Pressure Driven Lubrication Flow of a Bingham Fluid in a Channel: A Novel Approach

Pressure Driven Lubrication Flow of a Bingham Fluid in a Channel: A Novel Approach

[a] Fusi L.,[*] [a]Farina A., [a] Rosso F., [b]Roscani S.

[a] *Università degli Studi di Firenze*
Dipartimento di Matematica e Informatica "Ulisse Dini"
Viale Morgagni 67/a, 50134 Firenze, Italy

[b] *Universidad Nacional de Rosario*
CONICET - Departamento de Matemàtica, FCEIA
Pellegrini 250, S2000BTP, Rosario, Argentina

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Abstract

In this paper we present a novel approach for modelling the lubrication flow of a Bingham plastic in a channel with non uniform walls. The novelty consists in deriving the rigid plug equation using an integral approach based on Newton's second law, where the unyielded part is treated as an evolving non material volume. Such an approach leads to an integro-differential equation for the pressure that can be solved with an iterative procedure. We prove that a true unyielded plug exists even when the maximum width variation is not "small" and we find constraints on the amplitude of the channel that prevent the plug from "breaking". We show numerical results and comparisons with results obtained with different approaches. We also show how to extend our model in the case of a pressure-dependent viscosity.

1 Introduction

A Bingham plastic is a non-Newtonian fluid that behaves like a rigid body when a certain invariant of the stress is below a critical threshold and like a viscous fluid when the invariant is above (see [3], [4], or the original papers by E.C. Bingham [1], [2]). The typical way of proceeding when deriving the equation of motion for this kind of fluid is to write the balance of linear momentum[1]

$$\varrho^* \frac{D\mathbf{v}^*}{Dt^*} = -\nabla^* P^* + \nabla^* \cdot \mathbf{S}^*, \tag{1}$$

where ϱ^* is density, $\mathbf{v}^*$ is velocity, P^* is pressure and $\mathbf{S}^*$ is the deviatoric part of the stress. Equation (1) is written for the whole domain (rigid and liquid) and it is assumed that the velocity and the stress are continuous across the fluid/rigid interface. In the fluid region the constitutive equation is the one of a linear viscous fluid, while in the rigid part the stress is

[*] *Corresponding author:* fusi@math.unifi.it, Tel. +39552751437 Fax +39552751452.

[1] Throughout the whole paper the starred quantities and operators are dimensional, to distinguish them from their dimensionless equivalents.

undetermined and we only know that the strain rate vanishes, i.e. $\mathbf{D}^* = 0$ (this can be proved treating the unyielded part as a viscous fluid and letting the viscosity tend to infinite, see [7]).

In this paper we present a novel approach for modelling the flow of a Bingham fluid in a channel when the driving force is an applied pressure gradient (Poiseuille flow). We assume that the channel width is much smaller than the channel length, so that the lubrication approximation is suitable. When dealing with a lubrication flow equation (1) can be drastically simplified introducing the aspect ratio $\varepsilon \ll 1$ and rescaling the problem with quantities that contain ε. With this procedure we look for a solution that can be expressed as power series of ε and we study the problem at the leading order, i.e. neglecting all the terms containing ε. In doing this we are tacitly assuming that the rescaled variables and their derivatives are $\mathcal{O}(1)$ in both the liquid and solid domain. In particular the stress components S^*_{ij} are rescaled with the characteristic viscous stress and it is assumed that the non-dimensional components S_{ij} are everywhere $\mathcal{O}(1)$. The latter hypothesis can be checked "a posteriori" only in the liquid part, where the stress is determined, but not in the rigid domain, where the stress is not even defined. In other words we cannot verify if the order zero approximation of (1) is justified also in the unyielded part.

This point is of crucial importance, since we know that assuming $S_{ij} = \mathcal{O}(1)$ and using (1) to derive the motion in the rigid part leads to the well known "lubrication paradox", which consists in a plug velocity that depends on the longitudinal coordinate[2], see [5].

Motivated by this observation we have decided not to use equation (1) in the unyielded part and we have written the balance of linear momentum using an integral global approach similar to the one presented in [17] and in [11]. In practice we have considered the unyielded domain as an *evolving non material volume* Ω^*_t, whose dynamics is governed by Newton second law[3]

$$\frac{d}{dt^*}\int_{\Omega^*_t} (\varrho^* \mathbf{v}^*)\, dV^* = \int_{\partial\Omega^*_t} (\mathbf{T}^*\mathbf{n}) dS^*, \tag{2}$$

where $\mathbf{T}^* = -P^*\mathbf{I} + \mathbf{S}^*$ is the Cauchy stress tensor. Following this approach the knowledge of the stress tensor inside the rigid part is no longer needed and no guess has to be made on the order of magnitude of the stress components. We just need to know the stress acting on the exterior boundary of Ω^*_t, namely $\mathbf{T}^*|_{\partial\Omega^*_t}$, that is the forces responsible for the motion of the inner core. The external stress is composed by the one acting on the yield surface σ^* (see Fig. 1) and by the one acting on the lateral boundary (inlet and outlet of the channel, $x^* = 0, L^*$). On the yield surface σ^* this is simply the viscous stress evaluated on the interface (which is known once we solve the problem in the viscous domain). On the inlet and outlet it is just the applied pressure (which is a given datum of the problem).

At the leading order and in the case of non uniform channel width, equation (2) reduces to an integro-differential equation for the pressure P^*, whose solution allows to determine explicitly the velocity field $\mathbf{v}^*$ and the yield surface σ^* (which is a free boundary since it is not known in advance). In particular, in the rigid domain the longitudinal velocity v^*_1 is spatially uniform and the transversal velocity v^*_2 vanishes. Therefore the constraint of the rigid motion is fulfilled in the unyielded region and *no "lubrication paradox" arises.* These results are also extended to the case of fluids with constant density and pressure dependent viscosity (see [10] and the reference therein for an overview of these kind of fluids).

Our work *confirms* (with a completely different approach, based on (2)) the results presented in [5], [9], where it is proved that the central unyielded core persists for a sufficiently small perturbation (whose order of magnitude is ε) of the uniform walls, in contrast to the lubrication

[2]The paradox disappears when one considers a deformable core, see [6].
[3]In (2) we are neglecting body forces.

paradox (see point (1) in section 2.1 of [9]). Actually we extend such a result since we prove that a true plug persist even if the perturbation is $\mathcal{O}(1)$.

2 Derivation of the model

We consider the flow of an incompressible Bingham fluid in a channel of length L^* and amplitude $2h^*(x^*)$, as the one depicted in Fig. 1. Because of symmetry, we confine our analysis to the upper part of the layer, namely $[0, h^*(x^*)]$. We assume that the velocity field is given by

$$\mathbf{v}^* = v_1^*(x^*, y^*, t^*)\mathbf{i} + v_2^*(x^*, y^*, t^*)\mathbf{j},$$

where x^*, y^* are the longitudinal and transversal coordinate respectively.

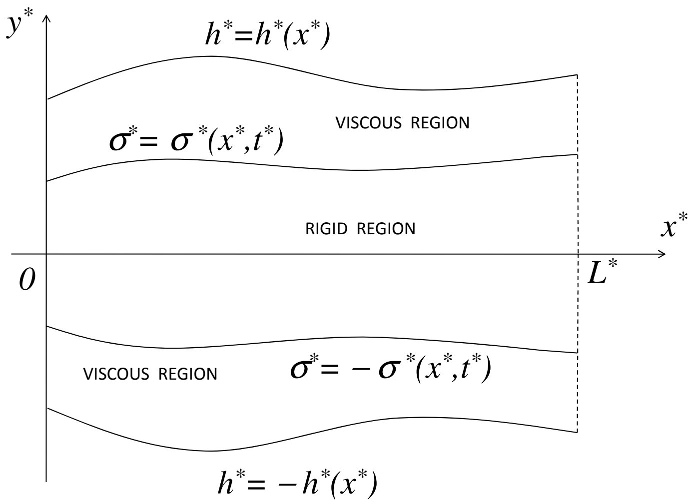

Figure 1: Sketch of the domain of the problem.

The Cauchy stress is $\mathbf{T}^* = -P^*\mathbf{I} + \mathbf{S}^*$, where $P^* = 1/3\text{tr}\mathbf{T}^*$, and $\mathbf{S}^*$ is the so-called extra-stress. The Bingham constitutive equation can be written as

$$\mathbf{S}^* = \left(2\eta_c^* + \frac{\tau_o^*}{II_{\mathbf{D}^*}}\right)\mathbf{D}^*, \tag{3}$$

or in the implicit form [14], [15]

$$\mathbf{D}^* = \left(\frac{II_{\mathbf{D}^*}}{2\eta_c^* II_{\mathbf{D}^*} + \tau_o^*}\right)\mathbf{S}^*, \tag{4}$$

where $\mathbf{D}^* = \frac{1}{2}\left(\nabla\mathbf{v}^* + \nabla\mathbf{v}^{*\mathrm{T}}\right)$, η_c^* is the viscosity, τ_o^* is the yield stress and where

$$II_{\mathbf{S}^*} = \sqrt{\frac{1}{2}\,\text{tr}\,\mathbf{S}^{*2}}, \qquad II_{\mathbf{D}^*} = \sqrt{\frac{1}{2}\,\text{tr}\,\mathbf{D}^{*2}}.$$

Equation (4) admits the solution $\mathbf{D}^* = 0$, corresponding to rigid body motion. On the other hand, if $\mathbf{D}^* \neq 0$, we can express $\mathbf{S}^*$ in terms of $\mathbf{D}^*$ and find $II_{\mathbf{S}^*} = 2\eta_c^* II_{\mathbf{D}^*} + \tau_o^*$, with $II_{\mathbf{S}^*} \geq \tau_o^*$.

