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Existence, uniqueness and concentration for a system of PDEs involving the Laplace-Beltrami operator

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Abstract

In this paper we derive a model for heat diffusion in a composite medium in which the different components are separated by thermally active interfaces. The previous result is obtained via a concentrated capacity procedure and leads to a non-stantard system of PDEs involving a Laplace-Beltrami operator acting on the interface. For such a system well-posedness is proved using contraction mapping and abstract parabolic problems theory. Finally, the exponential convergence (in time) of the solutions of our system to a steady state is proved.

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1 Introduction

In recent years there has been ongoing researches on new composite materials displaying more efficient thermal dispersion properties. This is required in many branches of technical applications in which some specific material characteristics, such as ductility, must be ensured together with an efficient heat dispersion. For example, that is the case in the encapsulation of electronic devices ([12, 18, 22, 27]). Typically, in this situation, the desired material properties are obtained by embedding some highly conductive nanoparticles in a rubber or polymer foam. In such a way the composite material retains the ductility of the host medium (say a rubber) and, hopefully, has an increased heat conductivity (because of the highly conductive inclusions). It turns out that, indeed, this is the case and the presence of the inclusions can increase fivefold conductivity. A similar situation is encountered in some newly developed engine coolants in which conductive nanoparticles are added to the fluid.

In some situations the embedded nanoparticles are covered by a membrane separating them from the surrounding medium ([22]). Such a membrane can be build up from the very beginning in the manufacture of the nanoparticles or, maybe, it can be a surfactant surrounding the particle which is added to avoid clotting.

Motivated by the previous considerations we have decided to investigate the overall behaviour of a composite material made of a hosting medium filled with nanoparticles enclosed in a membrane with the aim of determining its effects on the resulting conductivity. This is done in [9, 10] by means of a homogenization technique when the membrane is an $(N-1)$-dimensional surface. This approach is motivated by the obvious smallness of the thickness of the interfaces and leads to a description of their thermal behaviour in which the temperature is continuous across the membrane and solves a heat equation involving the Laplace-Beltrami operator having, as a source term, the jump of the heat flux.

In this regard, it remains to prove well-posedness for the system of equations modelling heat conduction in the composite materials with interfaces. At the same time, it is fundamental to provide a theoretical motivation of this model regarded as a concentration limit of a thick membrane. This paper is devoted to the proof of such results.

The main and most interesting feature of the well-posedness theorem of this paper is due to the fact that the system of PDE's involves a Laplace-Beltrami equation coupled with the heat equation in the surrounding media via the jump of the heat flux. A similar coupling on the boundary of the domain has been studied by many authors. Among the very wide literature on this topic, relevant papers are, for instance, [11, 20, 23, 26] (see also the references therein) for abstract parabolic equations, and [13, 14, 16, 24] for the Cahn-Hilliard or Allen-Cahn equation. On the other hand, our case differs from the previous ones since the coupling occurs on interfaces. Indeed, in our problem an evolutive equations is satisfied on both sides of the interface where the dynamical condition is assigned. The techniques of the previous papers are likely applicable to our case, nevertheless we believe that the merit of our approach lies in its simplicity which takes advantage of the linear structure of our system of equations. It is worthwhile to add that such a system of equations is interesting in itself and it calls for a careful tuning of the existence theorem, via contraction mapping and abstract parabolic problems theory as done, for instance, in [7].

The core part of the paper is devoted to the rigorous derivation of the phenomenological model used in our previous papers [9, 10]. Obviously, in this framework, the relevant physical quantities must be rescaled in a way such that they are conserved in the limit $\eta \to 0$, η being the thickness of the membrane. Our choice is to let the specific conductivity and capacity of the interface to blow up as η^{-1}. This is essential to allow thermal diffusion "along" the concentrated membrane, as required by the fact that in the no-thickness interface model we have a Laplace-Beltrami equation on the membrane.

In this framework, it is crucial to quote [20, 26], in which similar concentration results are proved, for a wide class of general nonlinear parabolic problems. However, in those papers, after concentration, only a domain Ω remains on whose boundary the "dynamical condition" is assigned; while, in this paper, we find, also in the limit, two domains separated by an active interface, where the two fluxes are coupled via a Laplace-Beltrami parabolic equation, as in [8].

On the other hand, scaling by the factor η should lead (as in [4]) to a concentrated model in which no tangential diffusion takes place, while the temperature has a transversal jump (see, for instance, [1, 2, 3, 5, 6] for a similar approach in the framework of electrical conduction). The proof of the concentration result relies on suitable a-priori estimates and proper identification of the limit function together with a description of the membrane in terms of proper curvilinear coordinates.

Finally, the last part of the paper is devoted to the study of the asymptotic properties of the solutions. Namely, well posedness for an elliptic counterpart of our original problem is proved together with the convergence in time of the solutions of the evolutive system of PDE's to the solution of such elliptic problem (as, for instance, in [15, 21]).

Such a convergence is proved to be exponential provided the source terms do not depend on time.

The paper is organized as follows: in Section 2 we recall the definition and some properties of the tangential operators (gradient, divergence, Laplace-Beltrami operator), we state our geometrical setting and we introduce our model. In Section 3 we prove the concentration result, while in Section 4 we prove the well-posedness for the concentrated problem. Finally, in Section 5, we prove the time-asymptotic result.

2 Preliminaries

2.1 Tangential derivatives

Let ϕ be a $\mathcal{C}^2$-function, $\mathbf{\Phi}$ be a $\mathcal{C}^2$-vector function and S a smooth surface in $\mathbb{R}^N$ with normal unit vector n. We recall that the tangential gradient of ϕ on S is given by

$$\nabla^B \phi = \nabla\phi - (n \cdot \nabla\phi) n \tag{2.1}$$

and the tangential divergence of $\mathbf{\Phi}$ on S is given by

$$\begin{aligned} \operatorname{div}^B \mathbf{\Phi} &= \operatorname{div}^B (\mathbf{\Phi} - (n \cdot \mathbf{\Phi}) n) = \operatorname{div} (\mathbf{\Phi} - (n \cdot \mathbf{\Phi}) n) \\ &= \operatorname{div} \mathbf{\Phi} - (n \cdot \nabla \mathbf{\Phi}_i) n_i - (\operatorname{div} n)(n \cdot \mathbf{\Phi}) \,, \end{aligned} \tag{2.2}$$

where, taking into account the smoothness of S, the normal vector n can be naturally defined in a small neighborhood of S as $\frac{\nabla d}{|\nabla d|}$, where d is the signed distance from S. Moreover, we define as usual the Laplace-Beltrami operator as

$$\Delta^B \phi = \mathrm{div}^B(\nabla^B \phi)\,. \tag{2.3}$$

Finally, we recall that on a regular surface S with no boundary (i.e. when $\partial S = \emptyset$) we have

$$\int_S \mathrm{div}^B\, \mathbf{\Phi}\, \mathrm{d}\sigma = 0\,. \tag{2.4}$$

2.2 Geometrical setting

Let Ω be a given open, connected and bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, such that $\Omega = \Omega_{\mathrm{int}} \cup \Omega_{\mathrm{out}} \cup \Gamma$, Ω_{int} and Ω_{out} are two disjoint open subsets of Ω, Ω_{out} is connected, $\Gamma = \partial\Omega_{\mathrm{int}} = \partial\Omega_{\mathrm{out}} \cap \Omega$, $\Gamma \cap \partial\Omega = \emptyset$ and we assume that Γ is of class $\mathcal{C}^\infty$. Let ν_Ω denote the normal unit vector to Γ pointing into Ω_{out} and $[u]$ be the jump of u across Γ as defined in (2.16).

Actually, in the physical framework, the interface is not an $(N-1)$-dimensional surface but it has a very small positive thickness. Hence, we need also to consider a more physical setting, in which it appears a small parameter which takes into account the thickness of the physical interface.

To this purpose, for $\eta > 0$, let us write Ω also as $\Omega = \Omega^\eta \cup \Gamma^\eta \cup \partial\Gamma^\eta$, where Ω^η and Γ^η are two disjoint open subsets of Ω, Γ^η is the tubular neighborhood of Γ with thickness η, and $\partial\Gamma^\eta$ is the boundary of Γ^η. Moreover, we assume also that $\Omega^\eta = \Omega^\eta_{\mathrm{int}} \cup \Omega^\eta_{\mathrm{out}}$, where $\Omega^\eta_{\mathrm{out}}$, $\Omega^\eta_{\mathrm{int}}$ correspond to the external region and to the internal one, respectively, and $\partial\Gamma^\eta = (\partial\Omega^\eta_{\mathrm{int}} \cup \partial\Omega^\eta_{\mathrm{out}}) \cap \Omega$. We assume that, for $\eta \to 0$ fixed, $|\Gamma^\eta| \sim \eta|\Gamma|_{N-1}$.

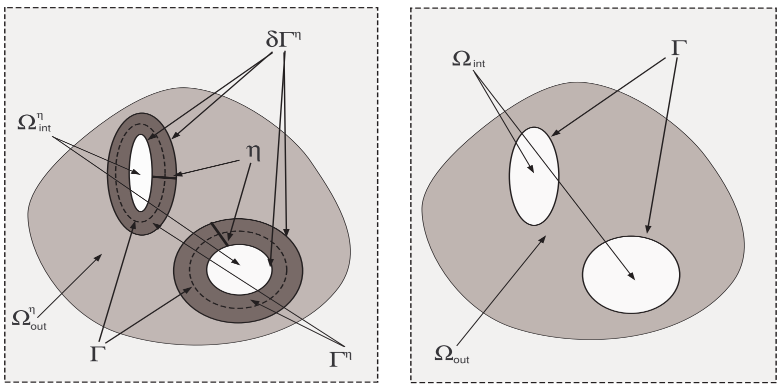

Figure 1: *Left*: before concentration; Γ^η is the dark grey region, $\Omega^\eta_{\mathrm{int}}$ is the white region and $\Omega^\eta_{\mathrm{out}}$ is the light grey region. *Right*: after concentration; Γ^η shrinks to Γ as $\eta \to 0$, Ω_{int} is the white region and Ω_{out} is the light grey region.

