}\|_{L^1(\mathbb{R}^N)}$. Hence, by iii) in Definition 1.5 we conclude that

L1=1

$$\lim_{n\to\infty} \|u_{n,\varrho}(t)\|_{L^1(\mathbb{R}^N)} = 1, \quad \text{for any } t > 0. \tag{6.2}$$

From Theorem 3.3 we know that

fund-bound

$$\|u_{n,\varrho}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le \boldsymbol{\gamma}(t-\tfrac{1}{n})^{-\frac{N}{\beta}} \|\psi_{n,\varrho}\|^{\frac{p}{\beta}}_{L^1(\mathbb{R}^N)} \le \boldsymbol{\gamma}(t-\tfrac{1}{n})^{-\frac{N}{\beta}} \tag{6.3}$$

holds true for any $t > \frac{1}{n}$ with $\beta = N(p-2)+p$ and a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N,p,\nu,L)$. Moreover, Theorem 3.4 ensures that for any $t > \frac{1}{n}$ we have

$$\operatorname{spt} u_{n,\varrho}(t) \subset B_{R_{n,\varrho}(t)}, \quad \text{with } R_{n,\varrho}(t) := 2\varrho + \boldsymbol{\gamma}_1 t^{\frac{1}{\beta}} \|\psi_{n,\varrho}\|^{\frac{p-2}{\beta}}_{L^1(\mathbb{R}^N)},$$

and a constant $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_1(N,p,\nu,L)$. We note that, again by iii) in Definition 1.5, we have

$$R_{n,\varrho}(t) \to R_\varrho(t) \text{ as } n\to\infty, \text{ where } R_\varrho(t) := 2\varrho + \boldsymbol{\gamma}_1 t^{\frac{1}{\beta}}.$$

Now, let $0 < \varepsilon < T < \infty$ and consider the sequence $\{u_{n,\varrho}\}_{n\ge n_o}^\infty$, for $\varrho > 0$ fixed, where $n_o \in \mathbb{N}$ is chosen such that $n_o > \frac{1}{\varepsilon}$. Then, by (6.3) the sequence $\{u_{n,\varrho}\}_{n\ge n_o}^\infty$ is uniformly bounded on $\mathbb{R}^N \times [\varepsilon,T]$. Moreover, by classical regularity results, cf. [13, §III.1, Theorem 1.1], the sequence $\{u_{n,\varrho}\}_{n\ge n_o}^\infty$ is also equi-Hölder continuous on $\mathbb{R}^N \times [\varepsilon,T]$. By Ascoli-Arzelá's theorem there exists a subsequence $\{u_{n_k,\varrho}\}_{n_k\ge n_o}^\infty$ converging uniformly to a Hölder-continuous function $w\colon \mathbb{R}^N \times [\varepsilon,T] \to \mathbb{R}$ with $w \le u$ and $\operatorname{spt} w(t) \subset B_{R(t)}$ for any $t \in [\varepsilon,T]$. Moreover, from (6.2) we know that $\|w(t)\|_{L^1(\mathbb{R}^N)} = 1$ for any $t \in [\varepsilon,T]$. Since also u is Hölder-continuous on $\mathbb{R}^N \times [\varepsilon,T]$ with $\|u(t)\|_{L^1(\mathbb{R}^N)} = 1$ for any $t \in [\varepsilon,T]$, we conclude that $u = w$ in $\mathbb{R}^N \times [\varepsilon,T)$, and hence $\operatorname{spt} u(t) \subset B_{R(t)}$ for any $t \in [\varepsilon,T]$. As this estimate holds for any choice of ε and T, we get that $\operatorname{spt} u(t) \subset B_{R_\varrho(t)}$ for any $t > 0$. Finally, since the previous inclusion holds for any $\varrho > 0$, we conclude that

sptu

$$\operatorname{spt} u \subset B_{r(t)}, \quad \text{for any } t > 0, \text{ where } r(t) := \boldsymbol{\gamma}_1 t^{\frac{1}{\beta}}. \tag{6.4}$$

Step 2. Estimates from above. By the estimate on the support of u from Step 1 and by Theorem 3.3, we find that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le \boldsymbol{\gamma}\, t^{-\frac{N}{\beta}}, \quad \text{for any } t > 0,$$

and a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N,p,\nu,L)$. This proves the upper estimate $u \le \mathcal{B}_{p,M_2}$ for some constant $M_2 > 0$ depending only on N,p,ν,L.

Step 3. Estimates from below. Our aim in this final step is to prove a reverse type inequality. Therefore, let $t > 0$ and $a \in (0,1)$ be a constant to be fixed later. From (6.4) and the fact that $\int_{\mathbb{R}^N} u(x,at)\,dx = 1$ we conclude that there exists a point $x_o \in B_{r(t)}$ such that

pot-lower

$$u(x_o,at) \ge \omega_N \boldsymbol{\gamma}^N (at)^{-\frac{N}{\beta}}, \tag{6.5}$$

where $\boldsymbol{\gamma}$ is the same constant as in (6.4). Now, we apply the Harnack estimates from Theorem 2.5 with $\theta = \frac{c_2}{u(x_o,at)}$ and $\tilde\varrho = (\frac{at}{2\theta^{p-2}})^{1/p}$ to get that

$$\omega_N \boldsymbol{\gamma}^N (at)^{-\frac{N}{\beta}} \le c_1 \inf_{B_{\tilde\varrho}(x_o)} u\big(x,\tfrac{3}{2}at\big),$$