Therefore, whenever $\mathbf{D}^* = 0$, we have $II_{\mathbf{S}^*} \leq \tau_o^*$. In other words, the stress is not determined as long as $II_{\mathbf{S}^*}$ is below the yield stress. We assume that the region where $II_{\mathbf{S}^*} \geq \tau_o^*$ (i.e. the viscous region) and the region where $II_{\mathbf{S}^*} \leq \tau_o^*$ (i.e. the rigid region) are separated by a sharp interface $y^* = \pm\sigma^*(x^*, t^*)$ (see Fig. 1).
The mechanical incompressibility yields

$$\mathrm{tr}\mathbf{D}^* = \frac{\partial v_1^*}{\partial x^*} + \frac{\partial v_2^*}{\partial y^*} = 0. \tag{5}$$

2.1 The viscous domain

The governing equations in the viscous part are[4]

$$\varrho^* \left(\frac{\partial v_1^*}{\partial t^*} + v_1^* \frac{\partial v_1^*}{\partial x^*} + v_2^* \frac{\partial v_1^*}{\partial y^*} \right) = -\frac{\partial P^*}{\partial x^*} + \frac{\partial S_{11}^*}{\partial x^*} + \frac{\partial S_{12}^*}{\partial y^*}, \tag{6}$$

$$\varrho^* \left(\frac{\partial v_2^*}{\partial t^*} + v_1^* \frac{\partial v_2^*}{\partial x^*} + v_2^* \frac{\partial v_2^*}{\partial y^*} \right) = -\frac{\partial P^*}{\partial y^*} + \frac{\partial S_{12}^*}{\partial x^*} + \frac{\partial S_{22}^*}{\partial y^*}, \tag{7}$$

$$\frac{\partial v_1^*}{\partial x^*} + \frac{\partial v_2^*}{\partial y^*} = 0 \tag{8}$$

where S_{ij}^* are the components of $\mathbf{S}^*$, given by (3).

2.2 The rigid domain

The inner rigid core $\Omega_{t^*}^*$ at some time $t^* > 0$ is given by

$$\Omega_{t^*}^* = \left\{ (x^*, y^*) : \quad x^* \in [0, L^*], \quad y^* \in [-\sigma^*, \sigma^*] \right\}.$$

Following (2), the integral momentum balance for the whole domain in the absence of body forces is given by

$$\int_{\Omega_t^*} \frac{\partial}{\partial t^*} (\varrho^* \mathbf{v}^*) \, dV^* + \int_{\partial\Omega_{t^*}^*} \varrho^* \mathbf{v}^* \left(\mathbf{w}^* \cdot \mathbf{n} \right) dS^* = \int_{\partial\Omega_{t^*}^*} (\mathbf{T}^* \mathbf{n}) dS^*, \tag{9}$$

where $\mathbf{w}^*$ is the velocity of the boundary $\partial\Omega_{t^*}^*$ and $\mathbf{n}$ its outward unit normal. We divide the boundary in four parts as depicted in Fig. 2 so that

$$\partial\Omega_{t^*}^* = \Gamma_{1,t^*}^* \cup \Gamma_{2,t^*}^* \cup \Gamma_{3,t^*}^* \cup \Gamma_{4,t^*}^* .$$

We have

$$\mathbf{n}_1 = \frac{(-\sigma_x^*, 1)}{\sqrt{1 + \sigma_x^{*2}}}, \qquad \mathbf{n}_2 = (1, 0), \qquad \mathbf{n}_3 = \frac{(-\sigma_x^*, -1)}{\sqrt{1 + \sigma_x^{*2}}}, \qquad \mathbf{n}_4 = (-1, 0).$$

$$\mathbf{w}^* \cdot \mathbf{n}_1 = \frac{\sigma_t^*}{\sqrt{1 + \sigma_x^{*2}}}, \qquad \mathbf{w}^* \cdot \mathbf{n}_2 = 0, \qquad \mathbf{w}^* \cdot \mathbf{n}_3 = \frac{\sigma_t^*}{\sqrt{1 + \sigma_x^{*2}}}, \qquad \mathbf{w}^* \cdot \mathbf{n}_4 = 0.$$

Now we evaluate the surface integral on the l.h.s. of (9) in each of the four part (consider that the velocity of lateral boundaries Γ_{2,t^*}^*, Γ_{4,t^*}^* is zero. The momentum balance (9) becomes

$$2\frac{\partial}{\partial t^*} (\varrho^* \mathbf{v}^*) \int_0^{L^*} \sigma^*(x^*, t^*) dx^* + 2\varrho^* \mathbf{v}^* \int_0^{L^*} \frac{\partial \sigma^*}{\partial t^*}(x^*, t^*) dx^* = \int_{\partial\Omega_{t^*}^*} \mathbf{T}^* \mathbf{n} \, dS^*,$$

[4] We neglect body forces.

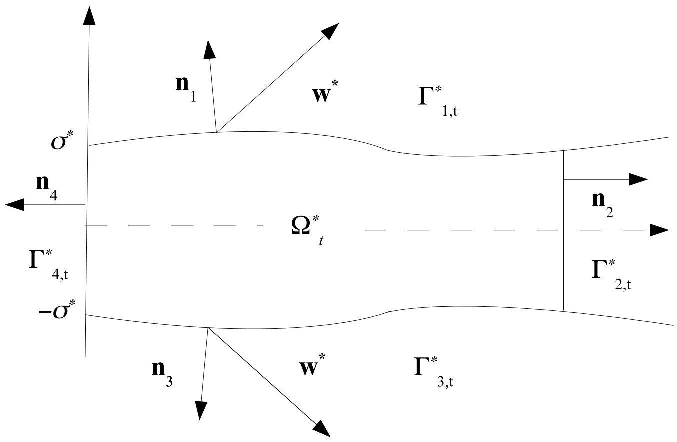

Figure 2: *Sketch of the inner rigid core*

where the surface integral on Γ^*_{1,t^*} and Γ^*_{3,t^*} on the left hand side are equal because of symmetry. Now we have to evaluate external forces acting on the boundary $\partial\Omega^*_{t^*}$ expressed by the surface integral on the r.h.s. of (9). We have

$$\mathbf{T}^*\mathbf{n}_1 = \frac{1}{\sqrt{1+\sigma_x^{*2}}}\left.\begin{pmatrix} -\sigma_x^* T_{11}^* + T_{12}^* \\ \\ -\sigma_x^* T_{12}^* + T_{22}^* \end{pmatrix}\right|_{y^*=\sigma^*} \qquad \mathbf{T}^*\mathbf{n}_2 = \begin{pmatrix} -P^*_{out} & 0 \\ \\ 0 & -P^*_{out} \end{pmatrix}\begin{pmatrix} 1 \\ \\ 0 \end{pmatrix}$$

$$\mathbf{T}^*\mathbf{n}_3 = \frac{1}{\sqrt{1+\sigma_x^{*2}}}\left.\begin{pmatrix} -\sigma_x^* T_{11}^* - T_{12}^* \\ \\ -\sigma_x^* T_{12}^* - T_{22}^* \end{pmatrix}\right|_{y^*=-\sigma^*} \qquad \mathbf{T}^*\mathbf{n}_4 = \begin{pmatrix} -P^*_{in} & 0 \\ \\ 0 & -P^*_{in} \end{pmatrix}\begin{pmatrix} -1 \\ \\ 0 \end{pmatrix}$$

where P^*_{in}, P^*_{out} ais the pressure applied at the lateral boundary that, for the sake of simplicity, are assumed that both P^*_{in}, P^*_{out} do not depend on y^*. Notice that the particular choice of $\mathbf{T}^*\mathbf{n}_2$, $\mathbf{T}^*\mathbf{n}_4$ means that only normal stress is imposed on the lateral boundaries. We observe that, still because of symmetry, the second component of $\mathbf{T}^*\mathbf{n}_1$ evaluated on σ^* must be the opposite of the second component of $\mathbf{T}^*\mathbf{n}_3$ evaluated on $-\sigma^*$

$$(-\sigma_x^* T_{12}^* + T_{22}^*)_{\sigma^*} = -(-\sigma_x^* T_{12}^* - T_{22}^*)_{-\sigma^*},$$

while the first component of $\mathbf{T}^*\mathbf{n}_1$ evaluated on σ^* must be equal to the first component of $\mathbf{T}^*\mathbf{n}_3$ evaluated on $-\sigma^*$

$$(-\sigma_x^* T_{11}^* + T_{12}^*)_{\sigma^*} = (-\sigma^* T_{11}^* - T_{12}^*)_{-\sigma^*},$$

