Tchebycheffian spline spaces over planar T-meshes: Dimension bounds and dimension instabilities

Tchebycheffian spline spaces over planar T-meshes: Dimension bounds and dimension instabilities

Cesare Bracco[a], Tom Lyche[b], Carla Manni[c], Hendrik Speleers[c]

[a] Department of Mathematics and Computer Science, University of Florence, Italy
[b] Department of Mathematics, University of Oslo, Norway
[c] Department of Mathematics, University of Rome "Tor Vergata", Italy

Abstract

We consider Tchebycheffian spline spaces over planar T-meshes and we study their dimension. We show that the structure of extended Tchebycheff spaces allows us to fully generalize the dimension upper bounds known in the literature for polynomial spline spaces over T-meshes. Moreover, we illustrate that the dimension of Tchebycheffian spline spaces over T-meshes can be unstable for certain configurations of the T-mesh, for any choice of the underlying extended Tchebycheff space.

Keywords: Extended Tchebycheff spaces, Extended complete Tchebycheff spaces, Tchebycheffian splines, T-meshes, dimension formula, dimension bounds, instability

1. Introduction

Tensor-product structures allow us to construct multivariate splines in a very simple and elegant way from univariate splines, and they have been applied in different contexts. However, such a multivariate structure lacks adequate local refinement, which is imperative for both geometric modeling and numerical simulation. This triggered the interest in alternative multivariate spline structures supporting local refinement but still preserving locally the simplicity of the tensor-product approach. T-splines [11, 26], hierarchical splines [7, 8] and locally refined (LR-) splines [6] are examples of such structures. All of them can be regarded as special instances of splines over T-meshes [5, 25].

Univariate Tchebycheffian splines are smooth piecewise functions with sections in extended Tchebycheff (ET-) spaces [19, 24]. They share many important properties with the classical (algebraic) polynomial splines but offer a more flexible framework, due to the wide variety of ET-spaces. As a consequence, besides their theoretical interest, spline spaces with sections in ET-spaces are attractive in several application areas ranging from geometric modeling to isogeometric analysis (see, e.g., [14, 19]). Multivariate extensions of Tchebycheffian splines can be easily obtained via (local) tensor-product structures.

Tchebycheffian spline spaces over T-meshes have been introduced in their full generality in [3]. Some earlier generalizations of the polynomial setting towards the Tchebycheffian setting were considered in [2, 4, 13]. Like in the polynomial case, a complete understanding of such Tchebycheffian spline spaces requires the knowledge of the dimension of the spline space defined on a prescribed T-mesh for a given smoothness. Unfortunately, the dimension of the spline space can be unstable (see [1, 9] for polynomial spline spaces and [3] for trigonometric and hyperbolic spline spaces). This means that the dimension may depend not only on combinatorial quantities of the T-mesh (such as number of vertices, edges and faces), on the smoothness and on the dimensions of the underlying section spaces, but also on the exact geometry of the T-mesh. Such instabilities complicate the derivation of an explicit dimension formula for any T-mesh configuration, and only lower and upper bounds can be given in the most general cases.

In [3] lower bounds for the dimension of Tchebycheffian spline spaces over T-meshes are provided by generalizing the homological techniques and the results presented in [22] for polynomials. Explicit upper bounds for the dimension are also obtained in [3] under a specific assumption on the underlying ET-spaces: the so-called $\boldsymbol{d}$-sum property.

Email addresses: cesare.bracco@unifi.it (Cesare Bracco), tom@math.uio.no (Tom Lyche), manni@mat.uniroma2.it (Carla Manni), speleers@mat.uniroma2.it (Hendrik Speleers)

In this paper, we deepen the analysis and we complete the results of [3], aiming to understand the influence of the considered ET-spaces on the dimension of Tchebycheffian splines over a given T-mesh. We show that, without any assumption on the underlying ET-spaces,

(a) the lower and upper bounds for the dimension of Tchebycheffian spline spaces over T-meshes provided in [3] hold. These bounds agree with those obtained in [22] for polynomial splines.
(b) there exist T-meshes such that the dimension of the "bi-quadratic" C^1 Tchebycheffian spline spaces is unstable and attains the lower or the upper bound depending on the exact geometry of the T-mesh.

Consequently, the provided lower and upper bounds for the dimension are sharp for any choice of the underlying ET-spaces. These bounds coincide on several relevant T-mesh configurations, resulting in explicit expressions for the dimension of the corresponding Tchebycheffian spline spaces. Moreover, the difference between the lower and the upper bound only depends on the T-mesh and not on the considered ET-space. This gives evidence that all Tchebycheffian spline spaces over the same T-mesh have a complete structural similarity, regardless of the ET-spaces we are dealing with.

A key ingredient in the paper is the possibility to represent any ET-space on a bounded closed interval in terms of the so-called generalized power functions [12], which generalize the classical Taylor basis for algebraic polynomials. Generalized power functions are used to obtain the instability results (b). Suitable sets of generalized power functions generate the so called ready-to-blossom-like bases, an important concept introduced in [16], whose existence turns out to be equivalent to the $\boldsymbol{d}$-sum property.

The crucial fact that ready-to-blossom-like bases exist in any ET-space has been proved in [16, Theorem 23] by using Wronskians, so as to establish existence and properties of blossoms [23] in the considered spaces under the weakest possible smoothness assumption. Existence of such bases in the space of algebraic polynomials has been previously proved in [21, Lemma 3] by linear algebra arguments, in the context of total positivity of certain polynomial bases. Recently, the same result has been proved again in [22, Proposition 1.8] by means of algebraic properties of the ring of polynomials.

The remainder of the paper is organized as follows. In Section 2 we define ET-spaces and focus on the subclass of extended complete Tchebycheff (ECT-) spaces. In particular, we define the generalized power functions and the ready-to-blossom-like bases, we recall some of their properties and we illustrate their relation with the $\boldsymbol{d}$-sum property. We note that it is not strictly necessary to mention ECT-spaces because only closed bounded intervals are of interest to define spline spaces and on such intervals any ET-space is an ECT-space [17]. However, for the sake of completeness and clarity, we prefer to present the material related to generalized powers in terms of ECT-spaces [24]. Section 3 is devoted to the definition of Tchebycheffian spline spaces over T-meshes. Section 4 contains the results on the dimension bounds and discusses some of their consequences. Generalized power functions are used in Section 5 to construct examples of instability in the dimension of C^1 spline spaces over T-meshes for any underlying ET-space of dimension 3. We end with some concluding remarks in Section 6.

Throughout the paper, the derivative operator will be denoted by D whenever it is clear on which variable it operates. If this is not the case, we will use the notation D_x or D_y to avoid any confusion.

2. Extended Tchebycheff and extended complete Tchebycheff spaces

We first define ET-spaces on a real interval J (see, e.g., [24]).

Definition 2.1 (Extended Tchebycheff space)**.** *Given an integer $p \geq 0$ and an interval J, a space $\mathbb{T}_p(J) \subset C^p(J)$ of dimension $p+1$ is an extended Tchebycheff (ET-) space on J if any Hermite interpolation problem with $p+1$ data on J has a unique solution in $\mathbb{T}_p(J)$. In other words, for any integer $m \geq 1$, let $\bar{x}_1, \ldots, \bar{x}_m$ be distinct points in J and let $d_1, \ldots, d_m$ be nonnegative integers such that $p+1 = \sum_{i=1}^{m}(d_i+1)$. Then, for any set $\{f_{i,j} \in \mathbb{R}\}_{i=1,\ldots,m,\, j=0,\ldots,d_i}$ there exists a unique $q \in \mathbb{T}_p(J)$ such that*

$$D^j q(\bar{x}_i) = f_{i,j}, \quad i = 1, \ldots, m, \quad j = 0, \ldots, d_i.$$

Example 2.1. *The space $\mathbb{P}_p$ of algebraic polynomials of degree less than or equal to p is an ET-space on the real line.*

Example 2.2. *The space $\langle \cos x, \sin x \rangle$ is an ET-space on any interval $[a, a+\pi)$.*

We now consider a particular subclass of ET-spaces, whose properties will be crucial later on.

Definition 2.2 (Extended complete Tchebycheff space)**.** *Given an integer $p \geq 0$ and an interval J, the space $\mathbb{T}_p(J) \subset C^p(J)$ of dimension $p+1$ is an extended complete Tchebycheff (ECT-) space if there exists a basis $\{u_0, \dots, u_p\}$ of $\mathbb{T}_p(J)$ such that every subspace $\langle u_0, \dots, u_k \rangle$ is an ET-space on J for $k = 0, \dots, p$. The basis $\{u_0, \dots, u_p\}$ is called an ECT-system.*

By [24, Theorem 9.1], the set $\{u_0, \dots, u_p\}$ of functions in $C^p(J)$ is an ECT-system if and only if their Wronskians

$$W(u_0, \dots, u_k)(x) := \det\bigl(D^i u_j(x)\bigr)_{i,j=0}^{k}, \quad k = 0, \dots, p$$

are positive for all $x \in J$.

Example 2.3. *The space $\mathbb{P}_p$ is an ECT-space on any interval of the real line. It can be regarded as the span of the ECT-system*

$$\left\{1, x-c, \frac{(x-c)^2}{2}, \dots, \frac{(x-c)^p}{p!}\right\}, \tag{2.1}$$

for any fixed $c \in \mathbb{R}$. Indeed, the Wronskians of this system are all equal to one. The functions in (2.1) form the classical Taylor basis for algebraic polynomials.

