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Higher-order Sobolev embeddings and isoperimetric inequalities

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Optimal higher-order Sobolev type embeddings are shown to follow via isoperimetric inequalities. This establishes a higher-order analogue of a well-known link between first-order Sobolev embeddings and isoperimetric inequalities. Sobolev type inequalities of any order, involving arbitrary rearrangement-invariant norms, on open sets in $\mathbb{R}^n$, possibly endowed with a measure density, are reduced to much simpler one-dimensional inequalities for suitable integral operators depending on the isoperimetric function of the relevant sets. As a consequence, the optimal target space in the relevant Sobolev embeddings can be determined both in standard and in non-standard classes of function spaces and underlying measure spaces. In particular, our results are applied to any-order Sobolev embeddings in regular (John) domains of the Euclidean space, in Maz'ya classes of (possibly irregular) Euclidean domains described in terms of their isoperimetric function, and in families of product probability spaces, of which the Gauss space is a classical instance.

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Contents

1. Introduction

Sobolev inequalities and isoperimetric inequalities had traditionally been investigated along independent lines of research, which had led to the cornerstone results by Sobolev [81,82], Gagliardo [43] and Nirenberg [73] on the one hand, and by De Giorgi [34] on the other hand, until their intimate connection was discovered some half a century ago. Such breakthrough goes back to the work of Maz'ya [68,69], who proved that quite general Sobolev inequalities are equivalent to either isoperimetric or isocapacitary inequalities. Independently, Federer and Fleming [41] also exploited De Giorgi's isoperimetric theorem to exhibit the best constant in the special case of the Sobolev inequality for functions whose gradient is integrable with power one in $\mathbb{R}^n$. These advances paved the way to an extensive research, along diverse directions, on the interplay between isoperimetric and Sobolev inequalities, and to a number of remarkable applications, such as the classics by Moser [72], Talenti [87], Aubin [2], Brézis and Lieb [13]. The contributions to this field now constitute the corpus of a vast literature, which includes the papers [1,3,5,9,10,14,15,19,21,23,26,29,30,32,37,39,44,47,48,51,52,54–56,61–63,71,77,86,88,91] and the monographs [16,18,20,45,49,70,80]. Needless to say, this list of references is by no means exhaustive.

The strength of the approach to Sobolev embeddings via isoperimetric inequalities stems from the fact that not only it applies to a broad range of situations, but also typically yields sharp results. The available results, however, essentially deal with first-order Sobolev inequalities, apart from few exceptions on quite specific issues concerning the higher-order case. Indeed, isoperimetric inequalities are usually considered ineffectual in proving optimal higher-order Sobolev embeddings. Customary techniques that are crucial in the derivation of first-order Sobolev inequalities from isoperimetric inequalities, such as symmetrization, or just truncation, cannot be adapted to the proof of higher-order Sobolev inequalities. A major drawback is that these operations do not preserve higher-order (weak) differentiability. A new approach to the sharp Sobolev inequality in $\mathbb{R}^n$, based on mass transportation techniques, has been introduced in [33], and has later been developed in various papers to attack other Sobolev type inequalities, but still in the first-order case. On the other hand, methods which can be employed to handle higher-order Sobolev inequalities, such as representation formulas, Fourier transforms, atomic decomposition, are not flexible enough to produce sharp conclusions in full generality. A paradigmatic instance in this connection is provided by the standard Sobolev embedding in $\mathbb{R}^n$ to which we alluded above, whose original proof via representation formulas [81,82] does not include the borderline case when derivatives are just integrable with power one. This case was restored in [43] and [73] through a completely different technique that rests upon one-dimensional integration combined with a clever use of Hölder's inequality.

One main purpose of the present paper is to show that, this notwithstanding, isoperimetric inequalities do imply optimal higher-order Sobolev embeddings in quite general frameworks. Sobolev embeddings for functions defined on underlying domains in $\mathbb{R}^n$, equipped with fairly general measures, are included in our discussion. Also, Sobolev-type norms built upon any rearrangement-invariant Banach function norm are considered. The use of isoperimetric inequalities is shown to allow for a unified approach to the relevant embeddings, which is based on the reduction to considerably simpler one-dimensional inequalities. Such reduction principle is crucial in a characterization of the best possible target for arbitrary-order Sobolev embeddings, in the class of all rearrangement-invariant Banach function spaces. As a consequence, the optimal target space in arbitrary-order Sobolev embeddings involving various customary and non-standard underlying domains and norms can be exhibited. In fact, establishing optimal higher-order Gaussian Sobolev embeddings, namely Sobolev embeddings in $\mathbb{R}^n$ endowed with the Gauss measure, was our original motivation for the present research. Failure of standard strategies in the solution of this problem led us to develop the general picture which is now the subject of this paper.

A key step in our proofs amounts to the development of a sharp iteration method involving subsequent applications of optimal Sobolev embeddings. We consider this method of independent interest for its possible use in different problems, where regularity properties of functions endowed with higher-order derivatives are in question.

2. An overview

We shall deal with Sobolev inequalities in an open connected set – briefly, a domain – Ω in $\mathbb{R}^n$, $n \geq 1$, equipped with a finite measure ν which is absolutely continuous with respect to the Lebesgue measure, with density ω. Namely,

$$d\nu(x) = \omega(x)\, dx, \tag{2.1}$$

where ω is a Borel function such that $\omega(x) > 0$ a.e. in Ω. Throughout the paper, we assume, for simplicity of notation, that ν is normalized in such a way that $\nu(\Omega) = 1$.

The basic case when ν is the Lebesgue measure will be referred to as Euclidean. Sobolev embeddings of arbitrary order for functions defined in Ω, with unconstrained values on $\partial\Omega$, will be considered. However, the even simpler case of functions vanishing (in the suitable sense) on $\partial\Omega$ together with their derivatives up to the order $m-1$ could be included in our discussion.

The isoperimetric inequality relative to (Ω,ν) tells us that

$$P_\nu(E,\Omega) \geq I_{\Omega,\nu}\big(\nu(E)\big), \tag{2.2}$$

where E is any measurable subset of Ω, and $P_\nu(E,\Omega)$ stands for its perimeter in Ω with respect to ν. Moreover, $I_{\Omega,\nu}$ denotes the largest non-decreasing function in $[0,\frac{1}{2}]$ for which (2.2) holds, called the isoperimetric function (or isoperimetric profile) of (Ω,ν), which was introduced in [68].

In the Euclidean case, (Ω,ν) will be simply denoted by Ω, and $I_{\Omega,\nu}$ by I_Ω. The isoperimetric function $I_{\Omega,\nu}$ is known only in few special instances, e.g. when Ω is a Euclidean ball [70], or agrees with the space $\mathbb{R}^n$ equipped with the Gauss measure [12]. However, the asymptotic behavior of $I_{\Omega,\nu}$ at 0 – the piece of information relevant in our applications – can be evaluated for various classes of domains, such as: Euclidean bounded domains whose boundary is locally a graph of a Lipschitz function [70], or, more generally, has a prescribed modulus of continuity [22,58]; Euclidean John domains, and even s-John domains; the space $\mathbb{R}^n$ equipped with the Gauss measure [12], or with product probability measures which generalize it [3,4]. The literature on isoperimetric inequalities is very rich. Let us limit ourselves to mentioning that, besides those quoted above, recent contributions on isoperimetric problems in (domains in) $\mathbb{R}^n$ endowed with a measure ν include [17,35,42,79].

Given a Banach function space $X(\Omega,\nu)$ of measurable functions on Ω, and a positive integer $m \in \mathbb{N}$, the m-th order Sobolev type space built upon $X(\Omega,\nu)$ is the normed linear space $V^m X(\Omega,\nu)$ of all functions on Ω whose m-th order weak derivatives belong to $X(\Omega,\nu)$, equipped with a natural norm induced by $X(\Omega,\nu)$.

A Sobolev embedding amounts to the boundedness of the identity operator from the Sobolev space $V^m X(\Omega,\nu)$ into another function space $Y(\Omega,\nu)$ and will be denoted by

$$V^m X(\Omega,\nu) \to Y(\Omega,\nu). \tag{2.3}$$

When $m=1$, we refer to (2.3) as a first-order embedding; otherwise, we call it a higher-order embedding.

Necessary and sufficient conditions for the validity of first-order Euclidean Sobolev embeddings with $X(\Omega) = L^1(\Omega)$ and $Y(\Omega) = L^q(\Omega)$ for some $q \geq 1$ can be given through the isoperimetric function I_Ω. Sufficient conditions for first-order Sobolev embeddings when $X(\Omega) = L^p(\Omega)$ for some $p > 1$ and $Y(\Omega) = L^q(\Omega)$, for some $q \geq 1$ can also be provided in terms of I_Ω. These results were established in [68,69], and are exposed in detail in [70, Section 6.4.3].

More recently, first-order Sobolev embeddings of the general form (2.3) (with $m = 1$), where $X(\Omega,\nu)$ and $Y(\Omega,\nu)$ are Banach function spaces whose norm depends only on the measure of level sets of functions, called rearrangement-invariant spaces in the literature, have been shown to follow from one-dimensional inequalities for suitable Hardy type operators which depend on the isoperimetric function $I_{\Omega,\nu}$, and involve the representation function norms $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ of $X(\Omega,\nu)$ and $Y(\Omega,\nu)$, respectively.

Although a reverse implication need not hold in very pathological settings (e.g. in Euclidean domains of Nikodým type [70, Remark 6.5.2]), first-order Sobolev inequalities are known to be equivalent to the associated one-dimensional Hardy inequalities in most situations of interest in applications. This is the case, for instance, when Ω is a regular Euclidean domain – specifically, a John domain in $\mathbb{R}^n$, $n \geq 2$ (see Section 6 for a definition). The class of John domains includes other more classical families of domains, such as Lipschitz domains, and domains with the cone property. The John domains arise in connection with the study of holomorphic dynamical systems and quasiconformal mappings. John domains are known to support a first-order Sobolev inequality with the same exponents as in the standard Sobolev inequality [11,48,54]. In fact, being a John domain is a necessary condition for such a Sobolev inequality to hold in the class of two-dimensional simply connected open sets, and in quite general classes of higher-dimensional domains [14]. The isoperimetric function I_Ω of any John domain is known to satisfy

$$I_\Omega(s) \approx s^{\frac{1}{n'}} \tag{2.4}$$

near 0, where $n' = \frac{n}{n-1}$. Here, and in what follows, the notation $\approx$ means that the two sides are bounded by each other up to multiplicative constants independent of appropriate quantities. For instance, in (2.4) such constants depend only on Ω.

As a consequence of (2.4), one can show that the first-order Sobolev embedding

$$V^1X(\Omega) \to Y(\Omega) \tag{2.5}$$

holds if and only if the Hardy type inequality

$$\left\| \int_t^1 f(s) s^{-1+\frac{1}{n}}\, ds \right\|_{Y(0,1)} \leq C\|f\|_{X(0,1)} \tag{2.6}$$

holds for some constant C, and for every nonnegative $f \in X(0,1)$. Results of this kind, showing that Sobolev embeddings follow from (and are possibly equivalent to) one-dimensional inequalities will be referred to as reduction principles or reduction theorems. The equivalence of (2.5) and (2.6) is a key tool in determining the optimal target $Y(\Omega)$ for $V^1X(\Omega)$ in (2.5) within families of rearrangement-invariant function spaces, such as Lebesgue, Lorentz, and Orlicz spaces, provided that such an optimal target space does exist [23,25,37]. An even more standard version of this reduction result, which holds for functions vanishing on $\partial\Omega$, and is called Pólya–Szegö symmetrization principle, is a crucial step in exhibiting the sharp constant in the classical Sobolev inequalities to which we alluded above [2,13,72,87].

A version of this picture for higher-order Sobolev inequalities is exhibited in the present paper. We show that any m-th order Sobolev embedding involving arbitrary rearrangement-invariant norms can be reduced to a suitable one-dimensional inequality for an integral operator, with a kernel depending on $I_{\Omega,\nu}$ and m.

Just to give an idea of the conclusions which follow from our results, let us mention that, if, for instance, Ω is a Euclidean John domain in $\mathbb{R}^n$, $n \geq 2$, then a full higher-order analogue of the equivalence of (2.5) and (2.6) holds. Namely, the m-th order Sobolev embedding

$$V^m X(\Omega) \to Y(\Omega)$$

holds if and only if the Hardy type inequality

$$\left\| \int_t^1 f(s) s^{-1+\frac{m}{n}}\, ds \right\|_{Y(0,1)} \leq C \|f\|_{X(0,1)} \tag{2.7}$$

holds for some constant C, and for every nonnegative $f \in X(0,1)$ (Theorem 6.1, Section 6).

Our approach to reduction principles for higher-order Sobolev embeddings relies on the iteration of first-order results. Loosely speaking, iteration is understood in the sense that, given a rearrangement-invariant space and $m \in \mathbb{N}$, a first-order optimal Sobolev embedding is applied to show that the $(m-1)$-th order derivatives of functions from the relevant Sobolev space belong to a suitable rearrangement-invariant space. Another first-order optimal Sobolev embedding is then applied to show that the $(m-2)$-th order derivatives belong to another rearrangement-invariant space, and so on. Eventually, m optimal first-order Sobolev embeddings are exploited to deduce that the functions themselves belong to a certain space.

Let us warn that, although this strategy is quite natural in principle, its implementation is not straightforward. Indeed, even in the basic setting when Ω is a Euclidean domain with a smooth boundary, and standard families of norms are considered, iteration of optimal first-order embeddings need not lead to optimal higher-order counterparts.

To see this, recall, for instance, that, if Ω is a regular domain in $\mathbb{R}^2$, then

$$V^2 L^1(\Omega) \to L^\infty(\Omega). \tag{2.8}$$

On the other hand, iterating twice the classical first-order Sobolev embedding only tells us that

$$V^2 L^1(\Omega) \to V^1 L^2(\Omega) \to L^q(\Omega) \tag{2.9}$$

for every $q < \infty$, and neither of the iterated embeddings can be improved in the framework of Lebesgue spaces. This shows that subsequent applications of optimal first-order Sobolev embeddings in the class of Lebesgue spaces do not necessarily yield optimal higher-order counterparts.

One might relate the loss of optimality in the chain of embeddings (2.9) to the lack of an optimal Lebesgue target space for the first-order Sobolev embedding of $V^1L^2(\Omega)$ when $n = 2$. However, non-optimal targets may appear after iteration even in situations where optimal first-order target spaces do exist. Consider, for example, Euclidean Sobolev embeddings involving Orlicz spaces. The optimal target in Sobolev embeddings of any order always exists in this class of spaces, and can be explicitly determined [23,27], see also [50]. In particular, Orlicz spaces naturally arise in the borderline case of the Sobolev embedding theorem. Indeed, if Ω is a regular domain in $\mathbb{R}^n$ and $1 \le m < n$, then

$$V^mL^{\frac{n}{m}}(\Omega) \to \exp L^{\frac{n}{n-m}}(\Omega) \tag{2.10}$$

[78,84,90]; see also [89] for $m = 1$. Here, $\exp L^{\alpha}(\Omega)$, with $\alpha > 0$, denotes the Orlicz space associated with the Young function given by $e^{t^{\alpha}} - 1$ for $t \ge 0$. Observe that the target space in (2.10) is actually optimal in the class of all Orlicz spaces [23,25]. Now, assume, for instance, that $n \ge 3$ and $m = 2$. Then (2.10) reduces to

$$V^2L^{\frac{n}{2}}(\Omega) \to \exp L^{\frac{n}{n-2}}(\Omega).$$

Via the iteration of optimal first-order embeddings, one gets

$$V^2L^{\frac{n}{2}}(\Omega) \to V^1L^n(\Omega) \to \exp L^{\frac{n}{n-1}}(\Omega) \supsetneqq \exp L^{\frac{n}{n-2}}(\Omega).$$

Thus, subsequent applications of optimal Sobolev embeddings even in the class of Orlicz spaces, where optimal target spaces always exist, need not result in optimal higher-order Sobolev embeddings.

The underlying idea behind the method that we shall introduce is that such a loss of optimality of the target space under iteration does not occur, provided that first-order (in fact, any-order) Sobolev embeddings whose targets are optimal among all rearrangement-invariant spaces are iterated. We thus proceed via a two-step argument, which can be outlined as follows. Firstly, given any function norm $\|\cdot\|_{X(0,1)}$ and the isoperimetric function $I_{\Omega,\nu}$ of (Ω,ν), the optimal target among all rearrangement-invariant function norms for the first-order Sobolev space $V^1X(\Omega,\nu)$ is characterized; secondly, first-order Sobolev embeddings with an optimal target are iterated to derive optimal targets in arbitrary-order Sobolev embeddings.

In order to grasp this procedure in a simple situation, observe that, when applied in the proof of embedding (2.8), it amounts to strengthening the chain in (2.9) by

$$V^2L^1(\Omega) \to V^1L^{2,1}(\Omega) \to L^{\infty}(\Omega), \tag{2.11}$$

where $L^{2,1}(\Omega)$ denotes a Lorentz space (strictly contained in $L^2(\Omega)$). We refer to [53,74,76] for standard Sobolev embeddings in Lorentz spaces. Note that both targets in the embeddings in (2.11) are actually optimal among all rearrangement-invariant spaces.

As mentioned above, our reduction principle asserts that the Sobolev embedding (2.3) follows from a suitable one-dimensional inequality for an integral operator depending on $I_{\Omega,\nu}$, m, $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$. Interestingly, in contrast with the first-order case, the relevant integral operator is not just of Hardy type, but involves a genuine kernel. The latter takes back the form of a basic (weighted) Hardy operator only if, loosely speaking, the isoperimetric function $I_{\Omega,\nu}(s)$ does not decay too fast to 0 when s tends to 0. This is the case, for instance, of (2.7). A major consequence of the reduction principle is a characterization of a target space $Y(\Omega,\nu)$ in embedding (2.3), depending on $X(\Omega,\nu)$, m, and $I_{\Omega,\nu}$, which turns out to be optimal among all rearrangement-invariant spaces whenever Sobolev embeddings and associated one-dimensional inequalities in the reduction principle are actually equivalent. This latter property depends on the geometry of (Ω,ν), and is fulfilled in most customary situations, to some of which a substantial part of this paper is devoted.

Besides regular Euclidean domains, namely the John domains which we have already briefly discussed, the implementations of our results that will be presented concern Maz'ya classes of (possibly irregular) Euclidean domains, and product probability spaces, of which the Gauss space and the Boltzmann spaces are distinguished instances.

The Maz'ya classes are defined as families of domains whose isoperimetric function is bounded from below by some fixed power. Sobolev embeddings in all domains from a class of this type take the same form, and a worst, in a sense, domain from the relevant class can be singled out to demonstrate the sharpness of the results.

The product probability spaces in $\mathbb{R}^n$ that are taken into account were analyzed in [3,4], and share common features with the Gauss space, namely $\mathbb{R}^n$ endowed with the probability measure $d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}}\,dx$. In particular, the Boltzmann spaces can be handled via our approach.

For the reader's convenience, we list at the end of the paper the main symbols employed throughout, with a reference to the equation where they are introduced.

3. Spaces of measurable functions

In this section, we briefly recall some basic facts from the theory of rearrangement-invariant spaces. For more details, a standard reference is [7].

Let (Ω,ν) be as in Section 2. Recall that we are assuming $\nu(\Omega) = 1$. The measure of any measurable set $E \subset \Omega$ is thus given by

$$\nu(E) = \int_E \omega(x)\,dx.$$

We set

$$\mathcal{M}(\Omega,\nu) = \big\{u : \Omega \to [-\infty,\infty] : \ u \text{ is } \nu\text{-measurable in } \Omega\big\}, \tag{3.1}$$

$$\mathcal{M}_+(\Omega,\nu) = \big\{u \in \mathcal{M}(\Omega,\nu) : u \ge 0\big\}, \tag{3.2}$$

and

$$\mathcal{M}_0(\Omega,\nu) = \big\{u \in \mathcal{M}(\Omega,\nu) : u \text{ is finite a.e. in } \Omega\big\}. \tag{3.3}$$

The *decreasing rearrangement* $u^* : [0,1] \to [0,\infty]$ of a function $u \in \mathcal{M}(\Omega,\nu)$ is defined as

$$u^*(s) = \inf\big\{t \ge 0 : \nu\big(\big\{x \in \Omega : \big|u(x)\big| > t\big\}\big) \le s\big\} \quad \text{for } s \in [0,1]. \tag{3.4}$$

The operation $u \mapsto u^*$ is monotone in the sense that

$$|u| \le |v| \quad \text{a.e. in } \Omega \quad \text{implies} \quad u^* \le v^* \quad \text{in } [0,1].$$

We also define $u^{**} : (0,1] \to [0,\infty]$ as

$$u^{**}(s) = \frac{1}{s}\int_0^s u^*(r)\,dr \quad \text{for } s \in (0,1]. \tag{3.5}$$

Note that u^{**} is also non-increasing, and $u^* \le u^{**}$ in $(0,1]$. Moreover,

$$\int_0^s (u+v)^*(r)\,dr \le \int_0^s u^*(r)\,dr + \int_0^s v^*(r)\,dr \quad \text{for } s \in [0,1], \tag{3.6}$$

for every $u, v \in \mathcal{M}_+(\Omega,\nu)$.

A basic property of rearrangements is the *Hardy–Littlewood inequality*, which tells us that, if $u, v \in \mathcal{M}(\Omega,\nu)$, then

$$\int_\Omega \big|u(x)v(x)\big|d\nu(x) \le \int_0^1 u^*(s)v^*(s)ds. \tag{3.7}$$

A special case of (3.7) states that for every $u \in \mathcal{M}(\Omega,\nu)$ and every measurable set $E \subset \Omega$,

$$\int_E \big|u(x)\big|d\nu(x) \le \int_0^{\nu(E)} u^*(s)\,ds.$$

We say that a functional

$$\|\cdot\|_{X(0,1)} : \mathcal{M}_+(0,1) \to [0,\infty] \tag{3.8}$$

is a *function norm*, if, for all f, g and $\{f_j\}_{j\in\mathbb{N}}$ in $\mathcal{M}_+(0,1)$, and every $\lambda \geq 0$, the following properties hold:

(P1) $\|f\|_{X(0,1)} = 0$ if and only if $f = 0$ a.e.; $\|\lambda f\|_{X(0,1)} = \lambda\|f\|_{X(0,1)}$; $\|f+g\|_{X(0,1)} \leq \|f\|_{X(0,1)} + \|g\|_{X(0,1)}$;
(P2) $f \leq g$ a.e. implies $\|f\|_{X(0,1)} \leq \|g\|_{X(0,1)}$;
(P3) $f_j \nearrow f$ a.e. implies $\|f_j\|_{X(0,1)} \nearrow \|f\|_{X(0,1)}$;
(P4) $\|1\|_{X(0,1)} < \infty$;
(P5) $\int_0^1 f(x)\,dx \leq C\|f\|_{X(0,1)}$ for some constant C independent of f.

If, in addition,

(P6) $\|f\|_{X(0,1)} = \|g\|_{X(0,1)}$ whenever $f^* = g^*$,

we say that $\|\cdot\|_{X(0,1)}$ is a *rearrangement-invariant function norm*.

With any rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, it is associated another functional on $\mathcal{M}_+(0,1)$, denoted by $\|\cdot\|_{X'(0,1)}$, and defined, for $g \in \mathcal{M}_+(0,1)$, as

$$\|g\|_{X'(0,1)} = \sup_{\substack{f\geq 0 \\ \|f\|_{X(0,1)}\leq 1}} \int_0^1 f(s)g(s)\,ds. \tag{3.9}$$

It turns out that $\|\cdot\|_{X'(0,1)}$ is also a rearrangement-invariant function norm, which is called the *associate function norm* of $\|\cdot\|_{X(0,1)}$. Moreover, for every rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$ and every function $f \in \mathcal{M}_+(0,1)$, we have

$$\|f\|_{X(0,1)} = \sup_{\substack{g\geq 0 \\ \|g\|_{X'(0,1)}\leq 1}} \int_0^1 f(s)g(s)\,ds. \tag{3.10}$$

We also introduce yet another functional on $\mathcal{M}_+(0,1)$, the *down associate function norm* of $\|\cdot\|_{X(0,1)}$. It is denoted by $\|\cdot\|_{X'_d(0,1)}$, and defined, for $g \in \mathcal{M}_+(0,1)$, as

$$\|g\|_{X'_d(0,1)} = \sup_{\|f\|_{X(0,1)}\leq 1} \int_0^1 f^*(t)g(t)\,dt. \tag{3.11}$$

Clearly, one has that $\|g\|_{X'_d(0,1)} \leq \|g\|_{X'(0,1)}$ for every $g \in \mathcal{M}_+(0,1)$, and $\|g\|_{X'_d(0,1)} = \|g\|_{X'(0,1)}$ if g is non-increasing.

Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, the space $X(\Omega,\nu)$ is defined as the collection of all functions $u \in \mathcal{M}(\Omega,\nu)$ such that the expression

$$\|u\|_{X(\Omega,\nu)} = \|u^*\|_{X(0,1)} \tag{3.12}$$

is finite. Such expression defines a norm on $X(\Omega,\nu)$, and the latter is a Banach space endowed with this norm, called a rearrangement-invariant space. Moreover, $X(\Omega,\nu) \subset \mathcal{M}_0(\Omega,\nu)$ for any rearrangement-invariant space $X(\Omega,\nu)$. The space $X(0,1)$ is called the *representation space* of $X(\Omega,\nu)$.

We also denote

$$X_{\mathrm{loc}}(\Omega,\nu) = \big\{u \in \mathcal{M}(\Omega,\nu) : u\chi_G \in X(\Omega,\nu) \text{ for every compact set } G \subset \Omega\big\}. \tag{3.13}$$

Here, χ_G denotes the characteristic function of G.

The rearrangement-invariant space $X'(\Omega,\nu)$ built upon the function norm $\|\cdot\|_{X'(0,1)}$ is called the *associate space* of $X(\Omega,\nu)$. It turns out that $X''(\Omega,\nu) = X(\Omega,\nu)$. Furthermore, the *Hölder inequality*

$$\int_\Omega |u(x)v(x)|\, d\nu(x) \le \|u\|_{X(\Omega,\nu)} \|v\|_{X'(\Omega,\nu)}$$

holds for every $u \in X(\Omega,\nu)$ and $v \in X'(\Omega,\nu)$.

For any rearrangement-invariant spaces $X(\Omega,\nu)$ and $Y(\Omega,\nu)$, we have that

$$X(\Omega,\nu) \to Y(\Omega,\nu) \quad \text{if and only if} \quad Y'(\Omega,\nu) \to X'(\Omega,\nu), \tag{3.14}$$

with the same embedding norms [7, Chapter 1, Proposition 2.10].

Given any $\lambda > 0$, the *dilation operator* E_λ, defined at $f \in \mathcal{M}(0,1)$ by

$$(E_\lambda f)(s) = \begin{cases} f(\lambda^{-1}s) & \text{if } 0 < s \le \lambda \\ 0 & \text{if } \lambda < s < 1, \end{cases} \tag{3.15}$$

is bounded on any rearrangement-invariant space $X(0,1)$, with norm not exceeding $\max\{1, \frac{1}{\lambda}\}$.

Hardy's lemma tells us that if $f_1, f_2 \in \mathcal{M}_+(0,1)$ satisfy

$$\int_0^s f_1(r)dr \le \int_0^s f_2(r)dr \quad \text{for every } s \in (0,1),$$

then

$$\int_0^1 f_1(r)h(r)dr \le \int_0^1 f_2(r)h(r)dr$$

for every non-increasing function $h : (0,1) \to [0,\infty]$. A consequence of this result is the *Hardy–Littlewood–Pólya principle* which asserts that if the functions $u, v \in \mathcal{M}(\Omega,\nu)$ satisfy

$$\int_0^s u^*(r)\,dr \le \int_0^s v^*(r)\,dr \quad \text{for } s \in (0,1),$$

then

$$\|u\|_{X(\Omega,\nu)} \le \|v\|_{X(\Omega,\nu)}$$

for every rearrangement-invariant space $X(\Omega,\nu)$.

