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Questa è la Versione finale referata (Post print/Accepted manuscript) della seguente pubblicazione:

Original Citation:

Inequalities à la Frölicher and cohomological decompositions / Angella, Daniele; Tomassini, Adriano. - In: JOURNAL OF NONCOMMUTATIVE GEOMETRY. - ISSN 1661-6952. - STAMPA. - 9:(2015), pp. 505-542. [10.4171/JNCG/199]

Availability:

This version is available at: 2158/1008940 since: 2021-04-02T16:18:10Z

Published version:

DOI: 10.4171/JNCG/199

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INEQUALITIES À LA FRÖLICHER AND COHOMOLOGICAL DECOMPOSITIONS

DANIELE ANGELLA AND ADRIANO TOMASSINI

ABSTRACT. We study Bott-Chern and Aeppli cohomologies of a vector space endowed with two anti-commuting endomorphisms whose square is zero. In particular, we prove an inequality *à la* Frölicher relating the dimensions of the Bott-Chern and Aeppli cohomologies to the dimensions of the Dolbeault cohomologies. We prove that the equality in such an inequality *à la* Frölicher characterizes the validity of the so-called cohomological property of satisfying the $\partial\bar{\partial}$ -Lemma. As an application, we study cohomological properties of compact either complex, or symplectic, or, more in general, generalized-complex manifolds.

INTRODUCTION

Given a compact complex manifold X , the *Bott-Chern cohomology*, $H_{BC}^{\bullet,\bullet}(X)$, [11], and the *Aeppli cohomology*, $H_A^{\bullet,\bullet}(X)$, [1], provide useful invariants, and have been studied by several authors in different contexts, see, e.g., [1, 11, 9, 17, 50, 2, 45, 33, 10, 49, 3, 6]. In the case of compact Kähler manifolds, or, more in general, of compact complex manifolds satisfying the $\partial\bar{\partial}$ -Lemma, the Bott-Chern and the Aeppli cohomology groups are naturally isomorphic to the Dolbeault cohomology groups. The $\partial\bar{\partial}$ -Lemma for compact complex manifolds has been studied by P. Deligne, Ph. A. Griffiths, J. Morgan, and D. P. Sullivan in [17], where it is proven that the validity of the $\partial\bar{\partial}$ -Lemma on a compact complex manifold X yields the formality of the differential graded algebra $(\wedge^\bullet X \otimes_{\mathbb{R}} \mathbb{C}, d)$, [17, Main Theorem]; in particular, a topological obstruction to the existence of Kähler structures on compact differentiable manifolds follows, [17, Lemma 5.11]. Furthermore, they showed that any compact manifold admitting a proper modification from a Kähler manifold (namely, a manifold in class \mathcal{C} of Fujiki, [21]) satisfies the $\partial\bar{\partial}$ -Lemma, [17, Corollary 5.23]. An adapted version of the $\partial\bar{\partial}$ -Lemma for differential graded Lie algebras has been considered also in [22] by W. M. Goldman and J. J. Millson, where they used a “principle of two types”, see [22, Proposition 7.3(ii)], as a key tool to prove formality of certain differential graded Lie algebras in the context of deformation theory, [22, Corollary page 84]. An algebraic approach to the $\partial\bar{\partial}$ -Lemma has been developed also by Y. I. Manin in [38] in the context of differential Gerstenhaber-Batalin-Vilkovisky algebras, in order to study Frobenius manifolds arising by means of solutions of Maurer-Cartan type equations. A generalized complex version of the $\partial\bar{\partial}$ -Lemma has been introduced and studied by G. R. Cavalcanti in [13, 14].

Since Bott-Chern and Aeppli cohomologies on compact Kähler manifolds coincide with Dolbeault cohomology, in [6], we were concerned in studying Bott-Chern cohomology of compact complex (possibly non-Kähler) manifolds X , showing the following *inequality à la Frölicher*, which relates the dimensions of the Bott-Chern and Aeppli cohomologies to the Betti numbers, [6, Theorem A]:

$$\text{for any } k \in \mathbb{Z}, \quad \sum_{p+q=k} (\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_{BC}^{q,p}(X)) \geq 2 \dim_{\mathbb{C}} H_{dR}^k(X; \mathbb{C});$$

furthermore, the authors showed that the equality in the above inequality holds for every $k \in \mathbb{Z}$ if and only if X satisfies the $\partial\bar{\partial}$ -Lemma, [6, Theorem B].

It turns out that such results depend actually on the structure of double complex of $(\wedge^{\bullet,\bullet} X, \partial, \bar{\partial})$. In this paper, we are concerned in a generalization of the inequality *à la* Frölicher in a more algebraic framework, so as to highlight the algebraic aspects. As an application, we recover the above results

2010 *Mathematics Subject Classification.* 32Q99, 53D05, 53D18.

Key words and phrases. $\partial\bar{\partial}$ -Lemma, Bott-Chern cohomology, Aeppli cohomology, generalized-complex, symplectic.

The first author is supported by the Project PRIN “Varietà reali e complesse: geometria, topologia e analisi armonica”, by the Project FIRB “Geometria Differenziale e Teoria Geometrica delle Funzioni”, by SIR2014 project RBSI14DYEB “Analytic aspects in complex and hypercomplex geometry”, and by GNSAGA of INdAM. The second author is supported by Project PRIN “Varietà reali e complesse: geometria, topologia e analisi armonica” and by GNSAGA of INdAM.

on the cohomology of compact complex manifolds, and we get results on the cohomology of compact symplectic manifolds and compact generalized complex manifolds: more precisely, characterizations of compact symplectic manifolds satisfying the Hard Lefschetz Condition and of compact generalized complex manifolds satisfying the $d\bar{d}^{\mathcal{J}}$ -Lemma are provided.

More precisely, consider a double complex $(B^{\bullet,\bullet}, \partial, \bar{\partial})$ of \mathbb{K} -vector spaces (namely, a \mathbb{Z}^2 -graded \mathbb{K} -vector space $B^{\bullet,\bullet}$ endowed with $\partial \in \text{End}^{1,0}(B^{\bullet,\bullet})$ and $\bar{\partial} \in \text{End}^{0,1}(B^{\bullet,\bullet})$ such that $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$). Several cohomologies can be studied: other than the *Dolbeault cohomologies*

$$H_{(\partial;\partial)}^{\bullet,\bullet}(B^{\bullet,\bullet}) := \frac{\ker \partial}{\text{im } \partial} \quad \text{and} \quad H_{(\bar{\partial};\bar{\partial})}^{\bullet,\bullet}(B^{\bullet,\bullet}) := \frac{\ker \bar{\partial}}{\text{im } \bar{\partial}},$$

and then the cohomology of the associated total complex, $(\text{Tot}^{\bullet} B^{\bullet,\bullet} := \bigoplus_{p+q=\bullet} B^{p,q}, d := \partial + \bar{\partial})$,

$$H_{(d;d)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet}) := \frac{\ker d}{\text{im } d},$$

one can consider also the *Bott-Chern cohomology* and the *Aeppli cohomology*, that is,

$$H_{(\partial,\bar{\partial};\partial\bar{\partial})}^{\bullet,\bullet}(B^{\bullet,\bullet}) := \frac{\ker \partial \cap \ker \bar{\partial}}{\text{im } \partial\bar{\partial}} \quad \text{and} \quad H_{(\partial\bar{\partial};\partial,\bar{\partial})}^{\bullet,\bullet}(B^{\bullet,\bullet}) := \frac{\ker \partial\bar{\partial}}{\text{im } \partial + \text{im } \bar{\partial}}.$$

The identity induces natural morphisms of (possibly \mathbb{Z} -graded, possibly \mathbb{Z}^2 -graded) \mathbb{K} -vector spaces:

$$\begin{array}{ccccc} & & H_{(\partial,\bar{\partial};\partial\bar{\partial})}^{\bullet,\bullet}(B^{\bullet,\bullet}) & & \\ & \swarrow & \downarrow & \searrow & \\ H_{(\partial;\partial)}^{\bullet,\bullet}(B^{\bullet,\bullet}) & & H_{(d;d)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet}) & & H_{(\bar{\partial};\bar{\partial})}^{\bullet,\bullet}(B^{\bullet,\bullet}) \\ & \searrow & \downarrow & \swarrow & \\ & & H_{(\partial\bar{\partial};\partial,\bar{\partial})}^{\bullet,\bullet}(B^{\bullet,\bullet}) & & \end{array}$$

In general, the above maps are neither injective nor surjective; actually, the map $H_{(\partial,\bar{\partial};\partial\bar{\partial})}^{\bullet,\bullet}(B^{\bullet,\bullet}) \rightarrow H_{(\partial\bar{\partial};\partial,\bar{\partial})}^{\bullet,\bullet}(B^{\bullet,\bullet})$ being injective is equivalent to all the above maps being isomorphisms, [17, Lemma 5.15, Remark 5.16, 5.21]. In such a case, one says that $(B^{\bullet,\bullet}, \partial, \bar{\partial})$ *satisfies the $\partial\bar{\partial}$ -Lemma*.

By considering the spectral sequence associated to the structure of double complex of $(B^{\bullet,\bullet}, \partial, \bar{\partial})$, one gets the *Frölicher inequality*, [20, Theorem 2],

$$\min \left\{ \dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\partial;\partial)}^{\bullet,\bullet}(B^{\bullet,\bullet}), \dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\bar{\partial};\bar{\partial})}^{\bullet,\bullet}(B^{\bullet,\bullet}) \right\} \geq \dim_{\mathbb{K}} H_{(d;d)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet}).$$

We prove an *inequality à la Frölicher* also for the Bott-Chern and Aeppli cohomologies. More precisely, we prove the following result.

Theorem 1 (see Theorem 2.4 and Corollary 2.6). *Let A^{\bullet} be a \mathbb{Z} -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \text{End}^{\hat{\delta}_1}(A^{\bullet})$ and $\delta_2 \in \text{End}^{\hat{\delta}_2}(A^{\bullet})$ such that $\delta_1^2 = \delta_2^2 = \delta_1\delta_2 + \delta_2\delta_1 = 0$. Suppose that*

$$\dim_{\mathbb{K}} H_{(\delta_1;\delta_1)}^{\bullet}(A^{\bullet}) < +\infty \quad \text{and} \quad \dim_{\mathbb{K}} H_{(\delta_2;\delta_2)}^{\bullet}(A^{\bullet}) < +\infty.$$

Then

$$\dim_{\mathbb{K}} H_{(\delta_1,\delta_2;\delta_1\delta_2)}^{\bullet}(A^{\bullet}) + \dim_{\mathbb{K}} H_{(\delta_1\delta_2;\delta_1,\delta_2)}^{\bullet}(A^{\bullet}) \geq \dim_{\mathbb{K}} H_{(\delta_1;\delta_1)}^{\bullet}(A^{\bullet}) + \dim_{\mathbb{K}} H_{(\delta_2;\delta_2)}^{\bullet}(A^{\bullet}).$$

In particular, given a bounded double complex $(B^{\bullet,\bullet}, \partial, \bar{\partial})$, and supposed that

$$\dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1;\delta_1)}^{\bullet,\bullet}(B^{\bullet,\bullet}) < +\infty \quad \text{and} \quad \dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_2;\delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) < +\infty,$$

then, for $\pm \in \{+, -\}$,

$$\dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1,\delta_2;\delta_1\delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) + \dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1\delta_2;\delta_1,\delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) \geq 2 \dim_{\mathbb{K}} H_{(\delta_1\pm\delta_2;\delta_1\pm\delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet}).$$

Furthermore, we provide a characterization of the equality in the above inequality *à la Frölicher* in terms of the validity of the $\delta_1\delta_2$ -Lemma.

Theorem 2 (see Theorem 3.3). *Let $(B^{\bullet,\bullet}, \delta_1, \delta_2)$ be a bounded double complex. Suppose that*

$$\dim_{\mathbb{K}} H_{(\delta_1; \delta_1)}^{\bullet,\bullet}(B^{\bullet,\bullet}) < +\infty \quad \text{and} \quad \dim_{\mathbb{K}} H_{(\delta_2; \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) < +\infty .$$

The following conditions are equivalent:

- (i) $B^{\bullet,\bullet}$ satisfies the $\delta_1\delta_2$ -Lemma;
- (ii) the equality

$$\begin{aligned} & \dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1, \delta_2; \delta_1, \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) + \dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1\delta_2; \delta_1, \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) \\ &= 2 \dim_{\mathbb{K}} H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet}) . \end{aligned}$$

holds.

Given a compact complex manifold X , one can apply Corollary 2.6 and Theorem 3.3 to the double complex $(\wedge^{\bullet,\bullet} X, \partial, \bar{\partial})$. More precisely, one recovers [6, Theorem A], getting that, on every compact complex manifold,

$$\dim_{\mathbb{C}} \text{Tot}^{\bullet} H_{BC}^{\bullet,\bullet}(X) + \dim_{\mathbb{C}} \text{Tot}^{\bullet} H_A^{\bullet,\bullet}(X) \geq 2 \dim_{\mathbb{C}} H_{dR}^{\bullet}(X; \mathbb{C}) ,$$

and the characterization of the $\partial\bar{\partial}$ -Lemma in terms of the Bott-Chern cohomology given in [6, Theorem B], namely, that the equality holds if and only if the $\partial\bar{\partial}$ -Lemma holds.

Furthermore, Corollary 2.6 and Theorem 3.3 allow also to study the cohomology of compact manifolds X endowed with symplectic forms ω . In this case, one considers the \mathbb{Z} -graded algebra $\wedge^{\bullet} X$ endowed with $d \in \text{End}^1(\wedge^{\bullet} X)$ and $d^{\Lambda} := [d, -\iota_{\omega^{-1}}] \in \text{End}^{-1}(\wedge^{\bullet} X)$, which satisfy $d^2 = (d^{\Lambda})^2 = d d^{\Lambda} + d^{\Lambda} d = 0$. The symplectic Bott-Chern and Aeppli cohomologies have been introduced and studied by L.-S. Tseng and S.-T. Yau in [47, 48, 49]. In particular, we get the following result.

Theorem 3 (see Theorem 4.4). *Let X be a compact manifold endowed with a symplectic structure ω . The inequality*

$$(5) \quad \dim_{\mathbb{R}} H_{(d, d^{\Lambda}; d, d^{\Lambda})}^{\bullet}(X) + \dim_{\mathbb{R}} H_{(d d^{\Lambda}; d, d^{\Lambda})}^{\bullet}(X) \geq 2 \dim_{\mathbb{R}} H_{dR}^{\bullet}(X; \mathbb{R})$$

holds. Furthermore, the equality in (5) holds if and only if X satisfies the Hard Lefschetz Condition.

We recall that a compact $2n$ -dimensional manifold X endowed with a symplectic form ω is said to satisfy the *Hard Lefschetz Condition* if $[\omega]^k \smile \cdot : H_{dR}^{n-k}(X; \mathbb{R}) \rightarrow H_{dR}^{n+k}(X; \mathbb{R})$ is an isomorphism for every $k \in \mathbb{Z}$.

Finally, Corollary 2.6 and Theorem 3.3 can be applied also to the study of the cohomology of generalized-complex manifolds. Generalized-complex geometry has been introduced by N. Hitchin in [30], and studied, among others, by M. Gualtieri, [25, 27, 26], and G. R. Cavalcanti, [13]. It provides a way to generalize both complex and symplectic geometry, since complex structures and symplectic structures appear as special cases of generalized-complex structures. See, e.g., [31] for an introduction to generalized-complex geometry; the cohomology of generalized-complex manifolds has been studied especially by G. R. Cavalcanti, [13, 14, 15]. On a manifold X endowed with an H -twisted generalized complex structure \mathcal{J} , (see §4.3 for the definitions,) one can consider the \mathbb{Z} -graduation $\text{Tot} \wedge^{\bullet} X \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{k \in \mathbb{Z}} U_{\mathcal{J}}^k$, and the endomorphisms $\partial_{\mathcal{J}, H} \in \text{End}^1(U_{\mathcal{J}}^{\bullet})$ and $\bar{\partial}_{\mathcal{J}, H} \in \text{End}^{-1}(U_{\mathcal{J}}^{\bullet})$, which satisfy $\partial_{\mathcal{J}, H}^2 = \bar{\partial}_{\mathcal{J}, H}^2 = \partial_{\mathcal{J}, H} \bar{\partial}_{\mathcal{J}, H} + \bar{\partial}_{\mathcal{J}, H} \partial_{\mathcal{J}, H} = 0$; then, let

$$GH_{\partial_{\mathcal{J}, H}}^{\bullet}(X) := \frac{\ker \partial_{\mathcal{J}, H}}{\text{im} \partial_{\mathcal{J}, H}} , \quad GH_{\bar{\partial}_{\mathcal{J}, H}}^{\bullet}(X) := \frac{\ker \bar{\partial}_{\mathcal{J}, H}}{\text{im} \bar{\partial}_{\mathcal{J}, H}} ,$$

and

$$GH_{BC_{\mathcal{J}, H}}^{\bullet}(X) := \frac{\ker \partial_{\mathcal{J}, H} \cap \ker \bar{\partial}_{\mathcal{J}, H}}{\text{im} \partial_{\mathcal{J}, H} \bar{\partial}_{\mathcal{J}, H}} , \quad GH_{A_{\mathcal{J}, H}}^{\bullet}(X) := \frac{\ker \partial_{\mathcal{J}, H} \bar{\partial}_{\mathcal{J}, H}}{\text{im} \partial_{\mathcal{J}, H} + \text{im} \bar{\partial}_{\mathcal{J}, H}} .$$

The above general results yield the following.