From Definition 2.2 it is clear that an ECT-space of dimension $p+1$ on J is an ET-space of dimension $p+1$ on J. The converse is not true. For instance, the space $\langle \cos x, \sin x \rangle$ is an ECT-space on the interval $(0, \pi)$ but not on $[0, \pi)$, where it is an ET-space, see Example 2.2. The next theorem shows that the two classes coincide in a very important case; for a proof we refer the reader to [17, Corollary 2.12], see also [23, Theorem 1.3] and [18].

Theorem 2.1. *If J is a bounded closed interval, then any ET-space on J is an ECT-space on J.*

We now detail some properties of ECT-spaces that will be useful in the later sections. In view of Theorem 2.1, any ET-space also enjoys such properties as long as the interval J we are interested in is bounded and closed. This is the case when dealing with spline spaces.

According to [12], suppose J is a bounded interval and $w_1, \dots, w_p$ are given continuous functions on J. For $x, y \in J$ we define the *repeated integrals* by

$$I_k(x, y; w_1, \dots, w_k) := \int_y^x w_1(t) I_{k-1}(t, y; w_2, \dots, w_k)\, \mathrm{d}t, \quad k \geq 1, \tag{2.2}$$

starting with $I_0(x, y) := 1$.

From [12, Proposition 3.1] it is known that

$$I_p(x, y; w_1, \dots, w_p) = \sum_{k=0}^{p} (-1)^{p-k} I_k(x, c; w_1, \dots, w_k) I_{p-k}(y, c; w_p, \dots, w_{k+1}), \quad c \in J. \tag{2.3}$$

Moreover, if $w_k \in C^{p-k}(J)$, $k = 1, \dots, p$, then

$$D_x^r I_k(x, y)|_{x=y} = 0, \quad r = 0, \dots, k-1, \tag{2.4}$$

$$D_x^k I_k(x, y)|_{x=y} \neq 0. \tag{2.5}$$

The repeated integrals are used to define the so-called *generalized power functions*,

$$u_k(x, c) := w_0(x) I_k(x, c; w_1, \dots, w_k), \quad k = 0, 1, \dots, p. \tag{2.6}$$

Example 2.4. *If $w_0 = \cdots = w_p = 1$, then for $k = 0, 1, \dots, p$,*

$$u_k(x, c) = I_k(x, c; w_1, \dots, w_k) = \frac{(x-c)^k}{k!}.$$

This motivates the term "generalized power functions". In this case, the expression (2.3) reads

$$\frac{(x-y)^p}{p!} = \sum_{k=0}^{p} (-1)^{p-k} \frac{(x-c)^k}{k!} \frac{(y-c)^{p-k}}{(p-k)!},$$

providing the well-known binomial expansion of $(x-y)^p/p!$.

The next theorem states that, with a proper choice of the functions $w_0, \dots, w_p$, the generalized power functions provide a characterization of any ECT-space, see [24, Chapter 9].

Theorem 2.2. *The space $\mathbb{T}_p(J)$ is an ECT-space of dimension $p+1$ on J if and only if there are positive functions $w_k \in C^{p-k}(J)$, $k = 0, \dots, p$ such that for any $c \in J$ the generalized power functions (2.6) span $\mathbb{T}_p(J)$.*

Example 2.5. *The space $\langle 1, \cos x, \sin x \rangle$ is an ECT-space on the interval $J = (-\pi, \pi)$, with*

$$w_0(x) = \cos^2\left(\frac{x}{2}\right), \quad w_1(x) = w_2(x) = 1 \Big/ \cos^2\left(\frac{x}{2}\right).$$

Indeed, for $x \in J$ and any fixed $y \in J$ we find the following ECT-system spanning $\langle 1, \cos x, \sin x \rangle$:

$$\begin{aligned} u_0(x,y) &= w_0(x) I_0(x,y) = \cos^2\left(\frac{x}{2}\right), \\ u_1(x,y) &= w_0(x) I_1(x,y;w_1) = \sin x - (1+\cos x)\tan\left(\frac{y}{2}\right), \\ u_2(x,y) &= w_0(x) I_2(x,y;w_1,w_2) = 2\sin^2\left(\frac{x-y}{2}\right) \Big/ \cos^2\left(\frac{y}{2}\right). \end{aligned}$$

From (2.4) and (2.5) we have that for any $c \in J$, $u_k(x,c)$ vanishes exactly k times at c. Moreover, denoting by Ψ_k^c any function in $\mathbb{T}_p$ vanishing exactly k times at c, (2.4) and (2.5) imply that

$$\Psi_k^c(x) = \sum_{j=k}^{p} a_j u_j(x,c), \quad a_j \in \mathbb{R}, \ a_k \neq 0, \tag{2.7}$$

in a complete similarity with the polynomial case. A useful "multipoint" generalization of the basis in (2.6) can be defined as follows, see [16]

Definition 2.3. *Let $\bar{x}_1, \dots, \bar{x}_m \in J$ be distinct points and $-1 \le d_i < p$, $d_i \in \mathbb{N}$, $i = 1, \dots, m$, such that $\sum_{i=1}^m (p - d_i) = p+1$. In a given $p+1$ dimensional space $\mathbb{E} \subset C^p(J)$ a ready-to-blossom-like basis relative to $\bar{x}_1, \cdots, \bar{x}_m$ is a basis of the form*

$$\Psi_{d_1+1}^{\bar{x}_1}, \dots, \Psi_p^{\bar{x}_1}, \dots, \Psi_{d_m+1}^{\bar{x}_m}, \dots, \Psi_p^{\bar{x}_m}. \tag{2.8}$$

In view of (2.7), for an ECT-space the existence of a ready-to blossom-like basis (2.8) is equivalent to the fact that the generalized powers $u_{d_1+1}(x,\bar{x}_i), \dots, u_p(x,\bar{x}_i), \dots, u_{d_m+1}(x,\bar{x}_m), \dots, u_p(x,\bar{x}_m)$ are linearly independent. In the polynomial case the ready-to-blossom-like bases exist and take the elementary form which is proved in the following result.

Proposition 2.1. *Let $-1 \le d_i < p$, $d_i \in \mathbb{N}$, $i = 1, \dots, m$, be given with $\sum_{i=1}^m (p - d_i) = p+1$. For any set of m distinct points $\bar{x}_1, \dots, \bar{x}_m \in \mathbb{R}$ the functions*

$$\Psi_k^{\bar{x}_i}(x) := \frac{(x-\bar{x}_i)^k}{k!}, \quad i = 1, \dots, m, \quad k = d_i + 1, \dots, p, \tag{2.9}$$

form a basis of $\mathbb{P}_p$.

Proof. It suffices to prove that the $p+1$ functions in (2.9) are linearly independent. Since $\sum_{i=1}^m (p-d_i) = p+1$, for $i = 1, \dots, m$, $j = 0, \dots, p-1-d_i$, there exist polynomials $\phi_{i,j} \in \mathbb{P}_p$, solving the following Hermite interpolation problems

$$D^l \phi_{i,j}(\bar{x}_k) = \delta_{i,k}\delta_{j,l}, \quad k = 1, \dots, m, \quad l = 0, \dots, p-1-d_k.$$

We define

$$u(x,y) := \sum_{i=1}^{m} \sum_{j=0}^{p-1-d_i} (-1)^j \phi_{i,j}(y) \Psi_{p-j}^{\bar{x}_i}(x).$$

Since $\Psi_{p-j}^{\bar{x}_i}(x) = D^j \frac{(x-\bar{x}_i)^p}{p!}$ we have that for $k = 1, \dots, m$ and $l = 0, \dots, p-1-d_k$,

$$D_y^l u(x,y)|_{y=\bar{x}_k} = \sum_{i=1}^{m} \sum_{j=0}^{p-1-d_i} (-1)^j D^l \phi_{i,j}(\bar{x}_k) \Psi_{p-j}^{\bar{x}_i}(x) = (-1)^l \Psi_{p-l}^{\bar{x}_k}(x) = D_y^l \frac{(x-y)^p}{p!}\Big|_{y=\bar{x}_k}.$$

These are $p+1$ Hermite conditions, thus from the uniqueness of Hermite interpolation in the y variable we deduce that

$$\frac{(x-y)^p}{p!} = u(x,y) = \sum_{i=1}^{m}\sum_{j=0}^{p-1-d_i}(-1)^j\phi_{i,j}(y)D^j\frac{(x-\bar{x}_i)^p}{p!} = \sum_{i=1}^{m}\sum_{j=0}^{p-1-d_i}(-1)^j\phi_{i,j}(y)\Psi_{p-j}^{\bar{x}_i}. \tag{2.10}$$

It can be easily verified that, for any sequence of $p+1$ distinct points $c_0,\dots,c_p$ in $\mathbb{R}$, the functions $(x-c_0)^p,\dots,(x-c_p)^p$ form a basis for $\mathbb{P}_p$, see also [21, Lemma 2]. Since relation (2.10) can be used to represent the above set of basis functions, this implies concludes the proof. □

Proposition 2.1 can be extended to any ECT space with the following result, which was proved in [16, Theorem 23].

Theorem 2.3. *An ET-space $\mathbb{T}_p(J)$ possesses a ready-to-blossom-like basis relative to each set of m distinct points belonging to J, with $1 \le m \le p+1$.*

Ready-to-blossom-like bases can be seen as a generalization of Bernstein-like bases [15] and are important tools for CAGD in the Tchebycheffian framework, because of their being so intimately connected with blossoms [16] which, after [23], have become a relevant way to handle Tchebycheffian spline curves. In particular, their use has been crucial for designing with ET-spaces under the weakest possible differentiability assumptions, also permitting to describe all possible weights which can be associated with a given ECT-space on a closed bounded interval [18]. They have since been extended to larger frameworks, to achieve many important results with reference to blossoms, e.g., they are strongly involved in the description of the largest class of geometrically continuous splines with pieces taken from different ET-spaces which can be used for design (see [19]), along with the many applications of this result in connection with [17] (see, e.g., [20] and references therein).