We stress the fact that the appearance of the small parameter η calls for a limit procedure; i.e., the concentration of the thick membrane Γ^η, in order to replace it with the $(N-1)$-dimensional surface Γ, then simplifying the geometry of the problem.

We will also use the following notation. Let $T > 0$ be a given time, for any spatial domain G, we will denote by $G_T = G \times (0,T)$ the corresponding space-time cylindrical domain.

2.3 Position of the problem

In this subsection, we give a complete formulation of the problems stated in the Introduction. We will present both the physical problem involving the thick membrane and the concentrated version involving only the $(N-1)$-dimensional interface. It will be the purpose of next section to show that the concentration limit ($\eta \to 0$) of the physical model actually gives rise to the interface problem.

We first state the physical problem in the framework of thin membranes. To this purpose, let $\mu_{\text{int}}, \mu_{\text{out}}, \alpha$ be strictly positive constants. Assume that $A, B \in \big(L^\infty(\Omega)\big)^{N\times N}$ are symmetric matrices satisfying

$$A(x)\xi \cdot \xi \geq \gamma_A |\xi|^2 \,, \quad \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^N \,; \tag{2.5}$$

$$B(x)\xi \cdot \xi \geq \gamma_B |\xi|^2 \,, \quad \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^N \,, \tag{2.6}$$

for suitable constants $\gamma_A, \gamma_B > 0$.

We set $a^\eta(x) = \mu_{\text{int}}$ in Ω^η_{int}, $a^\eta(x) = \mu_{\text{out}}$ in Ω^η_{out}, $a^\eta(x) = \alpha/\eta$ in Γ^η, $A^\eta(x) = A$ in $\Omega^\eta_{\text{int}} \cup \Omega^\eta_{\text{out}}$, $A^\eta(x) = \eta^{-1}B$ in Γ^η.

A meaningful case in the applications is the one in which A is a multiple of the identity by means of a scalar function $\lambda : \Omega \to \mathbb{R}^+$ such that

$$\lambda = \lambda_{\text{int}} \quad \text{in } \Omega_{\text{int}}, \qquad \lambda = \lambda_{\text{out}} \quad \text{in } \Omega_{\text{out}},$$

and $B = \beta I$, where I denotes the identity matrix and $\beta > 0$ (see for instance [3, 10]).

Let $\overline{u}_0 \in L^2(\Omega)$, $f, g \in L^2(\Omega_T)$ and set

$$f^\eta(x,t) = \begin{cases} f(x,t) & \text{if } (x,t) \in \Omega^\eta \times (0,T); \\ \dfrac{1}{\eta} g(x,t) & \text{if } (x,t) \in \Gamma^\eta \times (0,T). \end{cases}$$

For every $\eta > 0$, we consider the problem for $u^\eta(x,t) \in L^2\big(0,T; H^1_0(\Omega)\big)$ given by

$$a^\eta \frac{\partial u^\eta}{\partial t} - \operatorname{div}(A^\eta \nabla u^\eta) = f^\eta \,, \qquad \text{in } \Omega_T; \tag{2.7}$$

$$u^\eta(x,t) = 0 \,, \qquad \text{on } \partial\Omega \times (0,T); \tag{2.8}$$

$$u^\eta(x,0) = \overline{u}_0(x) \,, \qquad \text{in } \Omega; \tag{2.9}$$

which has the following standard weak formulation

$$-\int_0^T \!\! \int_\Omega a^\eta u^\eta \frac{\partial \phi}{\partial t} \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \!\! \int_\Omega A^\eta \nabla u^\eta \cdot \nabla \phi \, \mathrm{d}x \, \mathrm{d}t = \int_\Omega a^\eta \overline{u}_0 \phi(0) \, \mathrm{d}x + \int_0^T \!\! \int_\Omega f^\eta \phi \, \mathrm{d}x \, \mathrm{d}t \,, \tag{2.10}$$

for every test function $\phi \in \mathcal{C}^\infty(\Omega_T)$ such that ϕ has compact support in Ω for every $t \in (0,T)$ and $\phi(\cdot,T) = 0$ in Ω. Clearly, for any given $\eta > 0$, problem (2.7)–(2.9) (or (2.10)) is a classical parabolic problem and hence it has a unique solution $u^\eta(t) \in L^2\big(0,T;H^1_0(\Omega)\big) \cap \mathcal{C}^0\big([0,T];L^2(\Omega)\big)$.

Now, let us state the mathematical concentrated problem. To this purpose, we define μ as $\mu = \mu_{\text{int}}$ in Ω_{int} and $\mu = \mu_{\text{out}}$ in Ω_{out}. We assume also that $B \in L^\infty(\Gamma)$ and satisfies (2.6) for a.e. $x \in \Gamma$, $\overline{u}_0 \in L^2(\Gamma)$ and $g \in L^2(\Gamma_T)$. We consider the problem for $u(x,t)$ given by

$$\mu\frac{\partial u}{\partial t} - \operatorname{div}(A\nabla u) = f\,, \qquad \text{in } \Omega_T; \tag{2.11}$$

$$[u] = 0\,, \qquad \text{on } \Gamma_T; \tag{2.12}$$

$$\alpha\frac{\partial u}{\partial t} - \operatorname{div}^B(B\nabla^B u) = [A\nabla u\cdot\nu] + g\,, \qquad \text{on } \Gamma_T; \tag{2.13}$$

$$u(x,t) = 0\,, \qquad \text{on } \partial\Omega\times(0,T); \tag{2.14}$$

$$u(x,0) = \overline{u}_0(x)\,, \qquad \text{in } \Omega, \tag{2.15}$$

where we denote

$$[u] = u^{\text{out}} - u^{\text{int}}\,, \tag{2.16}$$

and the same notation is employed also for other quantities. Here the operators div and ∇, as well as div^B and ∇^B, act only with respect to the space variable x.

Since problem (2.11)–(2.15) is not standard, in order to define a proper notion of weak solution, we will need to introduce some suitable function spaces. To this purpose and for later use, we will denote by $H^1_B(\Gamma)$ the space of Lebesgue measurable functions $u : \Gamma \to \mathbb{R}$ such that $u \in L^2(\Gamma)$, $\nabla^B u \in L^2(\Gamma)$, endowed with the natural norm

$$\|u\|^2_{H^1_B(\Gamma)} = \int_\Gamma u^2\,\mathrm{d}\sigma + \int_\Gamma |\nabla^B u|^2\,\mathrm{d}\sigma\,. \tag{2.17}$$

Let us also set

$$\mathcal{X}_0(\Omega) := \{u \in H^1_0(\Omega)\ :\ tr\big|_\Gamma(u) \in H^1_B(\Gamma)\}\,. \tag{2.18}$$

We note that $\mathcal{X}_0(\Omega)$ is a Hilbert space endowed with the scalar product given by

$$\langle u,v\rangle_{\mathcal{X}_0(\Omega)} = \int_\Omega uv\,\mathrm{d}x + \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x + \int_\Gamma uv\,\mathrm{d}\sigma + \int_\Gamma \nabla^B u\cdot\nabla^B v\,\mathrm{d}\sigma\,, \tag{2.19}$$

for all $u,v \in \mathcal{X}_0(\Omega)$, where, for the sake of brevity, we identify u and v with their traces in the last two integrals of (2.19). Indeed, the only delicate point is to prove completeness. On the other hand, this last property is guaranteed by the continuity of the traces and by the completeness of $H^1_0(\Omega)$ and $H^1_B(\Gamma)$, which assure that a Cauchy sequence $\{u_n\} \in \mathcal{X}_0(\Omega)$ is such that $u_n \to \overline{u}$ strongly in $H^1_0(\Omega)$ and also $tr\big|_\Gamma(u_n) \to \overline{w}$ strongly in $H^1_B(\Gamma)$. Moreover, $tr\big|_\Gamma(u_n) \to tr\big|_\Gamma(\overline{u})$ strongly in $L^2(\Gamma)$ because of standard trace properties; hence, $tr\big|_\Gamma(\overline{u}) = \overline{w}$ and thus $\overline{u} \in \mathcal{X}_0(\Omega)$.

Definition 2.1. We say that $u \in L^2\big(0,T;\mathcal{X}_0(\Omega)\big)$ is a weak solution of problem (2.11)–(2.15) if

$$\begin{aligned}
&- \int_0^T \int_\Omega \mu u \frac{\partial \phi}{\partial \tau} \,\mathrm{d}x \,\mathrm{d}\tau + \int_0^T \int_\Omega A \nabla u \cdot \nabla \phi \,\mathrm{d}x \,\mathrm{d}\tau \\
&- \alpha \int_0^T \int_\Gamma u \frac{\partial \phi}{\partial \tau} \,\mathrm{d}\sigma \,\mathrm{d}\tau + \int_0^T \int_\Gamma B \nabla^B u \cdot \nabla^B \phi \,\mathrm{d}\sigma \,\mathrm{d}\tau \\
= &\int_\Omega \mu \overline{u}_0 \phi(x,0) \,\mathrm{d}x + \alpha \int_\Gamma \overline{u}_0 \phi(x,0) \,\mathrm{d}\sigma + \int_0^T \int_\Omega f \phi \,\mathrm{d}x \,\mathrm{d}\tau + \int_0^T \int_\Gamma g \phi \,\mathrm{d}\sigma \,\mathrm{d}\tau \,, \qquad (2.20)
\end{aligned}$$

for every test function $\phi \in \mathcal{C}^\infty(\Omega_T)$ such that ϕ has compact support in Ω for every $t \in (0,T)$ and $\phi(\cdot,T) = 0$ in Ω.

We will prove in Section 4 that problem (2.11)–(2.15) admits a unique solution $u \in L^2\big(0,T;\mathcal{X}_0(\Omega)\big) \cap \mathcal{C}^0\big([0,T];L^2(\Omega) \cap L^2(\Gamma)\big)$.