In conclusion we have found that

$$\int_{\partial\Omega^*_t} (\mathbf{T}^*\mathbf{n})dS^* = 2\int_0^{L^*} \begin{bmatrix} (-\sigma_x^* T_{11}^* + T_{12}^*)_{\sigma^*} \\ \\ 0 \end{bmatrix} dx^* +$$

$$+2\int_0^{\sigma^*_{out}} \begin{pmatrix} -P^*_{out} \\ \\ 0 \end{pmatrix} dy^* + 2\int_0^{\sigma^*_{in}} \begin{pmatrix} P^*_{in} \\ \\ 0 \end{pmatrix} dy^*,$$

where $\sigma^*_{in} = \sigma^*(0,t^*)$, $\sigma^*_{out} = \sigma^*(L^*,t^*)$. Recalling that in the rigid plug velocity is

$$\begin{cases} v_1^* = k_1^*(t^*), \\ \\ v_2^* = k_2^*(t^*) = 0 \quad (\text{by simmetry}), \end{cases} \tag{10}$$

the dynamics of the whole rigid region is expressed by the following equation[5]

$$\int_0^{L^*} \frac{\partial}{\partial t^*} \left(\varrho^* k_1^* \sigma^* \right) dx^* = \int_0^{L^*} \left[-\sigma_x^* T_{11}^* + T_{12}^* \right]_{\sigma^*+} \, dx^* + P_{in}^* \sigma_{in}^* - P_{out}^* \sigma_{out}^*, \tag{11}$$

where $\sigma_{in}^* = \sigma^*(0,t^*)$, $\sigma_{out}^* = \sigma^*(L^*,t^*)$ and where P_{in}^*, P_{out}^* are the applied pressures at the inlet and at the outlet of the channel, respectively. In particular, $P_{in}^* = P_{in}^*(t^*)$, while we assume that P_{out}^* is constant in time[6]. Hence the given pressure difference driving the flow is

$$\Delta P^*(t^*) = P_{in}^*(t^*) - P_{out}^*. \tag{12}$$

We set $P_c^* = \sup_{t^* \geq 0} \Delta P^*(t^*)$, which, essentially, is the order of magnitude of the applied pressure difference. Concerning the boundary conditions we impose

$$\mathbf{v}^*(x^*, h^*, t^*) = 0, \quad \text{i.e.} \quad \text{no} - \text{slip}, \tag{13}$$

while, following [6], we write[7]

$$[\![\mathbf{v}^* \cdot \mathbf{t}]\!]_{y^*=\sigma^*} = 0, \qquad [\![\mathbf{v}^* \cdot \mathbf{n}]\!]_{y^*=\sigma^*} = 0, \tag{14}$$

$$[\![\mathbf{T}^* \mathbf{n} \cdot \mathbf{t}]\!]_{y^*=\sigma^*} = 0, \qquad [\![\mathbf{T}^* \mathbf{n} \cdot \mathbf{n}]\!]_{y^*=\sigma^*} = 0, \tag{15}$$

which express the continuity of the velocity and of the stress across the yield surface $y^* = \sigma^*$ (in the expressions above $\mathbf{t}$ and $\mathbf{n}$ represent the tangent and normal unit vector to $y^* = \sigma^*$ respectively).

Remark 1 *In section 5 we extend our approach, considering also the case in which the viscosity depends on pressure, namely*

$$\eta^* = \eta_c^* \, \eta\left(P^*\right), \quad \text{with} \quad \eta\left(P^*\right) \in (0,1), \tag{16}$$

2.3 Non dimensional formulation

As stated in the introduction, we assume that the characteristic length of the channel L^* is by far greater than its characteristic height $2H^*$, where

$$H^* = \sup_{x^* \in [0,L^*]} h^*(x^*),$$

[5] We remark that

$$\left[-\sigma_x^* T_{11}^* + T_{12}^* \right]_{\sigma^*+} = \lim_{y^* \to \sigma^*+} \left(-\sigma_x^* T_{11}^* + T_{12}^* \right),$$

i.e. the limit is evaluated form the viscous domain. Indeed $\left[-\sigma_x^* T_{11}^* + T_{12}^* \right]_{\sigma^*+}$ represents the force exerted by the viscous region on the lateral side of the inner rigid core.

[6] Minor changes allow to treat also the case $P_{out}^* = P_{out}^*(t^*)$.

[7] The symbol $[\![...]\!]$ denotes the jump across the interface $y^* = \sigma^*$. We are also assuming $[\![\varrho^*]\!]_{y^*=\sigma^*} = 0$.

so that we may introduce the parameter

$$\varepsilon = \frac{H^*}{L^*} \ll 1,$$

which is crucial for applying the classical thin film approach (or lubrication approximation). We rescale the problem using the following non dimensional variables[8]

$$\begin{gathered}
x = \frac{x^*}{L^*}, \qquad y = \frac{y^*}{\varepsilon L^*}, \qquad \sigma = \frac{\sigma^*}{\varepsilon L^*}, \qquad h = \frac{h^*}{\varepsilon L^*}, \qquad t = \frac{t^*}{(L^*/U^*)}, \\
v_1 = \frac{v_1^*}{U^*}, \quad v_2 = \frac{v_2^*}{\varepsilon U^*}, \quad P = \frac{P^* - P^*_{out}}{P^*_c}, \quad \Delta P = \frac{\Delta P^*(t^*)}{P^*_c} \underset{(12)}{=} \frac{P^*_{in}(t^*) - P^*_{out}}{P^*_c}, \\
\mathbf{S} = \frac{\mathbf{S}^*}{(\eta^*_c U^*/H^*)}, \quad \mathbf{D} = \frac{\mathbf{D}^*}{(U^*/H^*)}, \quad II_D = \frac{II_{D^*}}{(U^*/H^*)}, \quad II_S = \frac{II_{S^*}}{(\eta^*_c U^*/H^*)},
\end{gathered} \tag{17}$$

where

$$U^* = \left(\frac{{H^*}^2}{\eta^*_c} \frac{P^*_c}{L^*} \right), \tag{18}$$

comes from Poiseuille formula. After some algebra we find

$$\mathbf{D} = \frac{1}{2} \left[\begin{array}{cc} 2\varepsilon \dfrac{\partial v_1}{\partial x} & \dfrac{\partial v_1}{\partial y} + \varepsilon^2 \dfrac{\partial v_2}{\partial x} \\ \dfrac{\partial v_1}{\partial y} + \varepsilon^2 \dfrac{\partial v_2}{\partial x} & 2\varepsilon \dfrac{\partial v_2}{\partial y} \end{array} \right], \qquad \mathbf{S} = \left(2 + \frac{\mathsf{Bi}}{II_D} \right) \mathbf{D},$$

where

$$\mathsf{Bi} = \frac{\tau^*_o H^*}{\eta^*_c U^*} \underset{(18)}{=} \frac{1}{\varepsilon} \frac{\tau^*_o}{P^*_c},$$

is the so-called Bingham number. Moreover

$$II_D = \sqrt{\varepsilon^2 \left(\frac{\partial v_1}{\partial x} \right)^2 + \frac{1}{4} \left(\frac{\partial v_1}{\partial y} + \varepsilon^2 \frac{\partial v_2}{\partial x} \right)^2}.$$

Equations (5)-(7) become

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0, \tag{19}$$

$$\varepsilon \mathsf{Re} \left(\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y} \right) = -\frac{\partial P}{\partial x} + \varepsilon \frac{\partial S_{11}}{\partial x} + \frac{\partial S_{12}}{\partial y}, \tag{20}$$

$$\varepsilon^3 \mathsf{Re} \left(\frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y} \right) = -\frac{\partial P}{\partial y} + \varepsilon^2 \frac{\partial S_{12}}{\partial x} + \varepsilon \frac{\partial S_{22}}{\partial y}, \tag{21}$$

where $\mathsf{Re} = \left(\dfrac{\varrho^* U^* H^*}{\eta^*_c} \right)$ is the Reynolds number. The inner core equation (11) becomes

$$\varepsilon \mathsf{Re} \int_0^1 \frac{\partial}{\partial t}(k_1 \sigma) dx = \int_0^1 \left[P\sigma_x - \varepsilon \sigma_x S_{11} + S_{12} \right]_{\sigma^+} \, dx + \Delta P \sigma_{in}, \tag{22}$$

[8]Recall that P^*_{out} is constant in time.

since $P|_{x=0} = \Delta P$, and $P|_{x=1} = 0$. The boundary conditions (13)-(15) become

$$\mathbf{v}(x,h,t) = 0, \tag{23}$$

$$[\![v_1]\!]_{y=\sigma} = [\![v_2]\!]_{y=\sigma} = 0, \tag{24}$$

$$\begin{cases} [\![P]\!] \left[1+\varepsilon^2\left(\dfrac{\partial\sigma}{\partial x}\right)^2\right]_{y=\sigma} + \left[\varepsilon^3 S_{11}\left(\dfrac{\partial\sigma}{\partial x}\right)^2 - 2\varepsilon^2 S_{12}\left(\dfrac{\partial\sigma}{\partial x}\right) + \varepsilon S_{22}\right]_{y=\sigma} = 0, \\[2ex] [\![S_{12}]\!]_{y=\sigma} + \varepsilon\left(\dfrac{\partial\sigma}{\partial x}\right)\left[S_{22} - S_{11} - \varepsilon S_{12}\dfrac{\partial\sigma}{\partial x}\right]_{y=\sigma} = 0. \end{cases} \tag{25}$$

In the rigid domain the non dimensional velocity field is

$$\begin{cases} v_1 = k_1(t), \\[2ex] v_2 = 0, \end{cases} \tag{26}$$

where $k_1 = k_1^*/U^*$.