Theorem 4 (see Theorem 4.10 and Theorem 4.11). *Let X be a compact differentiable manifold endowed with an H -twisted generalized complex structure \mathcal{J} . Then*

$$(6) \quad \dim_{\mathbb{C}} GH_{BC_{\mathcal{J}, H}}^{\bullet}(X) + \dim_{\mathbb{C}} GH_{A_{\mathcal{J}, H}}^{\bullet}(X) \geq \dim_{\mathbb{C}} GH_{\bar{\partial}_{\mathcal{J}, H}}^{\bullet}(X) + \dim_{\mathbb{C}} GH_{\partial_{\mathcal{J}, H}}^{\bullet}(X) .$$

Furthermore, X satisfies the $\partial_{\mathcal{J}, H} \bar{\partial}_{\mathcal{J}, H}$ -Lemma if and only if the Hodge and Frölicher spectral sequences associated to the canonical double complex $(U_{\mathcal{J}}^{\bullet, \bullet} \otimes \beta^{\bullet, \bullet}, \partial_{\mathcal{J}, H} \otimes_{\mathbb{C}} \text{id}, \bar{\partial}_{\mathcal{J}, H} \otimes_{\mathbb{C}} \beta)$ degenerate at the first level and the equality in (6) holds.

1. PRELIMINARIES AND NOTATION

Fix $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. In this section, we summarize some notation and results concerning graded \mathbb{K} -vector spaces endowed with two commuting differentials.

1.1. (Bi-)graded vector spaces. We set the notation, in constructing two functors in order to change over \mathbb{Z} -graduation and \mathbb{Z}^2 -graduation of a \mathbb{K} -vector space.

Consider a \mathbb{Z}^2 -graded \mathbb{K} -vector space $A^{\bullet, \bullet}$ endowed with two endomorphisms $\delta_1 \in \text{End}^{\hat{\delta}_1, 1, \hat{\delta}_1, 2}(A^{\bullet, \bullet})$ and $\delta_2 \in \text{End}^{\hat{\delta}_2, 1, \hat{\delta}_2, 2}(A^{\bullet, \bullet})$ such that $\delta_1^2 = \delta_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = 0$. Define the \mathbb{Z} -graded \mathbb{K} -vector space

$$\text{Tot}^\bullet(A^{\bullet, \bullet}) := \bigoplus_{p+q=\bullet} A^{p, q},$$

endowed with the endomorphisms

$$\delta_1 \in \text{End}^{\hat{\delta}_1, 1, \hat{\delta}_1, 2}(\text{Tot}^\bullet(A^{\bullet, \bullet})) \quad \text{and} \quad \delta_2 \in \text{End}^{\hat{\delta}_2, 1, \hat{\delta}_2, 2}(\text{Tot}^\bullet(A^{\bullet, \bullet}))$$

such that $\delta_1^2 = \delta_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = 0$.

Conversely, consider a \mathbb{Z} -graded \mathbb{K} -vector space A^\bullet endowed with two endomorphisms $\delta_1 \in \text{End}^{\hat{\delta}_1}(A^\bullet)$ and $\delta_2 \in \text{End}^{\hat{\delta}_2}(A^\bullet)$ such that $\delta_1^2 = \delta_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = 0$. Following [12, §1.3], [13, §4.2], see [23, §II.2], [16, §II], take an infinite cyclic multiplicative group $\{\beta^m : m \in \mathbb{Z}\}$ generated by some β , and consider the \mathbb{Z} -graded \mathbb{K} -vector space $\bigoplus_{\bullet \in \mathbb{Z}} \mathbb{K} \beta^\bullet$. Define the \mathbb{Z}^2 -graded \mathbb{K} -vector space

$$\text{Doub}^{\bullet, \bullet, \bullet}(A^\bullet) := A^{\hat{\delta}_1 \bullet + \hat{\delta}_2 \bullet} \otimes_{\mathbb{K}} \mathbb{K} \beta^\bullet,$$

endowed with the endomorphisms

$$\delta_1 \otimes_{\mathbb{K}} \text{id} \in \text{End}^{1, 0}(\text{Doub}^{\bullet, \bullet, \bullet}(A^\bullet)) \quad \text{and} \quad \delta_2 \otimes_{\mathbb{K}} \beta \in \text{End}^{1, 0}(\text{Doub}^{\bullet, \bullet, \bullet}(A^\bullet)),$$

which satisfy $(\delta_1 \otimes_{\mathbb{K}} \text{id})^2 = (\delta_2 \otimes_{\mathbb{K}} \beta)^2 = (\delta_1 \otimes_{\mathbb{K}} \text{id})(\delta_2 \otimes_{\mathbb{K}} \beta) + (\delta_2 \otimes_{\mathbb{K}} \beta)(\delta_1 \otimes_{\mathbb{K}} \text{id}) = 0$; following [12, §1.3], [13, §4.2], the double complex $(\text{Doub}^{\bullet, \bullet, \bullet}(A^\bullet), \delta_1 \otimes_{\mathbb{K}} \text{id}, \delta_2 \otimes_{\mathbb{K}} \beta)$ is called the *canonical double complex* associated to A^\bullet .

1.2. Cohomologies. Let A^\bullet be a \mathbb{Z} -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \text{End}^{\hat{\delta}_1}(A^\bullet)$ and $\delta_2 \in \text{End}^{\hat{\delta}_2}(A^\bullet)$ such that

$$\delta_1^2 = \delta_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = 0.$$

Since one has the \mathbb{Z} -graded \mathbb{K} -vector sub-spaces $\text{im } \delta_1 \delta_2 \subseteq \ker \delta_1 \cap \ker \delta_2$, and $\text{im } \delta_1 \subseteq \ker \delta_1$, and $\text{im } \delta_2 \subseteq \ker \delta_2$, and $\text{im } \delta_1 + \text{im } \delta_2 \subseteq \ker \delta_1 \delta_2$, one can define the \mathbb{Z} -graded \mathbb{K} -vector spaces

$$H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^\bullet(A^\bullet) := \frac{\ker \delta_1 \cap \ker \delta_2}{\text{im } \delta_1 \delta_2},$$

$$H_{(\delta_1; \delta_1)}^\bullet(A^\bullet) := \frac{\ker \delta_1}{\text{im } \delta_1}, \quad H_{(\delta_2; \delta_2)}^\bullet(A^\bullet) := \frac{\ker \delta_2}{\text{im } \delta_2},$$

$$H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet) := \frac{\ker \delta_1 \delta_2}{\text{im } \delta_1 + \text{im } \delta_2},$$

and, since one has the \mathbb{K} -vector sub-space $\text{im } (\delta_1 + \delta_2) \subseteq \ker (\delta_1 + \delta_2)$, one can define the \mathbb{K} -vector space

$$H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}(\text{Tot } A^\bullet) := \frac{\ker (\delta_1 + \delta_2)}{\text{im } (\delta_1 + \delta_2)};$$

we follow notation in [17, Remark 5.16]: more precisely, if maps $f_j : C_j \rightarrow A$ for $j \in \{1, \dots, r\}$ and $g_k : A \rightarrow B_k$ for $k \in \{1, \dots, s\}$ of \mathbb{K} -vector spaces are given, then $H_{(f_1, \dots, f_r; g_1, \dots, g_s)}$ denotes the quotient $\frac{\bigcap_{j=1}^r \ker f_j}{\sum_{k=1}^s \text{im } g_k}$. (Note that, up to consider $-\delta_2 \in \text{End}^{\hat{\delta}_2}(A^\bullet)$ instead of $\delta_2 \in \text{End}^{\hat{\delta}_2}(A^\bullet)$, one has the \mathbb{K} -vector sub-space $\text{im } (\delta_1 - \delta_2) \subseteq \ker (\delta_1 - \delta_2)$, and hence one can consider also the \mathbb{K} -vector space $H_{(\delta_1 - \delta_2; \delta_1 - \delta_2)}(\text{Tot } A^\bullet) := \frac{\ker (\delta_1 - \delta_2)}{\text{im } (\delta_1 - \delta_2)}$; note that, for $\sharp_{\delta_1, \delta_2} \in \{(\delta_1; \delta_1), (\delta_2; \delta_2), (\delta_1, \delta_2; \delta_1 \delta_2), (\delta_1 \delta_2; \delta_1, \delta_2)\}$, one has $H_{\sharp_{\delta_1, \delta_2}}^\bullet(A^\bullet) = H_{\sharp_{\delta_1, -\delta_2}}^\bullet(A^\bullet)$.)

Remark 1.1. Note that $H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}(A^\bullet)$ admits a $\left(\mathbb{Z} / \left(\hat{\delta}_1 - \hat{\delta}_2\right) \mathbb{Z}\right)$ -graduation; in particular, if $\hat{\delta}_1 = \hat{\delta}_2$, then $H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^\bullet(A^\bullet)$ is actually a \mathbb{Z} -graded \mathbb{K} -vector space.

Remark 1.2. Note that, for $\sharp \in \{(\delta_1, \delta_2; \delta_1\delta_2), (\delta_1; \delta_1), (\delta_2; \delta_2), (\delta_1\delta_2; \delta_1, \delta_2)\}$, if $A^{\bullet, \bullet}$ is actually \mathbb{Z}^2 -graded, then $H_{\sharp}^{\bullet}(A^{\bullet})$ admits a \mathbb{Z}^2 -graduation such that $\text{Tot}^{\bullet} H_{\sharp}^{\bullet, \bullet}(A^{\bullet, \bullet}) = H_{\sharp}^{\bullet}(\text{Tot}^{\bullet} A^{\bullet, \bullet})$. Furthermore, for $\delta_1 \in \text{End}^{\hat{\delta}_{1,1}, \hat{\delta}_{1,2}}(A^{\bullet, \bullet})$ and $\delta_2 \in \text{End}^{\hat{\delta}_{2,1}, \hat{\delta}_{2,2}}(A^{\bullet, \bullet})$, one has that $H_{(\delta_1+\delta_2; \delta_1+\delta_2)}(\text{Tot} A^{\bullet})$ admits a $\left(\left(\mathbb{Z}/\left(\hat{\delta}_{1,1} - \hat{\delta}_{2,1}\right)\mathbb{Z}\right) \times \left(\mathbb{Z}/\left(\hat{\delta}_{1,2} - \hat{\delta}_{2,2}\right)\mathbb{Z}\right)\right)$ -graduation; in particular, if $\hat{\delta}_{1,1} = \hat{\delta}_{2,1}$ and $\hat{\delta}_{1,2} = \hat{\delta}_{2,2}$, then $H_{(\delta_1+\delta_2; \delta_1+\delta_2)}(\text{Tot} A^{\bullet})$ is actually \mathbb{Z}^2 -graded.

Since $\ker \delta_1 \cap \ker \delta_2 \subseteq \ker(\delta_1 \pm \delta_2)$ and $\text{im} \delta_1 \delta_2 \subseteq \text{im}(\delta_1 \pm \delta_2)$ for $\pm \in \{+, -\}$, and $\ker \delta_1 \cap \ker \delta_2 \subseteq \ker \delta_1$ and $\text{im} \delta_1 \delta_2 \subseteq \text{im} \delta_1$, and $\ker \delta_1 \cap \ker \delta_2 \subseteq \ker \delta_2$ and $\text{im} \delta_1 \delta_2 \subseteq \text{im} \delta_2$, and $\ker(\delta_1 \pm \delta_2) \subseteq \ker \delta_1 \delta_2$ and $\text{im}(\delta_1 \pm \delta_2) \subseteq \text{im} \delta_1 + \text{im} \delta_2$ for $\pm \in \{+, -\}$, and $\ker \delta_1 \subseteq \ker \delta_1 \delta_2$ and $\text{im} \delta_1 \subseteq \text{im} \delta_1 + \text{im} \delta_2$, and $\ker \delta_2 \subseteq \ker \delta_1 \delta_2$ and $\text{im} \delta_2 \subseteq \text{im} \delta_1 + \text{im} \delta_2$, then the identity map induces natural morphisms of (possibly \mathbb{Z} -graded, possibly \mathbb{Z}^2 -graded) \mathbb{K} -vector spaces

$$\begin{array}{ccccc}
& & H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet}(A^{\bullet}) & & \\
& \swarrow & \downarrow & \searrow & \\
H_{(\delta_1; \delta_1)}^{\bullet}(A^{\bullet}) & \leftarrow & H_{(\delta_1+\delta_2; \delta_1+\delta_2)}(\text{Tot} A^{\bullet}) & \rightarrow & H_{(\delta_1-\delta_2; \delta_1-\delta_2)}(\text{Tot} A^{\bullet}) \\
& \searrow & \downarrow & \swarrow & \\
& & H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet}(A^{\bullet}) & &
\end{array}$$

(As a matter of notation, by writing, for example, $H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet}(A^{\bullet}) \rightarrow H_{(\delta_1+\delta_2; \delta_1+\delta_2)}(\text{Tot} A^{\bullet})$, we mean $\text{Tot} H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet}(A^{\bullet}) \rightarrow H_{(\delta_1+\delta_2; \delta_1+\delta_2)}(\text{Tot} A^{\bullet})$.)

1.3. $\delta_1\delta_2$ -Lemma. Let A^{\bullet} be a \mathbb{Z} -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \text{End}^{\hat{\delta}_1}(A^{\bullet})$ and $\delta_2 \in \text{End}^{\hat{\delta}_2}(A^{\bullet})$ such that $\delta_1^2 = \delta_2^2 = \delta_1\delta_2 + \delta_2\delta_1 = 0$, and consider the cohomologies introduced in §1.2. In general, the natural maps induced by the identity between such cohomologies are neither injective nor surjective: the following definition, [17], points out when they are actually isomorphisms.

Definition 1.3 ([17]). A \mathbb{Z} -graded \mathbb{K} -vector space A^{\bullet} endowed with two endomorphisms $\delta_1 \in \text{End}^{\hat{\delta}_1}(A^{\bullet})$ and $\delta_2 \in \text{End}^{\hat{\delta}_2}(A^{\bullet})$ such that $\delta_1^2 = \delta_2^2 = \delta_1\delta_2 + \delta_2\delta_1 = 0$ is said to satisfy the $\delta_1\delta_2$ -Lemma if and only if

$$\ker \delta_1 \cap \ker \delta_2 \cap (\text{im} \delta_1 + \text{im} \delta_2) = \text{im} \delta_1 \delta_2,$$

namely, if and only if the natural map $H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet}(A^{\bullet}) \rightarrow H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet}(A^{\bullet})$ induced by the identity is injective.

A \mathbb{Z}^2 -graded \mathbb{K} -vector space $A^{\bullet, \bullet}$ endowed with two endomorphisms $\delta_1 \in \text{End}^{\hat{\delta}_{1,1}, \hat{\delta}_{1,2}}(A^{\bullet, \bullet})$ and $\delta_2 \in \text{End}^{\hat{\delta}_{2,1}, \hat{\delta}_{2,2}}(A^{\bullet, \bullet})$ such that $\delta_1^2 = \delta_2^2 = \delta_1\delta_2 + \delta_2\delta_1 = 0$ is said to satisfy the $\delta_1\delta_2$ -Lemma if and only if $\text{Tot}^{\bullet}(A^{\bullet, \bullet})$ satisfies the $\delta_1\delta_2$ -Lemma.

We recall the following result, which provides further characterizations of the validity of the $\delta_1\delta_2$ -Lemma. (Note that, according to Remark 1.1 and Remark 1.2, the natural maps induced by the identity in Lemma 1.4 are maps of possibly \mathbb{Z} -graded, possibly \mathbb{Z}^2 -graded \mathbb{K} -vector spaces.)

Lemma 1.4 (see [17, Lemma 5.15]). *Let A^{\bullet} be a \mathbb{Z} -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \text{End}^{\hat{\delta}_1}(A^{\bullet})$ and $\delta_2 \in \text{End}^{\hat{\delta}_2}(A^{\bullet})$ such that $\delta_1^2 = \delta_2^2 = \delta_1\delta_2 + \delta_2\delta_1 = 0$. The following conditions are equivalent:*

- (i) A^{\bullet} satisfies the $\delta_1\delta_2$ -Lemma, namely, the natural map $H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet}(A^{\bullet}) \rightarrow H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet}(A^{\bullet})$ induced by the identity is injective;
- (ii) the natural map $H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet}(A^{\bullet}) \rightarrow H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet}(A^{\bullet})$ induced by the identity is surjective;
- (iii) both the natural map $H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet}(A^{\bullet}) \rightarrow H_{(\delta_1; \delta_1)}^{\bullet}(A^{\bullet})$ induced by the identity and the natural map $H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet}(A^{\bullet}) \rightarrow H_{(\delta_2; \delta_2)}^{\bullet}(A^{\bullet})$ induced by the identity are injective;
- (iv) both the natural map $H_{(\delta_1; \delta_1)}^{\bullet}(A^{\bullet}) \rightarrow H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet}(A^{\bullet})$ induced by the identity and the natural map $H_{(\delta_2; \delta_2)}^{\bullet}(A^{\bullet}) \rightarrow H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet}(A^{\bullet})$ induced by the identity are surjective.

Furthermore, suppose that the \mathbb{K} -vector space $\ker \delta_1 \delta_2$ admits a \mathbb{Z} -graduation

$$\ker \delta_1 \delta_2 = \bigoplus_{\ell \in \mathbb{Z}} \left(\ker \delta_1 \delta_2 \cap \tilde{A}^{\ell} \right)$$

with respect to which $\ker(\delta_1 \pm \delta_2) \cap \tilde{A}^\bullet = (\ker \delta_1 \cap \ker \delta_2) \cap \tilde{A}^\bullet$. (For example, if $\hat{\delta}_1 \neq \hat{\delta}_2$, then take the \mathbb{Z} -graduation given by A^\bullet . For example, if $A^{\bullet,\bullet}$ is actually \mathbb{Z}^2 -graded and $\delta_1 \in \text{End}^{\hat{\delta}_{1,1}, \hat{\delta}_{1,2}}(A^{\bullet,\bullet})$ and $\delta_2 \in \text{End}^{\hat{\delta}_{2,1}, \hat{\delta}_{2,2}}(A^{\bullet,\bullet})$ with $(\hat{\delta}_{1,1}, \hat{\delta}_{1,2}) \neq (\hat{\delta}_{2,1}, \hat{\delta}_{2,2})$, then take the \mathbb{Z} -graduation induced by the \mathbb{Z}^2 -graduation of $A^{\bullet,\bullet}$ by means of a chosen bijection $\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}^2$.) Then the previous conditions are equivalent to each of the following:

- (v) the natural map $\text{Tot } H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^\bullet(A^\bullet) \rightarrow H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}(\text{Tot } A^\bullet)$ induced by the identity is injective;
- (vi) the natural map $H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}(\text{Tot } A^\bullet) \rightarrow \text{Tot } H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet)$ induced by the identity is surjective;
- (vii) the natural map $\text{Tot } H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^\bullet(A^\bullet) \rightarrow H_{(\delta_1 - \delta_2; \delta_1 - \delta_2)}(\text{Tot } A^\bullet)$ induced by the identity is injective;
- (viii) the natural map $H_{(\delta_1 - \delta_2; \delta_1 - \delta_2)}(\text{Tot } A^\bullet) \rightarrow \text{Tot } H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet)$ induced by the identity is surjective.

