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A partial order structure on interval orders*

Filippo Disanto† Luca Ferrari‡ Simone Rinaldi§

Abstract

We introduce a partial order structure on the set of interval orders of a given size, and prove that such a structure is in fact a lattice. We also provide a way to compute meet and join inside this lattice. Finally, we show that, if we restrict to series parallel interval order, what we obtain is the classical Tamari poset.

1 Introduction

Interval orders are an interesting class of partial orders, introduced by Fishburn in [F1], which are especially important even in non strictly mathematical contexts, such as experimental psychology, economic theory, philosophical ontology and computer science [F2]. From a purely combinatorial point of view, some remarkable features of interval orders have been recently exploited in [BMCDK], where their connection with some interesting combinatorial structures, such as pattern avoiding permutations and chord diagrams, have been shown. Starting from that paper, a number of articles has been written, trying to go deeper in the combinatorial knowledge of interval orders.

In our work we will explore the possibility of introducing a suitable partial order structure on the set of interval orders (having ground set of fixed size). Our goal is twofold: the resulting poset should be as "natural" as possible, and it should be compatible with possible (already existing) partial orders on subsets of its ground set. We have been able to fulfill this goal, by defining a presumably new partial order structure which is easily defined in terms of a very natural labelling of the elements of the ground

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†Institut für Genetik, Universität zu Köln, Zülpicher Str. 47a, 50674 Köln, Germany

‡Universitá di Firenze, Dipartimento di Matematica e Informatica "U. Dini", viale Morgagni 65, 50134 Firenze, Italy luca.ferrari@unifi.it

§Universitá di Siena, Dipartimento di Scienze Matematiche ed Informatiche, Pian dei Mantellini, 44, 53100, Siena, Italy rinaldi@unisi.it

set, which has the additional features of being a lattice and of coinciding with the Tamari order when restricted to series parallel interval orders.

The article is organized as follows. In section 2 we recall those definitions and facts concerning interval orders and poset theory in general that will be useful throughout the paper. In section 3 we introduce a particular labelling of an interval order that will be crucial for the definition of our partial order on interval orders of the same size. Section 4 is the heart of the paper, and contains the proof that our poset is in fact a lattice. Finally, section 5 provides an argument to show that our partial order, restricted to series parallel interval orders, is isomorphic to the well-known Tamari order.

2 Notations and preliminaries

Let $P = (X, \leq)$ be a finite poset. A *linear extension of* P is a bijection $\lambda : X \to \{1, 2, \ldots, |X|\}$ such that $x < y$ in P implies $\lambda(x) < \lambda(y)$.

Given $Y \subseteq X$, the *up-set of* P *generated by* Y is the set $U_P(Y) = \{x \in X \mid \forall y \in Y, x > y\}$. Analogously, the *down-set of* P *generated by* Y is the set $D_P(Y) = \{x \in X \mid \forall y \in Y, x < y\}$. In particular, we will denote with $\mathcal{D}_P = \{D_P(\{x\}) \mid x \in X\}$ and $\mathcal{U}_P = \{U_P(\{x\}) \mid x \in X\}$ the sets of *principal down-sets* and *principal up-sets* of P, respectively. To simplify notations, we will often write $D(x)$ in place of $D_P(\{x\})$ and, analogously, $U(x)$ in place of $U_P(\{x\})$.

Observe that the above definitions of an up-set and of a down-set slightly differ from the usual ones which can be found in the literature. Indeed, in this work an up-set generated by Y does not contain the elements of Y (and the same convention holds for down-sets). We have preferred to give definitions in this way since this will help us in stating (and then proving) our main results.

Given $x, y \in X$, we say that x and y are *order equivalent* whenever $D(x) = D(y)$ and $U(x) = U(y)$. In this case, we will use the notation $x \sim y$.

We say that a poset P *avoids* a poset S when P has no subposet isomorphic to S. Borrowing notations from the theory of pattern avoiding permutations, we will refer to the class of posets avoiding the poset S using the symbol $AV(S)$; in particular, when we restrict ourselves to posets of cardinality n, we will write $AV_n(S)$.