3 Derivation of the concentrated problem

In this Section we will also assume that the initial datum $\overline{u}_0 \in W^{1,\infty}(\Omega)$ and that the source $g \in L^2\big(0,T;W^{1,\infty}(\Omega)\big)$. First we note that, by an integration by parts, one can derive from (2.10) the energy inequality

$$\begin{aligned}
&\sup_{t\in(0,T)} \int_{\Omega^\eta_{\mathrm{out}} \cup \Omega^\eta_{\mathrm{int}}} (u^\eta)^2(t) \,\mathrm{d}x + \sup_{t\in(0,T)} \frac{1}{\eta} \int_{\Gamma^\eta} (u^\eta)^2(t) \,\mathrm{d}x \\
&+ \int_0^T \int_{\Omega^\eta_{\mathrm{out}} \cup \Omega^\eta_{\mathrm{int}}} |\nabla u^\eta|^2 \,\mathrm{d}x \,\mathrm{d}\tau + \frac{1}{\eta} \int_0^T \int_{\Gamma^\eta} |\nabla u^\eta|^2 \,\mathrm{d}x \,\mathrm{d}\tau \le \gamma \,, \qquad (3.1)
\end{aligned}$$

where γ depends on $\mu_{\mathrm{int}}, \mu_{\mathrm{out}}, \alpha, \gamma_A, \gamma_B, |\Omega|, |\mathcal{G}|, \|\overline{u}_0\|_{L^\infty(\Omega)}, \|f\|_{L^2(\Omega_T)}, \|g\|_{L^2(0,T;L^\infty(\Gamma))}$, but not on η. As a consequence, as $\eta \to 0$, we may assume, extracting a subsequence if needed,

$$u^\eta \rightharpoonup u \,, \nabla u^\eta \rightharpoonup \nabla u \,, \qquad \text{weakly in } L^2(\Omega_T),$$

where $u \in L^2\big(0,T;H^1_0(\Omega)\big)$. We characterize u in Theorem 3.1 as the solution of the concentrated differential problem (2.11)–(2.15).

In order to proceed with the concentration of problem (2.7)–(2.9), we need to choose a suitable testing function in the weak formulation (2.10), before passing to the limit for $\eta \to 0$. To this purpose we recall that there exists an $\eta_0 > 0$, such that for $\eta < \eta_0$, the application

$$\psi : \Gamma \times [-\eta,\eta] \to \Gamma^{2\eta} \,, \qquad \psi(y_\Gamma, r) = y_\Gamma + r\nu(y_\Gamma) = y \in \Gamma^{2\eta}$$

is a diffeomorfism onto its image, where we denote by $\Gamma^{2\eta}$ the tubolar neighborhood of Γ with thickness 2η. Clearly, $\Gamma^{2\eta}$ can be considered as the union of surfaces denoted by Γ_r parallel to Γ and at distance $|r|$ from it, when r varies in $[-\eta,\eta]$. Hence, for $y \in \Gamma^{2\eta}$, there exists a unique

$(y_\Gamma, r) \in \Gamma \times [\eta, \eta]$ such that $y = y_\Gamma + r\nu(y_\Gamma)$ and then $y \in \Gamma_r$ and $\nu(y_\Gamma)$ coincides with the normal to the surface Γ_r at y. Moreover, we can locally parametrize Γ in such a way that there exist $\widehat{\Gamma} \subset \mathbb{R}^{N-1}$ and $y_\Gamma : \widehat{\Gamma} \to \Gamma$ such that $\Gamma \ni y_\Gamma = y_\Gamma(\xi)$, where $\xi = (\xi_1, \dots, \xi_{N-1}) \in \widehat{\Gamma}$ and, if we set $\mathrm{d}\sigma = \sqrt{g(\xi)}\,\mathrm{d}\xi$, we may assume that $\gamma_1 \le \sqrt{g(\xi)} \le \gamma_2$, for every $\xi \in \widehat{\Gamma}$, where γ_1, γ_2 are suitable strictly positive constants. As a consequence, we have obtained a change of coordinates in $\mathbb{R}^N$, whose Jacobian matrix will be denoted by $J(\xi, r)$, defined by

$$\Gamma^{2\eta} \ni y = (y_1, \dots, y_N) \longleftrightarrow (\xi, r) = (\xi_1, \dots, \xi_{N-1}, r) \in \widehat{\Gamma} \times [-\eta, \eta]\,.$$

By the assumed regularity of Γ, it follows that $J(\xi, r) = J(\xi, 0) + M_\eta$, where M_η denotes a suitable matrix such that $|M_\eta| \le \gamma\eta$, so that $|detJ(\xi, r)| = |detJ(\xi, 0)| + R_\eta$, where $|R_\eta| \le \gamma\eta$; moreover, by the choice of the coordinates (ξ, r), we have that $|det\big(J(\xi, 0)\big)| = \sqrt{g(\xi)}$ (recall that the volume element $\mathrm{d}y = |det\big(J(\xi, r)\big)|\,\mathrm{d}\xi\,\mathrm{d}r$ for $r = 0$, i.e. on Γ, becomes $\mathrm{d}y = |det\big(J(\xi, 0)\big)|\,\mathrm{d}\xi\,\mathrm{d}r = \mathrm{d}\sigma\,\mathrm{d}r = \sqrt{g(\xi)}\,\mathrm{d}\xi\,\mathrm{d}r$).

Finally we set $y_\Gamma = \pi_0(y)$, i.e. the orthogonal projection of $y \in \Gamma^{2\eta}$ on Γ, $r = \rho(y)$, i.e. the signed distance of $y \in \Gamma^{2\eta}$ from Γ; note that $|\nabla\rho(y)|$ is bounded.

In the sequel, we assume without loss of generality that the support of our testing functions is sufficiently small to allow for the representation introduced above. The general case can then be recovered by means of a standard partition of unity argument. Moreover, for the sake of brevity, we will use the same symbol for the same function even if written with respect to different variables.

Theorem 3.1. Assume that $\overline{u}_0 \in W^{1,\infty}(\Omega)$, $f \in L^2(\Omega_T)$, $g \in L^2\big(0, T; W^{1,\infty}(\Omega)\big)$, $B = bI$, where I is the identity matrix, $b \in L^\infty(\Gamma)$, $b(x) \ge \gamma_B$, for a.e. $x \in \Gamma$, and it is extended to the whole of $\Gamma^{2\eta}$ as $b(y) = b(\pi_0(y))$. Then, for $\eta \to 0$, $u^\eta \rightharpoonup u$ weakly in $L^2\big(0, T; H^1_0(\Omega)\big)$, where u is the solution of problem (2.20).

Proof. Let $\varphi \in \mathcal{C}^\infty(\Omega_T)$ such that φ has compact support (sufficiently small) in Ω for every $t \in (0, T)$ and $\varphi(\cdot, T) = 0$ in Ω, be the general testing function for the concentrated problem (2.11)–(2.15). Starting from φ, we construct a suitable test function φ^η for problem (2.10) in such a way that it does not depend on the transversal coordinate inside Γ^η (being constantly equal to its value on Γ) and it is linearly connected with φ in Ω^η_{int} and Ω^η_{out} along the r-direction. It is crucial in order to develop the concentration procedure to make this glueing where the diffusivity in equation (2.7) is stable with respect to η, i.e. inside the set $\Gamma^{2\eta} \setminus \Gamma^\eta \subset \Omega^\eta_{\text{int}} \cup \Omega^\eta_{\text{out}}$. To this purpose, define

$$\varphi^\eta(y, t) = \begin{cases} \varphi(y, t) & \text{if } (y, t) \in (\Omega^\eta_{\text{out}} \setminus \Gamma^{2\eta}) \times (0, T); \\ \varphi^\eta_{\text{out}}(y, t) & \text{if } (y, t) \in (\Omega^\eta_{\text{out}} \cap \Gamma^{2\eta}) \times (0, T); \\ \varphi\big(\pi_0(y), t\big) & \text{if } (y, t) \in \Gamma^\eta \times (0, T); \\ \varphi^\eta_{\text{int}}(y, t) & \text{if } (y, t) \in (\Omega^\eta_{\text{int}} \cap \Gamma^{2\eta}) \times (0, T); \\ \varphi(y, t) & \text{if } (y, t) \in (\Omega^\eta_{\text{int}} \setminus \Gamma^{2\eta}) \times (0, T); \end{cases} \tag{3.2}$$

where

$$\varphi^\eta_{\text{out}}(y,t) \quad = \quad \Big[\varphi\big(\pi_0(y) \;+\; \eta\nu(\pi_0(y)),t\big) \;-\; \varphi\big(\pi_0(y),t\big)\Big]\frac{2\rho(y)-\eta}{\eta} \;+\; \varphi\big(\pi_0(y),t\big)$$

and

$$\varphi^\eta_{\text{int}}(y,t) \quad = \quad \Big[\varphi\big(\pi_0(y),t\big) \;-\; \varphi\big(\pi_0(y) \;-\; \eta\nu(\pi_0(y)),t\big)\Big]\frac{2\rho(y)+\eta}{\eta} \;+\; \varphi\big(\pi_0(y),t\big).$$

