Peristaltic axisymmetric flow of a Bingham fluid

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Abstract

We model the peristaltic flow of a Bingham fluid in a tube in lubrication approximation. Following the procedure developed in [4] we derive the rigid plug equation using an integral formulation for the balance of linear momentum, modelling the unyielded domain as an evolving non-material volume. The mathematical problem is formulated for the yielded and unyielded part and appropriate boundary conditions are established at the pipe walls and at the yield surface. The zero order approximation leads to a system formed by an integral equation and an algebraic equation for the yield surface and for the plug velocity (which is uniform in space) respectively. Because of the integral approach adopted in the unyielded part of the flow, the leading order approximation does not give rise to the lubrication paradox. The problem is solved numerically and an analytical solution is found when the oscillating wall is given as a small perturbation of the uniform wall.

Keywords: Bingham fluids; lubrication approximation; peristaltic flows; asymptotic expansion; traveling wave; numerical simulations

1 Introduction

Peristaltic flows in channels or ducts are generated by the continuous periodic contraction and expansion of the flexible walls. The mechanism of peristalsis occurs in different branches of biomechanics and it is of prime importance when considering, for instance, how physiological fluids such as blood and urine are transported in the human body. Among the many biomechanical processes involving peristalsis we find the swallowing of food through the oesophagus, the movement of chyme through the intestine, spermatic flow in the male reproductive system, the movement of eggs in the fallopian tube and the transport of bile. Other non-biological important applications are the design of finger and roller pumps used in pumping fluids where contamination due to contact with the pumping machinery has to be avoided, the transport of highly viscous fluids or slurries, the design of dyalises machines, open-heart bypass, infusion pumps etc.

For all its potential applications peristalsis has been the subject of intensive theoretical and experimental studies in the past, see [8], [3], [12], [13]. When modelling peristaltic motion,

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the main scope is to characterize the basic fluid mechanics of the process and to determine the velocities and the pressure gradients that are generated by the wave. To obtain analytical solutions the models often rely on simplifying assumptions such as vanishingly small Reynolds number (creeping flow), infinitely long wavelength, small wave amplitude etc. Since the type of fluid commonly involved in peristaltic motion is non-Newtonian, a large part of the theoretical modelling is focussed on non-Newtonian fluids such as Jeffrey fluid, Casson fluid, Herschel-Bulkley fluid, Bingham fluid, Power-law fluid, etc. We refer the reader to the paper [7], where a schematic summary of key assumptions made by recent literatures related to peristaltic transport is available.

In this paper we investigate the peristaltic motion of a Bingham fluid flowing in a cylindrical deformable tube whose shape can be described by a periodic travelling wave. In particular we consider a flow which occurs in lubrication regime, that is we assume that the characteristic length of the tube is far larger than the typical radius. Although this kind of study is certainly not new (see for instance [9], [11], [10]), here we investigate the motion adopting a novel approach which consists in deriving the plug momentum equation using an integral formulation. This methodology has been introduced in [4] to model the planar flow of a Bingham plastic in a channel of varying width. The key feature is to treat the unyielded phase as an evolving non-material volume and to write the momentum balance globally and not locally. Indeed, it is well known that when modelling the plug with the classical differential formulation

$$\rho^* \frac{D\mathbf{v}^*}{Dt^*} = \nabla^* \cdot \boldsymbol{\sigma}^* \tag{1}$$

that requires the knowledge of the stress components σ^*_{ij} in the unyielded domain, we may end up with the so-called *lubrication* paradox, which consists in getting a plug velocity which depends on the longitudinal coordinate of the tube (see [2]). In [4] we have proved that this paradox arises because one cannot model the unyielded part using the stress components which are not even defined for that part of the fluid. To avoid this inconsistency we have treated the rigid part of the flow as an evolving non-material volume and we have written the momentum equation in the following integral form

$$\int_{\Omega_u^*} \frac{\partial}{\partial t^*}(\rho^* \mathbf{v}^*) dV^* + \int_{\partial\Omega_u^*} \rho^* \mathbf{v}^* (\mathbf{v}^* \cdot \mathbf{n}) dS = \int_{\partial\Omega_u^*} \boldsymbol{\sigma}^* \mathbf{n} dS, \tag{2}$$

where it is only required to know the stress $\boldsymbol{\sigma}^*$ exerted by the yielded part at the yield surface. Such a stress can be evaluated solving equation (1) in the yielded part. Using this strategy in [4] we have shown that it is possible to find an explicit expression for the yield surface and to determine the inner plug that moves with uniform velocity even for non-uniform channel walls. This method has been subsequently used to study the planar squeeze flow of a Bingham fluid [5] and the flow of Bingham fluid down an inclined channel [6]. Here we use the same procedure to model the peristaltic flow of a Bingham fluid in a tube where the walls evolve as a travelling wave.

The paper is organized as follows. We derive the model for a generic 3D setting using cylindrical coordinates (r^*, θ, z^*) and assuming that the main variables of the problem do not depend on θ. We rescale the problem assuming that the aspect ratio ε - corresponding to the ratio between the characteristic length of the tube and the characteristic radius - is small. Then we focus on the leading order approximation and we show that the mathematical problem reduces to a set of two equations (one of which is an integral equation) involving the velocity of the plug and the yield surface. Of course the general problem can be solved only numerically. Finally we assume that the oscillation amplitude of the wall is a *small parameter* too and we show that setting this parameter to zero corresponds to considering the classical Bingham model in cylindrical geometry with fixed walls. Finally we show that assuming that the oscillating wall is a small perturbation of the fixed wall, an analytical solution can be found as a *small perturbation* of the solution with uniform wall. For this latter case we plot the evolution of the yield surface, the plug velocity and the pressure gradient at different times. Moreover we study the dependence of the solution on the principal parameters of the model, i.e. the Bingham number and the prescribed inlet discharge.

2 Mathematical formulation of the problem

We consider the flow of a Bingham fluid in a pipe of circular-cross section whose amplitude[1] R^* evolves as a traveling wave. We suppose that the length of the pipe is L^*. The flow is modelled considering cylindrical coordinates (r^*, θ, z^*) and assuming that all the kinematical quantities appearing in the system do not depend on θ. The velocity field is of the form

$$\mathbf{v}^*(r^*, z^*, t^*) = v_r^*(r^*, z^*, t^*)\mathbf{e}_r + v_z^*(r^*, z^*, t^*)\mathbf{e}_z$$

Referring to Fig. 1 we assume that the travelling wave describing the motion of the wall is given by

$$R^*(z^*, t^*) = R^*_{ref}\left[1 + \delta\varphi\left(\frac{z^*}{\lambda^*} - \frac{t^*}{T^*}\right)\right] \tag{3}$$

where $R^*_{ref}\delta$ is semi-amplitude of the wave with $\delta \in (0, 1)$, $\varphi \in [-1, 1]$ is a smooth periodic function of period one, λ^* is the wavelength and T^* is the wave period. The stress $\boldsymbol{\sigma}^*$ can be decomposed in the following form

$$\boldsymbol{\sigma}^* = -p^*\boldsymbol{I} + \boldsymbol{\tau}^*,$$

where $\boldsymbol{\tau}^*$ is traceless deviatoric part of the stress and $p^* = -(1/3)\,\mathrm{tr}\,\boldsymbol{\tau}^*$ is the mean normal stress, or mechanical pressure. The Bingham stresses τ^*_{ij} are related to the strain rates $\dot{\gamma}^*_{ij}$ through the