3 Asymptotic expansion

Following again [6], we look for a solution in which the main variables of the problem can be expressed as power series of ε, namely

$$\mathbf{v} = \mathbf{v}^{(0)} + \varepsilon\mathbf{v}^{(1)} + \varepsilon^2\mathbf{v}^{(2)} + ...$$

$$P = P^{(0)} + \varepsilon P^{(1)} + \varepsilon^2 P^{(2)} + ...$$

$$\sigma = \sigma^{(0)} + \varepsilon\sigma^{(1)} + \varepsilon^2\sigma^{(2)} + ...$$

$$k_1 = k_1^{(0)} + \varepsilon k_1^{(1)} + \varepsilon^2 k_1^{(2)} + ...$$

$$\mathbf{S} = \mathbf{S}^{(0)} + \varepsilon\mathbf{S}^{(1)} + \varepsilon^2\mathbf{S}^{(2)} + ...$$

We further assume that $h(x)$ is sufficiently smooth[9] and limit our analysis to the leading order, considering $\mathsf{Bi} = \mathcal{O}(1)$ and $\mathsf{Re} \lesssim \mathcal{O}(1)$. We do not consider any converging issues.

3.1 The leading order approximation

In this section we determine the velocity field and the yield surface in terms of the pressure, where the latter satisfies an integro-differential equation of elliptic type (see (39)). We begin by observing that

$$S_{12}^{(0)} = \left[1 + \frac{\mathsf{Bi}}{|v_{1y}^{(0)}|}\right] v_{1y}^{(0)},$$

and since we are looking for a solution with $v_{1y}^{(0)} < 0$ in the upper part of the channel we get

$$S_{12}^{(0)} = v_{1y}^{(0)} - \mathsf{Bi}.$$

[9]Essentially we assume $\dfrac{\partial h}{\partial x} = O(1)$.

The problem reduces to

$$\begin{cases} \dfrac{\partial v_1^{(0)}}{\partial x} + \dfrac{\partial v_2^{(0)}}{\partial y} = 0, \\ \\ -\dfrac{\partial P^{(0)}}{\partial x} + \dfrac{\partial}{\partial y}\left(\dfrac{\partial v_1^{(0)}}{\partial y}\right) = 0, \\ \\ -\dfrac{\partial P^{(0)}}{\partial y} = 0, \end{cases} \tag{27}$$

with boundary conditions

$$\begin{cases} \left.\dfrac{\partial v_1^{(0)}}{\partial y}\right|_{y=\sigma^{(0)}} = 0, \\ \\ v_1^{(0)}(x,h,t) = 0. \end{cases} \tag{28}$$

The first comes from the condition $II_D = 0$ on $y = \sigma$, while the second is simply no-slip. From $(27)_3$ we get $P^{(0)} = P^{(0)}(x,t)$, so that $(27)_{1,2}$ can be used to find that[10]

$$v_1^{(0)} = -P_x^{(0)} \frac{(h-y)(y-2\sigma^{(0)}+h)}{6}. \tag{29}$$

Exploiting the continuity equation we find $v_2\,(x,y,t) = \displaystyle\int_y^h v_{1\ x} dy$, namely

$$v_2^{(0)} = -\frac{\partial}{\partial x}\left[P_x^{(0)} \frac{(y-h)^2(y-3\sigma^{(0)}+2h)}{6}\right]. \tag{30}$$

Evaluating $v_1^{(0)}$, $v_2^{(0)}$ on $\sigma^{(0)}$ and recalling conditions (24), (26), we obtain

$$\left. v_1^{(0)}\right|_{y=\sigma^{(0)}} = k_1^{(0)}(t) = -P_x^{(0)} \frac{(h-\sigma^{(0)})^2}{2}, \tag{31}$$

$$\left. v_2^{(0)}\right|_{y=\sigma^{(0)}} = \frac{\partial}{\partial x}\left[-P_x^{(0)} \frac{(h-\sigma^{(0)})^3}{3}\right] - \sigma_x^{(0)} P_x^{(0)} \frac{(h-\sigma^{(0)})^2}{2} = 0,$$

which entails

$$\underbrace{\left(-P_x^{(0)} \frac{(h-\sigma^{(0)})^2}{2}\right)}_{k_1^{(0)}} \cdot \frac{\partial}{\partial x}\left[\frac{2}{3}(h-\sigma^{(0)})\right] = -\sigma_x^{(0)} \underbrace{\left(-P_x^{(0)} \frac{(h-\sigma^{(0)})^2}{2}\right)}_{k_1^{(0)}}.$$

Hence, supposing $k_1^{(0)} \neq 0$, we get $\dfrac{\partial}{\partial x}\left[\dfrac{2}{3}(h-\sigma^{(0)}) + \sigma^{(0)}\right] = 0$. In conclusion

$$\sigma^{(0)}\,(x,t) = -2h\,(x) - C\,(t)\,, \tag{32}$$

[10] To keep notation light f_x, f_{xx} denote $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, respectively.

where C is unknown at this stage. Let us now consider the rigid core equation (22) at the zero order

$$\int_0^1 P^{(0)}\sigma_x^{(0)}dx - \mathsf{Bi} + \Delta P\sigma_{in}^{(0)} = 0,$$

which, after an integration by parts, reduces to

$$-\int_0^1 P_x^{(0)}\sigma^{(0)}dx = \mathsf{Bi}. \tag{33}$$

Substituting (32) into (33), we obtain

$$C\left(t\right) = \frac{2\displaystyle\int_0^1 P_x^{(0)}hdx - \mathsf{Bi}}{\Delta P\left(t\right)},$$

with $\Delta P\left(t\right)$ defined in (17). We have

$$\sigma^{(0)}\left(x,t\right) = -2h\left(x\right) + \frac{\mathsf{Bi}-2\displaystyle\int_0^1 P_x^{(0)}h\ dx}{\Delta P\left(t\right)}, \tag{34}$$

or equivalently

$$\sigma^{(0)}\left(x,t\right) = 2(h_{in} - h\left(x\right)) + \frac{\mathsf{Bi}}{\Delta P\left(t\right)} + \frac{2}{\Delta P\left(t\right)}\int_0^1 P^{(0)}h_x dx, \tag{35}$$

where $h_{in} = \left.h\right|_{x=0}$. Defining the viscous region width as

$$\ell^{(0)}\left(x,t\right) = h\left(x\right) - \sigma^{(0)}\left(x,t\right), \tag{36}$$

formula (34) entails

$$\ell^{(0)}\left(x,t\right) = 3h\left(x\right) + \frac{2\displaystyle\int_0^1 P_x^{(0)}h\ dx - \mathsf{Bi}}{\Delta P\left(t\right)}. \tag{37}$$

In particular, recalling (31) and (36), we have

$$k_1^{(0)} = -P_x^{(0)}\frac{\ell^{(0)2}}{2}. \tag{38}$$

Now, differentiating (38) with respect to[11] x, we obtain

$$P_{xx}^{(0)}\ \ell^{(0)\ 2} + 2\ell^{(0)}\ell_x^{(0)}P_x^{(0)} = 0, \quad \underset{(37)}{\Rightarrow} \quad P_{xx}^{(0)} + 6\frac{h_x}{\ell^{(0)}}P_x^{(0)} = 0,$$

i.e. such a integro-differential equation

$$P_{xx}^{(0)} + \frac{6h_x}{\left[3h + \dfrac{2\displaystyle\int_0^1 P_x^{(0)}h\ dx - \mathsf{Bi}}{\Delta P\left(t\right)}\right]}P_x^{(0)} = 0, \tag{39}$$

whose boundary conditions are $\left.P^{(0)}\right|_{x=0} = \Delta P\left(t\right)$, and $\left.P^{(0)}\right|_{x=1} = 0$. The solution $P^{(0)}(x,t)$ of (39) is then used to evaluate the $v_1^{(0)}$ via (29), $v_2^{(0)}$ via (30) and the yield surface $\sigma^{(0)}$ via (35).

[11] Recall that $k_{1x}^{(0)} = 0$.

Remark 2 *From (34) we see that* $\sigma_x^{(0)} = -2h_x$, *i.e. the core amplitude widens as the channel narrows, whereas it shrinks as the channel becomes wider. Such counterintuitive behavior has been already observed in section 3.1 of [9], where we can read: "An interesting feature of this solution is that the unyielded plug (i.e. the inner core) is wider in the narrower part of the channel. This is counterintuitive from the perspective of the stress, as we expect larger shear stresses in the narrower channel".*

3.2 Flow condition

It is interesting to investigate the so-called "flow condition": i.e. the condition on ΔP that prevent the system from coming to a stop. In case[12] $h(x) \equiv h_{in}$ we observe by (35) that:

- $\Delta P > \dfrac{\mathsf{Bi}}{h_{in}}, \implies \sigma^{(0)} < h_{in}$, i.e. the fluid is flowing.
- $\Delta P < \dfrac{\mathsf{Bi}}{h_{in}}, \implies \sigma^{(0)} > h_{in}$, i.e. the rigid core occupies the whole channel and there is no flow.

