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BANG–BANG TRAJECTORIES WITH A DOUBLE SWITCHING TIME IN THE MINIMUM TIME PROBLEM*

LAURA POGGIOLINI¹ AND MARCO SPADINI¹

Abstract. In this paper we deal with the strong local optimality of a triplet satisfying Pontryagin Maximum Principle in the minimum time problem between two fixed endpoints. The reference control is assumed to be bang-bang with a double switching time

Our method are based on a topological technique for the inversion of the projected maximised flow.

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1. INTRODUCTION

This paper is part of a project where Hamiltonian methods are applied to the study of sufficient second order conditions in optimal control. We consider the minimum time problem between two submanifolds of a finite dimensional manifold M in the case when the dynamics is affine with respect to the control and the latter takes values in a box of \mathbb{R}^m . Namely, we consider the following optimal control problem:

$$T \rightarrow \min, \tag{1.1a}$$

$$\dot{\xi}(t) = f_0(\xi(t)) + \sum_{s=1}^m u_s(t) f_s(\xi(t)) \quad \text{a.e. } t \in [0, T], \tag{1.1b}$$

$$\xi(0) \in N_0, \quad \xi(T) \in N_f, \tag{1.1c}$$

$$|u_s(t)| \leq 1 \quad s = 1, 2, \dots, m \quad \text{a.e. } t \in [0, T]. \tag{1.1d}$$

For such problem, we say that (T, ξ, u) is an *admissible triplet* if $T > 0$ and the couple $(\xi, u) \in W^{1,\infty}([0, T], M) \times L^\infty([0, T], \mathbb{R}^m)$ satisfies (1.1b), (1.1c) and (1.1d).

We assume we are given a reference triplet $(\widehat{T}, \widehat{\xi}, \widehat{u})$ which satisfies the necessary conditions for optimality, namely the Pontryagin Maximum Principle (PMP) with an associated covector $\widehat{\lambda}$, and where the reference control \widehat{u} is a regular bang-bang control with a double switching time $\widehat{\tau}$ and a finite number of simple switching times.

We are interested in strong local optimality. To be more precise, we are interested in proving state-local optimality of the reference triplet. In fact, as we are dealing with a free terminal-time problem, two different kinds of strong local optimality, defined according to different kinds of localisation, may be of interest.

Definition 1.1 ((time, state)-local optimality). The trajectory $\widehat{\xi}: [0, \widehat{T}] \rightarrow M$ is a (time, state)-local minimiser if there exist $\varepsilon > 0$ and a neighborhood \mathcal{U} of its graph in $\mathbb{R} \times M$ such that $\widehat{\xi}$ is a minimiser among the admissible trajectories whose graph is in \mathcal{U} and whose final time is greater than $\widehat{T} - \varepsilon$.

Definition 1.2 (state-local optimality). The trajectory $\widehat{\xi}$ is a state-local minimiser if there are neighborhoods \mathcal{U} of its range $\widehat{\xi}([0, \widehat{T}])$, \mathcal{U}_0 of $\widehat{\xi}(0)$ and \mathcal{U}_f of $\widehat{\xi}(\widehat{T})$ such that $\widehat{\xi}$ is a minimiser among the admissible trajectories whose range is in \mathcal{U} , whose initial point is in $N_0 \cap \mathcal{U}_0$ and whose final point is in $N_f \cap \mathcal{U}_f$.

Keywords and phrases: Hamiltonian methods, bang-bang controls, sufficient second order conditions

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Sufficient optimality conditions for state-local optimality in the case where only simple switches occur and the initial and final points are fixed were given in [7], while in [3, 5] the authors give sufficient second order conditions for (time, state)-local optimality in Bolza and in Mayer problems. State-local optimality for the Bolza problem with a control-affine running cost is considered in [2] in the case when only simple switches occur.

To keep the notation to the minimum we confine ourselves to the case when the state space is \mathbb{R}^n , the control is two-dimensional and only the double switch occurs. In fact, this case already contains most of the mathematical difficulties of the proof. Namely, the presence of a double switch gives rise to a piecewise C^1 (PC^1) maximised Hamiltonian flow where the number of *smooth pieces* around $\widehat{\lambda}(\widehat{\tau})$ is five, thus requiring a non trivial proof of the local invertibility of the projection of such flow on the state space. This kind of difficulty should be compared with the situation when at most four pieces are present, as in [6]. The more general case where the state space is a manifold and there are simple switches either preceding or following the double one can be treated, at the cost of a considerably heavier notation, with the same technique, see e.g. [5]. We recall that the definition of PC^1 maps is the following:

Definition 1.3 (PC^1 functions). Given two finite dimensional manifolds N_1 and N_2 , we say that a function $\gamma: N_1 \rightarrow N_2$ is a continuous selection of C^1 functions if γ is continuous and there exists a finite number of C^1 functions $\gamma_1, \dots, \gamma_k$ from N_1 in N_2 such that the *active index set* $I := \{i \in \{1, 2, \dots, k\}: \gamma(x) = \gamma_i(x)\}$ is nonempty for each $x \in N_1$. The functions γ_i 's are called *selection functions* of γ . A continuous function γ is called a PC^1 function if at every point $x \in N_1$ there exists a neighborhood V such that the restriction of γ to V is a continuous selection of C^1 functions.

2. THE PROBLEM

We consider the minimum time problem between two given submanifolds N_0 and N_f of the state space \mathbb{R}^n :

$$T \rightarrow \min, \tag{2.1a}$$

$$\dot{\xi}(t) = f_0(\xi(t)) + u_1(t)f_1(\xi(t)) + u_2(t)f_2(\xi(t)), \quad \text{a.e. } t \in [0, T] \tag{2.1b}$$

$$\xi(0) \in N_0, \quad \xi(T) \in N_f, \tag{2.1c}$$

$$|u_i(t)| \leq 1 \quad i = 1, 2, \quad \text{a.e. } t \in [0, T]. \tag{2.1d}$$

The data of the problem, i.e. the drift f_0 and the controlled vector fields, f_1 and f_2 are assumed to be smooth, let us say $C^\infty(\mathbb{R}^n)$.

Assume we are given an admissible reference triplet $(\widehat{T}, \widehat{\xi}, \widehat{u})$ satisfying the necessary optimality conditions (PMP) where the reference control \widehat{u} is

$$\widehat{u}(t) = (\widehat{u}_1(t), \widehat{u}_2(t)) = \begin{cases} (-1, -1) & t \in [0, \widehat{\tau}), \\ (1, 1) & t \in (\widehat{\tau}, \widehat{T}]. \end{cases}$$

(By an appropriate change of f_1 or f_2 with $-f_1$ and $-f_2$ one can always assume that this is the case.)

2.1. Notation

We are going to use some basic notions from symplectic geometry. For any manifold $N \subset \mathbb{R}^n$ and any $x \in N$, the tangent space and the cotangent space to N in x are denoted as $T_x N$ and $T_x^* N$, respectively. We recall that the cotangent bundle $T^*\mathbb{R}^n$ to \mathbb{R}^n can be identified with the Cartesian product $(\mathbb{R}^n)^* \times \mathbb{R}^n = T_x^*\mathbb{R}^n \times T_x\mathbb{R}^n$ for any $x \in \mathbb{R}^n$. The projection from $T^*\mathbb{R}^n$ onto \mathbb{R}^n is denoted as $\pi: \ell \in T^*\mathbb{R}^n \mapsto \pi\ell \in \mathbb{R}^n$. For the sake of clarity in several occasions we shall write $T_x\mathbb{R}^n$ in lieu of \mathbb{R}^n , to emphasize the fact that we are dealing with tangent vectors.

The canonical Liouville one-form \mathbf{s} on $T^*\mathbb{R}^n$ and the associated canonical symplectic two-form $\boldsymbol{\sigma} = d\mathbf{s}$ allow to associate to any, possibly time-dependent, smooth Hamiltonian $F_t: T^*\mathbb{R}^n \rightarrow \mathbb{R}$, the unique Hamiltonian vector field \vec{F}_t such that

$$\boldsymbol{\sigma}(v, \vec{F}_t(\ell)) = \langle dF_t(\ell), v \rangle, \quad \forall v \in T_\ell T^*\mathbb{R}^n.$$

Choosing coordinates $\ell = (p, x)$, we have

$$\vec{F}_t(p, x) = \left(\frac{-\partial F_t}{\partial x}, \frac{\partial F_t}{\partial p} \right) (p, x)$$

To any vector field $f: \mathbb{R}^n \rightarrow T\mathbb{R}^n$ we associate a Hamiltonian function F such that

$$F: \ell \in T^*\mathbb{R}^n \mapsto \langle \ell, f(\pi\ell) \rangle \in \mathbb{R},$$

so that

$$\vec{F}(p, x) = (-p \, df(x), f(x)). \quad (2.2)$$

We denote by \hat{f}_t the piecewise time-dependent vector field associated to the reference control:

$$\hat{f}_t := f_0 + \hat{u}_1(t)f_1 + \hat{u}_2(t)f_2$$

and by h_1, h_2 its restrictions to the time intervals $[0, \hat{\tau})$ and $(\hat{\tau}, \hat{T}]$, respectively:

$$h_1 := \hat{f}_t|_{[0, \hat{\tau})} = f_0 - f_1 - f_2, \quad h_2 := \hat{f}_t|_{(\hat{\tau}, \hat{T}]} = f_0 + f_1 + f_2.$$

For future reference we also define

$$\begin{aligned} k_1 &:= f_0 + f_1 - f_2 = h_1 + 2f_1 = h_2 - 2f_2, \\ k_2 &:= f_0 - f_1 + f_2 = h_1 + 2f_2 = h_2 - 2f_1. \end{aligned}$$

The associated Hamiltonian functions are denoted by the same letter, but capitalized. Namely

$$H_1(\ell) := \langle \ell, h_1(\pi\ell) \rangle, \quad H_2(\ell) := \langle \ell, h_2(\pi\ell) \rangle, \quad K_1(\ell) := \langle \ell, k_1(\pi\ell) \rangle, \quad K_2(\ell) := \langle \ell, k_2(\pi\ell) \rangle.$$

The maximised Hamiltonian of the control system (2.1) is well defined in the whole cotangent bundle $T^*\mathbb{R}^n$ and is denoted by H^{\max} :

$$H^{\max}(\ell) := \max \{ F_0(\ell) + u_1 F_1(\ell) + u_2 F_2(\ell) : (u_1, u_2) \in [-1, 1]^2 \} = F_0(\ell) + |F_1(\ell)| + |F_2(\ell)|.$$