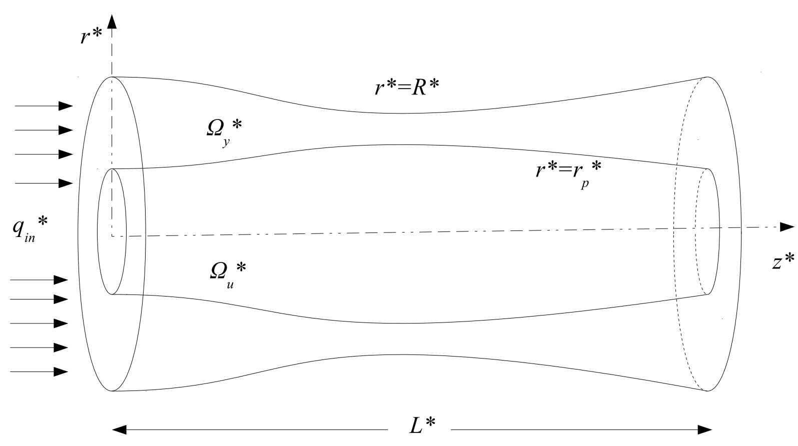

Figure 1: Sketch of the system

[1]The starred variables denote dimensional quantities.

constitutive equations

$$\begin{cases} \tau^*_{ij} = \left(2\mu^* + \dfrac{\tau^*_o}{\dot\gamma^*}\right)\dot\gamma^*_{ij} & \tau^* \geqslant \tau^*_o, \\ \\ \gamma^*_{ij} = 0 & \tau^* \leqslant \tau^*_o, \end{cases}$$

where τ^*_o is the yield stress, μ^* is the Bingham viscosity and where

$$\dot\gamma^* = \sqrt{\frac{1}{2}\sum \dot\gamma^*_{ij}\dot\gamma^*_{ij}}, \qquad \tau^* = \sqrt{\frac{1}{2}\sum \tau^*_{ij}\tau^*_{ij}},$$

are the second invariants of the strain and the stress respectively. In cylindrical coordinates

$$\tau^*_{rr} = \left(2\mu^* + \frac{\tau^*_o}{\dot\gamma^*}\right)\frac{\partial v^*_r}{\partial r^*}, \qquad \tau^*_{rz} = \left(2\mu^* + \frac{\tau^*_o}{\dot\gamma^*}\right)\frac{1}{2}\left[\frac{\partial v^*_r}{\partial z^*} + \frac{\partial v^*_z}{\partial r^*}\right]$$

$$\tau^*_{\theta\theta} = \left(2\mu^* + \frac{\tau^*_o}{\dot\gamma^*}\right)\frac{v^*_r}{r^*}, \qquad \tau^*_{zz} = \left(2\mu^* + \frac{\tau^*_o}{\dot\gamma^*}\right)\frac{\partial v^*_z}{\partial z^*}$$

We suppose that the yielded ($\tau^* \geqslant \tau^*_o$) and unyielded ($\tau^* \leqslant \tau^*_o$) phases are separated by a sharp interface $r^* = r^*_p(z^*, t^*)$, so that the whole domain can be divided into

$$\Omega^*_y = \left\{(r^*, z^*) : \quad z^* \in [0, L^*], \quad r^* \in [r^*_p, R^*]\right\}$$

$$\Omega^*_u = \left\{(r^*, z^*) : \quad z^* \in [0, L^*], \quad r^* \in [0, r^*_p]\right\}$$

where Ω^*_u represents the unyielded plug moving with uniform velocity. Assuming incompressibility, the velocity field satisfies

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* v^*_r\right) + \frac{\partial v^*_z}{\partial z^*} = 0, \tag{4}$$

in $\Omega^*_u \cup \Omega^*_y$. Following [4] we write the local differential form of the momentum balance in Ω^*_y, while in Ω^*_u we write the momentum balance using the global integral formulation, because in this latter region the stress is not defined and the local differential form of the momentum balance cannot be used [2]. Hence in Ω^*_y

$$\begin{cases} \rho^*\left[\dfrac{\partial v^*_r}{\partial t^*} + v^*_r\dfrac{\partial v^*_r}{\partial r^*} + v^*_z\dfrac{\partial v^*_r}{\partial z^*}\right] = -\dfrac{\partial p^*}{\partial r^*} + \dfrac{1}{r^*}\dfrac{\partial}{\partial r^*}\left(r^*\tau^*_{rr}\right) - \dfrac{\tau^*_{\theta\theta}}{r^*} + \dfrac{\partial \tau^*_{rz}}{\partial z^*} \\ \\ \rho^*\left[\dfrac{\partial v^*_z}{\partial t^*} + v^*_r\dfrac{\partial v^*_z}{\partial r^*} + v^*_z\dfrac{\partial v^*_z}{\partial z^*}\right] = -\dfrac{\partial p^*}{\partial z^*} + \dfrac{1}{r^*}\dfrac{\partial}{\partial r^*}\left(r^*\tau^*_{rz}\right) + \dfrac{\partial \tau^*_{zz}}{\partial z^*} \end{cases} \tag{5}$$

In Ω^*_u, following [4], we write

$$\int\limits_{\Omega^*_u} \frac{\partial}{\partial t^*}(\rho^*\mathbf{v}^*)dV^* + \int\limits_{\partial\Omega^*_u} \rho^*\mathbf{v}^*(\mathbf{v}^*\cdot\mathbf{n})dS = \int\limits_{\partial\Omega^*_u} \boldsymbol{\sigma}^*\mathbf{n}dS, \tag{6}$$

where $\mathbf{n}$ is the outward normal to $\partial\Omega^*_u$. For symmetry reasons the velocity field in the unyielded phase is

$$\mathbf{v}^* = v^*_p(t)\mathbf{e}_z,$$

[2] See [4] for detailed discussion on how to model the unyielded part of a Bingham fluid.

where $v_p^*(t)$ is unknown. Writing (6) component-wise we find

$$\begin{cases} 0 = \displaystyle\int_{\partial\Omega_u^*} \boldsymbol{\sigma}^* \mathbf{n} \cdot \mathbf{e}_r dS^*, \\ \rho^* \dot{v}_p^* \displaystyle\int_{\Omega_u^*} dV^* + \rho^* {v_p^*}^2 \int_{\partial\Omega_u^*} (\mathbf{e}_z \cdot \mathbf{n}) dS = \int_{\partial\Omega_u^*} \boldsymbol{\sigma}^* \mathbf{n} \cdot \mathbf{e}_z dS, \end{cases} \tag{7}$$

The boundary $\partial\Omega_u^*$ can be split as follows

$$\partial\Omega_u^* = \Sigma_\ell^* \cup \Sigma_{in}^* \cup \Sigma_{out}^*$$

where Σ_ℓ^* is the lateral boundary $r^* = r_p^*$ and Σ_{in}^*, Σ_{out}^* are the inlet and outlet boundaries. The outward normals are

$$\mathbf{n}_\ell = \frac{\mathbf{e}_r - \left(\dfrac{\partial r_p^*}{\partial z^*}\right) \mathbf{e}_z}{\sqrt{1 + \left(\dfrac{\partial r_p^*}{\partial z^*}\right)^2}}, \qquad \mathbf{n}_{in} = -\mathbf{e}_z, \qquad \mathbf{n}_{out} = \mathbf{e}_z.$$