The existence of ready-to-blossom-like bases is also closely related to an important ingredient for obtaining dimension results for Tchebycheffian spline spaces over T-meshes [3, Theorem 3.1], the so called $\boldsymbol{d}$-sum property.

Definition 2.4 ($\boldsymbol{d}$-sum property)**.** *Consider an ET-space $\mathbb{T}_p(J)$ of dimension $p+1$ on J. Let $\boldsymbol{d} := (d_1,\dots,d_m)$ with $0 \le d_i \le p$, $d_i \in \mathbb{N}$, $i=1,\dots,m$. We say that $\mathbb{T}_p(J)$ has the $\boldsymbol{d}$-sum property if for any set of m distinct points $\bar{x}_1,\dots,\bar{x}_m \in J$ we have*

$$\dim\left(\sum_{i=1}^{m}\mathbb{I}^{\mathbb{T}_p,d_i}(\bar{x}_i)\right) = \min\left(p+1,\ \sum_{i=1}^{m}p-d_i\right),$$

where

$$\mathbb{I}^{\mathbb{T}_p,d}(\bar{x}) := \left\{\, q \in \mathbb{T}_p(J) : D^l q(\bar{x}) = 0,\ l = 0,\dots,d \,\right\}.$$

For an ET-space, it easy to see that any set of $p-d$ functions of the form $\Psi_{d+1}^{\bar{x}},\cdots,\Psi_p^{\bar{x}}$ span $\mathbb{I}^{\mathbb{T}_p,d}(\bar{x})$, and that $\mathbb{I}^{\mathbb{T}_p,d}(\bar{x})$ is the trivial space if $d=p$. Therefore, from Theorem 2.3 we immediately obtain the following result.

Theorem 2.4. *An ET-space has the $\boldsymbol{d}$-sum property for any $\boldsymbol{d} := (d_1,\dots,d_m)$ with integers d_i such that $0 \le d_i \le p$ for $i=1,\dots,m$ and any $m \in \mathbb{N}$.*

3. Tchebycheffian spline spaces over T-meshes

We first recall the concepts and definitions related to T-meshes, using the notation given in [3] (see also [2, 22]). We consider a domain $\Omega \subset \mathbb{R}^2$ which is a finite union of closed axis-aligned rectangles, called *cells*, whose interiors are pairwise disjoint. This domain Ω is assumed to be simply connected and its interior Ω^o is connected. We denote by $[a_h,b_h]\times[a_v,b_v]$ the smallest rectangle containing Ω.

Definition 3.1 (T-mesh)**.** *A T-mesh $\mathcal{T} := (\mathcal{T}_2,\mathcal{T}_1,\mathcal{T}_0)$ on Ω is defined as:*

- *$\mathcal{T}_2$ is the collection of cells in Ω;*
- *$\mathcal{T}_1 = \mathcal{T}_1^h \cup \mathcal{T}_1^v$ is a finite set of closed axis-aligned horizontal and vertical segments in $\bigcup_{\sigma\in\mathcal{T}_2}\partial\sigma$, called* edges;

- *$\mathcal{T}_0 := \bigcup_{\tau \in \mathcal{T}_1} \partial\tau$ is a finite set of points, called vertices;*

such that

- *for each $\sigma \in \mathcal{T}_2$, $\partial\sigma$ is a finite union of elements of $\mathcal{T}_1$;*
- *for $\sigma, \sigma' \in \mathcal{T}_2$ with $\sigma \neq \sigma'$, $\sigma \cap \sigma' = \partial\sigma \cap \partial\sigma'$ is a finite union of elements of $\mathcal{T}_1 \cup \mathcal{T}_0$;*
- *for $\tau, \tau' \in \mathcal{T}_1$ with $\tau \neq \tau'$, $\tau \cap \tau' = \partial\tau \cap \partial\tau' \subset \mathcal{T}_0$;*
- *for each $\gamma \in \mathcal{T}_0$, $\gamma = \tau_h \cap \tau_v$ where τ_h is a horizontal edge and τ_v is a vertical edge.*

We denote by $\mathcal{T}_1^o$ the set of *interior edges*, i.e., the edges intersecting Ω^o. Analogously, $\mathcal{T}_0^o$ represents the set of *interior vertices*, i.e., the vertices in Ω^o. The elements of the sets $\mathcal{T}_1 \setminus \mathcal{T}_1^o$ and $\mathcal{T}_0 \setminus \mathcal{T}_0^o$ are the *boundary edges* and the *boundary vertices*, respectively. Moreover, $\mathcal{T}_1^{o,h}$ and $\mathcal{T}_1^{o,v}$ indicate the sets of the horizontal and vertical interior edges of $\mathcal{T}$, respectively, and we set $\mathcal{T}_1^o := \mathcal{T}_1^{o,h} \cup \mathcal{T}_1^{o,v}$. Then, the interior T-mesh is $\mathcal{T}^o := (\mathcal{T}_2, \mathcal{T}_1^o, \mathcal{T}_0^o)$.

A *segment* of $\mathcal{T}$ is a connected union of edges of $\mathcal{T}$ belonging to the same straight line. Given any $\tau \in \mathcal{T}_1^o$, we denote by $\rho(\tau)$ the maximal segment composed of edges of $\mathcal{T}_1^o$ containing τ. Moreover, we denote by $\textsc{ms}(\mathcal{T})$ the set of all such maximal segments, and by $\textsc{mis}(\mathcal{T})$ the set of all *maximal interior segments (MIS)*, that is the subset of $\textsc{ms}(\mathcal{T})$ whose elements do not intersect the boundary of the T-mesh. The set of all horizontal (respectively vertical) maximal interior segments is denoted by $\textsc{mis}_h(\mathcal{T})$ (respectively $\textsc{mis}_v(\mathcal{T})$). Given any $\gamma \in \mathcal{T}_0^o$, we define $\rho_h(\gamma) := \rho(\tau_h)$ and $\rho_v(\gamma) := \rho(\tau_v)$, such that $\gamma = \tau_h \cap \tau_v$ and $\tau_h \in \mathcal{T}_1^{o,h}$, $\tau_v \in \mathcal{T}_1^{o,v}$.

Since we are going to deal with Tchebycheffian spline spaces over T-meshes, we also need to formalize the concept of smoothness in this context.

Definition 3.2 (Smoothness)**.** *With each edge $\tau \in \mathcal{T}_1^o$, we associate an integer $r(\tau) \geq 0$. We say that $f \in C^{r(\tau)}(\tau)$ if the partial derivatives of f up to order $r(\tau)$ are continuous across the edge τ. We assume that $r(\tau) = r(\tau')$ for all τ, τ' lying on the same straight line, and we refer to this as the constant smoothness (along lines) assumption. A smoothness distribution on $\mathcal{T}$ is defined as*

$$\boldsymbol{r} := \{\, r(\tau), \forall \tau \in \mathcal{T}_1^o \,\},$$

and leads to the following class of smooth functions on Ω:

$$C^{\boldsymbol{r}}(\mathcal{T}) := \{\, f : \Omega \to \mathbb{R} : f \in C^{r(\tau)}(\tau), \forall \tau \in \mathcal{T}_1^o \,\}.$$

Given a smoothness distribution $\boldsymbol{r}$ on $\mathcal{T}$, with each vertex $\gamma \in \mathcal{T}_0^o$, we associate two integers $r_h(\gamma), r_v(\gamma)$, where $r_h(\gamma) := r(\tau_v)$ and $r_v(\gamma) := r(\tau_h)$ such that $\gamma = \tau_h \cap \tau_v$ and $\tau_h \in \mathcal{T}_1^{o,h}$, $\tau_v \in \mathcal{T}_1^{o,v}$. For each maximal segment $\rho \in \textsc{ms}(\mathcal{T})$ we set $r(\rho) := r(\tau)$, where τ is any interior edge belonging to ρ.

In the following, we denote by ℓ either h or v. Let $p_\ell \in \mathbb{N}$ with $p_\ell \geq 0$, and let $\mathbb{T}^\ell_{p_\ell}([a_\ell, b_\ell])$ be an ET-space of dimension $p_\ell + 1$ on $J_\ell := [a_\ell, b_\ell]$. Then, we define the tensor-product space

$$\mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}} := \mathbb{T}^h_{p_h}([a_h, b_h]) \otimes \mathbb{T}^v_{p_v}([a_v, b_v]), \tag{3.1}$$

where $\boldsymbol{p} := (p_h, p_v)$ and $\boldsymbol{T} := (T_h, T_v) := (\mathbb{T}^h_{p_h}, \mathbb{T}^v_{p_v})$. If the space (3.1) is the space of bivariate algebraic polynomials of bi-degree $\boldsymbol{p}$, then it is denoted by $\mathbb{P}_{\boldsymbol{p}}$. In analogy with the polynomial case, we call $\boldsymbol{p}$ the bi-degree of the space $\mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}}$.