Throughout the paper the symbol $\mathcal{O}(x)$ denotes a neighborhood of x in its ambient space. The flow starting at time 0 of the time-dependent vector field \hat{f}_t is defined in a neighborhood $\mathcal{O}(\hat{x}_0)$ for any $t \in [0, \hat{T}]$ and is denoted by $\hat{S}_t: \mathcal{O}(\hat{x}_0) \rightarrow \mathbb{R}^n$, i.e.

$$\frac{d}{dt} \hat{S}_t(x) = \hat{f}_t \circ \hat{S}_t(x) \quad \text{a.e. } t \in [0, \hat{T}], \quad \hat{S}_0(x) = x.$$

3. ASSUMPTIONS

We assume that the necessary conditions for optimality hold, namely the reference triplet $(\hat{T}, \hat{\xi}, \hat{u})$ satisfies Pontryagin Maximum Principle:

Assumption 3.1 (PMP). *There exist $p_0 \in \{0, 1\}$ and an absolutely continuous curve $\widehat{\lambda}: [0, \widehat{T}] \rightarrow T^*\mathbb{R}^n$ satisfying the following properties*

$$\begin{aligned} (p_0, \widehat{\lambda}(0)) &\neq (0, 0) \\ \pi\widehat{\lambda}(t) &= \widehat{\xi}(t) & \forall t \in [0, \widehat{T}] \\ \dot{\widehat{\lambda}}(t) &= \overrightarrow{F}_t(\widehat{\lambda}(t)) & \text{a.e. } t \in [0, \widehat{T}], \end{aligned} \quad (3.1)$$

$$\widehat{F}_t(\widehat{\lambda}(t)) = H^{\max}(\widehat{\lambda}(t)) = p_0 \quad \text{a.e. } t \in [0, \widehat{T}], \quad (3.2)$$

$$\widehat{\lambda}(0)\Big|_{T_{\widehat{x}_0}N_0} = 0, \quad \widehat{\lambda}(\widehat{T})\Big|_{T_{\widehat{x}_f}N_f} = 0. \quad (3.3)$$

We shall use the following notation:

$$\widehat{\ell}_0 := \widehat{\lambda}(0), \quad \widehat{\ell}_d := \widehat{\lambda}(\widehat{\tau}), \quad \widehat{\ell}_f := \widehat{\lambda}(\widehat{T}) \quad \text{and} \quad \widehat{x}_0 := \widehat{\xi}(0) = \pi\widehat{\ell}_0, \quad \widehat{x}_d := \widehat{\xi}(\widehat{\tau}) = \pi\widehat{\ell}_d, \quad \widehat{x}_f := \widehat{\xi}(\widehat{T}) = \pi\widehat{\ell}_f.$$

Remark 3.1. Notice that **(1)** by (3.1), $\widehat{\lambda}(t) = \widehat{\ell}_0 \widehat{S}_{t*}^{-1} \quad \forall t \in [0, \widehat{T}]$; **(2)** If $\widehat{\lambda}$ is a normal extremal (i.e. if $p_0 = 1$), then the transversality conditions (3.3) together with the maximality condition (3.2) yield $h_1(\widehat{x}_0) \notin T_{\widehat{x}_0}N_0$ and $h_2(\widehat{x}_f) \notin T_{\widehat{x}_f}N_f$.

Maximality condition (3.2) implies, for any $i = 1, 2$ and for almost every $t \in [0, \widehat{T}]$,

$$\widehat{u}_i(t)F_i(\widehat{\lambda}(t)) = \widehat{u}_i(t)\langle \widehat{\lambda}(t), f_i(\widehat{\xi}(t)) \rangle \geq 0.$$

We assume that the bang arcs of $\widehat{\lambda}$ are regular, i.e. we assume that in each point $\widehat{\lambda}(t)$, $t \neq \widehat{\tau}$, the maximum of the Hamiltonian is achieved only by $u = \widehat{u}(t)$ i.e.

$$F_0(\widehat{\lambda}(t)) + u_1F_1(\widehat{\lambda}(t)) + u_2F_2(\widehat{\lambda}(t)) < H^{\max}(\widehat{\lambda}(t)) \quad \forall (u_1, u_2) \in [-1, 1]^2 \setminus \{(\widehat{u}_1(t), \widehat{u}_2(t))\}.$$

In terms of the controlled Hamiltonians F_1 and F_2 this can be stated as follows:

Assumption 3.2 (Regularity along the bang arcs). *Let $i = 1, 2$. If $t \neq \widehat{\tau}$, then*

$$\widehat{u}_i(t)F_i(\widehat{\lambda}(t)) = \widehat{u}_i(t)\langle \widehat{\lambda}(t), f_i(\widehat{\xi}(t)) \rangle > 0.$$

From the necessary maximality condition (3.2) we get

$$\begin{aligned} \frac{d}{dt} 2F_i \circ \widehat{\lambda}(t) \Big|_{t=\widehat{\tau}-} &= \frac{d}{dt} (K_i - H_1) \circ \widehat{\lambda}(t) \Big|_{t=\widehat{\tau}-} \geq 0, \\ \frac{d}{dt} 2F_i \circ \widehat{\lambda}(t) \Big|_{t=\widehat{\tau}+} &= \frac{d}{dt} (H_2 - K_i) \circ \widehat{\lambda}(t) \Big|_{t=\widehat{\tau}+} \geq 0, \end{aligned} \quad i = 1, 2.$$

We assume that the strict inequalities hold:

Assumption 3.3 (Regularity at the double switching time).

$$\frac{d}{dt} (K_\nu - H_1) \circ \widehat{\lambda}(t) \Big|_{t=\widehat{\tau}-} > 0, \quad \frac{d}{dt} (H_2 - K_\nu) \circ \widehat{\lambda}(t) \Big|_{t=\widehat{\tau}+} > 0, \quad \nu = 1, 2. \quad (3.4)$$

Assumption 3.3 means that at time $\widehat{\tau}$ the reference adjoint $\widehat{\lambda}(t)$ arrives simultaneously at the hypersurfaces $F_1 = 0$ and $F_2 = 0$ with non-tangential velocity \overrightarrow{H}_1 and leaves with velocity \overrightarrow{H}_2 which is again non-tangential to both

the hypersurfaces. We shall call Assumption 3.3 the **STRONG BANG-BANG LEGENDRE CONDITION FOR DOUBLE SWITCHING TIMES**. Equivalently, this assumption can be expressed in terms of the Lie brackets of vector fields or in terms of the canonical symplectic structure $\sigma(\cdot, \cdot)$ on $T^*\mathbb{R}^n$. Recall that, given two smooth vector fields f and g , then the Lie bracket $[f, g]$ is given by the vector field $(Dg)f - (Df)g$.

Proposition 3.1. *Assumption 3.3 is equivalent to*

$$\begin{aligned} \langle \widehat{\ell}_d, [h_1, k_\nu](\widehat{x}_d) \rangle &= \sigma\left(\overrightarrow{H}_1, \overrightarrow{K}_\nu\right)(\widehat{\ell}_d) > 0, \\ \langle \widehat{\ell}_d, [k_\nu, h_2](\widehat{x}_d) \rangle &= \sigma\left(\overrightarrow{K}_\nu, \overrightarrow{H}_2\right)(\widehat{\ell}_d) > 0 \end{aligned} \quad \nu = 1, 2.$$

In what follows we shall also need to reformulate Assumption 3.3 in terms of the pull-backs along the reference flow \widehat{S}_t of the vector fields h_ν and k_ν . Define

$$g_\nu(x) := \widehat{S}_{\widehat{\tau}}^{-1} h_\nu \circ \widehat{S}_{\widehat{\tau}}(x), \quad j_\nu(x) := \widehat{S}_{\widehat{\tau}}^{-1} k_\nu \circ \widehat{S}_{\widehat{\tau}}(x), \quad \nu = 1, 2 \quad (3.5)$$

and let G_ν, J_ν be the associated Hamiltonians. Then a straightforward computation yields

Proposition 3.2. *Assumption 3.3 is equivalent to*

$$\begin{aligned} \langle \widehat{\ell}_0, [g_1, j_\nu](\widehat{x}_0) \rangle &= \sigma\left(\overrightarrow{G}_1, \overrightarrow{J}_\nu\right)(\widehat{\ell}_0) > 0, \\ \langle \widehat{\ell}_0, [j_\nu, g_2](\widehat{x}_0) \rangle &= \sigma\left(\overrightarrow{J}_\nu, \overrightarrow{G}_2\right)(\widehat{\ell}_0) > 0 \end{aligned} \quad \nu = 1, 2. \quad (3.6)$$

4. THE FINITE DIMENSIONAL SUB-PROBLEM

We now introduce a finite-dimensional sub-problem of (2.1) by keeping the same end-point constraints and restricting the set of admissible controls. Namely, we allow for independent variations of the switching times of each of the two reference control components \widehat{u}_1 and \widehat{u}_2 .

We then extend this sub-problem by allowing for variations of the initial points of trajectories on a neighborhood of \widehat{x}_0 in \mathbb{R}^n . We penalise the latter variations with a smooth cost α that vanishes on N_0 .

In order to write the second order approximation of this finite-dimensional problem, we first write (2.1) as a Mayer problem on the state space $\mathbb{R} \times \mathbb{R}^n$.

4.1. A second order approximation

We allow for perturbations of the final time, of the initial point of trajectories on N_0 , of the final point on N_f and of the switching time of either component of the reference control: Let $\tau_1 := \widehat{\tau} + \varepsilon_1$ and $\tau_2 := \widehat{\tau} + \varepsilon_2$ be the perturbed switching times of the first and of the second component of \widehat{u} , respectively, and let $\tau_3 := \widehat{T} + \varepsilon_3$ be the perturbation of the final time \widehat{T} . Two cases may occur depending on the sign of $\varepsilon_2 - \varepsilon_1$:

- If $\varepsilon_1 < \varepsilon_2$, the dynamics is given by

$$\dot{\xi}(t) = \begin{cases} h_1(\xi(t)) & t \in (0, \tau_1), \\ k_1(\xi(t)) & t \in (\tau_1, \tau_2), \\ h_2(\xi(t)) & t \in (\tau_2, \tau_3); \end{cases}$$

- If $\varepsilon_2 < \varepsilon_1$, the dynamics is given by

$$\dot{\xi}(t) = \begin{cases} h_1(\xi(t)) & t \in (0, \tau_2), \\ k_2(\xi(t)) & t \in (\tau_2, \tau_1), \\ h_2(\xi(t)) & t \in (\tau_1, \tau_3). \end{cases}$$