We assume that the stress at the inlet and outlet is given by

$$\boldsymbol{\sigma}_{in}^* = \begin{bmatrix} -p_{in}^* & 0 & 0 \\ 0 & -p_{in}^* & 0 \\ 0 & & -p_{in}^* \end{bmatrix} \qquad \boldsymbol{\sigma}_{out}^* = \begin{bmatrix} -p_{out}^* & 0 & 0 \\ 0 & -p_{out}^* & 0 \\ 0 & & -p_{out}^* \end{bmatrix} \tag{8}$$

where p_{in}^*, p_{out}^* may depend on time and are unknown. The particular form of (8) implies that the tangential components of the stress at $z^* = 0$ and $z^* = L^*$ are zero and the stress is directed along the normal $\mathbf{e}_z$, so that no torque is applied to the rigid plug. As a consequence $(7)_1$ can be rewritten as

$$0 = \underbrace{\int_{\Sigma_\ell^*} \boldsymbol{\sigma}^* \mathbf{n}_\ell \cdot \mathbf{e}_r dS}_{=0 \text{ symmetry}} + \underbrace{\int_{\Sigma_{in}^*} \boldsymbol{\sigma}_{in}^* \mathbf{n}_{in} \cdot \mathbf{e}_r dS}_{=0} + \underbrace{\int_{\Sigma_{out}^*} \boldsymbol{\sigma}_{out}^* \mathbf{n}_{in} \cdot \mathbf{e}_r dS}_{=0}.$$

Therefore the equation for the momentum in the unyielded plug reduces to the second of (7). We notice that

$$\int_{\partial\Omega_u^*} (\mathbf{e}_z \cdot \mathbf{n}) dS = \int_0^{2\pi} d\theta \int_0^{L^*} -r_p^* \frac{\partial r_p^*}{\partial z^*} dz^* - \pi {r_{p,in}^*}^2 + \pi {r_{p,out}^*}^2 = 0,$$

where $r_{p,in}^* = r_p^*(0, t^*)$, $r_{p,out}^* = r_p^*(L^*, t^*)$. Moreover

$$\int_{\partial\Omega_u^*} \boldsymbol{\sigma}^* \mathbf{n} \cdot \mathbf{e}_z dS = \int_0^{2\pi} d\theta \int_0^{L^*} \left(\tau_{rz}^* + p^* \frac{\partial r_p^*}{\partial z^*} - \tau_{zz}^* \frac{\partial r_p^*}{\partial z^*} \right)\Bigg|_{r_p^*} r_p^* dz^* + 2\pi \left[\int_0^{r_{p,in}^*} p_{in}^* r^* dr^* - \int_0^{r_{p,out}^*} p_{out}^* r^* dr^* \right]$$

Assuming that the pressure is continuous across Σ^*_{in}, Σ^*_{out} , we can integrate by parts the above and find

$$\int\limits_{\partial\Omega^*_u} \boldsymbol{\sigma}^* \mathbf{n}\cdot\mathbf{e}_z dS = 2\pi \int\limits_0^{L^*} \left(\tau^*_{rz} - \frac{r^*_p}{2}\frac{\partial p^*}{\partial z^*} - \tau^*_{zz}\frac{\partial r^*_p}{\partial z^*} \right)\Bigg|_{r^*_p} r^*_p dz^*$$

In conclusion $(7)_2$ reduces to

$$\rho^* \dot{v}^*_p \int\limits_0^{L^*} {r^*_p}^2 dz = \int\limits_0^{L^*} \left(2\tau^*_{rz} - r^*_p \frac{\partial p^*}{\partial z^*} - 2\tau^*_{zz}\frac{\partial r^*_p}{\partial z^*} \right)\Bigg|_{r^*_p} r^*_p dz^* \tag{9}$$

that represents the momentum balance of the unyielded phase. For what concerns the boundary conditions we assume no-slip on the wall $r^* = R^*$ so that

$$v^*_r(R^*, z^*, t^*) = \frac{\partial R^*}{\partial t^*} \qquad\qquad v^*_z(R^*, z^*, t^*) = 0. \tag{10}$$

Still following [4], we assume that the velocity and the stress are continuous across the yield surface $r^* = r^*_p$ so that

$$[\![\mathbf{v}^*\cdot\mathbf{n}_\ell]\!] = [\![\mathbf{v}^*\cdot\mathbf{t}_\ell]\!] = 0 \tag{11}$$

$$[\![\boldsymbol{\sigma}^*\mathbf{n}_\ell\cdot\mathbf{n}_\ell]\!] = -[\![p^*]\!]\left[1 + \left(\frac{\partial r^*_p}{\partial z^*}\right)^2\right] + [\![\tau^*_{rr} - 2\left(\frac{\partial r^*_p}{\partial z^*}\right)\tau^*_{rz} + \left(\frac{\partial r^*_p}{\partial z^*}\right)^2\tau^*_{zz}]\!] = 0 \tag{12}$$

$$[\![\boldsymbol{\sigma}^*\mathbf{n}_\ell\cdot\mathbf{t}_\ell]\!] = [\![\tau^*_{rz}]\!] + \left(\frac{\partial r^*_p}{\partial z^*}\right)[\![\tau^*_{rr} - \left(\frac{\partial r^*_p}{\partial z^*}\right)\tau^*_{rz} - \tau^*_{zz}]\!] = 0 \tag{13}$$

on $r^* = r^*_p$ where

$$\mathbf{t}_\ell = \frac{\left(\dfrac{\partial r^*_p}{\partial z^*}\right)\mathbf{e}_r + \mathbf{e}_z}{\sqrt{1 + \left(\dfrac{\partial r^*_p}{\partial z^*}\right)^2}},$$

is the tangential vector to r^*_p. Notice that (11) is equivalent to require $[\![v^*_r]\!] = [\![v^*_z]\!] = 0$. A further condition to be imposed on the yield surface is the yield criterion

$$\dot{\gamma}^* = 0, \tag{14}$$

or equivalently $\tau^* = \tau^*_o$. The flux at the inlet is given by

$$q^*_{in}(t^*) = \int\limits_{\Sigma^*_{in}} v^*_z dS = \int\limits_0^{2\pi} d\theta \int\limits_0^{R^*_{in}} r^* v^*_z(r^*, 0, t^*) dr^* \tag{15}$$

where

$$R^*_{in} = R^*(r^*, 0, t^*) = R^*_{ref}\left[1 + \delta\varphi\left(-\frac{t^*}{T^*}\right)\right]$$

The mathematical problem is therefore formed by by equations (4), (5), (9) coupled with conditions (10)-(15). To investigate the unsteady problem, we need also to indicate some appropriate initial conditions.

3 Lubrication scaling

In this section we rescale the variables to obtain a dimensionless formulation of the problem. We begin by making the assumption that the reference amplitude of the tube R^*_{ref} is far smaller that the tube length L^*, i.e.

$$\varepsilon = \frac{R^*_{ref}}{L^*} \ll 1. \tag{16}$$

The approach taken here is to exploit the small aspect ratio ε to expand the momentum equations in a perturbation series in powers of ε. In doing so we develop the so-called thin film or lubrication approximation. We rescale the variables to bring out balances that reflect the lubrication approximation, namely

$$z^* = L^* z \qquad r^* = R^*_{ref} r \qquad r^*_p = R^*_{ref} r_p \qquad R^* = R^*_{ref} R$$

$$v^*_z = V^*_{ref} v_z \qquad v^*_r = \varepsilon V^*_{ref} v_r \qquad v^*_p = V^*_{ref} v_p \qquad t^* = t^*_{ref} t$$

$$q^*_{in} = \left(\pi R^{*2}_{ref} V^*_{ref}\right) q_{in} \quad \tau^*_{ij} = \left(\frac{\mu^* V^*_{ref}}{R^*_{ref}}\right)\tau_{ij} \quad \dot{\gamma}^*_{ij} = \left(\frac{V^*_{ref}}{R^*_{ref}}\right)\dot{\gamma}_{ij} \quad p^* = \left(\frac{\mu^* L^* V^*_{ref}}{R^{*2}_{ref}}\right) p$$

where t^*_{ref} is a characteristic time, V^*_{ref} is the characteristic velocity in the z^* direction and where R is given by the term in square bracket in (3). Notice also that the pressure is rescaled using the Poiseuille formula. To detect the characteristic velocity V^*_{ref} we observe that

$$\frac{\partial R^*}{\partial t^*} = -\frac{R^*_{ref}\delta}{T^*}\varphi'\left(\frac{z^*}{\lambda^*} - \frac{t^*}{T^*}\right)$$

so that, recalling $(10)_1$, it seems reasonable to take

$$\varepsilon V^*_{ref} = \frac{R^*_{ref}}{T^*} \qquad \Longrightarrow \qquad V^*_{ref} = \frac{L^*}{T^*}. \tag{17}$$