Proof. For the sake of completeness, we recall here the proof in [17].

[(i) \Rightarrow (iii)]. By the hypothesis, $\ker \delta_1 \cap \ker \delta_2 \cap (\text{im } \delta_1 + \text{im } \delta_2) = \text{im } \delta_1 \delta_2$, and we have to prove that $\ker \delta_2 \cap \text{im } \delta_1 \subseteq \text{im } \delta_1 \delta_2$ and $\ker \delta_1 \cap \text{im } \delta_2 \subseteq \text{im } \delta_1 \delta_2$. Since $\text{im } \delta_1 \subseteq \text{im } \delta_1 + \text{im } \delta_2$ and $\text{im } \delta_2 \subseteq \text{im } \delta_1 + \text{im } \delta_2$, one gets immediately that the natural maps $H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^\bullet(A^\bullet) \rightarrow H_{(\delta_1; \delta_1)}^\bullet(A^\bullet)$ and $H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^\bullet(A^\bullet) \rightarrow H_{(\delta_2; \delta_2)}^\bullet(A^\bullet)$ are injective.

[(iii) \Rightarrow (iv)]. By the hypotheses, we have that $\ker \delta_2 \cap \text{im } \delta_1 = \text{im } \delta_1 \delta_2$ and $\ker \delta_1 \cap \text{im } \delta_2 = \text{im } \delta_1 \delta_2$, and we have to prove that $\ker \delta_1 + \text{im } \delta_2 \supseteq \ker \delta_1 \delta_2$ and $\ker \delta_2 + \text{im } \delta_1 \supseteq \ker \delta_1 \delta_2$. Let $x \in \ker \delta_1 \delta_2$. Then $\delta_1(x) \in \ker \delta_2 \cap \text{im } \delta_1 = \text{im } \delta_1 \delta_2$: let $y \in A^\bullet$ be such that $\delta_1(x) = \delta_1 \delta_2(y)$. Then $x = (x - \delta_2(y)) + \delta_2(y) \in \ker \delta_1 + \text{im } \delta_2$, since $\delta_1(x - \delta_2(y)) = 0$; it follows that the natural map $H_{(\delta_1; \delta_1)}^\bullet(A^\bullet) \rightarrow H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet)$ is surjective. Analogously, $\delta_2(x) \in \ker \delta_1 \cap \text{im } \delta_2 = \text{im } \delta_1 \delta_2$: let z be such that $\delta_2(x) = \delta_1 \delta_2(z)$. Then $x = (x + \delta_1(z)) - \delta_1(z) \in \ker \delta_2 + \text{im } \delta_1$, since $\delta_2(x + \delta_1(z)) = 0$; it follows that the natural map $H_{(\delta_2; \delta_2)}^\bullet(A^\bullet) \rightarrow H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet)$ is surjective.

[(iv) \Rightarrow (ii)]. By the hypothesis, $\ker \delta_1 + \text{im } \delta_2 = \ker \delta_1 \delta_2$ and $\ker \delta_2 + \text{im } \delta_1 = \ker \delta_1 \delta_2$, and we have to prove that $(\ker \delta_1 \cap \ker \delta_2) + \text{im } \delta_1 + \text{im } \delta_2 \supseteq \ker \delta_1 \delta_2$. Since $\ker \delta_1 \delta_2 = (\ker \delta_1 + \text{im } \delta_2) \cap (\ker \delta_2 + \text{im } \delta_1) \subseteq (\ker \delta_1 \cap \ker \delta_2) + \text{im } \delta_1 + \text{im } \delta_2$, one gets that the natural map $H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^\bullet(A^\bullet) \rightarrow H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet)$ is surjective.

[(ii) \Rightarrow (i)]. By the hypothesis, $(\ker \delta_1 \cap \ker \delta_2) + \text{im } \delta_1 + \text{im } \delta_2 = \ker \delta_1 \delta_2$, and we have to prove that $\ker \delta_1 \cap \ker \delta_2 \cap (\text{im } \delta_1 + \text{im } \delta_2) \subseteq \text{im } \delta_1 \delta_2$. Let $x := \delta_1(y) + \delta_2(z) \in \ker \delta_1 \cap \ker \delta_2 \cap (\text{im } \delta_1 + \text{im } \delta_2)$. Therefore $y \in \ker \delta_1 \delta_2 = (\ker \delta_1 \cap \ker \delta_2) + \text{im } \delta_1 + \text{im } \delta_2$ and $z \in \ker \delta_1 \delta_2 = (\ker \delta_1 \cap \ker \delta_2) + \text{im } \delta_1 + \text{im } \delta_2$. It follows that $\delta_1(y) \in \text{im } \delta_1 \delta_2$ and $\delta_2(z) \in \text{im } \delta_1 \delta_2$, and hence $x = \delta_1(y) + \delta_2(z) \in \text{im } \delta_1 \delta_2$, proving that the natural map $H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^\bullet(A^\bullet) \rightarrow H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet)$ is injective.

[(i) \Rightarrow (v), **and** (i) \Rightarrow (vii)]. By the hypothesis, $\ker \delta_1 \cap \ker \delta_2 \cap (\text{im } \delta_1 + \text{im } \delta_2) = \text{im } \delta_1 \delta_2$, and we have to prove that $\ker \delta_1 \cap \ker \delta_2 \cap \text{im } (\delta_1 \pm \delta_2) \subseteq \text{im } \delta_1 \delta_2$ for $\pm \in \{+, -\}$. Since $\ker \delta_1 \cap \ker \delta_2 \cap \text{im } (\delta_1 \pm \delta_2) \subseteq \ker \delta_1 \cap \ker \delta_2 \cap (\text{im } \delta_1 + \text{im } \delta_2)$, one gets immediately that the natural map $\text{Tot } H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^\bullet(A^\bullet) \rightarrow H_{(\delta_1 \pm \delta_2; \delta_1 \pm \delta_2)}(A^\bullet)$ is injective.

[(v) \Rightarrow (vi), **and** (vii) \Rightarrow (viii)]. Fix $\pm \in \{+, -\}$. By the hypothesis, $\ker \delta_1 \cap \ker \delta_2 \cap \text{im } (\delta_1 \pm \delta_2) = \text{im } \delta_1 \delta_2$, and we have to prove that $\ker (\delta_1 \pm \delta_2) + \text{im } \delta_1 + \text{im } \delta_2 \supseteq \ker \delta_1 \delta_2$. Let $x \in \ker \delta_1 \delta_2$. Then $(\delta_1 \pm \delta_2)(x) \in \ker \delta_1 \cap \ker \delta_2 \cap \text{im } (\delta_1 \pm \delta_2) = \text{im } \delta_1 \delta_2$; let $z \in \text{Tot } A^\bullet$ be such that $(\delta_1 \pm \delta_2)(x) = \delta_1 \delta_2(z)$. Since $(\delta_1 \pm \delta_2)(x \pm \frac{1}{2} \delta_1(z) - \frac{1}{2} \delta_2(z)) = 0$, one gets that $x = (x \pm \frac{1}{2} \delta_1(z) - \frac{1}{2} \delta_2(z)) - (\pm \frac{1}{2} \delta_1(z)) + \frac{1}{2} \delta_2(z) \in \ker (\delta_1 \pm \delta_2) + \text{im } \delta_1 + \text{im } \delta_2$, proving that the natural map $H_{(\delta_1 \pm \delta_2; \delta_1 \pm \delta_2)}(\text{Tot } A^\bullet) \rightarrow \text{Tot } H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet)$ is surjective.

To conclude the equivalences, we assume the additional hypothesis given in the statement.

[(vi) \Rightarrow (ii), **and** (viii) \Rightarrow (ii)]. Fix $\pm \in \{+, -\}$. By the hypothesis, $\ker (\delta_1 \pm \delta_2) + \text{im } \delta_1 + \text{im } \delta_2 = \ker \delta_1 \delta_2$, and we have to prove that $(\ker \delta_1 \cap \ker \delta_2) + \text{im } \delta_1 + \text{im } \delta_2 \supseteq \ker \delta_1 \delta_2$. By the additional hypothesis, we have that $\ker \delta_1 \delta_2$ admits a \mathbb{Z} -graduation $\ker \delta_1 \delta_2 = \bigoplus_{\ell \in \mathbb{Z}} (\ker \delta_1 \delta_2 \cap \tilde{A}^\ell)$ with respect to

which $\ker(\delta_1 \pm \delta_2) \cap \tilde{A}^\bullet = (\ker \delta_1 \cap \ker \delta_2) \cap \tilde{A}^\bullet$. Then one has that

$$\begin{aligned} \ker \delta_1 \delta_2 &= \bigoplus_{\ell \in \mathbb{Z}} \left(\ker \delta_1 \delta_2 \cap \tilde{A}^\ell \right) = \bigoplus_{\ell \in \mathbb{Z}} \left((\ker(\delta_1 \pm \delta_2) + \operatorname{im} \delta_1 + \operatorname{im} \delta_2) \cap \tilde{A}^\ell \right) \\ &\subseteq \bigoplus_{\ell \in \mathbb{Z}} \left((\ker(\delta_1 \pm \delta_2) \cap \tilde{A}^\ell) + \operatorname{im} \delta_1 + \operatorname{im} \delta_2 \right) = \bigoplus_{\ell \in \mathbb{Z}} \left(((\ker \delta_1 \cap \ker \delta_2) \cap \tilde{A}^\ell) + \operatorname{im} \delta_1 + \operatorname{im} \delta_2 \right) \\ &\subseteq (\ker \delta_1 \cap \ker \delta_2) + \operatorname{im} \delta_1 + \operatorname{im} \delta_2, \end{aligned}$$

proving that the natural map $H_{(\delta_1 \pm \delta_2; \delta_1 \pm \delta_2)}(\operatorname{Tot} A^\bullet) \rightarrow \operatorname{Tot} H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet)$ is surjective. \square

By noting that, for $\sharp_{\delta_1, \delta_2} \in \{\delta_1, \delta_2, \delta_1 \delta_2, \delta_1 + \delta_2, \delta_1 - \delta_2\}$,

$$(\ker \sharp_{\delta_1 \otimes_{\mathbb{K}} \operatorname{id}, \delta_2 \otimes_{\mathbb{K}} \beta})^{\bullet_1, \bullet_2} = (\ker \sharp_{\delta_1, \delta_2})^{\hat{\delta}_1 \bullet_1 + \hat{\delta}_2 \bullet_2} \otimes_{\mathbb{K}} \mathbb{K} \beta^{\bullet_2}$$

and

$$(\operatorname{im} \sharp_{\delta_1 \otimes_{\mathbb{K}} \operatorname{id}, \delta_2 \otimes_{\mathbb{K}} \beta})^{\bullet_1, \bullet_2} = (\operatorname{im} \sharp_{\delta_1, \delta_2})^{\hat{\delta}_1 \bullet_1 + \hat{\delta}_2 \bullet_2} \otimes_{\mathbb{K}} \mathbb{K} \beta^{\bullet_2},$$

we get the following lemmata.

Lemma 1.5. *Let A^\bullet be a \mathbb{Z} -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \operatorname{End}^{\hat{\delta}_1}(A^\bullet)$ and $\delta_2 \in \operatorname{End}^{\hat{\delta}_2}(A^\bullet)$ such that $\delta_1^2 = \delta_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = 0$. Then, there are natural isomorphisms of \mathbb{K} -vector spaces*

$$H_{\sharp_{\delta_1 \otimes_{\mathbb{K}} \operatorname{id}, \delta_2 \otimes_{\mathbb{K}} \beta}}^{\bullet_1, \bullet_2}(\operatorname{Doub}^{\bullet, \bullet} A^\bullet) \simeq \operatorname{Doub}^{\bullet_1, \bullet_2} H_{\sharp_{\delta_1, \delta_2}}^\bullet(A^\bullet),$$

where $\sharp_{\delta_1, \delta_2} \in \{(\delta_1, \delta_2; \delta_1 \delta_2), (\delta_1; \delta_1), (\delta_2; \delta_2), (\delta_1 \delta_2; \delta_1, \delta_2)\}$.

Lemma 1.6. *Let A^\bullet be a \mathbb{Z} -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \operatorname{End}^{\hat{\delta}_1}(A^\bullet)$ and $\delta_2 \in \operatorname{End}^{\hat{\delta}_2}(A^\bullet)$ such that $\delta_1^2 = \delta_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = 0$. Denote the greatest common divisor of $\hat{\delta}_1$ and $\hat{\delta}_2$ by $\operatorname{GCD}(\hat{\delta}_1, \hat{\delta}_2)$. The following conditions are equivalent:*

- (i) $A^{\operatorname{GCD}(\hat{\delta}_1, \hat{\delta}_2) \bullet}$ satisfies the $\delta_1 \delta_2$ -Lemma;
- (ii) $\operatorname{Doub}^{\bullet, \bullet}(A^\bullet)$ satisfies the $(\delta_1 \otimes_{\mathbb{K}} \operatorname{id})(\delta_2 \otimes_{\mathbb{K}} \beta)$ -Lemma.

Proof. Indeed,

$$(\ker(\delta_1 \otimes_{\mathbb{K}} \operatorname{id}) \cap \operatorname{im}(\delta_2 \otimes_{\mathbb{K}} \beta))^{\bullet_1, \bullet_2} = \left(\ker \delta_1 \cap \operatorname{im} \delta_2 \cap A^{\hat{\delta}_1 \bullet_1 + \hat{\delta}_2 \bullet_2} \right) \otimes_{\mathbb{K}} \mathbb{K} \beta^{\bullet_2}$$

and

$$(\operatorname{im}(\delta_1 \otimes_{\mathbb{K}} \operatorname{id})(\delta_2 \otimes_{\mathbb{K}} \beta))^{\bullet_1, \bullet_2} = \left(\operatorname{im} \delta_1 \delta_2 \cap A^{\hat{\delta}_1 \bullet_1 + \hat{\delta}_2 \bullet_2} \right) \otimes_{\mathbb{K}} \mathbb{K} \beta^{\bullet_2},$$

completing the proof. \square

2. AN INEQUALITY à la FRÖLICHER

Let $A^{\bullet, \bullet}$ be a bounded \mathbb{Z}^2 -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \operatorname{End}^{1,0}(A^{\bullet, \bullet})$ and $\delta_2 \in \operatorname{End}^{0,1}(A^{\bullet, \bullet})$ such that $\delta_1^2 = \delta_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = 0$. The bi-grading induces two natural bounded filtrations of the \mathbb{Z} -graded \mathbb{K} -vector space $\operatorname{Tot}^\bullet(A^{\bullet, \bullet})$ endowed with the endomorphism $\delta_1 + \delta_2 \in \operatorname{End}^1(\operatorname{Tot}^\bullet(A^{\bullet, \bullet}))$, namely,

$$\left\{ {}'F^p \operatorname{Tot}^\bullet(A^{\bullet, \bullet}) := \bigoplus_{\substack{r+s=\bullet \\ r \geq p}} A^{r,s} \hookrightarrow \operatorname{Tot}^\bullet(A^{\bullet, \bullet}) \right\}_{p \in \mathbb{Z}}$$

and

$$\left\{ {}''F^q \operatorname{Tot}^\bullet(A^{\bullet, \bullet}) := \bigoplus_{\substack{r+s=\bullet \\ s \geq q}} A^{r,s} \hookrightarrow \operatorname{Tot}^\bullet(A^{\bullet, \bullet}) \right\}_{q \in \mathbb{Z}}.$$

Such filtrations induce naturally two spectral sequences, respectively,

$$\left\{ {}'E_r^{\bullet, \bullet}(A^{\bullet, \bullet}, \delta_1, \delta_2) \right\}_{r \in \mathbb{Z}} \quad \text{and} \quad \left\{ {}''E_r^{\bullet, \bullet}(A^{\bullet, \bullet}, \delta_1, \delta_2) \right\}_{r \in \mathbb{Z}},$$

such that

$${}'E_1^{\bullet_1, \bullet_2}(A^{\bullet, \bullet}, \delta_1, \delta_2) \simeq H_{(\delta_2; \delta_2)}^{\bullet_1, \bullet_2}(A^{\bullet, \bullet}) \Rightarrow H_{(\hat{\delta}_1 + \hat{\delta}_2; \delta_1 + \delta_2)}^{\bullet_1 + \bullet_2}(\operatorname{Tot}^\bullet(A^{\bullet, \bullet})),$$

and

$${}''E_1^{\bullet_1, \bullet_2}(A^{\bullet, \bullet}, \delta_1, \delta_2) \simeq H_{(\delta_1; \delta_1)}^{\bullet_1, \bullet_2}(A^{\bullet, \bullet}) \Rightarrow H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet_1 + \bullet_2}(\text{Tot}^\bullet(A^{\bullet, \bullet})) ,$$

see, e.g., [40, §2.4], see also [24, §3.5].

By using these spectral sequences (and up to consider $-\delta_2$ instead of δ_2), one gets the classical *A. Frölicher inequality*.