Definition 3.3 (Tchebycheffian spline space over a T-mesh)**.** *Let $\mathcal{T}$ be a T-mesh with a smoothness distribution $\boldsymbol{r}$, and let $p_h, p_v \in \mathbb{N}$ with $p_h, p_v \geq 0$. The Tchebycheffian spline space over the T-mesh $\mathcal{T}$, denoted by $\mathbb{S}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T})$, is defined as the space of functions in $C^{\boldsymbol{r}}(\mathcal{T})$ such that, restricted to any cell $\sigma \in \mathcal{T}_2$, they belong to $\mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}}$, i.e.,*

$$\mathbb{S}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T}) := \left\{\, s \in C^{\boldsymbol{r}}(\mathcal{T}) : s_{|\sigma} \in \mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}},\ \sigma \in \mathcal{T}_2 \,\right\}.$$

In particular, in the case of bivariate algebraic polynomials,

$$\mathbb{S}^{\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T}) := \left\{\, s \in C^{\boldsymbol{r}}(\mathcal{T}) : s_{|\sigma} \in \mathbb{P}_{\boldsymbol{p}},\ \sigma \in \mathcal{T}_2 \,\right\}.$$

In the following, we will assume the usual condition on the smoothness (see [3, Section 2.2]):

$$r(\tau_v) < p_h, \quad \forall \tau_v \in \mathcal{T}_1^{o,v}, \quad r(\tau_h) < p_v, \quad \forall \tau_h \in \mathcal{T}_1^{o,h}.$$

For some detailed examples of T-meshes and the related concepts, we refer the reader to [3, Section 2] and [2, Section 2].

4. Dimension formula for Tchebycheffian spline spaces over T-meshes

In [3] homological techniques were employed to obtain bounds for the dimension of Tchebycheffian spline spaces defined over a T-mesh $\mathcal{T}$. In this section we improve those results. Indeed, thanks to Theorem 2.4, we are able to formulate the bounds, without any assumption on the relation between the bi-degree of the space and its smoothness. Let us first recall some preliminary definitions and concepts.

Definition 4.1 (**$\boldsymbol{r}$**-sum property on $\mathcal{T}$)**.** *Given a smoothness distribution $\boldsymbol{r}$ on $\mathcal{T}$, we say that $\boldsymbol{T} := (\mathbb{T}^h_{p_h}, \mathbb{T}^v_{p_v})$ has the $\boldsymbol{r}$-sum property on $\mathcal{T}$, if each of its components $\mathbb{T}^\ell_{p_\ell}([a_\ell, b_\ell])$ with $\ell = h, v$ has the $\boldsymbol{d}$-sum property (see Definition 2.4) for any subvector $\boldsymbol{d}$ of the vector $\boldsymbol{r}_\ell := (r_\ell(\gamma))_{\gamma\in\mathcal{T}^o_0}$.*

Let ι be an ordering of $\mathrm{MIS}(\mathcal{T})$. For any $\rho \in \mathrm{MIS}(\mathcal{T})$, we denote by $\Gamma_\iota(\rho)$ the set of vertices of ρ which do not belong to $\rho' \in \mathrm{MIS}(\mathcal{T})$ with $\iota(\rho') > \iota(\rho)$.

Definition 4.2 (Weight of MIS)**.** *Given an ordering ι of $\mathrm{MIS}(\mathcal{T})$, the weight of a maximal interior segment $\rho \in \mathrm{MIS}(\mathcal{T})$ is defined as*

$$\omega_\iota(\rho) := \begin{cases} \sum_{\gamma\in\Gamma_\iota(\rho)}(p_h - r_h(\gamma)), & \text{if } \rho \in \mathrm{MIS}_h(\mathcal{T}) \\ \sum_{\gamma\in\Gamma_\iota(\rho)}(p_v - r_v(\gamma)), & \text{if } \rho \in \mathrm{MIS}_v(\mathcal{T}) \end{cases}.$$

In the next theorem we collect the dimension results for Tchebycheffian spline spaces over T-meshes. This theorem improves the results of [3] and fully generalizes the results known for the polynomial case [22, Theorems 3.1 and 3.7].

Theorem 4.1. *Let $\mathbb{S}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T})$ be a Tchebycheffian spline space over a T-mesh $\mathcal{T}$. Then,*

$$\begin{aligned}\dim\big(\mathbb{S}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T})\big) = &\sum_{\sigma\in\mathcal{T}_2}(p_h+1)(p_v+1) - \sum_{\tau\in\mathcal{T}^{o,h}_1}(p_h+1)(r(\tau)+1) - \sum_{\tau\in\mathcal{T}^{o,v}_1}(r(\tau)+1)(p_v+1) \\ &+ \sum_{\gamma\in\mathcal{T}^o_0}(r_h(\gamma)+1)(r_v(\gamma)+1) + H_0,\end{aligned} \tag{4.1}$$

where

$$0 \le H_0 \le \sum_{\rho\in\mathrm{MIS}_h(\mathcal{T})}(p_h+1-\omega_\iota(\rho))_+\,(p_v - r(\rho)) + \sum_{\rho\in\mathrm{MIS}_v(\mathcal{T})}(p_h - r(\rho))\,(p_v+1-\omega_\iota(\rho))_+, \tag{4.2}$$

and $(x)_+ := \max(x, 0)$.

Proof. The dimension formula (4.1) was shown in [3, Theorem 4.1] with H_0 a specific homology term. Furthermore, in [3, Theorem 3.1], it was proved that such homology term can be bounded by (4.2) under the assumption that the considered couple of ET-spaces $\boldsymbol{T} = (\mathbb{T}^h_{p_h}, \mathbb{T}^v_{p_v})$ satisfies the $\boldsymbol{r}$-sum property on $\mathcal{T}$. From Theorem 2.4 we know that this is the case for any $\boldsymbol{T}$. This completes the proof. □

Example 4.1. *Consider the T-mesh $\mathcal{T}$ in Figure 1. This mesh has 4 maximal interior segments, namely $\mathrm{MIS}(\mathcal{T}) = \mathrm{MIS}_h(\mathcal{T}) \cup \mathrm{MIS}_v(\mathcal{T})$ with $\mathrm{MIS}_h(\mathcal{T}) = \{\rho_1, \rho_3\}$ and $\mathrm{MIS}_v(\mathcal{T}) = \{\rho_2, \rho_4\}$. The ordering ι of $\mathrm{MIS}(\mathcal{T})$ is given by $\iota(\rho_j) = j$, $j = 1, 2, 3, 4$. Note that*

$$|\mathcal{T}_2| = 24, \quad |\mathcal{T}^{o,h}_1| = |\mathcal{T}^{o,v}_1| = 22, \quad |\mathcal{T}^o_0| = 21.$$

Moreover, let $\boldsymbol{p} = (p_h, p_v) = (2, 2)$ and let $\boldsymbol{r}$ be the constant smoothness distribution where $r(\tau) = 1$ for all $\tau \in \mathcal{T}^o_1$. Then,

$$\omega_\iota(\rho_1) = 2, \quad \omega_\iota(\rho_2) = 3, \quad \omega_\iota(\rho_3) = 3, \quad \omega_\iota(\rho_4) = 4.$$

Theorem 4.1 implies that $0 \le H_0 \le 1$, and $36 \le \dim\big(\mathbb{S}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T})\big) \le 37$ for $\boldsymbol{T} = (\mathbb{T}^h_2, \mathbb{T}^v_2)$ where $\mathbb{T}^h_2, \mathbb{T}^v_2$ are any ET-spaces of dimension 3 on the intervals $[s_0, s_6]$ and $[t_0, t_6]$, respectively.

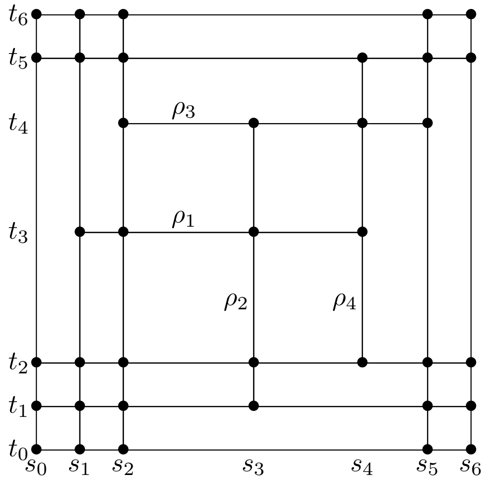


Figure 1: Example of an unstable T-mesh.

Theorem 4.1 gives an explicit and computable expression for the dimension of the space $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})$ when the upper bound in (4.2) is zero. It is evident that such configurations are of practical interest, and therefore the upper bound in (4.2) plays an important role in the design of T-mesh refinement algorithms. This upper bound depends on the bi-degree, on the smoothness, and on the weights of MIS with respect to any ordering of MIS. If the T-mesh is obtained by successive refinements, we can use the following algorithm (Algorithm 4.1) to generate a naturally induced ordering of MIS, which will be employed to construct a T-mesh refinement algorithm (Algorithm 4.2) ensuring $H_0 = 0$.