Let us select $t^*_{ref} = T^*$. We get

$$\begin{cases} \varepsilon^3 \mathsf{Re}\left[\dfrac{\partial v_r}{\partial t} + v_r\dfrac{\partial v_r}{\partial r} + v_z\dfrac{\partial v_r}{\partial z}\right] = -\dfrac{\partial p}{\partial r} + \dfrac{\varepsilon}{r}\dfrac{\partial}{\partial r}\left(r\tau_{rr}\right) - \varepsilon\dfrac{\tau_{\theta\theta}}{r} + \varepsilon^2\dfrac{\partial \tau_{rz}}{\partial z} \\[2ex] \varepsilon \mathsf{Re}\left[\dfrac{\partial v_z}{\partial t} + v_r\dfrac{\partial v_z}{\partial r} + v_z\dfrac{\partial v_z}{\partial z}\right] = -\dfrac{\partial p}{\partial z} + \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\tau_{rz}\right) + \varepsilon\dfrac{\partial \tau_{zz}}{\partial z} \end{cases} \tag{18}$$

where

$$\mathsf{Re} = \frac{\rho^* V^*_{ref} R^*_{ref}}{\mu^*}$$

is the Reynolds number. The equation in the rigid plug (9) becomes

$$\varepsilon \mathsf{Re}\Big[\dot{v}_p \int_0^1 r^{*2}_p \, dz\Big] = \int_0^1 \left(2\tau_{rz} - r_p\frac{\partial p}{\partial z} - 2\varepsilon\tau_{zz}\frac{\partial r_p}{\partial z}\right)\Big|_{r_p} r_p dz \tag{19}$$

while velocity acquires the form

$$\mathbf{v} = v_p(t)\mathbf{e}_z. \tag{20}$$

Mass conservation (4) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} = 0.$$

The boundary conditions on the tube walls are

$$v_r(R,z,t) = \frac{\partial R}{\partial t} = -\delta\varphi'\left(\frac{z}{\lambda} - t\right) \qquad v_z(R,z,t) = 0, \tag{21}$$

where

$$\lambda = \frac{\lambda^*}{L^*} \tag{22}$$

On the yield surface $r = r_p$ we have the yield criterion

$$\dot{\gamma} = \sqrt{\frac{\varepsilon^2}{2}\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right] + \frac{1}{4}\left(\varepsilon^2\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)^2} = 0,$$

and the continuity of the velocity and stress

$$[\![v_r]\!] = [\![v_z]\!] = 0, \tag{23}$$

$$-[\![p]\!]\left[1 + \varepsilon^2\left(\frac{\partial r_p}{\partial z}\right)^2\right] + [\![\varepsilon\tau_{rr} - 2\varepsilon^2\left(\frac{\partial r_p}{\partial z}\right)\tau_{rz} + \varepsilon^3\left(\frac{\partial r_p}{\partial z}\right)^2\tau_{zz}]\!] = 0 \tag{24}$$

$$[\![\tau_{rz}]\!] + \varepsilon\left(\frac{\partial r_p}{\partial z}\right)[\![\tau_{rr} - \varepsilon\left(\frac{\partial r_p}{\partial z}\right)\tau_{rz} - \tau_{zz}]\!] = 0 \tag{25}$$

The nondimensional stress components are

$$\tau_{ij} = \left(2 + \frac{\mathsf{Bn}}{\dot{\gamma}}\right)\dot{\gamma}_{ij}$$

where

$$\mathsf{Bn} = \left(\frac{\tau_o^* R_{ref}^*}{\mu^* V_{ref}^*}\right)$$

is the Bingham number. The nondimensional inlet flux is

$$q_{in}(t) = 2\int_0^{R_{in}} r v_z(r,0,t)dr. \tag{26}$$

Remark 1 *We notice that*

$$\frac{\partial R}{\partial t} = -\delta\varphi'\left(\frac{z}{\lambda} - t\right) \qquad \frac{\partial R}{\partial z} = \frac{\delta}{\lambda}\varphi'\left(\frac{z}{\lambda} - t\right)$$

so that

$$\frac{\partial R}{\partial t} + \lambda\frac{\partial R}{\partial z} = 0.$$

4 Thin-layer approximation: leading order

We exploit the small aspect ratio of the tube ($\varepsilon \ll 1$) to simplify the governing equations. The idea is the following: we substitute the asymptotic sequences

$$(\mathbf{v}, p, r_p) = (\mathbf{v}^{(0)}, p^{(0)}, r_p^{(0)}) + \varepsilon(\mathbf{v}^{(1)}, p^{(1)}, r_p^{(1)}) + \varepsilon^2(\mathbf{v}^{(2)}, p^{(2)}, r_p^{(2)}) + ... \tag{27}$$

into the governing equations/boundary conditions and we gather together terms of the same order obtaining a a hierarchy of equations that approximate the general problem. Due to the smallness of the parameter ε we may focus on the zero order of these approximated problems, sometimes referred to as the leading order, neglecting $O(\varepsilon)$ terms. This drastically simplifies the governing equations and analytic solutions may be found. We plug the expansion (27) into the governing equations and we retain only the leading order terms. Dropping the superscript $^{(0)}$, for simplicity of notation, we find that the equation in the yielded domain reduces to

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}(rv_r) + \dfrac{\partial v_z}{\partial z} = 0 \\ \\ -\dfrac{\partial p}{\partial r} = 0 \\ \\ -\dfrac{\partial p}{\partial z} + \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\tau_{rz}) = 0. \end{cases} \tag{28}$$

so that $p = p(z,t)$ in the yielded domain. In the unyielded domain

$$\int_0^1 \left(2\tau_{rz} - r_p\frac{\partial p}{\partial z}\right)\bigg|_{r_p} r_p dz = 0 \tag{29}$$

Mass balance is automatically satisfied in $[0, r_p]$ since the velocity field here is given by (20). The boundary conditions become

$$\text{On} \quad r = R \quad \begin{cases} v_r = -\delta\varphi'\left(\dfrac{z}{\lambda} - t\right) \\ \\ v_z = 0 \end{cases} \qquad \text{On} \quad r = r_p \quad \begin{cases} [\![p]\!] = 0 \\ \\ [\![\tau_{rz}]\!] = 0 \\ \\ \dot{\gamma} = \dfrac{1}{2}\left|\dfrac{\partial v_z}{\partial r}\right| = 0. \end{cases}$$