Note that the linearity is intended with respect to $\rho(y)$. By a density argument, we can use φ^η as a testing function in (2.10); then, it follows

$$\begin{aligned}
&-\int_0^T\!\!\int_{(\Omega^\eta_{\text{int}}\cup\Omega^\eta_{\text{out}})\setminus\Gamma^{2\eta}} \mu u^\eta \frac{\partial\varphi}{\partial\tau}\,\mathrm{d}y\,\mathrm{d}\tau - \frac{\alpha}{\eta}\int_0^T\!\!\int_{\Gamma^\eta} u^\eta \frac{\partial\varphi\big(\pi_0(y),\tau\big)}{\partial\tau}\,\mathrm{d}y\,\mathrm{d}\tau \\
&-\int_0^T\!\!\int_{\Omega^\eta_{\text{int}}\cap\Gamma^{2\eta}} \mu u^\eta \frac{\partial\varphi^\eta_{\text{int}}}{\partial\tau}\,\mathrm{d}y\,\mathrm{d}\tau - \int_0^T\!\!\int_{\Omega^\eta_{\text{out}}\cap\Gamma^{2\eta}} \mu u^\eta \frac{\partial\varphi^\eta_{\text{out}}}{\partial\tau}\,\mathrm{d}y\,\mathrm{d}\tau \\
&+\int_0^T\!\!\int_{(\Omega^\eta_{\text{int}}\cup\Omega^\eta_{\text{out}})\setminus\Gamma^{2\eta}} A\nabla u^\eta\cdot\nabla\varphi\,\mathrm{d}y\,\mathrm{d}\tau + \frac{1}{\eta}\int_0^T\!\!\int_{\Gamma^\eta} B\nabla u^\eta\cdot\nabla\varphi\big(\pi_0(y),\tau\big)\,\mathrm{d}y\,\mathrm{d}\tau \\
&+\int_0^T\!\!\int_{\Omega^\eta_{\text{int}}\cap\Gamma^{2\eta}} A\nabla u^\eta\cdot\nabla\varphi^\eta_{\text{int}}\,\mathrm{d}y\,\mathrm{d}\tau + \int_0^T\!\!\int_{\Omega^\eta_{\text{out}}\cap\Gamma^{2\eta}} A\nabla u^\eta\cdot\nabla\varphi^\eta_{\text{out}}\,\mathrm{d}y\,\mathrm{d}\tau \\
=&\int_{(\Omega^\eta_{\text{int}}\cup\Omega^\eta_{\text{out}})\setminus\Gamma^{2\eta}} \mu\overline{u}_0\varphi(y,0)\,\mathrm{d}y + \frac{\alpha}{\eta}\int_{\Gamma^\eta}\overline{u}_0\varphi\big(\pi_0(y),0\big)\,\mathrm{d}y + \int_{\Omega^\eta_{\text{int}}\cap\Gamma^{2\eta}}\mu\overline{u}_0\varphi^\eta_{\text{int}}(y,0)\,\mathrm{d}y \\
&+\int_{\Omega^\eta_{\text{out}}\cap\Gamma^{2\eta}}\mu\overline{u}_0\varphi^\eta_{\text{out}}(y,0)\,\mathrm{d}y + \int_0^T\!\!\int_{(\Omega^\eta_{\text{int}}\cup\Omega^\eta_{\text{out}})\setminus\Gamma^{2\eta}} f\varphi\,\mathrm{d}y\,\mathrm{d}\tau + \frac{1}{\eta}\int_0^T\!\!\int_{\Gamma^\eta} g\varphi\big(\pi_0(y),\tau\big)\,\mathrm{d}y\,\mathrm{d}\tau \\
&+\int_0^T\!\!\int_{\Omega^\eta_{\text{int}}\cap\Gamma^{2\eta}} f\varphi^\eta_{\text{int}}\,\mathrm{d}y\,\mathrm{d}\tau + \int_0^T\!\!\int_{\Omega^\eta_{\text{out}}\cap\Gamma^{2\eta}} f\varphi^\eta_{\text{out}}\,\mathrm{d}y\,\mathrm{d}\tau\,.
\end{aligned} \tag{3.3}$$

Due to estimate (3.1) and taking into account that

$$\nabla\varphi^\eta_{\text{out}}(y,t) = \Im(\eta) + \Big[\varphi\big(\pi_0(y)+\eta\nu(\pi_0(y)),t\big) - \varphi\big(\pi_0(y),t\big)\Big]\frac{2\nabla\rho(y)}{\eta}$$

where with $\Im(\eta)$ we denote a bounded quantity with respect to η (clearly, the same holds for $\varphi^\eta_{\text{int}}$) and

$$\Big|\Big[\varphi\big(\pi_0(y)+\eta\nu(\pi_0(y)),t\big) - \varphi\big(\pi_0(y),t\big)\Big]\Big| \le \gamma\eta\,, \tag{3.4}$$

with γ independent of η, it is easy to see that when $\eta\to 0$, the second, the fourth, the sixth and the eighth line in the equality (3.3) tend to 0; moreover,

$$
\begin{aligned}
\int_0^T\int_{(\Omega^\eta_{\text{int}}\cup\Omega^\eta_{\text{out}})\setminus\Gamma^{2\eta}} \mu u^\eta\frac{\partial\varphi}{\partial t}\,\mathrm{d}y\,\mathrm{d}\tau &\to \int_0^T\int_{\Omega_{\text{int}}\cup\Omega_{\text{out}}} \mu u\frac{\partial\varphi}{\partial t}\,\mathrm{d}y\,\mathrm{d}\tau\,,\\
\int_0^T\int_{(\Omega^\eta_{\text{int}}\cup\Omega^\eta_{\text{out}})\setminus\Gamma^{2\eta}} A\nabla u^\eta\cdot\nabla\varphi\,\mathrm{d}y\,\mathrm{d}\tau &\to \int_0^T\int_{\Omega_{\text{int}}\cup\Omega_{\text{out}}} A\nabla u\cdot\nabla\varphi\,\mathrm{d}y\,\mathrm{d}\tau\,,\\
\int_{(\Omega^\eta_{\text{int}}\cup\Omega^\eta_{\text{out}})\setminus\Gamma^{2\eta}} \mu\overline{u}_0\varphi(y,0)\,\mathrm{d}y &\to \int_{\Omega_{\text{int}}\cup\Omega_{\text{out}}} \mu\overline{u}_0\varphi(y,0)\,\mathrm{d}y\,,\\
\int_0^T\int_{(\Omega^\eta_{\text{int}}\cup\Omega^\eta_{\text{out}})\setminus\Gamma^{2\eta}} f\varphi\,\mathrm{d}y\,\mathrm{d}\tau &\to \int_0^T\int_{\Omega_{\text{int}}\cup\Omega_{\text{out}}} f\varphi\,\mathrm{d}y\,\mathrm{d}\tau\,.
\end{aligned}
$$

Finally, by the properties of the traces, it is not difficult to get also

$$
\begin{aligned}
\frac{\alpha}{\eta}\int_0^T\int_{\Gamma^\eta} u^\eta\frac{\partial\varphi\big(\pi_0(y),\tau\big)}{\partial\tau}\,\mathrm{d}y\,\mathrm{d}\tau = \alpha\int_0^T\left[\frac{1}{\eta}\int_{\Gamma^\eta} u^\eta\frac{\partial\varphi\big(\pi_0(y),\tau\big)}{\partial\tau}\,\mathrm{d}y\right]\mathrm{d}\tau &\to \alpha\int_0^T\int_\Gamma u\frac{\partial\varphi}{\partial\tau}\,\mathrm{d}\sigma\,\mathrm{d}\tau\\
\frac{\alpha}{\eta}\int_{\Gamma^\eta}\overline{u}_0\varphi\big(\pi_0(y),0\big)\,\mathrm{d}y = \alpha\left[\frac{1}{\eta}\int_{\Gamma^\eta}\overline{u}_0\varphi\big(\pi_0(y),0\big)\,\mathrm{d}y\right] &\to \alpha\int_\Gamma\overline{u}_0\varphi(y,0)\,\mathrm{d}\sigma\\
\frac{1}{\eta}\int_0^T\int_{\Gamma^\eta} g\varphi\big(\pi_0(y),\tau\big)\,\mathrm{d}y\,\mathrm{d}\tau = \int_0^T\left[\frac{1}{\eta}\int_{\Gamma^\eta} g\varphi\big(\pi_0(y),\tau\big)\,\mathrm{d}y\right]\mathrm{d}\tau &\to \int_0^T\int_\Gamma g\varphi\,\mathrm{d}y\,\mathrm{d}\tau\,.
\end{aligned}
$$

Hence the crucial limit is the sixth one in (3.3). To deal with this limit, we pass to the new coordinates (ξ,r) defined above, recalling that $J(\xi,r)$ denotes the Jacobian matrix of such a change of coordinates. Moreover, denoting by $\nabla^B_{\Gamma_r}$ the tangential gradient with respect to the surface Γ_r and recalling that the normal vector at $y\in\Gamma_r$ coincides with the normal at $\pi_0(y)\in\Gamma$, we have $\nabla^B_{\Gamma_r}u^\eta=\nabla u^\eta-(\nu(\pi_0(y))\cdot\nabla u^\eta)\nu(\pi_0(y))$, with $r=\rho(y)$. Also, since the test function does not depend on the normal coordinate r in Γ^η, we have that $\nabla\varphi(\pi_0(y),t)=\nabla^B\varphi(\pi_0(y),t)$ and hence $\nabla\varphi\cdot\nabla u^\eta=\nabla^B\varphi\cdot\nabla^B_{\Gamma_{\rho(y)}}u^\eta$. Then, setting for the sake of simplicity $\widetilde{J}(\xi):=\widetilde{J}(\xi,0)$ and taking into account the scalar nature of B, we can rewrite

$$
\begin{aligned}
&\frac{1}{\eta}\int_0^T\int_{\Gamma^\eta} B\big(\pi_0(y)\big)\nabla u^\eta\cdot\nabla\varphi\big(\pi_0(y),\tau\big)\,\mathrm{d}y\,\mathrm{d}\tau\\
=&\frac{1}{\eta}\int_0^T\int_{\Gamma^\eta} b\big(\pi_0(y)\big)\nabla^B\varphi\big(\pi_0(y),\tau\big)\cdot\nabla^B_{\Gamma_{\rho(y)}}u^\eta\,\mathrm{d}y\,\mathrm{d}\tau\\
=&\frac{1}{\eta}\int_0^T\int_{\widehat{\Gamma}}\int_{-\eta/2}^{\eta/2} b(\xi,0)(\widetilde{J}(\xi,r)\nabla_\xi\varphi(\xi,0,\tau))^T\widetilde{J}(\xi,r)\nabla_\xi u^\eta|detJ(\xi,r)|\,\mathrm{d}\xi\,\mathrm{d}r\,\mathrm{d}\tau\\
=:&I_1(\eta)+I_2(\eta)\,,
\end{aligned}
$$

where

$$
I_1(\eta):=\int_0^T\int_{\widehat{\Gamma}} b(\xi,0)\Big(\widetilde{J}(\xi)\nabla_\xi\varphi\big(\xi,0,\tau\big)\Big)^T\left(\frac{1}{\eta}\int_{-\eta/2}^{\eta/2}\widetilde{J}(\xi)\nabla_\xi u^\eta\big(\xi,r,\tau\big)\,\mathrm{d}r\right)\sqrt{g(\xi)}\,\mathrm{d}\xi\,\mathrm{d}\tau
$$

and $I_2(\eta)$ is the remaining part. Here, $\widetilde{J}(\xi,r)$ is the rectangular matrix such that, for every function $v(y)$, $\widetilde{J}(\xi,r)\nabla_\xi v(\xi,r) = \nabla^B_{\Gamma_{\rho(y)}} v(y)$, and the supscript ^{T} denotes the transposed vector. Obviously, due to the regularity of Γ, also the matrix $\widetilde{J}$ is regular, so that $\widetilde{J}(\xi,r) = \widetilde{J}(\xi,0) + O(\eta)$.