When $h(x)$ is not uniform we have to ensure that $\sigma^{(0)} < h(x)$, in order to prevent the flow from stopping (see (43)). Recalling (35), we have

$$\sigma^{(0)} = 2(h_{in} - h(x)) + \frac{\mathsf{Bi}}{\Delta P} + \frac{2}{\Delta P}\int_0^1 P^{(0)} h_x dx < h(x),$$

or, recalling (36),

$$\ell^{(0)}(x,t) = 3h(x) - 2h_{in} - \frac{\mathsf{Bi}}{\Delta P} - \frac{2}{\Delta P}\int_0^1 P^{(0)} h_x dx > 0.$$

Now, since

$$\ell^{(0)}(x,t) \geq 3\min_{x\in[0,1]} h - 2h_{in} - \frac{\mathsf{Bi}}{\Delta P} - \frac{2}{\Delta P}\int_0^1 P^{(0)} h_x dx, \tag{40}$$

we estimate $\int_0^1 P^{(0)} h_x dx$. To this end we remark that $P^{(0)}$ fulfils equation (39), which is of elliptic type. Maximum principle entails $0 \leq P^{(0)} \leq \Delta P$, $\forall\, x \in [0,1]$. So, rewriting $\int_0^1 P^{(0)} h_x dx$ as

$$\int_0^1 P^{(0)} h_x dx = \underbrace{\int_{\{h_x \leq 0\}} P^{(0)} h_x dx}_{\leq 0} + \underbrace{\int_{\{h_x \geq 0\}} P^{(0)} h_x dx}_{\geq 0},$$

we have[13]

$$\Delta P \min\{\underline{h}_{\,x}; 0\} \leq \int_{\{h_x \leq 0\}} P^{(0)} h_x dx \leq \int_0^1 P^{(0)} h_x dx \leq \int_{\{h_x \geq 0\}} P^{(0)} h_x dx \leq \Delta P \max\left\{\overline{h}_x; 0\right\},$$

where

$$\underline{h}_{\,x} = \min_{x\in[0,1]} h_x(x), \quad \text{and} \quad \overline{h}_x = \min_{x\in[0,1]} h_x(x).$$

[12] h_{in} is the inlet channel semi-amplitude, i.e. $h_{in} = h|_{x=0}$.

[13] Recall that $\max\{a;b\} = a$, if $a \geq b$, otherwise $\max\{a;b\} = b$. Something similar for $\min\{a;b\}$.

In conclusion

$$2\min\left\{\underline{h}_{\ x};0\right\} \leq \frac{2}{\Delta P}\int_0^1 P^{(0)}h_x dx \leq 2\max\left\{\overline{h}_x;0\right\}. \tag{41}$$

Therefore, recalling (40), we have

$$\ell^{(0)}(x,t) \geq 3h_{\min} - 2h_{in} - \frac{\mathsf{Bi}}{\Delta P} - \frac{2}{\Delta P}\int_0^1 P^{(0)}h_x dx \geq 3h_{\min} - 2\max\left\{\overline{h}_x;0\right\} - 2h_{in} - \frac{\mathsf{Bi}}{\Delta P}, \tag{42}$$

where $h_{\min} = \min_{x\in[0,1]} h$. So, if we assume $\left(3h_{\min} - 2\max\left\{\overline{h}_x;0\right\} - 2h_{in}\right) > 0$, and require that

$$3h_{\min} - 2\max\left\{\overline{h}_x;0\right\} - 2h_{in} - \frac{\mathsf{Bi}}{\Delta P} > 0, \quad \Leftrightarrow \quad \Delta P > \frac{\mathsf{Bi}}{3h_{\min} - 2\max\left\{\overline{h}_x;0\right\} - 2h_{in}}, \tag{43}$$

we are sure that the flow never ceases.

Example 3 *In case we consider "flat" channel with $h_x \equiv 0$, (43) reduces to*

$$\frac{\mathsf{Bi}}{\Delta P} < h_{in}, \quad \Leftrightarrow \quad \Delta P > \frac{\mathsf{Bi}}{h_{in}},$$

that is the flow condition for a channel with parallel walls.

Example 4 *If we consider a linear wall profile*

$$h(x) = h_{in} + \underbrace{(h_{out} - h_{in})}_{\Delta h} x,$$

where $h_{out} > 0$, there are two possibilities:

- $\Delta h > 0$, $\Rightarrow h_{\min} = h_{in}$, *and* $\max\left\{\overline{h}_x;0\right\} = \Delta h$. *Condition (43) yields*

$$\frac{\mathsf{Bi}}{\Delta P} < \underbrace{h_{in} - 2\Delta h}_{2h_{out}-3h_{in}}, \quad \Leftrightarrow \quad \Delta P > \frac{\mathsf{Bi}}{2h_{out} - 3h_{in}},$$

  where, of course, we assume $2h_{out} - 3h_{in} > 0$, namely $\dfrac{h_{out}}{h_{in}} > \dfrac{3}{2}$.

- $\Delta h < 0$, $\Rightarrow h_{\min} = h_{out}$, *and* $\max\left\{\overline{h}_x;0\right\} = 0$. *Inequality (43) entails*

$$\frac{\mathsf{Bi}}{\Delta P} < \underbrace{2\Delta h + h_{out}}_{3h_{out}-2h_{in}}, \quad \Leftrightarrow \quad \Delta P > \frac{\mathsf{Bi}}{3h_{out} - 2h_{in}},$$

  where now we require $\dfrac{h_{out}}{h_{in}} > \dfrac{2}{3}$.

Remark 5 *Actually condition (43) can be improved, estimating $\int_0^1 P^{(0)}h_x dx$ by means of the Cauchy-Schwarz inequality. Indeed, considering that*

$$-\left|\int_0^1 P^{(0)}h_x dx\right| \leq \int_0^1 P^{(0)}h_x dx \leq \left|\int_0^1 P^{(0)}h_x dx\right|,$$

Cauchy-Schwarz inequality yields[14]

$$-2\left\|h_x\right\|_{L^2} \leq \frac{2}{\Delta P}\int_0^1 P^{(0)} h_x dx \leq 2\left\|h_x\right\|_{L^2}, \tag{44}$$

since $P^{(0)} \in [0, \Delta P]$. *Hence (42) can be rewritten as*

$$\ell^{(0)}(x,t) \geq 3h_{\min} - 2\left\|h_x\right\|_{L^2} - 2h_{in} - \frac{\mathsf{Bi}}{\Delta P}.$$

So, introducing

$$\mathfrak{h} = \max\left\{\left(3h_{\min} - 2\max\left\{\ \overline{h}_x; 0\right\} - 2h_{in}\right);\ \left(3h_{\min} - 2\left\|h_x\right\|_{L^2} - 2h_{in}\right)\right\},$$

and assuming $\mathfrak{h} > 0$, *the flow condition can be rewritten as*

$$\Delta P > \frac{\mathsf{Bi}}{\mathfrak{h}}.$$

The latter is more general than (43), since it does not require that $\left(3h_{\min} - 2\max\left\{\overline{h}_x; 0\right\} - 2h_{in}\right)$ *is positive. Indeed we assume that at least one among* $\left(3h_{\min} - 2\max\left\{\overline{h}_x; 0\right\} - 2h_{in}\right)$ *and* $\left(3h_{\min} - 2\left\|h_x\right\|_{L^2} - 2h_{in}\right)$, *is positive.*

3.3 Inner core appearance or disappearance

A non uniform channel profile may cause the appearance/disappearance of the rigid plug. These phenomena (highlighted also in [5] and [9] and therein referred to as "breaking of the plug") are not possible when the channel profile is uniform, namely when $h(x) \equiv h_{in}$, since $\sigma^{(0)}(t) = \dfrac{\mathsf{Bi}}{\Delta P(t)}$. Recalling (35), we set

$$\sigma^{(0)}(x,t) = \max\left\{0;\ 2(h_{in} - h(x)) + \frac{\mathsf{Bi}}{\Delta P} + \frac{2}{\Delta P}\int_0^1 P^{(0)} h_x dx\right\},$$

in order to avoid physical inconsistencies. Hence, $\sigma^{(0)}(x,t)$ vanishes when

$$h(x) \geq h_{in} + \frac{\mathsf{Bi}}{2\Delta P} + \frac{1}{\Delta P}\int_0^1 P^{(0)} h_x dx. \tag{45}$$

The r.h.s. of (45) is a critical value, that we denote as h_{crt}, such that, whenever $h(x) \geq h_{crt}$ the core disappears.