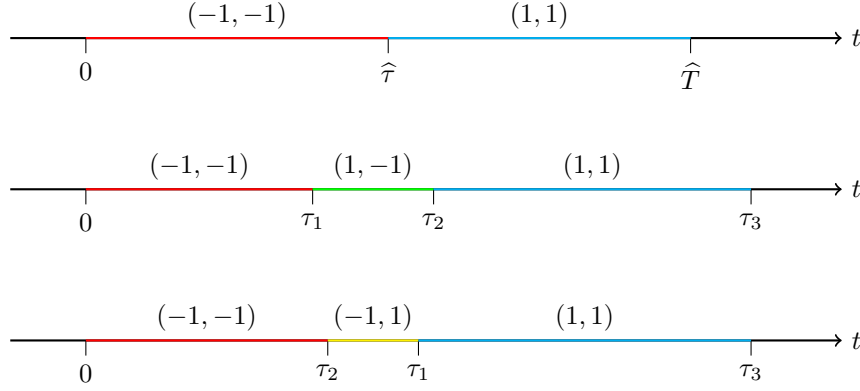


FIGURE 1. Variations of the reference control

We can write the given minimum time problem as a Mayer one, with state space $\mathbb{R} \times \mathbb{R}^n$. Let $\xi_0: t \in [0, T] \mapsto \xi_0(t) \in \mathbb{R}$ be such that

$$\dot{\xi}_0(t) = 1, \quad \xi_0(0) = 0.$$

Also let the boldfaced vector fields \mathbf{f}_i be the extended vector fields associated to f_i , $i = 0, 1, 2$, as follows:

$$\mathbf{f}_0 = \begin{pmatrix} 1 \\ f_0 \end{pmatrix}, \quad \mathbf{f}_i = \begin{pmatrix} 0 \\ f_i \end{pmatrix}, \quad i = 1, 2.$$

An analogous definition holds for the boldfaced vector fields $\mathbf{h}_\nu, \mathbf{k}_\nu$, $\nu = 1, 2$. Finally, define $\boldsymbol{\xi}(t) := (\xi_0(t), \xi(t)) \in \mathbb{R} \times \mathbb{R}^n$. Then the minimum time problem (2.1) is equivalent to

$$\begin{aligned} & \xi_0(T) \rightarrow \min \\ & \dot{\boldsymbol{\xi}} = \mathbf{f}_0(\boldsymbol{\xi}(t)) + u_1(t)\mathbf{f}_1(\boldsymbol{\xi}(t)) + u_2(t)\mathbf{f}_2(\boldsymbol{\xi}(t)) \quad \text{a.e. } t \in [0, T], \\ & \boldsymbol{\xi}(0) \in \{0\} \times N_0, \quad \boldsymbol{\xi}(T) \in \mathbb{R} \times N_f, \\ & |u_i(t)| \leq 1 \quad i = 1, 2, \text{ a.e. } t \in [0, T]. \end{aligned}$$

We now restrict the control variations by allowing only perturbations of the switching and final times, thus obtaining the finite-dimensional minimisation problem

$$\xi_0(\widehat{T} + \delta_3) \rightarrow \min \tag{4.1a}$$

$$\dot{\boldsymbol{\xi}} = \begin{cases} \mathbf{h}_1(\boldsymbol{\xi}(t)) & t \in (0, \widehat{\tau} + \delta_1), \\ \mathbf{k}_\nu(\boldsymbol{\xi}(t)) & t \in (\widehat{\tau} + \delta_1, \widehat{\tau} + \delta_2), \\ \mathbf{h}_2(\boldsymbol{\xi}(t)) & t \in (\widehat{\tau} + \delta_2, \widehat{T} + \delta_3), \end{cases} \tag{4.1b}$$

$$\boldsymbol{\xi}(0) \in \{0\} \times N_0, \quad \boldsymbol{\xi}(\widehat{T} + \delta_3) \in \mathbb{R} \times N_f, \tag{4.1c}$$

$$\delta_1 := \min\{\varepsilon_1, \varepsilon_2\}, \quad \delta_2 := \max\{\varepsilon_1, \varepsilon_2\}, \quad \delta_3 := \varepsilon_3, \tag{4.1d}$$

$$\nu = \begin{cases} 1 & \text{if } \varepsilon_1 \leq \varepsilon_2, \\ 2 & \text{if } \varepsilon_1 > \varepsilon_2. \end{cases} \tag{4.1e}$$

Let $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth nonnegative function vanishing on N_0 . We remove the constraint on the initial point $\xi(0)$ introducing the penalty cost α on such point. We thus obtain the following Mayer problem

$$\alpha(\xi(0)) + \xi_0(\widehat{T} + \delta_3) \rightarrow \min \quad (4.2a)$$

$$\dot{\xi} = \begin{cases} \mathbf{h}_1(\xi(t)) & t \in (0, \widehat{\tau} + \delta_1), \\ \mathbf{k}_\nu(\xi(t)) & t \in (\widehat{\tau} + \delta_1, \widehat{\tau} + \delta_2), \\ \mathbf{h}_2(\xi(t)) & t \in (\widehat{\tau} + \delta_2, \widehat{T} + \delta_3), \end{cases} \quad (4.2b)$$

$$\xi(0) \in \{0\} \times \mathbb{R}^n, \quad \xi(\widehat{T} + \delta_3) \in \mathbb{R} \times N_f \quad (4.2c)$$

$$\delta_1 := \min\{\varepsilon_1, \varepsilon_2\}, \quad \delta_2 := \max\{\varepsilon_1, \varepsilon_2\}, \quad \delta_3 := \varepsilon_3, \quad (4.2d)$$

$$\nu = \begin{cases} 1 & \text{if } \varepsilon_1 \leq \varepsilon_2, \\ 2 & \text{if } \varepsilon_1 > \varepsilon_2. \end{cases} \quad (4.2e)$$

Let $g_\nu, j_\nu, \nu = 1, 2$ be the pullbacks along the reference flow of the vector fields h_ν and k_ν , as defined in equation (3.5). Let \widehat{N}_f be the pullback of N_f along the reference flow:

$$\widehat{N}_f := \widehat{S}_{\widehat{T}}^{-1}(N_f)$$

and let $T_{\widehat{x}_0}\widehat{N}_f = \widehat{S}_{\widehat{T}*}^{-1}(T_{\widehat{x}_f}N_f)$ be its tangent space at \widehat{x}_0 .

By the transversality condition (3.3) at the reference final time \widehat{T} , there exists a smooth function $\beta: \mathbb{R}^n \rightarrow \mathbb{R}$ that vanishes on N_f and such that $d\beta(\widehat{x}_f) = -\widehat{\ell}_f$. Also let $\widehat{\beta}$ be the pull-back of β along the reference flow, $\widehat{\beta} := \beta \circ \widehat{S}_{\widehat{T}}$ so that, by Remark 3.1 (1),

$$\widehat{\beta}: \mathcal{O}(\widehat{x}_0) \rightarrow \mathbb{R}, \quad \widehat{\beta}|_{\mathcal{O}(\widehat{x}_0) \cap \widehat{N}_f} \equiv 0, \quad d\widehat{\beta}(\widehat{x}_0) = -\widehat{\ell}_0.$$

Let us set

$$a_1 := \delta_1, \quad b := \delta_2 - \delta_1 = |\varepsilon_2 - \varepsilon_1|, \quad a_2 := \delta_3 - \delta_2;$$

then the second order approximations of problems (4.2), for $\nu = 1, 2$, are defined on the closed half-spaces

$$V_\nu^+ := \left\{ (\delta x, a_1, b, a_2) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} : \delta x + a_1 g_1(\widehat{x}_0) + b j_\nu(\widehat{x}_0) + a_2 g_2(\widehat{x}_0) \in T_{\widehat{x}_0}\widehat{N}_f \right\}$$

and are given by

$$\begin{aligned} J_\nu''[\delta x, a_1, b, a_2] = & D^2(\alpha + \widehat{\beta})(\widehat{x}_0)[\delta x]^2 + 2 \delta x \cdot (a_1 g_1 + b j_\nu + a_2 g_2) \cdot \widehat{\beta}(\widehat{x}_0) + (a_1 g_1 + b j_\nu + a_2 g_2)^2 \cdot \widehat{\beta}(\widehat{x}_0) \\ & + a_1 b [g_1, j_\nu] \cdot \widehat{\beta}(\widehat{x}_0) + a_1 a_2 [g_1, g_2] \cdot \widehat{\beta}(\widehat{x}_0) + b a_2 [j_\nu, g_2] \cdot \widehat{\beta}(\widehat{x}_0), \end{aligned} \quad (4.3)$$

see [5] for the construction. The restrictions of J_ν'' to the sets

$$V_{0,\nu}^+ := \left\{ (\delta x, a_1, b, a_2) \in T_{\widehat{x}_0}N_0 \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} : \delta x + a_1 g_1(\widehat{x}_0) + b j_\nu(\widehat{x}_0) + a_2 g_2(\widehat{x}_0) \in T_{\widehat{x}_0}\widehat{N}_f \right\}, \quad \nu = 1, 2,$$

are indeed the second order approximation of (4.1).

We are now in a position to state our last assumption.

Assumption 4.1. For each $\nu = 1, 2$, J_ν'' is coercive on $V_{0,\nu}^+$.

Since both J_1'' and J_2'' are quadratic forms, we may as well remove the constraint $b \geq 0$ and let them be defined on the spaces

$$V_\nu := \left\{ (\delta x, a_1, b, a_2) \in \mathbb{R}^n \times \mathbb{R}^3 : \delta x + a_1 g_1(\hat{x}_0) + b j_\nu(\hat{x}_0) + a_2 g_2(\hat{x}_0) \in T_{\hat{x}_0} \hat{N}_f \right\}, \quad \nu = 1, 2. \quad (4.4)$$

Also let

$$V_{0,\nu} := \left\{ (\delta x, a_1, b, a_2) \in T_{\hat{x}_0} N_0 \times \mathbb{R}^3 : \delta x + a_1 g_1(\hat{x}_0) + b j_\nu(\hat{x}_0) + a_2 g_2(\hat{x}_0) \in T_{\hat{x}_0} \hat{N}_f \right\}, \quad \nu = 1, 2.$$

By [1] we obtain the following:

Theorem 4.1. *If both the second order approximations J_1'' and J_2'' are coercive on $V_{0,1}$ and $V_{0,2}$ respectively, then there exists a smooth function $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\alpha|_{N_0} \equiv 0$, $d\alpha(\hat{x}_0) = \hat{\ell}_0$ and both J_1'' and J_2'' are coercive quadratic forms on V_1 and V_2 , respectively.*

The main result of this paper is the following

Theorem 4.2. *Assume $(\hat{T}, \hat{\xi}, \hat{u})$ is an admissible triplet for the minimum time problem (2.1). Assume the triplet is bang-bang with only one switching time which is a double switching time. Assume the triplet satisfies PMP (Assumption 3.1), the regularity assumption along the bang arcs (Assumption 3.2), the regularity assumption at the double switching time (Assumption 3.3) and the coercivity assumption (Assumption 4.1). Moreover assume the trajectory $\hat{\xi}$ is injective. Then, $\hat{\xi}$ is a strict state-locally optimal trajectory. In particular, if $p_0 = 0$, then $\hat{\xi}$ is isolated among admissible trajectories.*

5. HAMILTONIAN METHODS

In this section we describe the procedure we are going to follow in order to prove Theorem 4.2, namely the Hamiltonian approach to state-local optimality.