Clearly, using the energy estimate (3.1),

$$|I_2(\eta)| \le \gamma \frac{\eta}{\sqrt{\eta}} \left(\frac{1}{\eta} \int_0^T \int_{\Gamma^\eta} |\nabla u^\eta|^2 \, \mathrm{d}y \, \mathrm{d}\tau \right)^{1/2} \sqrt{\eta} \le \gamma\eta \to 0 \qquad \text{as } \eta \to 0.$$

On the other hand, by Holder's inequality and again the energy estimate (3.1), it follows that $\frac{1}{\eta}\int_{-\eta/2}^{\eta/2} \widetilde{J}(\xi)\nabla_\xi u^\eta\big(\xi,r,\tau\big)\,\mathrm{d}r$ is bounded uniformly with respect to η so that there exists a vector function $\mathbf{V} \in L^2(0,T;L^2(\widehat{\Gamma}))$ such that, up to a subsequence,

$$\frac{1}{\eta} \int_{-\eta/2}^{\eta/2} \widetilde{J}(\xi)\nabla_\xi u^\eta(\xi,r,\tau)\,\mathrm{d}r \rightharpoonup \mathbf{V}\,, \qquad \text{weakly in } L^2(0,T;L^2(\widehat{\Gamma})).$$

Hence, we obtain

$$\begin{aligned} I_1(\eta) &= \int_0^T \int_{\widehat{\Gamma}} b(\xi,0)\Big(\widetilde{J}(\xi)\nabla_\xi\varphi\big(\xi,0,\tau\big)\Big)^T \left(\frac{1}{\eta}\int_{-\eta/2}^{\eta/2} \widetilde{J}(\xi)\nabla_\xi u^\eta\big(\xi,r,\tau\big)\,\mathrm{d}r\right) \sqrt{g(\xi)}\,\mathrm{d}\xi\,\mathrm{d}\tau \\ &\to \int_0^T \int_{\widehat{\Gamma}} b(\xi,0)\Big(\widetilde{J}(\xi)\nabla_\xi\varphi(\xi,0,\tau)\Big)^T \mathbf{V}\sqrt{g(\xi)}\,\mathrm{d}\xi\,\mathrm{d}\tau = \int_0^T \int_\Gamma b\nabla^B\varphi\cdot\mathbf{V}\,\mathrm{d}\sigma\,\mathrm{d}\tau\,. \end{aligned}$$

It remains to identify $\mathbf{V}$ as the tangential gradient of the limit u; i.e., $\mathbf{V} = \nabla^B u$ on Γ. To this aim we consider a vector test function $\Psi \in \mathcal{C}^1_c(\Omega_T)$; we obtain

$$\begin{aligned} &\int_0^T \int_\Gamma \operatorname{div}^B \Psi\, u \,\mathrm{d}\sigma\,\mathrm{d}\tau \longleftarrow \int_0^T \int_\Gamma \operatorname{div}^B \Psi \left(\frac{1}{\eta}\int_{-\eta/2}^{\eta/2} u^\eta(y_\Gamma + r\nu(y_\Gamma),\tau)\,\mathrm{d}r\right)\mathrm{d}\sigma\,\mathrm{d}\tau \\ &= -\int_0^T \int_\Gamma \Psi\cdot\nabla^B \left(\frac{1}{\eta}\int_{-\eta/2}^{\eta/2} u^\eta(y_\Gamma + r\nu(y_\Gamma),\tau)\,\mathrm{d}r\right)\mathrm{d}\sigma\,\mathrm{d}\tau \\ &= -\int_0^T \int_{\widehat{\Gamma}} \Psi\cdot\left(\frac{1}{\eta}\int_{-\eta/2}^{\eta/2} \widetilde{J}(\xi)\nabla_\xi u^\eta(\xi,r,\tau)\,\mathrm{d}r\right)\sqrt{g(\xi)}\,\mathrm{d}\xi\,\mathrm{d}\tau \\ &\longrightarrow -\int_0^T \int_{\widehat{\Gamma}} \Psi\cdot\mathbf{V}\sqrt{g(\xi)}\,\mathrm{d}\xi\,\mathrm{d}\tau = -\int_0^T \int_\Gamma \Psi\cdot\mathbf{V}\,\mathrm{d}\sigma\,\mathrm{d}\tau\,, \end{aligned}$$

which implies that $\mathbf{V} = \nabla^B u$. This proves that the limit for $\eta \to 0$ of equality (3.3) gives rise to (2.20), where $B = bI$; i.e., the concentration limit of u^η is the weak solution of system (2.11)–(2.15). □

Remark 3.2. Notice that, even if in the physical applications we have in mind the capacitive coefficients μ and α are constant in each phase (see [9, 10]), the results in Theorem 3.1 can be generalized to the case in which $\mu \in L^\infty(\Omega)$ and $\alpha \in L^\infty(\Gamma)$, with $\mu(x) \ge \mu_0$ a.e. in Ω and $\alpha(x) \ge \alpha_0$ a.e. on Γ, for proper constants $\mu_0, \alpha_0 > 0$. In this case, we assume that μ and α are extended to the whole of $\Gamma^{2\eta}$ constantly along the transversal direction; i.e. $\mu(y) = \mu(\pi_0(y))$ and $\alpha(y) = \alpha(\pi_0(y))$.

4 Well-posedness of the concentrated problem

In this section we consider the following nonlinear version of problem (2.11)–(2.15)

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \mathrm{div}(A\nabla u) &= f(x,t,u)\,, && \text{in } (\Omega_{\text{int}}\cup\Omega_{\text{out}})\times(0,T); && (4.1)\\
[u] &= 0\,, && \text{on } \Gamma_T; && (4.2)\\
\frac{\partial u}{\partial t} - \mathrm{div}^B(B\nabla^B u) &= [A\nabla u\cdot\nu] + g(x,t,u)\,, && \text{on } \Gamma_T; && (4.3)\\
u(x,t) &= 0\,, && \text{on } \partial\Omega\times(0,T); && (4.4)\\
u(x,0) &= \overline{u}_0(x)\,, && \text{in } \Omega, && (4.5)
\end{aligned}
$$

where, with no loss of generality, we have assumed that $\mu_{\text{int}} = \mu_{\text{out}} = \alpha = 1$ (see Remark 4.3 below). The weak formulation of the previous problem is clearly the same as in (2.20), replacing f and g with their nonlinear versions.

The main result of this section is the following theorem.

Theorem 4.1. Let $A \in \big(L^\infty(\Omega)\big)^{N\times N}$ be a symmetric matrix satisfying (2.5) and $B \in \big(L^\infty(\Gamma)\big)^{N\times N}$ be a symmetric matrix satisfying

$$
B(x)\xi\cdot\xi \geq \gamma_B|\xi|^2\,, \quad \text{for a.e. } x\in\Gamma \text{ and every } \xi\in\mathbb{R}^N\,. \tag{4.6}
$$

Assume that $\overline{u}_0 \in H^1_0(\Omega)$. Assume also that $f \in L^2\big(\Omega_T;\mathcal{C}^0(\mathbb{R})\big)$ and $g \in L^2\big(\Gamma_T;\mathcal{C}^0(\mathbb{R})\big)$ are two given functions such that there exist $\ell_f, \ell_g > 0$ with

$$
\begin{aligned}
|f(x,t,s_1)-f(x,t,s_2)| &\leq \ell_f|s_1-s_2|\,, && \text{for a.e. } (x,t)\in\Omega_T \text{ and } \forall s_1,s_2\in\mathbb{R}; && (4.7)\\
|g(x,t,r_1)-g(x,t,r_2)| &\leq \ell_g|r_1-r_2|\,, && \text{for a.e. } (x,t)\in\Gamma_T \text{ and } \forall r_1,r_2\in\mathbb{R}. && (4.8)
\end{aligned}
$$

Then problem (4.1)–(4.5) admits a unique solution $u \in L^2\big(0,T;\mathcal{X}_0(\Omega)\big)\cap\mathcal{C}^0\big([0,T];L^2(\Omega)\cap L^2(\Gamma)\big)$.