Notation 2.1. Given two \mathbb{Z} -graded \mathbb{K} -vector spaces A^\bullet and B^\bullet , writing, for example, $\dim_{\mathbb{K}} A^\bullet \geq \dim_{\mathbb{K}} B^\bullet$, we mean that, for any $k \in \mathbb{Z}$, the inequality $\dim_{\mathbb{K}} A^k \geq \dim_{\mathbb{K}} B^k$ holds.

Proposition 2.2 ([20, Theorem 2]). *Let $A^{\bullet, \bullet}$ be a bounded \mathbb{Z}^2 -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \text{End}^{1,0}(A^{\bullet, \bullet})$ and $\delta_2 \in \text{End}^{0,1}(A^{\bullet, \bullet})$ such that $\delta_1^2 = \delta_2^2 = \delta_1\delta_2 + \delta_2\delta_1 = 0$. Then, for $\pm \in \{+, -\}$,*

$$\min \left\{ \dim_{\mathbb{K}} \text{Tot}^\bullet H_{(\delta_1; \delta_1)}^{\bullet, \bullet}(A^{\bullet, \bullet}), \dim_{\mathbb{K}} \text{Tot}^\bullet H_{(\delta_2; \delta_2)}^{\bullet, \bullet}(A^{\bullet, \bullet}) \right\} \geq \dim_{\mathbb{K}} H_{(\delta_1 \pm \delta_2; \delta_1 \pm \delta_2)}^\bullet(\text{Tot}^\bullet A^{\bullet, \bullet}) .$$

As a straightforward consequence, the following result holds in the \mathbb{Z} -graded case.

Corollary 2.3. *Let A^\bullet be a bounded \mathbb{Z} -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \text{End}^{\hat{\delta}_1}(A^\bullet)$ and $\delta_2 \in \text{End}^{\hat{\delta}_2}(A^\bullet)$ such that $\delta_1^2 = \delta_2^2 = \delta_1\delta_2 + \delta_2\delta_1 = 0$. Then, for $\pm \in \{+, -\}$,*

$$\begin{aligned} & \min \left\{ \sum_{p+q=\bullet} \dim_{\mathbb{K}} H_{(\delta_1; \delta_1)}^{\hat{\delta}_1 p + \hat{\delta}_2 q}(A^\bullet), \sum_{p+q=\bullet} \dim_{\mathbb{K}} H_{(\delta_2; \delta_2)}^{\hat{\delta}_1 p + \hat{\delta}_2 q}(A^\bullet) \right\} \\ & \geq \dim_{\mathbb{K}} H_{((\delta_1 \otimes_{\mathbb{K}} \text{id}) \pm (\delta_2 \otimes_{\mathbb{K}} \beta); (\delta_1 \otimes_{\mathbb{K}} \text{id}) \pm (\delta_2 \otimes_{\mathbb{K}} \beta))}^\bullet(\text{Tot}^\bullet \text{Doub}^{\bullet, \bullet} A^\bullet) . \end{aligned}$$

Proof. By Lemma 1.5, one has that, for $\sharp_{\delta_1, \delta_2} \in \{(\delta_1, \delta_2; \delta_1\delta_2), (\delta_1; \delta_1), (\delta_2; \delta_2), (\delta_1\delta_2; \delta_1, \delta_2)\}$,

$$\dim_{\mathbb{K}} H_{\sharp_{\delta_1 \otimes_{\mathbb{K}} \text{id}, \delta_2 \otimes_{\mathbb{K}} \beta}}^{\bullet_1, \bullet_2}(\text{Doub}^{\bullet, \bullet} A^\bullet) = \dim_{\mathbb{K}} H_{\sharp_{\delta_1, \delta_2}}^{\hat{\delta}_1 \bullet_1 + \hat{\delta}_2 \bullet_2}(A^\bullet) .$$

Hence, by applying the classical Frölicher inequality, Proposition 2.2, to $\text{Doub}^{\bullet, \bullet}$ endowed with $\delta_1 \otimes_{\mathbb{K}} \text{id}$ and $\delta_2 \otimes_{\mathbb{K}} \beta$, one gets, for $\pm \in \{+, -\}$,

$$\begin{aligned} & \min \left\{ \sum_{p+q=\bullet} \dim_{\mathbb{K}} H_{(\delta_1; \delta_1)}^{\hat{\delta}_1 p + \hat{\delta}_2 q}(A^\bullet), \sum_{p+q=\bullet} \dim_{\mathbb{K}} H_{(\delta_2; \delta_2)}^{\hat{\delta}_1 p + \hat{\delta}_2 q}(A^\bullet) \right\} \\ & = \min \left\{ \sum_{p+q=\bullet} \dim_{\mathbb{K}} H_{(\delta_1 \otimes_{\mathbb{K}} \text{id}; \delta_1 \otimes_{\mathbb{K}} \text{id})}^{p, q}(\text{Doub}^{\bullet, \bullet} A^\bullet), \sum_{p+q=\bullet} \dim_{\mathbb{K}} H_{(\delta_2 \otimes_{\mathbb{K}} \beta; \delta_2 \otimes_{\mathbb{K}} \beta)}^{p, q}(\text{Doub}^{\bullet, \bullet} A^\bullet) \right\} \\ & = \min \left\{ \dim_{\mathbb{K}} \text{Tot}^\bullet H_{(\delta_1 \otimes_{\mathbb{K}} \text{id}; \delta_1 \otimes_{\mathbb{K}} \text{id})}^{\bullet, \bullet}(\text{Doub}^{\bullet, \bullet} A^\bullet), \dim_{\mathbb{K}} \text{Tot}^\bullet H_{(\delta_2 \otimes_{\mathbb{K}} \beta; \delta_2 \otimes_{\mathbb{K}} \beta)}^{\bullet, \bullet}(\text{Doub}^{\bullet, \bullet} A^\bullet) \right\} \\ & \geq \dim_{\mathbb{K}} H_{((\delta_1 \otimes_{\mathbb{K}} \text{id}) \pm (\delta_2 \otimes_{\mathbb{K}} \beta); (\delta_1 \otimes_{\mathbb{K}} \text{id}) \pm (\delta_2 \otimes_{\mathbb{K}} \beta))}^\bullet(\text{Tot}^\bullet \text{Doub}^{\bullet, \bullet} A^\bullet) , \end{aligned}$$

completing the proof. \square

We prove the following inequality *à la* Frölicher involving the cohomologies $H_{(\delta_1, \delta_2; \delta_1\delta_2)}^\bullet(A^\bullet)$ and $H_{(\delta_1\delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet)$, other than $H_{(\delta_1; \delta_1)}^\bullet(A^\bullet)$ and $H_{(\delta_2; \delta_2)}^\bullet(A^\bullet)$.

Theorem 2.4. *Let A^\bullet be a \mathbb{Z} -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \text{End}^{\hat{\delta}_1}(A^\bullet)$ and $\delta_2 \in \text{End}^{\hat{\delta}_2}(A^\bullet)$ such that $\delta_1^2 = \delta_2^2 = \delta_1\delta_2 + \delta_2\delta_1 = 0$. Suppose that*

$$\dim_{\mathbb{K}} H_{(\delta_1; \delta_1)}^\bullet(A^\bullet) < +\infty \quad \text{and} \quad \dim_{\mathbb{K}} H_{(\delta_2; \delta_2)}^\bullet(A^\bullet) < +\infty .$$

Then

$$(1) \quad \dim_{\mathbb{K}} H_{(\delta_1, \delta_2; \delta_1\delta_2)}^\bullet(A^\bullet) + \dim_{\mathbb{K}} H_{(\delta_1\delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet) \geq \dim_{\mathbb{K}} H_{(\delta_1; \delta_1)}^\bullet(A^\bullet) + \dim_{\mathbb{K}} H_{(\delta_2; \delta_2)}^\bullet(A^\bullet) .$$

Proof. If either $H_{(\delta_1, \delta_2; \delta_1\delta_2)}^\bullet(A^\bullet)$ or $H_{(\delta_1\delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet)$ is not finite-dimensional, then the inequality holds trivially; hence, we are reduced to suppose that also

$$\dim_{\mathbb{K}} H_{(\delta_1, \delta_2; \delta_1\delta_2)}^\bullet(A^\bullet) < +\infty \quad \text{and} \quad \dim_{\mathbb{K}} H_{(\delta_1\delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet) < +\infty .$$

Following J. Varouchas, [50, §3.1], consider the exact sequences

$$\begin{aligned}
0 &\rightarrow \frac{\operatorname{im} \delta_1 \cap \operatorname{im} \delta_2}{\operatorname{im} \delta_1 \delta_2} \rightarrow \frac{\ker \delta_1 \cap \operatorname{im} \delta_2}{\operatorname{im} \delta_1 \delta_2} \rightarrow \frac{\ker \delta_1}{\operatorname{im} \delta_1} \rightarrow \frac{\ker \delta_1 \delta_2}{\operatorname{im} \delta_1 + \operatorname{im} \delta_2} \rightarrow \frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \operatorname{im} \delta_2} \rightarrow 0, \\
0 &\rightarrow \frac{\operatorname{im} \delta_1 \cap \operatorname{im} \delta_2}{\operatorname{im} \delta_1 \delta_2} \rightarrow \frac{\ker \delta_2 \cap \operatorname{im} \delta_1}{\operatorname{im} \delta_1 \delta_2} \rightarrow \frac{\ker \delta_2}{\operatorname{im} \delta_2} \rightarrow \frac{\ker \delta_1 \delta_2}{\operatorname{im} \delta_1 + \operatorname{im} \delta_2} \rightarrow \frac{\ker \delta_1 \delta_2}{\ker \delta_2 + \operatorname{im} \delta_1} \rightarrow 0, \\
0 &\rightarrow \frac{\operatorname{im} \delta_1 \cap \ker \delta_2}{\operatorname{im} \delta_1 \delta_2} \rightarrow \frac{\ker \delta_1 \cap \ker \delta_2}{\operatorname{im} \delta_1 \delta_2} \rightarrow \frac{\ker \delta_1}{\operatorname{im} \delta_1} \rightarrow \frac{\ker \delta_1 \delta_2}{\ker \delta_2 + \operatorname{im} \delta_1} \rightarrow \frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \ker \delta_2} \rightarrow 0, \\
0 &\rightarrow \frac{\operatorname{im} \delta_2 \cap \ker \delta_1}{\operatorname{im} \delta_1 \delta_2} \rightarrow \frac{\ker \delta_1 \cap \ker \delta_2}{\operatorname{im} \delta_1 \delta_2} \rightarrow \frac{\ker \delta_2}{\operatorname{im} \delta_2} \rightarrow \frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \operatorname{im} \delta_2} \rightarrow \frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \ker \delta_2} \rightarrow 0
\end{aligned}$$

of \mathbb{Z} -graded \mathbb{K} -vector spaces.

Note that all the \mathbb{K} -vector spaces appearing in the exact sequences have finite dimension. Indeed, since $H_{(\delta_1, \delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet)$ has finite dimension, then

$$\dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \operatorname{im} \delta_2} < +\infty \quad \text{and} \quad \dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\ker \delta_2 + \operatorname{im} \delta_1} < +\infty;$$

since $H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^\bullet(A^\bullet)$ has finite dimension, then

$$\dim_{\mathbb{K}} \frac{\operatorname{im} \delta_1 \cap \ker \delta_2}{\operatorname{im} \delta_1 \delta_2} < +\infty \quad \text{and} \quad \dim_{\mathbb{K}} \frac{\operatorname{im} \delta_2 \cap \ker \delta_1}{\operatorname{im} \delta_1 \delta_2} < +\infty.$$

Furthermore, note that the natural maps $\frac{\ker \delta_1 \cap \operatorname{im} \delta_2}{\operatorname{im} \delta_1 \delta_2} \rightarrow H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^\bullet(A^\bullet)$ and $\frac{\ker \delta_2 \cap \operatorname{im} \delta_1}{\operatorname{im} \delta_1 \delta_2} \rightarrow H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^\bullet(A^\bullet)$ induced by the identity are injective, and hence

$$\dim_{\mathbb{K}} \frac{\ker \delta_1 \cap \operatorname{im} \delta_2}{\operatorname{im} \delta_1 \delta_2} < +\infty \quad \text{and} \quad \dim_{\mathbb{K}} \frac{\ker \delta_2 \cap \operatorname{im} \delta_1}{\operatorname{im} \delta_1 \delta_2} < +\infty;$$

it follows also that

$$\dim_{\mathbb{K}} \frac{\operatorname{im} \delta_1 \cap \operatorname{im} \delta_2}{\operatorname{im} \delta_1 \delta_2} < +\infty.$$

Analogously, since the natural maps $H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet) \rightarrow \frac{\ker \delta_1 \delta_2}{\ker \delta_2 + \operatorname{im} \delta_1}$ and $H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet) \rightarrow \frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \operatorname{im} \delta_2}$ induced by the identity are surjective, then

$$\dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\ker \delta_2 + \operatorname{im} \delta_1} < +\infty \quad \text{and} \quad \dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \operatorname{im} \delta_2} < +\infty,$$

and hence also

$$\dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \ker \delta_2} < +\infty.$$

By using the above exact sequences, it follows that

$$\begin{aligned}
\dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\operatorname{im} \delta_1 + \operatorname{im} \delta_2} &= \dim_{\mathbb{K}} \frac{\operatorname{im} \delta_1 \cap \operatorname{im} \delta_2}{\operatorname{im} \delta_1 \delta_2} - \dim_{\mathbb{K}} \frac{\ker \delta_1 \cap \operatorname{im} \delta_2}{\operatorname{im} \delta_1 \delta_2} + \dim_{\mathbb{K}} \frac{\ker \delta_1}{\operatorname{im} \delta_1} + \dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \operatorname{im} \delta_2}, \\
\dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\operatorname{im} \delta_1 + \operatorname{im} \delta_2} &= \dim_{\mathbb{K}} \frac{\operatorname{im} \delta_1 \cap \operatorname{im} \delta_2}{\operatorname{im} \delta_1 \delta_2} - \dim_{\mathbb{K}} \frac{\ker \delta_2 \cap \operatorname{im} \delta_1}{\operatorname{im} \delta_1 \delta_2} + \dim_{\mathbb{K}} \frac{\ker \delta_2}{\operatorname{im} \delta_2} + \dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\ker \delta_2 + \operatorname{im} \delta_1}, \\
\dim_{\mathbb{K}} \frac{\ker \delta_1 \cap \ker \delta_2}{\operatorname{im} \delta_1 \delta_2} &= \dim_{\mathbb{K}} \frac{\operatorname{im} \delta_1 \cap \ker \delta_2}{\operatorname{im} \delta_1 \delta_2} + \dim_{\mathbb{K}} \frac{\ker \delta_1}{\operatorname{im} \delta_1} - \dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\ker \delta_2 + \operatorname{im} \delta_1} + \dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \ker \delta_2}, \\
\dim_{\mathbb{K}} \frac{\ker \delta_1 \cap \ker \delta_2}{\operatorname{im} \delta_1 \delta_2} &= \dim_{\mathbb{K}} \frac{\operatorname{im} \delta_2 \cap \ker \delta_1}{\operatorname{im} \delta_1 \delta_2} + \dim_{\mathbb{K}} \frac{\ker \delta_2}{\operatorname{im} \delta_2} - \dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \operatorname{im} \delta_2} + \dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \ker \delta_2},
\end{aligned}$$

from which, by summing up, one gets

$$\begin{aligned}
(2) \quad &2 \dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\operatorname{im} \delta_1 + \operatorname{im} \delta_2} + 2 \dim_{\mathbb{K}} \frac{\ker \delta_1 \cap \ker \delta_2}{\operatorname{im} \delta_1 \delta_2} \\
&= 2 \dim_{\mathbb{K}} \frac{\ker \delta_1}{\operatorname{im} \delta_1} + 2 \dim_{\mathbb{K}} \frac{\ker \delta_2}{\operatorname{im} \delta_2} + 2 \dim_{\mathbb{K}} \frac{\operatorname{im} \delta_1 \cap \operatorname{im} \delta_2}{\operatorname{im} \delta_1 \delta_2} + 2 \dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \ker \delta_2} \\
&\geq 2 \dim_{\mathbb{K}} \frac{\ker \delta_1}{\operatorname{im} \delta_1} + 2 \dim_{\mathbb{K}} \frac{\ker \delta_2}{\operatorname{im} \delta_2},
\end{aligned}$$

yielding

$$\dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\operatorname{im} \delta_1 + \operatorname{im} \delta_2} + \dim_{\mathbb{K}} \frac{\ker \delta_1 \cap \ker \delta_2}{\operatorname{im} \delta_1 \delta_2} \geq \dim_{\mathbb{K}} \frac{\ker \delta_1}{\operatorname{im} \delta_1} + \dim_{\mathbb{K}} \frac{\ker \delta_2}{\operatorname{im} \delta_2},$$

and hence the theorem. \square

Remark 2.5. Note that the proof of Theorem 2.4 works also for \mathbb{Z}^2 -graded \mathbb{K} -vector spaces, since in this case J. Varouchas' exact sequences are in fact exact sequences of \mathbb{Z}^2 -graded \mathbb{K} -vector spaces. More precisely, one gets that, given a \mathbb{Z}^2 -graded \mathbb{K} -vector space $A^{\bullet, \bullet}$ endowed with two endomorphisms $\delta_1 \in \operatorname{End}^{\delta_{1,1}, \delta_{1,2}}(A^{\bullet, \bullet})$ and $\delta_2 \in \operatorname{End}^{\delta_{2,1}, \delta_{2,2}}(A^{\bullet, \bullet})$ such that $\delta_1^2 = \delta_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = 0$, and supposed that

$$\dim_{\mathbb{K}} H_{(\delta_1; \delta_1)}^{\bullet, \bullet}(A^{\bullet, \bullet}) < +\infty \quad \text{and} \quad \dim_{\mathbb{K}} H_{(\delta_2; \delta_2)}^{\bullet, \bullet}(A^{\bullet, \bullet}) < +\infty,$$

then

$$\dim_{\mathbb{K}} H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet, \bullet}(A^{\bullet, \bullet}) + \dim_{\mathbb{K}} H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet, \bullet}(A^{\bullet, \bullet}) \geq \dim_{\mathbb{K}} H_{(\delta_1; \delta_1)}^{\bullet, \bullet}(A^{\bullet, \bullet}) + \dim_{\mathbb{K}} H_{(\delta_2; \delta_2)}^{\bullet, \bullet}(A^{\bullet, \bullet}).$$

As a consequence of Theorem 2.4 and Proposition 2.2, one gets the following inequality *à la* Frölicher for double complexes, namely, \mathbb{Z}^2 -graded \mathbb{K} -vector spaces $B^{\bullet, \bullet}$ endowed with two endomorphisms $\delta_1 \in \operatorname{End}^{1,0}(B^{\bullet, \bullet})$ and $\delta_2 \in \operatorname{End}^{0,1}(B^{\bullet, \bullet})$ such that $\delta_1^2 = \delta_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = 0$.