An important class of posets is that of interval orders [BMCDK, EZ, F1, Kh]. A poset $P = (X, \leq)$ is called an *interval order* when there exists a function J mapping each element $x \in X$ into a closed interval $J(x) = [a_x, b_x] \subseteq \mathbf{R}$ in such a way that, for all $x, y \in X$, $x < y$ in P if and only if $b_x < a_y$ in $\mathbf{R}$. We call J an *(interval) representation* of P. If the interval order P is finite, then we can obviously find a representation of P such that, for every element x, the values a_x and b_x are integers.

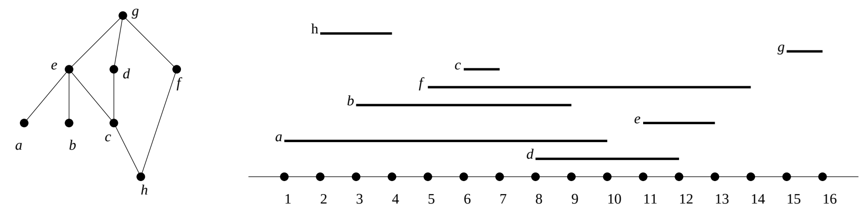

Figure 1: An interval order and one of its representations.

In [F1] Fishburn gives the following characterization for the class of interval orders in terms of avoiding subposets. Recall that the poset $2+2$ is the disjoint union on two chains each having two elements (see Figure 2).

Theorem 2.1 *A poset $P = (X, \leq)$ is an interval order if and only if $P \in AV(2+2)$.* ■

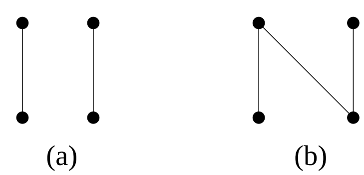

Figure 2: (a) The poset $2+2$; (b) The fence of order four.

The following proposition, stated in [Kh], gives a characterization for the class of interval orders in terms of principal down-sets and principal up-sets.

Proposition 2.2 *The following statements are equivalent:*

i) P is an interval order;

ii) any two distinct sets in $\mathcal{D}_P$ are ordered by inclusion;

iii) any two distinct sets in $\mathcal{U}_P$ are ordered by inclusion. ■

3 The admissible labelling of an interval order

Let $P(X, \leq)$ be a poset. The following proposition gives an immediate characterization of order equivalent elements, whose easy proof is left to the reader.

Proposition 3.1 *Two elements x and y of a poset P are order equivalent if and only if the map from P to itself exchanging x and y is an automorphism of P.* ■

From now on in this section, the poset P will denote an interval order.

A linear extension λ of P is called an *admissible labelling* of P whenever, for all $x, y \in X$, if $\lambda(x) < \lambda(y)$ then either $D(x) \subset D(y)$, or $D(x) = D(y)$ and $U(x) \subset U(y)$, or $x \sim y$.

Such a labelling has been defined in [DPPR], in the context of a recursive construction of interval orders, where it is also shown that each interval order admits at least one admissible labelling.

A trivial property of an admissible labelling of an interval order (which will be useful in the next section) is the following.

Proposition 3.2 *Let λ be an admissible labelling of P. Given $x, y, z \in X$ such that $x < y$ and $\lambda(y) \leq \lambda(z)$, then $x < z$.*

Proof. From the definition of admissible labelling, $\lambda(y) \leq \lambda(z)$ implies that $D(y) \subseteq D(z)$. Since $x \in D(y)$, we have that $x \in D(z)$, that is $x < z$. ■

The rest of this section is devoted showing that an interval order admits a unique admissible labelling (up to automorphisms).