Algorithm 4.1. *Given a T-mesh $\mathcal{T}$ with an ordering ι of $\textsc{mis}(\mathcal{T})$, let $\tilde{\mathcal{T}}$ be the T-mesh obtained after inserting a new segment τ (i.e., one or more consecutive edges). The ordering $\tilde{\iota}$ of $\textsc{mis}(\tilde{\mathcal{T}})$ is computed as follows:*

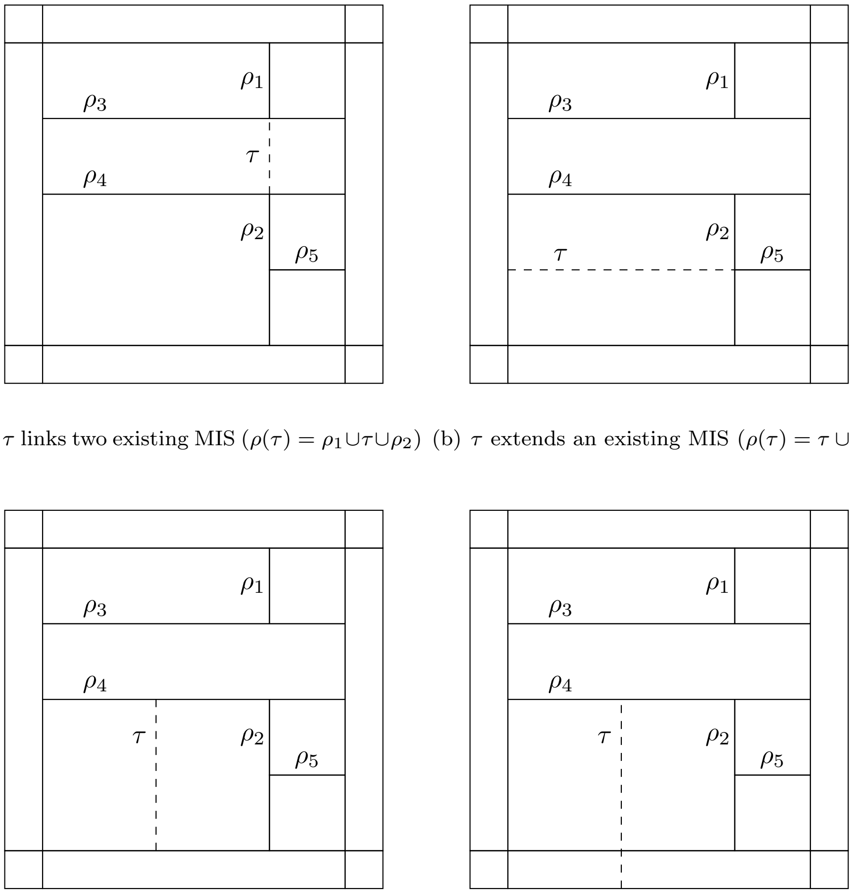
1. *set $\tilde{\iota}(\tilde{\rho}) := \iota(\tilde{\rho})$ for all $\tilde{\rho} \in \textsc{mis}(\tilde{\mathcal{T}}) \setminus \{\rho(\tau)\}$ where $\rho(\tau) \in \textsc{ms}(\tilde{\mathcal{T}})$ is the maximal segment containing τ;*
2. *if $\rho(\tau) \in \textsc{mis}(\tilde{\mathcal{T}})$:*
   (a) *if $\rho(\tau) = \rho' \cup \tau \cup \rho''$ with $\rho', \rho'' \in \textsc{mis}(\mathcal{T})$, then $\tilde{\iota}(\rho(\tau)) := \min(\iota(\rho'), \iota(\rho''))$;*
   (b) *if $\rho(\tau) = \tau \cup \rho'$ with $\rho' \in \textsc{mis}(\mathcal{T})$, then $\tilde{\iota}(\rho(\tau)) := \iota(\rho')$;*
   (c) *if $\rho(\tau) = \tau$, then $\tilde{\iota}(\tau) := 1 + \max_{\rho' \in \textsc{mis}(\mathcal{T})} \iota(\rho')$.*

Figure 2 illustrates the different cases of inserting a new segment τ in Algorithm 4.1. Note that the indices of the ordering of MIS produced by Algorithm 4.1 are not necessarily consecutive. As shown in the next lemma, Algorithm 4.1 ensures that the weights of the already existing MIS do not decrease.

Lemma 4.1. *Given a T-mesh $\mathcal{T}$ with an ordering ι of $\textsc{mis}(\mathcal{T})$, let $\tilde{\mathcal{T}}$ be the T-mesh obtained after inserting a new segment τ and let $\tilde{\iota}$ be the ordering of $\textsc{mis}(\tilde{\mathcal{T}})$ computed by Algorithm 4.1. Then, for all $\tilde{\rho} \in \textsc{mis}(\tilde{\mathcal{T}})$ such that $M(\tilde{\rho}) := \{\rho \in \textsc{mis}(\mathcal{T}) : \rho \subseteq \tilde{\rho}\} \neq \emptyset$, there exists $\bar{\rho} \in M(\tilde{\rho})$ with $\iota(\bar{\rho}) = \tilde{\iota}(\tilde{\rho})$ and $\omega_\iota(\bar{\rho}) \leq \omega_{\tilde{\iota}}(\tilde{\rho})$.*

Proof. Let us first focus on Step 1 in Algorithm 4.1. Since $\textsc{mis}(\tilde{\mathcal{T}}) \setminus \{\rho(\tau)\} = \textsc{mis}(\tilde{\mathcal{T}}) \cap \textsc{mis}(\mathcal{T})$, we have $M(\tilde{\rho}) = \{\tilde{\rho}\}$ for all $\tilde{\rho} \in \textsc{mis}(\tilde{\mathcal{T}}) \setminus \{\rho(\tau)\}$. In such case, we set $\bar{\rho} = \tilde{\rho}$. When arriving at Step 2(a) and considering $\tilde{\rho} = \rho(\tau) \in \textsc{mis}(\tilde{\mathcal{T}})$, we have $M(\tilde{\rho}) = \{\rho', \rho''\}$ and we set $\bar{\rho} = \rho'$ if $\iota(\rho') < \iota(\rho'')$ and $\bar{\rho} = \rho''$ otherwise. Similarly, at Step 2(b), we have $M(\tilde{\rho}) = \{\rho'\}$ and we set $\bar{\rho} = \rho'$. Note that we can ignore Step 2(c) since $M(\tilde{\rho}) = \emptyset$.

For the above choices of $\bar{\rho}$ it is clear from the algorithm that $\iota(\bar{\rho}) = \tilde{\iota}(\tilde{\rho})$. In the last part of the proof, we will show that $\Gamma_\iota(\bar{\rho}) \subseteq \Gamma_{\tilde{\iota}}(\tilde{\rho})$, and this immediately implies $\omega_\iota(\bar{\rho}) \leq \omega_{\tilde{\iota}}(\tilde{\rho})$. Inserting τ results in adding possible vertices belonging to the elements of $\textsc{mis}(\mathcal{T})$. Therefore, by the definition of $\Gamma_\iota, \Gamma_{\tilde{\iota}}$ and $\iota(\bar{\rho}) = \tilde{\iota}(\tilde{\rho})$, we have $\Gamma_\iota(\bar{\rho}) = \Gamma_{\tilde{\iota}}(\tilde{\rho})$ if $\tilde{\iota}(\tilde{\rho}) < \tilde{\iota}(\rho(\tau))$, and $\Gamma_\iota(\bar{\rho}) \subseteq \Gamma_{\tilde{\iota}}(\tilde{\rho})$ otherwise, see also Figure 2. $\square$



(a) τ links two existing MIS ($\rho(\tau) = \rho_1 \cup \tau \cup \rho_2$) (b) τ extends an existing MIS ($\rho(\tau) = \tau \cup \rho_5$)

(c) τ introduces a new MIS ($\rho(\tau) = \tau$) (d) τ does not modify the set of MIS ($\rho(\tau) = \tau$)

Figure 2: Examples of the different cases of inserting a segment τ (indicated by dashed lines) in Algorithm 4.1.

We now detail a refinement strategy that generates a sequence of refined T-meshes such that the upper bound in (4.2) is kept to be zero throughout the entire refinement process. It is a particular implementation of the so-called (p_h+1, p_v+1)-*weighted subdivision rule* described in [22, Algorithm 4.4]. This rule was developed in the context of polynomial splines but is also valid in our more general Tchebycheffian spline setting.

Algorithm 4.2. *Given a T-mesh $\mathcal{T}$ with an ordering ι of $\text{MIS}(\mathcal{T})$, two positive integers p_h, p_v, and a smoothness distribution $\boldsymbol{r}$ such that $\omega_\iota(\rho) \geq p_h + 1$ for any $\rho \in \text{MIS}_h(\mathcal{T})$ and $\omega_\iota(\rho) \geq p_v + 1$ for any $\rho \in \text{MIS}_v(\mathcal{T})$, we construct the refinement as follows for a segment τ to be inserted in $\mathcal{T}$:*

1. *if τ does not extend an existing edge, then extend τ so that the maximal segment containing τ, say $\rho(\tau)$, intersects $\partial\Omega$ or satisfies*

$$\sum_{\rho' \in \Gamma_{\rho(\tau)}} (p_\tau - r(\rho')) \geq p_\tau + 1, \quad p_\tau := \begin{cases} p_h, & \text{if } \tau \text{ is horizontal,} \\ p_v, & \text{if } \tau \text{ is vertical,} \end{cases} \tag{4.3}$$

   where $\Gamma_{\rho(\tau)}$ is the set of maximal segments $\rho' \in \text{MS}(\mathcal{T})$ intersecting $\rho(\tau)$;
2. *update the ordering ι according to Algorithm 4.1.*

The next proposition shows that Algorithm 4.2 can be applied successively once a valid initial T-mesh configuration is constructed. For instance, any tensor-product mesh leads to a valid initial configuration because it does not contain MIS. Moreover, Algorithm 4.2 guarantees that $H_0 = 0$ in Theorem 4.1.