Corollary 2.6. *Let $B^{\bullet, \bullet}$ be a bounded \mathbb{Z}^2 -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \operatorname{End}^{1,0}(B^{\bullet, \bullet})$ and $\delta_2 \in \operatorname{End}^{0,1}(B^{\bullet, \bullet})$ such that $\delta_1^2 = \delta_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = 0$. Suppose that*

$$\dim_{\mathbb{K}} \operatorname{Tot}^{\bullet} H_{(\delta_1; \delta_1)}^{\bullet, \bullet}(B^{\bullet, \bullet}) < +\infty \quad \text{and} \quad \dim_{\mathbb{K}} \operatorname{Tot}^{\bullet} H_{(\delta_2; \delta_2)}^{\bullet, \bullet}(B^{\bullet, \bullet}) < +\infty.$$

Then, for $\pm \in \{+, -\}$,

$$(3) \quad \dim_{\mathbb{K}} \operatorname{Tot}^{\bullet} H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet, \bullet}(B^{\bullet, \bullet}) + \dim_{\mathbb{K}} \operatorname{Tot}^{\bullet} H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet, \bullet}(B^{\bullet, \bullet}) \geq 2 \dim_{\mathbb{K}} H_{(\delta_1 \pm \delta_2; \delta_1 \pm \delta_2)}^{\bullet}(\operatorname{Tot}^{\bullet} B^{\bullet, \bullet}).$$

3. A CHARACTERIZATION OF $\delta_1 \delta_2$ -LEMMA BY MEANS OF THE INEQUALITY *à la* FRÖLICHER

With the aim to characterize the validity of the $\delta_1 \delta_2$ -Lemma in terms of the dimensions of the cohomologies $H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet}(A^{\bullet})$ and $H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet}(A^{\bullet})$, we need the following lemmata.

Lemma 3.1. *Let $B^{\bullet, \bullet}$ be a \mathbb{Z}^2 -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \operatorname{End}^{1,0}(B^{\bullet, \bullet})$ and $\delta_2 \in \operatorname{End}^{0,1}(B^{\bullet, \bullet})$ such that $\delta_1^2 = \delta_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = 0$. If*

$$\frac{\operatorname{im} \delta_1 \cap \operatorname{im} \delta_2}{\operatorname{im} \delta_1 \delta_2} = \{0\},$$

then the natural map $\iota: \operatorname{Tot}^{\bullet} H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet, \bullet}(B^{\bullet, \bullet}) \rightarrow H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\operatorname{Tot}^{\bullet} B^{\bullet, \bullet})$ induced by the identity is surjective.

Proof. Let $\mathbf{a} := [x] \in H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\operatorname{Tot}^{\bullet} B^{\bullet, \bullet})$. Since $\delta_1(x) + \delta_2(x) = 0$ and $\operatorname{im} \delta_1 \cap \operatorname{im} \delta_2 = \operatorname{im} \delta_1 \delta_2$, then we have $\delta_1(x) = -\delta_2(x) \in \operatorname{im} \delta_1 \cap \operatorname{im} \delta_2 = \operatorname{im} \delta_1 \delta_2$; let $y \in \operatorname{Tot}^{\bullet-1} B^{\bullet, \bullet}$ be such that $\delta_1(x) = \delta_1 \delta_2(y) = -\delta_2(x)$. Hence, consider $\mathbf{a} = [x] = [x - (\delta_1 + \delta_2)(y)] \in H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\operatorname{Tot}^{\bullet} B^{\bullet, \bullet})$, and note that $\mathbf{a} = \iota([x - (\delta_1 + \delta_2)(y)])$ where $[x - (\delta_1 + \delta_2)(y)] \in \operatorname{Tot}^{\bullet} H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet, \bullet}(B^{\bullet, \bullet})$, since $\delta_1(x - (\delta_1 + \delta_2)(y)) = 0$ and $\delta_2(x - (\delta_1 + \delta_2)(y)) = 0$. \square

Lemma 3.2. *Let $B^{\bullet, \bullet}$ be a \mathbb{Z}^2 -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \operatorname{End}^{1,0}(B^{\bullet, \bullet})$ and $\delta_2 \in \operatorname{End}^{0,1}(B^{\bullet, \bullet})$ such that $\delta_1^2 = \delta_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = 0$. If*

$$\frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \ker \delta_2} = \{0\},$$

then the natural map $\iota: H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\operatorname{Tot}^{\bullet} B^{\bullet, \bullet}) \rightarrow \operatorname{Tot}^{\bullet} H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet, \bullet}(B^{\bullet, \bullet})$ induced by the identity is injective.

Proof. Let $\mathbf{a} := [x] \in H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\operatorname{Tot}^{\bullet} B^{\bullet, \bullet})$. Suppose that $\iota(\mathbf{a}) = [0] \in \operatorname{Tot}^{\bullet} H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet, \bullet}(B^{\bullet, \bullet})$, that is, there exist $y \in \operatorname{Tot}^{\bullet-1} B^{\bullet, \bullet}$ and $z \in \operatorname{Tot}^{\bullet-1} B^{\bullet, \bullet}$ such that $x = \delta_1(y) + \delta_2(z)$. Since $(\delta_1 + \delta_2)(x) = 0$ and $\ker \delta_1 \delta_2 = \ker \delta_1 + \ker \delta_2$, it follows that $\delta_1 \delta_2(z - y) = 0$, that is, $z - y \in \ker \delta_1 \delta_2 = \ker \delta_1 + \ker \delta_2$. Let $u \in \ker \delta_1$ and $v \in \ker \delta_2$ be such that $z - y = u + v$. Then, one has that $x = \delta_1(y) + \delta_2(z) = \delta_1(y) + \delta_2(y + u + v) = (\delta_1 + \delta_2)(y + u) \in \operatorname{im}(\delta_1 + \delta_2)$, proving that $\mathbf{a} = [0] \in H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\operatorname{Tot}^{\bullet} B^{\bullet, \bullet})$. \square

We can now prove the following characterization of the $\delta_1 \delta_2$ -Lemma for double complexes in terms of the equality in (3).

Theorem 3.3. *Let $B^{\bullet,\bullet}$ be a bounded \mathbb{Z}^2 -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \text{End}^{1,0}(B^{\bullet,\bullet})$ and $\delta_2 \in \text{End}^{0,1}(B^{\bullet,\bullet})$ such that $\delta_1^2 = \delta_2^2 = \delta_1\delta_2 + \delta_2\delta_1 = 0$. Suppose that*

$$\dim_{\mathbb{K}} H_{(\delta_1, \delta_1)}^{\bullet,\bullet}(B^{\bullet,\bullet}) < +\infty \quad \text{and} \quad \dim_{\mathbb{K}} H_{(\delta_2, \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) < +\infty .$$

The following conditions are equivalent:

- (i) $B^{\bullet,\bullet}$ satisfies the $\delta_1\delta_2$ -Lemma;
- (ii) the equality

$$\begin{aligned} & \dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) + \dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) \\ &= 2 \dim_{\mathbb{K}} H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet}) . \end{aligned}$$

holds.

Proof. [(i) \Rightarrow (ii)]. By Lemma 1.4, it follows that

$$\dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) \leq \dim_{\mathbb{K}} H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet})$$

and

$$\dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) \leq \dim_{\mathbb{K}} H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet}) .$$

By Corollary 2.6, it follows that

$$\dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) + \dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) \geq 2 \dim_{\mathbb{K}} H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet}) .$$

Hence actually the equality holds.

[(ii) \Rightarrow (i)]. Since, by (2) and Proposition 2.2, it holds

$$\begin{aligned} & \dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) + \dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) \\ &= \dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1, \delta_1)}^{\bullet,\bullet}(B^{\bullet,\bullet}) + \dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_2, \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) \\ & \quad + \dim_{\mathbb{K}} \frac{\text{im } \delta_1 \cap \text{im } \delta_2}{\text{im } \delta_1 \delta_2} + \dim_{\mathbb{K}} \frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \ker \delta_2} \\ & \geq 2 \dim_{\mathbb{K}} H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet}) , \end{aligned}$$

then, by the hypothesis, it follows that

$$\frac{\text{im } \delta_1 \cap \text{im } \delta_2}{\text{im } \delta_1 \delta_2} = \{0\} \quad \text{and} \quad \frac{\ker \delta_1 \delta_2}{\ker \delta_1 + \ker \delta_2} = \{0\} .$$

By Lemma 3.1, one gets that the natural map $H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet}) \rightarrow H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet})$ induced by the identity is surjective; by Lemma 3.2, one gets that the natural map $H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet}) \rightarrow H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet})$ induced by the identity is injective. In particular, one has that

$$\dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) \geq \dim_{\mathbb{K}} H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet})$$

and that

$$\dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) \geq \dim_{\mathbb{K}} H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet}) .$$

Hence, by the hypothesis, it holds in fact that

$$\dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) = \dim_{\mathbb{K}} H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet}) = \dim_{\mathbb{K}} \text{Tot}^{\bullet} H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet}) .$$

Since $H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet})$ and $H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet,\bullet}(B^{\bullet,\bullet})$ are finite-dimensional by hypothesis, it follows that the natural maps $H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet}) \rightarrow H_{(\delta_1 + \delta_2; \delta_1 + \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet})$ and $H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet}) \rightarrow H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\bullet}(\text{Tot}^{\bullet} B^{\bullet,\bullet})$ induced by the identity are in fact isomorphisms. By Lemma 1.4, one gets the theorem. \square

In order to apply Theorem 3.3 to \mathbb{Z} -graded \mathbb{K} -vector spaces to get geometric applications, e.g., for compact symplectic manifolds, we need to record the following corollaries.

Corollary 3.4. *Let A^\bullet be a bounded \mathbb{Z} -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \text{End}^{\hat{\delta}_1}(A^\bullet)$ and $\delta_2 \in \text{End}^{\hat{\delta}_2}(A^\bullet)$ such that $\delta_1^2 = \delta_2^2 = \delta_1\delta_2 + \delta_2\delta_1 = 0$. Denote the greatest common divisor of $\hat{\delta}_1$ and $\hat{\delta}_2$ by $\text{GCD}(\hat{\delta}_1, \hat{\delta}_2)$. Suppose that*

$$\dim_{\mathbb{K}} H_{(\delta_1; \delta_1)}^\bullet(A^\bullet) < +\infty \quad \text{and} \quad \dim_{\mathbb{K}} H_{(\delta_2; \delta_2)}^\bullet(A^\bullet) < +\infty .$$

The following conditions are equivalent:

- (i) $A^{\text{GCD}(\hat{\delta}_1, \hat{\delta}_2)\bullet}$ satisfies the $\delta_1\delta_2$ -Lemma;
- (ii) the equality

$$\begin{aligned} & \sum_{p+q=\bullet} \left(\dim_{\mathbb{K}} H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^{\hat{\delta}_1 p + \hat{\delta}_2 q}(A^\bullet) + \dim_{\mathbb{K}} H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^{\hat{\delta}_1 p + \hat{\delta}_2 q}(A^\bullet) \right) \\ &= 2 \dim_{\mathbb{K}} H_{(\delta_1 \otimes_{\mathbb{K}} \text{id} + \delta_2 \otimes_{\mathbb{K}} \beta; \delta_1 \otimes_{\mathbb{K}} \text{id} + \delta_2 \otimes_{\mathbb{K}} \beta)}^\bullet(\text{Tot}^\bullet \text{Doub}^{\bullet, \bullet} A^\bullet) . \end{aligned}$$

holds.

Proof. The Corollary follows from Lemma 1.6, Theorem 3.3, and Lemma 1.5. \square

Corollary 3.5. *Let A^\bullet be a bounded \mathbb{Z} -graded \mathbb{K} -vector space endowed with two endomorphisms $\delta_1 \in \text{End}^{\hat{\delta}_1}(A^\bullet)$ and $\delta_2 \in \text{End}^{\hat{\delta}_2}(A^\bullet)$ such that $\delta_1^2 = \delta_2^2 = \delta_1\delta_2 + \delta_2\delta_1 = 0$. Suppose that the greatest common divisor of $\hat{\delta}_1$ and $\hat{\delta}_2$ is $\text{GCD}(\hat{\delta}_1, \hat{\delta}_2) = 1$, and that*

$$\dim_{\mathbb{K}} H_{(\delta_1; \delta_1)}^\bullet(A^\bullet) < +\infty \quad \text{and} \quad \dim_{\mathbb{K}} H_{(\delta_2; \delta_2)}^\bullet(A^\bullet) < +\infty .$$

The following conditions are equivalent:

- (i) A^\bullet satisfies the $\delta_1\delta_2$ -Lemma;
- (ii) (a) both the Hodge and Frölicher spectral sequences of $(\text{Doub}^{\bullet, \bullet} A^\bullet, \delta_1 \otimes_{\mathbb{K}} \text{id}, \delta_2 \otimes_{\mathbb{K}} \beta)$ degenerate at the first level, equivalently, the equalities

$$\begin{aligned} & \dim_{\mathbb{K}} H_{(\delta_1 \otimes_{\mathbb{K}} \text{id} + \delta_2 \otimes_{\mathbb{K}} \beta; \delta_1 \otimes_{\mathbb{K}} \text{id} + \delta_2 \otimes_{\mathbb{K}} \beta)}^\bullet(\text{Tot}^\bullet \text{Doub}^{\bullet, \bullet} A^\bullet) \\ &= \dim_{\mathbb{K}} \text{Tot}^\bullet H_{(\delta_2 \otimes_{\mathbb{K}} \beta; \delta_2 \otimes_{\mathbb{K}} \beta)}^{\bullet, \bullet}(\text{Doub}^{\bullet, \bullet} A^\bullet) \\ &= \dim_{\mathbb{K}} \text{Tot}^\bullet H_{(\delta_1 \otimes_{\mathbb{K}} \text{id}; \delta_1 \otimes_{\mathbb{K}} \text{id})}^{\bullet, \bullet}(\text{Doub}^{\bullet, \bullet} A^\bullet) \end{aligned}$$

hold;

- (b) the equality

$$\begin{aligned} & \dim_{\mathbb{K}} H_{(\delta_1, \delta_2; \delta_1 \delta_2)}^\bullet(A^\bullet) + \dim_{\mathbb{K}} H_{(\delta_1 \delta_2; \delta_1, \delta_2)}^\bullet(A^\bullet) \\ &= \dim_{\mathbb{K}} H_{(\delta_1; \delta_1)}^\bullet(A^\bullet) + \dim_{\mathbb{K}} H_{(\delta_2; \delta_2)}^\bullet(A^\bullet) \end{aligned}$$

holds.

Proof. The Corollary follows from Corollary 3.4, Proposition 2.2, Theorem 2.4, and Lemma 1.5. \square

4. APPLICATIONS

In this section, we prove or recover applications of the inequality *à la* Frölicher, Theorem 2.4 and Theorem 3.3, to the complex, symplectic, and generalized complex cases.

4.1. Complex structures. Let X be a compact complex manifold. Consider the \mathbb{Z}^2 -graded \mathbb{C} -vector space $\wedge^{\bullet, \bullet} X$ of bi-graded complex differential forms endowed with the endomorphisms $\partial \in \text{End}^{1,0}(\wedge^{\bullet, \bullet} X)$ and $\bar{\partial} \in \text{End}^{0,1}(\wedge^{\bullet, \bullet} X)$, which satisfy $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$. As usual, define the Dolbeault cohomologies as

$$H_{\partial}^{\bullet, \bullet}(X) := H_{(\partial; \partial)}^{\bullet, \bullet}(\wedge^{\bullet, \bullet} X) , \quad H_{\bar{\partial}}^{\bullet, \bullet}(X) := H_{(\bar{\partial}; \bar{\partial})}^{\bullet, \bullet}(\wedge^{\bullet, \bullet} X) ,$$

and the *Bott-Chern cohomology* and the *Aeppli cohomology* as, respectively, [11, 1],

$$H_{BC}^{\bullet, \bullet}(X) := H_{(\partial, \bar{\partial}; \partial \bar{\partial})}^{\bullet, \bullet}(\wedge^{\bullet, \bullet} X) , \quad H_A^{\bullet, \bullet}(X) := H_{(\partial \bar{\partial}; \partial, \bar{\partial})}^{\bullet, \bullet}(\wedge^{\bullet, \bullet} X) .$$

Note that, since X is a compact manifold, $\dim_{\mathbb{C}} \text{Tot}^\bullet H_{\bar{\partial}}^{\bullet, \bullet}(X) < +\infty$: indeed, for any Hermitian metric g with \mathbb{C} -linear Hodge- $*$ -operator $*_g: \wedge^{\bullet_1, \bullet_2} X \rightarrow \wedge^{\dim_{\mathbb{C}} X - \bullet_2, \dim_{\mathbb{C}} X - \bullet_1}$, one has an isomorphism $\ker \left[\bar{\partial}, \bar{\partial}^* \right] \xrightarrow{\cong} H_{\bar{\partial}}^{\bullet, \bullet}(X)$, where $\bar{\partial}^*$ is the adjoint operator of $\bar{\partial}$ with respect to the inner product induced

on $\wedge^{\bullet, \bullet} X$ by g , and the 2nd-order self-adjoint differential operator $[\bar{\partial}, \bar{\partial}^*]$ is elliptic. Furthermore, $\dim_{\mathbb{C}} \text{Tot}^{\bullet} H_{\bar{\partial}}^{\bullet, \bullet}(X) = \dim_{\mathbb{C}} \text{Tot}^{\bullet} H_{\bar{\partial}}^{\bullet, \bullet}(X) < +\infty$, since conjugation induces the (\mathbb{C} -anti-linear) isomorphism $H_{\bar{\partial}}^{\bullet 1, \bullet 2}(X) \simeq H_{\bar{\partial}}^{\bullet 2, \bullet 1}(X)$ of \mathbb{R} -vector spaces.