In order to achieve this result, we first prove the well-posedness of a linear version of problem (2.11)–(2.15); i.e.,

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \mathrm{div}(A\nabla u) &= f(x,t)\,, && \text{in } (\Omega_{\text{int}}\cup\Omega_{\text{out}})\times(0,T); && (4.9)\\
[u] &= 0\,, && \text{on } \Gamma_T; && (4.10)\\
\frac{\partial u}{\partial t} - \mathrm{div}^B(B\nabla^B u) &= [A\nabla u\cdot\nu] + g(x,t)\,, && \text{on } \Gamma_T; && (4.11)\\
u(x,t) &= 0\,, && \text{on } \partial\Omega\times(0,T); && (4.12)\\
u(x,0) &= \overline{u}_0(x)\,, && \text{in } \Omega. && (4.13)
\end{aligned}
$$

Theorem 4.2. Let $A \in \big(L^\infty(\Omega)\big)^{N\times N}$ and $B \in \big(L^\infty(\Gamma)\big)^{N\times N}$ be two given symmetric matrices satisfying (2.5) and (4.6), respectively. Assume that $\overline{u}_0 \in H^1_0(\Omega)$, $f \in L^2(\Omega_T)$, $g \in L^2(\Gamma_T)$. Then problem (4.9)–(4.13) admits a unique solution $u \in L^2\big(0,T;\mathcal{X}_0(\Omega)\big)\cap\mathcal{C}^0\big([0,T];L^2(\Omega)\cap L^2(\Gamma)\big)$.

Proof. Let us consider problem (4.9)–(4.13) in an abstract parabolic setting, as for instance in [25] and [28]. To this purpose, let us set

$$
\begin{aligned}
H &= \{\hat{u} := (u, \widetilde{u}) \in L^2(\Omega) \times L^2(\Gamma)\}\,,\\
V &= \{\hat{u} := (u, \widetilde{u}) \in H^1_0(\Omega) \times H^1_B(\Gamma)\,, \widetilde{u} = tr\big|_{\Gamma}(u)\}\,;
\end{aligned}
\tag{4.14}
$$

and notice that H and V are Hilbert spaces if we define

$$
\begin{aligned}
\langle \hat{u}, \hat{v}\rangle_H &= \int_\Omega uv \,\mathrm{d}x + \int_\Gamma \widetilde{u}\widetilde{v}\,\mathrm{d}\sigma\,;\\
\langle \hat{u}, \hat{v}\rangle_V &= \langle \hat{u}, \hat{v}\rangle_H + \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x + \int_\Gamma \nabla^B\widetilde{u}\cdot\nabla^B\widetilde{v}\,\mathrm{d}\sigma\\
&= \int_\Omega uv \,\mathrm{d}x + \int_\Gamma \widetilde{u}\widetilde{v}\,\mathrm{d}\sigma + \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x + \int_\Gamma \nabla^B\widetilde{u}\cdot\nabla^B\widetilde{v}\,\mathrm{d}\sigma\,.
\end{aligned}
\tag{4.15}
$$

Indeed, H is a product of two Hilbert spaces and V, which is a linear space strictly contained in $H^1_0(\Omega)\times H^1_B(\Gamma)$, is complete and hence a Hilbert space, too. The completeness of V is obtained as done in Subsection 2.3 below formula (2.19).

Moreover, $V \subset H$ with compact and dense injection. Define also the bilinear and symmetric form $a : V\times V \to \mathbb{R}$ as

$$
a(\hat{u},\hat{v}) = \int_\Omega A\nabla u\cdot\nabla v\,\mathrm{d}x + \int_\Gamma B\nabla^B\widetilde{u}\cdot\nabla^B\widetilde{v}\,\mathrm{d}\sigma\,,
\tag{4.16}
$$

which satisfies

$$
\begin{aligned}
|a(\hat{u},\hat{v})| &\le C\Big(\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \|\nabla^B\widetilde{u}\|_{L^2(\Gamma)}\|\nabla^B\widetilde{v}\|_{L^2(\Gamma)}\Big) \le C\|\hat{u}\|_V\|\hat{v}\|_V\,,\\
a(\hat{u},\hat{u}) &\ge c\Big(\|\nabla u\|^2_{L^2(\Omega)} + \|\nabla^B\widetilde{u}\|^2_{L^2(\Gamma)}\Big) \ge c\|\hat{u}\|^2_V\,,
\end{aligned}
\tag{4.17}
$$

where we use both the Poincaré and the trace inequalities and c, C are positive constants depending on Ω, Γ, γ_A, γ_B, $\|A\|_\infty$ and $\|B\|_\infty$. Indeed, since $u\in H^1_0(\Omega)$ and $\widetilde{u} = tr\big|_\Gamma(u)$, the norm $\|\nabla u\|_{L^2(\Omega)}$ controls both $\|u\|_{L^2(\Omega)}$ and $\|\widetilde{u}\|_{L^2(\Gamma)}$. Hence, a is a continuous and coercive bilinear form on $V\times V$.

Now, let us rewrite problem (2.11)–(2.15) in the following abstract form:

$$
\begin{aligned}
&\text{find } \hat{u}\in L^2(0,T;V)\cap\mathcal{C}^0([0,T];H) \text{ such that } \hat{u}(0) = \overline{u}_0 \text{ and}\\
&\qquad \frac{d}{dt}\langle\hat{u}(t),\hat{\phi}\rangle_H + a(\hat{u}(t),\hat{\phi}) = \langle\hat{F}(t),\hat{\phi}\rangle_H\,, \qquad \forall\hat{\phi}\in V
\end{aligned}
\tag{4.18}
$$

in the sense of distribution in $(0,T)$, where we set $\hat{F}(t) = \big(f(\cdot,t), g(\cdot,t)\big)\in H$. Indeed, the weak formulation (2.20) coincides with the distributional formulation of the abstract parabolic equation in (4.18), when we take into account the density of the test functions in $L^2\big(0,T;V\big)\cap\mathcal{C}^0\big([0,T];H\big)$. By [25, Theorem 7.2.1] problem (4.18) admits a unique solution and this concludes the proof. □

Proof of Theorem 4.1. Let us consider the space $\mathcal{S} = L^2(\Omega_{\overline{T}})\times L^2(\Gamma_{\overline{T}})$, endowed with the norm $\|(s,\widetilde{s})\|_{\mathcal{S}} = \sqrt{\|s\|^2_{L^2(\Omega_{\overline{T}})} + \|\widetilde{s}\|^2_{L^2(\Gamma_{\overline{T}})}}$, where $\overline{T}\le T$ will be chosen later. Let us define the operator

$L:\mathcal{S}\to\mathcal{S}$ as $L(s,\widetilde{s})=(r,\widetilde{r})$ where $r\in L^2\big(0,T;\mathcal{X}_0(\Omega)\big)\cap\mathcal{C}^0\big([0,T];L^2(\Omega)\cap L^2(\Gamma)\big)$ is the unique solution of (4.9)–(4.13) with f,g replaced by $f(x,t,s(x,t))$ and $g(x,t,\widetilde{s}(x,t))$, respectively, and $\widetilde{r}=r\,|_{\Gamma_{\overline{T}}}$.

We claim that, if $\overline{T}$ is chosen sufficiently small depending on $\gamma_A,\gamma_B,\ell_f,\ell_g$, but not on the initial datum $\overline{u}_0$, then the operator L is a contraction mapping. Indeed, setting $(R,\widetilde{R})=(r_1-r_2,\widetilde{r}_1-\widetilde{r}_2)\in\mathcal{S}$, where $(r_i,\widetilde{r}_i)=L(s_i,\widetilde{s}_i)$, $i=1,2$, we obtain that R satisfies

$$\frac{\partial R}{\partial t}-\operatorname{div}(A\nabla R)=f(x,t,s_1)-f(x,t,s_2)\,,\qquad \text{in } (\Omega_{\text{int}}\cup\Omega_{\text{out}})\times(0,T); \tag{4.19}$$

$$[R]=0\,,\qquad \text{on } \Gamma_T; \tag{4.20}$$

$$\frac{\partial R}{\partial t}-\operatorname{div}^B(B\nabla^B R)=[A\nabla R\cdot\nu]+g(x,t,\widetilde{s}_1)-g(x,t,\widetilde{s}_2)\,,\quad \text{on } \Gamma_T; \tag{4.21}$$

$$R(x,t)=0\,,\qquad \text{on } \partial\Omega\times(0,T); \tag{4.22}$$

$$R(x,0)=0\,,\qquad \text{in } \Omega. \tag{4.23}$$

Hence, multiplying (4.19) by R, integrating by parts in $\Omega_{\overline{T}}$, and taking into account (4.20)–(4.23), we obtain

$$\begin{aligned}
&\sup_{(0,\overline{T})}\int_\Omega R^2(t)\,\mathrm{d}x+\sup_{(0,\overline{T})}\int_\Gamma R^2(t)\,\mathrm{d}\sigma+\int_0^{\overline{T}}\int_\Omega|\nabla R|^2\,\mathrm{d}x\,\mathrm{d}\tau+\int_0^{\overline{T}}\int_\Gamma|\nabla^B R|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\\
&\le\gamma\left(\int_0^{\overline{T}}\int_\Omega[f(x,t,s_1)-f(x,t,s_2)]R\,\mathrm{d}x\,\mathrm{d}\tau+\int_0^{\overline{T}}\int_\Gamma[g(x,t,\widetilde{s}_1)-g(x,t,\widetilde{s}_2)]R\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)\\
&\le\gamma\left(\ell_f\int_0^{\overline{T}}\int_\Omega(s_1-s_2)R\,\mathrm{d}x\,\mathrm{d}\tau+\ell_g\int_0^{\overline{T}}\int_\Gamma(\widetilde{s}_1-\widetilde{s}_2)R\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)\\
&\le\gamma\left(\frac{\ell_f}{2}\|s_1-s_2\|^2_{L^2(\Omega_{\overline{T}})}+\frac{\ell_f}{2}\|R\|^2_{L^2(\Omega_{\overline{T}})}+\frac{\ell_g}{2}\|\widetilde{s}_1-\widetilde{s}_2\|^2_{L^2(\Gamma_{\overline{T}})}+\frac{\ell_g}{2}\|R\|^2_{L^2(\Gamma_{\overline{T}})}\right),
\end{aligned}\tag{4.24}$$

where $\gamma=\frac{1}{\min(2^{-1},\gamma_A,\gamma_B)}$. Dropping the last two integrals in the first line of (4.24) and integrating in $(0,\overline{T})$ we get

$$\begin{aligned}
&\int_0^{\overline{T}}\int_\Omega R^2(\tau)\,\mathrm{d}x\,\mathrm{d}\tau+\int_0^{\overline{T}}\int_\Gamma R^2(\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau\\
&\le\frac{\gamma}{2}\max(\ell_f,\ell_g)\overline{T}\left(\|s_1-s_2\|^2_{L^2(\Omega_{\overline{T}})}+\|R\|^2_{L^2(\Omega_{\overline{T}})}+\|\widetilde{s}_1-\widetilde{s}_2\|^2_{L^2(\Gamma_{\overline{T}})}+\|R\|^2_{L^2(\Gamma_{\overline{T}})}\right).
\end{aligned}\tag{4.25}$$