Proposition 4.1. *Let $\tilde{\mathcal{T}}$ be the T-mesh generated by Algorithm 4.2 and let $\tilde{\iota}$ be the corresponding ordering of* $\text{MIS}(\tilde{\mathcal{T}})$. *We have*

- $\omega_{\tilde{\iota}}(\tilde{\rho}) \geq p_h + 1$ *for any* $\tilde{\rho} \in \text{MIS}_h(\tilde{\mathcal{T}})$ *and* $\omega_{\tilde{\iota}}(\tilde{\rho}) \geq p_v + 1$ *for any* $\tilde{\rho} \in \text{MIS}_v(\tilde{\mathcal{T}})$*;*
- $H_0 = 0$.

Proof. Let τ be the segment to be inserted in $\mathcal{T}$ by means of Algorithm 4.2. If τ does not extend an existing edge and $\rho(\tau) \in \text{MIS}(\tilde{\mathcal{T}})$, then Algorithm 4.1 (Step 2(c)) gives $\tilde{\iota}(\tau) := 1 + \max_{\rho' \in \text{MIS}(\mathcal{T})} \iota(\rho')$. Hence, $\Gamma_{\tilde{\iota}}(\rho(\tau))$ is the set of interior vertices of $\tilde{\mathcal{T}}$ belonging to $\rho(\tau)$, i.e., the intersections of $\rho(\tau)$ with the elements in $\Gamma_{\rho(\tau)}$. By Definition 4.2 and by the condition (4.3), we get $\omega_{\tilde{\iota}}(\rho(\tau)) \geq p_h + 1$ if $\rho(\tau) \in \text{MIS}_h(\tilde{\mathcal{T}})$ and $\omega_{\tilde{\iota}}(\rho(\tau)) \geq p_v + 1$ if $\rho(\tau) \in \text{MIS}_v(\tilde{\mathcal{T}})$. Moreover, from the properties of Algorithm 4.1 in Lemma 4.1 we know that the weight of any other MIS in $\tilde{\mathcal{T}}$, say $\tilde{\rho}$, is not decreased (because $M(\tilde{\rho}) \neq \emptyset$). It follows that $\omega_{\tilde{\iota}}(\tilde{\rho}) \geq p_h + 1$ for any $\tilde{\rho} \in \text{MIS}_h(\tilde{\mathcal{T}})$ and $\omega_{\tilde{\iota}}(\tilde{\rho}) \geq p_v + 1$ for any $\tilde{\rho} \in \text{MIS}_v(\tilde{\mathcal{T}})$. By using the bounds in (4.2), this also implies that $H_0 = 0$. □

The refinement strategy detailed in Algorithm 4.2 and the corresponding explicit dimension formula is completely in agreement with similar results given for polynomial LR B-splines in [6, Section 5]. Furthermore, the dimension formula is in agreement with the results and the examples for polynomial spline spaces over T-meshes provided in [10, Section 4].

Theorem 4.1 also strengthens the relationship between Tchebycheffian and polynomial spline spaces over T-meshes. For any couple of ET-spaces $\boldsymbol{T}$, the difference between the dimensions of the Tchebycheffian spline space $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})$ and the related polynomial spline space $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{r}}(\mathcal{T})$ is bounded in terms of a quantity that only depends on the mesh, see (4.2). In particular, such difference is zero if the mesh $\mathcal{T}$ has been obtained by applying Algorithm 4.2.

5. Instability

In this section we show that the dimension of the Tchebycheffian spline space $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})$ can depend not only on the topological information of $\mathcal{T}$ but also on the geometry of the T-mesh. This particular behavior is usually referred to as *instability in the dimension* of the considered space. Examples of instability in the dimension of spline spaces over T-meshes are known for polynomial spline spaces [1, 9] and for trigonometric and hyperbolic spline spaces [3].

We focus on the T-mesh in Figure 1, which is a mirrored version of the one already considered in [3, 9]. We consider $\boldsymbol{p} = (2, 2)$ and a constant smoothness distribution $\boldsymbol{r}$ such that $r(\tau) = 1$ for all $\tau \in \mathcal{T}_1^o$. Moreover, we set $\boldsymbol{T} = (\mathbb{T}_2, \mathbb{T}_2)$, where $\mathbb{T}_2$ is any ET-space of dimension 3 on a given bounded closed interval. Note that $\mathbb{T}_2$ is an ECT-space on the same interval identified by some positive weights w_0, w_1, w_2 (see Theorem 2.1 and Theorem 2.2). From Example 4.1 we get

$$\dim\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})\big) = 36 + H_0, \quad 0 \leq H_0 \leq 1. \tag{5.1}$$

Following the same reasoning as in [3, Section 5], we also know that $H_0 = 12 - K_0$, where K_0 is the dimension of the space spanned by the rows of the following matrix

$$M := \begin{pmatrix} \psi_{s_4,2}(x) & 0 & 0 & \psi_{t_3,2}(y) \\ \psi_{s_1,2}(x) & 0 & 0 & 0 \\ \psi_{s_2,2}(x) & 0 & 0 & 0 \\ \psi_{s_3,2}(x) & \psi_{t_3,2}(y) & 0 & 0 \\ 0 & \psi_{t_1,2}(y) & 0 & 0 \\ 0 & \psi_{t_2,2}(y) & 0 & 0 \\ 0 & \psi_{t_4,2}(y) & \psi_{s_3,2}(x) & 0 \\ 0 & 0 & \psi_{s_2,2}(x) & 0 \\ 0 & 0 & \psi_{s_5,2}(x) & 0 \\ 0 & 0 & \psi_{s_4,2}(x) & \psi_{t_4,2}(y) \\ 0 & 0 & 0 & \psi_{t_2,2}(y) \\ 0 & 0 & 0 & \psi_{t_5,2}(y) \end{pmatrix}. \tag{5.2}$$

In this matrix, $\{\psi_{\bar{z},i}\}_{i=0}^2$ is a Taylor-like basis of the space $\mathbb{T}_2$, i.e.,

$$D_z^k \psi_{\bar{z},i}(\bar{z}) = \delta_{ik}, \quad i, k = 0, 1, 2.$$

For instance, in the polynomial case (i.e., $\mathbb{T}_2 = \mathbb{P}_2$) we have $\psi_{\bar{z},i}(z) = (z-\bar{z})^i/i!$, $i = 0,1,2$, and K_0 is given by the dimension of the space spanned by the rows of the following matrix

$$M_{\mathbb{P}_2}(s_1,\ldots,s_5;t_1,\ldots,t_5) := \begin{pmatrix}
1 & s_4 & {s_4}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_3 & {t_3}^2 \\
1 & s_1 & {s_1}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & s_2 & {s_2}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & s_3 & {s_3}^2 & 1 & t_3 & {t_3}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & t_1 & {t_1}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & t_2 & {t_2}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & t_4 & {t_4}^2 & 1 & s_3 & {s_3}^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & s_2 & {s_2}^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & s_5 & {s_5}^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & s_4 & {s_4}^2 & 1 & t_4 & {t_4}^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_2 & {t_2}^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_5 & {t_5}^2
\end{pmatrix}.$$

It is clear that $\operatorname{rank}(M_{\mathbb{P}_2}(s_1,\ldots,s_5;t_1,\ldots,t_5)) \geq 11$. The matrix $M_{\mathbb{P}_2}(s_1,\ldots,s_5;t_1,\ldots,t_5)$ has been analyzed in [9] where it has been proved that it is singular if and only if

$$\frac{(s_3-s_1)(s_5-s_4)}{(t_3-t_1)(t_5-t_4)} = \frac{(s_4-s_1)(s_5-s_3)}{(t_4-t_1)(t_5-t_3)}, \tag{5.3}$$

and, in particular, if $s_i = t_i$ for all i. Hence, for $\boldsymbol{T} = (\mathbb{P}_2, \mathbb{P}_2)$ we get

$$H_0 = \begin{cases} 1, & \text{if (5.3) holds,} \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence, the dimension of the C^1 bi-quadratic polynomial spline space over the T-mesh in Figure 1 depends on the geometry of $\mathcal{T}$ according to the validity of (5.3).

Let us now consider a general ET-space $\mathbb{T}_2$. From Theorem 2.2 and from (2.4), (2.6) it follows that (possibly up to a constant) the Taylor-like basis function $\psi_{\bar{z},2}$ is given by

$$u_2(x,\bar{z}) = w_0(x) I_2(x,\bar{z};w_1,w_2).$$

Since $w_0 > 0$, the matrix in (5.2) has the same rank as the matrix

$$M := \begin{pmatrix}
I_2(x,s_4;w_1,w_2) & 0 & 0 & I_2(y,t_3;w_1,w_2) \\
I_2(x,s_1;w_1,w_2) & 0 & 0 & 0 \\
I_2(x,s_2;w_1,w_2) & 0 & 0 & 0 \\
I_2(x,s_3;w_1,w_2) & I_2(y,t_3;w_1,w_2) & 0 & 0 \\
0 & I_2(y,t_1;w_1,w_2) & 0 & 0 \\
0 & I_2(y,t_2;w_1,w_2) & 0 & 0 \\
0 & I_2(y,t_4;w_1,w_2) & I_2(x,s_3;w_1,w_2) & 0 \\
0 & 0 & I_2(x,s_2;w_1,w_2) & 0 \\
0 & 0 & I_2(x,s_5;w_1,w_2) & 0 \\
0 & 0 & I_2(x,s_4;w_1,w_2) & I_2(y,t_4;w_1,w_2) \\
0 & 0 & 0 & I_2(y,t_2;w_1,w_2) \\
0 & 0 & 0 & I_2(y,t_5;w_1,w_2)
\end{pmatrix}.$$