Let H^{\max} be the maximised Hamiltonian of the control system. Also assume that there exist $\varepsilon > 0$ and a neighborhood $\mathcal{O}(\hat{\ell}_0)$ of $\hat{\ell}_0$ such that the flow \mathcal{H}^{\max} of the associated Hamiltonian vector field \vec{H}^{\max} is well defined and PC^1 in $(-\varepsilon, \hat{T} + \varepsilon) \times \mathcal{O}(\hat{\ell}_0)$; denote it as

$$\mathcal{H}^{\max}: (t, \ell) \in (-\varepsilon, \hat{T} + \varepsilon) \times \mathcal{O}(\hat{\ell}_0) \mapsto \mathcal{H}_t^{\max}(\ell) \in T^*\mathbb{R}^n.$$

Let $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}$ be the smooth function of Theorem 4.1. Let us assume there exists a neighborhood $\mathcal{O}(\hat{x}_0)$ of \hat{x}_0 in \mathbb{R}^n such that

$$\Lambda_0 := \left\{ \ell \in T^*\mathbb{R}^n : H^{\max}(\ell) = p_0, \ell = d\alpha(x), x \in \mathcal{O}(\hat{x}_0) \right\}$$

is a $(n-1)$ -dimensional manifold in $T^*\mathbb{R}^n$ which satisfies the following properties:

- The one-form $\omega := (\mathcal{H}^{\max})^*(p dq)$ is exact on $(-\varepsilon, \hat{T} + \varepsilon) \times \Lambda_0$;
- The flow $\pi \mathcal{H}^{\max}: (t, \ell) \in (-\varepsilon, \hat{T} + \varepsilon) \times \Lambda_0 \mapsto \pi \mathcal{H}_t^{\max}(\ell) \in \mathbb{R}^n$ is one-to-one onto a neighborhood \mathcal{V} of the range of $\hat{\xi}$.

Notice that $\hat{\xi}(t) = \pi \mathcal{H}_t^{\max}(\hat{\ell}_0)$ for any $t \in [0, \hat{T}]$ so that a necessary condition for the invertibility of the map $\pi \mathcal{H}^{\max}: [0, T] \times \Lambda_0 \rightarrow \mathcal{V}$ is the injectivity of $\hat{\xi}$ as required in Theorem 4.2. See also Theorem 3 in [8].

Define

$$\psi := (\pi \mathcal{H}^{\max})^{-1} = (\psi^{\mathbb{R}}, \psi^{\Lambda_0}) : \mathcal{V} \rightarrow (-\varepsilon, \hat{T} + \varepsilon) \times \Lambda_0 \quad (5.1)$$

and let (T, ξ, u) be an admissible triplet such that the range of ξ is in \mathcal{V} . Assume, by contradiction, that $T < \hat{T}$. We can obtain a closed path in \mathcal{V} by concatenating in sequence the curve ξ , a path γ_f from $\xi(T)$ to \hat{x}_f , the curve

$\widehat{\xi}$ ran backward in time and a path γ_0 from \widehat{x}_0 to $\xi(0)$. From the exactness of the one-form $\omega := (\mathcal{H}^{\max})^*(p dq)$ we get

$$0 = \oint \omega = \int_{\psi(\widehat{\xi})} \omega + \int_{\psi(\gamma_f)} \omega - \int_{\psi(\widehat{\xi})} \omega + \int_{\psi(\gamma_0)} \omega = I_1 + I_2 - I_3 + I_4. \quad (5.2)$$

Computing each of these integrals we get

$$I_1: \int_{\psi(\widehat{\xi})} \omega = \int_0^T \langle \mathcal{H}^{\max}(\psi(\xi(t))), \dot{\xi}(t) \rangle dt \leq \int_0^T H^{\max}(\mathcal{H}^{\max}(\psi(\xi(t)))) dt = p_0 T;$$

I_2 : Parametrise $\psi \circ \gamma_f$ as

$$\psi \circ \gamma_f: s \in [0, 1] \mapsto (t(s), d\alpha(q(s)), q(s)) \in (-\varepsilon, \widehat{T} + \varepsilon) \times \Lambda_0$$

where $(t(0), d\alpha(q(0)), q(0)) = (t_f, d\alpha(q_f), q_f) = \psi(\xi(T))$ and $(t(1), d\alpha(q(1)), q(1)) = (\widehat{T}, d\alpha(\widehat{x}_0), \widehat{x}_0) = \psi(\widehat{\xi}(\widehat{T}))$. By the regularity Assumption 3.2 we can assume that $H^{\max} \equiv H_2$ along $\mathcal{H}^{\max}(\psi(\gamma_f))$, hence

$$\begin{aligned} \int_{\psi(\gamma_f)} \omega &= \int_0^1 \langle \mathcal{H}_{t(s)}^{\max}(d\alpha(q(s)), q(s)), \dot{t}(s) h_2(\pi \mathcal{H}_{t(s)}^{\max}(d\alpha(q(s)), q(s))) + \pi_* \mathcal{H}_{t(s)}^{\max} \dot{q}(s) \rangle ds \\ &= \int_0^1 \dot{t}(s) H_2(\mathcal{H}_{t(s)}^{\max}(d\alpha(q(s)), q(s))) + \langle \mathcal{H}_{t(s)}^{\max}(d\alpha(q(s)), q(s)), \pi_* \mathcal{H}_{t(s)}^{\max} \dot{q}(s) \rangle ds \\ &= \int_0^1 p_0 \dot{t}(s) ds + \int_0^1 \langle d\alpha(q(s)), \dot{q}(s) \rangle ds = p_0(\widehat{T} - t_f) + \alpha(\widehat{x}_0) - \alpha(q_f); \end{aligned}$$

$$I_3: \int_{\psi(\widehat{\xi})} \omega = \int_0^{\widehat{T}} \langle \widehat{\lambda}(t), \dot{\widehat{\xi}}(t) \rangle dt = p_0 \widehat{T};$$

I_4 : Parametrise $\psi \circ \gamma_0$ as

$$\psi \circ \gamma_0: s \in [0, 1] \mapsto (t(s), d\alpha(q(s)), q(s)) \in (-\varepsilon, \widehat{T} + \varepsilon) \times \Lambda_0$$

where $(t(0), d\alpha(q(0)), q(0)) = (0, d\alpha(\widehat{x}_0), \widehat{x}_0) = \psi(\widehat{x}_0)$ and $(t(1), d\alpha(q(1)), q(1)) = (t_0, d\alpha(q_0), q_0) = \psi(\xi(0))$. By the regularity Assumption 3.2 we can assume that $H^{\max} \equiv H_1$ along $\mathcal{H}^{\max}(\psi(\gamma_0))$, hence we get

$$\begin{aligned} \int_{\psi(\gamma_0)} \omega &= \int_0^1 \langle \mathcal{H}_{t(s)}^{\max}(d\alpha(q(s)), q(s)), \dot{t}(s) h_1(\pi \mathcal{H}_{t(s)}^{\max}(d\alpha(q(s)), q(s))) + \pi_* \mathcal{H}_{t(s)}^{\max} \dot{q}(s) \rangle ds \\ &= \int_0^1 \dot{t}(s) H_1(\mathcal{H}_{t(s)}^{\max}(d\alpha(q(s)), q(s))) + \langle \mathcal{H}_{t(s)}^{\max}(d\alpha(q(s)), q(s)), \pi_* \mathcal{H}_{t(s)}^{\max} \dot{q}(s) \rangle ds \\ &= \int_0^1 p_0 \dot{t}(s) ds + \int_0^1 \langle d\alpha(q(s)), \dot{q}(s) \rangle ds = p_0 t_0 + \alpha(q_0) - \alpha(\widehat{x}_0). \end{aligned} \quad (5.3)$$

Since $\xi(0) \in N_0$ and $\pi \mathcal{H}_{t_0}^{\max}(d\alpha(q_0), q_0) = \exp t_0 h_1(q_0)$, with $(d\alpha(q_0), q_0) \in \Lambda_0$, we have

$$\begin{aligned} 0 &= \alpha(\xi(0)) - \alpha(\widehat{x}_0) = \alpha(\xi(0)) - \alpha(q_0) - p_0 t_0 + \alpha(q_0) - \alpha(\widehat{x}_0) + p_0 t_0 \\ &= (\alpha(\pi \mathcal{H}_{t_0}^{\max}(d\alpha(q_0), q_0)) - \alpha(q_0) - p_0 t_0) + (\alpha(q_0) - \alpha(\widehat{x}_0) + p_0 t_0) \\ &= t_0 (h_1 \cdot \alpha(q_0) - p_0) + \frac{t_0^2}{2} h_1 \cdot h_1 \cdot \alpha(\bar{q}) + (\alpha(q_0) - \alpha(\widehat{x}_0) + p_0 t_0) \\ &= \frac{t_0^2}{2} h_1 \cdot h_1 \cdot \alpha(\bar{q}) + (\alpha(q_0) - \alpha(\widehat{x}_0) + p_0 t_0) \end{aligned}$$

where $\bar{q} = \exp \bar{s} h_1(q_0)$ for some s between 0 and t_0 . Hence, substituting in (5.3) we get

$$\int_{\psi(\gamma_0)} \omega = -\frac{t_0^2}{2} h_1 \cdot h_1 \cdot \alpha(\bar{q}).$$

Thus, substituting in (5.2) we get

$$0 = \oint \omega \leq p_0 T + p_0(\widehat{T} - t_f) + \alpha(\widehat{x}_0) - \alpha(q_f) - p_0 \widehat{T} - \frac{t_0^2}{2} h_1 \cdot h_1 \cdot \alpha(\bar{q})$$

so that

$$\begin{aligned} p_0(T - \widehat{T}) &\geq p_0(t_f - \widehat{T}) + \alpha(q_f) - \alpha(\widehat{x}_0) + \frac{t_0^2}{2} h_1 \cdot h_1 \cdot \alpha(\bar{q}) \\ &= C(\xi(T)) - C(\widehat{x}_f) + \frac{t_0^2}{2} h_1 \cdot h_1 \cdot \alpha(\bar{q}) \end{aligned} \tag{5.4}$$

where $C(x) := p_0(\psi^{\mathbb{R}}(x)) + \alpha(\pi\psi^{\Lambda_0}(x))$.