We can easily check that

$$\dot{\gamma} = \frac{1}{2}\left|\frac{\partial v_z}{\partial r}\right| \qquad r \in [r_p, R]$$

so that the only non-zero component of the stress in the yielded phase is

$$\tau_{rz} = \frac{\partial v_z}{\partial r} - \mathsf{Bn}. \tag{30}$$

The minus sign in front of the Bingham number Bn is taken since we expect that the velocity v_z is decreasing in the interval $[r_p, R]$. As a consequence $\tau_{rz}|_{r_p} = -\mathsf{Bn}$ and (29) can be rewritten as

$$\int_0^1 \left(2\mathsf{Bn}r_p + \frac{\partial p}{\partial z}r_p^2\right) dz = 0. \tag{31}$$

Let us now integrate $(28)_3$ between r_p and $r > r_p$ recalling that τ_{rz} is given by (30). We find

$$\frac{\partial v_z}{\partial r} = \mathsf{Bn}\left(1 - \frac{r_p}{r}\right) + \frac{1}{2}\frac{\partial p}{\partial z}\left(r - \frac{r_p^2}{r}\right) \qquad r \in [r_p, R]. \tag{32}$$

Integrating the above with the no-slip condition $v_z = 0$ on $r = R$ we find

$$v_z = \mathsf{Bn}\left[r - R + r_p \ln\left(\frac{R}{r}\right)\right] + \frac{1}{2}\frac{\partial p}{\partial z}\left[\frac{r^2 - R^2}{2} + r_p^2 \ln\left(\frac{R}{r}\right)\right] \qquad r \in [r_p, R].$$

Therefore

$$v_p(t) = v_z\Big|_{r_p} = \mathsf{Bn}\left[r_p - R + r_p \ln\left(\frac{R}{r_p}\right)\right] + \frac{1}{2}\frac{\partial p}{\partial z}\left[\frac{r_p^2 - R^2}{2} + r_p^2 \ln\left(\frac{R}{r_p}\right)\right].$$

Now, integrating mass balance $(28)_1$ between r and R we find

$$R\, v_r(R, z, t) - r\, v_r(r, z, t) = -\int_r^R \frac{\partial}{\partial z}\Big[\xi v_z(\xi, z, t)\Big] d\xi \tag{33}$$

We observe that

$$\frac{\partial}{\partial z}\left[\int_r^R \xi v_z(\xi, z, t) d\xi\right] = R\frac{\partial R}{\partial z}\underbrace{v_z(R, z, t)}_{=0} + \int_r^R \frac{\partial}{\partial z}\Big[\xi v_z(\xi, z, t)\Big] d\xi$$

Hence (33) can be rewritten as

$$r\, v_r(r, z, t) = -\delta\varphi'(1 + \delta\varphi) + \frac{\partial}{\partial z}\underbrace{\left[\int_r^R \xi v_z(\xi, z, t) d\xi\right]}_{=:J(r,z,t)}$$

Now

$$\frac{\partial}{\partial z}\Big[J(r, z, t)\Big]\Big|_{r=r_p} = \frac{\partial}{\partial z}\Big[J(r_p, z, t)\Big] - \frac{\partial J}{\partial r}(r_p, z, t)\frac{\partial r_p}{\partial z}$$

so that, recalling that $v_r = 0$ on $r = r_p$ (because of symmetry) and that $\partial J/\partial r = -r v_z(r, z, t)$, we get

$$-\delta\varphi'(1 + \delta\varphi) + \frac{\partial}{\partial z}\left[\int_{r_p}^R \xi v_z(\xi, z, t) d\xi\right] + r_p \underbrace{v_z(r_p, z, t)}_{=v_p(t)}\frac{\partial r_p}{\partial z} = 0$$

The above can be rewritten as

$$\frac{\partial}{\partial z}\left[-\frac{\lambda R^2}{2} + \int_{r_p}^R \xi v_z(\xi, z, t) d\xi + v_p\frac{r_p^2}{2}\right] = \frac{\partial}{\partial z}\left[-\frac{\lambda R^2}{2} + \int_0^R \xi v_z(\xi, z, t) d\xi\right] = 0$$

implying

$$-\frac{\lambda R^2}{2} + \underbrace{\int_0^R \xi v_z(\xi, z, t) d\xi}_{\text{adimensional flux} \quad q(z,t)/2} = \mathcal{C}(t)$$

Since the above must hold for every z we get

$$\mathcal{C}(t) = -\lambda \frac{R_{in}^2}{2} + \frac{q_{in}}{2}$$

and

$$\int_0^R \xi v_z(\xi, z, t) d\xi = \int_{r_p}^R \xi v_z(\xi, z, t) d\xi + v_p \frac{r_p^2}{2} = \frac{q_{in}}{2} - \lambda \left(\frac{R_{in}^2 - R^2}{2} \right)$$

Integrating by parts we find

$$\int_{r_p}^R \xi v_z(\xi, z, t) d\xi + v_p \frac{r_p^2}{2} = \underbrace{v_z \frac{\xi^2}{2}\Big|_{r_p}^R + v_p \frac{r_p^2}{2}}_{=0} - \int_{r_p}^R \frac{\xi^2}{2} \frac{\partial v_z}{\partial \xi} d\xi,$$

so that

$$\int_{r_p}^R \xi^2 \frac{\partial v_z}{\partial \xi} d\xi = -q_{in} + \lambda \left(R_{in}^2 - R^2 \right). \tag{34}$$

Now, plugging (32) into (34), we find

$$\int_{r_p}^R \left[\mathsf{Bn}(\xi^2 - \xi r_p) + \frac{\partial p}{\partial z} \left(\frac{\xi^3 - \xi r_p^2}{2} \right) \right] d\xi = -q_{in} + \lambda \left(R_{in}^2 - R^2 \right)$$

which yields

$$\left[\mathsf{Bn}(\frac{\xi^3}{3} - r_p \frac{\xi^2}{2}) + \frac{\partial p}{\partial z} \left(\frac{\xi^4}{8} - r_p^2 \frac{\xi^2}{4} \right) \right] \Big|_{r_p}^R = -q_{in} + \lambda \left(R_{in}^2 - R^2 \right).$$

After some calculations we find

$$\frac{\mathsf{Bn}}{6}(R - r_p)^2(2R + r_p) + \frac{1}{8}\frac{\partial p}{\partial z}(R^2 - r_p^2)^2 = -q_{in} + \lambda \left(R_{in}^2 - R^2 \right),$$

or equivalently

$$\frac{\partial p}{\partial z} = \frac{-8q_{in} + 8\lambda(R_{in}^2 - R^2) - \frac{4\mathsf{Bn}}{3}(R - r_p)^2(2R + r_p)}{(R^2 - r_p^2)^2}. \tag{35}$$

In conclusion, recalling that $v_z(r_p, z, t) = v_p(t)$ we have that the system to be solved is the following

$$\begin{cases} \displaystyle\int_0^1 \left(2\mathsf{Bn} r_p + \frac{\partial p}{\partial z} r_p^2 \right) dz = 0, \\ \\ v_p(t) = \mathsf{Bn} \left[r_p - R + r_p \ln\left(\frac{R}{r_p} \right) \right] + \dfrac{1}{2}\dfrac{\partial p}{\partial z} \left[\dfrac{r_p^2 - R^2}{2} + r_p^2 \ln\left(\dfrac{R}{r_p} \right) \right], \end{cases} \tag{36}$$

where $\partial p/\partial z$ is given by (35). The problem is formally closed since we have two equations for the unknowns (v_p, r_p). From (35) we notice that in our problem we can either specify the inlet flux q_{in} or the pressure drop between the inlet and the outlet $\Delta p = p_{in} - p_{out}$. Indeed, if we suppose to know q_{in}, then the pressure gradient comes directly from (35). If, on the other hand, we suppose that the pressure difference Δp is given, then integrating (35) in z between 0 and 1 we find

$$q_{in}(t) = \frac{\Delta p + \int_0^1 \frac{8\lambda(R_{in}^2 - R^2) - \frac{4\mathsf{Bn}}{3}(R - r_p)^2(2R + r_p)}{(R^2 - r_p^2)^2} dz}{8\int_0^1 \frac{dz}{(R^2 - r_p^2)^2}}. \tag{37}$$

Now, inserting (37) into (35), we find $\partial p/\partial z$ in terms of Δp.