Let $X(\Omega,\nu)$ and $Y(\Omega,\nu)$ be rearrangement-invariant spaces. By [7, Chapter 1, Theorem 1.8],

$$X(\Omega,\nu) \subset Y(\Omega,\nu) \quad \text{if and only if} \quad X(\Omega,\nu) \to Y(\Omega,\nu).$$

For every rearrangement-invariant space $X(\Omega,\nu)$, one has that

$$L^\infty(\Omega,\nu) \to X(\Omega,\nu) \to L^1(\Omega,\nu). \tag{3.16}$$

An embedding of the form

$$X_{\mathrm{loc}}(\Omega,\nu) \to Y_{\mathrm{loc}}(\Omega,\mu),$$

where μ is a measure enjoying the same properties as ν, means that, for every compact set $G \subset \Omega$, there exists a constant C such that

$$\|u\chi_G\|_{Y(\Omega,\mu)} \le C\|u\chi_G\|_{X(\Omega,\nu)},$$

for every $u \in X_{\mathrm{loc}}(\Omega,\nu)$.

Throughout, we use the convention that $\frac{1}{\infty} = 0$, and $0 \cdot \infty = 0$.

A basic example of a function norm is the standard *Lebesgue norm* $\|\cdot\|_{L^p(0,1)}$, for $p \in [1,\infty]$, upon which the Lebesgue spaces $L^p(\Omega,\nu)$ are built.

The *Lorentz spaces* yield an extension of the Lebesgue spaces. Assume that $1 \le p, q \le \infty$. We define the functionals $\|\cdot\|_{L^{p,q}(0,1)}$ and $\|\cdot\|_{L^{(p,q)}(0,1)}$ as

$$\begin{aligned} \|f\|_{L^{p,q}(0,1)} &= \left\| s^{\frac{1}{p}-\frac{1}{q}} f^*(s) \right\|_{L^q(0,1)} \quad \text{and} \\ \|f\|_{L^{(p,q)}(0,1)} &= \left\| s^{\frac{1}{p}-\frac{1}{q}} f^{**}(s) \right\|_{L^q(0,1)}, \end{aligned} \tag{3.17}$$

respectively, for $f \in \mathcal{M}_+(0,1)$. One can show that

$$L^{p,q}(\Omega,\nu)=L^{(p,q)}(\Omega,\nu)\quad \text{if } 1<p\le\infty, \tag{3.18}$$

with equivalent norms. If one of the conditions

$$\begin{cases} 1<p<\infty, \quad 1\le q\le\infty, \\ p=q=1, \\ p=q=\infty, \end{cases} \tag{3.19}$$

is satisfied, then $\|\cdot\|_{L^{p,q}(0,1)}$ is equivalent to a rearrangement-invariant function norm. The corresponding rearrangement-invariant space $L^{p,q}(\Omega,\nu)$ is called a *Lorentz space*.

Let us recall that $L^{p,p}(\Omega,\nu)=L^p(\Omega,\nu)$ for every $p\in[1,\infty]$ and that $1\le q\le r\le\infty$ implies $L^{p,q}(\Omega,\nu)\to L^{p,r}(\Omega,\nu)$ with equality if and only if $q=r$.

Assume now that $1\le p,q\le\infty$, and a third parameter $\alpha\in\mathbb{R}$ is called into play. We define the functionals $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ and $\|\cdot\|_{L^{(p,q;\alpha)}(0,1)}$ as

$$\begin{cases} \|f\|_{L^{p,q;\alpha}(0,1)}=\left\|s^{\frac{1}{p}-\frac{1}{q}}\log^{\alpha}\left(\frac{2}{s}\right)f^*(s)\right\|_{L^q(0,1)}, \\ \|f\|_{L^{(p,q;\alpha)}(0,1)}=\left\|s^{\frac{1}{p}-\frac{1}{q}}\log^{\alpha}\left(\frac{2}{s}\right)f^{**}(s)\right\|_{L^q(0,1)}, \end{cases} \tag{3.20}$$

respectively, for $f\in\mathcal{M}_+(0,1)$. If one of the following conditions

$$\begin{cases} 1<p<\infty, \quad 1\le q\le\infty, \quad \alpha\in\mathbb{R}; \\ p=1, \quad q=1, \quad \alpha\ge 0; \\ p=\infty, \quad q=\infty, \quad \alpha\le 0; \\ p=\infty, \quad 1\le q<\infty, \quad \alpha+\frac{1}{q}<0, \end{cases} \tag{3.21}$$

is satisfied, then $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ is equivalent to a rearrangement-invariant function norm, called a *Lorentz–Zygmund function norm*. The corresponding rearrangement-invariant space $L^{p,q;\alpha}(\Omega,\nu)$ is a *Lorentz–Zygmund space*. At a few occasions, we shall need also the so-called *generalized Lorentz–Zygmund space* $L^{p,q;\alpha,\beta}(\Omega,\nu)$, where $p,q\in[1,\infty]$ and $\alpha,\beta\in\mathbb{R}$. It is the space built upon the functional given by

$$\|f\|_{L^{p,q;\alpha,\beta}(0,1)}=\left\|s^{\frac{1}{p}-\frac{1}{q}}\log^{\alpha}\left(\frac{2}{s}\right)\log^{\beta}\left(1+\log\left(\frac{2}{s}\right)\right)f^*(s)\right\|_{L^q(0,1)} \tag{3.22}$$

for $f\in\mathcal{M}_+(0,1)$. The values of p,q,α and β, for which $\|\cdot\|_{L^{p,q;\alpha,\beta}(0,1)}$ is actually equivalent to a rearrangement-invariant function norm, are characterized in [40]. For more details on (generalized) Lorentz–Zygmund spaces, see e.g. [6,40,75]. Assume that one of the conditions in (3.21) is satisfied. Then the associate space $(L^{p,q;\alpha})'(\Omega,\nu)$ of the Lorentz–Zygmund space $L^{p,q;\alpha}(\Omega,\nu)$ satisfies (up to equivalent norms)

$$
\left(L^{p,q;\alpha}\right)'(\Omega,\nu) = \begin{cases} L^{p',q';-\alpha}(\Omega,\nu) & \text{if } 1<p<\infty,\ 1\le q\le\infty,\ \alpha\in\mathbb{R}; \\ L^{\infty,\infty;-\alpha}(\Omega,\nu) & \text{if } p=1,\ q=1,\ \alpha\ge 0; \\ L^{1,1;-\alpha}(\Omega,\nu) & \text{if } p=\infty,\ q=\infty,\ \alpha\le 0; \\ L^{(1,q';-\alpha-1)}(\Omega,\nu) & \text{if } p=\infty,\ 1\le q<\infty,\ \alpha+\frac{1}{q}<0 \end{cases} \tag{3.23}
$$

[75, Theorems 6.11 and 6.12]. Moreover,

$$
L^{(p,q;\alpha)}(\Omega,\nu) = \begin{cases} L^{p,q;\alpha}(\Omega,\nu) & \text{if } 1<p\le\infty; \\ L^{1,1;\alpha+1}(\Omega,\nu) & \text{if } p=q=1,\ \alpha>-1, \end{cases} \tag{3.24}
$$

and

$$
L^p(\Omega,\nu) \to L^{(1,q)}(\Omega,\nu) \quad \text{for every } 1<p\le\infty,\ 1\le q\le\infty
$$

[75, Theorem 3.16 (i), (ii)].

A generalization of the Lebesgue spaces in a different direction is provided by the Orlicz spaces. Let $A : [0,\infty) \to [0,\infty]$ be a Young function, namely a convex (non-trivial), left-continuous function vanishing at 0. Any such function takes the form

$$
A(t) = \int_0^t a(\tau)d\tau \quad \text{for } t\ge 0, \tag{3.25}
$$

for some non-decreasing, left-continuous function $a : [0,\infty) \to [0,\infty]$ which is neither identically equal to 0, nor to ∞. The Orlicz space $L^A(\Omega,\nu)$ is the rearrangement-invariant space associated with the *Luxemburg function norm* defined as

$$
\|f\|_{L^A(0,1)} = \inf\left\{\lambda>0 : \int_0^1 A\left(\frac{f(s)}{\lambda}\right)ds \le 1\right\} \tag{3.26}
$$

for $f \in \mathcal{M}_+(0,1)$. In particular, $L^A(\Omega,\nu) = L^p(\Omega,\nu)$ if $A(t) = t^p$ for some $p \in [1,\infty)$, and $L^A(\Omega,\nu) = L^\infty(\Omega,\nu)$ if $A(t) = \infty\chi_{(1,\infty)}(t)$.

A Young function A is said to dominate another Young function B near infinity if positive constants c and t_0 exist such that

$$
B(t) \le A(ct) \quad \text{for } t\ge t_0.
$$

The functions A and B are called equivalent near infinity if they dominate each other near infinity. One has that

$$
L^A(\Omega,\nu) \to L^B(\Omega,\nu) \quad \text{if and only if} \quad A \text{ dominates } B \text{ near infinity.} \tag{3.27}
$$

We denote by $L^p\log^\alpha L(\Omega,\nu)$ the Orlicz space associated with a Young function equivalent to $t^p(\log t)^\alpha$ near infinity, where either $p>1$ and $\alpha\in\mathbb{R}$, or $p=1$ and $\alpha\ge 0$. The notation $\exp L^\beta(\Omega,\nu)$ will be used for the Orlicz space built upon a Young function equivalent to e^{t^β} near infinity, where $\beta > 0$. Also, $\exp\exp L^\beta(\Omega,\nu)$ stands for the Orlicz space associated with a Young function equivalent to $e^{e^{t^\beta}}$ near infinity.

The classes of Orlicz and (generalized) Lorentz–Zygmund spaces overlap, up to equivalent norms. For instance, if $1 \le p < \infty$ and $\alpha \in \mathbb{R}$, then

$$L^{p,p;\alpha}(\Omega,\nu) = L^p \log^{p\alpha} L(\Omega,\nu).$$

Moreover, if $\beta > 0$, then

$$L^{\infty,\infty;-\beta}(\Omega,\nu) = \exp L^{\frac{1}{\beta}}(\Omega,\nu)$$

and [40, Lemma 2.2]

$$L^{\infty,\infty;0,-\beta}(\Omega,\nu) = \exp\exp L^{\frac{1}{\beta}}(\Omega,\nu).$$

A common extension of the Orlicz and Lorentz spaces is provided by a family of Orlicz–Lorentz spaces defined as follows. Given $p \in (1,\infty)$, $q \in [1,\infty)$ and a Young function D such that

$$\int^\infty \frac{D(t)}{t^{1+p}}\,dt < \infty,$$

we denote by $L(p,q,D)(\Omega,\nu)$ the Orlicz–Lorentz space associated with the rearrangement-invariant function norm defined, for $f \in \mathcal{M}_+(0,1)$, as

$$\|f\|_{L(p,q,D)(0,1)} = \left\|s^{-\frac{1}{p}} f^*\left(s^{\frac{1}{q}}\right)\right\|_{L^D(0,1)}. \tag{3.28}$$

The fact that $\|\cdot\|_{L(p,q,D)(0,1)}$ is actually a function norm follows via easy modifications in the proof of [25, Proposition 2.1]. Observe that the class of the spaces $L(p,q,D)(\Omega,\nu)$ actually includes (up to equivalent norms) Orlicz spaces and various instances of Lorentz and Lorentz–Zygmund spaces.

4. Spaces of Sobolev type and the isoperimetric function

Let (Ω,ν) be as in Section 2. Define the perimeter of a measurable set E in (Ω,ν)

$$P_\nu(E,\Omega) = \int_{\Omega\cap\partial^M E} \omega(x)\,d\mathcal{H}^{n-1}(x), \tag{4.1}$$

where $\partial^M E$ denotes the essential boundary of E, in the sense of geometric measure theory [70,92]. The isoperimetric function $I_{\Omega,\nu} : [0,1] \to [0,\infty]$ of (Ω,ν) is then given by

$$I_{\Omega,\nu}(s) = \inf\left\{P_\nu(E,\Omega) : E \subset \Omega, s \le \nu(E) \le \frac{1}{2}\right\} \quad \text{if } s \in \left[0, \frac{1}{2}\right], \tag{4.2}$$

and $I_{\Omega,\nu}(s) = I_{\Omega,\nu}(1-s)$ if $s \in (\frac{1}{2}, 1]$. The isoperimetric inequality (2.2) in (Ω,ν) is a straightforward consequence of this definition and of the fact that $P_\nu(E,\Omega) = P_\nu(\Omega \setminus E, \Omega)$ for every set $E \subset \Omega$.

Let us observe that, actually, $I_{\Omega,\nu}(s) < \infty$ for $s \in [0, \frac{1}{2})$. To verify this fact, fix any $x_0 \in \Omega$, and let $R > 0$ be such that $\nu(\Omega \cap B_R(x_0)) = \frac{1}{2}$. Here, $B_R(x_0)$ denotes the ball, centered at x_0, with radius R. By the polar-coordinates formula for integrals,

$$\frac{1}{2} = \int\limits_{\Omega\cap B_R(x_0)} \omega(x)\,dx = \int\limits_0^R \int\limits_{\Omega\cap\partial B_\rho(x_0)} \omega(x)\,d\mathcal{H}^{n-1}(x)\,d\rho = \int\limits_0^R P_\nu\big(\Omega \cap B_\rho(x_0), \Omega\big)\,d\rho, \tag{4.3}$$

whence $P_\nu(\Omega \cap B_\rho(x_0), \Omega) < \infty$ for a.e. $\rho \in (0, R)$. The finiteness of $I_{\Omega,\nu}$ in $[0, \frac{1}{2})$ now follows by its very definition.

The next result shows that the best possible behavior of an isoperimetric function at 0 is that given by (2.4), in the sense that $I_{\Omega,\nu}(s)$ cannot decay more slowly than $s^{\frac{1}{n'}}$ as $s \to 0$, whatever (Ω,ν) is.

Proposition 4.1. *There exists a positive constant $C = C(\Omega,\nu)$ such that*

$$I_{\Omega,\nu}(s) \le C s^{\frac{1}{n'}} \quad \textit{near } 0. \tag{4.4}$$

Proof. Let x_0 be any Lebesgue point of ω, namely a point such that

$$\lim_{r\to 0^+} \frac{1}{|B_r(x_0)|} \int\limits_{B_r(x_0)} \omega(x)\,dx \tag{4.5}$$

exists and is finite. Here, $|E|$ denotes the Lebesgue measure of a set $E \subset \mathbb{R}^n$. By (4.5), there exists $r_0 > 0$ and $C > 0$ such that

$$\int\limits_{B_r(x_0)} \omega(x)\,dx \le C r^n \quad \text{if } 0 < r < r_0. \tag{4.6}$$

By an analogous chain as in (4.3),

$$\int\limits_{B_r(x_0)} \omega(x)\,dx = \int\limits_0^r P_\nu\big(B_\rho(x_0), \Omega\big)\,d\rho \ge \frac{r}{2} \inf\left\{P_\nu\big(B_\rho(x_0), \Omega\big) : \frac{r}{2} \le \rho \le r\right\} \tag{4.7}$$

if $0 < r < r_0$. From (4.6) and (4.7) we deduce that there exists a constant C such that

$$
\begin{aligned}
C\big|B_r(x_0)\big|^{\frac{1}{n'}} &\geq \inf\left\{P_\nu\big(B_\rho(x_0),\Omega\big) : \frac{r}{2} \leq \rho \leq r\right\} \\
&= \inf\left\{P_\nu\big(B_\rho(x_0),\Omega\big) : \frac{1}{2^n}\big|B_r(x_0)\big| \leq \big|B_\rho(x_0)\big| \leq \big|B_r(x_0)\big|\right\} \quad \text{if } 0 < r < r_0.
\end{aligned}
$$

Thus, there exists a constant C such that

$$
Cs^{\frac{1}{n'}} \geq \inf\left\{P_\nu(E,\Omega) : s \leq |E| \leq \frac{1}{2}\right\},
$$

provided that s is sufficiently small, and hence (4.4) follows. □

Let $m \in \mathbb{N}$ and let $X(\Omega,\nu)$ be a rearrangement-invariant space. We define the m-th order Sobolev space $V^mX(\Omega,\nu)$ as

$$
\begin{aligned}
V^mX(\Omega,\nu) = \big\{u : u &\text{ is } m\text{-times weakly differentiable in } \Omega, \\
&\text{and } \big|\nabla^m u\big| \in X(\Omega,\nu)\big\}.
\end{aligned} \tag{4.8}
$$

Here, $\nabla^m u$ denotes the vector of all m-th order weak derivatives of u. We shall also set $\nabla^0 u = u$. Let us notice that in the definition of $V^mX(\Omega,\nu)$ it is only required that the derivatives of the highest order m of u belong to $X(\Omega,\nu)$. This assumption does not entail, in general, that also u and its derivatives up to the order $m-1$ belong to $X(\Omega,\nu)$, or even to $L^1(\Omega,\nu)$. Thus, it may happen that $V^mX(\Omega,\nu) \not\subseteq V^kX(\Omega,\nu)$ for $m > k$. Such inclusion indeed fails, for instance, when $(\Omega,\nu) = (\mathbb{R}^n,\gamma_n)$, the Gauss space, and $\|\cdot\|_{X(0,1)} = \|\cdot\|_{L^\infty(0,1)}$ (or $\|\cdot\|_{X(0,1)} = \|\cdot\|_{\exp L^\beta(0,1)}$ for some $\beta > 0$). Examples of Euclidean domains for which $V^mX(\Omega) \not\subseteq L^1(\Omega)$ are those of Nikodým type, see, e.g., [70, Sections 5.2 and 5.4].

However, if $I_{\Omega,\nu}(s)$ does not decay at 0 faster than linearly, namely if there exists a positive constant C such that

$$
I_{\Omega,\nu}(s) \geq Cs \quad \text{for } s \in \left[0,\frac{1}{2}\right], \tag{4.9}
$$

then any function $u \in V^mX(\Omega,\nu)$ does at least belong to $L^1(\Omega,\nu)$, together with all its derivatives up to the order $m-1$. This is a consequence of the next result. Such result in the case when ν is the Lebesgue measure is established in [70, Theorem 5.2.3]; the general case rests upon an analogous argument. We provide a proof for completeness.

Proposition 4.2 *(Condition for $V^1L^1(\Omega,\nu) \subset L^1(\Omega,\nu)$). Assume that (4.9) holds. Then $V^1L^1(\Omega,\nu) \subset L^1(\Omega,\nu)$, and*

$$
\frac{C}{2}\left\|u - \int_\Omega u\,d\nu\right\|_{L^1(\Omega,\nu)} \leq \|\nabla u\|_{L^1(\Omega,\nu)} \tag{4.10}
$$

for every $u \in V^1L^1(\Omega,\nu)$, where C is the same constant as in (4.9).

Proof. Let $\mathrm{med}(u)$ denote the median of a function $u \in \mathcal{M}(\Omega, \nu)$, given by

$$\mathrm{med}(u) = \sup\left\{t \in \mathbb{R} : \nu\big(\{x \in \Omega :\, u(x) > t\}\big) > \frac{1}{2}\right\}. \tag{4.11}$$

We begin by showing that

$$C\big\|u - \mathrm{med}(u)\big\|_{L^1(\Omega,\nu)} \le \|\nabla u\|_{L^1(\Omega,\nu)} \tag{4.12}$$

for every $u \in V^1L^1(\Omega, \nu)$. On replacing, if necessary, u by $u - \mathrm{med}(u)$, we may assume, without loss of generality, that $\mathrm{med}(u) = 0$. Let us set $u_+ = \frac{1}{2}(|u| + u)$ and $u_- = \frac{1}{2}(|u| - u)$, the positive and the negative parts of u, respectively. Thus,

$$\nu\big(\{u_\pm > t\}\big) \le \frac{1}{2} \quad \text{for } t > 0. \tag{4.13}$$

By (2.2) and (4.9),

$$P_\nu\big(\{u_\pm > t\}, \Omega\big) \ge I_{\Omega,\nu}\big(\nu\big(\{u_\pm > t\}\big)\big) \ge C\nu\big(\{u_\pm > t\}\big).$$

Therefore, owing to (4.13), and to the coarea formula, we have that

$$\begin{aligned} C\|u_\pm\|_{L^1(\Omega,\nu)} &= C\int_0^\infty \nu\big(\{u_\pm > t\}\big)\, dt \le \int_0^\infty P_\nu\big(\{u_\pm > t\}, \Omega\big)\, dt \\ &= \int_0^\infty \int_{\partial^M\{u_\pm > t\}\cap\Omega} \omega(x)\, d\mathcal{H}^{n-1}(x)\, dt = \int_\Omega |\nabla u_\pm|\, d\nu. \end{aligned}$$

Hence, (4.12) follows. In particular, (4.12) tells us that $V^1L^1(\Omega, \nu) \subset L^1(\Omega, \nu)$. Inequality (4.10) is a consequence of (4.12) and of the fact that

$$\left\|u - \int_\Omega u\, d\nu\right\|_{L^1(\Omega,\nu)} \le 2\big\|u - \mathrm{med}(u)\big\|_{L^1(\Omega,\nu)}$$

for every $u \in L^1(\Omega, \nu)$. □

Corollary 4.3. *Assume that (4.9) holds. Let $m \ge 1$. Let $X(\Omega, \nu)$ be any rearrangement-invariant space. Then $V^mX(\Omega, \nu) \subset V^kL^1(\Omega, \nu)$ for every $k = 0, \dots, m-1$.*

Proof. By property (P5) of rearrangement-invariant spaces, $V^mX(\Omega, \nu) \to V^mL^1(\Omega, \nu)$. Thus, the conclusion follows from an iterated use of Proposition 4.2. □

Under (4.9), an assumption which will always be kept in force hereafter, $V^m X(\Omega,\nu)$ is easily seen to be a normed linear space, equipped with the norm

$$\|u\|_{V^m X(\Omega,\nu)} = \sum_{k=0}^{m-1} \|\nabla^k u\|_{L^1(\Omega,\nu)} + \|\nabla^m u\|_{X(\Omega,\nu)}. \tag{4.14}$$

Standard arguments show that $V^m X(\Omega,\nu)$ is complete, and hence a Banach space, under the additional assumption that

$$L^1_{\mathrm{loc}}(\Omega,\nu) \to L^1_{\mathrm{loc}}(\Omega).$$

We also define the subspace $V^m_\perp X(\Omega,\nu)$ of $V^m X(\Omega,\nu)$ as

$$V^m_\perp X(\Omega,\nu) = \left\{ u \in V^m X(\Omega,\nu) : \int_\Omega \nabla^k u \, d\nu = 0, \text{ for } k = 0,\dots,m-1 \right\}. \tag{4.15}$$

The Sobolev embedding (2.3) turns out to be equivalent to a Poincaré type inequality for functions in $V^m_\perp X(\Omega,\nu)$.

Proposition 4.4 *(Equivalence of Sobolev and Poincaré inequalities). Assume that (Ω,ν) fulfills (4.9) and that $m \geq 1$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then*

$$V^m X(\Omega,\nu) \to Y(\Omega,\nu) \tag{4.16}$$

if and only if there exists a constant C such that

$$\|u\|_{Y(\Omega,\nu)} \leq C \|\nabla^m u\|_{X(\Omega,\nu)} \tag{4.17}$$

for every $u \in V^m_\perp X(\Omega,\nu)$.

Proof. Assume that (4.16) holds. Thus, there exists a constant C such that

$$\|u\|_{Y(\Omega,\nu)} \leq C \left(\sum_{k=0}^{m-1} \|\nabla^k u\|_{L^1(\Omega,\nu)} + \|\nabla^m u\|_{X(\Omega,\nu)} \right) \tag{4.18}$$

for every $u \in V^m X(\Omega,\nu)$. Iterating inequality (4.10) implies that there exist constants $C_1,\dots,C_m$ such that

$$\|u\|_{L^1(\Omega,\nu)} \leq C_1 \|\nabla u\|_{L^1(\Omega,\nu)} \leq C_2 \|\nabla^2 u\|_{L^1(\Omega,\nu)} \leq \dots \leq C_m \|\nabla^m u\|_{L^1(\Omega,\nu)} \tag{4.19}$$

for every $u \in V^m_\perp X(\Omega,\nu)$. By property (P5) of rearrangement-invariant function norms, there exists a constant C, independent of u, such that $\|\nabla^m u\|_{L^1(\Omega,\nu)} \leq C\|\nabla^m u\|_{X(\Omega,\nu)}$. Thus, (4.17) follows from (4.18) and (4.19).

Suppose next that (4.17) holds. Given $k \in \mathbb{N}$, denote by $\mathcal{P}^k$ the space of polynomials whose degree does not exceed k. Observe that $\mathcal{P}^k \subset L^1(\Omega, \nu)$ for every $k \in \mathbb{N}$. Indeed, $\nabla^h P = 0$ for every $P \in \mathcal{P}^k$, provided that $h > k$, and hence $\mathcal{P}^k \subset V^h X(\Omega, \nu)$ for any rearrangement-invariant space $X(\Omega, \nu)$. The inclusion $\mathcal{P}^k \subset L^1(\Omega, \nu)$ thus follows via Corollary 4.3. Next, it is not difficult to verify that, for each $u \in V^m X(\Omega, \nu)$, there exists a (unique) polynomial $P_u \in \mathcal{P}^{m-1}$ such that $u - P_u \in V^m_\perp X(\Omega, \nu)$. Moreover, the coefficients of P_u are linear combinations of the components of $\int_\Omega \nabla^k u \, d\nu$, for $k = 0, \dots, m-1$, with coefficients depending on n, m and (Ω, ν). Now, we claim that

$$\mathcal{P}^m \subset Y(\Omega, \nu). \tag{4.20}$$

This inclusion is trivial in the case when Ω is bounded, owing to axioms (P2) and (P4) of the definition of rearrangement-invariant function norms, since any polynomial is bounded in Ω. To verify (4.20) in the general case, consider, for each $i = 1, \dots, n$, the polynomial $Q(x) = x_i^m \in \mathcal{P}^m$. Let $P_Q \in \mathcal{P}^{m-1}$ be the polynomial associated with Q as above, such that $Q - P_Q \in V^m_\perp X(\Omega, \nu)$. Note that the polynomial P_Q also depends only on x_i. From (4.17) applied with $u = Q - P_Q$ we deduce that $Q - P_Q \in Y(\Omega, \nu)$. This inclusion and the inequality $|Q - P_Q| \geq C|x_i|^m$, which holds, for a suitable positive constant C, if $|x_i|$ is sufficiently large, tell us, via axiom (P2) of the definition of rearrangement-invariant function norms, that $|x_i|^m \in Y(\Omega, \nu)$ as well. Thus, $|x|^m \in Y(\Omega, \nu)$, and by axiom (P2) again, any polynomial of degree not exceeding m also belongs to $Y(\Omega, \nu)$. Hence, (4.20) follows. Thus, given any $u \in V^m X(\Omega, \nu)$, we have that

$$\begin{aligned}
\|u\|_{Y(\Omega,\nu)} &\leq \|u - P_u\|_{Y(\Omega,\nu)} + \|P_u\|_{Y(\Omega,\nu)} \\
&\leq C\big\|\nabla^m u\big\|_{X(\Omega,\nu)} + \sum_{k=0}^{m-1} C \int_\Omega \big|\nabla^k u\big| \, d\nu \sum_{\alpha_1 + \dots + \alpha_n = k} \big\||x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n}\big\|_{Y(\Omega,\nu)} \\
&\leq C\big\|\nabla^m u\big\|_{X(\Omega,\nu)} + C' \sum_{k=0}^{m-1} \big\|\nabla^k u\big\|_{L^1(\Omega,\nu)},
\end{aligned}$$

for some constants C and C' independent of u. Hence, embedding (4.16) follows. □

Let us incidentally mention that more customary Sobolev type spaces $W^m X(\Omega, \nu)$ can be defined as

$$\begin{aligned}
W^m X(\Omega, \nu) = \big\{u : u \text{ is } m\text{-times weakly differentiable in } \Omega, \\
\big|\nabla^k u\big| \in X(\Omega, \nu) \text{ for } k = 0, \dots, m\big\},
\end{aligned} \tag{4.21}$$

and equipped with the norm

$$\|u\|_{W^m X(\Omega,\nu)} = \sum_{k=0}^{m} \big\|\nabla^k u\big\|_{X(\Omega,\nu)}. \tag{4.22}$$

The space $W^m X(\Omega,\nu)$ is a normed linear space, and it is a Banach space if

$$X_{\mathrm{loc}}(\Omega,\nu) \to L^1_{\mathrm{loc}}(\Omega).$$

By the second embedding in (3.16),

$$W^m X(\Omega,\nu) \to V^m X(\Omega,\nu) \tag{4.23}$$

for every (Ω,ν) fulfilling (4.9), but, in general, $W^m X(\Omega,\nu) \subsetneqq V^m X(\Omega,\nu)$. For instance, if $(\Omega,\nu) = (\mathbb{R}^n,\gamma_n)$, the Gauss space, and $\|\cdot\|_{X(0,1)} = \|\cdot\|_{L^\infty(0,1)}$ (or $\|\cdot\|_{X(0,1)} = \|\cdot\|_{\exp L^\beta(0,1)}$ for some $\beta > 0$), then $V^m X(\Omega,\nu) \neq W^m X(\Omega,\nu)$. However, the spaces $W^m X(\Omega,\nu)$ and $V^m X(\Omega,\nu)$ agree if condition (4.9) is slightly strengthened to

$$\int_0 \frac{ds}{I_{\Omega,\nu}(s)} < \infty. \tag{4.24}$$

Note that (4.24) indeed implies (4.9), since $\frac{1}{I_{\Omega,\nu}}$ is a non-increasing function.