Taking advantage of the coercivity of the second variation of the problem, in the following sections we shall show that the function α given by Theorem 4.1 is such that the manifold Λ_0 satisfies the properties required for the construction given above. Also we shall show that $h_1 \cdot h_1 \cdot \alpha(\bar{q}) > 0$ and that $C|_{N_f}$ has a strict local minimum in \widehat{x}_f . If $p_0 = 1$, this yields the state-local optimality of $\widehat{\xi}$. If either $T = \widehat{T}$ or $p_0 = 0$, then equalities must hold throughout (5.4). We will show how this fact implies $\xi \equiv \widehat{\xi}$, i.e. we shall prove that the minimum is strict and in particular, if $p_0 = 0$, then $\widehat{\xi}$ is isolated among admissible trajectories.

6. THE MAXIMISED FLOW

We are now going to prove the properties of the maximised Hamiltonian H^{\max} and of the flow of the associated Hamiltonian vector field $\overrightarrow{H}^{\max}$. Such flow will turn out to be Lipschitz continuous and PC^1 . In our construction we shall use only the regularity assumptions 3.2-3.3 and not the coercivity of the second order approximations.

In order to define the maximized Hamiltonian $H^{\max}(\ell)$ in a neighborhood of the range of $\widehat{\lambda}$ we decouple the double switching time. In this we depart from [7] in that we introduce the new vector fields k_1, k_2 in the sequence of values assumed by the reference vector field. We proceed in four steps:

- For $\nu = 1, 2$ let $\tau_\nu(\ell)$ be the unique solution to

$$2F_\nu \circ \exp \tau_\nu(\ell) \overrightarrow{H}_1(\ell) = (K_\nu - H_1) \circ \exp \tau_\nu(\ell) \overrightarrow{H}_1(\ell) = 0 \tag{6.1}$$

defined by the implicit function theorem (see below) in a neighborhood of $(\widehat{\tau}, \widehat{\ell}_0)$;

- Choose

$$\theta_1(\ell) := \min \{ \tau_1(\ell), \tau_2(\ell) \};$$

- For $\nu = 1, 2$, let $\tau_\nu^2(\ell)$ be the unique solution to

$$\begin{aligned} 2F_{3-\nu} \circ \exp(\tau_\nu^2(\ell) - \tau_\nu(\ell)) \overrightarrow{K}_\nu \circ \exp \tau_\nu(\ell) \overrightarrow{H}_1(\ell) = \\ = (H_2 - K_\nu) \circ \exp(\tau_\nu^2(\ell) - \tau_\nu(\ell)) \overrightarrow{K}_\nu \circ \exp \tau_\nu(\ell) \overrightarrow{H}_1(\ell) = 0 \end{aligned} \tag{6.2}$$

defined by the implicit function theorem (see below) in a neighborhood of $(\widehat{\tau}, \widehat{\ell}_0)$;

- Choose

$$\theta_2(\ell) = \begin{cases} \tau_1^2(\ell) & \text{if } \tau_1(\ell) \leq \tau_2(\ell), \\ \tau_2^2(\ell) & \text{if } \tau_2(\ell) < \tau_1(\ell). \end{cases}$$

Notice that if $\tau_1(\ell) = \tau_2(\ell)$, then $\tau_1^2(\ell) = \tau_2^2(\ell) = \tau_1(\ell) = \tau_2(\ell)$ so that $\theta_2(\cdot)$ is continuous. In Proposition 6.1 we will show that in general $\theta_1(\ell) \leq \theta_2(\ell)$. To be more precise, the functions $\theta_1(\cdot)$, $\theta_2(\cdot)$ are Lipschitz continuous on their domain and are actually C^1 on their domain with the only possible exception of the set $\{\ell \in T^*\mathbb{R}^n : \tau_1(\ell) = \tau_2(\ell)\}$.

To justify the previous procedure we have to show that we can actually apply the implicit function theorem to define the switching times τ_ν , $\tau_\nu^2(\ell)$, $\nu = 1, 2$. Let

$$\Phi_\nu(t, \ell) := (K_\nu - H_1) \circ \exp t\vec{H}_1(\ell) \quad \nu = 1, 2.$$

Then

$$\frac{\partial \Phi_\nu}{\partial t} \Big|_{(\hat{\tau}, \hat{\ell}_0)} = \sigma \left(\vec{H}_1, \vec{K}_\nu \right) (\hat{\ell}_d), \quad \nu = 1, 2,$$

which are positive by Assumption 3.3, so that $\tau_1(\cdot)$ and $\tau_2(\cdot)$ are both well defined by means of the implicit function theorem. Now let

$$\Phi_\nu^2(t, \ell) := (H_2 - K_\nu) \circ \exp(t - \tau_\nu(\ell))\vec{K}_\nu \circ \exp \tau_\nu(\ell)\vec{H}_1(\ell), \quad \nu = 1, 2,$$

then

$$\frac{\partial \Phi_\nu^2}{\partial t} \Big|_{(\hat{\tau}, \hat{\ell}_0)} = \sigma \left(\vec{K}_\nu, \vec{H}_2 \right) (\hat{\ell}_d), \quad \nu = 1, 2,$$

which are positive again by Assumption 3.3, and the same argument applies.

Proposition 6.1. *There exists a neighborhood $\mathcal{O}(\hat{\ell}_0)$ of $\hat{\ell}_0$ in $T^*\mathbb{R}^n$ such that $\theta_1(\ell) \leq \theta_2(\ell)$ for any $\ell \in \mathcal{O}(\hat{\ell}_0)$.*

Proof. If ℓ is such that $\tau_1(\ell) = \tau_2(\ell)$, then $\theta_1(\ell) = \theta_2(\ell)$. Assume ℓ is such that $\theta_1(\ell) = \tau_1(\ell) < \tau_2(\ell)$. Since $\Phi_2(\tau_2(\ell), \ell) = 0$ one has

$$\Phi_2(t, \ell) = \frac{\partial \Phi_2}{\partial t}(\tau_2(\ell), \ell)(t - \tau_2(\ell)) + o(t - \tau_2(\ell)) = (t - \tau_2(\ell)) \left(\sigma \left(\vec{H}_1, \vec{K}_2 \right) \Big|_{\exp \tau_2(\ell)\vec{H}_1(\ell)} + o(1) \right).$$

In particular, choosing $t = \theta_1(\ell) = \tau_1(\ell)$, by Assumption 3.3 and by continuity, when ℓ is sufficiently close to $\hat{\ell}_0$, we get

$$\Phi_2(\theta_1(\ell), \ell) = (K_2 - H_1) \circ \exp \theta_1(\ell)\vec{H}_1(\ell) < 0. \quad (6.3)$$

Since $K_2 - H_1 = 2F_2 = H_2 - K_1$, inequality (6.3) can be written as

$$0 > (H_2 - K_1) \circ \exp 0\vec{K}_1 \circ \exp \theta_1(\ell)\vec{H}_1(\ell),$$

i.e. the switch of the component u_2 has not yet occurred at time $\tau_1(\ell)$, so that $\theta_2(\ell) - \tau_1(\ell) = \tau_1^2(\ell) - \tau_1(\ell) > 0$. An analogous proof holds if $\theta_1(\ell) = \tau_2(\ell) < \tau_1(\ell)$. \square

The construction above shows that the flow of the maximized Hamiltonian coincides with the flow of the Hamiltonian $H : (t, \ell) \in [0, T] \times T^*\mathbb{R}^n \mapsto H_t(\ell) \in \mathbb{R}$:

$$H_t(\ell) := \begin{cases} H_1(\ell) & t \in [-\varepsilon, \theta_1(\ell)], \\ K_\nu(\ell) & t \in (\theta_1(\ell), \theta_2(\ell)], \text{ when } \nu \text{ is such that } \theta_1(\ell) = \tau_\nu(\ell), \\ H_2(\ell) & t \in (\theta_2(\ell), \hat{T} + \varepsilon]. \end{cases}$$

Namely the maximised flow $\mathcal{H}_t^{\max}(\ell)$ is given by:

if $(t, \ell) \in S_0 := \{(t, \ell) : t \in [-\varepsilon, \theta_1(\ell)]\}$ then

$$\mathcal{H}_t^{\max}(\ell) = \exp t\vec{H}_1(\ell); \quad (6.4a)$$

if $(t, \ell) \in S_1 := \{(t, \ell) : \theta_1(\ell) = \tau_1(\ell), t \in (\theta_1(\ell), \theta_2(\ell))\}$ then

$$\mathcal{H}_t^{\max}(\ell) = \exp(t - \theta_1(\ell))\overrightarrow{K}_1 \circ \exp \theta_1(\ell)\overrightarrow{H}_1(\ell); \quad (6.4b)$$

if $(t, \ell) \in S_1^2 := \{(t, \ell) : \theta_1(\ell) = \tau_1(\ell), t \in (\theta_2(\ell), \widehat{T} + \varepsilon]\}$ then

$$\mathcal{H}_t^{\max}(\ell) = \exp(t - \theta_2(\ell))\overrightarrow{H}_2(\ell) \circ \exp(\theta_2(\ell) - \theta_1(\ell))\overrightarrow{K}_1 \circ \exp \theta_1(\ell)\overrightarrow{H}_1(\ell); \quad (6.4c)$$

if $(t, \ell) \in S_2 := \{(t, \ell) : \theta_1(\ell) = \tau_2(\ell), t \in (\theta_1(\ell), \theta_2(\ell))\}$ then

$$\mathcal{H}_t^{\max}(\ell) = \exp(t - \theta_1(\ell))\overrightarrow{K}_2 \circ \exp \theta_1(\ell)\overrightarrow{H}_1(\ell); \quad (6.4d)$$

if $(t, \ell) \in S_2^2 := \{(t, \ell) : \theta_1(\ell) = \tau_2(\ell), t \in (\theta_2(\ell), \widehat{T} + \varepsilon]\}$ then

$$\mathcal{H}_t^{\max}(\ell) = \exp(t - \theta_2(\ell))\overrightarrow{H}_2(\ell) \circ \exp(\theta_2(\ell) - \theta_1(\ell))\overrightarrow{K}_2 \circ \exp \theta_1(\ell)\overrightarrow{H}_1(\ell). \quad (6.4e)$$