Example 6 *Let us consider the channel profile*

$$h(x) = \frac{\arctan\left[5\left(\dfrac{1}{2} - x\right)\right]}{4\arctan\left(\dfrac{5}{2}\right)} + \frac{3}{4}. \tag{46}$$

[14] $\| f \|_{L^2} = \left[\int_0^1 f^{\,2}(x)\, dx\right]^{1/2}$.

depicted with the dashed line in Fig. 7. We now estimate h_{crt} exploiting (45), when $\Delta P = 10$, and $\mathsf{Bi} = 5$*,*

$$\begin{aligned} h\left(x\right) &\geq 1 + \frac{\mathsf{Bi}}{2\Delta P} - \frac{1}{\Delta P}\int_0^1 P^{(0)}\left|h_x\right| dx \\ &\geq 1 + \frac{\mathsf{Bi}}{2\Delta P} - \left\|h_x\right\|_{L^2} \gtrsim 1 + \frac{\mathsf{Bi}}{2\Delta P} - 0.58 \approx 0.67. \end{aligned}$$

The "core free" region is thus obtained solving $h\left(x\right) \geq h_{crt}$, which we approximate with $h\left(x\right) \geq 0.67$, whose solution is the interval $1 \leq x \leq 0.58$. Looking at Figure 7 the actual "core free" region is $1 \leq x \lesssim 0.55$, which substantially agrees with the above estimate.

3.4 Solution for a channel whose width is almost uniform

When $h = h_{in}$ (i.e. uniform channel amplitude) equation (35) gives

$$\sigma^{(0)}\left(t\right) = \frac{\mathsf{Bi}}{\Delta P\left(t\right)}, \tag{47}$$

and (39) reduces to

$$\begin{cases} P^{(0)}_{xx} = 0, & 0 < x < 1, \\ \left.P^{(0)}\right|_{x=0} = \Delta P\left(t\right), \quad \text{and} & \left.P^{(0)}\right|_{x=1} = 0, \end{cases} \implies P^{(0)}\left(x,t\right) = \left(1-x\right)\Delta P\left(t\right),$$

and the velocity field becomes[15]

$$\begin{cases} v_1^{(0)} &= -\Delta P\left(t\right)\left[\dfrac{(y-\sigma^{(0)})^2}{2} - \dfrac{(1-\sigma^{(0)})^2}{2}\right], \\ v_2^{(0)} &= 0, \end{cases} \tag{48}$$

and we also find $k_1^{(0)}(t) = \dfrac{\Delta P\left(t\right)}{2}(1-\sigma^{(0)})^2$.

Let us now consider a non-uniform channel profile $h\left(x\right)$. We set

$$h\left(x\right) = \langle h\rangle + \phi\left(x\right), \tag{49}$$

where $\langle h\rangle$ denotes the spatial average along the channel, i.e. $\langle\,\cdot\,\rangle = \displaystyle\int_0^1 (\,\cdot\,)\,dx$, and assume $\max\left|\phi\left(x\right)\right|$ "small" (in other words we consider an almost "flat" channel). We notice that $\int_0^1 \phi(x)dx = 0$. We look for $P^{(0)}$ in the form

$$P^{(0)}\left(x,t\right) = \left(1-x\right)\Delta P\left(t\right) + \Pi\left(x,t\right), \tag{50}$$

where[16] $\left.\Pi\right|_{x=0} = \left.\Pi\right|_{x=1} = 0$, and where we expect that both $\max\left|\Pi\right|$, $\max\left|\Pi_x\right|$ are also "small". Inserting (49) and (50) into (34) we obtain

$$\sigma^{(0)}\left(x,t\right) = \frac{\mathsf{Bi}}{\Delta P} - 2\phi\left(x\right) - \frac{2}{\Delta P}\int_0^1 \Pi_x\phi\; dx \approx \frac{\mathsf{Bi}}{\Delta P\left(t\right)} - 2\phi\left(x\right). \tag{51}$$

[15] We set, for the sake of simplicity, $h_{in} = 1$.
[16] Recall that $\left.P\right|_{x=0} = \Delta P$, $\left.P\right|_{x=1} = 0$.

Notice that $\left\langle\sigma^{(0)}\right\rangle = \dfrac{\mathsf{Bi}}{\Delta P}$, i.e. the average width of the rigid core is the one corresponding to the flat channel. Concerning $\ell^{(0)}$, form (36) we have

$$\ell^{(0)}\left(x,t\right) \approx \langle h\rangle - \frac{\mathsf{Bi}}{\Delta P\left(t\right)} + 3\phi\left(x\right). \qquad (52)$$

Exploiting then (39) we compute the pressure field solving

$$\Pi_{xx} + \frac{6\phi_x}{\ell^{(0)}}\left(-\Delta P + \Pi_x\right) = 0.$$

Neglecting $\phi_x\Pi_x$, and considering (52), we have

$$\begin{cases} \Pi_{xx} - 2\Delta P\left[\dfrac{\phi_x}{\phi+\mathcal{A}}\right] = 0, \quad \text{where} \quad \mathcal{A} = \dfrac{< h >}{3} - \dfrac{\mathsf{Bi}}{3\Delta P}, \\ \\ \Pi|_{x=0} = \Pi|_{x=1} = 0, \end{cases}$$

so that

$$\Pi_x = (const.) + 2\Delta P \ln\left[1 + \frac{\phi\left(x\right)}{\mathcal{A}}\right] \approx (const.) + 2\Delta P\frac{\phi\left(x\right)}{\mathcal{A}}.$$

In conclusion

$$\Pi\left(x,t\right) = \frac{2\Delta P\left(t\right)}{\mathcal{A}}\int_0^x \phi(x^{'})dx^{'},$$

which yields

$$P^{(0)}\left(x,t\right) = \Delta P\left(t\right)\left(1-x\right) + \frac{6\Delta P^2}{\langle h\rangle\Delta P - \mathsf{Bi}}\int_0^x \phi\left(x'\right)dx'. \qquad (53)$$

Example 7 *Let us consider $h\left(x\right) = 1 + mx$, with m "small", that we write also as*

$$h\left(x\right) = \underbrace{1 + \frac{m}{2}}_{\langle h\rangle} + \underbrace{m\left(x - \frac{1}{2}\right)}_{\phi(x)}.$$

Equations (51), (53) yield

$$\sigma^{(0)} = \frac{\mathsf{Bi}}{\Delta P} - 2m\left(x - \frac{1}{2}\right),$$

$$P^{(0)}\left(x,t\right) = \Delta P\left(t\right)\left(1-x\right) + \frac{3m\Delta P^2}{\langle h\rangle\Delta P - \mathsf{Bi}}x\left(x-1\right),$$

respectively. We see that $\sigma_x^{(0)} = -2m$, i.e. the core amplitude widens for $m < 0$, whereas it shrinks for $m > 0$.

Example 8 *We consider a wavy channel as the one of [5]*

$$h(x) = 1 - \theta\cos\left[2\pi\delta\left(x - \frac{1}{2}\right)\right], \qquad (54)$$

where $\delta > 0$, $\theta \in (0,1)$. We thus write

$$h\left(x\right) = \underbrace{\left[1 - \frac{\theta}{\pi\delta}\sin\left(\pi\delta\right)\right]}_{\langle h\rangle} + \underbrace{\left[\frac{\theta\sin\left(\pi\delta\right)}{\pi\delta} - \theta\cos\left(2\pi\delta\left(x - \frac{1}{2}\right)\right)\right]}_{\phi(x)},$$

with $\max|\phi| = \mathcal{O}(\theta)$. *Exploiting (51) we obtain*

$$\sigma^{(0)} \approx \frac{\mathsf{Bi}}{\Delta P} - 2\theta\left[\frac{\sin(\pi\delta)}{\pi\delta} - \cos\left(2\pi\delta\left(x - \frac{1}{2}\right)\right)\right], \tag{55}$$

The behavior for $\theta = 0.1$, *and* $\delta = 1/5$ *is shown in Fig. 3-4. In particular in Fig. 4 a close-up showing the difference between the approximated solution (55) and the computed one (see next section) is displayed. We emphasize that Fig. 3 essentially agrees with the behavior shown in Fig. 3.(a) of [9].*

4 Numerical simulation and comparison with results from the existing literature

We note that, setting $F = P_x^{(0)}$, the elliptic problem (39)

$$\begin{cases} P_{xx}^{(0)} + 6\dfrac{h_x}{\ell^{(0)}}P_x^{(0)} = 0, \\ \\ P^{(0)}\big|_{x=0} = \Delta P(t), \quad \text{and} \quad P^{(0)}\big|_{x=1} = 0, \end{cases}$$

can be transformed in the following integral equation

$$F = -\Delta P \frac{\exp\left\{-\displaystyle\int_0^x \frac{6h_{x'}}{\ell_F}dx'\right\}}{\displaystyle\int_0^1 \exp\left\{-\int_0^x \frac{6h_{x'}}{\ell_F}dx'\right\}dx}, \tag{56}$$

where, recalling (37),

$$\ell_F = \min\left\{h(x),\ 3h(x) + \frac{2\displaystyle\int_0^1 Fh\ dx - \mathsf{Bi}}{\Delta P}\right\}.$$

Now, if the conditions ensuring that $\ell^{(0)}$ is strictly positive (Section 3.2) are fulfilled, we can solve (56) through the following iterative procedure:

Step $j = 0$. We set $F_0 = -\Delta P$, and $\ell_{F,\,0} = \min\left\{h(x),\ 3h(x) - \dfrac{\mathsf{Bi}}{\Delta P} - 2\displaystyle\int_0^1 h\ dx\right\}$.