Now, choosing $\overline{T}=\frac{1}{2\gamma\max(\ell_f,\ell_g)}$, after simple computations, it follows

$$\int_0^{\overline{T}}\int_\Omega R^2\,\mathrm{d}x\,\mathrm{d}\tau+\int_0^{\overline{T}}\int_\Gamma R^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\le\frac{1}{3}\left(\|s_1-s_2\|^2_{L^2(\Omega_{\overline{T}})}+\|\widetilde{s}_1-\widetilde{s}_2\|^2_{L^2(\Gamma_{\overline{T}})}\right),\tag{4.26}$$

which implies

$$\|L(s_1,\widetilde{s}_1)-L(s_2,\widetilde{s}_2)\|_{\mathcal{S}}=\|(R,\widetilde{R})\|_{\mathcal{S}}\le\frac{1}{\sqrt{3}}\|(s_1-s_2,\widetilde{s}_1-\widetilde{s}_2)\|_{\mathcal{S}}.\tag{4.27}$$

Hence the claim is proved. Therefore, by the Contraction Mapping Theorem there exists a unique fixed point of L in $\mathcal{S}$ given by $(u\mid_{\Omega_{\overline{T}}}, u\mid_{\Gamma_{\overline{T}}})$, where $u\in L^2\big(0,T;\mathcal{X}_0(\Omega)\big)\cap\mathcal{C}^0\big([0,T];L^2(\Omega)\cap L^2(\Gamma)\big)$ and satisfies

$$\frac{\partial u}{\partial t}-\operatorname{div}(A\nabla u)=f(x,t,u)\,,\qquad \text{in } (\Omega_{\text{int}}\cup\Omega_{\text{out}})\times(0,\overline{T}); \tag{4.28}$$

$$[u]=0\,,\qquad \text{on } \Gamma\times(0,\overline{T}); \tag{4.29}$$

$$\frac{\partial u}{\partial t}-\operatorname{div}^B(B\nabla^B u)=[A\nabla u\cdot\nu]+g(x,t,u)\,,\qquad \text{on } \Gamma\times(0,\overline{T}); \tag{4.30}$$

$$u(x,t)=0\,,\qquad \text{on } \partial\Omega\times(0,\overline{T}); \tag{4.31}$$

$$u(x,0)=\overline{u}_0\,,\qquad \text{in } \Omega. \tag{4.32}$$

Since $\overline{T}$ is independent of the initial datum $\overline{u}_0$, the previous procedure can be iterated step by step in intervals with amplitude $\overline{T}$, thus covering the whole time interval $(0,T)$ in the statement of the theorem. □

Remark 4.3. Notice that the results in Theorems 4.1 and 4.2 can be generalized to the case in which capacitive coefficients $\mu\in L^\infty(\Omega)$ and $\alpha\in L^\infty(\Gamma)$ (with $\mu(x)\geq\mu_0$ a.e. in Ω and $\alpha(x)\geq\alpha_0$ a.e. on Γ, for proper constants $\mu_0,\alpha_0>0$) appear in front of the time derivative in (4.1) (or in (4.9)) and in (4.3) (or in (4.11)). Indeed it is enough to redefine in the proof of Theorem 4.2 the scalar product on the space H as

$$\langle\hat{u},\hat{v}\rangle_H=\int_\Omega\mu(x)uv\,\mathrm{d}x+\int_\Gamma\alpha(x)\widetilde{u}\widetilde{v}\,\mathrm{d}\sigma\,.$$

5 Time-asymptotic limit

In this section we will prove that, for $t\to+\infty$, the solution of problem (4.9)–(4.13), with $f\in L^2(\Omega)$ and $g\in L^2(\Gamma)$ independent of time, converges in a suitable way to the solution u_∞ of the following elliptic system

$$-\operatorname{div}(A\nabla u_\infty)=f\,,\qquad \text{in } \Omega_{\text{int}},\cup\Omega_{\text{out}}; \tag{5.1}$$

$$[u_\infty]=0\,,\qquad \text{on } \Gamma; \tag{5.2}$$

$$-\operatorname{div}^B(B\nabla^B u_\infty)=[A\nabla u_\infty\cdot\nu]+g\,,\qquad \text{on } \Gamma; \tag{5.3}$$

$$u_\infty=0\,,\qquad \text{on } \partial\Omega. \tag{5.4}$$

In order to achieve this goal, we first state an existence and uniqueness theorem for the previous elliptic system. It is a quite standard result, based on the Lax-Milgram lemma, but for the sake of completeness, we prefer to give here the complete proof.

Theorem 5.1. Let $A\in\big(L^\infty(\Omega)\big)^{N\times N}$ be a symmetric matrix satisfying (2.5) and $B\in\big(L^\infty(\Gamma)\big)^{N\times N}$ be a symmetric matrix satisfying (4.6). Assume also that $f\in L^2(\Omega)$ and $g\in L^2(\Gamma)$. Then problem (5.1)–(5.4) admits a unique solution $u_\infty\in\mathcal{X}_0(\Omega)$.

We recall that the weak formulation of problem (5.1)–(5.4) is the following

$$
\begin{aligned}
&\text{find } u_\infty \in \mathcal{X}_0(\Omega) \text{ such that}\\
&\int_\Omega A\nabla u_\infty\cdot\nabla\phi\,\mathrm{d}x+\int_\Gamma B\nabla^B u_\infty\cdot\nabla^B\phi\,\mathrm{d}\sigma=\int_\Omega f\phi\,\mathrm{d}x+\int_\Gamma g\phi\,\mathrm{d}\sigma\,,\qquad \forall\phi\in\mathcal{X}_0(\Omega)\,.
\end{aligned}
\tag{5.5}
$$

Moreover, the following energy estimate holds

$$
\int_\Omega |\nabla u_\infty|^2\,\mathrm{d}x+\int_\Gamma |\nabla^B u_\infty|^2\,\mathrm{d}\sigma\le C\,,
\tag{5.6}
$$

where the positive constant C depends on $\gamma_A,\gamma_B,\Omega,\Gamma$, $\|f\|_{L^2(\Omega)}$ and $\|g\|_{L^2(\Gamma)}$.

Proof. Let us consider the Hilbert space $\mathcal{X}_0(\Omega)$ endowed with the scalar product defined by

$$
\langle u,v\rangle_{\mathcal{X}_0(\Omega)}=\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x+\int_\Gamma \nabla^B u\cdot\nabla^B v\,\mathrm{d}\sigma\,.
$$

Taking into account the Poincaré and the standard trace inequalities, we get that the previous scalar product is equivalent (and also more convenient in this context) to the one defined in (2.19).

Consider the bilinear and symmetric form $a:\mathcal{X}_0(\Omega)\times\mathcal{X}_0(\Omega)\to\mathbb{R}$ defined by

$$
a(u,v)=\int_\Omega A\nabla u\cdot\nabla v\ \mathrm{d}x+\int_\Gamma B\nabla^B u\cdot\nabla^B v\ \mathrm{d}\sigma\,,
$$

which satisfies

$$
\begin{aligned}
&|a(u,v)|\le C\Big(\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}+\|\nabla^B u\|_{L^2(\Gamma)}\|\nabla^B v\|_{L^2(\Gamma)}\Big)\le C\|u\|_{\mathcal{X}_0(\Omega)}\|v\|_{\mathcal{X}_0(\Omega)}\,,\\
&a(u,u)\ge c\Big(\|\nabla u\|^2_{L^2(\Omega)}+\|\nabla^B u\|^2_{L^2(\Gamma)}\Big)=c\|u\|^2_{\mathcal{X}_0(\Omega)}\,.
\end{aligned}
\tag{5.7}
$$

Therefore a is a continuous and coercive bilinear form on $\mathcal{X}_0(\Omega)\times\mathcal{X}_0(\Omega)$. Moreover, defining the linear functional $L:\mathcal{X}_0(\Omega)\to\mathbb{R}$ as

$$
L(u)=\int_\Omega fu\,\mathrm{d}x+\int_\Gamma gu\,\mathrm{d}\sigma\,,
$$

it follows that L is continuous on $\mathcal{X}_0(\Omega)$, since

$$
|L(u)|\le\|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}+\|g\|_{L^2(\Gamma)}\|u\|_{L^2(\Gamma)}\le C\|u\|_{\mathcal{X}_0(\Omega)}\,,
$$

where C is a positive constant depending on $\|f\|_{L^2(\Omega)},\|g\|_{L^2(\Gamma)}$, the Poincaré constant and the constant in the standard trace inequality. Finally, notice that the weak formulation (5.5) can be written in the form

$$
a(u_\infty,\phi)=L(\phi)\,,\qquad\forall\phi\in\mathcal{X}_0(\Omega)\,.
\tag{5.8}
$$

Hence, the stated result is achieved applying Lax-Milgram lemma to (5.8) and this concludes the proof. □

Remark 5.2. Clearly, if we assume that $f \in \mathcal{C}^\infty(\Omega)$ and $g \in \mathcal{C}^\infty(\Gamma)$, then the solution u_∞ to problem (5.1)–(5.4) belongs to $\mathcal{C}^\infty(\Omega)$. Indeed, since by our assumptions in Subsection 2.2, Γ is of class $\mathcal{C}^\infty$, the proof is quite standard and it relies on a local rectification of Γ and on an iterated use of energy estimates (similar to the one in (5.6)) applied to higher order derivatives of u_∞.