In what follows we will need the differentials of τ_ν and τ_ν^2 , $\nu = 1, 2$, in $\widehat{\ell}_0$. For ease of reading we shall write $d\tau_\nu$ and $d\tau_\nu^2$ instead of $d\tau_\nu(\widehat{\ell}_0)$ and $d\tau_\nu^2(\widehat{\ell}_0)$. Formulas from the differentials easily follow from equations (6.1)-(6.2). In particular when $\delta\ell = d\alpha_* dx$ we have the following simplified formulas for $\langle d\tau_\nu, \delta\ell \rangle$ and $\langle d\tau_\nu^2, \delta\ell \rangle$, $\nu = 1, 2$:

$$\langle d\tau_\nu, \delta\ell \rangle = \frac{-\delta x \cdot j_\nu \cdot \alpha(\widehat{x}_0)}{[g_1, j_\nu] \cdot \alpha(\widehat{x}_0)}, \quad (6.5)$$

$$\langle d(\tau_\nu^2 - \tau_\nu), \delta\ell \rangle = \frac{1}{[j_\nu, g_2] \cdot \alpha(\widehat{x}_0)} \left\{ -\delta x \cdot j_{3-\nu} \cdot \alpha(\widehat{x}_0) + \delta x \cdot j_\nu \cdot \alpha(\widehat{x}_0) \frac{[g_1, j_{3-\nu}] \cdot \alpha(\widehat{x}_0)}{[g_1, j_\nu] \cdot \alpha(\widehat{x}_0)} \right\}. \quad (6.6)$$

In order to apply the invertibility results of [6] to the projected maximised Hamiltonian flow $\pi\mathcal{H}^{\max}$, we need to write its first order approximation $\pi_*\mathcal{H}_*^{\max}$ in a neighborhood of the point $(t, \ell) = (\widehat{\tau}, \widehat{\ell}_0)$. Clearly this first order approximation is a piecewise linear map which we specify by giving its form in the polyhedral cones $C_0, C_1, C_1^2, C_2, C_2^2$ tangent to the sectors $S_0, S_1, S_1^2, S_2, S_2^2$ defined in (6.4). We recall that the vector fields g_ν, j_ν , $\nu = 1, 2$, defined in (3.5), are the pull-backs of the vector fields h_ν, k_ν , respectively.

In $C_0 := \{(\delta t, \delta\ell) \in \mathbb{R} \times T_{\widehat{\ell}_0} T^*\mathbb{R}^n : \delta t < \min\{\langle d\tau_1, \delta\ell \rangle, \langle d\tau_2, \delta\ell \rangle\}\}$

$$\pi_*\mathcal{H}_*^{\max}(\delta t, \delta\ell) = L_0(\delta t, \delta\ell) := \widehat{S}_{\widehat{\tau}*} \left(\delta t g_1(\widehat{x}_0) + \pi_*\delta\ell \right), \quad (6.7a)$$

in $C_1 := \{(\delta t, \delta\ell) \in \mathbb{R} \times T_{\widehat{\ell}_0} T^*\mathbb{R}^n : \langle d\tau_1, \delta\ell \rangle < \delta t < \langle d\tau_1^2, \delta\ell \rangle, \langle d\tau_1, \delta\ell \rangle < \langle d\tau_2, \delta\ell \rangle\}$

$$\pi_*\mathcal{H}_*^{\max}(\delta t, \delta\ell) = L_1(\delta t, \delta\ell) := \widehat{S}_{\widehat{\tau}*} \left((\delta t - \langle d\tau_1, \delta\ell \rangle) j_1(\widehat{x}_0) + \langle d\tau_1, \delta\ell \rangle g_1(\widehat{x}_0) + \pi_*\delta\ell \right),$$

in $C_1^2 := \{(\delta t, \delta\ell) \in \mathbb{R} \times T_{\widehat{\ell}_0} T^*\mathbb{R}^n : \langle d\tau_1, \delta\ell \rangle < \langle d\tau_1^2, \delta\ell \rangle < \delta t, \langle d\tau_1, \delta\ell \rangle < \langle d\tau_2, \delta\ell \rangle\}$

$$\begin{aligned} \pi_*\mathcal{H}_*^{\max}(\delta t, \delta\ell) &= L_1^2(\delta t, \delta\ell) \\ &:= \widehat{S}_{\widehat{\tau}*} \left((\delta t - \langle d\tau_1^2, \delta\ell \rangle) g_2(\widehat{x}_0) + \langle d(\tau_1^2 - \tau_1)(\widehat{\ell}_0), \delta\ell \rangle j_1(\widehat{x}_0) + \langle d\tau_1, \delta\ell \rangle g_1(\widehat{x}_0) + \pi_*\delta\ell \right), \end{aligned} \quad (6.7b)$$

in $C_2 := \{(\delta t, \delta\ell) \in \mathbb{R} \times T_{\widehat{\ell}_0} T^*\mathbb{R}^n : \langle d\tau_2, \delta\ell \rangle < \delta t < \langle d\tau_2^2, \delta\ell \rangle, \langle d\tau_2, \delta\ell \rangle < \langle d\tau_1, \delta\ell \rangle\}$

$$\pi_*\mathcal{H}_*^{\max}(\delta t, \delta\ell) = L_2(\delta t, \delta\ell) := \widehat{S}_{\widehat{\tau}*} \left((\delta t - \langle d\tau_2, \delta\ell \rangle) j_2(\widehat{x}_0) + \langle d\tau_2, \delta\ell \rangle g_1(\widehat{x}_0) + \pi_*\delta\ell \right), \quad (6.7c)$$

in $C_2^2 := \{(\delta t, \delta \ell) \in \mathbb{R} \times T_{\widehat{\ell}_0} T^* \mathbb{R}^n : \langle d\tau_2, \delta \ell \rangle < \langle d\tau_2^2, \delta \ell \rangle < \delta t, \langle d\tau_2, \delta \ell \rangle < \langle d\tau_1, \delta \ell \rangle\}$

$$\begin{aligned} \pi_* \mathcal{H}_*^{\max}(\delta t, \delta \ell) &= L_2^2(\delta t, \delta \ell) \\ &:= \widehat{S}_{\tau_*} \left((\delta t - \langle d\tau_2^2, \delta \ell \rangle) g_2(\widehat{x}_0) + \langle d(\tau_2^2 - \tau_2)(\widehat{\ell}_0), \delta \ell \rangle j_2(\widehat{x}_0) + \langle d\tau_2, \delta \ell \rangle g_1(\widehat{x}_0) + \pi_* \delta \ell \right). \end{aligned} \quad (6.7d)$$

7. PROPERTIES OF THE SECOND ORDER APPROXIMATION

7.1. Exploiting the coercivity of the second order approximation

We first prove the invertibility of the first order approximation of the projected maximised flow $\pi_* \mathcal{H}_*^{\max}$. In order to take advantage of the invertibility results of [6] for the continuous and piecewise linear map $\pi_* \mathcal{H}_*^{\max}$, we must first exploit the coercivity of the second order approximation of sub-problem (4.2). This is achieved by examining J_ν'' on certain subspaces of the space V_ν defined in (4.4). For $\nu = 1, 2$ let

$$\begin{aligned} V_\nu^1 &:= \{\delta e = (\delta x, a_1, b, a_2) \in V_\nu : b = a_2 = 0, \delta x + a_1 g_1(\widehat{x}_0) = 0\}, \\ V_\nu^2 &:= \{\delta e = (\delta x, a_1, b, a_2) \in V_\nu : a_2 = 0, \delta x + a_1 g_1(\widehat{x}_0) + b j_\nu(\widehat{x}_0) = 0\}, \\ V_\nu^3 &:= \{\delta e = (\delta x, a_1, b, a_2) \in V_\nu : \delta x + a_1 g_1(\widehat{x}_0) + b j_\nu(\widehat{x}_0) + a_2 g_2(\widehat{x}_0) = 0\}, \end{aligned}$$

and let Q_ν be the bilinear form associated to J_ν'' , see [4], i.e. if $\delta e = (\delta x, a_1, b, a_2)$ and $\delta f = (\delta y, c_1, d, c_2)$ then

$$\begin{aligned} Q_\nu[\delta e, \delta f] &= D^2(\alpha + \widehat{\beta})(\widehat{x}_0)(\delta x, \delta y) + \delta y \cdot (a_1 g_1 + b j_\nu + a_2 g_2) \cdot \widehat{\beta}(\widehat{x}_0) \\ &\quad + \delta x \cdot (c_1 g_1 + d j_\nu + c_2 g_2) \cdot \widehat{\beta}(\widehat{x}_0) + (c_1 g_1 + d j_\nu + c_2 g_2) \cdot (a_1 g_1 + b j_\nu + a_2 g_2) \cdot \widehat{\beta}(\widehat{x}_0) \\ &\quad + da_1 [g_1, j_\nu] \cdot \widehat{\beta}(\widehat{x}_0) + c_2 a_1 [g_1, g_2] \cdot \widehat{\beta}(\widehat{x}_0) + c_2 b [j_\nu, g_2] \cdot \widehat{\beta}(\widehat{x}_0). \end{aligned} \quad (7.1)$$

For any subspace $W \subset V_\nu$ denote as $W^{\perp\nu}$ the subspace of V_ν orthogonal to W with respect to Q_ν i.e.

$$W^{\perp\nu} := \{\delta e \in V_\nu : Q_\nu[\delta e, \delta f] = 0 \quad \forall \delta f \in W\}.$$

Clearly $V_1^1 = V_2^1$, so we shall simply denote this subspace as V^1 . Moreover, for any $\nu = 1, 2$ we have

$$V^1 \subseteq V_\nu^2 \subseteq V_\nu^3 \subseteq V_\nu,$$

so that J_ν'' is coercive on V_ν if and only if it is coercive on the four subspaces $V^1, V_\nu^2 \cap (V^1)^{\perp\nu}, V_\nu^3 \cap (V^2)^{\perp\nu}$ and $V_\nu \cap (V_\nu^3)^{\perp\nu}$.

The following proposition gathers the properties of the above subspaces of V_ν and characterises the coercivity of J_ν'' on such subspaces. The proposition should be compared to Lemmas 4.1-4.4 of [5] where analogous conditions are obtained for the weaker kind of coercivity needed to prove (time, state)-local optimality.