Lemma 3.3 *Suppose that λ_1 and λ_2 are two admissible labellings of P. If $\lambda_1(x) > \lambda_1(y)$ and $\lambda_2(x) < \lambda_2(y)$, then $x \sim y$.*

Proof. This follows immediately from the definition of an admissible labelling. ■

Proposition 3.4 *Suppose that λ_1 and λ_2 are two admissible labellings of P. If $\lambda_1(x) = \lambda_2(y)$, then $x \sim y$.*

Proof. Let $\ell = \lambda_1(x) = \lambda_2(y)$. Concerning the values of the two labels $\lambda_2(x)$ and $\lambda_1(y)$ we have essentially two different cases.

1. If $\lambda_2(x), \lambda_1(y) < \ell$, then we can simply apply the above lemma. The same argument can be used in the case $\lambda_2(x), \lambda_1(y) > \ell$.

2. Suppose, without loss of generality, that $\lambda_1(y) < \ell < \lambda_2(x)$. We then claim that there exists $z \in X$ such that $z \sim x$ and $z \sim y$ (whence the claim will easily follow by transitivity). Indeed, we observe that

$$|\{z \in X \setminus \{x, y\} \mid \lambda_2(z) > \ell\}| < |\{z \in X \setminus \{x, y\} \mid \lambda_1(z) > \ell\}|$$

(since, in the labelling λ_2, the label of x is greater than ℓ). Thus there exists an element $z \in X \setminus \{x, y\}$ such that $\lambda_1(z) > \ell$ and $\lambda_2(z) < \ell$. Since we are assuming that $\ell < \lambda_2(x)$, we have that $\lambda_1(z) > \lambda_1(x)$ and $\lambda_2(z) < \lambda_2(x)$. Therefore we can apply once again Lemma 3.3 to obtain that $z \sim x$. An analogous argument shows that $z \sim y$. ■

Corollary 3.5 *Let λ_1, λ_2 be two admissible labellings of P. Then there exists an automorphism f of P such that, for all $x \in P$, $\lambda_1(x) = \lambda_2(f(x))$.*

Proof. Given $x \in X$, let $f(x)$ be the (unique) element $y \in X$ such that $\lambda_1(x) = \lambda_2(y)$. According to Proposition 3.4, we have $x \sim y$, and so the map f is an automorphism of P (since it is the composition of automorphisms, by Proposition 3.1). ■

The last corollary states that there exists a unique admissible labelling of a given interval order *up to order automorphism*. This uniqueness result will be frequently used in the rest of the paper.

4 The poset $AV_n(2+2)$

In the present section, which is the heart of our work, we endow each set $AV_n(2+2)$ with a partial order structure. We then prove that the resulting poset is in fact a lattice, which provides a generalization of the *Tamari lattice*. This partial order on $AV_n(2+2)$ is believed to be new.

Before proceeding further, we briefly recall that a *lattice* is a poset in which every finite set has least upper bound (called *join*) and greatest lower bound (called *meet*). In particular, for any two elements x and y of a lattice, their join is denoted by $x \vee y$ and their meet is denoted by $x \wedge y$.

In the sequel we will consider *interval orders endowed with their admissible labelling*, and we will identify an element x with its label $\lambda(x)$ (this can be done by Corollary 3.5). Moreover, for an interval order $P = (X, \leq)$, we will write $x \leq y$ to indicate that the element x is less than or equal to y

with respect to the partial order of P, whereas we will write $x \prec y$ to mean that the label of x is less than the label of y. See Figure 4 for an example.

Given two interval orders P_1 and P_2 on the same ground set $X = [n] = \{1, 2, \dots, n\}$, we declare $P_1 \leq_T P_2$ whenever $P_1 \supseteq P_2$, i.e. the (partial order) relation P_2 is a subset of the (partial order) relation P_1.

The following proposition characterizes the order relation $\leq_T$ in terms of both the principal up-sets and the principal down-sets of the elements of $AV_n(2+2)$. The proof is an easy consequence of the notations and results previously recalled, so it is left to the reader.