Proposition 4.5 *(Condition for $W^m X(\Omega,\nu) = V^m X(\Omega,\nu)$). Let (Ω,ν) be as above, and let $m \in \mathbb{N}$. Assume that (4.24) holds. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then*

$$W^m X(\Omega,\nu) = V^m X(\Omega,\nu), \tag{4.25}$$

up to equivalent norms.

A proof of this proposition relies upon one of our main results, and can be found at the end of Section 9.

5. Main results

The present section contains the main results of this paper, which link embeddings and Poincaré inequalities for Sobolev-type spaces of arbitrary order to isoperimetric inequalities. The relevant results depend only on a lower bound for the isoperimetric function $I_{\Omega,\nu}$ of (Ω,ν) in terms of some other non-decreasing function $I : [0,1] \to [0,\infty)$; precisely, on the existence of a positive constant c such that

$$I_{\Omega,\nu}(s) \ge cI(cs) \quad \text{for } s \in \left[0, \frac{1}{2}\right]. \tag{5.1}$$

As mentioned in Proposition 4.2 and the preceding remarks, it is reasonable to suppose that the function $I_{\Omega,\nu}$ satisfies the estimate (4.9). In the light of this fact, in what follows we shall assume that

$$\inf_{t\in(0,1)} \frac{I(t)}{t} > 0. \tag{5.2}$$

Theorem 5.1 *(Reduction principle). Assume that (Ω, ν) fulfills (5.1) for some non-decreasing function I satisfying (5.2). Let $m \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. If there exists a constant C_1 such that*

$$\left\| \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds \right\|_{Y(0,1)} \le C_1 \|f\|_{X(0,1)} \tag{5.3}$$

for every nonnegative $f \in X(0,1)$, then

$$V^m X(\Omega,\nu) \to Y(\Omega,\nu), \tag{5.4}$$

and there exists a constant C_2 such that

$$\|u\|_{Y(\Omega,\nu)} \le C_2 \left\|\nabla^m u\right\|_{X(\Omega,\nu)} \tag{5.5}$$

for every $u \in V_\perp^m X(\Omega,\nu)$.

Remark 5.2. It turns out that inequality (5.3) holds for every nonnegative $f \in X(0,1)$ if and only if it just holds for every nonnegative and non-increasing $f \in X(0,1)$. This fact will be proved in Corollary 9.8, Section 9, and can be of use in concrete applications of Theorem 5.1. Indeed, the available criteria for the validity of one-dimensional inequalities for integral operators take, in general, different forms according to whether trial functions are arbitrary, or just monotone.

As already stressed in Sections 1 and 2, the first-order case ($m = 1$) of Theorem 5.1 is already well known; the novelty here amounts to the higher-order case when $m > 1$. To be more precise, when $m = 1$, a version of Theorem 5.1 in the standard Euclidean case, for functions vanishing on $\partial\Omega$, is by now classical, and has been exploited in the proof of Sobolev inequalities with sharp constants, including [2,13,72,87]. An argument showing that (5.3) with $m = 1$ implies (5.4) and (5.5), for functions with arbitrary boundary values, for Orlicz norms, on regular Euclidean domains, or, more generally, on domains in Maz'ya classes, is presented [23, Proof of Theorem 2 and Remark 2]. A proof for arbitrary rearrangement-invariant norms, in Gauss space, is given in [32]. The same proof translates verbatim to general measure spaces (Ω, ν) as in Theorem 5.1 – see e.g. [66].

A major feature of Theorem 5.1 is the difference occurring in (5.3) between the first-order case ($m = 1$) and the higher-order case ($m > 1$). Indeed, the integral operator appearing in (5.3) when $m = 1$ is just a weighted Hardy-type operator, namely a primitive of f times a weight, whereas, in the higher-order case, a genuine kernel, with a more complicated structure, comes into play. In fact, this seems to be the first known instance where such a kernel operator is needed in a reduction result for Sobolev-type embeddings. Of course, this makes the proof of inequalities of the form (5.3) more challenging, although several contributions on one-dimensional inequalities for kernel operators are fortunately available in the literature (see e.g. the survey papers [57,67,83], and the monographs [36,38]).

Remark 5.3. As we shall see, the Sobolev embedding (5.4) (or the Poincaré inequality (5.5)) and inequality (5.3), with a function I equivalent to the isoperimetric function $I_{\Omega,\nu}$ on some neighborhood of zero, are actually equivalent in customary families of measure spaces (Ω, ν), and hence, Theorem 5.4 below will enable us to determine the optimal rearrangement-invariant target spaces in Sobolev embeddings for these measure spaces. Incidentally, let us mention that when $m = 1$, this is the case whenever the geometry of (Ω, ν) allows the construction of a family of trial functions u in (5.4) or (5.5) characterized by the following properties: the level sets of u are isoperimetric (or almost isoperimetric) in (Ω, ν); $|\nabla u|$ is constant (or almost constant) on the boundary of the level sets of u. If $m > 1$, then the latter requirement has to be complemented by requiring that the derivatives of u up to the order m restricted to the boundary of the level sets satisfy certain conditions depending on I. The relevant conditions have, however, a technical nature, and it is not worth to state them explicitly. In fact, heuristically speaking, properties (5.3), (5.5) and (5.4) turn out to be equivalent for every $m \geq 1$ on the same measure spaces (Ω, ν) as they are equivalent for $m = 1$. Such equivalence certainly holds in any customary, non-pathological situation, including the three frameworks to which our results will be applied, namely John domains, Euclidean domains from Maz'ya classes, and product probability spaces in $\mathbb{R}^n$ extending the Gauss space.

Now we are in a position to characterize the space which, in the situation discussed in Remark 5.3, is the optimal rearrangement-invariant target space in the Sobolev embedding (5.4). Such an optimal space is the one built upon the rearrangement-invariant function norm $\|\cdot\|_{X_{m,I}(0,1)}$, whose associate norm is defined as

$$\|f\|_{X'_{m,I}(0,1)} = \left\| \frac{1}{I(t)} \int_0^t \left(\int_s^t \frac{dr}{I(r)} \right)^{m-1} f^*(s)ds \right\|_{X'(0,1)} \tag{5.6}$$

for $f \in \mathcal{M}_+(0,1)$.

Theorem 5.4 *(Optimal target). Assume that (Ω, ν), m, I and $\|\cdot\|_{X(0,1)}$ are as in Theorem 5.1. Then the functional $\|\cdot\|_{X'_{m,I}(0,1)}$, given by (5.6), is a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X_{m,I}(0,1)}$ satisfies*

$$V^m X(\Omega,\nu) \to X_{m,I}(\Omega,\nu), \tag{5.7}$$

and there exists a constant C such that

$$\|u\|_{X_{m,I}(\Omega,\nu)} \leq C\big\|\nabla^m u\big\|_{X(\Omega,\nu)} \tag{5.8}$$

for every $u \in V_\perp^m X(\Omega,\nu)$.

Moreover, if (Ω,ν) is such that (5.4), or equivalently (5.5), implies (5.3), and hence (5.3), (5.4) and (5.5) are equivalent, then the function norm $\|\cdot\|_{X_{m,I}(0,1)}$ is optimal in (5.7) and (5.8) among all rearrangement-invariant norms.

An important special case of Theorems 5.1 and 5.4 is enucleated in the following corollary.

Corollary 5.5 *(Sobolev embeddings into L^∞). Assume that (Ω,ν), m, I and $\|\cdot\|_{X(0,1)}$ are as in Theorem 5.1. If*

$$\left\|\frac{1}{I(s)}\left(\int_0^s \frac{dr}{I(r)}\right)^{m-1}\right\|_{X'(0,1)} < \infty, \tag{5.9}$$

then

$$V^m X(\Omega,\nu) \to L^\infty(\Omega,\nu), \tag{5.10}$$

and there exists a constant C such that

$$\|u\|_{L^\infty(\Omega,\nu)} \leq C\big\|\nabla^m u\big\|_{X(\Omega,\nu)} \tag{5.11}$$

for every $u \in V_\perp^m X(\Omega,\nu)$.

Moreover, if (Ω,ν) is such that (5.4), or equivalently (5.5), implies (5.3), and hence (5.3), (5.4) and (5.5) are equivalent, then (5.9) is necessary for (5.10) or (5.11) to hold.

Remark 5.6. If (Ω,ν) is such that (5.4), or equivalently (5.5), implies (5.3), and hence (5.3), (5.4) and (5.5) are equivalent, then (5.10) cannot hold, whatever $\|\cdot\|_{X(0,1)}$ is, if I decays so fast at 0 that

$$\int_0 \frac{dr}{I(r)} = \infty.$$

Our last main result concerns the preservation of optimality in targets among all rearrangement-invariant spaces under iteration of Sobolev embeddings of arbitrary order.

Theorem 5.7 *(Iteration principle). Assume that* (Ω, ν), I *and* $\|\cdot\|_{X(0,1)}$ *are as in Theorem 5.1. Let* $k, h \in \mathbb{N}$. *Then*

$$(X_{k,I})_{h,I}(\Omega, \nu) = X_{k+h,I}(\Omega, \nu),$$

up to equivalent norms.

We now focus on the case when

$$\int_0^s \frac{dr}{I(r)} \approx \frac{s}{I(s)} \quad \text{for } s \in (0,1). \tag{5.12}$$

If the function I satisfies (5.12), then the results of Theorems 5.1, 5.4 and 5.7 can be somewhat simplified. This is the content of the next three corollaries. Let us preliminarily observe that, since the right-hand side of (5.12) does not exceed its left-hand side for any non-decreasing function I, only the estimate in the reverse direction is relevant in (5.12).

Corollary 5.8 *(Reduction principle under (5.12)). Let* (Ω, ν), m, I, $\|\cdot\|_{X(0,1)}$ *and* $\|\cdot\|_{Y(0,1)}$ *be as in Theorem 5.1. Assume, in addition, that I fulfills (5.12). If there exists a constant* C_1 *such that*

$$\left\| \int_t^1 f(s) \frac{s^{m-1}}{I(s)^m}\, ds \right\|_{Y(0,1)} \le C_1 \|f\|_{X(0,1)} \tag{5.13}$$

for every nonnegative $f \in X(0,1)$, *then*

$$V^m X(\Omega, \nu) \to Y(\Omega, \nu), \tag{5.14}$$

and there exists a constant C_2 *such that*

$$\|u\|_{Y(\Omega,\nu)} \le C_2 \big\|\nabla^m u\big\|_{X(\Omega,\nu)} \tag{5.15}$$

for every $u \in V_\perp^m X(\Omega, \nu)$.

Let us notice that a remark parallel to Remark 5.2 applies on the equivalence of the validity of (5.13) for any f, or for any non-increasing f (see Proposition 8.6, Section 8).

The next corollary tells us that, under the extra condition (5.12), the optimal rearrangement-invariant target space takes a simplified form. Namely, it can be equivalently defined via the rearrangement-invariant function norm $\|\cdot\|_{X^\sharp_{m,I}(0,1)}$ obeying

$$\|f\|_{(X^\sharp_{m,I})'(0,1)} = \left\| \frac{t^{m-1}}{I(t)^m} \int\limits_0^t f^*(s)\,ds \right\|_{X'(0,1)} \tag{5.16}$$

for every $f \in \mathcal{M}_+(0,1)$.

Corollary 5.9 *(Optimal target under (5.12)). Assume that (Ω,ν), m, I and $\|\cdot\|_{X(0,1)}$ are as in Corollary 5.8. Then the functional $\|\cdot\|_{(X^\sharp_{m,I})'(0,1)}$, given by (5.16), is a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X^\sharp_{m,I}(0,1)}$ satisfies*

$$V^m X(\Omega,\nu) \to X^\sharp_{m,I}(\Omega,\nu), \tag{5.17}$$

and there exists a constant C such that

$$\|u\|_{X^\sharp_{m,I}(\Omega,\nu)} \le C \big\|\nabla^m u\big\|_{X(\Omega,\nu)} \tag{5.18}$$

for every $u \in V^m_\perp X(\Omega,\nu)$.

Moreover, if (Ω,ν) is such that the validity of (5.14), or equivalently (5.15), implies (5.13), and hence (5.13), (5.14) and (5.15) are equivalent, then the function norm $\|\cdot\|_{X^\sharp_{m,I}(0,1)}$ is optimal in (5.17) and (5.18) among all rearrangement-invariant norms.

We conclude this section with a stability result for the iterated embeddings under the additional condition (5.12).

Corollary 5.10 *(Iteration principle under (5.12)). Assume that (Ω,ν), I and $\|\cdot\|_{X(0,1)}$ are as in Corollary 5.8. Let $k,h \in \mathbb{N}$. Then*

$$\big(X^\sharp_{k,I}\big)^\sharp_{h,I}(\Omega,\nu) = X^\sharp_{k+h,I}(\Omega,\nu),$$

up to equivalent norms.

6. Euclidean–Sobolev embeddings

The main results of this section are reduction theorems and their consequences for Euclidean Sobolev embeddings, of arbitrary order m, on John domains, and on domains from Maz'ya classes.

We begin with the reduction theorem for John domains. Recall that a bounded open set Ω in $\mathbb{R}^n$ is called a *John domain* if there exist a constant $c \in (0,1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\varpi : [0,l] \to \Omega$, parameterized by arclength, such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\operatorname{dist}\big(\varpi(r), \partial\Omega\big) \ge cr \quad \text{for } r \in [0,l].$$

Theorem 6.1 *(Reduction principle for John domains). Let $n \in \mathbb{N}$, $n \geq 2$, and let $m \in \mathbb{N}$. Assume that Ω is a John domain in $\mathbb{R}^n$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following assertions are equivalent.*

(i) *The Hardy type inequality*

$$\left\| \int_t^1 f(s) s^{-1+\frac{m}{n}} \, ds \right\|_{Y(0,1)} \leq C_1 \|f\|_{X(0,1)} \tag{6.1}$$

holds for some constant C_1, and for every nonnegative $f \in X(0,1)$.

(ii) *The Sobolev embedding*

$$V^m X(\Omega) \to Y(\Omega) \tag{6.2}$$

holds.

(iii) *The Poincaré inequality*

$$\|u\|_{Y(\Omega)} \leq C_2 \left\| \nabla^m u \right\|_{X(\Omega)} \tag{6.3}$$

holds for some constant C_2 and every $u \in V_\perp^m X(\Omega)$.

Forerunners of Theorem 6.1 are known. The first order case ($m = 1$) on Lipschitz domains was obtained in [37]. In the case when $m = 2$, and functions vanishing on $\partial\Omega$ are considered, the equivalence of (6.1) and (6.3) was proved in [26], as a consequence of a non-standard rearrangement inequality for second-order derivatives (see also [24] for a related one-dimensional second-order rearrangement inequality). The equivalence of (6.1) and (6.2), when $m \leq n-1$ and Ω is a Lipschitz domain, was established in [53] by a method relying upon interpolation techniques. Such a method does not carry over to the more general setting of Theorem 6.1, since it requires that Ω be an extension domain.

Let us also warn that results reducing higher-order Sobolev embeddings to one-dimensional inequalities can be obtained via more standard methods, such as, for instance, representation formulas of convolution type combined with O'Neil rearrangement estimates for convolutions, or plain iteration of certain first-order pointwise rearrangement estimates [64]. However, these approaches lead to optimal Sobolev embeddings only under additional technical assumptions on the involved rearrangement-invariant function norms $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$.

Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$ and $m \in \mathbb{N}$, we define $\|\cdot\|_{X_{m,\mathrm{John}}(0,1)}$ as the rearrangement-invariant function norm, whose associate function norm is given by

$$\|f\|_{X'_{m,\mathrm{John}}(0,1)} = \left\| s^{-1+\frac{m}{n}} \int_0^s f^*(r) dr \right\|_{X'(0,1)} \tag{6.4}$$

for $f \in \mathcal{M}_+(0,1)$. The function norm $\|\cdot\|_{X_{m,\mathrm{John}}(0,1)}$ is optimal, as a target, for Sobolev embeddings of $V^m X(\Omega)$.

Theorem 6.2 *(Optimal target for John domains). Let n, m, Ω and $\|\cdot\|_{X(0,1)}$ be as in Theorem 6.1. Then the functional $\|\cdot\|_{X'_{m,\mathrm{John}}(0,1)}$, given by (6.4), is a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X_{m,\mathrm{John}}(0,1)}$ satisfies*

$$V^m X(\Omega) \to X_{m,\mathrm{John}}(\Omega), \tag{6.5}$$

and

$$\|u\|_{X_{m,\mathrm{John}}(\Omega)} \le C \big\|\nabla^m u\big\|_{X(\Omega)} \tag{6.6}$$

for some constant C and every $u \in V_\perp^m X(\Omega)$.

Moreover, the function norm $\|\cdot\|_{X_{m,\mathrm{John}}(0,1)}$ is optimal in (6.5) and (6.6) among all rearrangement-invariant norms.

The iteration principle for optimal target norms in Sobolev embeddings on John domains reads as follows.

Theorem 6.3 *(Iteration principle for John domains). Let $n \in \mathbb{N}$, Ω and $\|\cdot\|_{X(0,1)}$ be as in Theorem 6.1. Let $k, h \in \mathbb{N}$. Then*

$$(X_{k,\mathrm{John}})_{h,\mathrm{John}}(\Omega) = X_{k+h,\mathrm{John}}(\Omega),$$

up to equivalent norms.

Let us now focus on Maz'ya classes of domains. Given $\alpha \in [\frac{1}{n'}, 1]$, we denote by $\mathcal{J}_\alpha$ the *Maz'ya class* of all Euclidean domains Ω satisfying (5.1), with $I(s) = s^\alpha$ for $s \in [0,1]$, namely

$$\mathcal{J}_\alpha = \left\{ \Omega : I_\Omega(s) \ge C s^\alpha \text{ for some constant } C > 0 \text{ and for } s \in \left[0, \frac{1}{2}\right] \right\}. \tag{6.7}$$

Thanks to (2.4), any John domain belongs to the class $\mathcal{J}_{\frac{1}{n'}}$.

The reduction theorem in the class $\mathcal{J}_\alpha$ takes the following form.

Theorem 6.4 *(Reduction principle for Maz'ya classes). Let $n \in \mathbb{N}$, $n \ge 2$, $m \in \mathbb{N}$ and $\alpha \in [\frac{1}{n'}, 1]$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Assume that either $\alpha \in [\frac{1}{n'}, 1)$ and there exists a constant C_1 such that*

$$\left\| \int_t^1 f(s) s^{-1+m(1-\alpha)}\, ds \right\|_{Y(0,1)} \le C_1 \|f\|_{X(0,1)} \tag{6.8}$$

for every nonnegative $f \in X(0,1)$, or $\alpha = 1$ and there exists a constant C_1 such that

$$\left\| \int_t^1 f(s) \frac{1}{s} \left(\log \frac{s}{t} \right)^{m-1} ds \right\|_{Y(0,1)} \le C_1 \|f\|_{X(0,1)} \tag{6.9}$$

for every nonnegative $f \in X(0,1)$. Then the Sobolev embedding

$$V^m X(\Omega) \to Y(\Omega) \tag{6.10}$$

holds for every $\Omega \in \mathcal{J}_\alpha$ and, equivalently, the Poincaré inequality

$$\|u\|_{Y(\Omega)} \le C_2 \left\| \nabla^m u \right\|_{X(\Omega)} \tag{6.11}$$

holds for every $\Omega \in \mathcal{J}_\alpha$, for some constant C_2, depending on Ω, m, X and Y, and every $u \in V_\perp^m X(\Omega)$.

Conversely, if the Sobolev embedding (6.10), or, equivalently, the Poincaré inequality (6.11), holds for every $\Omega \in \mathcal{J}_\alpha$, then either inequality (6.8), or (6.9) holds, according to whether $\alpha \in [\frac{1}{n'}, 1)$ or $\alpha = 1$.

A major consequence of Theorem 6.4 is the identification of the optimal rearrangement-invariant target space $Y(\Omega)$ associated with a given domain $X(\Omega)$ in embedding (6.10), as Ω is allowed to range among all domains in the class $\mathcal{J}_\alpha$. This is the content of the next result. The rearrangement-invariant function norm yielding such an optimal space will be denoted by $\| \cdot \|_{X_{m,\alpha}(0,1)}$. Given a rearrangement-invariant function norm $\| \cdot \|_{X(0,1)}$, $m \in \mathbb{N}$, and $\alpha \in [\frac{1}{n'}, 1]$, it is characterized through its associate function norm defined by

$$\|f\|_{X'_{m,\alpha}(0,1)} = \begin{cases} \|s^{-1+m(1-\alpha)} \int_0^s f^*(r) dr\|_{X'(0,1)} & \text{if } \alpha \in [\frac{1}{n'}, 1), \\ \|\frac{1}{s} \int_0^s (\log \frac{s}{r})^{m-1} f^*(r) dr\|_{X'(0,1)} & \text{if } \alpha = 1, \end{cases} \tag{6.12}$$

for $f \in \mathcal{M}_+(0,1)$.

Theorem 6.5 *(Optimal target for Maz'ya classes). Let $n \in \mathbb{N}$, $n \ge 2$, $m \in \mathbb{N}$, α and $\| \cdot \|_{X(0,1)}$ be as in Theorem 6.4. Then the functional $\| \cdot \|_{X'_{m,\alpha}(0,1)}$, given by (6.12), is a rearrangement-invariant function norm, whose associate norm $\| \cdot \|_{X_{m,\alpha}(0,1)}$ satisfies*

$$V^m X(\Omega) \to X_{m,\alpha}(\Omega) \tag{6.13}$$

for every $\Omega \in \mathcal{J}_\alpha$, and

$$\|u\|_{X_{m,\alpha}(\Omega)} \le C \left\| \nabla^m u \right\|_{X(\Omega)} \tag{6.14}$$

for every $\Omega \in \mathcal{J}_\alpha$, for some constant C, depending on Ω, m, X and Y, and every $u \in V_\perp^m X(\Omega)$.

Moreover, the function norm $\|\cdot\|_{X_{m,\alpha}(0,1)}$ is optimal in (6.13) and (6.14) among all rearrangement-invariant norms, as Ω ranges in $\mathcal{J}_\alpha$.

Theorem 6.5 is a straightforward consequence of Theorem 6.4, and either Corollary 5.9 or Theorem 5.4, according to whether $\alpha \in [\frac{1}{n'}, 1)$ or $\alpha = 1$.

The stability of the process of finding optimal rearrangement-invariant targets in Euclidean Sobolev embeddings on Maz'ya domains under iteration is the object of the last main result of the present section. This is the key ingredient which bridges the first-order case of Theorems 6.4 and 6.5 to their higher-order versions.

Theorem 6.6 *(Iteration principle for Maz'ya classes). Let $n \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1]$ and $\|\cdot\|_{X(0,1)}$ be as in Theorem 6.4. Let $k, h \in \mathbb{N}$. Assume that $\Omega \in \mathcal{J}_\alpha$. Then,*

$$(X_{k,\alpha})_{h,\alpha}(\Omega) = X_{k+h,\alpha}(\Omega),$$

up to equivalent norms.

Theorem 6.6 follows from a specialization of Corollary 5.10 ($\alpha \in [\frac{1}{n'}, 1)$), or Theorem 5.7 ($\alpha = 1$).

Remark 6.7. Note that there is one important difference between the reduction and the optimal-target theorems concerning John domains on the one hand, and their counterparts for general Maz'ya domains on the other hand. Namely, the equivalence in Theorem 6.1 and the optimality result in Theorem 6.2 are valid *for each single John domain*, whereas the necessity of condition (6.8) or (6.9) for (6.10) (and (6.11)) in Theorem 6.4 as well as the optimality of the target space in Theorem 6.5 are valid in the *class of all $\Omega \in \mathcal{J}_\alpha$*. This is inevitable, since, of course, each class $\mathcal{J}_\alpha$ contains all regular domains, and for such domains Sobolev embeddings with stronger target norms hold.

The remaining part of this section is devoted to applications of Theorems 6.4–6.6 to customary function norms. Consider first the case when Lebesgue or Lorentz norms are concerned. Our conclusions take a different form, according to whether $\alpha \in [\frac{1}{n'}, 1)$, or $\alpha = 1$.

We begin by assuming that $\alpha \in [\frac{1}{n'}, 1)$. Note that results for regular (i.e. John) domains are covered by the choice $\alpha = \frac{1}{n'}$.

Sobolev embeddings involving usual Lebesgue norms are contained in the following theorem.

Theorem 6.8. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\Omega \in \mathcal{J}_\alpha$ for some $\alpha \in [\frac{1}{n'}, 1)$. Let $m \in \mathbb{N}$ and $p \in [1, \infty]$. Then*

$$V^m L^p(\Omega) \to \begin{cases} L^{\frac{p}{1-mp(1-\alpha)}}(\Omega) & \textit{if } m(1-\alpha)<1 \textit{ and } 1 \le p < \frac{1}{m(1-\alpha)}, \\ L^r(\Omega) & \textit{for any } r \in [1,\infty), \textit{ if } m(1-\alpha)<1 \textit{ and } p = \frac{1}{m(1-\alpha)}, \\ L^\infty(\Omega) & \textit{otherwise.} \end{cases} \tag{6.15}$$

Moreover, in the first and the third cases, the target spaces in (6.15) are optimal among all Lebesgue spaces, as Ω *ranges in* $\mathcal{J}_\alpha$*.*

Although the target spaces in (6.15) cannot be improved in the class of Lebesgue spaces, the first two embeddings in (6.15) can be strengthened if more general rearrangement-invariant spaces are employed. Such a strengthening can be obtained as a special case of a Sobolev embedding for Lorentz spaces which reads as follows.