Step $j = 1$. $F_1 = -\Delta P \dfrac{\exp\left\{-\displaystyle\int_0^x \frac{6h_{x'}}{\ell_{F,\,0}}dx'\right\}}{\displaystyle\int_0^1 \exp\left\{-\int_0^x \frac{6h_{x'}}{\ell_{F,\,0}}dx'\right\}dx}$.

......

Step $j > 1$. $F_j = -\Delta P \dfrac{\exp\left\{-\displaystyle\int_0^x \frac{6h_{x'}}{\ell_{F,\,j-1}}dx'\right\}}{\displaystyle\int_0^1 \exp\left\{-\int_0^x \frac{6h_{x'}}{\ell_{F,\,j-1}}dx'\right\}dx}$, with

$$\ell_{F,\,j-1} = \min\left\{h(x),\ 3h(x) + \frac{2\displaystyle\int_0^1 F_{j-1}\ h\ dx - \mathsf{Bi}}{\Delta P}\right\}.$$

Iterating the procedure until the desired tolerance is reached, we determine the solution $F = P_x^{(0)}$. Integration then provides the pressure field $P^{(0)}$. We can show that, under suitable hypotheses, the solution of (56) exists and is unique (this will be the subject of a forthcoming paper).

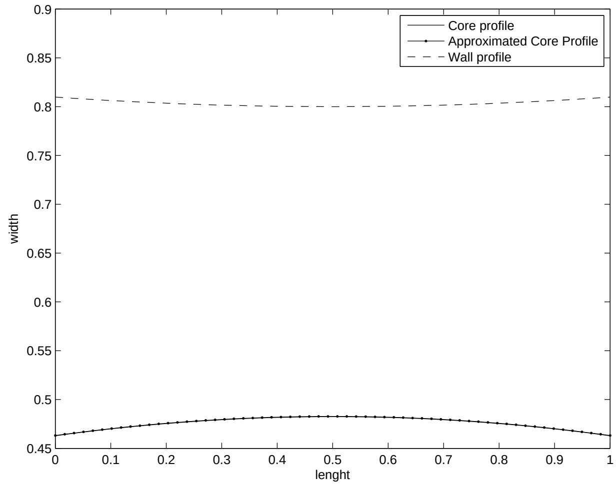

Figure 3: The channel profile $h(x)$ is (54) and of $\sigma^{(0)}$ given by (34), (55), with $\mathsf{Bi} = 5$, $\Delta P = 10.5$, $\delta = 0.2$, $\theta = 0.1$.

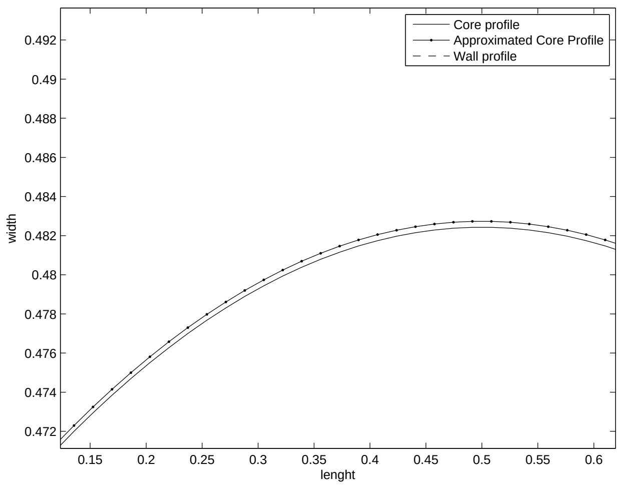

Figure 4: Close up for the difference between $\sigma^{(0)}$ given by (55) and $\sigma^{(0)}$ given by (34).

In Figures 3, 4 we have plotted $h(x)$ and $\sigma^{(0)}(x)$ for the wavy channel profile given by (54). Comparing Fig. 3 with the numerical simulation of [5], we notice a good qualitative agreement, even if the problem studied in [5] is substantially different. Indeed in [5] the problem is solved in a "periodic" portion of the channel imposing a constant flux, differently from our case in which a pressure gradient is applied. In Figures 5, 6 we have reported the contour plots of $v_1^{(0)}$, and $v_2^{(0)}$, when $h(x)$ is given by (54), with $\delta = 0.1$, $\theta = 0.02$, and $\mathsf{Bi} = 5$.
The solid colored regions of Fig. 5-6 denote the core, with vanishing transversal velocity and

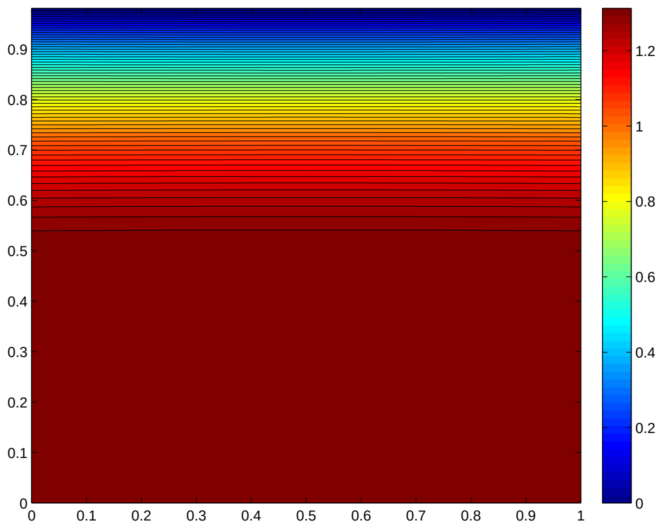

Figure 5: Plot of x-component of the velocity, h given by (54), $\delta = 0.1$, $\theta = 0.02$, and $\mathsf{Bi} = 5$.

uniform longitudinal velocity. Notice also the symmetry of the transversal velocity shown in Fig. 6. In Fig. 7, 8, 9 we have considered the profile (46). The yield surface $\sigma^{(0)}$ and the velocities $v_1^{(0)}$, $v_2^{(0)}$ are reported respectively.

5 Model with pressure dependent viscosity

Differently from the classical constitutive model here we assume that viscosity depends monotonically on pressure (see, e.g. [13] and [16]). Hence (4) rewrites in this way

$$\mathbf{D}^* = \frac{II_{\mathbf{D}^*}}{2\eta^*\left(P^*\right) II_{\mathbf{D}^*} + \tau_o^*}\mathbf{S}^*.$$

In particular, recalling (16), the viscosity is expanded considering

$$\eta(P) = \eta(P^{(0)} + \varepsilon P^{(1)} + \varepsilon^2 P^{(2)} + ...),$$

so that, around $\varepsilon = 0$ we get $\eta = \eta^{(0)} + \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + ...$, where

$$\eta^{(0)} = \eta(P^{(0)}), \qquad \eta^{(1)} = \frac{d\eta}{dP}(P^{(0)})\ P^{(1)}. \tag{57}$$

Following the same procedure described in the section 3.1, problem (27) can be rewritten as

$$\begin{cases} \dfrac{\partial v_1^{(0)}}{\partial x} + \dfrac{\partial v_2^{(0)}}{\partial y} = 0, \\[2ex] -\dfrac{\partial P^{(0)}}{\partial x} + \dfrac{\partial}{\partial y}\left(\eta^{(0)}\left(P^{(0)}\right)\dfrac{\partial v_1^{(0)}}{\partial y}\right) = 0, \\[2ex] -\dfrac{\partial P^{(0)}}{\partial y} = 0, \end{cases}$$

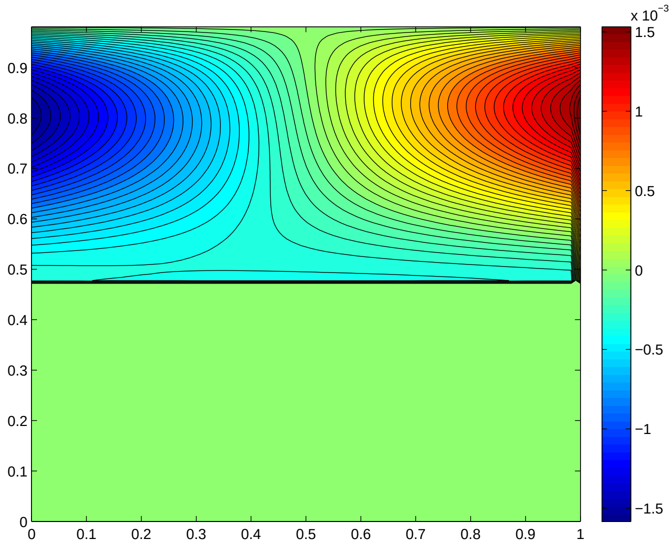

Figure 6: Plot of y-component of the velocity, h given by (54), $\delta = 0.1$, $\theta = 0.02$, and $\mathsf{Bi} = 5$.

whose boundary conditions are still given by (28). Similarly to what we found in section 3.1, we have

$$\begin{cases} v_1^{(0)} = \dfrac{P_x^{(0)}}{\eta^{(0)}\left(P^{(0)}\right)} \dfrac{(y-h^{(0)})(y-2\sigma^{(0)}+h^{(0)})}{6}, \\[2ex] v_2^{(0)} = \dfrac{\partial}{\partial x}\left[\dfrac{P_x^{(0)}}{\eta^{(0)}\left(P^{(0)}\right)} \dfrac{(y-h^{(0)})^2(y-3\sigma^{(0)}+2h^{(0)})}{6}\right], \end{cases}$$

and

$$k_1^{(0)}(t) = -\frac{P_x^{(0)}}{\eta(P^{(0)})} \frac{(h^{(0)}-\sigma^{(0)})^2}{2}.$$

The interface $\sigma^{(0)}$ is still given by (34), while equation (39) modifies in this way

$$\left(\frac{P_x^{(0)}}{\eta(P^{(0)})}\right)_x + 6\frac{h_x}{\ell^{(0)}} \frac{P_x^{(0)}}{\eta(P^{(0)})} = 0, \tag{58}$$

where $\ell^{(0)}$ is given by (37).