Proposition 4.1 *Let P_1, P_2 be two interval orders on $X = \{1, 2, \dots, n\}$. The following are equivalent:*

i) for each $x \in X$, $U_{P_1}(x) \supseteq U_{P_2}(x)$;

ii) for each $x \in X$, $D_{P_1}(x) \supseteq D_{P_2}(x)$;

iii) $P_1 \leq_T P_2$. ■

Let $\Gamma = \{(i, j) \in [n] \times [n] \mid i \leq j\}$. Then the set of interval orders on $[n]$ is clearly what is usually called a *family of subsets of* Γ. We recall here a classical definition which can be found, for instance, in [DP]. A family of subsets $\mathcal{L}$ of a set Γ is called a *closure system* on Γ when it is closed under arbitrary intersections and it contains Γ. Analogously, when $\mathcal{L}$ is closed under arbitrary unions and it contains the empty set, it will be called a *dual closure system* on Γ.

The following result (recorded in [DP] as well) gives an important property of closure systems.

Theorem 4.2 *Any closure system $\mathcal{L}$ is a complete lattice, in which*

$$\begin{aligned} \bigwedge_{i \in I} A_i &= \bigcap_{i \in I} A_i, \\ \bigvee_{i \in I} A_i &= \bigcap \{B \in \mathcal{L} \mid B \supseteq \bigcup_{i \in I} A_i\}, \end{aligned} \tag{1}$$

for all $\{A_i\}_{i \in I} \subseteq \mathcal{L}$.

Analogously, any dual closure system $\mathcal{L}$ is a complete lattice, in which

$$\begin{aligned} \bigwedge_{i \in I} A_i &= \bigcup_{i \in I} A_i, \\ \bigvee_{i \in I} A_i &= \bigcup \{B \in \mathcal{L} \mid B \subseteq \bigcap_{i \in I} A_i\}, \end{aligned} \tag{2}$$

for all $\{A_i\}_{i \in I} \subseteq \mathcal{L}$. ■

In view of the above theorem, the following result is trivial, so it is stated without proof.

Lemma 4.3 *Let $\mathcal{L}$ be a family of subsets of a set Γ. Suppose that there exists $A \in \mathcal{L}$ such that $A \subseteq B$ for all $B \in \mathcal{L}$ and $\mathcal{L}$ is closed under arbitrary nonempty unions. Then $\mathcal{L}$ is a complete lattice, in which the meet and join operations are computed as in (2).* ■

The above facts allow us to formulate our main result concerning the order structure of $AV_n(2+2)$.

Theorem 4.4 *$(AV_n(2+2), \leq_T)$ is a (complete) lattice, in which the meet and join operations are expressed as follows:*

$$\begin{aligned} P_1 \wedge P_2 &= P_1 \cup P_2, \\ P_1 \vee P_2 &= \bigcup \{P \in AV_n(2+2) \mid P \subseteq P_1 \cap P_2\}. \end{aligned}$$

Proof. We start by observing that $D = \{(i,i) \mid i \in [n]\}$ is an interval order on $[n]$ (since it is the discrete poset on $[n]$) and that any interval order on $[n]$ clearly contains D. Thus $AV_n(2+2)$ is a family of subsets of Γ having D as the minimum. Therefore, since $AV_n(2+2)$ is finite, in view of Lemma 4.3 it will be enough to prove that, for any $P_1, P_2 \in AV_n(2+2)$, $P = P_1 \cup P_2 \in AV_n(2+2)$. In what follows, we will denote by $\leq_P$ the partial order relation on P, and by $\leq_{P_i}$ the partial order relation on each P_i, for $i = 1, 2$.