4.1 The special case of a flat channel profile

Suppose that the tube profile is flat, so that $R = 1$ and suppose that q_{in} is given. In this case it is clear from (35), $(36)_2$ that

$$r_p = r_p(t) \qquad \frac{\partial^2 p}{\partial z^2} = 0,$$

meaning that the yield surface is flat and that the pressure gradient is a function of time only. From $(36)_1$ we find that

$$\frac{\partial p}{\partial z} = -\frac{2\mathsf{Bn}}{r_p},$$

while, from (35),

$$\frac{2\mathsf{Bn}}{r_p} = \frac{8q_{in} + \frac{4\mathsf{Bn}}{3}(1 - r_p)^2(2 + r_p)}{(1 - r_p^2)^2}.$$

Rearranging the above we get

$$\underbrace{(1 - r_p^2)^2}_{=:L(r_p)} = \underbrace{\left(\frac{2}{3}\right) r_p\Big[\alpha + (1 - r_p)^2(2 + r_p)\Big]}_{=:M(r_p)} \qquad \alpha = \frac{6q_{in}}{\mathsf{Bn}} \tag{38}$$

Hence the solution to our problem is given by some $\bar{r}_p(t)$ such that $M(\bar{r}_p) = L(\bar{r}_p)$. Looking at Fig. 2 we see that for every $\alpha > 0$ (i.e. for every choice of positive Bn and q_{in}) there exists only one $\bar{r}_p$ satisfying (38). Moreover it is easy to check that

$$\bar{v}_z(r,t) = \begin{cases} \frac{\mathsf{Bn}}{2\bar{r}_p}(1 - \bar{r}_p)^2 & r \in [0, \bar{r}_p] \\ \frac{\mathsf{Bn}}{2\bar{r}_p}(1 - r)(1 + r - 2\bar{r}_p) & r \in [\bar{r}_p, 1] \end{cases} \qquad \bar{v}_r(r,t) = 0.$$

Notice that $\bar{v}_z(r,t)$ is continuously differentiable in $[0,1]$ with the classical parabolic profile in the yielded part. Suppose now that the pressure drop $\Delta p(t) > 0$ is given. Then the yield surface and pressure gradient are given by

$$\bar{r}_p(t) = \frac{2\mathsf{Bn}}{\Delta p}, \qquad \frac{\partial p}{\partial z} = -\Delta p,$$

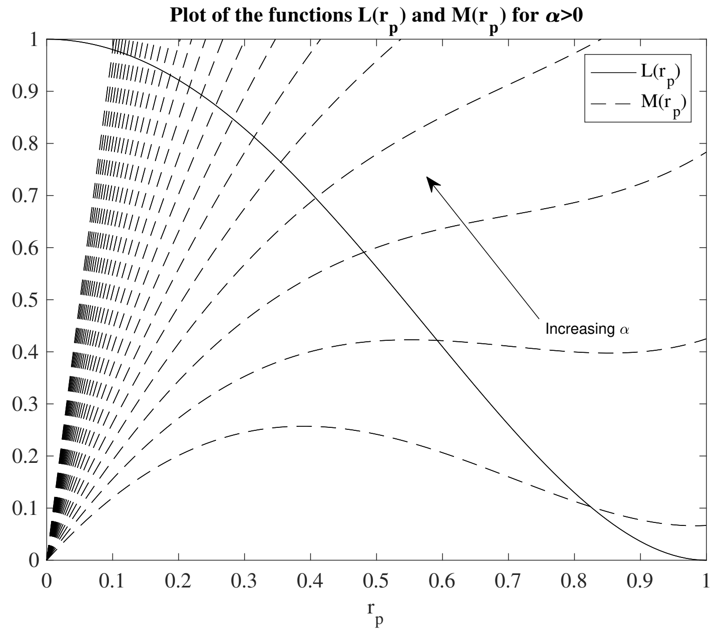

Figure 2: Plot of the functions $M(r_p)$, $L(r_p)$ for increasing $\alpha > 0$.

respectively. In this case the constraint $\Delta p > 2\mathsf{Bn}$ has to be enforced to guarantee that $\bar{r}_p(t) < 1$. The velocity field is given by

$$\bar{v}_z(r,t) = \begin{cases} \dfrac{\Delta p}{4}\left(1 - \dfrac{2\mathsf{Bn}}{\Delta p}\right)^2 & r \in \left[0, \dfrac{2\mathsf{Bn}}{\Delta p}\right] \\ \dfrac{\Delta p}{4}(1-r)\left(1 + r - \dfrac{4\mathsf{Bn}}{\Delta p}\right) & r \in \left[\dfrac{2\mathsf{Bn}}{\Delta p}, 1\right] \end{cases} \qquad \bar{v}_r(r,t) = 0.$$

Also in this case $\bar{v}_z(r,t)$ is continuously differentiable in $[0,1]$ with the parabolic profile in the yielded part. The inlet flux is found from (37)

$$q_{in}(t) = \frac{\Delta p}{4}\left[1 - \left(\frac{2}{3}\right)\left(\frac{2\mathsf{Bn}}{\Delta p}\right)\left(1 - \frac{2\mathsf{Bn}}{\Delta p}\right)^2\left(2 + \frac{2\mathsf{Bn}}{\Delta p}\right)\right]$$

where one can easily verify that the term in square bracket is always positive for $2\mathsf{Bn}/\Delta p \in (0,1)$.

4.2 An almost flat tube profile

In this section we invesigate the case in which the wall is a *small perturbation* of the flat tube profile, whose solution has been pinpointed in the previous section. To this aim we look for a solution of the form

$$\begin{cases} p = p^{(0)} + \delta p^{(1)} & \dfrac{\partial p}{\partial z} = \dfrac{\partial p^{(0)}}{\partial z} + \delta \dfrac{\partial p^{(1)}}{\partial z} \\ v_p = v_p^{(0)} + \delta v_p^{(1)} & r_p = r_p^{(0)} + \delta r_p^{(1)} \\ v_z = v_z^{(0)} + \delta v_z^{(1)} & v_r = v_r^{(0)} + \delta v_r^{(1)} \end{cases} \tag{39}$$

with $R = 1 + \delta\varphi$, $R_{in} = 1 + \delta\varphi_{in}$. The superscript $^{(0)}$, $^{(1)}$ are now intended w.r.t. the expansion in δ and must not be confused with the ones used for the lubrication approximation. Plugging (39) into (35), (36) and gathering the terms of the same order we find the problems at the zero and first order w.r.t. to δ. The leading order problem is exactly that of Section 4.1 with the explicit solution $r_p^{(0)} = \bar{r}_p$ and with

$$\frac{\partial p^{(0)}}{\partial z} = -\frac{2\mathsf{Bn}}{r_p^{(0)}}. \tag{40}$$

Hence, we focus on the first order problem. Equation (35) becomes

$$\frac{\partial p^{(1)}}{\partial z} = r_p^{(1)} A(t) + B(z,t), \tag{41}$$

where

$$A(t) = \frac{4\mathsf{Bn}}{(r_p^{(0)^2} - 1)} \qquad B(z,t) = \frac{8\mathsf{Bn}\varphi(1 - r_p^{(0)}) + 16\lambda r_p^{(0)}(\varphi_{in} - \varphi)}{r_p^{(0)}(r_p^{(0)^2} - 1)^2}.$$