Using (2.3) we can write for $i = 1,\ldots,5$,

$$I_2(x,s_i;w_1,w_2) = I_2(s_i,s_1;w_2,w_1) - I_1(x,s_1;w_1) I_1(s_i,s_1;w_2) + I_2(x,s_1;w_1,w_2),$$

and a similar expression holds for $I_2(y,t_i;w_1,w_2)$. Since $\{1, I_1(x,s_1;w_1), I_2(x,s_1;w_1,w_2)\}$ (and also $\{1, I_1(y,t_1;w_1), I_2(y,t_1;w_1,w_2)\}$) are linearly independent, K_0 is given by the dimension of the space spanned by the rows of the following matrix

$$M_{\mathbb{T}_2}(s_1,\ldots,s_5;t_1,\ldots,t_5) := \begin{pmatrix} 1 & S_4^{(1)} & S_4^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & T_3^{(1)} & T_3^{(2)} \\ 1 & S_1^{(1)} & S_1^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & S_2^{(1)} & S_2^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & S_3^{(1)} & S_3^{(2)} & 1 & T_3^{(1)} & T_3^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & T_1^{(1)} & T_1^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & T_2^{(1)} & T_2^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & T_4^{(1)} & T_4^{(2)} & 1 & S_3^{(1)} & S_3^{(2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & S_2^{(1)} & S_2^{(2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & S_5^{(1)} & S_5^{(2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & S_4^{(1)} & S_4^{(2)} & 1 & T_4^{(1)} & T_4^{(2)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & T_2^{(1)} & T_2^{(2)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & T_5^{(1)} & T_5^{(2)} \end{pmatrix}, \tag{5.4}$$

where

$$S_i^{(1)} := I_1(s_i,s_1;w_2) = \int_{s_1}^{s_i} w_2(v)\,\mathrm{d}v, \quad S_i^{(2)} := I_2(s_i,s_1;w_2,w_1) = \int_{s_1}^{s_i} w_2(v)\int_{s_1}^{v} w_1(u)\,\mathrm{d}u\mathrm{d}v,$$
$$T_i^{(1)} := I_1(t_i,t_1;w_2) = \int_{t_1}^{t_i} w_2(v)\,\mathrm{d}v, \quad T_i^{(2)} := I_2(t_i,t_1;w_2,w_1) = \int_{t_1}^{t_i} w_2(v)\int_{t_1}^{v} w_1(u)\,\mathrm{d}u\mathrm{d}v.$$

In the case $w_1 = w_2$, the rank of this matrix behaves very similar to the polynomial case.

Proposition 5.1. *If $w_1 = w_2$ then the matrix (5.4) is singular if and only if*

$$\frac{\big(S_3^{(1)}-S_1^{(1)}\big)\big(S_5^{(1)}-S_4^{(1)}\big)}{\big(T_3^{(1)}-T_1^{(1)}\big)\big(T_5^{(1)}-T_4^{(1)}\big)} = \frac{\big(S_4^{(1)}-S_1^{(1)}\big)\big(S_5^{(1)}-S_3^{(1)}\big)}{\big(T_4^{(1)}-T_1^{(1)}\big)\big(T_5^{(1)}-T_3^{(1)}\big)}. \tag{5.5}$$

Proof. Let $w := w_1 = w_2$. By a symmetry argument we immediately get

$$S_i^{(2)} := \int_{s_1}^{s_i} w(v)\int_{s_1}^{v} w(u)\,\mathrm{d}u\mathrm{d}v = \frac{1}{2}\left(\int_{s_1}^{s_i} w(v)\,\mathrm{d}v\right)^2 = \frac{1}{2}\left(S_i^{(1)}\right)^2.$$

The result follows by comparing the matrices $M_{\mathbb{T}_2}(s_1,\ldots,s_5;t_1,\ldots,t_5)$ and $M_{\mathbb{P}_2}(s_1,\ldots,s_5;t_1,\ldots,t_5)$ and from (5.3). □

Proposition 5.1 shows that, in the case $w_1 = w_2$, the dimension of the Tchebycheffian spline space over the T-mesh in Figure 1 depends on the geometry of $\mathcal{T}$ according to the validity of (5.5). We now deal with the general case of (possibly different) positive weights w_1, w_2.

Proposition 5.2. *We have*

$$\det\big(M_{\mathbb{T}_2}(s_1,s_2,s_3,s_4,s_5;s_1,s_2,s_3,s_4,s_5)\big) = 0.$$

Moreover, for any pair w_1, w_2 there exists $\tilde{t}_5(w_1,w_2) > s_5$ such that for any t_5 with $s_5 < t_5 < \tilde{t}_5(w_1,w_2)$ we have

$$\det\big(M_{\mathbb{T}_2}(s_1,s_2,s_3,s_4,s_5;s_1,s_2,s_3,s_4,t_5)\big) \neq 0.$$

Proof. Since $s_i = t_i$, we have $T_i^{(1)} = S_i^{(1)}$, $T_i^{(2)} = S_i^{(2)}$, for $i = 1,2,3,4$. Assuming $t_5 = s_5 + \epsilon_5$ for some $\epsilon_5 \geq 0$, we get

$$T_5^{(1)} = S_5^{(1)} + \epsilon^{(1)}, \quad T_5^{(2)} = S_5^{(2)} + \epsilon^{(2)},$$

where

$$\epsilon^{(1)} = \int_{s_5}^{s_5+\epsilon_5} w_2(v)\,\mathrm{d}v, \quad \epsilon^{(2)} = \int_{s_5}^{s_5+\epsilon_5} w_2(v)\,\mathrm{d}v\int_{s_1}^{s_5} w_1(u)\,\mathrm{d}u + \int_{s_5}^{s_5+\epsilon_5} w_2(v)\int_{s_5}^{v} w_1(u)\,\mathrm{d}u\mathrm{d}v. \tag{5.6}$$

A direct computation gives

$$\begin{aligned}
&\det\big(M_{\mathbb{T}_2}(s_1,s_2,s_3,s_4,s_5;s_1,s_2,s_3,s_4,t_5)\big) = \\
&\qquad\big(S_1^{(1)}S_3^{(2)} - S_2^{(1)}S_3^{(2)} + S_2^{(1)}S_1^{(2)} - S_1^{(1)}S_2^{(2)} - S_3^{(1)}S_1^{(2)} + S_3^{(1)}S_2^{(2)}\big)\\
&\qquad\big(-S_4^{(1)}S_1^{(2)} + S_4^{(1)}S_2^{(2)} + S_1^{(1)}S_4^{(2)} - S_2^{(1)}S_4^{(2)} + S_2^{(1)}S_1^{(2)} - S_1^{(1)}S_2^{(2)}\big)\\
&\qquad\big(-S_3^{(1)}S_4^{(2)} + S_2^{(1)}S_4^{(2)} + S_4^{(1)}S_3^{(2)} - S_2^{(1)}S_3^{(2)} + S_3^{(1)}S_2^{(2)} - S_4^{(1)}S_2^{(2)}\big)\\
&\qquad\big(\epsilon^{(1)}\big(S_5^{(2)} - S_2^{(2)}\big) - \epsilon^{(2)}\big(S_5^{(1)} - S_2^{(1)}\big)\big),
\end{aligned}$$

and so $\det\big(M_{\mathbb{T}_2}(s_1,s_2,s_3,s_4,s_5;s_1,s_2,s_3,s_4,s_5)\big) = 0$.

Suppose now $\epsilon_5 > 0$. We can rewrite the determinant as follows

$$\begin{aligned}
\det\big(M_{\mathbb{T}_2}(s_1,s_2,s_3,s_4,s_5;s_1,s_2,s_3,s_4,t_5)\big) = &-\big(S_2^{(1)} - S_1^{(1)}\big)\big(S_3^{(1)} - S_1^{(1)}\big)\big(R_{1,3} - R_{1,2}\big)\\
&\big(S_2^{(1)} - S_1^{(1)}\big)\big(S_4^{(1)} - S_1^{(1)}\big)\big(R_{1,4} - R_{1,2}\big)\\
&\big(S_3^{(1)} - S_2^{(1)}\big)\big(S_4^{(1)} - S_2^{(1)}\big)\big(R_{2,4} - R_{2,3}\big)\\
&\epsilon^{(1)}\big(S_5^{(1)} - S_2^{(1)}\big)\big(R_{2,5} - \epsilon^{(2)}/\epsilon^{(1)}\big),
\end{aligned}$$

where

$$R_{i,j} := \frac{S_j^{(2)} - S_i^{(2)}}{S_j^{(1)} - S_i^{(1)}}, \quad i \neq j.$$

Note that $R_{i,j}$ is well defined because $w_2(x) > 0$, and

$$\begin{aligned}
R_{i,j} =& \frac{\int_{s_1}^{s_j} w_2(v)\int_{s_1}^{v} w_1(u)\,\mathrm{d}u\mathrm{d}v - \int_{s_1}^{s_i} w_2(v)\int_{s_1}^{v} w_1(u)\,\mathrm{d}u\mathrm{d}v}{\int_{s_i}^{s_j} w_2(v)\,\mathrm{d}v}\\
=& \frac{\int_{s_i}^{s_j} w_2(v)\,\mathrm{d}v\int_{s_1}^{s_i} w_1(u)\,\mathrm{d}u + \int_{s_i}^{s_j} w_2(v)\int_{s_i}^{v} w_1(u)\,\mathrm{d}u\mathrm{d}v}{\int_{s_i}^{s_j} w_2(v)\,\mathrm{d}v} = \int_{s_1}^{s_i} w_1(u)\,\mathrm{d}u + F(s_i,s_j),
\end{aligned}$$

where

$$F(x,y) := \frac{\int_x^y w_2(v)\int_x^v w_1(u)\,\mathrm{d}u\mathrm{d}v}{\int_x^y w_2(v)\,\mathrm{d}v}. \tag{5.7}$$