The first thing to prove is that P is a poset. In fact, P is trivially reflexive (since it contains D). Moreover, suppose that $x \leq_P y$ and $y \leq_P x$. If the two relations hold in the same poset P_i (that is, if $x \leq_{P_i} y$ and $y \leq_{P_i} x$ for $i = 1$ or $i = 2$), then trivially $x = y$. Otherwise, suppose (without loss of generality) that $x \leq_{P_1} y$ and $y \leq_{P_2} x$. Since the admissible labelling is a linear extension of its interval order, then necessarily $x \preceq y$ and $y \preceq x$, whence immediately $x = y$, and so $\leq_P$ is antisymmetric. Finally, suppose that $x \leq_P y$ and $y \leq_P z$. Also in this case, the only nontrivial case arises when (without loss of generality) $x <_{P_1} y$ and $y <_{P_2} z$. In particular, this implies that $y \preceq z$. Thus, by Proposition 3.2, we can conclude that $x <_{P_1} z$, whence $x <_P z$, that is $\leq_P$ is transitive.

Our next goal is to show that P is an interval order. By Proposition 2.2, we will achieve this by showing that, if $x \preceq y$, then $D_P(x) \subseteq D_P(y)$. Indeed, let $z \in D_P(x)$, i.e. $z <_P x$. Without loss of generality, this means that $z <_{P_1} x$. Together with $x \preceq y$ and Proposition 3.2, this implies that $z <_{P_1} y$, and so $z <_P y$, i.e. $z \in D_P(y)$.

Finally, we observe that the labelling of the elements of P induced by P_1 and P_2 is an admissible labelling. Indeed, it is easy to show (and so is left to the reader) that such a labelling is a linear extension of P, and that, for each x, $D_P(x) = D_{P_1}(x) \cup D_{P_2}(x)$ and $U_P(x) = U_{P_1}(x) \cup U_{P_2}(x)$. ■

5 The Tamari lattice on series parallel interval orders

In this final section we will consider the restriction of the poset $(AV_n(2+2), \leq_T)$ to the set of series parallel interval orders. This means that we will focus on the poset $(AV_n(2+2, N), \leq_T)$, where N denotes the *fence having four elements* (see Figure 2). In particular we will show that, for any positive integer n, $(AV_n(2+2, N), \leq_T)$ is the Tamari lattice of order n.

We point out that, in the literature, there are several extensions of the Tamari lattice, see for instance [R, S, T]. However, to the best of our knowledge, the extension we propose in this paper does not match any of them.

From now on, we will consider planar rooted trees whose nodes are labelled according to the preorder visit (with the root labelled 0) and we will systematically identify a node of a tree with its label (as in the tree represented in Figure 4). Moreover, we will write $x \prec y$ to mean that the label of the node x is less than the label of the node y. Finally, we will depict trees with their root at the bottom; so, words like *left* or *right* will refer to this representation (in particular, the sons of a node will be canonically ordered from left to right). Given a planar tree T and one of its nodes k, let $u_T(k)$ be the set of *descendants* of k in the tree T. Now recall that, according to [Kn], the *Tamari order* is defined on the set of planar trees of the same size by saying that a planar tree T_1 is less than or equal to a planar tree T_2 whenever, for every node k, $|u_{T_1}(k)| \leq |u_{T_2}(k)|$.

In order to define a lattice structure on series parallel interval orders which is isomorphic to the Tamari lattice, we make use of a suitable map sending such partial orders into planar trees, along the lines of what have been done in [DFPR]. Let T be a planar tree and $[n] = \{1, 2, \dots, n\}$ the set of its nodes different from the root. We define a binary relation R on $[n]$ by setting xRy whenever either $x = y$ or the following two facts hold: $y \notin u(x)$ and $x \prec y$. Figure 4 shows an instance of such a map. In the above cited paper, the authors proved a bunch of results concerning this map; we collect them in the next proposition.