Note also that $\dim_{\mathbb{C}} \text{Tot}^{\bullet} H_{BC}^{\bullet, \bullet}(X) = \dim_{\mathbb{C}} \text{Tot}^{2 \dim_{\mathbb{C}} X - \bullet} H_A^{\bullet, \bullet}(X) < +\infty$, [45, Corollaire 2.3, §2.c]: indeed, for any Hermitian metric g on X , the \mathbb{C} -linear Hodge- $*$ -operator $*_g: \wedge^{\bullet 1, \bullet 2} X \rightarrow \wedge^{\dim_{\mathbb{C}} X - \bullet 2, \dim_{\mathbb{C}} X - \bullet 1} X$ induces the isomorphism $*_g: H_{BC}^{\bullet 1, \bullet 2}(X) \xrightarrow{\simeq} H_A^{\dim_{\mathbb{C}} X - \bullet 2, \dim_{\mathbb{C}} X - \bullet 1}(X)$, [45, §2.c], and $\ker \tilde{\Delta}_{BC} \xrightarrow{\simeq} H_{BC}^{\bullet, \bullet}(X)$, [45, Théorème 2.2], where $\tilde{\Delta}_{BC} := (\partial \bar{\partial}) (\partial \bar{\partial})^* + (\partial \bar{\partial})^* (\partial \bar{\partial}) + (\bar{\partial}^* \partial) (\bar{\partial}^* \partial)^* + (\bar{\partial}^* \partial)^* (\bar{\partial}^* \partial) + \bar{\partial}^* \bar{\partial} + \partial^* \partial$ is a 4th-order self-adjoint elliptic differential operator, [32, Proposition 5], see also [45, §2.b].

By abuse of notation, one says that X satisfies the $\partial \bar{\partial}$ -Lemma if the double complex $(\wedge^{\bullet, \bullet} X, \partial, \bar{\partial})$ satisfies the $\partial \bar{\partial}$ -Lemma, and one says that X satisfies the $d d^c$ -Lemma if the \mathbb{Z} -graded \mathbb{C} -vector space $\wedge^{\bullet} X \otimes_{\mathbb{R}} \mathbb{C}$ endowed with the endomorphisms $d \in \text{End}^1(\wedge^{\bullet} X \otimes_{\mathbb{R}} \mathbb{C})$ and $d^c := -i(\partial - \bar{\partial}) \in \text{End}^1(\wedge^{\bullet} X \otimes_{\mathbb{R}} \mathbb{C})$ such that $d^2 = (d^c)^2 = [d, d^c] = 0$ satisfies the $d d^c$ -Lemma. Actually, it turns out that X satisfies the $d d^c$ -Lemma if and only if X satisfies the $\partial \bar{\partial}$ -Lemma, [17, Remark 5.14]: indeed, note that $\partial = \frac{1}{2}(d + i d^c)$ and $\bar{\partial} = \frac{1}{2}(d - i d^c)$, and $\partial \bar{\partial} = -\frac{i}{2} d d^c$.

From Corollary 2.6 and Theorem 3.3, one gets straightforwardly the following inequality *à la* Frölicher for the Bott-Chern cohomology of a compact complex manifolds and the corresponding characterization of the $\partial \bar{\partial}$ -Lemma by means of the Bott-Chern cohomology, first proved by the authors in [6].

Corollary 4.1 ([6, Theorem A, Theorem B]). *Let X be a compact complex manifold. The inequality*

$$(4) \quad \dim_{\mathbb{C}} \text{Tot}^{\bullet} H_{BC}^{\bullet, \bullet}(X) + \dim_{\mathbb{C}} \text{Tot}^{\bullet} H_A^{\bullet, \bullet}(X) \geq 2 \dim_{\mathbb{C}} H_{dR}^{\bullet}(X; \mathbb{C})$$

holds. Furthermore, the equality in (4) holds if and only if X satisfies the $\partial \bar{\partial}$ -Lemma.

4.2. Symplectic structures. Let X be a $2n$ -dimensional compact manifold endowed with a *symplectic structure* ω , namely, a non-degenerate d -closed 2-form on X . The symplectic form ω induces a natural isomorphism $I: TX \xrightarrow{\simeq} T^*X$; more precisely, $I(\cdot)(\cdot) := \omega(\cdot, \cdot)$. Set $\Pi :=: \omega^{-1} := \omega(I^{-1} \cdot, I^{-1} \cdot) \in \wedge^2 TX$ the *canonical Poisson bi-vector* associated to ω , namely, in a Darboux chart with local coordinates $\{x^1, \dots, x^n, y^1, \dots, y^n\}$ such that $\omega \stackrel{\text{loc}}{=} \sum_{j=1}^n dx^j \wedge dy^j$, one has $\omega^{-1} \stackrel{\text{loc}}{=} \sum_{j=1}^n \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial y^j}$. One gets a bi- \mathbb{R} -linear form on $\wedge^k X$, denoted by $(\omega^{-1})^k$, by defining it on the simple elements $\alpha^1 \wedge \dots \wedge \alpha^k \in \wedge^k X$ and $\beta^1 \wedge \dots \wedge \beta^k \in \wedge^k X$ as

$$(\omega^{-1})^k (\alpha^1 \wedge \dots \wedge \alpha^k, \beta^1 \wedge \dots \wedge \beta^k) := \det (\omega^{-1} (\alpha^\ell, \beta^m))_{\ell, m \in \{1, \dots, k\}} ;$$

note that $(\omega^{-1})^k$ is skew-symmetric, respectively symmetric, according to k is odd, respectively even.

We recall that the operators

$$\begin{aligned} L &\in \text{End}^2(\wedge^{\bullet} X) , & L(\alpha) &:= \omega \wedge \alpha , \\ \Lambda &\in \text{End}^{-2}(\wedge^{\bullet} X) , & \Lambda(\alpha) &:= -\iota_{\Pi} \alpha , \\ H &\in \text{End}^0(\wedge^{\bullet} X) , & H(\alpha) &:= \sum_{k \in \mathbb{Z}} (n - k) \pi_{\wedge^k X} \alpha , \end{aligned}$$

yield an $\mathfrak{sl}(2; \mathbb{R})$ -representation on $\wedge^{\bullet} X$ (where $\iota_{\xi}: \wedge^{\bullet} X \rightarrow \wedge^{\bullet-2} X$ denotes the interior product with $\xi \in \wedge^2(TX)$, and $\pi_{\wedge^k X}: \wedge^{\bullet} X \rightarrow \wedge^k X$ denotes the natural projection onto $\wedge^k X$, for $k \in \mathbb{Z}$).

Define the *symplectic co-differential operator* as

$$d^{\Lambda} := [d, \Lambda] \in \text{End}^{-1}(\wedge^{\bullet} X) ;$$

one has that $(d^{\Lambda})^2 = [d, d^{\Lambda}] = 0$, see [34, page 266, page 265], [12, Proposition 1.2.3, Theorem 1.3.1].

As a matter of notation, for $\sharp \in \left\{ (d, d^{\Lambda}; d d^{\Lambda}), (d; d), (d^{\Lambda}; d^{\Lambda}), (d d^{\Lambda}; d, d^{\Lambda}) \right\}$, we shorten $H_{\sharp}^{\bullet}(X) := H_{\sharp}^{\bullet}(\wedge^{\bullet} X)$. Note that $H_{(d; d)}^{\bullet}(X) = H_{dR}^{\bullet}(X; \mathbb{R})$. As regards notation introduced by L.-S. Tseng and S.-T. Yau in [47, §3], note that $H_{(d^{\Lambda}; d^{\Lambda})}^{\bullet}(X) = H_{d^{\Lambda}}^{\bullet}(X)$, and that $H_{(d, d^{\Lambda}; d d^{\Lambda})}^{\bullet}(X) = H_{d + d^{\Lambda}}^{\bullet}(X)$, and that $H_{(d d^{\Lambda}; d, d^{\Lambda})}^{\bullet}(X) = H_{d d^{\Lambda}}^{\bullet}(X)$.

Note also that, as a consequence of the Hodge theory developed by L.-S. Tseng and S.-T. Yau in [47, Proposition 3.3, Theorem 3.5, Theorem 3.16], one has that, [47, Corollary 3.6, Corollary 3.17], X being compact, for $\sharp \in \left\{ \left(d, d^\Lambda; d, d^\Lambda \right), (d; d), \left(d^\Lambda; d^\Lambda \right), \left(d, d^\Lambda; d, d^\Lambda \right) \right\}$,

$$\dim_{\mathbb{R}} H_{\sharp}^{\bullet}(X) < +\infty.$$

With the aim to develop a symplectic counterpart of Riemannian Hodge theory for compact symplectic manifolds, J.-L. Brylinski defined the *symplectic- \star -operator*, [12, §2],

$$\star_{\omega}: \wedge^{\bullet} X \rightarrow \wedge^{2n-\bullet} X$$

requiring that, for every $\alpha, \beta \in \wedge^k X$,

$$\alpha \wedge \star_{\omega} \beta = (\omega^{-1})^k (\alpha, \beta) \omega^n.$$

Since $d^{\Lambda} \lfloor_{\wedge^k X} = (-1)^{k+1} \star_{\omega} d \star_{\omega}$, [12, Theorem 2.2.1], and $\star_{\omega}^2 = \text{id}$, [12, Lemma 2.1.2], then one gets that \star_{ω} induces the isomorphism

$$\star_{\omega}: H_{(d;d)}^{\bullet}(X) \xrightarrow{\cong} H_{(d^{\Lambda};d^{\Lambda})}^{2n-\bullet}(X).$$

In particular, by the Poincaré duality, it follows that

$$\dim_{\mathbb{R}} H_{(d;d)}^{\bullet}(X) = \dim_{\mathbb{R}} H_{(d^{\Lambda};d^{\Lambda})}^{\bullet}(X) < +\infty.$$

Furthermore, by choosing an almost-complex structure J compatible with ω (namely, such that $\omega(\cdot, J\cdot)$ is positive definite and $\omega(J\cdot, J\cdot) = \omega$), and by considering the J -Hermitian metric $g := \omega(\cdot, J\cdot)$, one gets that, [47, Corollary 3.25], the Hodge- \ast -operator $\ast_g: \wedge^{\bullet} X \rightarrow \wedge^{2n-\bullet} X$ associated to g induces the isomorphism, [12, Corollary 2.2.2],

$$\ast_g: H_{(d,d^{\Lambda};d,d^{\Lambda})}^{\bullet}(X) \xrightarrow{\cong} H_{(d^{\Lambda};d,d^{\Lambda})}^{2n-\bullet}(X).$$

In particular, it follows that

$$\dim_{\mathbb{R}} H_{(d,d^{\Lambda};d,d^{\Lambda})}^{\bullet}(X) = \dim_{\mathbb{R}} H_{(d^{\Lambda};d,d^{\Lambda})}^{2n-\bullet}(X) < +\infty.$$

Recall that one says that the *Hard Lefschetz Condition* holds on X if

$$\text{(HLC)} \quad \text{for every } k \in \mathbb{N}, \quad L^k: H_{dR}^{n-k}(X; \mathbb{R}) \xrightarrow{\cong} H_{dR}^{n+k}(X; \mathbb{R}).$$

As in [7], and miming [35] in the almost-complex case, define, for $r, s \in \mathbb{N}$,

$$H_{\omega}^{(r,s)}(X; \mathbb{R}) := \left\{ \left[L^r \gamma^{(s)} \right] \in H_{dR}^{2r+s}(X; \mathbb{R}) : \Lambda \gamma^{(s)} = 0 \right\} \subseteq H_{dR}^{2r+s}(X; \mathbb{R});$$

one has that

$$\sum_{2r+s=\bullet} H_{\omega}^{(r,s)}(X; \mathbb{R}) \subseteq H_{dR}^{\bullet}(X; \mathbb{R}),$$

but in general neither the sum is direct, nor the inclusion is an equality.

As proved by Y. Lin in [36, Proposition A.5], if the Hard Lefschetz Condition holds on X , then

$$H_{\omega}^{(0,\bullet)}(X; \mathbb{R}) = PH_d^{\bullet}(X; \mathbb{R}),$$

where

$$PH_d^{\bullet}(X; \mathbb{R}) := \frac{\ker d \cap \ker d^{\Lambda} \cap \ker \Lambda}{\text{im } d \lfloor_{\ker d^{\Lambda} \cap \ker \Lambda}}$$

is the *primitive cohomology* introduced by L.-S. Tseng and S.-T. Yau in [47, §4.1].

We recall the following result.

Theorem 4.2 ([39, Corollary 2], [51, Theorem 0.1], [41, Proposition 1.4], [28], [47, Proposition 3.13], [13, Theorem 5.4], [7, Remark 2.3]). *Let X be a compact manifold endowed with a symplectic structure ω . The following conditions are equivalent:*

- (i) *every de Rham cohomology class of X admits a representative being both d -closed and d^{Λ} -closed, namely, Brylinski's conjecture [12, Conjecture 2.2.7] holds on X ;*
- (ii) *the Hard Lefschetz Condition holds on X ;*
- (iii) *the natural map $H_{(d,d^{\Lambda};d,d^{\Lambda})}^{\bullet}(\wedge^{\bullet} X) \rightarrow H_{dR}^{\bullet}(X; \mathbb{R})$ induced by the identity is surjective;*
- (iv) *the natural map $H_{(d,d^{\Lambda};d,d^{\Lambda})}^{\bullet}(\wedge^{\bullet} X) \rightarrow H_{dR}^{\bullet}(X; \mathbb{R})$ induced by the identity is an isomorphism;*

- (v) the bounded \mathbb{Z} -graded \mathbb{R} -vector space $\wedge^\bullet X$ endowed with the endomorphisms $d \in \text{End}^1(\wedge^\bullet X)$ and $d^\Lambda \in \text{End}^{-1}(\wedge^\bullet X)$ satisfies the dd^Λ -Lemma;
- (vi) the decomposition

$$H_{dR}^\bullet(X; \mathbb{R}) = \bigoplus_{r \in \mathbb{N}} L^r H_\omega^{(0, \bullet-2r)}(X; \mathbb{R}),$$

holds.

In order to apply Corollary 3.5 to the \mathbb{Z} -graded \mathbb{R} -vector space $\wedge^\bullet X$ endowed with the endomorphisms $d \in \text{End}^1(\wedge^\bullet X)$ and $d^\Lambda \in \text{End}^{-1}(\wedge^\bullet X)$, satisfying $d^2 = (d^\Lambda)^2 = [d, d^\Lambda] = 0$, we need the following result.

Lemma 4.3 ([12, Theorem 2.3.1], [18, Theorem 2.5]; see also [18, Theorem 2.9], [14, Theorem 5.2]). *Let X be a compact manifold endowed with a symplectic structure ω . Consider the \mathbb{Z}^2 -graded \mathbb{R} -vector space $\wedge^\bullet X$ endowed with the endomorphisms $d \in \text{End}^1(\wedge^\bullet X)$ and $d^\Lambda \in \text{End}^{-1}(\wedge^\bullet X)$. Both the spectral sequences associated to the canonical double complex $(\text{Doub}^{\bullet, \bullet} \wedge^\bullet X, d \otimes_{\mathbb{R}} \text{id}, d^\Lambda \otimes_{\mathbb{R}} \beta)$ degenerate at the first level.*

Hence, by applying Theorem 2.4 and Corollary 3.5 to the \mathbb{Z} -graded \mathbb{R} -vector space $\wedge^\bullet X$ endowed with $d \in \text{End}^1(\wedge^\bullet X)$ and $d^\Lambda \in \text{End}^{-1}(\wedge^\bullet X)$, we get the following result.

Theorem 4.4. *Let X be a compact manifold endowed with a symplectic structure ω . The inequality*

$$(5) \quad \dim_{\mathbb{R}} H_{(d, d^\Lambda; d, d^\Lambda)}^\bullet(X) + \dim_{\mathbb{R}} H_{(d, d^\Lambda; d, d^\Lambda)}^\bullet(X) \geq 2 \dim_{\mathbb{R}} H_{dR}^\bullet(X; \mathbb{R})$$

holds. Furthermore, the equality in (5) holds if and only if X satisfies the Hard Lefschetz Condition.

Consider $X = \Gamma \backslash G$ a solvmanifold endowed with a G -left-invariant symplectic structure ω ; in particular, ω induces a linear symplectic structure on \mathfrak{g} ; therefore the endomorphisms $d \in \text{End}^1(\wedge^\bullet X)$ and $d^\Lambda \in \text{End}^{-1}(\wedge^\bullet X)$ yield endomorphisms $d \in \text{End}^1(\wedge^\bullet \mathfrak{g}^*)$ and $d^\Lambda \in \text{End}^{-1}(\wedge^\bullet \mathfrak{g}^*)$ on the \mathbb{Z} -graded \mathbb{R} -vector sub-space $\wedge^\bullet \mathfrak{g}^* \hookrightarrow \wedge^\bullet X$, where we identify objects on \mathfrak{g} with G -left-invariant objects on X by means of left-translations. For $\sharp \in \left\{ (d, d^\Lambda; d, d^\Lambda), (d; d), (d^\Lambda; d^\Lambda), (d, d^\Lambda; d, d^\Lambda) \right\}$, one has the natural map $\iota: H_\sharp^\bullet(\wedge^\bullet \mathfrak{g}^*) \rightarrow H_\sharp^\bullet(X)$. We recall the following result, which allows to compute the cohomologies of a completely-solvable solvmanifold by using just left-invariant forms; recall, e.g., that, by A. Hattori's theorem [29, Corollary 4.2], if G is *completely-solvable* (that is, for any $g \in G$, all the eigenvalues of $\text{Ad}_g := d(\psi_g)_e \in \text{Aut}(\mathfrak{g})$ are real, equivalently, if, for any $X \in \mathfrak{g}$, all the eigenvalues of $\text{ad}_X := [X, \cdot] \in \text{End}(\mathfrak{g})$ are real, where $\psi: G \ni g \mapsto (\psi_g: h \mapsto ghg^{-1}) \in \text{Aut}(G)$ and e is the identity element of G), then the natural map $H_{dR}^\bullet(\wedge^\bullet \mathfrak{g}^*) \rightarrow H_{dR}^\bullet(X; \mathbb{R})$ is an isomorphism.