Similarly, from (5.6) we obtain

$$\frac{\epsilon^{(2)}}{\epsilon^{(1)}} = \int_{s_1}^{s_5} w_1(u)\,\mathrm{d}u + F(s_5, s_5+\epsilon_5).$$

It is clear that $S_j^{(1)} - S_i^{(1)} > 0$ for $s_i < s_j$. Moreover, since $F(x,y)$ is increasing with respect to y for any fixed x, $x < y$ (see Lemma 5.1), we deduce for $s_i < s_j < s_k$,

$$R_{i,k} - R_{i,j} = F(s_i,s_k) - F(s_i,s_j) > 0,$$

and for sufficiently small but positive ϵ_5, we get

$$R_{2,5} - \frac{\epsilon^{(2)}}{\epsilon^{(1)}} = F(s_2,s_5) - \int_{s_2}^{s_5} w_1(u)\,\mathrm{d}u - F(s_5,s_5+\epsilon_5) \neq 0.$$

Summarizing, there exists $\tilde{t}_5(w_1,w_2) > s_5$ such that for any t_5 with $s_5 < t_5 < \tilde{t}_5(w_1,w_2)$ all the factors in $\det\big(M_{\mathbb{T}_2}(s_1,s_2,s_3,s_4,s_5;s_1,s_2,s_3,s_4,t_5)\big)$ are different from 0. This concludes the proof. □

Lemma 5.1. *Let F be defined as in (5.7) and $x < y$. Then, for any fixed x we have*

$$\lim_{y\to x} F(x,y) = 0,$$

and F is a positive, monotone increasing function of y.

Proof. Using L'Hôpital's rule and taking into account that the weights are positive, we get

$$\lim_{y\to x} F(x,y) = \lim_{y\to x} \frac{w_2(y)\int_x^y w_1(u)\,\mathrm{d}u}{w_2(y)} = 0.$$

Moreover, for $x < y$, it is clear that $F(x,y) > 0$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y}F(x,y) =& \frac{w_2(y)\int_x^y w_1(u)\,\mathrm{d}u\int_x^y w_2(v)\,\mathrm{d}v - w_2(y)\int_x^y w_2(v)\int_x^v w_1(u)\,\mathrm{d}u\mathrm{d}v}{\left(\int_x^y w_2(v)\,\mathrm{d}v\right)^2}\\ =& \frac{w_2(y)\int_x^y w_2(v)\int_v^y w_1(u)\,\mathrm{d}u\mathrm{d}v}{\left(\int_x^y w_2(v)\,\mathrm{d}v\right)^2} > 0.\end{aligned}$$

□

From Proposition 5.2 it follows that the T-mesh in Figure 1 is an example of unstable T-mesh for C^1 Tchebycheffian spline spaces with $\boldsymbol{T} = (\mathbb{T}_2, \mathbb{T}_2)$ and $\mathbb{T}_2$ is any ET-space of dimension 3.

Taking into account Example 4.1, the results of this section show that the bounds for the dimension of Tchebycheffian spline spaces provided in Theorem 4.1 are sharp.

6. Conclusions

By exploiting the properties of ET-spaces on closed bounded intervals we have improved and completed the dimension results of [3]. More precisely, we have stated explicit upper bounds for the dimension of any Tchebycheffian spline space over a planar T-mesh, without any assumption on the underlying ET-spaces. The provided bounds lead to explicit expressions for the dimension of Tchebycheffian spline spaces on several relevant T-mesh configurations (where lower and upper bounds coincide). Besides their intrinsic theoretical interest, this opens the door for a full generalization to the Tchebycheffian setting of the construction of LR-splines by providing a proper tool to analyze their linear independence (see, e.g., [2, 6]).

Furthermore, we have analyzed instability in the dimension of Tchebycheffian spline spaces over T-meshes, and we have shown that there exist T-meshes such that the corresponding C^1 spline spaces have unstable dimension for any underlying ET-space of dimension 3. This shows that the provided dimension bounds are sharp, regardless of the ET-spaces we are dealing with.

Acknowledgements

The authors thank M.L. Mazure and the anonymous referees for their comments and suggestions which greatly improved the content and presentation of the paper. This work was partially supported by the Gruppo Nazionale Calcolo Scientifico – Istituto Nazionale di Alta Matematica (GNCS-INdAM), by the "Mission Sustainability 2017" program of the University of Rome Tor Vergata through the project IDEAS, and by MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006. The Centre International de Rencontres Mathématiques (CIRM) in Luminy is acknowledged for the Research-in-Pairs support.

This is a post-print version of the paper which can be found at https://doi.org/10.1016/j.cam.2018.10.026

References

References

[1] D. Berdinsky, M.-J. Oh, T.-W. Kim, and B. Mourrain. On the problem of instability in the dimension of a spline space over a T-mesh. *Comput. Graph.* **36** (2012), 507–513.

[2] C. Bracco, T. Lyche, C. Manni, F. Roman, and H. Speleers. Generalized spline spaces over T-meshes: Dimension formula and locally refined generalized B-splines. *Appl. Math. Comput.* **272** (2016), 187–198.

[3] C. Bracco, T. Lyche, C. Manni, F. Roman, and H. Speleers. On the dimension of Tchebycheffian spline spaces over planar T-meshes. *Comput. Aided Geom. Design* **45** (2016), 151–173.

[4] C. Bracco and F. Roman. Spaces of generalized splines over T-meshes. *J. Comput. Appl. Math.* **294** (2016), 102–123.

[5] J. Deng, F. Chen, and Y. Feng. Dimensions of spline spaces over T-meshes. *J. Comput. Appl. Math.* **194** (2006), 267–283.

[6] T. Dokken, T. Lyche, and K.F. Pettersen. Polynomial splines over locally refined box-partitions. *Comput. Aided Geom. Design* **30** (2013), 331–356.

[7] D.R. Forsey and R.H. Bartels. Hierarchical B-spline refinement. *Comput. Graph.* **22** (1988), 205–212.

[8] C. Giannelli, B. Jüttler, and H. Speleers. THB-splines: The truncated basis for hierarchical splines. *Comput. Aided Geom. Design* **29** (2012), 485–498.

[9] X. Li and F. Chen. On the instability in the dimension of spline spaces over T-meshes. *Comput. Aided Geom. Design* **28** (2011), 420–426.

[10] X. Li and J. Deng. On the dimension of spline spaces over T-meshes with smoothing cofactor-conformality method. *Comput. Aided Geom. Design* **41** (2016), 76–86.

[11] X. Li and M.A. Scott. Analysis-suitable T-splines: Characterization, refineability, and approximation. *Math. Mod. Meth. Appl. Sci.* **24** (2014), 1141–1164.

[12] T. Lyche. A recurrence relation for Chebyshevian B-splines. *Constr. Approx.* **1** (1985), 155–173.

[13] C. Manni, F. Pelosi, and H. Speleers. Local hierarchical h-refinements in IgA based on generalized B-splines. In: M. Floater et al. (eds.), Mathematical Methods for Curves and Surfaces 2012, *LNCS* **8177** (2014), 341–363.

[14] C. Manni, F. Roman, and H. Speleers. Generalized B-splines in isogeometric analysis. In: G.E. Fasshauer and L.L. Schumaker (eds.), Approximation Theory XV: San Antonio 2016, *Springer Proc. Math. Stat.* **201** (2017), 239–267.

[15] M.L. Mazure. Chebyshev spaces and Bernstein bases. *Constr. Approx.* **22** (2005), 347–363.

[16] M.L. Mazure. Ready-to-blossom bases in Chebyshev spaces. *Studies Comput. Math.* **12** (2006), 109–148.

[17] M.L. Mazure. Extended Chebyshev piecewise spaces characterised via weight functions. *J. Approx. Theory* **145** (2007), 33–54.

[18] M.L. Mazure. Finding all systems of weight functions associated with a given extended Chebyshev space. *J. Approx. Theory* **163** (2011), 363–376.

[19] M.L. Mazure. How to build all Chebyshevian spline spaces good for geometric design? *Numer. Math.* **119** (2011), 517–556.

[20] M.L. Mazure. Constructing totally positive piecewise Chebyshevian B-spline bases *J. Comput. Appl. Math.* **342** (2018), 550–586.

[21] K. Mørken. Total Positivity and Splines. In: M. Gasca and C.A. Micchelli (eds.), Total Positivity and its Applications, *Mathematics and Its Applications* **359**, Springer (1996), 47–84.

[22] B. Mourrain. On the dimension of spline spaces on planar T-meshes. *Math. Comp.* **83** (2014), 847–871.

[23] H. Pottmann. The geometry of Tchebycheffian splines. *Comput. Aided Geom. Design* **10** (1993), 181–210.

[24] L.L. Schumaker. Spline Functions: Basic Theory, Third Edition. Cambridge University Press (2007).

[25] L.L. Schumaker and L. Wang. Approximation power of polynomial splines on T-meshes. *Comput. Aided Geom. Design* **29** (2012), 599–612.

[26] T. Sederberg, J. Zheng, A. Bakenov, and A. Nasri. T-splines and T-NURCCs. *ACM Trans. Graphics* **22** (2003), 477–484.