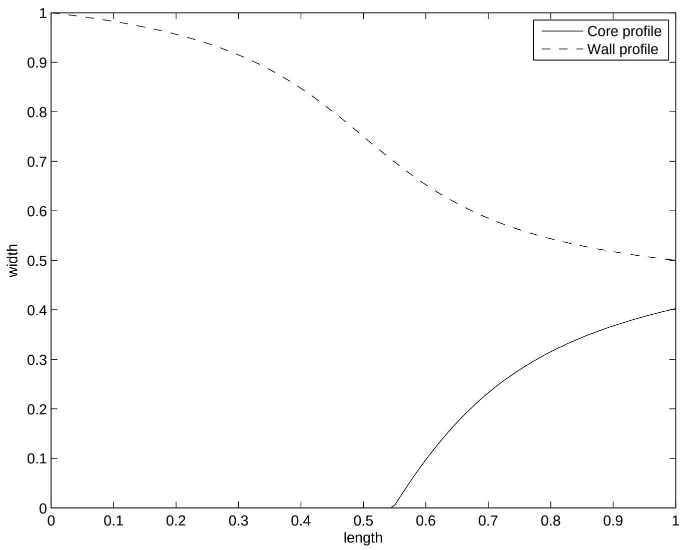

Figure 7: Plot of $\sigma^{(0)}$ and h, when h is given by (46).

5.0.1 Solution for an almost flat channel with exponential viscosity

In case $\eta(P) = e^{\gamma P}$, and $h \equiv 1$, we get (see [8])

$$\begin{cases} v_1^{(0)} = \dfrac{[e^{-\gamma P_{in}} - e^{-\gamma P_{out}}]}{\gamma} \left[\dfrac{(y-\sigma^{(0)})^2}{2} - \dfrac{(1-\sigma^{(0)})^2}{2} \right], \\ \\ v_2^{(0)} = 0, \\ \\ \sigma^{(0)} = \dfrac{\mathsf{Bi}}{\Delta P}, \\ \\ P(x) = P_{in} - \dfrac{1}{\gamma} \ln \left[1 + \left(e^{\gamma \Delta P} - 1 \right) x \right]. \end{cases} \tag{59}$$

We now consider $h^{(0)} = 1 + mf(x)$, with m "small" perturbation. We look for a solution of (58) of the form

$$P^{(0)}(x,t) = P_{in} - \frac{1}{\gamma} \ln \left[1 + \left(e^{\gamma \Delta P} - 1 \right) x \right] + m\Pi, (x,t), \tag{60}$$

with $\Pi(x=0,t) = \Pi(x=1,t) = 0$. After replacing (60) into (58) and neglecting the m^2, we find

$$\Pi(x,t) = -\frac{6}{\gamma} \frac{\left(e^{\gamma \Delta P} - 1 \right)}{1 + (e^{\gamma \Delta P} - 1)x} \left[x \int_0^1 f(\xi) d\xi - \int_0^x f(\xi) d\xi \right],$$

and

$$\sigma^{(0)} = \frac{\mathsf{Bi}}{\Delta P} - m \left[2f(x) - \frac{2}{\gamma \Delta P} \int_0^1 \frac{f(\xi) \left(e^{\gamma \Delta P} - 1 \right)}{1 + (e^{\gamma \Delta P} - 1)\xi} d\xi \right].$$

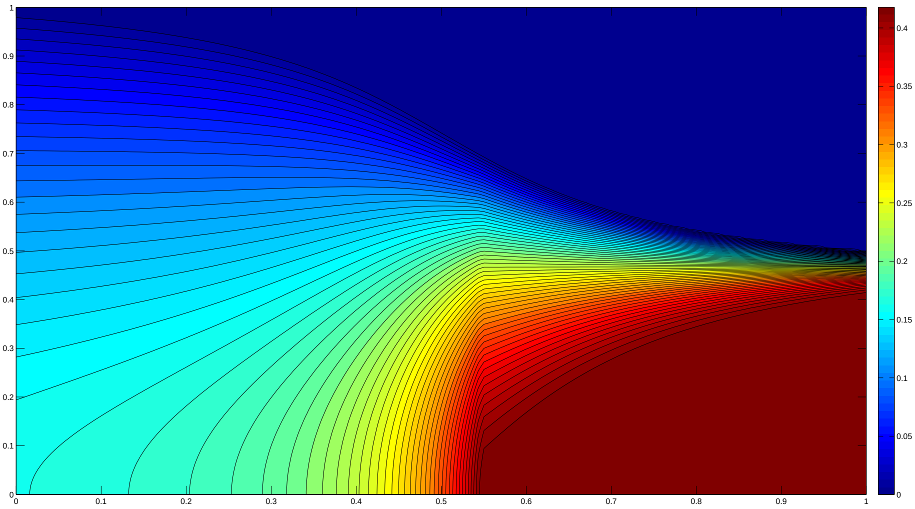

Figure 8: Plot of x-component of the velocity, h given by (46).

6 Conclusion

In this paper we studied the Poiseuille flow of a Bingham fluid in a channel whose walls are not flat. We used a lubrication approximation assuming that the channel length is much larger than its width. The novelty of our approach lies in the motion equation of the inner rigid core which was derived applying the momentum conservation (essentially Newton's second law) to the whole core. The latter is a body of variable mass whose boundary is not material. The idea, that traces its roots back to the paper by Safronchik [17] and Rubistein [11], has never been applied to this kind of problem.

By developing the model at the leading order, we were able to express both components of the velocity and the core surface σ in terms of the pressure, which is governed by a boundary value problem of elliptic type. We actually ended up with a integro-differential equation, whose numerical solution can be obtained by an iterative method. We also provided an approximated explicit solution in case the channel width is almost uniform.

The main results are the following:

- The method that we developed provides the classical Bingham solution when the channel walls are parallel.
- We predict that the rigid core expands where the channel narrows and vice versa (as observed in [9]).
- We proved that our approach does not provide any "paradox", even when the maximum oscillation of the channel walls is $\mathcal{O}(1)$.
- We predicted the possibility of a vanishing core (the so-called "breaking of the plug"). We indeed showed an example in which a "core free" region is located at the channel inlet. This phenomenon, as already remarked in [9] and [5], is peculiar to these kind of flows: it can not occur when the flow runs through two parallel planes.

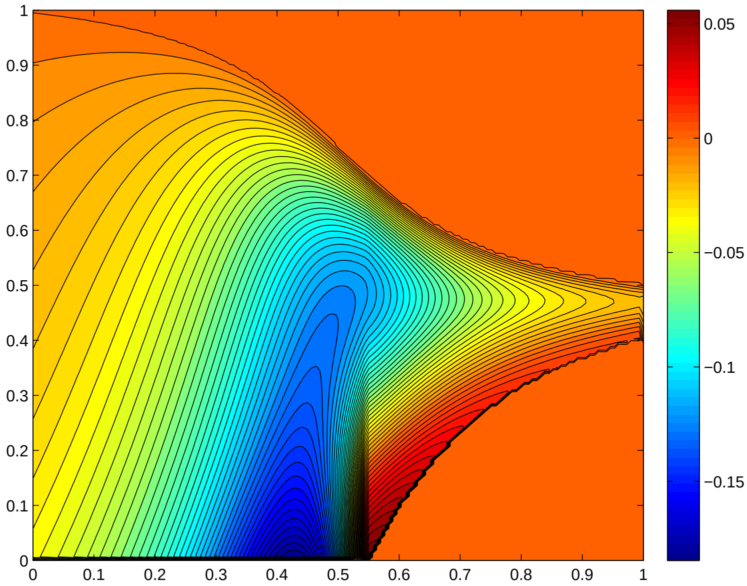

Figure 9: Plot of y-component of the velocity, h given by (46).

- We provided estimates on the pressure difference ensuring that the flow does not stop (i.e. that the core is detached from the channel's walls).

In the last part of the article we generalized the model also to the case of viscosity depending on the pressure. The equation for the pressure is still a integro-differential equation of elliptic type (rather similar to the one with constant viscosity).

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