Remark 5.3. Notice that we can prove also a periodic version of Theorem 5.1; i.e., we can prove an existence and uniqueness result for the periodic problem

$$-\operatorname{div}(A\nabla \mathrm{v}) = f\,, \qquad \text{in } E_{\text{int}} \cup E_{\text{out}}; \tag{5.9}$$

$$[\mathrm{v}] = 0\,, \qquad \text{on } \mathcal{G}; \tag{5.10}$$

$$-\operatorname{div}^B(B\nabla^B \mathrm{v}) = [A\nabla \mathrm{v}\cdot\nu] + g\,, \qquad \text{on } \mathcal{G}; \tag{5.11}$$

$$\mathrm{v} \quad \text{is } Y\text{-periodic}; \tag{5.12}$$

$$\int_Y \mathrm{v}\,\mathrm{d}y = 0\,; \tag{5.13}$$

where the requirements (5.12)–(5.13) replace the previous boundary condition (5.4). Here, we have denoted by Y the unit open cell $(0,1)^N \subset \mathbb{R}^N$, we have assumed that $Y = E_{\text{out}} \cup E_{\text{int}} \cup \mathcal{G}$, where E_{int} and E_{out} are two disjoint open subsets of Y, E_{out} are connected, $\mathcal{G} = \partial E_{\text{int}} = \partial E_{\text{out}} \cap Y$, $\mathcal{G} \cap \partial Y = \emptyset$ and $\mathcal{G}$ is of class $\mathcal{C}^\infty$, and we have also denoted by ν the normal unit vector to $\mathcal{G}$ pointing into E_{out}. However, in this case, in order to apply a suitable version of Lax-Milgram lemma (see, for instance, [19, Lemma 2.1], we need to assume also that the compatibility condition

$$\int_Y f\,\mathrm{d}y + \int_{\mathcal{G}} g\,\mathrm{d}\sigma = 0 \tag{5.14}$$

is satisfied, as it is common in periodic problems.

Theorem 5.4. Assume that $A \in \big(L^\infty(\Omega)\big)^{N\times N}$ and $B \in \big(L^\infty(\Gamma)\big)^{N\times N}$ are two given symmetric matrices satisfying (2.5) and (4.6), respectively, and that the initial datum $\overline{u}_0 \in H^1_0(\Omega)$. Assume that $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$ are independent of time. Let $u \in L^2\big(0,T;\mathcal{X}_0(\Omega)\big) \cap \mathcal{C}^0\big([0,T];L^2(\Omega)\cap L^2(\Gamma)\big)$ be the unique solution of problem (4.9)–(4.13) and $u_\infty \in \mathcal{X}_0(\Omega)$ be the unique solution of problem (5.1)–(5.4). Then, there exist $\theta,\gamma > 0$ such that

$$\|u(t) - u_\infty\|_{H^1(\Omega)} + \|u(t) - u_\infty\|_{H^1_B(\Gamma)} \le \gamma \mathrm{e}^{-\theta t}, \qquad \forall t \ge 1\,. \tag{5.15}$$

Proof. We first prove that

$$\|u_t(t)\|^2_{L^2(\Omega)} + \|u_t(t)\|^2_{L^2(\Gamma)} \le \widetilde{\gamma}\mathrm{e}^{-2\theta t}, \qquad \forall t \ge 1\,, \tag{5.16}$$

for suitable $\theta,\widetilde{\gamma} > 0$. Indeed, by [25, Theorem 7.2.1], we get that there exists a strictly increasing sequence of nonnegative eigenvalues λ_j and a sequence of eigenfunctions $\hat{w}_j := (w_j, \widetilde{w}_j) \in V$ (recall the definition of V given in (4.14)) such that

$$\hat{u}(x,t) := (u,\widetilde{u}) = \sum_{j=1}^{+\infty}\left[a_j - \frac{F_j}{\lambda_j}\right]\hat{w}_j(x)\mathrm{e}^{-\lambda_j t} + \sum_{j=1}^{+\infty}\frac{F_j}{\lambda_j}\hat{w}_j(x)\,, \tag{5.17}$$

with $a_j = \langle \hat{u}(0), \hat{w}_j \rangle_H$ and $F_j = \langle \hat{F}, \hat{w}_j \rangle_H$, where $\hat{F} = (f, g) \in H$. In (5.17), $\widetilde{u}$ stands for $tr|_\Gamma(u)$. We claim that the first eigenvalue λ_1 is different from zero. Indeed, if this is not the case, we have that $\hat{w}_1 = (w_1, \widetilde{w}_1)$ is a nonzero solution of the following eigenvalue problem

$$
\begin{aligned}
-\operatorname{div}(A\nabla w_1) &= 0\,, && \text{in } \Omega_{\text{int}}, \cup \Omega_{\text{out}};\\
[w_1] &= 0\,, && \text{on } \Gamma;\\
-\operatorname{div}^B(B\nabla^B w_1) &= [A\nabla w_1 \cdot \nu]\,, && \text{on } \Gamma;\\
w_1 &= 0\,, && \text{on } \partial\Omega;
\end{aligned}
$$

and this is a contradiction thanks to the uniqueness property stated in Theorem 5.1. Recall that $\widetilde{w}_1$ is the trace of w_1 on Γ; therefore, the second and the third equations above should be written in terms of $\widetilde{w}_1$. However, with abuse of notation, we prefer not to invoke $\widetilde{w}_1$, thus following the same notation as in (5.1)–(5.4).

Differentiating (5.17) with respect to t, we obtain

$$
\hat{u}_t(x,t) = -\sum_{j=1}^{+\infty} [\lambda_j a_j - F_j] \hat{w}_j(x) \mathrm{e}^{-\lambda_j t}\,, \tag{5.18}
$$

where, for $t \geq 1$, we have

$$
\lambda_j^2 \mathrm{e}^{-2(\lambda_j - \lambda_1)t} \leq \lambda_j^2 \mathrm{e}^{-2(\lambda_j - \lambda_1)} \leq \lambda_1^2 + \mathrm{e}^{-2(1-\lambda_1)} =: \widetilde{\lambda}\,, \qquad \forall j \geq 1\,.
$$

Notice that, in the last inequality, we have used the fact that, for $j > 1$, the function $\lambda_j \mapsto \lambda_j^2 \mathrm{e}^{-2(\lambda_j - \lambda_1)}$ reaches its maximum value for $\lambda_j = 1$. Therefore, taking into account that $[\lambda_j a_j - F_j]^2 \leq 2(\lambda_j^2 a_j^2 + F_j^2)$ and

$$
\sum_{j=1}^{+\infty} a_j^2 = \|\overline{u}_0\|_H^2 \qquad \text{and} \qquad \sum_{j=1}^{+\infty} F_j^2 = \|\hat{F}\|_H^2\,,
$$

it follows

$$
\|u_t(t)\|^2_{L^2(\Omega)} + \|u_t(t)\|^2_{L^2(\Gamma)} = \|u_t(t)\|^2_H \leq 2\mathrm{e}^{-2\lambda_1 t}(\widetilde{\lambda}\|\overline{u}_0\|^2_H + \|\hat{F}\|^2_H)\,,
$$

and hence (5.16) holds with $\theta = \lambda_1$ and the constant $\widetilde{\gamma} = 2(\widetilde{\lambda}\,\|\overline{u}_0\|^2_H + \|\hat{F}\|^2_H)$.

Now, for a.e. $t \geq 1$, set $U(t) = u(t) - u_\infty$ and notice that it solves the system

$$
\begin{aligned}
-\operatorname{div}(A\nabla U(t)) &= -u_t(t)\,, && \text{in } \Omega_{\text{int}}, \cup \Omega_{\text{out}};\\
[U(t)] &= 0\,, && \text{on } \Gamma;\\
-\operatorname{div}^B(B\nabla^B U(t)) &= [A\nabla U(t) \cdot \nu] - u_t(t)\,, && \text{on } \Gamma;\\
U(t) &= 0\,, && \text{on } \partial\Omega.
\end{aligned}
$$

By standard computations, we obtain

$$\int_\Omega A\nabla U(t)\cdot\nabla U(t)\,\mathrm{d}x+\int_\Gamma B\nabla^B U(t)\cdot\nabla^B U(t)\,\mathrm{d}\sigma=-\int_\Omega u_t(t)U(t)\,\mathrm{d}x-\int_\Gamma u_t(t)U(t)\,\mathrm{d}\sigma\,,$$

which, applying Young's inequality, implies

$$\begin{aligned}&\int_\Omega |\nabla U(t)|^2\,\mathrm{d}x+\int_\Gamma |\nabla^B U(t)|^2\,\mathrm{d}\sigma\\ &\leq \gamma\left(\frac{1}{2\delta}\int_\Omega |u_t(t)|^2\,\mathrm{d}x+\frac{\delta}{2}\int_\Omega |U(t)|^2\,\mathrm{d}x+\frac{1}{2\delta}\int_\Gamma |u_t(t)|^2\,\mathrm{d}\sigma+\frac{\delta}{2}\int_\Gamma |U(t)|^2\,\mathrm{d}\sigma\right).\end{aligned}$$

Then, using Poincaré's inequality and (5.16) and choosing δ sufficiently small, we get

$$\int_\Omega |\nabla U(t)|^2\,\mathrm{d}x+\int_\Gamma |\nabla^B U(t)|^2\,\mathrm{d}\sigma\leq\gamma\mathrm{e}^{-2\lambda_1 t}\,.$$

A further application of Poincaré's inequality leads to (5.15) with $\theta=\lambda_1$ and γ a positive constant independent of u and u_∞. □

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