Proposition 5.1 *We have the following results:*

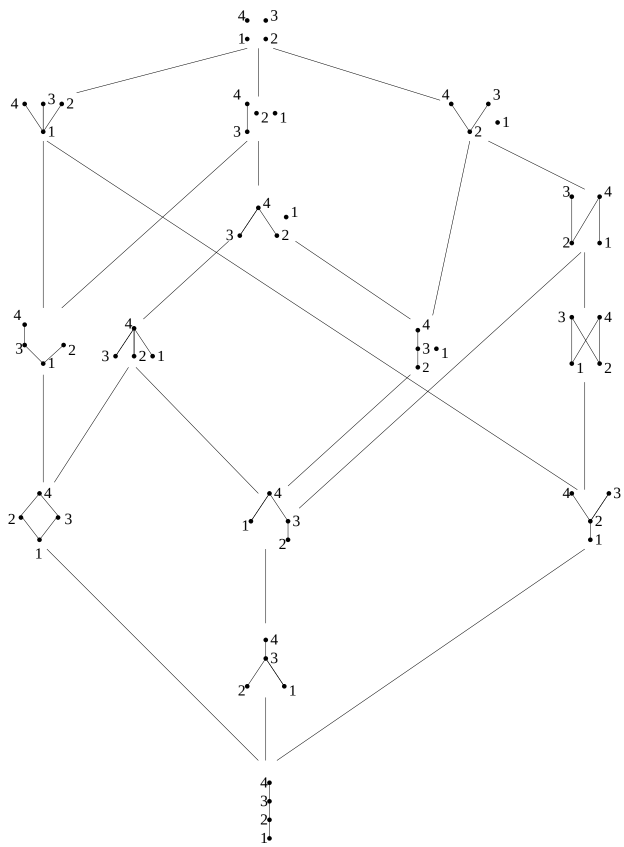

Figure 3: The Hasse diagram of the lattice $(AV_4(2+2), \leq_T)$.

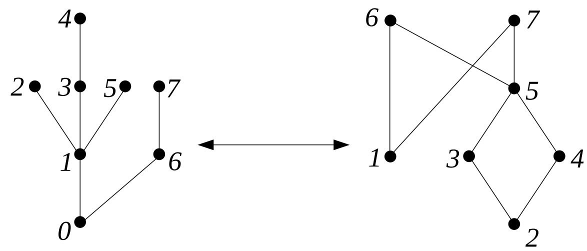

Figure 4: A planar tree with its preorder labelling and the associated series parallel interval order with its admissible labelling.

1. *The structure* $P_T = ([n], R)$ *is a series parallel interval order, and the labelling of its elements is a linear extension of* R.

2. *Every series parallel interval order of size* n *is isomorphic to* P_T*, for some planar tree* T.

3. *Set* $P_1 = P_{T_1}$ *and* $P_2 = P_{T_2}$*, if we define* $P_1 \leq_t P_2$ *when,* $\forall x \in [n]$, $D_{P_1}(x) \supseteq D_{P_2}(x)$*, then* $(AV_n(2+2, N), \leq_t)$ *is the Tamari lattice of order* n. ■

Our goal is now to show that, if we restrict the order relation $\leq_T$ defined in the previous section to the set of series parallel interval orders, we obtain precisely the Tamari poset.

Proposition 5.2 *The labelling of the poset* P_T *determined by the preorder visit on the associated tree* T *coincides with the admissible labelling of* R.

Proof. We start by observing that the labelling of P_T determined by the preorder visit on T is indeed a linear extension of R (due to Proposition 5.1), so the statement of this proposition makes sense.

Consider $x, y \in [n]$, with $x \preceq y$. We first observe that, in P_T, $D(x) \subseteq D(y)$, that is, for all $z \in [n]$, zRx implies zRy (the proof of this assertion is very easy, so we leave it to the reader). Now suppose that $x \prec y$ and $D(x) = D(y)$. This implies that $y \in u(x)$, since otherwise we would have xRy, i.e. $x \in D(y)$, which is not possible (recall that $x \notin D(x)$). If xRz, then necessarily $z \notin u(x)$, and so a fortiori $z \notin u(y)$; moreover, it is immediate to see that $y \prec z$. Therefore we have yRz, thus proving that $U(x) \subseteq U(y)$. ■

The above proposition shows that the notion of admissible labelling, when restricted to series parallel interval orders, coincides with the notion of *preorder linear extension* introduced in [DFPR]. Thus, using 3 of Proposition 5.1, we can finally state the main theorem of this section.