Theorem 6.9. *Let* $n \in \mathbb{N}$*,* $n \ge 2$*, and let* $\Omega \in \mathcal{J}_\alpha$ *for some* $\alpha \in [\frac{1}{n'}, 1)$*. Let* $m \in \mathbb{N}$ *and* $p, q \in [1,\infty]$*. Assume that one of the conditions in (3.19) holds. Then*

$$V^m L^{p,q}(\Omega) \to \begin{cases} L^{\frac{p}{1-mp(1-\alpha)},q}(\Omega) & \textit{if } m(1-\alpha)<1 \textit{ and } 1 \le p < \frac{1}{m(1-\alpha)}, \\ L^{\infty,q;-1}(\Omega) & \textit{if } m(1-\alpha)<1,\ p = \frac{1}{m(1-\alpha)} \textit{ and } q>1, \\ L^\infty(\Omega) & \textit{otherwise.} \end{cases} \tag{6.16}$$

Moreover, the target spaces in (6.16) are optimal among all rearrangement-invariant spaces, as Ω *ranges in* $\mathcal{J}_\alpha$*.*

The particular choice of parameters $p = q$, $1 \le p < \frac{1}{m(1-\alpha)}$ in Theorem 6.9 shows that

$$V^m L^p(\Omega) \to L^{\frac{p}{1-mp(1-\alpha)},p}(\Omega).$$

This is a non-trivial strengthening of the first embedding in (6.15), since $L^{\frac{p}{1-mp(1-\alpha)},p}(\Omega) \subsetneq L^{\frac{p}{1-mp(1-\alpha)}}$. Likewise, the choice $m(1-\alpha)<1$ and $p = q = \frac{1}{m(1-\alpha)}$ shows that also the second embedding in (6.15) can be essentially improved by

$$V^m L^p(\Omega) \to L^{\infty,p;-1}(\Omega).$$

Assume now that $\alpha = 1$. The embedding theorem in Lebesgue spaces takes the following form.

Theorem 6.10. *Let* $n \in \mathbb{N}$*,* $n \ge 2$*, and let* $\Omega \in \mathcal{J}_1$*. Let* $m \in \mathbb{N}$ *and* $p \in [1,\infty]$*. Then*

$$V^m L^p(\Omega) \to \begin{cases} L^p(\Omega) & \textit{if } 1 \le p < \infty, \\ L^r(\Omega) & \textit{for any } r \in [1,\infty), \textit{ if } p = \infty. \end{cases} \tag{6.17}$$

Moreover, in the former case of (6.17), the target space is optimal among all Lebesgue spaces, as Ω ranges in $\mathcal{J}_1$.

Optimal embeddings for Lorentz–Sobolev spaces are provided in the next theorem.

Theorem 6.11. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\Omega \in \mathcal{J}_1$. Let $m \in \mathbb{N}$ and $p, q \in [1, \infty]$. Assume that one of the conditions in (3.19) holds. Then*

$$V^m L^{p,q}(\Omega) \to \begin{cases} L^{p,q}(\Omega) & \text{if } 1 \leq p < \infty, \\ \exp L^{\frac{1}{m}}(\Omega) & \text{if } p = q = \infty. \end{cases} \tag{6.18}$$

The target spaces are optimal in (6.18) among all rearrangement-invariant spaces, as Ω ranges in $\mathcal{J}_1$.

Our last application in this section concerns Orlicz–Sobolev spaces. Let $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1)$, and let A be a Young function. If $m < \frac{1}{1-\alpha}$, we may assume, without loss of generality, that

$$\int_0 \left(\frac{t}{A(t)}\right)^{\frac{m(1-\alpha)}{1-m(1-\alpha)}} dt < \infty. \tag{6.19}$$

Indeed, by (3.27), the function A can be modified near 0, if necessary, in such a way that (6.19) is fulfilled, on leaving the space $V^m L^A(\Omega)$ unchanged (up to equivalent norms).

If $m < \frac{1}{1-\alpha}$ and the integral

$$\int^\infty \left(\frac{t}{A(t)}\right)^{\frac{m(1-\alpha)}{1-m(1-\alpha)}} dt \tag{6.20}$$

diverges, we define the function $H_{m,\alpha} : [0, \infty) \to [0, \infty)$ as

$$H_{m,\alpha}(s) = \left(\int_0^s \left(\frac{t}{A(t)}\right)^{\frac{m(1-\alpha)}{1-m(1-\alpha)}} dt\right)^{1-m(1-\alpha)} \quad \text{for } s \geq 0,$$

and the Young function $A_{m,\alpha}$ as

$$A_{m,\alpha}(t) = A\big(H_{m,\alpha}^{-1}(t)\big) \quad \text{for } t \geq 0. \tag{6.21}$$

Theorem 6.12. *Assume that $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1)$ and $\Omega \in \mathcal{J}_\alpha$. Let A be a Young function fulfilling (6.19). Then*

$$V^m L^A(\Omega) \to \begin{cases} L^{A_{m,\alpha}}(\Omega) & \text{if } m < \frac{1}{1-\alpha}, \text{ and the integral (6.20) diverges,} \\ L^\infty(\Omega) & \text{if either } m \geq \frac{1}{1-\alpha}, \text{ or } m < \frac{1}{1-\alpha} \\ & \text{and the integral (6.20) converges.} \end{cases} \tag{6.22}$$

Moreover, the target spaces in (6.22) are optimal among all Orlicz spaces, as Ω ranges in $\mathcal{J}_\alpha$.

Theorem 6.12 follows from Theorem 6.4, via [28, Theorem 4].

The first case of embedding (6.22) can be enhanced, on replacing the optimal Orlicz target spaces with the optimal rearrangement-invariant target spaces. The latter turn out to belong to the family of Orlicz–Lorentz spaces defined in Section 3.

Assume that $m < \frac{1}{1-\alpha}$, and the integral (6.20) diverges. Let a be the left-continuous function appearing in (3.25), and let B be the Young function given by

$$B(t) = \int_0^t b(\tau)d\tau \quad \text{for } t \geq 0,$$

where b is the non-decreasing, left-continuous function in $[0, \infty)$ obeying

$$b^{-1}(s) = \left(\int_{a^{-1}(s)}^{\infty} \left(\int_0^{\tau} \left(\frac{1}{a(t)} \right)^{\frac{m(1-\alpha)}{1-m(1-\alpha)}} dt \right)^{-\frac{1}{m(1-\alpha)}} \frac{d\tau}{a(\tau)^{\frac{1}{1-m(1-\alpha)}}} \right)^{\frac{m(1-\alpha)}{m(1-\alpha)-1}} \quad \text{for } s \geq 0.$$

Here, a^{-1} and b^{-1} denote the (generalized) left-continuous inverses of a and b, respectively.

Recall from Section 3 that $L(\frac{1}{m(1-\alpha)}, 1, B)(\Omega)$ is the Orlicz–Lorentz space built upon the function norm given by

$$\|f\|_{L(\frac{1}{m(1-\alpha)},1,B)(0,1)} = \left\| s^{-m(1-\alpha)} f^*(s) \right\|_{L^B(0,1)}$$

for $f \in \mathcal{M}_+(0, 1)$.

Theorem 6.13. *Assume that $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1)$ and $\Omega \in \mathcal{J}_\alpha$. Let A be a Young function fulfilling (6.19). Assume that $m < \frac{1}{1-\alpha}$, and the integral in (6.20) diverges. Then*

$$V^m L^A(\Omega) \to L\left(\frac{1}{m(1-\alpha)}, 1, B \right)(\Omega), \tag{6.23}$$

and the target space in (6.23) is optimal among all rearrangement-invariant spaces, as Ω ranges in $\mathcal{J}_\alpha$.

Embedding (6.23) is a consequence of Theorem 6.4, and of [25, inequality (3.1)].

Example 6.14. Consider the case when

$$A(t) \approx t^p (\log t)^\beta \quad \text{near infinity, where either } p > 1 \text{ and } \beta \in \mathbb{R}, \text{ or } p = 1 \text{ and } \beta \geq 0.$$

Hence, $L^A(\Omega) = L^p \log^\beta L(\Omega)$. An application of Theorem 6.12 tells us that

$$V^m L^p \log^\beta L(\Omega) \to \begin{cases} L^{\frac{p}{1-pm(1-\alpha)}} \log^{\frac{\beta}{1-pm(1-\alpha)}} L(\Omega) \quad \text{if } mp(1-\alpha) < 1, \\ \exp L^{\frac{1}{1-(1+\beta)m(1-\alpha)}}(\Omega) \\ \quad \text{if } mp(1-\alpha) = 1 \text{ and } \beta < \frac{1-m(1-\alpha)}{m(1-\alpha)}, \\ \exp\exp L^{\frac{1}{1-m(1-\alpha)}}(\Omega) \\ \quad \text{if } mp(1-\alpha) = 1 \text{ and } \beta = \frac{1-m(1-\alpha)}{m(1-\alpha)}, \\ L^\infty(\Omega) \quad \text{if either } mp(1-\alpha) > 1, \\ \quad \text{or } mp(1-\alpha) = 1 \text{ and } \beta > \frac{1-m(1-\alpha)}{m(1-\alpha)}. \end{cases} \tag{6.24}$$

Moreover, the target spaces in (6.24) are optimal among all Orlicz spaces, as Ω ranges in $\mathcal{J}_\alpha$.

The first three embeddings in (6.24) can be improved on allowing more general rearrangement-invariant target spaces. Indeed, we have that

$$V^m L^p \log^\beta L(\Omega) \to \begin{cases} L^{\frac{p}{1-pm(1-\alpha)},p;\frac{\beta}{p}}(\Omega) & \text{if } mp(1-\alpha) < 1, \\ L^{\infty,\frac{1}{m(1-\alpha)};m(1-\alpha)\beta-1}(\Omega) & \text{if } mp(1-\alpha) = 1 \text{ and } \beta < \frac{1-m(1-\alpha)}{m(1-\alpha)}, \\ L^{\infty,\frac{1}{m(1-\alpha)};-m(1-\alpha),-1}(\Omega) & \text{if } mp(1-\alpha) = 1 \text{ and } \beta = \frac{1-m(1-\alpha)}{m(1-\alpha)}, \end{cases} \tag{6.25}$$

the targets being optimal among all rearrangement-invariant spaces in (6.25) as Ω ranges among all domains in $\mathcal{J}_\alpha$. This is a consequence of Theorem 6.13, and of the fact that the Orlicz–Lorentz spaces $L(\frac{1}{m(1-\alpha)}, 1, B)(\Omega)$ associated with the present choices of the function A agree (up to equivalent norms) with the (generalized) Lorentz–Zygmund spaces appearing on the right-hand side of (6.25).

7. Sobolev embeddings in product probability spaces

The class of product probability measures in $\mathbb{R}^n$, $n \geq 1$, which we consider in this section, arises in connection with the study of generalized hypercontractivity theory and integrability properties of the associated heat semigroups. The isoperimetric problem in the corresponding probability spaces was studied in [4] – see also [3,8,9,31,59,60].

Assume that $\Phi : [0,\infty) \to [0,\infty)$ is a strictly increasing convex function, twice continuously differentiable in $(0,\infty)$, such that $\sqrt{\Phi}$ is concave and $\Phi(0) = 0$. Let μ_Φ be the probability measure on $\mathbb{R}$ given by

$$d\mu_\Phi(x) = c_\Phi e^{-\Phi(|x|)}\, dx, \tag{7.1}$$

where c_Φ is a constant chosen in such a way that $\mu_\Phi(\mathbb{R}) = 1$. The product measure $\mu_{\Phi,n}$ on $\mathbb{R}^n$, $n \geq 1$, generated by μ_Φ, is then defined as

$$\mu_{\Phi,n} = \underbrace{\mu_\Phi \times \cdots \times \mu_\Phi}_{n\text{-times}}. \tag{7.2}$$

Clearly, $\mu_{\Phi,1} = \mu_\Phi$, and $(\mathbb{R}^n, \mu_{\Phi,n})$ is a probability space for every $n \in \mathbb{N}$.

The main example of a measure μ_Φ is obtained by taking

$$\Phi(t) = \frac{1}{2}t^2. \tag{7.3}$$

This choice yields $\mu_{\Phi,n} = \gamma_n$, the Gauss measure which obeys

$$d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx. \tag{7.4}$$

More generally, given any $\beta \in [1,2]$, the Boltzmann measure $\gamma_{n,\beta}$ in $\mathbb{R}^n$, associated with

$$\Phi(t) = \frac{1}{\beta}t^\beta, \tag{7.5}$$

satisfies the above assumptions.

Let $H : \mathbb{R} \to (0,1)$ be defined as

$$H(t) = \int_t^\infty c_\Phi e^{-\Phi(|r|)}\, dr \quad \text{for } t \in \mathbb{R}, \tag{7.6}$$

and let $F_\Phi : [0,1] \to [0,\infty)$ be given by

$$F_\Phi(s) = c_\Phi e^{-\Phi(|H^{-1}(s)|)} \quad \text{for } s \in (0,1), \quad \text{and} \quad F_\Phi(0) = F_\Phi(1) = 0. \tag{7.7}$$

Since μ_Φ is a probability measure and $\mu_{\Phi,n}$ is defined by (7.2), it is easily seen that, for each $i = 1, \dots, n$,

$$\mu_{\Phi,n}\big(\big\{(x_1,\dots,x_n) : x_i > t\big\}\big) = H(t) \quad \text{for } t \in \mathbb{R}, \tag{7.8}$$

and

$$P_{\mu_{\Phi,n}}\big(\big\{(x_1,\dots,x_n) : x_i > t\big\}, \mathbb{R}^n\big) = c_\Phi e^{-\Phi(|t|)} = -H'(t) \quad \text{for } t \in \mathbb{R}.$$

Hence, $F_\Phi(s)$ agrees with the perimeter of any half-space of the form $\{x_i > t\}$, whose measure is s.

Next, define $L_\Phi : [0,1] \to [0,\infty)$ as

$$L_\Phi(s) = s\Phi'\left(\Phi^{-1}\left(\log\left(\frac{2}{s}\right)\right)\right) \quad \text{for } s \in (0,1], \quad \text{and} \quad L_\Phi(0) = 0. \tag{7.9}$$

Then the isoperimetric function of $(\mathbb{R}^n, \mu_{\Phi,n})$ satisfies

$$I_{\mathbb{R}^n,\mu_{\Phi,n}}(s) \approx F_\Phi(s) \approx L_\Phi(s) \quad \text{for } s \in \left[0, \frac{1}{2}\right] \tag{7.10}$$

(see [4, Proposition 13 and Theorem 15]; note that the second equivalence in (7.10) also relies upon Lemma 11.1(ii) of Section 11). Furthermore, half-spaces, whose boundary is orthogonal to a coordinate axis, are "approximate solutions" to the isoperimetric problem in $(\mathbb{R}^n, \mu_{\Phi,n})$ in the sense that there exist constants C_1 and C_2, depending on n, such that, for every $s \in (0,1)$, any such half-space V with measure s satisfies

$$C_1 P_{\mu_{\Phi,n}}\big(V, \mathbb{R}^n\big) \le I_{\mathbb{R}^n,\mu_{\Phi,n}}(s) \le C_2 P_{\mu_{\Phi,n}}\big(V, \mathbb{R}^n\big).$$

In the special case when $\mu_{\Phi,n} = \gamma_n$, the Gauss measure, Eq. (7.10) yields

$$I_{\mathbb{R}^n,\gamma_n}(s) \approx s\left(\log \frac{2}{s}\right)^{\frac{1}{2}} \quad \text{for } s \in \left(0, \frac{1}{2}\right].$$

Moreover, any half-space is, in fact, an exact minimizer in the isoperimetric inequality [12,85].

Our reduction theorem for Sobolev embeddings in product probability spaces reads as follows.

Theorem 7.1 *(Reduction principle for product probability spaces). Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, let $\mu_{\Phi,n}$ be the probability measure defined by (7.2), and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following facts are equivalent.*

(i) *The inequality*

$$\left\| \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}}\right)^m \int_t^1 \frac{f(s)}{s}\left(\log \frac{s}{t}\right)^{m-1} ds \right\|_{Y(0,1)} \le C_1 \|f\|_{X(0,1)} \tag{7.11}$$

holds for some constant C_1, and for every nonnegative $f \in X(0,1)$.

(ii) *The embedding*

$$V^m X\big(\mathbb{R}^n, \mu_{\Phi,n}\big) \to Y\big(\mathbb{R}^n, \mu_{\Phi,n}\big) \tag{7.12}$$

holds.

(iii) *The Poincaré inequality*

$$\|u\|_{Y(\mathbb{R}^n,\mu_{\Phi,n})} \le C_2 \big\|\nabla^m u\big\|_{X(\mathbb{R}^n,\mu_{\Phi,n})} \tag{7.13}$$

holds for some constant C_2 and every $u \in V_\perp^m X(\mathbb{R}^n, \mu_{\Phi,n})$.

Let us notice that inequality (7.11) is not just a specialization of (5.3), but even a further simplification of such specialization.

Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm, and let $n, m \in \mathbb{N}$. The rearrangement-invariant function norm $\|\cdot\|_{X_{m,\Phi}(0,1)}$ which yields the optimal rearrangement-invariant target space $Y(\mathbb{R}^n, \mu_{\Phi,n})$ in embedding (7.12) is defined as follows. Consider the rearrangement-invariant function norm $\|\cdot\|_{\widetilde{X}_m(0,1)}$ whose associate norm fulfills

$$\|g\|_{\widetilde{X}'_m(0,1)} = \left\| \frac{1}{s} \int_0^s \left(\log \frac{s}{r}\right)^{m-1} g^*(r)\, dr \right\|_{X'(0,1)} \tag{7.14}$$

for $g \in \mathcal{M}_+(0,1)$. Then $\|\cdot\|_{X_{m,\Phi}(0,1)}$ is given by

$$\|f\|_{X_{m,\Phi}(0,1)} = \left\| \left(\frac{\log \frac{2}{s}}{\Phi^{-1}(\log \frac{2}{s})} \right)^m f^*(s) \right\|_{\widetilde{X}_m(0,1)} \tag{7.15}$$

for $f \in \mathcal{M}_+(0,1)$.

Remark 7.2. Note that if $\Phi(t) = t$, and $m \in \mathbb{N}$, we have that

$$X_{m,\Phi}(0,1) = \widetilde{X}_m(0,1)$$

for every rearrangement-invariant norm $\|\cdot\|_{X(0,1)}$.

Theorem 7.3 *(Optimal target for product probability spaces). Let n, m, $\mu_{\Phi,n}$ and $\|\cdot\|_{X(0,1)}$ be as in Theorem 7.1. Then the functional $\|\cdot\|_{X_{m,\Phi}(0,1)}$, given by (7.15), is a rearrangement-invariant function norm satisfying*

$$V^m X(\mathbb{R}^n, \mu_{\Phi,n}) \to X_{m,\Phi}(\mathbb{R}^n, \mu_{\Phi,n}), \tag{7.16}$$

and there exists a constant C such that

$$\|u\|_{X_{m,\Phi}(\mathbb{R}^n,\mu_{\Phi,n})} \le C \big\|\nabla^m u\big\|_{X(\mathbb{R}^n,\mu_{\Phi,n})}, \tag{7.17}$$

for every $u \in V^m_\perp X(\mathbb{R}^n, \mu_{\Phi,n})$.

Moreover, the function norm $\|\cdot\|_{X_{m,\Phi}(0,1)}$ is optimal in (7.16) and in (7.17) among all rearrangement-invariant norms.

Remark 7.4. Let us emphasize that inequality (7.11) implies embedding (7.12) with a norm independent of n, and the Poincaré inequality (7.13) with constant C_2 independent of n. The norm of the optimal embedding (7.16), and the constant C in the corresponding Poincaré inequality (7.17) are independent of n as well.

For a broad class of rearrangement-invariant function norms $\|\cdot\|_{X(0,1)}$ the expression of the associated optimal Sobolev target norm $\|\cdot\|_{X_{m,\Phi}(0,1)}$ can be substantially simplified, as observed in the next proposition.

Proposition 7.5. *Let $m \in \mathbb{N}$ and let Φ be as in (7.1). Suppose that $\| \cdot \|_{X(0,1)}$ is a rearrangement-invariant function norm such that the operator*

$$f \mapsto f^{**}$$

is bounded on $X'(0,1)$. Then

$$\|f\|_{X_{m,\Phi}(0,1)} \approx \left\| \left(\frac{\log \frac{2}{s}}{\Phi^{-1}(\log \frac{2}{s})} \right)^m f^*(s) \right\|_{X(0,1)}$$

up to multiplicative constants independent of $f \in \mathcal{M}_+(0,1)$.

The rearrangement-invariant spaces on which the operator "**" is bounded are fully characterized in terms of their upper Boyd index. In particular, the assumptions of Proposition 7.5 are satisfied if and only if the upper Boyd index of $X'(0,1)$ is strictly smaller that 1 [7, Theorem 5.15].

The iteration principle for Sobolev embeddings on product probability measure spaces, on which Theorem 7.1 rests, reads as follows.

Theorem 7.6 *(Iteration principle for product probability spaces). Let n, $\mu_{\Phi,n}$ and $\|\cdot\|_{X(0,1)}$ be as in Theorem 7.1, and let $k, h \in \mathbb{N}$. Then,*

$$(X_{k,\Phi})_{h,\Phi}\big(\mathbb{R}^n, \mu_{\Phi,n}\big) = X_{k+h,\Phi}\big(\mathbb{R}^n, \mu_{\Phi,n}\big),$$

up to equivalent norms.

Specialization of Theorems 7.1, 7.3 and 7.6 to the case of (7.3) easily leads to the following results for Gaussian Sobolev embeddings of any order.

Theorem 7.7 *(Reduction principle in Gauss space). Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, and let $\| \cdot \|_{X(0,1)}$ and $\| \cdot \|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following facts are equivalent.*

(i) *The inequality*

$$\left\| \frac{1}{(\log \frac{2}{s})^{\frac{m}{2}}} \int_s^1 \frac{f(r)}{r} \left(\log \frac{r}{s} \right)^{m-1} dr \right\|_{Y(0,1)} \leq C_1 \|f\|_{X(0,1)}$$

holds for some constant C_1, and for every nonnegative $f \in X(0,1)$.

(ii) *The embedding*

$$V^m X\big(\mathbb{R}^n, \gamma_n\big) \to Y\big(\mathbb{R}^n, \gamma_n\big)$$

holds.

(iii) *The Poincaré inequality*

$$\|u\|_{Y(\mathbb{R}^n,\gamma_n)} \le C_2 \big\|\nabla^m u\big\|_{X(\mathbb{R}^n,\gamma_n)}$$

holds for some constant C_2, and for every $u \in V_\perp^m X(\mathbb{R}^n,\gamma_n)$.

Given $n, m \in \mathbb{N}$, and a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, define the rearrangement-invariant function norm $\|\cdot\|_{X_{m,G}(0,1)}$ by

$$\|f\|_{X_{m,G}(0,1)} = \left\| \left(\log \frac{2}{s}\right)^{\frac{m}{2}} f^*(s) \right\|_{\widetilde{X}_m(0,1)} \tag{7.18}$$

for $f \in \mathcal{M}_+(0,1)$.

Theorem 7.8 *(Optimal target in Gauss space). Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then the functional $\|\cdot\|_{X_{m,G}(0,1)}$, given by (7.18), is a rearrangement-invariant function norm satisfying*

$$V^m X\big(\mathbb{R}^n,\gamma_n\big) \to X_{m,G}\big(\mathbb{R}^n,\gamma_n\big) \tag{7.19}$$

and

$$\|u\|_{X_{m,G}(\mathbb{R}^n,\gamma_n)} \le C \big\|\nabla^m u\big\|_{X(\mathbb{R}^n,\gamma_n)} \tag{7.20}$$

for some constant C and every $u \in V_\perp^m X(\mathbb{R}^n,\gamma_n)$.

Moreover, the function norm $\|\cdot\|_{X_{m,G}(0,1)}$ is optimal in (7.19) and (7.20) among all rearrangement-invariant norms.

Observe that, even for $m = 1$, Theorems 7.7 and 7.8 provide us with a characterization of Gaussian Sobolev embeddings which somewhat simplifies earlier results in a similar direction [32,65].

Theorem 7.9 *(Iteration principle in Gauss space). Let $n, k, h \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then,*

$$(X_{k,G})_{h,G}\big(\mathbb{R}^n,\gamma_n\big) = X_{k+h,G}\big(\mathbb{R}^n,\gamma_n\big),$$

up to equivalent norms.

Of course, versions of Theorems 7.7–7.9, with the Gauss measure replaced with the Boltzmann measure, given by the choice (7.5), can similarly be deduced from Theorems 7.1, 7.3 and 7.6. The reduction principle and the optimal target space then take the following form.

Theorem 7.10 *(Reduction principle in Boltzmann spaces). Assume that $n, m \in \mathbb{N}$, and $\beta \in [1,2]$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following facts are equivalent.*

(i) *The inequality*

$$\left\| \frac{1}{(\log \frac{2}{s})^{\frac{m(\beta-1)}{\beta}}} \int_s^1 \frac{f(r)}{r} \left(\log \frac{r}{s}\right)^{m-1} dr \right\|_{Y(0,1)} \le C_1 \|f\|_{X(0,1)}$$

holds for some constant C_1, and for every nonnegative $f \in X(0,1)$.

(ii) *The embedding*

$$V^m X\big(\mathbb{R}^n, \gamma_{n,\beta}\big) \to Y\big(\mathbb{R}^n, \gamma_{n,\beta}\big)$$

holds.

(iii) *The Poincaré inequality*

$$\|u\|_{Y(\mathbb{R}^n,\gamma_{n,\beta})} \le C_2 \big\|\nabla^m u\big\|_{X(\mathbb{R}^n,\gamma_{n,\beta})}$$

holds for some constant C_2 and for every $u \in V_\perp^m X(\mathbb{R}^n, \gamma_{n,\beta})$.

Given $n, m \in \mathbb{N}$, $\beta \in [1,2]$, and a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, define the rearrangement-invariant function norm $\|\cdot\|_{X_{m,B,\beta}(0,1)}$ by

$$\|f\|_{X_{m,B,\beta}(0,1)} = \left\| \left(\log \frac{2}{s}\right)^{\frac{m(\beta-1)}{\beta}} f^*(s) \right\|_{\widetilde{X}_m(0,1)} \tag{7.21}$$

for $f \in \mathcal{M}_+(0,1)$.

Theorem 7.11 *(Optimal target in Boltzmann spaces). Let $n, m \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then the functional $\|\cdot\|_{X_{m,B,\beta}(0,1)}$, given by (7.21), is a rearrangement-invariant function norm satisfying*

$$V^m X\big(\mathbb{R}^n, \gamma_{n,\beta}\big) \to X_{m,B,\beta}\big(\mathbb{R}^n, \gamma_{n,\beta}\big) \tag{7.22}$$

and

$$\|u\|_{X_{m,B,\beta}(\mathbb{R}^n,\gamma_{n,\beta})} \le C \big\|\nabla^m u\big\|_{X(\mathbb{R}^n,\gamma_{n,\beta})} \tag{7.23}$$

for some constant C and every $u \in V_\perp^m X(\mathbb{R}^n, \gamma_{n,\beta})$.

Moreover, the function norm $\|\cdot\|_{X_{m,B,\beta}(0,1)}$ is optimal in (7.22) and (7.23) among all rearrangement-invariant norms.