Proposition 7.1. *The following properties hold:*

(1) *if $\delta e = (\delta x, a_1, b, a_2) \in V^1$ then*

$$J_\nu''[\delta e]^2 = a_1^2 g_1 \cdot g_1 \cdot \alpha(\widehat{x}_0) = a_1^2 h_1 \cdot h_1 \cdot \alpha(\widehat{x}_0), \quad (7.2)$$

(2) *if $\delta e = (\delta x, a_1, b, a_2) \in V_\nu^2 \cap (V^1)^{\perp\nu}$ then*

$$\delta x \cdot g_1 \cdot \alpha(\widehat{x}_0) = \delta x \cdot h_1 \cdot \alpha(\widehat{x}_0) = 0 \quad (7.3)$$

$$J_\nu''[\delta e]^2 = \frac{1}{2} b [g_1, j_\nu] \cdot \alpha(\widehat{x}_0) (\langle d\tau_\nu, d\alpha_* \delta x \rangle - a_1), \quad (7.4)$$

(3) if $\delta e = (\delta x, a_1, b, a_2) \in V_\nu^3 \cap (V_\nu^2)^\perp$ then

$$\delta x \cdot g_1 \cdot \alpha(\widehat{x}_0) = 0, \quad a_1 = \langle d\tau_\nu, d\alpha_* \delta x \rangle, \quad (7.5)$$

$$J_\nu''[\delta e]^2 = \frac{1}{2} a_2 [j_\nu, g_2] \cdot \alpha(\widehat{x}_0) (\langle d(\tau_\nu^2 - \tau_\nu), d\alpha_* \delta x \rangle - b), \quad (7.6)$$

(4) if $\delta e = (\delta x, a_1, b, a_2) \in V_\nu \cap (V_\nu^3)^\perp$ then

$$\delta x \cdot g_1 \cdot \alpha(\widehat{x}_0) = 0, \quad a_1 = \langle d\tau_\nu, d\alpha_* \delta x \rangle, \quad b = \langle d(\tau_\nu^2 - \tau_\nu), d\alpha_* \delta x \rangle, \quad (7.7)$$

$$\begin{aligned} J_\nu''[\delta e]^2 &= \frac{1}{2} D^2(\alpha + \widehat{\beta})(\widehat{x}_0)[\delta x, \delta x + a_1 g_1 + b j_\nu + a_2 g_2] \\ &\quad + \frac{1}{2} (\delta x + a_1 g_1 + b j_\nu + a_2 g_2) \cdot (a_1 g_1 + b j_\nu + a_2 g_2) \cdot \widehat{\beta}(\widehat{x}_0). \end{aligned} \quad (7.8)$$

Proof. (1), (2), (3) and (7.7) are obtained as straightforward computations from (7.1). In order to prove (7.8) it suffices to take into account (7.7), (6.5) and (6.6). \square

7.2. Invertibility of the projected maximised flow

In this section we prove that the function α defined in Theorem 4.1 satisfies the properties required for the construction of Section 5. Namely, let

$$\Lambda := \{d\alpha(x) : x \in \mathbb{R}^n\}$$

be the Lagrangian manifold defined by the function α of Theorem 4.1. Also let

$$\Lambda_0 := \{\ell \in \Lambda : H_1(\ell) = p_0\}, \quad M_0 := \pi\Lambda_0 = \{x \in \mathbb{R}^n : h_1 \cdot \alpha(x) = p_0\}. \quad (7.9)$$

By Theorem 3 of [8] it suffices to show that $\pi\mathcal{H}^{\max}$ is locally invertible at $(t, \widehat{\ell}_0)$ for any $t \in [0, \widehat{T}]$. If $t < \widehat{\tau}$ then $\pi\mathcal{H}^{\max}$ is smooth at $(t, \widehat{\ell}_0)$ and $\pi_*\mathcal{H}_*^{\max}(\delta t, \delta \ell) = \widehat{S}_{t*}(\delta t g_1(\widehat{x}_0) + \pi_* \delta \ell)$. So, in order to prove the local invertibility of $\pi\mathcal{H}^{\max}$ at $(t, \widehat{\ell}_0)$, $t \in [0, \widehat{\tau})$, it suffices to show that $g_1(\widehat{x}_0)$ is not tangent to $\pi\Lambda_0$. Indeed the following lemma holds:

Lemma 7.2. *The sets M_0 and Λ_0 are $(n-1)$ -dimensional submanifolds of \mathbb{R}^n and $T^*\mathbb{R}^n$, respectively. Moreover $h_1(\widehat{x}_0)$ is not tangent to M_0 .*

Proof. By (7.9) it suffices to show that M_0 is a submanifold i.e., it suffices to show that there exists $\delta x \in T_{\widehat{x}_0}\mathbb{R}^n$ such that $\delta x \cdot h_1 \cdot \alpha(\widehat{x}_0) \neq 0$. By the coercivity of the second variation, see equation (7.2), we get the claim by choosing $\delta x = h_1(\widehat{x}_0)$. \square

For $\nu = 1, 2$, let

$$M_\nu := \left\{ \pi\mathcal{H}_{\tau_\nu(\ell)}^{\max}(\ell) : \ell \in \Lambda_0 \right\}, \quad M_\nu^2 := \left\{ \pi\mathcal{H}_{\tau_\nu^2(\ell)}^{\max}(\ell) : \ell \in \Lambda_0 \right\}.$$

Proposition 7.3. *For $\nu = 1, 2$, M_ν and M_ν^2 are $(n-1)$ -dimensional submanifolds of \mathbb{R}^n . Moreover:*

- (1) h_1 and k_ν are not tangent to M_ν and there exist $c_\nu > 0$, $\delta x_\nu \in T_{\widehat{x}_d}M_\nu$ such that $k_\nu(\widehat{x}_d) = \delta x_\nu + c_\nu h_1(\widehat{x}_d)$;
- (2) h_2 and k_ν are not tangent to M_ν^2 and there exist $c_\nu^2 > 0$, $\delta x_\nu^2 \in T_{\widehat{x}_d}M_\nu^2$ such that $h_2(\widehat{x}_d) = \delta x_\nu^2 + c_\nu^2 k_\nu(\widehat{x}_d)$.

Proof. Let $\delta x \in T_{\widehat{x}_d}\mathbb{R}^n$. Then δx is tangent to M_ν if and only if there exists $\delta \ell_0 \in \Lambda_0$ such that $\delta x = \pi_*\widehat{\mathcal{H}}_{\widehat{\tau}*}\delta \ell_0 + \langle d\tau_\nu, \delta \ell_0 \rangle h_1(\widehat{x}_d) = \widehat{S}_{\widehat{\tau}*}(\pi_*\delta \ell_0 + \langle d\tau_\nu, \delta \ell_0 \rangle g_1(\widehat{x}_0))$. As $\pi_*\delta \ell_0 \in T_{\widehat{x}_0}M_0$ while $g_1(\widehat{x}_0) \notin T_{\widehat{x}_0}M_0$ we get $\delta x = 0$ if and only if $\pi_*\delta \ell_0 = 0$, i.e., if and only if $\delta \ell_0 = d\alpha_*\pi_*\delta \ell_0 = 0$. This proves that M_ν is a $(n-1)$ -dimensional submanifold of the state space \mathbb{R}^n .

Let us now prove (1): assume $h_1(\widehat{x}_d)$ is tangent to $T_{\widehat{x}_d}M_\nu$. Then there exists $\delta x_0 \in T_{\widehat{x}_0}M_0$ such that $g_1(\widehat{x}_0) = \delta x_0 + \langle d\tau_\nu, d\alpha_*\delta x_0 \rangle g_1(\widehat{x}_0)$, i.e., $\delta x_0 + (\langle d\tau_\nu, d\alpha_*\delta x_0 \rangle - 1)g_1(\widehat{x}_0) = 0$ so that $\delta x_0 = 0$ while $\langle d\tau_\nu, d\alpha_*\delta x_0 \rangle = 1$, a contradiction.

Let $\delta x_\nu \in T_{\hat{x}_d} M_\nu$ and $c_\nu \in \mathbb{R}$ such that $k_\nu(\hat{x}_d) = \delta x_\nu + c_\nu h_1(\hat{x}_d)$ and let $\delta x_0 \in T_{\hat{x}_0} M_0$ be such that $\delta x_\nu = \widehat{S}_{\hat{\tau}_*} \delta x_0 + \langle d\tau_\nu, d\alpha_* \delta x_0 \rangle h_1(\hat{x}_d)$, so that $j_\nu(\hat{x}_0) = \delta x_0 + (\langle d\tau_\nu, d\alpha_* \delta x_0 \rangle + c_\nu) g_1(\hat{x}_0)$. Thus, by (7.3), $\delta e := (\delta x_0, \langle d\tau_\nu, d\alpha_* \delta x_0 \rangle + c_\nu, -1, 0) \in V_2^\nu \cap (V_1)^\perp$ and, by (7.4), $0 < J_\nu''[\delta e]^2 = \frac{1}{2}(-1)[g_1, j_\nu] \cdot \alpha(\hat{x}_0)(-c_\nu)$ so that $c_\nu > 0$.

Let us now turn to M_ν^2 : $\delta x \in T_{\hat{x}_d} \mathbb{R}^n$ is tangent to M_ν^2 if and only if there exists $\delta \ell_0 \in T_{\hat{\ell}_0} \Lambda_0$ such that

$$\begin{aligned} \delta x &= \pi_* \widehat{\mathcal{H}}_{\hat{\tau}_*} \delta \ell_0 + \langle d\tau_\nu, \delta \ell_0 \rangle h_1(\hat{x}_d) + \langle d(\tau_\nu^2 - \tau_\nu), \delta \ell_0 \rangle k_\nu(\hat{x}_d) \\ &= \widehat{S}_{\hat{\tau}_*} \left(\pi_* \delta \ell_0 + \langle d\tau_\nu, \delta \ell_0 \rangle g_1(\hat{x}_0) + \langle d(\tau_\nu^2 - \tau_\nu), \delta \ell_0 \rangle j_\nu(\hat{x}_0) \right). \end{aligned}$$

Thus, if $\delta x = 0$, then $\delta e := (\pi_* \delta \ell_0, \langle d\tau_\nu, \delta \ell_0 \rangle, \langle d(\tau_\nu^2 - \tau_\nu), \delta \ell_0 \rangle, 0) \in V_\nu^2 \cap (V_\nu^2)^\perp$ by (7.5), which is $\{0\}$ by the coercivity assumption. Thus $\pi_* \delta \ell_0 = 0$ and $\delta \ell_0 = d\alpha_* \pi_* \delta \ell_0 = 0$. Thus M_ν^2 is a $(n-1)$ -dimensional submanifold of \mathbb{R}^n .

Let us now prove (2): $k_\nu(\hat{x}_d)$ is tangent to M_ν^2 if and only if there exists $\delta x_0 \in T_{\hat{x}_0} M_0$ such that $j_\nu(\hat{x}_0) = \delta x_0 + \langle d\tau_\nu, d\alpha_* \delta x_0 \rangle g_1(\hat{x}_0) + \langle d(\tau_\nu^2 - \tau_\nu), d\alpha_* \delta x_0 \rangle j_\nu(\hat{x}_0)$, i.e., $\delta e := (\delta x_0, \langle d\tau_\nu, d\alpha_* \delta x_0 \rangle, \langle d(\tau_\nu^2 - \tau_\nu), d\alpha_* \delta x_0 \rangle - 1, 0)$, by (7.5) is in $V_\nu^2 \cap (V_\nu^2)^\perp$ which is $\{0\}$ by the coercivity assumption. So that $\delta x_0 = 0$ while $\langle d(\tau_\nu^2 - \tau_\nu), d\alpha_* \delta x_0 \rangle$ is equal to 1, a contradiction.