Theorem 4.5 ([37, Theorem 3, Remark 4], see also [4]). *Let $X = \Gamma \backslash G$ be a completely-solvable solvmanifold endowed with a G -left-invariant symplectic structure ω . Then, for $\sharp \in \left\{ (d, d^\Lambda; d, d^\Lambda), (d; d), (d^\Lambda; d^\Lambda), (d, d^\Lambda; d, d^\Lambda) \right\}$, the natural map*

$$\iota: H_\sharp^\bullet(\wedge^\bullet \mathfrak{g}^*) \rightarrow H_\sharp^\bullet(X)$$

is an isomorphism.

Example 4.6. Let $\mathbb{I}_3 := \mathbb{Z}[i]^3 \backslash (\mathbb{C}^3, *)$ be the *Iwasawa manifold*, where the group structure $*$ on \mathbb{C}^3 is defined by

$$(z_1, z_2, z_3) * (w_1, w_2, w_3) := (z_1 + w_1, z_2 + w_2, z_3 + z_1 w_2 + w_3).$$

There exists a $(\mathbb{C}^3, *)$ -left-invariant co-frame $\{e^j\}_{j \in \{1, \dots, 6\}}$ of T^*X such that

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = -e^{13} + e^{24}, \quad de^6 = -e^{14} - e^{23}$$

(in order to simplify notation, we shorten, e.g., $e^{12} := e^1 \wedge e^2$).

Consider the $(\mathbb{C}^3, *)$ -left-invariant almost-Kähler structure (J, ω, g) on \mathbb{I}_3 defined by

$$Je^1 := -e^6, \quad Je^2 := -e^5, \quad Je^3 := -e^4, \quad \omega := e^{16} + e^{25} + e^{34}, \quad g := \omega(\cdot, J\cdot);$$

it has been studied in [8, §4] as an example of an almost-Kähler structure non-inducing a decomposition in cohomology according to the almost-complex structure, [8, Proposition 4.1].

The symplectic cohomologies of the Iwasawa manifold \mathbb{I}_3 endowed with the $(\mathbb{C}^3, *)$ -left-invariant symplectic structure ω can be computed using just $(\mathbb{C}^3, *)$ -left-invariant forms. This follows from [37,

Theorem 3], see also [5, Theorem 3.2], thanks to [43, Theorem 1]. The real dimensions of the symplectic cohomologies are summarized in Table 1.

Note that, from [47, Proposition 3.24], on a compact symplectic manifold of dimension $2n$, the symplectic cohomologies $H_{(d,d^\Lambda;dd^\Lambda)}^k$, $H_{(d,d^\Lambda;dd^\Lambda)}^{2n-k}$, $H_{(dd^\Lambda;d,d^\Lambda)}^k$, $H_{(dd^\Lambda;d,d^\Lambda)}^{2n-k}$ are all (non-naturally) isomorphic. This is done by means of the wedge product with powers of the symplectic forms and of the Hodge- $*$ -operator associated to a compatible metric, which both induce isomorphisms between cohomologies.

(The computations have been performed with the aid of Sage [44]. Further examples are studied in [5].)

$\dim_{\mathbb{C}} H_{\sharp}^{\bullet}(\mathbb{I}_3)$	$(d; d)$	$(d^\Lambda; d^\Lambda)$	$(d, d^\Lambda; d d^\Lambda)$	$(d d^\Lambda; d, d^\Lambda)$
0	1	1	1	1
1	4	4	4	4
2	8	8	10	10
3	10	10	11	11
4	8	8	10	10
5	4	4	4	4
6	1	1	1	1

TABLE 1. The symplectic cohomologies of the Iwasawa manifold $\mathbb{I}_3 := \mathbb{Z}[i]^3 \setminus (\mathbb{C}^3, *)$ endowed with the symplectic structure $\omega := e^1 \wedge e^6 + e^2 \wedge e^5 + e^3 \wedge e^4$.

In particular, note that

$$\begin{aligned} \dim_{\mathbb{K}} H_{(d,d^\Lambda;d d^\Lambda)}^1(X) + \dim_{\mathbb{K}} H_{(d d^\Lambda;d,d^\Lambda)}^1(X) - 2 \dim_{\mathbb{K}} H_{dR}^1(X; \mathbb{R}) &= 0, \\ \dim_{\mathbb{K}} H_{(d,d^\Lambda;d d^\Lambda)}^2(X) + \dim_{\mathbb{K}} H_{(d d^\Lambda;d,d^\Lambda)}^2(X) - 2 \dim_{\mathbb{K}} H_{dR}^2(X; \mathbb{R}) &= 3, \\ \dim_{\mathbb{K}} H_{(d,d^\Lambda;d d^\Lambda)}^3(X) + \dim_{\mathbb{K}} H_{(d d^\Lambda;d,d^\Lambda)}^3(X) - 2 \dim_{\mathbb{K}} H_{dR}^3(X; \mathbb{R}) &= 2. \end{aligned}$$

Remark 4.7. More in general, let X be a compact manifold endowed with a Poisson bracket $\{\cdot, \cdot\}$, and denote by G the Poisson tensor associated to $\{\cdot, \cdot\}$. By following J.-L. Koszul, [34], one defines $\delta := [\iota_G, d] \in \text{End}^{-1}(\wedge^\bullet X)$. One has that $\delta^2 = 0$ and $[d, \delta] = 0$, [34, page 266, page 265], see also [12, Proposition 1.2.3, Theorem 1.3.1].

One has that, on any compact Poisson manifold, the first spectral sequence $'E_r^{\bullet, \bullet}$ associated to the canonical double complex $(\text{Doub}^{\bullet, \bullet} \wedge^\bullet X, d \otimes_{\mathbb{R}} \text{id}, \delta \otimes_{\mathbb{R}} \beta)$ degenerates at the first level, [18, Theorem 2.5].

On the other hand, M. Fernández, R. Ibáñez, and M. de León provided an example of a compact Poisson manifold (more precisely, of a nilmanifold endowed with a co-symplectic structure) such that the second spectral sequence $''E_r^{\bullet, \bullet}(\text{Doub}^{\bullet, \bullet} \wedge^\bullet X, d \otimes_{\mathbb{R}} \text{id}, \delta \otimes_{\mathbb{R}} \beta)$ does not degenerate at the first level, [18, Theorem 5.1].

In fact, on a compact $2n$ -dimensional manifold X endowed with a symplectic structure ω , the symplectic- \star -operator $\star_\omega: \wedge^\bullet X \rightarrow \wedge^{2n-\bullet} X$ induces the isomorphism $\star_\omega: 'E_r^{\bullet_1, \bullet_2} \xrightarrow{\cong} ''E_r^{\bullet_2, 2n+\bullet_1}$, [18, Theorem 2.9]; it follows that, on a compact symplectic manifold, also the second spectral sequence $''E_r^{\bullet, \bullet}(\text{Doub}^{\bullet, \bullet} \wedge^\bullet X, d \otimes_{\mathbb{R}} \text{id}, \delta \otimes_{\mathbb{R}} \beta)$ actually degenerates at the first level, [12, Theorem 2.3.1], see also [18, Theorem 2.8].

4.3. Generalized complex structures. Let X be a compact differentiable manifold of dimension $2n$. Consider the bundle $TX \oplus T^*X$ endowed with the natural symmetric pairing

$$\langle \cdot | \cdot \rangle : (TX \oplus T^*X) \times (TX \oplus T^*X) \rightarrow \mathbb{R}, \quad \langle X + \xi | Y + \eta \rangle := \frac{1}{2} (\xi(Y) + \eta(X)).$$

Fix a d-closed 3-form H on X . On the space $\mathcal{C}^\infty(X; TX \oplus T^*X)$ of smooth sections of $TX \oplus T^*X$ over X , define the H -twisted Courant bracket as

$$\begin{aligned} [\cdot, \cdot]_H &: \mathcal{C}^\infty(X; TX \oplus T^*X) \times \mathcal{C}^\infty(X; TX \oplus T^*X) \rightarrow \mathcal{C}^\infty(X; TX \oplus T^*X), \\ [X + \xi, Y + \eta]_H &:= [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi) + \iota_Y \iota_X H \end{aligned}$$

(where $\iota_X \in \text{End}^{-1}(\wedge^\bullet X)$ denotes the interior product with $X \in \mathcal{C}^\infty(X; TX)$ and $\mathcal{L}_X := [\iota_X, d] \in \text{End}^0(\wedge^\bullet X)$ denotes the Lie derivative along $X \in \mathcal{C}^\infty(X; TX)$); the H -twisted Courant bracket can be seen also as a derived bracket induced by the H -twisted differential $d_H := d + H \wedge \cdot$, see [25, §3.2], [27, §2].

Furthermore, consider the *Clifford action* of $TX \oplus T^*X$ on the space of differential forms with respect to $\langle \cdot | \cdot \rangle$,

$$\text{Cliff}(TX \oplus T^*X) \times \wedge^\bullet X \rightarrow \wedge^{\bullet-1} X \oplus \wedge^{\bullet+1} X, \quad (X + \xi) \cdot \varphi := \iota_X \varphi + \xi \wedge \varphi,$$

and its bi- \mathbb{C} -linear extension $\text{Cliff}((TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C}) \times (\wedge^\bullet X \otimes_{\mathbb{R}} \mathbb{C}) \rightarrow (\wedge^{\bullet-1} X \otimes_{\mathbb{R}} \mathbb{C}) \oplus (\wedge^{\bullet+1} X \otimes_{\mathbb{R}} \mathbb{C})$, where $\text{Cliff}(TX \oplus T^*X) := \left(\bigoplus_{k \in \mathbb{Z}} \bigotimes_{j=1}^k (TX \oplus T^*X) \right) / \{v \otimes_{\mathbb{R}} v - \langle v | v \rangle : v \in TX \oplus T^*X\}$ is the Clifford algebra associated to $TX \oplus T^*X$ and $\langle \cdot | \cdot \rangle$.

Recall that an H -twisted generalized complex structure on X , [25, Definition 4.14, Definition 4.18], [27, Definition 3.1] is an endomorphism $\mathcal{J} \in \text{End}(TX \oplus T^*X)$ such that (i) $\mathcal{J}^2 = -\text{id}_{TX \oplus T^*X}$, and (ii) \mathcal{J} is orthogonal with respect to $\langle \cdot | \cdot \rangle$, and (iii) the Nijenhuis tensor

$\text{Nij}_{\mathcal{J}, H} := -[\mathcal{J} \cdot, \mathcal{J} \cdot]_H + \mathcal{J}[\mathcal{J} \cdot, \cdot]_H + \mathcal{J}[\cdot, \mathcal{J} \cdot]_H + \mathcal{J}[\cdot, \cdot]_H \in (TX \oplus T^*X) \otimes_{\mathbb{R}} (TX \oplus T^*X) \otimes_{\mathbb{R}} (TX \oplus T^*X)^*$ of \mathcal{J} with respect to the H -twisted Courant bracket vanishes identically.

Equivalently, [25, Proposition 4.3], (by setting $L := L_{\mathcal{J}}$ the i-eigen-bundle of the \mathbb{C} -linear extension of \mathcal{J} to $(TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C}$), a generalized complex structure on X is identified by a sub-bundle L of $(TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C}$ such that (i) L is maximal isotropic with respect to $\langle \cdot | \cdot \rangle$, and (ii) L is involutive with respect to the H -twisted Courant bracket, and (iii) $L \cap \bar{L} = \{0\}$.

Equivalently, [25, Theorem 4.8], (by choosing a complex form ρ whose Clifford annihilator

$$L_\rho := \{v \in (TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C} : v \cdot \rho = 0\}$$

is the i-eigen-bundle $L_{\mathcal{J}}$ of the \mathbb{C} -linear extension of \mathcal{J} to $(TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C}$), a generalized complex structure on X is identified by a sub-bundle $U := U_{\mathcal{J}}$ (which is called the *canonical bundle*, [25, §4.1], [27, Definition 3.7]) of complex rank 1 of $\wedge^\bullet X \otimes_{\mathbb{R}} \mathbb{C}$ being locally generated by a form $\rho = \exp(B + i\omega) \wedge \Omega$, where $B \in \wedge^2 X$, and $\omega \in \wedge^2 X$, and $\Omega = \theta^1 \wedge \dots \wedge \theta^k \in \wedge^k X \otimes_{\mathbb{R}} \mathbb{C}$ with $\{\theta^1, \dots, \theta^k\}$ a set of linearly independent complex 1-forms, such that (i) $\Omega \wedge \bar{\Omega} \wedge \omega^{n-k} \neq 0$, and (ii) there exists $v \in (TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C}$ such that $d_H \rho = v \cdot \rho$, where $d_H := d + H \wedge \cdot$.

By definition, the *type* of a generalized complex structure \mathcal{J} on X , [25, §4.3], [27, Definition 3.5], is the upper-semi-continuous function

$$\text{type}(\mathcal{J}) := \frac{1}{2} \dim_{\mathbb{R}}(T^*X \cap \mathcal{J}T^*X)$$

on X , equivalently, [27, Definition 1.1], the degree of the form Ω .

A generalized complex structure \mathcal{J} on X induces a \mathbb{Z} -graduation on the space of complex differential forms on X , [25, §4.4], [27, Proposition 3.8]. Namely, define, for $k \in \mathbb{Z}$,

$$U^k := U_{\mathcal{J}}^k := \wedge^{n-k} \bar{L}_{\mathcal{J}} \cdot U_{\mathcal{J}} \subseteq \wedge^\bullet X \otimes_{\mathbb{R}} \mathbb{C},$$

where $L_{\mathcal{J}}$ is the i-eigenspace of the \mathbb{C} -linear extension of \mathcal{J} to $(TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C}$ and $U_{\mathcal{J}}^n := U_{\mathcal{J}}$ is the canonical bundle of \mathcal{J} .

For a $\langle \cdot | \cdot \rangle$ -orthogonal endomorphism $\mathcal{J} \in \text{End}(TX \oplus T^*X)$ satisfying $\mathcal{J}^2 = -\text{id}_{TX \oplus T^*X}$, the \mathbb{Z} -graduation $U_{\mathcal{J}}^\bullet$ still makes sense, and the condition that $\text{Nij}_{\mathcal{J}, H} = 0$ turns out to be equivalent, [25, Theorem 4.3], [27, Theorem 3.14], to

$$d_H: U_{\mathcal{J}}^\bullet \rightarrow U_{\mathcal{J}}^{\bullet+1} \oplus U_{\mathcal{J}}^{\bullet-1}.$$

Therefore, on a compact differentiable manifold endowed with a generalized complex structure \mathcal{J} , one has, [25, §4.4], [27, §3],

$$d_H = \partial_{\mathcal{J}, H} + \bar{\partial}_{\mathcal{J}, H} \quad \text{where} \quad \partial_{\mathcal{J}, H}: U_{\mathcal{J}}^\bullet \rightarrow U_{\mathcal{J}}^{\bullet+1} \quad \text{and} \quad \bar{\partial}_{\mathcal{J}, H}: U_{\mathcal{J}}^\bullet \rightarrow U_{\mathcal{J}}^{\bullet-1}.$$

Define also, [25, page 52], [27, Remark at page 97],

$$d_H^{\mathcal{J}} := -i(\partial_{\mathcal{J},H} - \bar{\partial}_{\mathcal{J},H}) : U_{\mathcal{J}}^{\bullet} \rightarrow U_{\mathcal{J}}^{\bullet+1} \oplus U_{\mathcal{J}}^{\bullet-1} .$$

By abuse of notation, one says that X satisfies the $\partial_{\mathcal{J},H}\bar{\partial}_{\mathcal{J},H}$ -Lemma if $(U^{\bullet}, \partial_{\mathcal{J},H}, \bar{\partial}_{\mathcal{J},H})$ satisfies the $\partial_{\mathcal{J},H}\bar{\partial}_{\mathcal{J},H}$ -Lemma, and one says that X satisfies the $d_H d_H^{\mathcal{J}}$ -Lemma if $(U^{\bullet}, d_H, d_H^{\mathcal{J}})$ satisfies the $d_H d_H^{\mathcal{J}}$ -Lemma. Actually, it turns out that X satisfies the $d_H d_H^{\mathcal{J}}$ -Lemma if and only if X satisfies the $\partial_{\mathcal{J},H}\bar{\partial}_{\mathcal{J},H}$ -Lemma, [14, Remark at page 129]: indeed, note that $\ker \partial_{\mathcal{J},H}\bar{\partial}_{\mathcal{J},H} = \ker d_H d_H^{\mathcal{J}}$, and $\ker \partial_{\mathcal{J},H} \cap \ker \bar{\partial}_{\mathcal{J},H} = \ker d_H \cap \ker d_H^{\mathcal{J}}$, and $\text{im } \partial_{\mathcal{J},H} + \text{im } \bar{\partial}_{\mathcal{J},H} = \text{im } d_H + \text{im } d_H^{\mathcal{J}}$.

Moreover, the following result by G. R. Cavalcanti holds.