We present an application of the results of this section to the particular case when $\mu_{\Phi,n}$ is a Boltzmann measure, and the norms are of Lorentz–Zygmund type.

Theorem 7.12. *Let $n, m \in \mathbb{N}$, let $\beta \in [1,2]$ and let $p, q \in [1,\infty]$ and $\alpha \in \mathbb{R}$ be such that one of the conditions in (3.21) is satisfied. Then*

$$V^m L^{p,q;\alpha}(\mathbb{R}^n, \gamma_{n,\beta}) \to \begin{cases} L^{p,q;\alpha+\frac{m(\beta-1)}{\beta}}(\mathbb{R}^n, \gamma_{n,\beta}) & \text{if } p < \infty; \\ L^{\infty,q;\alpha-\frac{m}{\beta}}(\mathbb{R}^n, \gamma_{n,\beta}) & \text{if } p = \infty. \end{cases}$$

Moreover, in both cases, the target space is optimal among all rearrangement-invariant spaces.

When $\beta = 2$, Theorem 7.12 yields the following sharp Sobolev type embeddings in Gauss space.

Theorem 7.13. *Let $n, m \in \mathbb{N}$, and let $p, q \in [1,\infty]$ and $\alpha \in \mathbb{R}$ be such that one of the conditions in (3.21) is satisfied. Then*

$$V^m L^{p,q;\alpha}(\mathbb{R}^n, \gamma_n) \to \begin{cases} L^{p,q;\alpha+\frac{m}{2}}(\mathbb{R}^n, \gamma_n) & \text{if } p < \infty; \\ L^{\infty,q;\alpha-\frac{m}{2}}(\mathbb{R}^n, \gamma_n) & \text{if } p = \infty. \end{cases}$$

Moreover, in both cases, the target space is optimal among all rearrangement-invariant spaces.

A further specialization of the indices p, q, α appearing in Theorem 7.13 leads to the following basic embeddings. In particular, when $m = 1$ we recover a classical Gaussian Sobolev embedding [46].

Corollary 7.14. *Let $n, m \in \mathbb{N}$.*

(i) *Assume that $p \in [1,\infty)$. Then*

$$V^m L^p(\mathbb{R}^n, \gamma_n) \to L^p \log^{\frac{mp}{2}} L(\mathbb{R}^n, \gamma_n),$$

and the target space is optimal among all rearrangement-invariant spaces.

(ii) *Assume that $\gamma > 0$. Then*

$$V^m \exp L^\gamma(\mathbb{R}^n, \gamma_n) \to \exp L^{\frac{2\gamma}{2+m\gamma}}(\mathbb{R}^n, \gamma_n),$$

and the target space is optimal among all rearrangement-invariant spaces.

(iii)

$$V^m L^\infty(\mathbb{R}^n, \gamma_n) \to \exp L^{\frac{2}{m}}(\mathbb{R}^n, \gamma_n),$$

and the target space is optimal among all rearrangement-invariant spaces.

Note that the target space in the second embedding of Theorem 7.13, and in the embeddings (ii) and (iii) of Corollary 7.14 increases in m. This is related to the fact that $V^m L^{\infty,q;\alpha}(\mathbb{R}^n,\gamma_n) \not\subseteq V^k L^{\infty,q;\alpha}(\mathbb{R}^n,\gamma_n)$ if $m > k$.

8. Optimal target function norms

In this section we collect some basic properties about certain one-dimensional operators playing a role in the proofs of our main results.

Let $T : \mathcal{M}_+(0,1) \to \mathcal{M}_+(0,1)$ be a sublinear operator, namely an operator such that

$$T(\lambda f) = \lambda T f, \quad \text{and} \quad T(f+g) \leq C(Tf + Tg),$$

for some positive constant C, and for every $\lambda \geq 0$ and $f, g \in \mathcal{M}_+(0,1)$.

Given two rearrangement-invariant spaces $X(0,1)$ and $Y(0,1)$, we say that T is bounded from $X(0,1)$ into $Y(0,1)$, and write

$$T : X(0,1) \to Y(0,1), \tag{8.1}$$

if the quantity

$$\|T\| = \sup\bigl\{\|Tf\|_{Y(0,1)}; f \in X(0,1) \cap \mathcal{M}_+(0,1), \|f\|_{X(0,1)} \leq 1\bigr\}$$

is finite. Such a quantity will be called the norm of T. The space $Y(0,1)$ will be called optimal, within a certain class, in (8.1) if, whenever $Z(0,1)$ is another rearrangement-invariant space, from the same class, such that $T : X(0,1) \to Z(0,1)$, we have that $Y(0,1) \to Z(0,1)$. Equivalently, the function norm $\|\cdot\|_{Y(0,1)}$ will be said to be optimal in (8.1) in the relevant class.

Two operators T and T' from $\mathcal{M}_+(0,1)$ into $\mathcal{M}_+(0,1)$ will be called mutually associate if

$$\int_0^1 Tf(s)g(s)\,ds = \int_0^1 f(s)T'g(s)\,ds$$

for every $f, g \in \mathcal{M}_+(0,1)$.

Lemma 8.1. *Let T and T' be mutually associate operators, and let $X(0,1)$ and $Y(0,1)$ be rearrangement-invariant spaces. Then,*

$$T : X(0,1) \to Y(0,1) \quad \textit{if and only if} \quad T' : Y'(0,1) \to X'(0,1),$$

and

$$\|T\| = \|T'\|.$$

Proof. The conclusion is a consequence of the following chain:

$$\|T\| = \sup_{\substack{f \geq 0 \\ \|f\|_{X(0,1)} \leq 1}} \|Tf\|_{Y(0,1)} = \sup_{\substack{f \geq 0 \\ \|f\|_{X(0,1)} \leq 1}} \sup_{\substack{g \geq 0 \\ \|g\|_{Y'(0,1)} \leq 1}} \int_0^1 Tf(s)g(s)\,ds$$

$$= \sup_{\substack{g \geq 0 \\ \|g\|_{Y'(0,1)} \leq 1}} \sup_{\substack{f \geq 0 \\ \|f\|_{X(0,1)} \leq 1}} \int_0^1 f(s)T'g(s)\,ds = \sup_{\substack{g \geq 0 \\ \|g\|_{Y'(0,1)} \leq 1}} \|T'g\|_{X'(0,1)} = \|T'\|. \qquad \square$$

Let $I : [0,1] \to [0,\infty)$ be a measurable function satisfying (5.2). We define the operators H_I and R_I from $\mathcal{M}_+(0,1)$ into $\mathcal{M}_+(0,1)$ by

$$H_I f(t) = \int_t^1 \frac{f(s)}{I(s)}\,ds \quad \text{for } t \in (0,1], \tag{8.2}$$

and

$$R_I f(t) = \frac{1}{I(t)} \int_0^t f(s)\,ds \quad \text{for } t \in (0,1], \tag{8.3}$$

for $f \in \mathcal{M}_+(0,1)$. Moreover, given $j \in \mathbb{N}$, we set

$$H_I^j = \underbrace{H_I \circ H_I \circ \ldots \circ H_I}_{j\text{-times}} \quad \text{and} \quad R_I^j = \underbrace{R_I \circ R_I \circ \ldots \circ R_I}_{j\text{-times}}. \tag{8.4}$$

We also set $H_I^0 = R_I^0 = \mathrm{Id}$.

Remarks 8.2. (i) The operators H_I and R_I are mutually associate. Hence, H_I^j and R_I^j are also mutually associate for $j \in \mathbb{N}$.

(ii) By the Hardy–Littlewood inequality (3.7), we have, for every $f \in \mathcal{M}_+(0,1)$,

$$R_I f(t) \leq R_I f^*(t) \quad \text{for } t \in (0,1].$$

More generally, for every $f \in \mathcal{M}_+(0,1)$ and $j \in \mathbb{N}$, one has that

$$R_I^j f(t) \leq R_I^j f^*(t) \quad \text{for } t \in (0,1]. \tag{8.5}$$

(iii) For every $j \in \mathbb{N}$ and $f \in \mathcal{M}_+(0,1)$, we have that

$$H_I^j f(t) = \frac{1}{(j-1)!} \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{j-1} ds \quad \text{for } t \in (0,1]. \tag{8.6}$$

Equation (8.6) holds for $j = 1$ by the very definition of H_I. On the other hand, if (8.6) is assumed to hold for some $j \in \mathbb{N}$, then

$$H_I^{j+1} f(t) = \int_t^1 \frac{H_I^j f(s)}{I(s)}\, ds = \frac{1}{(j-1)!} \int_t^1 \frac{1}{I(s)} \int_s^1 \frac{f(r)}{I(r)} \left(\int_s^r \frac{d\tau}{I(\tau)} \right)^{j-1} dr\, ds$$
$$= \frac{1}{(j-1)!} \int_t^1 \frac{f(r)}{I(r)} \int_t^r \frac{1}{I(s)} \left(\int_s^r \frac{d\tau}{I(\tau)} \right)^{j-1} ds\, dr$$
$$= \frac{1}{j!} \int_t^1 \frac{f(r)}{I(r)} \left(\int_t^r \frac{d\tau}{I(\tau)} \right)^{j} dr \quad \text{for } t \in (0,1].$$

Hence, (8.6) follows by induction. Similarly, for every $j \in \mathbb{N}$ and $f \in \mathcal{M}_+(0,1)$, we also have that

$$R_I^j f(t) = \frac{1}{(j-1)!} \frac{1}{I(t)} \int_0^t f(s) \left(\int_s^t \frac{dr}{I(r)} \right)^{j-1} ds \quad \text{for } t \in (0,1]. \tag{8.7}$$

Given any $j \in \mathbb{N}$ and any rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, Eq. (8.7) implies that

$$\|f\|_{X'_{j,I}(0,1)} = (j-1)! \big\| R_I^j f^* \big\|_{X'(0,1)}, \tag{8.8}$$

for $f \in \mathcal{M}_+(0,1)$, where $\|\cdot\|_{X'_{j,I}(0,1)}$ is the functional introduced in (5.6). We also formally set $\|\cdot\|_{X'_{0,I}} = \|\cdot\|_{X'(0,1)}$.

Proposition 8.3. *Let $I : [0,1] \to [0,\infty)$ be a measurable function satisfying (5.2). Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm and let $j \in \mathbb{N}$. Then the functional $\|\cdot\|_{X'_{j,I}(0,1)}$ defined in (8.8) is a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X_{j,I}(0,1)}$ fulfills*

$$H_I^j : X(0,1) \to X_{j,I}(0,1). \tag{8.9}$$

Moreover, the space $X_{j,I}(0,1)$ is the optimal target in (8.9) among all rearrangement-invariant spaces.

Proof. We begin by showing that the functional $\|\cdot\|_{X'_{j,I}(0,1)}$ is a rearrangement-invariant function norm. Let $f, g \in \mathcal{M}_+(0,1)$. By (3.6), $\int_0^t (f+g)^*(s)\, ds \le \int_0^t f^*(s)\, ds + \int_0^t g^*(s)\, ds$ for $t \in (0,1)$. Thus, by Hardy's lemma (see Section 3) applied, for each fixed $t \in (0,1)$, with $f_1(s) = (f+g)^*(s)$, $f_2(s) = f^*(s) + g^*(s)$ and $h(s) = \chi_{(0,t)}(s) (\int_s^t \frac{dr}{I(r)})^{j-1}$, we obtain the triangle inequality

$$\|f+g\|_{X'_{j,I}(0,1)} \leq \|f\|_{X'_{j,I}(0,1)} + \|g\|_{X'_{j,I}(0,1)}.$$

Other properties in the axiom (P1) of the definition of rearrangement-invariant function norm, as well as the axioms (P2), (P3) and (P6) are obviously satisfied. Next, it follows from (5.2) that there exists a positive constant C such that $\frac{1}{I(t)} \leq \frac{C}{t}$ for $t \in (0,1)$. Therefore,

$$\begin{aligned}\|1\|_{X'_{j,I}(0,1)} &= \left\| \frac{1}{I(t)} \int_0^t \left(\int_s^t \frac{dr}{I(r)} \right)^{j-1} ds \right\|_{X'(0,1)} \leq C^j \left\| \frac{1}{t} \int_0^t \left(\int_s^t \frac{dr}{r} \right)^{j-1} ds \right\|_{X'(0,1)} \\ &= C^j \left\| \frac{1}{t} \int_0^t \left(\log \frac{t}{s} \right)^{j-1} ds \right\|_{X'(0,1)} = (j-1)! C^j \|1\|_{X'(0,1)},\end{aligned}$$

and (P4) follows. As far as (P5) is concerned, note that

$$\int_0^1 f^*(s)\,ds \leq 2 \int_0^{\frac{1}{2}} f^*(s)\,ds$$

for every $f \in \mathcal{M}_+(0,1)$. Thus, by (P5) for the norm $\|\cdot\|_{X'(0,1)}$, there exists a positive constant C such that, if $f \in \mathcal{M}_+(0,1)$, then

$$\begin{aligned}&\left\| \frac{1}{I(t)} \int_0^t f^*(s) \left(\int_s^t \frac{dr}{I(r)} \right)^{j-1} ds \right\|_{X'(0,1)} \\ &\quad \geq C \int_0^1 \frac{1}{I(t)} \int_0^t f^*(s) \left(\int_s^t \frac{dr}{I(r)} \right)^{j-1} ds\,dt \\ &\quad = \frac{C}{j} \int_0^1 f^*(s) \left(\int_s^1 \frac{dr}{I(r)} \right)^j ds \geq \frac{C}{j} \left(\int_{\frac{1}{2}}^1 \frac{dr}{I(r)} \right)^j \int_0^{\frac{1}{2}} f^*(s)\,ds \geq C' \|f\|_{L^1(0,1)}\end{aligned}$$

where $C' = \frac{C}{2j} (\int_{\frac{1}{2}}^1 \frac{dr}{I(r)})^j$. Hence, property (P5) follows.

To prove (8.9), note that, by (8.5) and (8.8), we have

$$\|R_I^j f\|_{X'(0,1)} \leq \|R_I^j f^*\|_{X'(0,1)} = \frac{1}{(j-1)!} \|f\|_{X'_{j,I}(0,1)}$$

for $f \in \mathcal{M}_+(0,1)$. Hence,

$$R_I^j : X'_{j,I}(0,1) \to X'(0,1).$$

Since R_I^j and H_I^j are mutually associate, Eq. (8.9) follows via Lemma 8.1.

It remains to prove that $X_{j,I}(0,1)$ is optimal in (8.9) among all rearrangement-invariant spaces. To this purpose, assume that $Y(0,1)$ is another rearrangement-invariant space such that $H_I^j : X(0,1) \to Y(0,1)$. Then, by Lemma 8.1 again, $R_I^j : Y'(0,1) \to X'(0,1)$, namely

$$\left\|R_I^j f\right\|_{X'(0,1)} \le C\|f\|_{Y'(0,1)}$$

for some positive constant C, and every $f \in \mathcal{M}_+(0,1)$. Thus, in particular, by (8.8),

$$\|f\|_{X'_{j,I}(0,1)} = (j-1)!\left\|R_I^j f^*\right\|_{X'(0,1)} \le (j-1)!C\left\|f^*\right\|_{Y'(0,1)} = (j-1)!C\|f\|_{Y'(0,1)}$$

for every $f \in \mathcal{M}_+(0,1)$. Hence, $Y'(0,1) \to X'_{j,I}(0,1)$, and, equivalently, $X_{j,I}(0,1) \to Y(0,1)$. This shows that $X_{j,I}(0,1)$ is optimal in (8.9) among all rearrangement-invariant spaces. □

We introduce one more sequence of function norms, based on the iteration of the first-order function norm $\|\cdot\|_{X'_{1,I}(0,1)}$. Let $I : [0,1] \to [0,\infty)$ be a measurable function satisfying (5.2). Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Let $j \in \mathbb{N} \cup \{0\}$. We define $\|\cdot\|_{X_j(0,1)}$ as the rearrangement-invariant function norm whose associate norm $\|\cdot\|_{X'_j(0,1)}$ is given, via iteration, by $\|\cdot\|_{X'_0(0,1)} = \|\cdot\|_{X'(0,1)}$, and, for $j \ge 1$, by

$$\|f\|_{X'_j(0,1)} = \left\|R_I f^*\right\|_{X'_{j-1}(0,1)} \tag{8.10}$$

for $f \in \mathcal{M}_+(0,1)$. Note that

$$\|f\|_{X_1(0,1)} = \|f\|_{X_{1,I}(0,1)}. \tag{8.11}$$

Remark 8.4. By Proposition 8.3, applied j times, with $j = 1$, we obtain that, for every $j \in \mathbb{N} \cup \{0\}$, the functional $\|\cdot\|_{X'_j(0,1)}$ is actually a rearrangement-invariant function norm. Moreover, its associate function norm $\|\cdot\|_{X_j(0,1)}$ fulfills

$$H_I : X_j(0,1) \to X_{j+1}(0,1), \tag{8.12}$$

and $\|\cdot\|_{X_{j+1}(0,1)}$ is the optimal target function norm in (8.12) among all rearrangement-invariant function norms. By Lemma 8.1, we also have

$$R_I : X'_{j+1}(0,1) \to X'_j(0,1).$$

Remark 8.5. Note that, by the very definition of $X_j(0,1)$,

$$X_j(0,1) = \underbrace{\big(\dots(X_{1,I})_{1,I}\dots\big)_{1,I}}_{j\text{-times}}(0,1)$$

for $j \in \mathbb{N}$. In particular,

$$(X_k)_h(0,1) = X_{k+h}(0,1) \tag{8.13}$$

for every $k, h \in \mathbb{N}$.

We now turn our attention to the special situation when I satisfies, in addition, condition (5.12). In this case, most of the results take a simpler form. We start with a result concerned with the equivalence of two couples of functionals under (5.12).

Proposition 8.6. *Let $I : [0,1] \to [0,\infty)$ be a non-decreasing function satisfying (5.12) and let $\|\cdot\|_{X(0,1)}$ be any rearrangement-invariant function norm. Then the following assertions hold.*

(i) *For every $j \in \mathbb{N}$, and $f \in \mathcal{M}_+(0,1)$,*

$$\left\| \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{j-1} ds \right\|_{X(0,1)} \approx \left\| \int_t^1 f(s) \frac{s^{j-1}}{I(s)^j}\, ds \right\|_{X(0,1)}, \tag{8.14}$$

up to multiplicative constants independent of $\|\cdot\|_{X(0,1)}$ and f.

(ii) *For every $j \in \mathbb{N}$, and $f \in \mathcal{M}_+(0,1)$,*

$$\left\| \frac{1}{I(s)} \int_0^s f(t) \left(\int_t^s \frac{dr}{I(r)} \right)^{j-1} dt \right\|_{X(0,1)} \approx \left\| \frac{s^{j-1}}{I(s)^j} \int_0^s f(t)dt \right\|_{X(0,1)},$$

up to multiplicative constants independent of $\|\cdot\|_{X(0,1)}$ and f.

Proof. We first note that, owing to the monotonicity of I, we have, for every $j \in \mathbb{N}$,

$$\left(\frac{s}{I(s)} \right)^{j-1} = \frac{2^{j-1}}{I(s)^{j-1}} \left(\int_{\frac{s}{2}}^s dr \right)^{j-1} \le 2^{j-1} \left(\int_{\frac{s}{2}}^s \frac{dr}{I(r)} \right)^{j-1} \quad \text{for } s \in (0,1).$$

Thus,

$$\begin{aligned} \int_{2t}^1 \frac{f(s)}{I(s)} \left(\frac{s}{I(s)} \right)^{j-1} ds &\le 2^{j-1} \int_{2t}^1 \frac{f(s)}{I(s)} \left(\int_{\frac{s}{2}}^s \frac{dr}{I(r)} \right)^{j-1} ds \\ &\le 2^{j-1} \int_{2t}^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{j-1} ds \end{aligned}$$

$$\leq 2^{j-1} \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{j-1} ds \quad \text{for } t \in \left(0, \frac{1}{2}\right].$$

Hence, the right-hand side of (8.14) does not exceed a constant times its left-hand side, owing to the boundedness of the dilation operator in rearrangement-invariant spaces. Note that this inequality holds even without assumption (5.12). On the other hand, (5.12) implies

$$\begin{aligned} \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{j-1} ds &\leq \int_t^1 \frac{f(s)}{I(s)} \left(\int_0^s \frac{dr}{I(r)} \right)^{j-1} ds \\ &\leq C^{j-1} \int_t^1 \frac{f(s)}{I(s)} \left(\frac{s}{I(s)} \right)^{j-1} ds \quad \text{for } t \in (0,1), \end{aligned}$$

hence the converse inequality in (8.14) follows. This proves (i).

The proof of (ii) is similar. $\square$

Given $j \in \mathbb{N}$ and a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, let $\|\cdot\|_{(X^\sharp_{j,I})'(0,1)}$ be the functional defined as in (5.16).

Remark 8.7. It follows from Proposition 8.6 and its proof that for every rearrangement-invariant norm $\|\cdot\|_{X(0,1)}$ and every $j \in \mathbb{N}$, we have

$$\left(X^\sharp_{j,I}\right)'(0,1) \to X'_{j,I}(0,1),$$

and if moreover (5.12) is satisfied, then, in fact,

$$\left(X^\sharp_{j,I}\right)'(0,1) = X'_{j,I}(0,1).$$

This observation has a straightforward consequence.

Proposition 8.8. *Let $I : [0,1] \to [0,\infty)$ be a non-decreasing function satisfying (5.12) and let $\|\cdot\|_{X(0,1)}$ be any rearrangement-invariant function norm. Then*

$$X_{j,I}(0,1) = X^\sharp_{j,I}(0,1),$$

up to equivalent norms.

The following result is a counterpart of Proposition 8.3 under (5.12). It follows from Proposition 8.3, with $j = 1$ and I replaced with the function $(0,1) \ni t \mapsto \frac{I(t)^j}{t^{j-1}}$, which obviously satisfies (5.2).

Proposition 8.9. *Let $I : [0,1] \to [0,\infty)$ be a non-decreasing function satisfying (5.12). Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant norm and let $j \in \mathbb{N}$. Then the functional $\|\cdot\|_{(X^\sharp_{j,I})'(0,1)}$ defined as in (5.16) is a rearrangement-invariant function norm. Moreover,*

$$H^j_I : X(0,1) \to X^\sharp_{j,I}(0,1),$$

and $X^\sharp_{j,I}(0,1)$ is optimal in (8.9) among all rearrangement-invariant spaces.

9. Proofs of the main results

Here we are concerned with the proof of the results of Section 5. In what follows, R^m_I denotes the operator defined as in (8.4).

Lemma 9.1. *Let $I : [0,1] \to [0,\infty)$ be a non-decreasing function fulfilling (5.2), and let $m \in \mathbb{N} \cup \{0\}$. Then, for every $f \in \mathcal{M}_+(0,1)$,*

$$R^m_I f^*(t) \le 2^m R^m_I f^*(s) \quad \text{if } 0 < \frac{t}{2} \le s \le t \le 1. \tag{9.1}$$

Consequently, for every $f \in \mathcal{M}_+(0,1)$,

$$(d-c) R^m_I f^*(d) \le 2^{m+1} \int_c^d R^m_I f^*(s)\, ds \quad \text{if } 0 \le c < d \le 1. \tag{9.2}$$

Proof. We prove inequality (9.1) by induction. Fix any $f \in \mathcal{M}_+(0,1)$. If $m = 0$, then (9.1) is satisfied thanks to the monotonicity of f^*. Next, let $m \ge 1$, and assume that (9.1) is fulfilled with m replaced with $m-1$. If $0 < \frac{t}{2} \le s \le t \le 1$, then

$$\begin{aligned} R^m_I f^*(t) &= \frac{1}{I(t)} \int_0^t R^{m-1}_I f^*(r)\, dr \le \frac{2^{m-1}}{I(s)} \int_0^t R^{m-1}_I f^*\left(\frac{r}{2}\right) dr \\ &= \frac{2^m}{I(s)} \int_0^{\frac{t}{2}} R^{m-1}_I f^*(r)\, dr \\ &\le \frac{2^m}{I(s)} \int_0^s R^{m-1}_I f^*(r)\, dr = 2^m R^m_I f^*(s), \end{aligned}$$

where the first inequality holds according to the induction assumption and to the fact that I is non-decreasing on $[0,1]$. Inequality (9.1) follows.

Now, let $0 \le c < d \le 1$, $m \in \mathbb{N} \cup \{0\}$ and $f \in \mathcal{M}_+(0,1)$. Thanks to (9.1),

$$\int_c^d R_I^m f^*(d)\,ds = 2\int_{\frac{c+d}{2}}^d R_I^m f^*(d)\,ds \le 2^{m+1}\int_{\frac{c+d}{2}}^d R_I^m f^*(s)\,ds \le 2^{m+1}\int_c^d R_I^m f^*(s)\,ds.$$

This proves (9.2). $\square$

Given $m \in \mathbb{N}$ and a non-decreasing function $I : [0,1] \to [0,\infty)$ fulfilling (5.2), we define the operator G_I^m at every $f \in \mathcal{M}_+(0,1)$ by

$$G_I^m f(t) = \sup_{t \le s \le 1} R_I^m f^*(s) \quad \text{for } t \in (0,1). \tag{9.3}$$

When $m = 1$ we simply denote G_I^1 by G_I. Note that, trivially, $R_I^m f^* \le G_I^m f$ for every $f \in \mathcal{M}_+(0,1)$. Moreover, $G_I^m f$ is a non-increasing function, and hence $(R_I^m f^*)^* \le G_I^m f$ as well.

The following lemma tells us that the operator G_I^m does not essentially change if I is replaced with its left-continuous representative.

Lemma 9.2. *Let $m \in \mathbb{N}$, let $I : [0,1] \to [0,\infty)$ be a non-decreasing function fulfilling* (5.2)*, and let $I_0 : [0,1] \to [0,\infty)$ be the left-continuous function which agrees with I a.e. in $[0,1]$. Then, for every $f \in \mathcal{M}_+(0,1)$,*

$$G_I^m f = G_{I_0}^m f$$

up to a countable subset of $(0,1)$.