In deriving (41) we have exploited (40). The plug integral equation $(36)_1$ becomes

$$\int_0^1 \left[r_p^{(1)} C(t) + r_p^{(0)^2} B(z,t) \right] dz = 0, \tag{42}$$

where

$$C(t) = \frac{2\mathsf{Bn}(r_p^{(0)^2} + 1)}{(r_p^{(0)^2} - 1)}.$$

Finally the velocity of the plug at the first order is

$$v_p^{(1)} = \frac{\partial p^{(1)}}{\partial z} D(t) + r_p^{(1)} E(t) + F(z,t), \tag{43}$$

where

$$D(t) = \frac{r_p^{(0)^2} - 1 - 2r_p^{(0)^2} \log\left(r_p^{(0)}\right)}{4} \qquad E(t) = \mathsf{Bn} \log\left(r_p^{(0)}\right),$$

$$F(z,t) = \mathsf{Bn}\varphi \frac{(1 - r_p^{(0)})}{r_p^{(0)}}.$$

Inserting (41) into (43) we find

$$r_p^{(1)} = \frac{v_p^{(1)} - (BD + F)}{(AD + E)}. \tag{44}$$

Then, inserting (44) into (42) we find

$$v_p^{(1)} = \frac{1}{C} \int_0^1 \left[(BD + F)C - r_p^{(0)^2} B(AD + E) \right] dz.$$

In conclusion we have found

$$r_p^{(1)} = \frac{\int_0^1 \left[(BD + F)C - r_p^{(0)^2} B(AD + E) \right] dz - C(BD + F)}{C(AD + E)}. \tag{45}$$

The first order component of the yield surface is therefore determined in terms of the solution at the zero order. Recalling the definitions of coefficients A, B, C, D, E, F one can show, after some calculations that

$$v_p^{(1)}(t) = \left[\frac{4\lambda}{(r_p^{(0)^2}+1)} - \frac{\mathsf{Bn}(r_p^{(0)}-1)^2(r_p^{(0)}+1)}{r_p^{(0)}(r_p^{(0)^2}+1)} \right] \int_0^1 \varphi dz - \frac{4\lambda\varphi_{in}}{(r_p^{(0)^2}+1)}$$

5 Numerics

To illustrate the behaviour of the solution we perform some numerical simulations based on the results of Section 4.1 and 4.2. In particular, using the explicit formulas derived at the zero and first order approximation in δ, we plot the evolution of the yield surface $r_p(z,t)$, of the pressure gradient $\partial p(z,t)/\partial z$ and of the plug velocity $v_p(t)$. We begin by showing the evolution of the yield surface $r_p = r_p^{(0)} + \delta r_p^{(1)}$ when δ is *small*. We assume that the travelling wave φ representing the evolving wall of the tube is given by

$$\varphi\left(\frac{z}{\lambda} - t\right) = \sin\left[2\pi\left(\frac{z}{\lambda} - t\right)\right]. \tag{46}$$

We take

$$q_{in} = 1, \qquad \delta = 0.1 \qquad \lambda = 0.8 \qquad \mathsf{Bn} = 0.5.$$

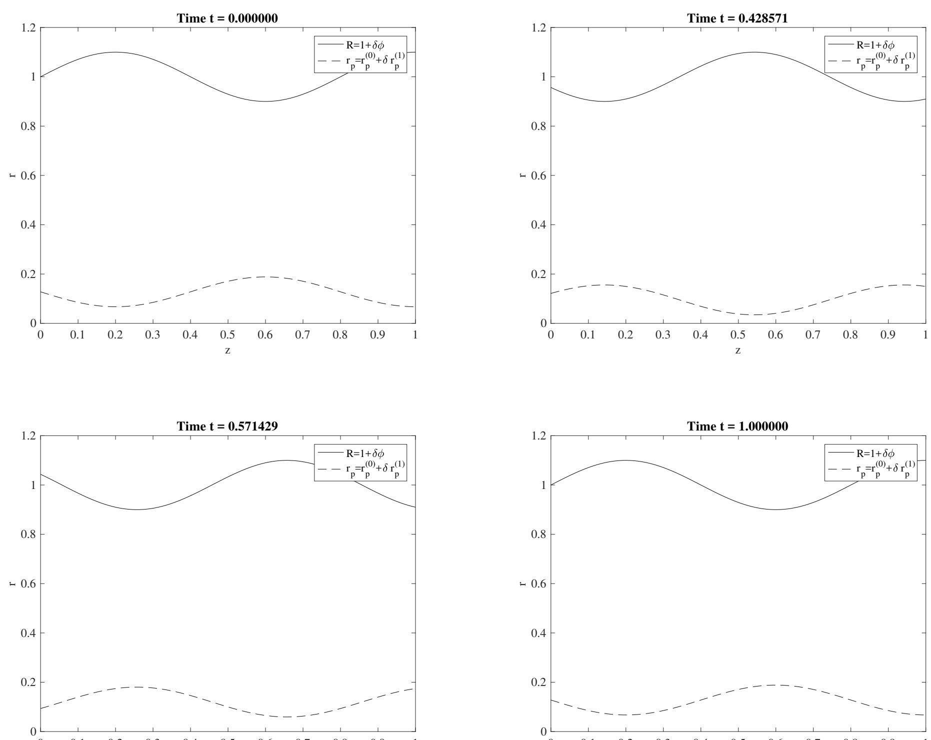

Figure 3: Evolution of the tube wall R and of yield surface r_p with $t \in [0,1]$ and R given by (46).

In Fig. 3 the plots of the yield surface (dashed) and of the tube wall (solid) are shown for increasing time. In particular $t \in [0,1]$ so that the figures describe the evolution of the system during the period of the oscillation. We notice that, at each time, the plug profile decreases as the wall radius increases, i.e. the yield surface and the wall are out of phase. This result is consistent with what found in [4] and in [2], where the same type of behaviour was observed for the case of a static wall.

In Fig. 4 we plot the pressure gradient defined in (39) as a function of z and the pressure drop $\Delta p = p_{in} - p_{out}$ which is a function of time only. We notice that the pressure gradient have a sinusoidal behaviour with fixed maximum oscillation. The pressure drop changes with time but remains always positive since the pressure gradient is always negative.

In Fig. 5 we plot the behaviour of the pressure gradient for $z = 0.5$ as a function of time for different values of the Bingham number Bn and of the prescribed inlet discharge q_{in}. We notice that the increase of the Bingham number and of the inlet discharge produces an increase of the modulus of the pressure gradient.

In Fig. 6 we plot the behaviour of the yield surface at $z = 0.5$ for different values of the Bingham number Bn and of the prescribed inlet discharge q_{in}. From Fig. 6_2 we notice that there is a q_{crit} below which the characteristic behaviour according to which to a narrowing tube corresponds an expanding plug stops. Therefore there's a q_{crit} below which the plug and the wall are in phase. Indeed, as one can see, if $0.8 \gtrsim q_{in}$ the monotonicity of the yield surface r_p is the same of the tube wall R. The same does not occur with Bn. Indeed, looking at Fig. 6_1, we see that, independently of Bn, the tube wall R and the yield surface r_p have opposite monotonic character. Clearly the increase of the Bingham number, and hence of the yield stress, produces an increase of the thickness of the rigid plug.