Theorem 5.3 *The Tamari lattice of order* n *is the restriction of the lattice* $(AV_n(2+2), \leq_T)$ *to the set of series parallel interval orders* $AV_n(2+2, N)$.
■

6 Further work

The main aim of the present work has been the definition of a (presumably new) lattice structure on interval orders which, restricted to series parallel interval orders, turns out to be isomorphic to the classical Tamari lattice structure. However, concerning the general (order-theoretic) properties of such a structure, we have only scratched the surface, and we believe that it would be very interesting to go deeper into the knowledge of these lattices. As an example of what could be done, we close our paper with a structural result which gives some insight on the relationship between the poset of interval orders and its subposet of series parallel interval orders.

Proposition 6.1 *For every* $n \in \mathbf{N}$, $AV_n(2+2, N)$ *is a meet subsemilattice (but not in general a join subsemilattice) of* $AV_n(2+2)$.

Proof. The fact that $AV_n(2+2, N)$ is not in general a join subsemilattice of $AV_n(2+2)$ can be easily verified, for instance, by inspecting Figure 3 (and by noticing that the fence of order four can be obtained as the join of two series parallel interval orders).

In order to prove that $AV_n(2+2, N)$ is a meet subsemilattice of $AV_n(2+2)$, we argue by contradiction and suppose that, given $P = P_1 \wedge P_2$, with $P_1, P_2 \in AV_n(2+2, N)$, there exists a subposet of P isomorhpic to N (P cannot contain any subposet isomorphic to $2+2$, of course). To fix notations, suppose that $\{a, b, c, d\}$ is an occurrence of the poset N inside P, with $a \leq_P c, b \leq_P c$ and $b \leq_P d$. It is clear that there cannot exist $i \in \{1, 2\}$ such that the above listed inequalities hold in P_i. Thus we have three essentially distinct cases.

a) $a \leq_{P_1} c, b \leq_{P_2} c$ and $b \leq_{P_1} d$. This case is plainly impossible, otherwise $\{a, b, c, d\}$ would be an occurrence of $2+2$ inside P_1.

b) $a \leq_{P_1} c, b \leq_{P_1} c$ and $b \leq_{P_2} d$. In this case, in P_1 d is incomparable with any of the remaining three elements (otherwise there would be an occurrence of N in P_1). We claim that, in P_1, $D(d) \subseteq D(b)$. Indeed, if we had $x \in D(d)$ and $x \notin D(b)$, then there would exist x such that $x < d$ and $x \not< b$. A simple argument then shows that $\{b, c, x, d\}$ would constitute an occurrence of either N or $2 + 2$ in P_1 (depending on whether $x < c$ or not), which is not possible. From $D(d) \subseteq D(b)$ and $U(b) \nsubseteq U(d)$ we deduce that $d \prec b$, but this leads to a contradiction, since we are assuming that $b \leq_{P_2} d$ (which implies that $b \prec d$).

c) $a \leq_{P_1} c, b \leq_{P_2} c$ and $b \leq_{P_2} d$. In this case, we can assume that both b and d are incomparable with any of the three remaining elements in P_1 (otherwise one of the above cases would occur). We claim that, in P_1, $D(b) \subseteq D(a)$. Indeed, if we had $x \in D(b)$ and $x \notin D(a)$, then there would exist x such that $x < b$ and $x \not< a$. A simple argument then shows that $\{a, b, c, x\}$ would constitute an occurrence of either N or $2 + 2$ in P_1 (depending on whether $x < c$ or not), which is not possible. From $D(b) \subseteq D(a)$ we deduce that $b \prec a$. Moreover, in P_2, a is easily seen to be incomparable with the remaining three elements; from this fact, using an argument similar to the previous ones (and whose details are then left to the reader), we deduce that $D(a) \subseteq D(b)$, whence we get $a \prec b$ (since also $U(b) \nsubseteq U(a)$), which contradicts to what was previously shown.

Thus we have shown that in all cases, P cannot contain any occurrence of the subposet N, which completes our proof. ■

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