Proof. Define $M = \{t \in (0,1) : I(t) \ne I_0(t)\}$. The set M is at most countable. We shall prove that, for every $g \in \mathcal{M}_+(0,1)$,

$$\sup_{t \le s \le 1} \frac{1}{I(s)} \int_0^s g(r)\,dr = \sup_{t \le s \le 1} \frac{1}{I_0(s)} \int_0^s g(r)\,dr \quad \text{for } t \in (0,1) \setminus M. \tag{9.4}$$

The conclusion will then follow by applying (9.4) to the function $g = R_{I_0}^{m-1} f^*$, and by the fact that $\frac{1}{I(s)} \int_0^s (R_I^{m-1} f^*)(r)\,dr = \frac{1}{I(s)} \int_0^s (R_{I_0}^{m-1} f^*)(r)\,dr$ for $s \in (0,1]$. Fix $g \in \mathcal{M}_+(0,1)$ and $t \in (0,1)$. Given $s \in (t,1]$, we have that

$$\begin{aligned}\frac{1}{I(s)} \int_0^s g(r)\,dr &\le \left(\lim_{\tau \to s_-} \frac{1}{I(\tau)}\right) \int_0^s g(r)\,dr \\ &= \lim_{\tau \to s_-} \frac{1}{I(\tau)} \int_0^\tau g(r)\,dr \le \sup_{t < \tau \le 1} \frac{1}{I(\tau)} \int_0^\tau g(r)\,dr.\end{aligned}$$

On taking the supremum over all $s \in (t,1]$, we get that

$$\sup_{t<s\leq 1} \frac{1}{I(s)} \int_0^s g(r)\,dr \leq \sup_{t<s\leq 1} \left(\lim_{\tau\to s_-} \frac{1}{I(\tau)} \right) \int_0^s g(r)\,dr \leq \sup_{t<\tau\leq 1} \frac{1}{I(\tau)} \int_0^\tau g(r)\,dr.$$

Hence, since $I_0(s) = \lim_{\tau\to s_-} I(\tau)$ for $s \in (0,1]$,

$$\sup_{t<s\leq 1} \frac{1}{I(s)} \int_0^s g(r)\,dr = \sup_{t<s\leq 1} \frac{1}{I_0(s)} \int_0^s g(r)\,dr \quad \text{for } t \in (0,1).$$

This yields (9.4). □

Proposition 9.3. *Let $m \in \mathbb{N}$, let $I : [0,1] \to [0,\infty)$ be a left-continuous non-decreasing function fulfilling (5.2), and let $f \in \mathcal{M}_+(0,1)$. Define*

$$E = \{t \in (0,1) : R_I^m f^*(t) < G_I^m f(t)\}. \tag{9.5}$$

Then E is an open subset of $(0,1)$. Hence, there exists an at most countable collection $\{(c_k,d_k)\}_{k\in S}$ of pairwise disjoint open intervals in $(0,1)$ such that

$$E = \bigcup_{k\in S} (c_k,d_k). \tag{9.6}$$

Moreover,

$$G_I^m f(t) = R_I^m f^*(t) \quad \textit{if } t \in (0,1) \setminus E, \tag{9.7}$$

and

$$G_I^m f(t) = R_I^m f^*(d_k) \quad \textit{if } t \in (c_k,d_k) \textit{ for some } k \in S. \tag{9.8}$$

Proof. Fix $t \in (0,1)$. If $G_I^m f(t) = \infty$, then both functions $G_I^m f$ and $R_I^m f^*$ are identically equal to ∞, and hence there is nothing to prove. Assume that $G_I^m f(t) < \infty$. Then we claim that $\sup_{t\leq s\leq 1} R_I^m f^*(s)$ is attained. This follows from the fact that the function $R_I^m f^*(s)$ is upper-semicontinuous, since $I(s)R_I^m f^*(s)$ is continuous, and $\frac{1}{I(s)}$ is upper-semicontinuous. Notice that this latter property holds since I is left-continuous and non-decreasing, and hence lower-semicontinuous.

Suppose now that $t \in E$. Then, due to the upper-semicontinuity of $R_I^m f^*$, there exists $\delta > 0$ such that

$$R_I^m f^*(r) < G_I^m f(t) \quad \text{if } r \in (t-\delta, t+\delta). \tag{9.9}$$

Let $c \in [t, 1]$ be such that $R_I^m f^*(c) = G_I^m f(t)$. Then, thanks to (9.9), $c \in [t + \delta, 1]$. It easily follows that $G_I^m f(t) = G_I^m f(r)$ for every $r \in (t - \delta, t + \delta)$, a piece of information that, combined with (9.9), yields $r \in E$. This shows that E is an open set. Assertion (9.7) is trivial and (9.8) is an easy consequence of the definition of $G_I^m f$. □

Proposition 9.4. *Let $m \in \mathbb{N}$, let $I : [0, 1] \to [0, \infty)$ be a left-continuous non-decreasing function fulfilling (5.2), and let $f \in \mathcal{M}_+(0, 1)$. Then*

$$G_I^m G_I f \approx G_I^{m+1} f, \tag{9.10}$$

up to multiplicative constants depending on m.

Proof. Fix any $f \in \mathcal{M}_+(0, 1)$. Since $R_I f^* \le G_I f$, for every $m \in \mathbb{N}$

$$\begin{aligned} G_I^{m+1} f(t) &= \sup_{t \le s \le 1} R_I^m R_I f^*(s) \\ &\le \sup_{t \le s \le 1} R_I^m G_I f^*(s) = G_I^m G_I f(t) \quad \text{for } t \in (0, 1). \end{aligned} \tag{9.11}$$

This shows that the right-hand side of (9.10) does not exceed the left-hand side. To show a converse inequality, consider the set E defined as in (9.5), with $m = 1$. By Proposition 9.3, the set E is open. Let $\{(c_k, d_k)\}_{k \in S}$ be open intervals as in (9.6). If $t \in (c_k, d_k)$ for some $k \in S$, then, by (9.8) with $m = 1$,

$$\frac{d_k}{I(d_k)} f^{**}(d_k) = R_I f^*(d_k) = G_I f(t) \ge R_I f^*(t) \ge \frac{t}{I(t)} f^{**}(d_k). \tag{9.12}$$

Observe that $f^{**}(d_k) > 0$. Indeed, if $f^{**}(d_k) = 0$, then $R_I f^*(t) = R_I f^*(d_k) = G_I f(t) = 0$, and hence $t \notin E$, a contradiction. Thus, we obtain from (9.12)

$$\frac{d_k}{I(d_k)} \ge \frac{t}{I(t)} \quad \text{for } t \in (c_k, d_k). \tag{9.13}$$

We shall now prove by induction that, given $m \in \mathbb{N} \cup \{0\}$, there exists a constant $C = C(m)$ such that

$$R_I^m G_I f(t) \le C \left(R_I^{m+1} f^*(t) + \sum_{k \in S} \chi_{(c_k, d_k)}(t) R_I^{m+1} f^*(d_k) \right) \quad \text{for } t \in (0, 1). \tag{9.14}$$

Let $m = 0$. Then (9.14) holds with $C = 1$, by (9.7) and (9.8) (with $m = 1$). Next, suppose that (9.14) holds for some $m \in \mathbb{N} \cup \{0\}$. Fix any $t \in (0, 1)$. Then

$$
\begin{aligned}
R_I^{m+1} G_I f(t) &= \frac{1}{I(t)} \int_0^t R_I^m G_I f(r)\,dr \le \frac{C}{I(t)} \int_0^t R_I^{m+1} f^*(r)\,dr \\
&\quad + \frac{C}{I(t)} \sum_{\{\ell\in S: d_\ell \le t\}} \int_{c_\ell}^{d_\ell} R_I^{m+1} f^*(d_\ell)\,dr \\
&\quad + \frac{C}{I(t)} \sum_{k\in S} \chi_{(c_k,d_k)}(t) \int_{c_k}^{t} R_I^{m+1} f^*(d_k)\,dr \\
&\le C R_I^{m+2} f^*(t) + \frac{2^{m+2} C}{I(t)} \sum_{\{\ell\in S: d_\ell \le t\}} \int_{c_\ell}^{d_\ell} R_I^{m+1} f^*(r)\,dr \\
&\quad + C \frac{t}{I(t)} \sum_{k\in S} \chi_{(c_k,d_k)}(t) R_I^{m+1} f^*(d_k) \quad \text{(by (9.2))} \\
&\le C R_I^{m+2} f^*(t) + \frac{2^{m+2} C}{I(t)} \int_0^t R_I^{m+1} f^*(r)\,dr \\
&\quad + C \sum_{k\in S} \chi_{(c_k,d_k)}(t) \frac{d_k}{I(d_k)} R_I^{m+1} f^*(d_k) \quad \text{(by (9.13))} \\
&\le \big(C + 2^{m+2} C\big) R_I^{m+2} f^*(t) \\
&\quad + C 2^{m+2} \sum_{k\in S} \chi_{(c_k,d_k)}(t) \frac{1}{I(d_k)} \int_0^{d_k} R_I^{m+1} f^*(r)\,dr \quad \text{(by (9.2))} \\
&= \big(C + 2^{m+2} C\big) R_I^{m+2} f^*(t) + C 2^{m+2} \sum_{k\in S} \chi_{(c_k,d_k)}(t) R_I^{m+2} f^*(d_k) \\
&\le C' \bigg(R_I^{m+2} f^*(t) + \sum_{k\in S} \chi_{(c_k,d_k)}(t) R_I^{m+2} f^*(d_k) \bigg),
\end{aligned}
$$

where $C' = C + 2^{m+2} C$. This proves (9.14).

Owing to (9.14), for every $m \in \mathbb{N}$ we have that

$$
G_I^m G_I f(t) = \sup_{t \le s \le 1} R_I^m G_I f(s) \le 2C G_I^{m+1} f(t) \quad \text{for } t \in (0,1).
$$

Combining this inequality with (9.11) yields (9.10). $\square$

Theorem 9.5. *Let $I : [0,1] \to [0,\infty)$ be a non-decreasing function satisfying (5.2) and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Let $m \in \mathbb{N} \cup \{0\}$. Then*

$$\|R_I^{m+1}f^*\|_{X'(0,1)} \approx \|R_I^m\big((R_If^*)^*\big)\|_{X'(0,1)} \approx \|G_I^{m+1}f\|_{X'(0,1)} \approx \|R_I^{m+1}f^*\|_{X'_d(0,1)} \tag{9.15}$$

for every $f \in \mathcal{M}_+(0,1)$, up to multiplicative constants depending on m.

Proof. We may assume, without loss of generality, that I is left-continuous. Indeed, equation (9.15) is not affected by a replacement of I with its left-continuous representative, since the latter can differ from I at most on a countable subset of $[0,1]$, and since Lemma 9.2 holds.

Fix any $f \in \mathcal{M}_+(0,1)$, and let $m \geq 1$. By (8.5) and Proposition 9.4, there exists a constant $C = C(m)$ such that

$$R_I^{m+1}f^*(t) \leq R_I^m\big((R_If^*)^*\big)(t) \leq R_I^m(G_If)(t) \leq G_I^mG_If(t) \leq CG_I^{m+1}f(t) \quad \text{for } t \in (0,1).$$

Hence,

$$\|R_I^{m+1}f^*\|_{X'(0,1)} \leq \|R_I^m\big((R_If^*)^*\big)\|_{X'(0,1)} \leq C\|G_I^{m+1}f\|_{X'(0,1)}. \tag{9.16}$$

Observe that (9.16) trivially holds also when $m = 0$.

Let E be defined as in (9.5), with m replaced with $m+1$, and let $\{(c_k, d_k)\}_{k\in S}$ be as in (9.6). For every $g \in X(0,1)$, define

$$A(g) = \chi_{(0,1)\setminus E}\,g^* + \sum_{k\in S} \chi_{(c_k,d_k)} \frac{1}{d_k - c_k} \int_{c_k}^{d_k} g^*(t)\,dt. \tag{9.17}$$

Then $A(g)$ is non-increasing on $(0,1)$. Moreover, if $\|g\|_{X(0,1)} \leq 1$, then by [7, Theorem 4.8, Chapter 2],

$$\|A(g)\|_{X(0,1)} \leq \|g^*\|_{X(0,1)} = \|g\|_{X(0,1)} \leq 1. \tag{9.18}$$

Therefore,

$$\begin{aligned}
\int_0^1 g^*(t)G_I^{m+1}f(t)\,dt &= \int_{(0,1)\setminus E} g^*(t)R_I^{m+1}f^*(t)\,dt + \sum_{k\in S}\int_{c_k}^{d_k} g^*(t)R_I^{m+1}f^*(d_k)\,dt \\
&= \int_{(0,1)\setminus E} g^*(t)R_I^{m+1}f^*(t)\,dt \\
&\quad + \sum_{k\in S} \frac{1}{d_k - c_k}\left(\int_{c_k}^{d_k} g^*(t)\,dt\right)(d_k - c_k)R_I^{m+1}f^*(d_k)
\end{aligned}$$

$$
\begin{aligned}
&\le \int\limits_{(0,1)\setminus E} A(g)(t) R_I^{m+1} f^*(t)\, dt \\
&\quad + 2^{m+2} \sum_{k\in S} \int\limits_{c_k}^{d_k} A(g)(t) R_I^{m+1} f^*(t)\, dt \quad \text{(by (9.2))} \\
&\le 2^{m+2} \int\limits_0^1 A(g)(t) R_I^{m+1} f^*(t)\, dt \\
&\le 2^{m+2} \sup_{\|h\|_{X(0,1)}\le 1} \int\limits_0^1 h^*(t) R_I^{m+1} f^*(t)\, dt \quad \text{(by (9.18))} \\
&= 2^{m+2} \big\| R_I^{m+1} f^* \big\|_{X'_d(0,1)}.
\end{aligned}
$$

On taking the supremum over all g from the unit ball of $X(0,1)$, we get

$$
\big\| G_I^{m+1} f \big\|_{X'(0,1)} = \big\| G_I^{m+1} f \big\|_{X'_d(0,1)} \le 2^{m+2} \big\| R_I^{m+1} f^* \big\|_{X'_d(0,1)}. \tag{9.19}
$$

On the other hand, by the very definition of $\|\cdot\|_{X'_d(0,1)}$,

$$
\big\| R_I^{m+1} f^* \big\|_{X'_d(0,1)} \le \big\| R_I^{m+1} f^* \big\|_{X'(0,1)}. \tag{9.20}
$$

Eq. (9.15) follows from (9.16), (9.19) and (9.20). $\square$

Corollary 9.6. *Let $I : [0,1] \to [0,\infty)$ be a non-decreasing function satisfying (5.2), and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Let $m \in \mathbb{N}$. Then*

$$
(X_{m,I})_1(0,1) = X_{m+1,I}(0,1) \tag{9.21}
$$

(up to equivalent norms).

Proof. By (8.10) and (8.8), if $f \in \mathcal{M}_+(0,1)$, then

$$
\|f\|_{((X_{m,I})_1)'(0,1)} = \big\| R_I f^* \big\|_{X'_{m,I}(0,1)} = (m-1)! \big\| R_I^m \big((R_I f^*)^* \big) \big\|_{X'(0,1)},
$$

and

$$
\|f\|_{X'_{m+1,I}(0,1)} = m! \big\| R_I^{m+1} f^* \big\|_{X'(0,1)}.
$$

Hence, it follows from Theorem 9.5 that

$$\|f\|_{((X_{m,I})_1)'(0,1)} \approx \|f\|_{X'_{m+1,I}(0,1)}.$$

By (3.14), this establishes (9.21). □

Theorem 9.7. *Let $I : [0,1] \to [0,\infty)$ be a non-decreasing function satisfying (5.2) and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then, for every $m \in \mathbb{N}$,*

$$X_{m,I}(0,1) = X_m(0,1). \tag{9.22}$$

Proof. We argue by induction. As noted in (8.11), we have $X_1(0,1) = X_{1,I}(0,1)$. Assume now that (9.22) holds for some $m \in \mathbb{N}$. By (8.13), the induction assumption, and (9.21),

$$X_{m+1}(0,1) = (X_m)_1(0,1) = (X_{m,I})_1(0,1) = X_{m+1,I}(0,1).$$

The conclusion follows. □

One consequence of Theorem 9.5, specifically of the equivalence of the leftmost and the rightmost side of (9.15), is the following feature of inequality (5.3), which was already mentioned in Remark 5.2.

Corollary 9.8. *Assume that (Ω, ν) fulfills (5.1) for some non-decreasing function I satisfying (5.2). Let $m \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following two assertions are equivalent:*

(i) *There exists a constant C_1 such that inequality (5.3) holds for every nonnegative $f \in X(0,1)$.*

(ii) *There exists a constant C_1' such that inequality (5.3) holds for every nonnegative non-increasing $f \in X(0,1)$.*

Proof. The fact that (i) implies (ii) is trivial. Conversely, assume that (ii) holds. Fix $f \in \mathcal{M}_+(0,1)$. Equation (8.6) with $j = m$ reads

$$\int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds = (m-1)! H_I^m f(t) \quad \text{for } t \in (0,1). \tag{9.23}$$

Now, the function $H_I^m f$ is non-increasing on $(0,1)$. Therefore, it follows from (3.10) and the Hardy–Littlewood inequality (3.7) that

$$\|H_I^m f\|_{Y(0,1)} = \sup_{\|g\|_{Y'(0,1)} \le 1} \int_0^1 g^*(t) H_I^m f(t)\, dt.$$

Consequently, by Fubini's theorem, we have

$$\left\|H_I^m f\right\|_{Y(0,1)} = \sup_{\|g\|_{Y'(0,1)} \le 1} \int_0^1 f(t) R_I^m g^*(t)\, dt. \tag{9.24}$$

Owing to (9.23) and to the rearrangement-invariance of the norm $\|\cdot\|_{X(0,1)}$, assertion (ii) tells us that

$$C_1' \ge (m-1)! \sup_{\|f\|_{X(0,1)} \le 1} \left\|H_I^m f^*\right\|_{Y(0,1)}.$$

Hence, on applying (9.24) with f replaced with f^*, interchanging the suprema and recalling the definition of the norm $\|\cdot\|_{X_d'(0,1)}$, we get

$$\begin{aligned} C_1' &\ge (m-1)! \sup_{\|f\|_{X(0,1)} \le 1} \sup_{\|g\|_{Y'(0,1)} \le 1} \int_0^1 f^*(t) R_I^m g^*(t)\, dt \\ &= (m-1)! \sup_{\|g\|_{Y'(0,1)} \le 1} \sup_{\|f\|_{X(0,1)} \le 1} \int_0^1 f^*(t) R_I^m g^*(t)\, dt \\ &= (m-1)! \sup_{\|g\|_{Y'(0,1)} \le 1} \left\|R_I^m g^*\right\|_{X_d'(0,1)}. \end{aligned}$$

It follows from the equivalence of the first and the last term in (9.15) that there exists a constant C such that

$$\left\|R_I^m g^*\right\|_{X'(0,1)} \le C \left\|R_I^m g^*\right\|_{X_d'(0,1)}.$$

Therefore,

$$C_1' C \ge (m-1)! \sup_{\|g\|_{Y'(0,1)} \le 1} \left\|R_I^m g^*\right\|_{X'(0,1)},$$

namely, by the definition of the norm $\|\cdot\|_{X'(0,1)}$,

$$C_1' C \ge (m-1)! \sup_{\|g\|_{Y'(0,1)} \le 1} \sup_{\|f\|_{X(0,1)} \le 1} \int_0^1 f(t) R_I^m g^*(t)\, dt.$$

Interchanging suprema again and using Fubini's theorem and (9.24) yields

$$\begin{aligned} C_1' C &\ge (m-1)! \sup_{\|f\|_{X(0,1)} \le 1} \sup_{\|g\|_{Y'(0,1)} \le 1} \int_0^1 g^*(t) H_I^m f(t)\, dt \\ &= (m-1)! \sup_{\|f\|_{X(0,1)} \le 1} \left\|H_I^m f\right\|_{Y(0,1)}. \end{aligned}$$

Hence, inequality (5.3), or equivalently assertion (i), follows. □

Proof of Theorem 5.1. As observed in Section 5, the case when $m = 1$ is already well-known, and is in fact the point of departure of our approach. We thus focus on the case when $m \geq 2$. On applying Proposition 8.3 with $j = 1$, we get that

$$\left\| \int_t^1 \frac{f(s)}{I(s)}\, ds \right\|_{X_{1,I}(0,1)} \leq \|f\|_{X(0,1)}$$

for every $f \in \mathcal{M}_+(0,1)$. Thus, (5.3) holds with $m = 1$ and $Y(0,1) = X_{1,I}(0,1)$. Hence, by the result for $m = 1$,

$$V^1 X(\Omega, \nu) \to X_1(\Omega, \nu). \tag{9.25}$$

Note that here we have also made use of (8.11). By embedding (9.25) applied to each of the spaces $X_j(\Omega, \nu)$, for $j = 0, \dots, m-1$, we get

$$V^1 X_j(\Omega, \nu) \to X_{j+1}(\Omega, \nu),$$

whence

$$V^m X(\Omega, \nu) \to V^{m-1} X_1(\Omega, \nu) \to V^{m-2} X_2(\Omega, \nu) \to \ldots \to V^1 X_{m-1}(\Omega, \nu) \to X_m(\Omega, \nu). \tag{9.26}$$

Inequality (5.3) tells us that

$$H_I^m : X(0,1) \to Y(0,1). \tag{9.27}$$

The optimality of the space $X_{m,I}(0,1)$ as a target in (9.27), proved in Proposition 8.3, entails that

$$X_{m,I}(0,1) \to Y(0,1). \tag{9.28}$$

A combination of (9.26), (9.22) and (9.28) yields

$$V^m X(\Omega, \nu) \to X_m(\Omega, \nu) = X_{m,I}(\Omega, \nu) \to Y(\Omega, \nu), \tag{9.29}$$

and (5.4) follows.

Finally, (5.5) is equivalent to (5.4) by Proposition 4.4. Note that assumption (4.9) of that Proposition is satisfied, owing to (5.2). □

Proof of Theorem 5.4. Embedding (5.7) is a straightforward consequence of (9.29). In turn, Proposition 4.4 yields the Poincaré inequality (5.8).

Assume now that the validity of (5.4) implies (5.3). Let $\|\cdot\|_{Y(0,1)}$ be any rearrangement-invariant function norm such that (5.4) holds. Then, by our assumption, inequality (5.3) holds as well, namely

$$H_I^m : X(0,1) \to Y(0,1). \tag{9.30}$$

Since, by Proposition 8.3, $X_{m,I}(0,1)$ is the optimal rearrangement-invariant target space in (9.30), we necessarily have

$$X_{m,I}(0,1) \to Y(0,1).$$

This implies the optimality of the norm $\|\cdot\|_{X_{m,I}(0,1)}$ in (5.7). □

Proof of Corollary 5.5. Observe that

$$\begin{aligned}\sup_{\substack{f\ge 0\\ \|f\|_{X(0,1)}\le 1}} \left\| \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds \right\|_{L^\infty(0,1)} &= \sup_{\substack{f\ge 0\\ \|f\|_{X(0,1)}\le 1}} \int_0^1 \frac{f(s)}{I(s)} \left(\int_0^s \frac{dr}{I(r)} \right)^{m-1} ds \\ &= \left\| \frac{1}{I(s)} \left(\int_0^s \frac{dr}{I(r)} \right)^{m-1} \right\|_{X'(0,1)}.\end{aligned}$$

Hence, (5.9) is equivalent to (5.3) with $Y(0,1) = L^\infty(0,1)$. The assertion thus follows from Theorem 5.1. □

Proof of Theorem 5.7. By Theorem 9.7 and (8.13),

$$(X_{k,I})_{h,I}(0,1) = (X_k)_h(0,1) = X_{k+h}(0,1) = X_{k+h,I}(0,1),$$

and the claim follows. □

Corollaries 5.8, 5.9 and 5.10 follow from Theorems 5.1, 5.4 and 5.7, respectively (via Propositions 8.6–8.9).

Proof of Proposition 4.5. Owing to (4.23), Eq. (4.25) will follow if we show that

$$V^m X(\Omega,\nu) \to W^m X(\Omega,\nu). \tag{9.31}$$

The isoperimetric function $I_{\Omega,\nu}$ is non-decreasing on $[0, \frac{1}{3}]$ by definition. Let us define the function I by

$$I(s) = \begin{cases} I_{\Omega,\nu}(s) & \text{if } s \in [0, \frac{1}{3}], \\ I_{\Omega,\nu}(\frac{1}{3}) & \text{if } s \in [\frac{1}{3}, 1]. \end{cases} \tag{9.32}$$

Then I is non-decreasing on $[0,1]$. Moreover, by (4.24), it satisfies (5.2). Let H_I be the operator defined as in (8.2), with I given by (9.32). Then,

$$\|H_I f\|_{L^1(0,1)} \leq \int_0^1 f(t)\frac{t}{I(t)}\,dt \leq C\|f\|_{L^1(0,1)},$$

and

$$\|H_I f\|_{L^\infty(0,1)} \leq \int_0^1 \frac{f(t)}{I(t)}\,dt \leq \int_0^1 \frac{ds}{I(s)}\|f\|_{L^\infty(0,1)} \leq C\|f\|_{L^\infty(0,1)},$$

for every $f \in \mathcal{M}_+(0,1)$. Thus, H_I is well defined and bounded both on $L^1(0,1)$ and on $L^\infty(0,1)$. Owing to an interpolation theorem of Calderón [7, Chapter 3, Theorem 2.12], the operator H_I is bounded on every rearrangement-invariant space $X(0,1)$. Hence, from Theorem 5.1 applied with $Y(0,1) = X(0,1)$ and $m = 1$, we obtain that

$$V^1X(\Omega,\nu) \to X(\Omega,\nu). \tag{9.33}$$

Iterating (9.33) tells us that there exists a constant C such that

$$\big\|\nabla^h u\big\|_{X(\Omega,\nu)} \leq C\left(\sum_{k=h}^{m-1}\big\|\nabla^k u\big\|_{L^1(\Omega,\nu)} + \big\|\nabla^m u\big\|_{X(\Omega,\nu)}\right) \tag{9.34}$$

for every $h = 0,\dots,m-1$, and $u \in V^mX(\Omega,\nu)$. Embedding (9.31) is a consequence of (9.34). $\square$

10. Proofs of the Euclidean Sobolev embeddings

In what follows, we shall make use of the fact that the function $I(t) = t^\alpha$ satisfies (5.12) if $\alpha \in (0,1)$.

Proof of Theorem 6.1. If the one-dimensional inequality (6.1) holds, then the Sobolev embedding (6.2) and the Poincaré inequality (6.3) hold as well, owing to (2.4) and to Corollary 5.8. This shows that (i) implies (ii) and (iii). The equivalence of (ii) and (iii) is a consequence of Proposition 4.4.

It thus only remains to prove that (ii) implies (i). Assume that the Sobolev embedding (6.2) holds. If $m \geq n$, then there is nothing to prove, since (6.1) holds for every rearrangement-invariant spaces $X(0,1)$ and $Y(0,1)$. Indeed,

$$\left\|\int_t^1 f(s)s^{-1+\frac{m}{n}}\,ds\right\|_{L^\infty(0,1)} = \int_0^1 f(s)s^{-1+\frac{m}{n}}\,ds \leq \|f\|_{L^1(0,1)}$$

for every nonnegative $f \in L^1(0,1)$, and hence (6.1) follows from (3.16). In the case when $m \le n-1$, the validity of (6.1) was proved in [53, Theorem A]. Note that the proof is given in [53] for Lipschitz domains, and with the space $W^m X(\Omega)$ in the place of $V^m X(\Omega)$. However, by Proposition 4.5, $W^m X(\Omega) = V^m X(\Omega)$ if Ω is a John domain, since (4.24) is fulfilled for any such domain. Moreover, the Lipschitz property of the domain is immaterial, since the proof does not involve any property of the boundary and hence applies, in fact, to any open set Ω. □

Proof of Theorem 6.2. By Theorem 6.1, every John domain has the property that (5.14) implies (5.13). Consequently, the conclusion follows from Corollary 5.9. □

Proof of Theorem 6.3. The assertion is a consequence of Corollary 5.10. □

The following result provides us with model Euclidean domains of revolution in $\mathbb{R}^n$ in the class $\mathcal{J}_\alpha$. It is an easy consequence of a special case of [70, Section 5.3.3]. In the statement, ω_{n-1} denotes the Lebesgue measure of the unit ball in $\mathbb{R}^{n-1}$.