Finally, in Fig. 7, we plot the velocity of the plug $v_p(t) = v_p^{(0)}(t) + \delta v_p^{(1)}(t)$ for different values of the Bingham number Bn and of the prescribed inlet discharge q_{in}. We observe that the velocity $v_p(t)$ increases with the Bingham number and with the inlet discharge for each time t.

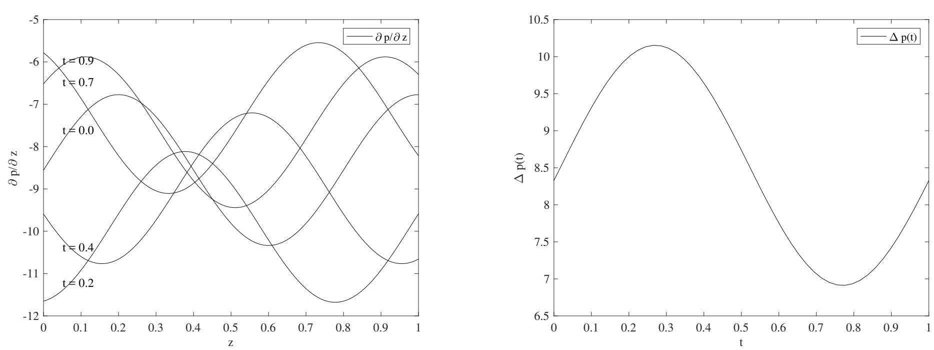

Figure 4: Pressure gradient $\partial p/\partial z$ as a function of space z and Δp as a function of time t.

To investigate the dependence of the formation of the plug on δ, we introduce the quantity $\mathsf{Bn}_{min} = \mathsf{Bn}_{min}(\delta)$, representing the minimum value of Bn (for a given δ) below which the plug breaks. This value is actually a function of q_{in} too, but here, for simplicity, we have set $q_{in} = 1$. In Fig. 8 we have plotted the relation $\delta - \mathsf{Bn}_{min}$ with δ ranging from 0.05 to 0.15. As one can see, the relation is linear in this range. The region where the plug does not break, i.e. the region where the model is consistent, is the one to the right of the line, as indicated.

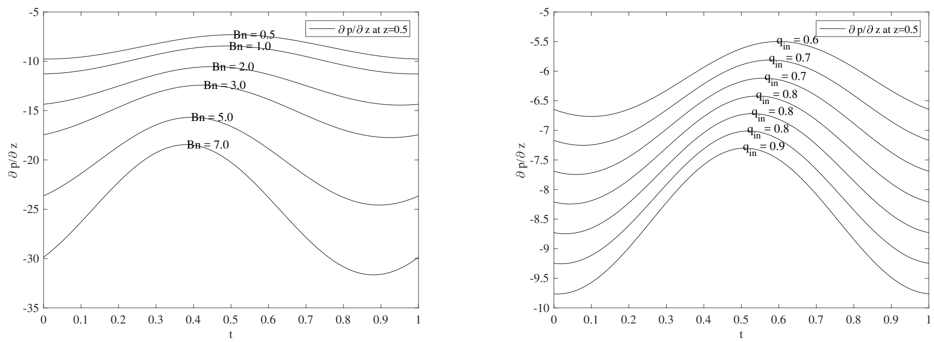

Figure 5: Pressure gradient $\partial p/\partial z$ as a function of time at $z = 0.5$ for different Bn and q_{in}.

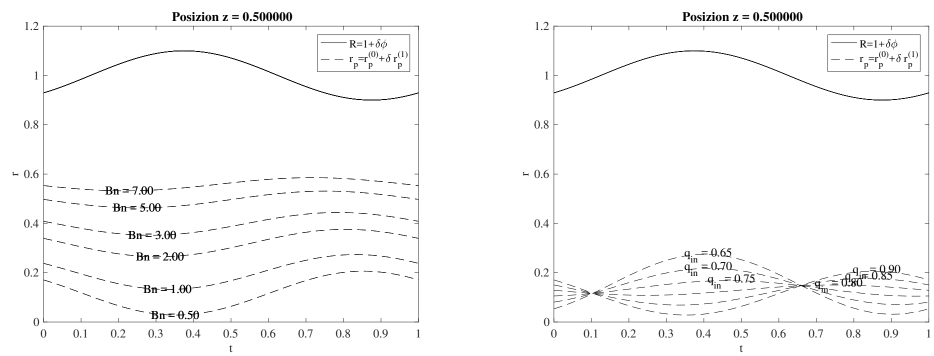

Figure 6: Yield surface $r_p = r_p^{(0)} + \delta r_p^{(1)}$ as a function of time at $z = 0.5$ for different Bn and q_{in}.

6 Conclusions

We have proposed a model for the peristaltic motion of a Bingham fluid in a cylindrical tube whose walls evolve as a periodic travelling wave. We have derived the model in a general 3D framework using a cylindrical coordinate system. Following [4], the momentum equation for the rigid part of the fluid has been written using an integral formulation in which the unyielded part is treated as an evolving material surface. We have rescaled the problem exploiting the small aspect ratio ε representing the ratio between the tube length and the typical radius and we have focussed on the leading order approximation. For this specific case we have shown that it is possible to find an explicit expression for the pressure gradient as a function of the (unknown) yield surface $r_p(z,t)$, of the prescribed inlet discharge $q_{in}(t)$ and of the wall radius $R(z,t)$. We have proved that the mathematical problem eventually reduces to a set of two equation (one of which is integral) for the yield surface r_p and for the rigid plug velocity v_p. We have shown that when the oscillation amplitude δ of the wall is null we recover the classical one dimensional Bingham model in cylindrical geometry. Then we have considered an oscillating wall which is given as a small perturbation of the flat wall and we have proved that also in this case we may determine explicit solutions that involve the solutions obtained when $\delta = 0$.

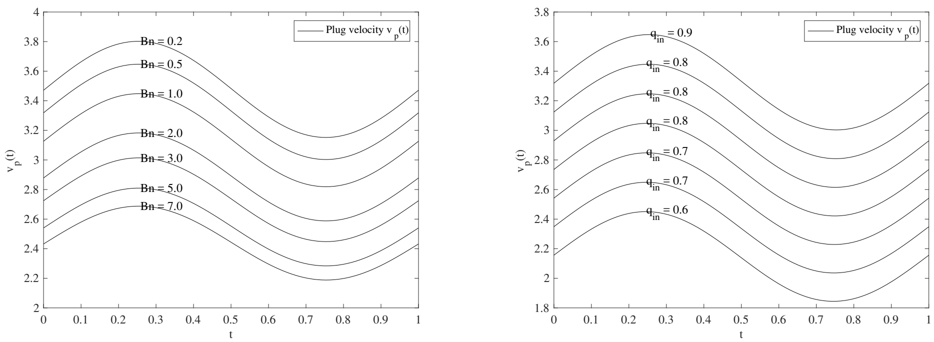

Figure 7: Velocity of the plug $v_p = v_p^{(0)} + \delta v_p^{(1)}$ as a function of time for different Bn and q_{in}.

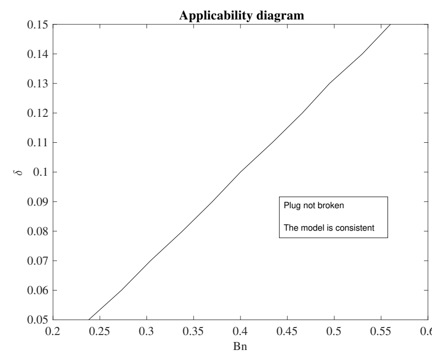

Figure 8: Bn_{min} as a function of δ.

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