Theorem 4.8 ([13, Theorem 4.2], [14, Theorem 4.1, Corollary 2]). *A manifold X endowed with an H -twisted generalized complex structure \mathcal{J} satisfies the $d_H d_H^{\mathcal{J}}$ -Lemma if and only if $(\ker d_H^{\mathcal{J}}, d) \hookrightarrow (U^{\bullet}, d_H)$ is a quasi-isomorphism of differential \mathbb{Z} -graded \mathbb{C} -vector spaces. In this case, it follows that the splitting $\wedge^{\bullet} X \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{k \in \mathbb{Z}} U^k$ gives rise to a decomposition in cohomology.*

An application of [17, Proposition 5.17, 5.21] yields the following result.

Theorem 4.9 ([13, Theorem 4.4], [14, Theorem 5.1]). *A manifold X endowed with an H -twisted generalized complex structure \mathcal{J} satisfies the $dd^{\mathcal{J}}$ -lemma if and only if the canonical spectral sequence degenerates at the first level and the decomposition of complex forms into sub-bundles U^k induces a decomposition in cohomology.*

Given a compact complex manifold X endowed with an H -twisted generalized complex structure, consider the following cohomologies:

$$GH_{dR_H}(X) := H_{(d_H; d_H)}(\text{Tot } U_{\mathcal{J}}^{\bullet}) ,$$

and

$$GH_{\bar{\partial}_{\mathcal{J},H}}^{\bullet}(X) := H_{(\bar{\partial}_{\mathcal{J},H}; \bar{\partial}_{\mathcal{J},H})}^{\bullet}(U_{\mathcal{J}}^{\bullet}) , \quad GH_{\partial_{\mathcal{J},H}}^{\bullet}(X) := H_{(\partial_{\mathcal{J},H}; \partial_{\mathcal{J},H})}^{\bullet}(U_{\mathcal{J}}^{\bullet}) ,$$

and

$$GH_{BC_{\mathcal{J},H}}^{\bullet}(X) := H_{(\partial_{\mathcal{J},H}, \bar{\partial}_{\mathcal{J},H}; \partial_{\mathcal{J},H}, \bar{\partial}_{\mathcal{J},H})}^{\bullet}(U_{\mathcal{J}}^{\bullet}) , \quad GH_{A_{\mathcal{J},H}}^{\bullet}(X) := H_{(\partial_{\mathcal{J},H}, \bar{\partial}_{\mathcal{J},H}; \partial_{\mathcal{J},H}, \bar{\partial}_{\mathcal{J},H})}^{\bullet}(U_{\mathcal{J}}^{\bullet}) .$$

Note that, for $H = 0$, one has $GH_{dR_0}(X) = \text{Tot } H_{dR}^{\bullet}(X; \mathbb{R})$.

By [25, Proposition 5.1], [27, Proposition 3.15], it follows that $\dim_{\mathbb{C}} GH_{\partial_{\mathcal{J},H}}^{\bullet}(X) < +\infty$ and $\dim_{\mathbb{C}} GH_{\bar{\partial}_{\mathcal{J},H}}^{\bullet}(X) < +\infty$.

As an application of Theorem 2.4, we get the following result.

Theorem 4.10. *Let X be a compact differentiable manifold endowed with an H -twisted generalized complex structure \mathcal{J} . Then*

$$(6) \quad \dim_{\mathbb{C}} GH_{BC_{\mathcal{J},H}}^{\bullet}(X) + \dim_{\mathbb{C}} GH_{A_{\mathcal{J},H}}^{\bullet}(X) \geq \dim_{\mathbb{C}} GH_{\bar{\partial}_{\mathcal{J},H}}^{\bullet}(X) + \dim_{\mathbb{C}} GH_{\partial_{\mathcal{J},H}}^{\bullet}(X) .$$

As an application of Corollary 3.5, we get the following result; compare it also with [13, Theorem 4.4].

Theorem 4.11. *Let X be a compact differentiable manifold endowed with an H -twisted generalized complex structure \mathcal{J} . The following conditions are equivalent:*

- X satisfies the $\partial_{\mathcal{J},H}\bar{\partial}_{\mathcal{J},H}$ -Lemma;
- the Hodge and Frölicher spectral sequences associated to the canonical double complex $(\text{Doub}^{\bullet, \bullet} U_{\mathcal{J}}^{\bullet}, \partial_{\mathcal{J},H} \otimes_{\mathbb{C}} \text{id}, \bar{\partial}_{\mathcal{J},H} \otimes_{\mathbb{C}} \beta)$ degenerate at the first level and the equality in (6),

$$\dim_{\mathbb{C}} GH_{BC_{\mathcal{J},H}}^{\bullet}(X) + \dim_{\mathbb{C}} GH_{A_{\mathcal{J},H}}^{\bullet}(X) = \dim_{\mathbb{C}} GH_{\bar{\partial}_{\mathcal{J},H}}^{\bullet}(X) + \dim_{\mathbb{C}} GH_{\partial_{\mathcal{J},H}}^{\bullet}(X) ,$$

holds.

Symplectic structures and complex structures provide the fundamental examples of generalized complex structures; in fact, the following generalized Darboux theorem by M. Gualtieri holds. (Recall that a *regular point* of a generalized complex manifold is a point at which the type of the generalized complex structure is locally constant.)

Theorem 4.12 ([25, Theorem 4.35], [27, Theorem 3.6]). *For any regular point of a $2n$ -dimensional generalized complex manifold with type equal to k , there is an open neighbourhood endowed with a set of local coordinates such that the generalized complex structure is a B -field transform of the standard generalized complex structure of $\mathbb{C}^k \times \mathbb{R}^{2n-2k}$.*

The standard generalized complex structure of constant type n (that is, locally equivalent to the standard complex structure of \mathbb{C}^n), the generalized complex structure of constant type 0 (that is, locally equivalent to the standard symplectic structure of \mathbb{R}^{2n}), and the B -field transform of a generalized complex structure are recalled in the following examples. See also [25, Example 4.12].

Example 4.13 (Generalized complex structures of type n , [25, Example 4.11, Example 4.25]). Let X be a compact $2n$ -dimensional manifold endowed with a complex structure $J \in \text{End}(TX)$. Consider the (0-twisted) generalized complex structure

$$\mathcal{J}_J := \left(\begin{array}{c|c} -J & 0 \\ \hline 0 & J^* \end{array} \right) \in \text{End}(TX \oplus T^*X) ,$$

where $J^* \in \text{End}(T^*X)$ denotes the dual endomorphism of $J \in \text{End}(TX)$. Note that the i -eigenspace of the \mathbb{C} -linear extension of \mathcal{J}_J to $(TX \oplus T^*X) \otimes_{\mathbb{C}} \mathbb{C}$ is

$$L_{\mathcal{J}_J} = T_J^{0,1}X \oplus \left(T_J^{1,0}X \right)^* ,$$

and the canonical bundle is

$$U_{\mathcal{J}_J}^n = \wedge_J^{n,0}X .$$

Hence, one gets that, [25, Example 4.25],

$$U_{\mathcal{J}_J}^\bullet = \bigoplus_{p-q=\bullet} \wedge_J^{p,q}X ,$$

and that

$$\partial_{\mathcal{J}_J} = \partial_J \quad \text{and} \quad \bar{\partial}_{\mathcal{J}_J} = \bar{\partial}_J ;$$

note that $d^{\mathcal{J}_J}$ is the operator $d_J^c := -i(\partial - \bar{\partial})$, [25, Remark 4.26]. Note also that X satisfies the $d d^{\mathcal{J}_J}$ -Lemma if and only if X satisfies the $d d_J^c$ -Lemma, and that the Hodge and Frölicher spectral sequence associated to the canonical double complex $(\text{Doub}^{\bullet,\bullet} U_{\mathcal{J}_J}^\bullet, \partial_{\mathcal{J}_J} \otimes_{\mathbb{R}} \text{id}, \bar{\partial}_{\mathcal{J}_J} \otimes_{\mathbb{R}} \beta)$ degenerates at the first level if and only if the Hodge and Frölicher spectral sequence associated to the double complex $(\wedge_J^{\bullet,\bullet} X, \partial_J, \bar{\partial}_J)$ does, [13, Remark at page 76].

In particular, it follows that, for $\sharp \in \{\bar{\partial}, \partial, BC, A\}$,

$$GH_{\sharp \mathcal{J}_J}^\bullet(X) = \text{Tot}^\bullet H_{\sharp J}^{\bullet,\bullet}(X) = \bigoplus_{p-q=\bullet} H_{\sharp J}^{p,q}(X) .$$

Therefore, by Theorem 4.10 and Theorem 4.11, and by using the equalities $\dim_{\mathbb{C}} H_{BCJ}^{\bullet_1, \bullet_2}(X) = \dim_{\mathbb{C}} H_{AJ}^{n-\bullet_2, n-\bullet_1}(X)$ and $\dim_{\mathbb{C}} H_{\bar{\partial}_J}^{\bullet_1, \bullet_2}(X) = \dim_{\mathbb{C}} H_{\partial_J}^{n-\bullet_2, n-\bullet_1}(X)$, one gets the following result, compare Corollary 4.1, [6, Theorem A, Theorem B].

Corollary 4.14. *Let X be a compact complex manifold. Then the inequality*

$$\sum_{p-q=\bullet} \dim_{\mathbb{C}} H_{BCJ}^{p,q}(X) \geq \sum_{p-q=\bullet} \dim_{\mathbb{C}} H_{\bar{\partial}_J}^{p,q}(X)$$

holds. Furthermore, X satisfies the $\partial_J \bar{\partial}_J$ -Lemma if and only if (i) the Hodge and Frölicher spectral sequence of X degenerates at the first level, namely,

$$\dim_{\mathbb{C}} H_{dR}^\bullet(X; \mathbb{C}) = \dim_{\mathbb{C}} \text{Tot}^\bullet H_{\bar{\partial}_J}^{\bullet,\bullet}(X) ,$$

and (ii) the equality

$$\sum_{p-q=\bullet} \dim_{\mathbb{C}} H_{BCJ}^{p,q}(X) = \sum_{p-q=\bullet} \dim_{\mathbb{C}} H_{\bar{\partial}_J}^{p,q}(X)$$

holds.

$\mathbb{I}_3 := \mathbb{Z}[1^3] \setminus \mathbb{C}^3$ classes	$\dim_{\mathbb{C}} \text{Tot}^{-3} H_{\sharp}^{\bullet, -\bullet}(X)$		$\dim_{\mathbb{C}} \text{Tot}^{-2} H_{\sharp}^{\bullet, -\bullet}(X)$		$\dim_{\mathbb{C}} \text{Tot}^{-1} H_{\sharp}^{\bullet, -\bullet}(X)$		$\dim_{\mathbb{C}} \text{Tot}^0 H_{\sharp}^{\bullet, -\bullet}(X)$		$\dim_{\mathbb{C}} \text{Tot}^1 H_{\sharp}^{\bullet, -\bullet}(X)$		$\dim_{\mathbb{C}} \text{Tot}^2 H_{\sharp}^{\bullet, -\bullet}(X)$		$\dim_{\mathbb{C}} \text{Tot}^3 H_{\sharp}^{\bullet, -\bullet}(X)$							
	$\bar{\partial}$	∂	BC	A	$\bar{\partial}$	∂	BC	A	$\bar{\partial}$	∂	BC	A	$\bar{\partial}$	∂	BC	A				
(i)	1	1	1	1	5	5	5	11	11	11	11	11	11	5	5	5	1	1	1	1
(ii.a)	1	1	1	1	4	4	4	9	9	11	11	13	13	4	4	4	1	1	1	1
(ii.b)	1	1	1	1	4	4	4	9	9	11	11	12	12	4	4	4	1	1	1	1
(iii.a)	1	1	1	1	3	3	3	8	8	11	11	12	12	3	3	3	1	1	1	1
(iii.b)	1	1	1	1	3	3	3	8	8	11	11	12	12	3	3	3	1	1	1	1
	b₀ = 1		b₁ = 4		b₂ = 8		b₃ = 10		b₄ = 8		b₅ = 4		b₆ = 1							

TABLE 2. Generalized complex cohomologies of the Iwasawa manifold.

Example 4.15 (Generalized complex structures of type 0, [25, Example 4.10]). Let X be a compact $2n$ -dimensional manifold endowed with a symplectic structure $\omega \in \wedge^2 X \simeq \text{Hom}(TX; T^*X)$. Consider the (0-twisted) generalized complex structure

$$\mathcal{J}_\omega := \left(\begin{array}{c|c} 0 & -\omega^{-1} \\ \hline \omega & 0 \end{array} \right),$$

where $\omega^{-1} \in \text{Hom}(T^*X; TX)$ denotes the inverse of $\omega \in \text{Hom}(TX; T^*X)$. Note that the \mathbb{C} -linear extension of \mathcal{J}_ω to $(TX \otimes_{\mathbb{R}} \mathbb{C}) \oplus (T^*X \otimes_{\mathbb{R}} \mathbb{C})$ is

$$L_{\mathcal{J}_\omega} = \{X - i\omega(X, \cdot) : X \in TX \otimes_{\mathbb{R}} \mathbb{C}\},$$

which has Clifford annihilator $\exp(i\omega)$, and the canonical bundle is

$$U_{\mathcal{J}_\omega}^n = \mathbb{C} \langle \exp(i\omega) \rangle.$$

In particular, one gets that, [14, Theorem 2.2],

$$U_{\mathcal{J}_\omega}^{n-\bullet} = \exp(i\omega) \left(\exp\left(\frac{\Lambda}{2i}\right) (\wedge^\bullet X \otimes_{\mathbb{R}} \mathbb{C}) \right),$$

where $\Lambda := -\iota_{\omega^{-1}}$. Note that, [14, §2.2],

$$d^{\mathcal{J}_\omega} = d^\Lambda.$$

By considering the natural isomorphism

$$\varphi: \wedge^\bullet X \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \wedge^\bullet X \otimes_{\mathbb{R}} \mathbb{C}, \quad \varphi(\alpha) := \exp(i\omega) \left(\exp\left(\frac{\Lambda}{2i}\right) \alpha \right),$$

one gets that, [14, Corollary 1],

$$\varphi(\wedge^\bullet X \otimes_{\mathbb{R}} \mathbb{C}) \simeq U^{n-\bullet}, \quad \text{and} \quad \varphi d = \bar{\partial}_{\mathcal{J}_\omega} \varphi \quad \text{and} \quad \varphi d^{\mathcal{J}_\omega} = -2i \partial_{\mathcal{J}_\omega} \varphi;$$

in particular,

$$GH_{\bar{\partial}_{\mathcal{J}_\omega}}^\bullet(X) \simeq H_{dR}^{n-\bullet}(X; \mathbb{C}).$$

In particular, one recovers Theorem 4.4, namely,

$$\dim_{\mathbb{R}} H_{(d, d^\Lambda; d, d^\Lambda)}^\bullet(X) + \dim_{\mathbb{R}} H_{(d, d^\Lambda; d, d^\Lambda)}^\bullet(X) \geq 2 \dim_{\mathbb{R}} H_{dR}^\bullet(X; \mathbb{R}),$$

and the equality holds if and only if X satisfies the Hard Lefschetz Condition.

Example 4.16 (B -transform, [25, §3.3]). Let X be a compact $2n$ -dimensional manifold endowed with an H -twisted generalized complex structure \mathcal{J} , and let B be a d -closed 2-form. Consider the H -twisted generalized complex structure

$$\mathcal{J}^B := \exp(-B) \mathcal{J} \exp B \quad \text{where} \quad \exp B = \left(\begin{array}{c|c} \text{id}_{TX} & 0 \\ \hline B & \text{id}_{T^*X} \end{array} \right).$$

Note that the \mathbb{C} -linear extension of \mathcal{J} to $(TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C}$ is, [13, Example 2.3],

$$L_{\mathcal{J}^B} = \{X + \xi - \iota_X B : X + \xi \in L_{\mathcal{J}}\},$$

and the canonical bundle is, [13, Example 2.6],

$$U_{\mathcal{J}^B}^n = \exp B \wedge U_{\mathcal{J}}^n.$$

Hence one gets that, [14, §2.3],

$$U_{\mathcal{J}^B}^\bullet = \exp B \wedge U_{\mathcal{J}}^\bullet.$$

and that, [14, §2.3],

$$\partial_{\mathcal{J}^B} = \exp(-B) \partial_{\mathcal{J}} \exp B \quad \text{and} \quad \bar{\partial}_{\mathcal{J}^B} = \exp(-B) \bar{\partial}_{\mathcal{J}} \exp B.$$

In particular, one gets that \mathcal{J} satisfies the $\partial_{\mathcal{J}} \bar{\partial}_{\mathcal{J}}$ -Lemma if and only if \mathcal{J}^B satisfies the $\partial_{\mathcal{J}^B} \bar{\partial}_{\mathcal{J}^B}$ -Lemma.

Remark 4.17. We recall that, given a d -closed 3-form H on a manifold X , an H -twisted generalized Kähler structure on X is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of H -twisted generalized complex structures on X such that (i) \mathcal{J}_1 and \mathcal{J}_2 commute, and (ii) the symmetric pairing $\langle \mathcal{J}_1 \cdot, \mathcal{J}_2 \cdot \rangle$ is positive definite. Generalized Kähler geometry is equivalent to a bi-Hermitian geometry with torsion, [26, Theorem 2.18].

We recall that a compact manifold X endowed with an H -twisted generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ satisfies both the $d_H d_H^{\mathcal{J}_1}$ -Lemma and the $d_H d_H^{\mathcal{J}_2}$ -Lemma, [26, Corollary 4.2].

Any Kähler structure provide an example of a 0-twisted generalized Kähler structure. A left-invariant non-trivial twisted generalized Kähler structure on a (non-completely solvable) solvmanifold (which is the

total space of a \mathbb{T}^2 -bundle over the Inoue surface, [19, Proposition 3.2]) has been constructed by A. Fino and the second author, [19, Theorem 3.5].

Remark 4.18. Note that A. Tomasiello proved in [46, §B] that satisfying the $dd^{\mathcal{J}}$ -Lemma is a stable property under small deformations.

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