Proposition 10.1. (i) *Given $\alpha \in [\frac{1}{n'}, 1)$, define $\eta_\alpha : [0, \frac{1}{1-\alpha}] \to [0,\infty)$ as*

$$\eta_\alpha(r) = \omega_{n-1}^{-\frac{1}{n-1}} \big(1-(1-\alpha)r\big)^{\frac{\alpha}{(1-\alpha)(n-1)}} \quad \text{for } r \in \left[0, \frac{1}{1-\alpha}\right].$$

Let Ω be the Euclidean domain in $\mathbb{R}^n$ given by

$$\Omega = \left\{ \big(x', x_n\big) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, 0 < x_n < \frac{1}{1-\alpha}, \big|x'\big| < \eta_\alpha(x_n) \right\}.$$

Then $|\Omega| = 1$, and

$$I_\Omega(s) \approx s^\alpha \quad \text{for } s \in \left[0, \frac{1}{2}\right]. \tag{10.1}$$

(ii) *Define $\eta_1 : [0,\infty) \to [0,\infty)$ as*

$$\eta_1(r) = \omega_{n-1}^{-\frac{1}{n-1}} e^{-\frac{r}{n-1}} \quad \text{for } r \ge 0.$$

Let Ω be the Euclidean domain in $\mathbb{R}^n$ given by

$$\Omega = \big\{ \big(x', x_n\big) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > 0, \big|x'\big| < \eta_1(x_n) \big\}.$$

Then $|\Omega| = 1$, and

$$I_\Omega(s) \approx s \quad \text{for } s \in \left[0, \frac{1}{2}\right]. \tag{10.2}$$

Proof of Theorem 6.4. The Sobolev embedding (6.10) and the Poincaré inequality (6.11) are equivalent, owing to Theorem 4.4. If $\alpha \in [\frac{1}{n'}, 1)$, then inequality (6.8) implies (6.10) and (6.11), via Corollary 5.8, whereas if $\alpha = 1$, then inequality (6.9) implies (6.10) and (6.11) via Theorem 5.1.

It thus remains to exhibit a domain $\Omega \in \mathcal{J}_\alpha$ such that the Sobolev embedding (6.10) implies either (6.8), or (6.9), according to whether $\alpha \in [\frac{1}{n'}, 1)$ or $\alpha = 1$.
If $\alpha \in [\frac{1}{n'}, 1)$, let Ω be the set given by Proposition 10.1, Part (i), whereas, if $\alpha = 1$, let Ω be the set given by Proposition 10.1, Part (ii). By either (10.1) or (10.2), one has that $\Omega \in \mathcal{J}_\alpha$. Consequently, embedding (6.10) entails that there exists a constant C such that

$$\|u\|_{Y(\Omega)} \leq C\left(\|\nabla^m u\|_{X(\Omega)} + \sum_{k=0}^{m-1} \|\nabla^k u\|_{L^1(\Omega)}\right) \tag{10.3}$$

for every $u \in V^m X(\Omega)$. Let us fix any nonnegative function $f \in X(0,1)$, and define $u : \Omega \to [0, \infty)$ as

$$u(x) = \int_{M_\alpha(x_n)}^{1} \frac{1}{r_1^\alpha} \int_{r_1}^{1} \frac{1}{r_2^\alpha} \cdots \int_{r_{m-1}}^{1} \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1} \ldots dr_1 \quad \text{for } x \in \Omega,$$

where M_α is given by

$$M_\alpha(r) = \begin{cases} (1-(1-\alpha)r)^{\frac{1}{1-\alpha}} & \text{for } r \in [0, \frac{1}{1-\alpha}], \text{ if } \alpha \in [\frac{1}{n'}, 1), \\ e^{-r} & \text{for } r \in [0, \infty), \text{ if } \alpha = 1. \end{cases}$$

The function u is m-times weakly differentiable in Ω, and, since $-M_\alpha' = (M_\alpha)^\alpha$,

$$|\nabla^k u(x)| = \frac{\partial^k u}{\partial x_n^k}(x) = \int_{M_\alpha(x_n)}^{1} \frac{1}{r_{k+1}^\alpha} \int_{r_{k+1}}^{1} \frac{1}{r_{k+2}^\alpha} \cdots \int_{r_{m-1}}^{1} \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1} \ldots dr_{k+1}$$
$$\text{for a.e. } x \in \Omega,$$

for $k = 1, \ldots, m-1$, and

$$|\nabla^m u(x)| = \frac{\partial^m u}{\partial x_n^m}(x) = f(M_\alpha(x_n)) \quad \text{for a.e. } x \in \Omega.$$

Moreover, on setting $L_\alpha = \frac{1}{1-\alpha}$ if $\alpha \in [\frac{1}{n'}, 1)$, and $L_\alpha = \infty$ if $\alpha = 1$, we have that

$$\left|\left\{(x', x_n) \in \Omega : x_n > t\right\}\right| = \omega_{n-1} \int_t^{L_\alpha} \eta_\alpha(r)^{n-1}\, dr = \int_t^{L_\alpha} M_\alpha(r)^\alpha\, dr$$
$$= \int_t^{L_\alpha} -M'_\alpha(r)\, dr = M_\alpha(t) \quad \text{for } t \in (0, L_\alpha).$$

Thus,

$$u^*(s) = \int_s^1 \frac{1}{r_1^\alpha} \int_{r_1}^1 \frac{1}{r_2^\alpha} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1} \ldots dr_1 \quad \text{for } s \in (0,1), \tag{10.4}$$

$$\left|\nabla^k u\right|^*(s) = \int_s^1 \frac{1}{r_{k+1}^\alpha} \int_{r_{k+1}}^1 \frac{1}{r_{k+2}^\alpha} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1} \ldots dr_{k+1} \quad \text{for } s \in (0,1), \tag{10.5}$$

for $1 \le k \le m-1$, and

$$\left|\nabla^m u\right|^*(s) = f^*(s) \quad \text{for } s \in (0,1). \tag{10.6}$$

Eq. (10.6) ensures that $u \in V^m X(\Omega)$. On the other hand, by (10.3) and (10.4)–(10.6),

$$\left\| \int_s^1 \frac{1}{r_1^\alpha} \int_{r_1}^1 \frac{1}{r_2^\alpha} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1} \ldots dr_1 \right\|_{Y(0,1)}$$
$$\le C\|f\|_{X(0,1)} + C \sum_{k=0}^{m-1} \int_0^1 \int_s^1 \frac{1}{r_{k+1}^\alpha} \int_{r_{k+1}}^1 \frac{1}{r_{k+2}^\alpha} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1} \ldots dr_{k+1}\, ds. \tag{10.7}$$

Subsequent applications of Fubini's theorem tell us that

$$\int_s^1 \frac{1}{r_{k+1}^\alpha} \int_{r_{k+1}}^1 \frac{1}{r_{k+2}^\alpha} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1} \ldots dr_{k+1}$$
$$= \frac{1}{(m-k-1)!} \int_s^1 \frac{1}{r^\alpha} \left(\int_s^r \frac{dt}{t^\alpha} \right)^{m-k-1} f(r) dr \quad \text{for } s \in (0,1). \tag{10.8}$$

By (10.8), (3.16) and (8.9) applied with $I(t) = t^\alpha$, one has that

$$\int_0^1 \int_s^1 \frac{1}{r_{k+1}^\alpha} \int_{r_{k+1}}^1 \frac{1}{r_{k+2}^\alpha} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1} \ldots dr_{k+1}\, ds$$
$$= \frac{1}{(m-k-1)!} \int_0^1 \int_s^1 \frac{1}{r^\alpha} \left(\int_s^r \frac{dt}{t^\alpha} \right)^{m-k-1} f(r) dr\, ds = \left\| H_I^{m-k} f \right\|_{L^1(0,1)}$$
$$\le C \left\| H_I^{m-k} f \right\|_{(L^1)_{m-k,I}(0,1)} \le C' \|f\|_{L^1(0,1)} \le C'' \|f\|_{X(0,1)} \tag{10.9}$$

for $k = 0, \ldots, m-1$, for some constants C, C' and C''.

When $\alpha = 1$, inequality (6.9) follows from (10.7), (10.8) and (10.9). When $\alpha \in [\frac{1}{n'}, 1)$, inequality (6.8) follows from (10.7), (10.8) and (10.9), via Proposition 8.6, Part (i). □

Theorem 10.2. *Let $p, q \in [1, \infty]$ and $\alpha \in \mathbb{R}$ be such that one of the conditions in (3.21) is satisfied. Let $I : [0,1] \to [0, \infty)$ be a non-decreasing function such that $\frac{t}{I(t)}$ is non-decreasing. Then*

$$\left\| R_I f^* \right\|_{(L^{p,q;\alpha})'(0,1)} \approx \left\| t^{\frac{1}{p'} - \frac{1}{q'}} \left(\log \frac{2}{t} \right)^{-\alpha} R_I f^*(t) \right\|_{L^{q'}(0,1)}$$

for every $f \in \mathcal{M}_+(0,1)$, up to multiplicative constants depending on p, q, α.

Proof. Fix $f \in \mathcal{M}_+(0,1)$. By Theorem 9.5, applied with $m = 0$ and $X(0,1) = L^{p,q;\alpha}(0,1)$, and Hölder's inequality, there exists a universal constant C such that

$$\left\| R_I f^*(t) \right\|_{(L^{p,q;\alpha})'(0,1)} \le C \left\| R_I f^*(t) \right\|_{(L^{p,q;\alpha})'_d(0,1)}$$
$$= C \sup_{\|g\|_{L^{p,q;\alpha}(0,1)} \le 1} \int_0^1 g^*(t) R_I f^*(t)\, dt$$
$$= C \sup_{\|g\|_{L^{p,q;\alpha}(0,1)} \le 1} \int_0^1 g^*(t) t^{\frac{1}{p} - \frac{1}{q}} \left(\log \frac{2}{t} \right)^{\alpha} t^{\frac{1}{p'} - \frac{1}{q'}} \left(\log \frac{2}{t} \right)^{-\alpha} R_I f^*(t)\, dt$$
$$\le C \left\| t^{\frac{1}{p'} - \frac{1}{q'}} \left(\log \frac{2}{t} \right)^{-\alpha} R_I f^*(t) \right\|_{L^{q'}(0,1)}.$$

In order to prove the reverse inequality, assume first that either $1 < p < \infty$ or $p = q = 1$ and $\alpha \ge 0$ or $p = q = \infty$ and $\alpha \le 0$. By Theorem 9.5 (with $m = 0$) and (3.23),

$$\left\| t^{\frac{1}{p'} - \frac{1}{q'}} \left(\log \frac{2}{t} \right)^{-\alpha} R_I f^*(t) \right\|_{L^{q'}(0,1)}$$
$$\le \left\| t^{\frac{1}{p'} - \frac{1}{q'}} \left(\log \frac{2}{t} \right)^{-\alpha} \sup_{t \le s \le 1} R_I f^*(t) \right\|_{L^{q'}(0,1)}$$

$$= \|G_I f\|_{L^{p',q';-\alpha}(0,1)} \approx \|G_I f\|_{(L^{p,q;\alpha})'(0,1)}$$
$$\approx \left\|R_I f^*\right\|_{(L^{p,q;\alpha})'(0,1)},$$

where the last but one equivalence holds up to multiplicative constants depending on p, q, α.

It remains to consider the case when $p = \infty$, $q \in [1, \infty)$ and $\alpha + \frac{1}{q} < 0$. We have that

$$\begin{aligned}
&\left\|R_I f^*(t)\right\|_{(L^{p,q;\alpha})'(0,1)} \approx \left\|R_I f^*(t)\right\|_{L^{(1,q';-\alpha-1)}(0,1)} \\
&= \left\| t^{1-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha-1} \frac{1}{t} \int\limits_0^t \left(R_I f^*\right)^*(s)\, ds \right\|_{L^{q'}(0,1)} \\
&\geq \left\| t^{1-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha-1} \frac{1}{t} \int\limits_0^t R_I f^*(s)\, ds \right\|_{L^{q'}(0,1)} \\
&= \left\| t^{-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha-1} \int\limits_0^t f^*(r) \int\limits_r^t \frac{ds}{I(s)}\, dr \right\|_{L^{q'}(0,1)} \\
&\geq \left\| t^{-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha-1} \int\limits_0^{t^2} f^*(r) \int\limits_r^t \frac{ds}{I(s)}\, dr \right\|_{L^{q'}(0,1)} \\
&\geq \left\| t^{-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha-1} \int\limits_0^{t^2} f^*(r)\, dr \int\limits_{t^2}^t \frac{s}{I(s)} \frac{ds}{s} \right\|_{L^{q'}(0,1)} \\
&\geq \left\| t^{2-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha-1} f^{**}(t^2) \frac{t^2}{I(t^2)} \left(\log \frac{1}{t}\right) \right\|_{L^{q'}(0,1)} \\
&\geq \frac{1}{2} \left\| \chi_{(0,\frac{1}{2})}(t) t^{2-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha} \frac{t^2}{I(t^2)} f^{**}(t^2) \right\|_{L^{q'}(0,1)} \\
&\geq C \left\| t^{1-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha} R_I f^*(t) \right\|_{L^{q'}(0,1)},
\end{aligned}$$

for some constant $C = C(\alpha, q)$. The proof is complete. $\square$

Proof of Theorem 6.8. By Corollary 5.9,

$$\|f\|_{((L^p)_{m,\alpha})'(0,1)} = \left\|s^{m(1-\alpha)} f^{**}(s)\right\|_{L^{p'}(0,1)}$$

for $f \in \mathcal{M}_+(0,1)$. If $m(1-\alpha) < 1$ and $p < \frac{1}{m(1-\alpha)}$ and r is given by $\frac{1}{r} = \frac{1}{p} - m(1-\alpha)$ (note that $1 < r < \infty$), then this equality of norms, (3.24) and (3.23) yield

$$\big(L^p\big)_{m,\alpha}(0,1) = \big(L^{(r',p')}\big)'(0,1) = \big(L^{r',p'}\big)'(0,1) = L^{r,p}(0,1).$$

Since $p < r$, we have $L^{r,p}(0,1) \to L^r(0,1)$, and the claim follows. The optimality of $L^r(0,1)$ is a consequence of the fact that $L^q(0,1) \subsetneq L^{r,p}(0,1)$ if $q > r$. If $m(1-\alpha) < 1$ and $p = \frac{1}{m(1-\alpha)}$, then $L^{r'}(0,1) \to L^{(1,p)}(0,1)$ for every $r \in [1,\infty)$, and hence

$$\big(L^p\big)_{m,\alpha}(0,1) = \big(L^{(1,p')}\big)'(0,1) \to L^r(0,1).$$

Finally, if either $m(1-\alpha) \geq 1$, or $m(1-\alpha) < 1$ and $p > \frac{1}{m(1-\alpha)}$, then (5.9) is satisfied. The conclusion thus follows from Corollary 5.5. □

Proof of Theorem 6.9. First, assume that either $m(1-\alpha) \geq 1$, or $m(1-\alpha) < 1$, $p = \frac{1}{m(1-\alpha)}$ and $q = 1$, or $m(1-\alpha) < 1$ and $p > \frac{1}{m(1-\alpha)}$. In each of these cases, condition (5.9) is satisfied with $I(t) = t^\alpha$ and $X(0,1) = L^{p,q}(0,1)$. Hence, by Corollary 5.5, $V^m L^{p,q}(\Omega) \to L^\infty(\Omega)$.

Next, assume that $m(1-\alpha) < 1$, and either $1 \leq p < \frac{1}{m(1-\alpha)}$, or $p = \frac{1}{m(1-\alpha)}$ and $q > 1$. Set $J(t) = t^{-m(1-\alpha)+1}$ for $t \in [0,1]$. Then J is a non-decreasing function such that $\frac{t}{J(t)}$ is non-decreasing on $(0,1)$. Given $f \in \mathcal{M}_+(0,1)$, by Theorem 10.2 (with $\alpha = 0$),

$$\begin{aligned}
\|f\|_{(L^{p,q}_{m,\alpha})'(0,1)} &= \left\| s^{-1+m(1-\alpha)} \int_0^s f^*(r)\,dr \right\|_{(L^{p,q})'(0,1)} = \big\|R_J f^*\big\|_{(L^{p,q})'(0,1)} \\
&\approx \big\|t^{\frac{1}{p'}-\frac{1}{q'}} R_J f^*(t)\big\|_{L^{q'}(0,1)} = \big\|t^{\frac{1}{p'}-\frac{1}{q'}+m(1-\alpha)} f^{**}(t)\big\|_{L^{q'}(0,1)} \\
&= \|f\|_{L^{(r',q')}(0,1)},
\end{aligned}$$

where the equivalence holds up to constants depending on p and q, and r' satisfies $\frac{1}{r'} = \frac{1}{p'} + m(1-\alpha)$. Owing to (3.24), (3.23) and (3.18), $L^{(r',q')}(0,1) = (L^{\frac{p}{1-mp(1-\alpha)},q})'(0,1)$ if $m(1-\alpha) < 1$ and $1 \leq p < \frac{1}{m(1-\alpha)}$, and $L^{(r',q')}(0,1) = (L^{\infty,q;-1})'(0,1)$ if $m(1-\alpha) < 1$, $p = \frac{1}{m(1-\alpha)}$ and $q > 1$. The conclusion follows. □

Proof of Theorem 6.10, sketched. Since (5.1) holds with $I(t) = t$, in the case when $1 \leq p < \infty$ the assertion follows from an analogous argument as in the proof of Theorem 7.12 applied with $\beta = 1$ (see Section 11 below for the proof of Theorem 7.12). If $p = \infty$, Theorem 7.12 has to be combined with an appropriate embedding between Lebesgue spaces. □

Proof of Theorem 6.11, sketched. Since (5.1) holds with $I(t) = t$, the conclusion is a consequence of an analogous argument as in the proof of Theorem 7.12 applied with $\beta = 1$. □

11. Proofs of the Sobolev embeddings in product probability spaces

This final section is devoted to the proof of the results of Section 7.

Lemma 11.1. *Let Φ be as in (7.1). Then:*

(i) *The function L_Φ defined by (7.9) is non-decreasing on $[0,1]$;*

(ii) *The inequality*

$$s\Phi'\left(\Phi^{-1}\left(\log\left(\frac{1}{s}\right)\right)\right) \le L_\Phi(s) \le 2s\Phi'\left(\Phi^{-1}\left(\log\left(\frac{1}{s}\right)\right)\right) \tag{11.1}$$

holds for every $s \in (0, \frac{1}{2}]$;

(iii) *The inequality*

$$\frac{\Phi^{-1}(s)}{2s} \le \frac{1}{\Phi'(\Phi^{-1}(s))} \le \frac{\Phi^{-1}(s)-\Phi^{-1}(t)}{s-t} \le \frac{\Phi^{-1}(s)}{s} \tag{11.2}$$

holds whenever $0 \le t < s < \infty$.

Proof. (i) The convexity of Φ and the concavity of $\sqrt{\Phi}$ imply that

$$0 \le \Phi''(t) \le \frac{\Phi'(t)^2}{2\Phi(t)} \quad \text{for } t > 0.$$

Therefore,

$$L'_\Phi(s) = \Phi'\left(\Phi^{-1}\left(\log\frac{2}{s}\right)\right) - \frac{\Phi''(\Phi^{-1}(\log\frac{2}{s}))}{\Phi'(\Phi^{-1}(\log\frac{2}{s}))} \ge \frac{\Phi''(\Phi^{-1}(\log\frac{2}{s}))}{\Phi'(\Phi^{-1}(\log\frac{2}{s}))}\left(2\log\frac{2}{s}-1\right) > 0$$

for $s \in (0,1)$.

Hence, (i) follows.

(ii) The first inequality in (11.1) trivially holds, since both Φ' and Φ^{-1} are non-decreasing functions. The second inequality follows from (i) and from the fact that

$$2s\Phi'\left(\Phi^{-1}\left(\log\left(\frac{1}{s}\right)\right)\right) = L_\Phi(2s) \quad \text{for } s \in \left(0, \frac{1}{2}\right].$$

(iii) Let $0 \le r_1 < r_2 < \infty$. Owing to the convexity of Φ and to the fact that $\Phi(0) = 0$, we obtain that

$$\frac{\Phi(r_2)}{r_2} \le \frac{\Phi(r_2)-\Phi(r_1)}{r_2-r_1} \le \Phi'(r_2). \tag{11.3}$$

Furthermore, by the concavity of $\sqrt{\Phi}$ and the fact that $\sqrt{\Phi(0)} = 0$,

$$(\sqrt{\Phi})'(r_2) = \frac{\Phi'(r_2)}{2\sqrt{\Phi(r_2)}} \leq \frac{\sqrt{\Phi(r_2)}}{r_2},$$

and, therefore,

$$\Phi'(r_2) \leq \frac{2\Phi(r_2)}{r_2}. \tag{11.4}$$

Let $0 \leq t < s < \infty$. If we set $r_1 = \Phi^{-1}(t)$, $r_2 = \Phi^{-1}(s)$, then $0 \leq r_1 < r_2 < \infty$. Hence, inequalities (11.3) and (11.4) yield

$$\frac{s}{\Phi^{-1}(s)} \leq \frac{s-t}{\Phi^{-1}(s) - \Phi^{-1}(t)} \leq \Phi'\big(\Phi^{-1}(s)\big) \leq \frac{2s}{\Phi^{-1}(s)}.$$

Assertion (11.2) follows. $\square$

Let $m \in \mathbb{N}$. We define the operator P_Φ^m from $\mathcal{M}_+(0,1)$ into $\mathcal{M}_+(0,1)$ by

$$P_\Phi^m f(t) = \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}}\right)^m \int_t^1 \frac{f(s)}{s}\left(\log \frac{s}{t}\right)^{m-1} ds \quad \text{for } t \in (0,1), \tag{11.5}$$

and for $f \in \mathcal{M}_+(0,1)$. Moreover, let $H_{L_\Phi}^m$ be the operator defined as in (8.4) (see also (8.6)), with $I = L_\Phi$, namely

$$H_{L_\Phi}^m f(t) = \frac{1}{(m-1)!} \int_t^1 \frac{f(s)}{L_\Phi(s)} \left(\int_t^s \frac{dr}{L_\Phi(r)}\right)^{m-1} ds \quad \text{for } t \in (0,1), \tag{11.6}$$

and for $f \in \mathcal{M}_+(0,1)$. Observe that, by the change of variables $\tau \mapsto \Phi^{-1}(\log \frac{2}{t})$, we have

$$\begin{aligned} \frac{1}{L_\Phi(r)}\left(\int_s^r \frac{dt}{L_\Phi(t)}\right)^{m-1} &= \frac{1}{r\Phi'(\Phi^{-1}(\log \frac{2}{r}))}\left(\int_s^r \frac{dt}{t\Phi'(\Phi^{-1}(\log \frac{2}{t}))}\right)^{m-1} \\ &= \frac{(\Phi^{-1}(\log \frac{2}{s}) - \Phi^{-1}(\log \frac{2}{r}))^{m-1}}{r\Phi'(\Phi^{-1}(\log \frac{2}{r}))} \quad \text{for } 0 < s \leq r < 1. \end{aligned} \tag{11.7}$$

In particular, this yields

$$H_{L_\Phi}^m f(t) = \frac{1}{(m-1)!} \int_t^1 \frac{f(s)}{s\Phi'(\Phi^{-1}(\log \frac{2}{s}))} \left(\Phi^{-1}\left(\log \frac{2}{t}\right) - \Phi^{-1}\left(\log \frac{2}{s}\right)\right)^{m-1} ds$$
$$\text{for } t \in (0,1), \tag{11.8}$$

and $f \in \mathcal{M}_+(0,1)$.

A connection between the operators P_Φ^m and $H_{L_\Phi}^m$ is described in the following proposition.

Proposition 11.2. *Suppose that Φ is as in (7.1), $m \in \mathbb{N}$ and $f \in \mathcal{M}_+(0,1)$. Then*

$$\frac{1}{2^m(m-1)!} P_\Phi^m f(t) \leq H_{L_\Phi}^m f(t) \quad \textit{for } t \in (0,1). \tag{11.9}$$

Moreover, if f is non-increasing on $(0,1)$, then

$$H_{L_\Phi}^m f(t) \leq \frac{1}{(m-1)!} P_\Phi^m f(t) \quad \textit{for } t \in (0,1). \tag{11.10}$$

Proof. Let $f \in \mathcal{M}_+(0,1)$. Since the function $s \mapsto \frac{1}{\Phi'(\Phi^{-1}(\log \frac{2}{s}))}$ is non-decreasing on $(0,1)$, from the first inequality in (11.2) we obtain that

$$\begin{aligned}
H_{L_\Phi}^m f(t) &= \frac{1}{(m-1)!} \int_t^1 \frac{f(s)}{s\Phi'(\Phi^{-1}(\log \frac{2}{s}))} \left(\int_t^s \frac{dr}{r\Phi'(\Phi^{-1}(\log \frac{2}{r}))} \right)^{m-1} ds \\
&\geq \frac{1}{(m-1)!} \frac{1}{(\Phi'(\Phi^{-1}(\log \frac{2}{t})))^m} \int_t^1 \frac{f(s)}{s} \left(\int_t^s \frac{dr}{r} \right)^{m-1} ds \\
&= \frac{1}{(m-1)!} \frac{1}{(\Phi'(\Phi^{-1}(\log \frac{2}{t})))^m} \int_t^1 \frac{f(s)}{s} \left(\log \frac{s}{t} \right)^{m-1} ds \\
&\geq \frac{1}{(m-1)!} \frac{1}{2^m} \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}} \right)^m \int_t^1 \frac{f(s)}{s} \left(\log \frac{s}{t} \right)^{m-1} ds \\
&= \frac{1}{(m-1)!} \frac{1}{2^m} P_\Phi^m f(t) \quad \text{for } t \in (0,1).
\end{aligned}$$

Now, assume that f is non-increasing on $(0,1)$. In the special case when f is a characteristic function of an open interval, namely, $f = \chi_{(0,b)}$ for some $b \in (0,1]$, Eq. (11.8) tells us that

$$H_{L_\Phi}^m (\chi_{(0,b)})(t) = \frac{1}{m!} \chi_{(0,b)}(t) \left(\Phi^{-1}\left(\log \frac{2}{t}\right) - \Phi^{-1}\left(\log \frac{2}{b}\right) \right)^m$$

and

$$P_\Phi^m (\chi_{(0,b)})(t) = \chi_{(0,b)}(t) \frac{1}{m} \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}} \right)^m \left(\log \frac{2}{t} - \log \frac{2}{b} \right)^m$$

for $t \in (0,1)$. By the last inequality in (11.2),

$$\left(\frac{\Phi^{-1}(\log\frac{2}{t})-\Phi^{-1}(\log\frac{2}{b})}{\log\frac{2}{t}-\log\frac{2}{b}}\right)^m \le \left(\frac{\Phi^{-1}(\log\frac{2}{t})}{\log\frac{2}{t}}\right)^m \quad \text{for } t\in(0,b).$$

Hence,

$$H^m_{L_\Phi}(\chi_{(0,b)}) \le \frac{1}{(m-1)!}P^m_\Phi(\chi_{(0,b)}). \tag{11.11}$$

Assume next that f is a nonnegative non-increasing simple function on $(0,1)$. Then there exist $k\in\mathbb{N}$, nonnegative numbers $a_1,a_2,\dots,a_k\in\mathbb{R}$ and $0<b_1<b_2<\dots<b_k\le 1$ such that $f=\sum_{i=1}^k a_i\chi_{(0,b_i)}$ a.e. on $(0,1)$. Hence, owing to (11.11),

$$\begin{aligned} H^m_{L_\Phi}f(t) &= \sum_{i=1}^k a_i H^m_{L_\Phi}(\chi_{(0,b_i)})(t) \le \frac{1}{(m-1)!}\sum_{i=1}^k a_i P^m_\Phi(\chi_{(0,b_i)})(t) \\ &= \frac{1}{(m-1)!}P^m_\Phi f(t) \quad \text{for } t\in(0,1). \end{aligned}$$

Finally, if $f\in\mathcal{M}_+(0,1)$ is non-increasing on $(0,1)$, then there exists a sequence f_k of nonnegative non-increasing simple functions on $(0,1)$ such that $f_n\uparrow f$. Clearly,

$$H^m_{L_\Phi}f(t) = \lim_{n\to\infty} H^m_{L_\Phi}f_n(t) \le \frac{1}{(m-1)!}\lim_{n\to\infty} P^m_\Phi f_n(t) = \frac{1}{(m-1)!}P^m_\Phi f(t) \quad \text{for } t\in(0,1),$$

whence (11.10) follows. □

Proposition 11.2 has an important consequence.