where c_1 and c_2 denote the constants from Theorem 2.5 depending only on N, p, ν, L. In particular, the choice of $\tilde{\varrho}$ and (6.5) impliy that $\tilde{\varrho} \geq \varrho := \boldsymbol{\gamma}_2 (at)^{\frac{1}{\beta}}$ for a constant $\boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_2(N,p,\nu,L)$. Next, we let $M := \frac{2\boldsymbol{\gamma}_1}{\boldsymbol{\gamma}_2}$, ensuring that $M\varrho \geq 2r(at)$ and hence $B_{r(at)} \subset B_{M\varrho}(x_o)$. We note that M depends only on N, p, ν, L. By ε_o and k_o we denote the corresponding constants from Lemma 2.4 depending on N, p, ν, L, M and by the dependencies of M mentioned before, ε_o and k_o depend only on the data N, p, ν, L. The application of Lemma 2.4 with this choice of M and $\delta := c_1^{-1} \alpha_n \boldsymbol{\gamma}^N (at)^{-\frac{N}{\beta}}$ yields that

$$u\big(x, \tfrac{3}{2}at + k_o \delta^{2-p} \varrho^p\big) \geq \varepsilon_o \delta, \qquad \text{for any } x \in B_{r(at)}.$$

We compute

$$\tfrac{3}{2}at + k_o \delta^{2-p}\varrho^p = \bigg[\tfrac{3}{2} + k_o \boldsymbol{\gamma}_2^p \Big(\frac{\alpha_n \boldsymbol{\gamma}^N}{c_1}\Big)^{2-p}\bigg] at.$$

Therefore, choosing

$$a := \bigg[\tfrac{3}{2} + k_o \boldsymbol{\gamma}_2^p \Big(\frac{\alpha_n \boldsymbol{\gamma}^N}{c_1}\Big)^{2-p}\bigg]^{-1} \in (0,1),$$

depending on N, p, ν, L, we find that

$$u(x,t) \geq \boldsymbol{\gamma} t^{-\frac{N}{\beta}}, \qquad \text{for any } x \in B_{r(at)},$$

with a constant $\boldsymbol{\gamma} = \boldsymbol{\gamma}(N,p,\nu,L)$. Noting that $B_{r(at)} = \boldsymbol{\gamma}_1 a^{\frac{1}{\beta}} t^{\frac{1}{\beta}}$, this proves the lower bound $u \geq \mathcal{B}_{p,M_1}$ for some constant $M_1 > 0$, depending only on N, p, ν, L. □

Remark 6.1. The previous estimates should be very useful to study the asymptotic behavior of the fundamental solution. For the prototype equation see [25, 36, 37, 38]. □

sec:final

7. Final considerations

We point out that the estimates derived for the p-Laplacian type equations hold also for the porous medium type equations. Let us consider the quasilinear degenerate porous medium type problem

porousmedium

$$\begin{cases} \partial_t u = \operatorname{div} \mathbf{A}(x,t,u,Du^m), & \text{in } \mathbb{R}^N \times (0,\infty), \\ w(\cdot,0) = \delta_0, & \text{in } \mathbb{R}^N, \end{cases} \tag{7.1}$$

where the the vector field $\mathbf{A}\colon \mathbb{R}^N \times (0,\infty) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function satisfies the following growth and ellipticity conditions:

ellipticporous

$$\begin{cases} \mathbf{A}(x,t,u,\xi)\cdot\xi \geq m\nu |u|^{m-1}|\xi|^2, \\ |\mathbf{A}(x,t,u,\xi)| \leq mL|u|^{m-1}|\xi|, \end{cases} \tag{7.2}$$

for almost all $(x,t) \in \mathbb{R}^N \times (0,\infty)$, all $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, for some constants $0 < \nu \leq L < \infty$ and with $m > 1$. Moreover, $\mathbf{A}$ is required to be monotone in the variable ξ and Lipschitz continuous in the variable $|u|^{m-1}u$, in the sense that

monotoneporous

$$\begin{cases} \big(\mathbf{A}(x,t,u,\xi_1) - \mathbf{A}(x,t,u,\xi_2)\big)\cdot(\xi_1 - \xi_2) \geq 0, \\ |\mathbf{A}(x,t,u_1,\xi) - \mathbf{A}(x,t,u_2,\xi)| \leq \Lambda \big||u_1|^{m-1}u_1 - |u_2|^{m-1}u_2\big|(1+|\xi|), \end{cases} \tag{7.3}$$

for some $\Lambda > 0$ and for almost all $(x,t) \in \mathbb{R}^N \times (0,\infty)$ and all $u, u_i \in \mathbb{R}$ and $\xi, \xi_i \in \mathbb{R}^N$, $i = 1,2$. The estimates derived in Theorems 1.2, 1.4, and 1.7, can be extended to the solutions of (7.1) under the assumptions (7.2) and (7.3), since solutions to (7.1) satisfy the main ingredients for our estimates, i.e. the expansion of positivity and the Harnack inequality; see [25] and the references therein.

Moreover, these estimates hold also for the Fokker-Planck equation. Actually, let us consider the following equation

EqFokkerPlanck

$$\partial_t u = \operatorname{div} \mathbf{A}(x,t,u,Du) + \operatorname{div}(x \cdot u), \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{7.4}$$

where the operator $\mathbf{A}$ satisfies conditions (1.2) and (1.3). As proved by Carrillo-Toscani [10] (see also [37] and references therein), equation (7.4) can be transformed to equation (7.1) by the change of variables

$$w(x,t) = \alpha(t)^N u(\alpha(t)x, \beta(t)),$$

where $\alpha(t) = e^t$ and $\beta(t) = \frac{1}{k}(e^{kt} - 1)$.

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