Proposition 11.3. *Let Φ be as in (7.1), let $m\in\mathbb{N}$ and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then*

$$P^m_\Phi : X(0,1)\to Y(0,1) \quad \textit{if and only if} \quad H^m_{L_\Phi} : X(0,1)\to Y(0,1).$$

Proof. By (11.9), the boundedness of the operator $H^m_{L_\Phi}$ implies the boundedness of P^m_Φ. Conversely, if P^m_Φ is bounded from $X(0,1)$ into $Y(0,1)$ then, in particular, there exists a constant C such that

$$\left\|P^m_\Phi f\right\|_{Y(0,1)} \le C\|f\|_{X(0,1)}$$

for every nonnegative non-increasing function $f\in X(0,1)$. Combining this inequality with (11.10), we obtain that

$$\left\|H^m_{L_\Phi} f\right\|_{Y(0,1)} \le C\|f\|_{X(0,1)}$$

for every nonnegative non-increasing $f \in X(0,1)$. In view of Corollary 9.8, this is equivalent to the boundedness of $H^m_{L_\Phi}$ from $X(0,1)$ into $Y(0,1)$. $\square$

Proof of Theorem 7.1. Properties (ii) and (iii) are equivalent, by Proposition 4.4. Let us show that (i) and (ii) are equivalent as well. First, assume that (i) is satisfied. Owing to Proposition 11.3, there exists a constant C such that

$$\left\| \int_t^1 \frac{f(s)}{L_\Phi(s)} \left(\int_t^s \frac{dr}{L_\Phi(r)} \right)^{m-1} ds \right\|_{Y(0,1)} \le C \|f\|_{X(0,1)}$$

for every nonnegative $f \in X(0,1)$. By Lemma 11.1(i), the function L_Φ is non-decreasing on $[0,1]$. Furthermore, condition (5.2) is clearly satisfied with $I = L_\Phi$. Thanks to these facts and to (7.10), the assumptions of Theorem 5.1 are fulfilled with $(\Omega, \nu) = (\mathbb{R}^n, \mu_{\Phi,n})$ and $I = L_\Phi$. Hence, (ii) follows.

It only remains to prove that (ii) implies (i). Assume that (ii) holds, namely, there exists a constant C, such that

$$\|u\|_{Y(\mathbb{R}^n,\mu_{\Phi,n})} \le C \left(\|\nabla^m u\|_{X(\mathbb{R}^n,\mu_{\Phi,n})} + \sum_{k=0}^{m-1} \|\nabla^k u\|_{L^1(\mathbb{R}^n,\mu_{\Phi,n})} \right) \tag{11.12}$$

for every $u \in V^m X(\mathbb{R}^n, \mu_{\Phi,n})$.

Given any nonnegative function $f \in X(0,1)$ such that $f(s) = 0$ if $s \in (\frac{1}{2}, 1)$, consider the function $u : \mathbb{R}^n \to \mathbb{R}$ defined as

$$u(x) = \int_{H(x_1)}^1 \frac{1}{F_\Phi(r_1)} \int_{r_1}^1 \frac{1}{F_\Phi(r_2)} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{F_\Phi(r_m)} \, dr_m \, dr_{m-1} \dots dr_1 \qquad \text{for } x \in \mathbb{R}^n,$$

where H is given by (7.6). Note that, since $H'(t) = -F_\Phi(H(t))$, then

$$\begin{aligned} |\nabla^k u(x)| &= \frac{\partial^k u}{\partial x_1^k}(x) \\ &= \int_{H(x_1)}^1 \frac{1}{F_\Phi(r_{k+1})} \int_{r_{k+1}}^1 \frac{1}{F_\Phi(r_{k+2})} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{F_\Phi(r_m)} \, dr_m \, dr_{m-1} \dots dr_{k+1} \end{aligned}$$

for a.e. $x \in \mathbb{R}^n$, for $k = 1, \dots, m-1$, and

$$|\nabla^m u(x)| = \frac{\partial^m u}{\partial x_1^m}(x) = f\big(H(x_1)\big) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Thus, by (7.8),

$$\left|\nabla^k u\right|^*(s) = \int_s^1 \frac{1}{F_\Phi(r_{k+1})} \int_{r_{k+1}}^1 \frac{1}{F_\Phi(r_{k+2})} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{F_\Phi(r_m)}\, dr_m\, dr_{m-1} \ldots dr_{k+1} \quad \text{for } s \in (0,1), \tag{11.13}$$

for $k = 0, \ldots, m-1$, and

$$\left|\nabla^m u\right|^*(s) = f^*(s) \quad \text{for } s \in (0,1). \tag{11.14}$$

By (11.14), $u \in V^m X(\mathbb{R}^n, \mu_{\Phi,n})$. From (11.12), (11.13) and (11.14) we thus deduce that

$$\begin{aligned} &\left\| \int_s^1 \frac{1}{F_\Phi(r_1)} \int_{r_1}^1 \frac{1}{F_\Phi(r_2)} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{F_\Phi(r_m)}\, dr_m\, dr_{m-1} \ldots dr_1 \right\|_{Y(0,1)} \\ &\quad \le C\|f\|_{X(0,1)} \\ &\qquad + C \sum_{k=0}^{m-1} \int_0^1 \int_s^1 \frac{1}{F_\Phi(r_{k+1})} \int_{r_{k+1}}^1 \frac{1}{F_\Phi(r_{k+2})} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{F_\Phi(r_m)}\, dr_m\, dr_{m-1} \ldots dr_{k+1}\, ds. \end{aligned} \tag{11.15}$$

Owing to Fubini's Theorem, (7.10) and (11.7),

$$\begin{aligned} &\int_s^1 \frac{1}{F_\Phi(r_1)} \int_{r_1}^1 \frac{1}{F_\Phi(r_2)} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{F_\Phi(r_m)}\, dr_m\, dr_{m-1} \ldots dr_1 \\ &\approx \int_s^1 \frac{f(r)}{F_\Phi(r)} \left(\int_s^r \frac{dt}{F_\Phi(t)} \right)^{m-1} dr \\ &\approx \int_s^1 \frac{f(r)}{L_\Phi(r)} \left(\int_s^r \frac{dt}{L_\Phi(t)} \right)^{m-1} dr \\ &= \int_s^1 f(r) \frac{(\Phi^{-1}(\log \frac{2}{s}) - \Phi^{-1}(\log \frac{2}{r}))^{m-1}}{r\Phi'(\Phi^{-1}(\log \frac{2}{r}))}\, dr \quad \text{for } s \in (0,1). \end{aligned} \tag{11.16}$$

Note that the second equivalence makes use of the fact that f vanishes in $(\frac{1}{2}, 1)$. On the other hand, by (11.16) (with m replaced with $m-k$), (3.16), and (8.9) (with I replaced with L_Φ),

$$\int_0^1 \int_s^1 \frac{1}{F_\Phi(r_{k+1})} \int_{r_{k+1}}^1 \frac{1}{F_\Phi(r_{k+2})} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{F_\Phi(r_m)}\, dr_m\, dr_{m-1} \ldots dr_{k+1}\, ds$$

$$\approx \int_0^1 \int_s^1 \frac{f(r)}{L_\Phi(r)} \left(\int_s^r \frac{dt}{L_\Phi(t)} \right)^{m-k-1} dr\, ds \approx \|H_{L_\Phi}^{m-k} f\|_{L^1(0,1)}$$

$$\leq C \big\| H_{L_\Phi}^{m-k} f \big\|_{(L^1)_{m-k,L_\Phi}(0,1)} \leq C' \|f\|_{L^1(0,1)} \leq C'' \|f\|_{X(0,1)} \tag{11.17}$$

for some constants C, C' and C''. From inequalities (11.15)–(11.17), we deduce that there exists a constant C such that

$$\left\| \int_s^1 f(r) \frac{(\Phi^{-1}(\log \frac{2}{s}) - \Phi^{-1}(\log \frac{2}{r}))^{m-1}}{r\Phi'(\Phi^{-1}(\log \frac{2}{r}))} \, dr \right\|_{Y(0,1)} \leq C\|f\|_{X(0,1)}$$

for every nonnegative function $f \in X(0,1)$ such that $f(s) = 0$ if $s \in (\frac{1}{2}, 1)$. By Proposition 11.2, for each such function f we also have

$$\left\| \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}} \right)^m \int_t^1 \frac{f(s)}{s} \left(\log \frac{s}{t} \right)^{m-1} ds \right\|_{Y(0,1)} \leq 2^m C \|f\|_{X(0,1)}. \tag{11.18}$$

Finally, assume that f is any nonnegative function from $X(0,1)$ (which need not vanish in $(\frac{1}{2}, 1)$). Then, by the boundedness of the dilation operator on $Y(0,1)$, there exists a constant C such that

$$\left\| \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}} \right)^m \int_t^1 \frac{f(s)}{s} \left(\log \frac{s}{t} \right)^{m-1} ds \right\|_{Y(0,1)}$$

$$\leq C \left\| \chi_{(0,\frac{1}{2})}(t) \left(\frac{\Phi^{-1}(\log \frac{1}{t})}{\log \frac{1}{t}} \right)^m \int_{2t}^1 \frac{f(s)}{s} \left(\log \frac{s}{2t} \right)^{m-1} ds \right\|_{Y(0,1)}. \tag{11.19}$$

Furthermore, since

$$\frac{\Phi^{-1}(\log \frac{1}{t})}{\log \frac{1}{t}} \leq \frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{1}{t}} \leq \frac{2\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}} \quad \text{for } t \in \left(0, \frac{1}{2}\right),$$

from inequality (11.18) with f replaced with $\chi_{(0,\frac{1}{2})}(t) f(2t)$, and the boundedness of the dilation operator, we obtain

$$\left\| \chi_{(0,\frac{1}{2})}(t) \left(\frac{\Phi^{-1}(\log \frac{1}{t})}{\log \frac{1}{t}} \right)^m \int_{2t}^1 \frac{f(s)}{s} \left(\log \frac{s}{2t} \right)^{m-1} ds \right\|_{Y(0,1)}$$

$$\leq 2^m \left\| \chi_{(0,\frac{1}{2})}(t) \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}} \right)^m \int_t^{\frac{1}{2}} \frac{f(2s)}{s} \left(\log \frac{s}{t} \right)^{m-1} ds \right\|_{Y(0,1)}$$

$$\leq C' \big\| \chi_{(0,\frac{1}{2})}(t) f(2t) \big\|_{X(0,1)} \leq C'' \|f\|_{X(0,1)} \tag{11.20}$$

for some constants C' and C'' independent of f. Coupling (11.19) with (11.20) yields (7.11). $\square$

Proof of Theorem 7.3. Set $J(s) = s$ for $s \in [0,1]$. Then condition (5.2) is obviously fulfilled with $I = J$. The norm $\|\cdot\|_{\widetilde{X}_{m,J}(0,1)}$ is thus well defined and, moreover, $\|\cdot\|_{\widetilde{X}_m(0,1)} = \|\cdot\|_{X_{m,J}(0,1)}$. Therefore, Proposition 8.3 tells us that $\|\cdot\|_{\widetilde{X}_m(0,1)}$ is a rearrangement-invariant function norm. We shall now verify that $\|\cdot\|_{X_{m,\Phi}(0,1)}$ is a rearrangement-invariant function norm as well. The first two properties in (P1) and properties (P2) and (P3) are straightforward consequences of the corresponding properties for $\|\cdot\|_{\widetilde{X}_m(0,1)}$. To prove the triangle inequality, fix $f, g \in \mathcal{M}_+(0,1)$. By (3.6), $\int_0^s (f+g)^*(r)\,dr \le \int_0^s (f^*(r)+g^*(r))\,dr$ for $s \in (0,1)$. We observe that for each $t \in (0,1)$, the function $s \mapsto \chi_{(0,t)}(s)(\frac{\log\frac{2}{s}}{\Phi^{-1}(\log\frac{2}{s})})^m$ is nonnegative and non-increasing on $(0,1)$. Hardy's lemma therefore yields that

$$\int_0^t \left(\frac{\log\frac{2}{s}}{\Phi^{-1}(\log\frac{2}{s})}\right)^m (f+g)^*(s)\,ds$$
$$\le \int_0^t \left(\left(\frac{\log\frac{2}{s}}{\Phi^{-1}(\log\frac{2}{s})}\right)^m f^*(s) + \left(\frac{\log\frac{2}{s}}{\Phi^{-1}(\log\frac{2}{s})}\right)^m g^*(s)\right) ds$$

for $t \in (0,1)$. The triangle inequality now follows using the Hardy–Littlewood–Pólya principle and the triangle inequality for $\|\cdot\|_{\widetilde{X}_m(0,1)}$.

One has that

$$\exp L^{\frac{1}{m}}(0,1) = \left(L^\infty\right)_m(0,1) \to \widetilde{X}_m(0,1),$$

where the equality is a consequence of Theorem 6.11. Thus, there exists a constant C such that

$$\|1\|_{X_{m,\Phi}(0,1)} = \left\|\left(\frac{\log\frac{2}{s}}{\Phi^{-1}(\log\frac{2}{s})}\right)^m\right\|_{\widetilde{X}_m(0,1)} \le C\left\|\left(\frac{\log\frac{2}{s}}{\Phi^{-1}(\log\frac{2}{s})}\right)^m\right\|_{\exp L^{\frac{1}{m}}(0,1)}$$
$$\approx \left\|\frac{1}{(\Phi^{-1}(\log\frac{2}{s}))^m}\right\|_{L^\infty(0,1)} = \frac{1}{(\Phi^{-1}(\log 2))^m} < \infty.$$

This proves (P4).

Finally, by property (P5) for $\|\cdot\|_{\widetilde{X}_m(0,1)}$, there exists a positive constant C such that for all $f \in \mathcal{M}_+(0,1)$,

$$\|f\|_{X_{m,\Phi}(0,1)} \ge \left(\frac{\log 2}{\Phi^{-1}(\log 2)}\right)^m \|f^*\|_{\widetilde{X}_m(0,1)} \ge \left(\frac{C\log 2}{\Phi^{-1}(\log 2)}\right)^m \int_0^1 f^*(s)\,ds.$$

Therefore, $\|\cdot\|_{X_{m,\Phi}(0,1)}$ satisfies (P5). Since the property (P6) holds trivially, $\|\cdot\|_{X_{m,\Phi}(0,1)}$ is actually a rearrangement-invariant norm.

It follows from the proof of Theorem 7.1 that the assumptions of Theorem 5.4 are fulfilled with $(\Omega,\nu)=(\mathbb{R}^n,\mu_{\Phi,n})$ and $I=L_\Phi$. Therefore, $\|\cdot\|_{X_{m,L_\Phi}(0,1)}$ is the optimal rearrangement-invariant target function norm for $\|\cdot\|_{X(0,1)}$ in the Sobolev embedding (7.12). Thus, the proof will be complete if we show that $X_{m,\Phi}(0,1)=X_{m,L_\Phi}(0,1)$. We have that

$$\begin{aligned}
\|f\|_{X'_{m,L_\Phi}(0,1)} &= (m-1)!\big\|R^m_{L_\Phi}f^*\big\|_{X'(0,1)} \approx \big\|R^m_{L_\Phi}f^*\big\|_{X'_d(0,1)} \quad \text{(by Theorem 9.5)}\\
&= \sup_{\|g\|_{X(0,1)}\le 1}\int_0^1 g^*(t)R^m_{L_\Phi}f^*(t)\,dt\\
&= \sup_{\|g\|_{X(0,1)}\le 1}\int_0^1 f^*(t)H^m_{L_\Phi}g^*(t)\,dt\\
&\approx \sup_{\|g\|_{X(0,1)}\le 1}\int_0^1 f^*(t)P^m_{\Phi}g^*(t)\,dt \quad \text{(by Proposition 11.2)}\\
&\approx \sup_{\|g\|_{X(0,1)}\le 1}\int_0^1 f^*(t)\left(\frac{\Phi^{-1}(\log\frac{2}{t})}{\log\frac{2}{t}}\right)^m H^m_J g^*(t)\,dt\\
&= \sup_{\|g\|_{X(0,1)}\le 1}\int_0^1 g^*(t)R^m_J\left(f^*(s)\left(\frac{\Phi^{-1}(\log\frac{2}{s})}{\log\frac{2}{s}}\right)^m\right)(t)\,dt\\
&= \left\|R^m_J\left(f^*(s)\left(\frac{\Phi^{-1}(\log\frac{2}{s})}{\log\frac{2}{s}}\right)^m\right)\right\|_{X'_d(0,1)} \quad \text{for } f\in L^1(0,1),
\end{aligned} \tag{11.21}$$

up to multiplicative constants depending on m.

We now claim that, given $f\in L^1(0,1)$, there exists a non-decreasing function I on $[0,1]$ fulfilling (5.2) and a function $h\in\mathcal{M}_+(0,1)$ such that

$$f^*(s)\left(\frac{\Phi^{-1}(\log\frac{2}{s})}{\log\frac{2}{s}}\right)^m \approx R_I h^*(s) \quad \text{for } s\in(0,1), \tag{11.22}$$

up to multiplicative constants depending on m. Indeed, let $s_0\in(0,1)$ be chosen in such a way that the function $s\mapsto s(\log\frac{2}{s})^{m+1}$ is non-decreasing on $(0,s_0)$. Then we set

$$I(s)=\frac{1}{f^*(s)} \quad \text{for } s\in(0,1] \quad \text{and} \quad I(0)=0,$$

and

$$h(s)=\begin{cases}\dfrac{(\Phi^{-1}(\log\frac{2}{s}))^{m-1}(\Phi^{-1}(\log\frac{2}{s})-\frac{\log\frac{2}{s}}{\Phi'(\Phi^{-1}(\log\frac{2}{s}))})}{s(\log\frac{2}{s})^{m+1}}, & s\in(0,s_0]\\ \dfrac{(\Phi^{-1}(\log\frac{2}{s}))^{m-1}(\Phi^{-1}(\log\frac{2}{s})-\frac{\log\frac{2}{s}}{\Phi'(\Phi^{-1}(\log\frac{2}{s}))})}{s_0(\log\frac{2}{s_0})^{m+1}}, & s\in(s_0,1).\end{cases}$$

It follows from (11.2) that the function h is non-negative on $(0,1)$. To verify (11.22) we first show that h is non-increasing on $(0,1)$. The function Φ^{-1} is clearly non-decreasing on $(0,\infty)$. Furthermore, we deduce from the convexity of Φ that the function $s\mapsto \Phi^{-1}(s)-\frac{s}{\Phi'(\Phi^{-1}(s))}$ is non-decreasing on $(0,\infty)$. Altogether, this ensures that the function

$$s\mapsto\left(\Phi^{-1}\left(\log\frac{2}{s}\right)\right)^{m-1}\left(\Phi^{-1}\left(\log\frac{2}{s}\right)-\frac{\log\frac{2}{s}}{\Phi'(\Phi^{-1}(\log\frac{2}{s}))}\right)$$

is non-increasing on $(0,1)$. By the definition of s_0, the function

$$s\mapsto\begin{cases}s(\log\frac{2}{s})^{m+1}, & s\in(0,s_0]\\ s_0(\log\frac{2}{s_0})^{m+1}, & s\in(s_0,1)\end{cases}$$

is non-decreasing (and continuous) on $(0,1)$, and therefore, in particular, $h=h^*$.

Consequently, we have

$$\begin{aligned}f^*(s)\left(\frac{\Phi^{-1}(\log\frac{2}{s})}{\log\frac{2}{s}}\right)^m&=\frac{m}{I(s)}\int_0^s\frac{(\Phi^{-1}(\log\frac{2}{r}))^{m-1}(\Phi^{-1}(\log\frac{2}{r})-\frac{\log\frac{2}{r}}{\Phi'(\Phi^{-1}(\log\frac{2}{r}))})}{r(\log\frac{2}{r})^{m+1}}\,dr\\&\approx\frac{1}{I(s)}\int_0^s h^*(r)\,dr=R_Ih^*(s)\quad\text{for } s\in(0,1),\end{aligned}$$

up to multiplicative constants depending on m. This proves (11.22). Furthermore, it can be easily verified that the function I fulfills also the remaining required properties.

Coupling (11.21) with (11.22) entails that

$$\|f\|_{X'_{m,L_\Phi}(0,1)}\approx\left\|R_J^mR_Ih^*\right\|_{X'_d(0,1)},\tag{11.23}$$

up to multiplicative constants depending on m.

Now, the same proof as that of Theorem 9.5 yields that

$$\left\|R_J^mR_Ih^*\right\|_{X'_d(0,1)}\approx\left\|R_J^m\left(R_Ih^*\right)^*\right\|_{X'(0,1)},\tag{11.24}$$

up to multiplicative constants still depending only on m.

On combining (11.23), (11.24) and (11.22), we obtain that for every $f\in L^1(0,1)$,

$$\|f\|_{X'_{m,L_\Phi}(0,1)}\approx\left\|R_J^m\left(f^*(\cdot)\left(\frac{\Phi^{-1}(\log\frac{2}{(\cdot)})}{\log\frac{2}{(\cdot)}}\right)^m\right)^*(t)\right\|_{X'(0,1)}$$

$$\approx \left\| f^*(t)\left(\frac{\varPhi^{-1}(\log\frac{2}{t})}{\log\frac{2}{t}}\right)^m \right\|_{\widetilde{X}'_m(0,1)} \tag{11.25}$$

up to multiplicative constants depending on m. Consequently, by (11.25), we have that, for every $g \in \mathcal{M}_+(0,1)$,

$$\begin{aligned}
\|g\|_{X_{m,L_\varPhi}(0,1)} &= \sup\left\{ \int_0^1 f^*(s)g^*(s)\,ds : \|f\|_{X'_{m,L_\varPhi}(0,1)} \le 1 \right\} \\
&\approx \sup\left\{ \int_0^1 f^*(s)g^*(s)\,ds : \left\| f^*(t)\left(\frac{\varPhi^{-1}(\log\frac{2}{t})}{\log\frac{2}{t}}\right)^m \right\|_{\widetilde{X}'_m(0,1)} \le 1 \right\} \\
&\le \left\| g^*(t)\left(\frac{\log\frac{2}{t}}{\varPhi^{-1}(\log\frac{2}{t})}\right)^m \right\|_{\widetilde{X}_m(0,1)},
\end{aligned}$$

up to multiplicative constants depending on m.

Conversely,

$$\begin{aligned}
&\left\| g^*(t)\left(\frac{\log\frac{2}{t}}{\varPhi^{-1}(\log\frac{2}{t})}\right)^m \right\|_{\widetilde{X}_m(0,1)} \\
&= \sup\left\{ \int_0^1 g^*(t)\left(\frac{\log\frac{2}{t}}{\varPhi^{-1}(\log\frac{2}{t})}\right)^m f^*(t)\,dt : \|f\|_{\widetilde{X}'_m(0,1)} \le 1 \right\} \\
&\approx \sup\left\{ \int_0^1 g^*(t)\left(\frac{\log\frac{2}{t}}{\varPhi^{-1}(\log\frac{2}{t})}\right)^m f^*(t)\,dt : \left\| f^*(t)\left(\frac{\log\frac{2}{t}}{\varPhi^{-1}(\log\frac{2}{t})}\right)^m \right\|_{X'_{m,L_\varPhi}(0,1)} \le 1 \right\} \\
&\le \|g\|_{X_{m,L_\varPhi}(0,1)},
\end{aligned} \tag{11.26}$$

up to multiplicative constants depending on m. Note that the equivalence in (11.26) holds by (11.25) and the fact that the function $t \mapsto (\frac{\log\frac{2}{t}}{\varPhi^{-1}(\log\frac{2}{t})})^m$ is non-increasing. Hence, $X_{m,\varPhi}(0,1) = X_{m,L_\varPhi}(0,1)$. The proof is complete. □

Proof of Proposition 7.5. Since the m-th iteration of the double-star operator $g \mapsto g^{**}$ associates a function g with $\frac{1}{s}\int_0^s (\log\frac{s}{r})^{m-1} g^*(r)\,dr$ for $s \in (0,1)$, we obtain from the boundedness of the double-star operator on $X'(0,1)$ that

$$\|g\|_{\widetilde{X}'_m(0,1)} \approx \|g\|_{X'(0,1)}.$$

Thus, $\widetilde{X}_m(0,1) = X(0,1)$. Consequently, the assertion follows from (7.15). □

Proof of Theorem 7.6. This is a consequence of Theorem 5.7 and of the fact that $X_{m,\varPhi}(0,1) = X_{m,L_\varPhi}(0,1)$. □

Proof of Theorem 7.12. Set $X(0,1) = L^{p,q;\alpha}(0,1)$. We claim that

$$\widetilde{X}_m(0,1) = \begin{cases} L^{p,q;\alpha}(0,1) & \text{if } p < \infty, \\ L^{\infty,q;\alpha-m}(0,1) & \text{if } p = \infty. \end{cases}$$

Indeed, let $p < \infty$ and set $\Phi(t) = t$ for $t \in [0,\infty)$. Then, by Remark 7.2,

$$\widetilde{X}_m(0,1) = X_{m,\Phi}(0,1).$$

By (3.23) and (3.24), the operator $f \mapsto f^{**}$ is bounded on $X'(0,1)$. Therefore, by Proposition 7.5,

$$X_{m,\Phi}(0,1) = X(0,1) = L^{p,q;\alpha}(0,1).$$

Now, let $p = \infty$, and set $I(s) = s$ for $s \in [0,1]$. Then $R_I f^* = f^{**}$, whence, by Theorem 10.2,

$$\|f\|_{(\widetilde{X}_1)'(0,1)} = \|f^{**}\|_{X'(0,1)} \approx \left\| t^{1-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha} f^{**}(t) \right\|_{L^{q'}(0,1)} = \|f\|_{L^{(1,q';-\alpha)}(0,1)}.$$

Owing to (3.23) and (3.24),

$$\left(L^{(1,q';-\alpha)}\right)'(0,1) = L^{\infty,q;\alpha-1}(0,1).$$

Thus,

$$\widetilde{X}_1(0,1) = L^{\infty,q;\alpha-1}(0,1).$$

By making use of Theorem 7.6 combined with Remark 7.2, we obtain that

$$\widetilde{X}_m(0,1) = L^{\infty,q;\alpha-m}(0,1).$$

The conclusion is now a consequence of Theorem 7.11. □

12. List of symbols

Function spaces

$\mathcal{M}(\Omega,\nu)$ (3.1)
$\mathcal{M}_+(\Omega,\nu)$ (3.2)
$\mathcal{M}_0(\Omega,\nu)$ (3.3)
$X(\Omega,\nu)$ (3.12)
$X_{\mathrm{loc}}(\Omega,\nu)$ (3.13)
$L^p \log^\alpha L(\Omega,\nu)$ below (3.27)

Function norms

Operators

Miscellaneous

u^{**} (3.5)
$P_\nu(E,\Omega)$ (4.1)
$I_{\Omega,\nu}$ (4.2)
$\mathrm{med}(u)$ (4.11)
$\mathcal{J}_\alpha$ (6.7)
$A_{m,\alpha}$ (6.21)
Φ above (7.1)
μ_Φ (7.1)
c_Φ (7.1)
$\mu_{\Phi,n}$ (7.2)
$d\gamma_n$ (7.4)
$d\gamma_{n,\beta}$ (7.5)
F_Φ (7.7)
L_Φ (7.9)
ω_{n-1} above Proposition 10.1

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