Let $\delta x_\nu^2 \in T_{\hat{x}_d} M_\nu^2$ and $c_\nu^2 \in \mathbb{R}$ such that $h_2(\hat{x}_d) = \delta x_\nu^2 + c_\nu^2 k_\nu(\hat{x}_d)$ and let $\delta x_0 \in T_{\hat{x}_0} M_0$ such that

$$\delta x_\nu^2 = \widehat{S}_{\hat{\tau}_*} \delta x_0 + \langle d\tau_\nu, d\alpha_* \delta x_0 \rangle h_1(\hat{x}_d) + \langle d(\tau_\nu^2 - \tau_\nu), d\alpha_* \delta x_0 \rangle k_\nu(\hat{x}_d)$$

so that $g_2(\hat{x}_0) = \delta x_0 + \langle d\tau_\nu, d\alpha_* \delta x_0 \rangle g_1(\hat{x}_0) + (\langle d(\tau_\nu^2 - \tau_\nu), d\alpha_* \delta x_0 \rangle + c_\nu^2) j_\nu(\hat{x}_0)$ i.e., by (7.5)

$$\delta e := (\delta x_0, \langle d\tau_\nu, d\alpha_* \delta x_0 \rangle, \langle d(\tau_\nu^2 - \tau_\nu), d\alpha_* \delta x_0 \rangle + c_\nu^2, -1) \in V_\nu^3 \cap (V_\nu^2)^\perp.$$

Thus, by the coercivity assumption and (7.6) we get $0 < J_\nu''[\delta e]^2 = \frac{1}{2}(-1)[j_\nu, g_2] \cdot \alpha(\hat{x}_0)(-c_\nu^2)$ so that $c_\nu^2 > 0$. \square

We now prove that the determinants of the linear maps defined in (6.7) have the same sign. This is equivalent to proving that the images of each pair of adjacent sectors do not overlap (see Proposition 3.1 in [6]).

Further on we will show that

- if $d\tau_1|_{T_{\hat{\ell}_0} \Lambda_0}$ and $d\tau_2|_{T_{\hat{\ell}_0} \Lambda_0}$ do not coincide, then the conditions of Theorem 4.1 of [6] are satisfied, see section 7.2.3,
- if $d\tau_1|_{T_{\hat{\ell}_0} \Lambda_0} \equiv d\tau_2|_{T_{\hat{\ell}_0} \Lambda_0}$, then Clarke's inverse map theorem can be applied, see section 7.2.4.

7.2.1. Sectors C_0 and C_1 .

Assume, by contradiction, there exist $(\delta t_0, \delta \ell_0) \in C_0 \cap (\mathbb{R} \times T_{\hat{\ell}_0} \Lambda_0)$ and $(\delta t_1, \delta \ell_1) \in C_1 \cap (\mathbb{R} \times T_{\hat{\ell}_0} \Lambda_0)$ such that

$$L_0(\delta t_0, \delta \ell_0) = L_1(\delta t_1, \delta \ell_1), \quad (7.10)$$

Equation (7.10) is equivalent to

$$\pi_* (\delta \ell_1 - \delta \ell_0) + (\langle d\tau_1, \delta \ell_1 \rangle - \delta t_0) g_1(\hat{x}_0) + (\delta t_1 - \langle d\tau_1, \delta \ell_1 \rangle) j_1(\hat{x}_0) = 0.$$

Let

$$\delta x := \pi_* (\delta \ell_1 - \delta \ell_0), \quad a_1 := \langle d\tau_1, \delta \ell_1 \rangle - \delta t_0, \quad b := \delta t_1 - \langle d\tau_1, \delta \ell_1 \rangle.$$

Notice that

$$(\delta t_0 - \langle d\tau_1, \delta \ell_0 \rangle) b = (\delta t_0 - \langle d\tau_1, \delta \ell_0 \rangle) (\delta t_1 - \langle d\tau_1, \delta \ell_1 \rangle) < 0$$

because $(\delta t_0, \delta \ell_0) \in C_0$ and $(\delta t_1, \delta \ell_1) \in C_1$. Thus, using (3.6), (7.4) and (6.5) we get

$$\begin{aligned} 0 &\geq [g_1, j_1] \cdot \alpha(\widehat{x}_0) (\delta t_0 - \langle d\tau_1, \delta \ell_0 \rangle) (\delta t_1 - \langle d\tau_1, \delta \ell_1 \rangle) \\ &= b(-a_1 [g_1, j_1] \cdot \alpha(\widehat{x}_0) - \delta x \cdot j_1 \cdot \alpha(\widehat{x}_0)) = J_1''[a_1, b, 0]^2 > 0, \end{aligned}$$

a contradiction.

7.2.2. Sectors C_1 and C_1^2

Assume, by contradiction, there exist $(\delta t_1, \delta \ell_1) \in C_1 \cap (\mathbb{R} \times T_{\widehat{\ell}_0} \Lambda_0)$ and $(\delta t_1^2, \delta \ell_1^2) \in C_1^2 \cap (\mathbb{R} \times T_{\widehat{\ell}_0} \Lambda_0)$ such that

$$L_1(\delta t_1, \delta \ell_1) = L_1^2(\delta t_1^2, \delta \ell_1^2), \quad (7.11)$$

Equation (7.11) is equivalent to

$$\begin{aligned} \pi_* (\delta \ell_1^2 - \delta \ell_1) + \langle d\tau_1, \delta \ell_1^2 - \delta \ell_1 \rangle g_1(\widehat{x}_0) + \\ + (\langle d\tau_1^2, \delta \ell_1^2 \rangle - \langle d\tau_1, \delta \ell_1^2 - \delta \ell_1 \rangle - \delta t_1) j_1(\widehat{x}_0) + (\delta t_1^2 - \langle d\tau_1^2, \delta \ell_1^2 \rangle) g_2(\widehat{x}_0) = 0. \end{aligned}$$

Let

$$\begin{aligned} \delta \ell &:= \delta \ell_1^2 - \delta \ell_1, & \delta x &:= \pi_* \delta \ell, & a_1 &:= \langle d\tau_1, \delta \ell \rangle, \\ b &:= \langle d\tau_1^2, \delta \ell_1^2 \rangle - \langle d\tau_1, \delta \ell \rangle - \delta t_1, & a_2 &:= \delta t_1^2 - \langle d\tau_1^2, \delta \ell_1^2 \rangle. \end{aligned} \quad (7.12)$$

Notice that

$$(\delta t_1 - \langle d\tau_1^2, \delta \ell_1 \rangle) a_2 = (\delta t_1 - \langle d\tau_1^2, \delta \ell_1 \rangle) (\delta t_1^2 - \langle d\tau_1^2, \delta \ell_1^2 \rangle) < 0$$

because $(\delta t_1, \delta \ell_1) \in C_1$ and $(\delta t_1^2, \delta \ell_1^2) \in C_1^2$.

Thus, using (3.6), (7.4) and (6.6) with $\nu = 1$ we get

$$\begin{aligned} 0 &\geq [j_1, g_2] \cdot \alpha(\widehat{x}_0) (\delta t_1^2 - \langle d\tau_1^2, \delta \ell_1^2 \rangle) (\delta t_1 - \langle d\tau_1^2, \delta \ell_1 \rangle) = a_2 [j_1, g_2] \cdot \alpha(\widehat{x}_0) (\delta t_1 - \langle d\tau_1^2, \delta \ell_1 \rangle + b - b) \\ &= a_2 [j_1, g_2] \cdot \alpha(\widehat{x}_0) \left(\langle d(\tau_1^2 - \tau_1)(\widehat{\ell}_0), d\alpha_* \delta x \rangle - b \right) \\ &= a_2 \left([g_1, j_2] \cdot \alpha(\widehat{x}_0) \left(\frac{\delta x \cdot j_1 \cdot \alpha(\widehat{x}_0)}{[g_1, j_1] \cdot \alpha(\widehat{x}_0)} - \frac{\delta x \cdot j_2 \cdot \alpha(\widehat{x}_0)}{[g_1, j_2] \cdot \alpha(\widehat{x}_0)} \right) - b [j_1, g_2] \cdot \alpha(\widehat{x}_0) \right) \end{aligned}$$

as $j_2 - g_1 = g_2 - j_1$, $[g_1, j_2] = [g_1, j_2 - g_1] = [g_1, g_2 - j_1]$ and since $\delta x \cdot g_1 \cdot \alpha(\widehat{x}_0) = 0$, so that $\delta x \cdot j_2 \cdot \alpha(\widehat{x}_0) = \delta x \cdot (g_2 - j_1) \cdot \alpha(\widehat{x}_0)$ we get:

$$\begin{aligned} &= a_2 \left([g_1, g_2 - j_1] \cdot \alpha(\widehat{x}_0) \frac{\delta x \cdot j_1 \cdot \alpha(\widehat{x}_0)}{[g_1, j_1] \cdot \alpha(\widehat{x}_0)} - \delta x \cdot (g_2 - j_1) \cdot \alpha(\widehat{x}_0) - b [j_1, g_2] \cdot \alpha(\widehat{x}_0) \right) \\ &= a_2 \left([g_1, g_2] \cdot \alpha(\widehat{x}_0) \frac{\delta x \cdot j_1 \cdot \alpha(\widehat{x}_0)}{[g_1, j_1] \cdot \alpha(\widehat{x}_0)} - \delta x \cdot g_2 \cdot \alpha(\widehat{x}_0) - b [j_1, g_2] \cdot \alpha(\widehat{x}_0) \right) \\ &= a_2 [g_1, g_2] \cdot \alpha(\widehat{x}_0) \frac{\delta x \cdot j_1 \cdot \alpha(\widehat{x}_0)}{[g_1, j_1] \cdot \alpha(\widehat{x}_0)} - \delta x \cdot a_2 g_2 \cdot \alpha(\widehat{x}_0) - a_2 b [j_1, g_2] \cdot \alpha(\widehat{x}_0) \end{aligned}$$

by (6.5) with $\nu = 1$ and the definition of a_1 given in (7.12):

$$= -a_1 a_2 [g_1, g_2] \cdot \alpha(\widehat{x}_0) - \delta x \cdot (a_1 g_1 + b j_1 - b j_1 + a_2 g_2) \cdot \alpha(\widehat{x}_0) - a_2 b [j_1, g_2] \cdot \alpha(\widehat{x}_0)$$

again by (6.5) with $\nu = 1$ and the definition of a_1 given in (7.12):

$$\begin{aligned} &= -a_1 a_2 [g_1, g_2] \cdot \alpha(\widehat{x}_0) - a_2 b [j_1, g_2] \cdot \alpha(\widehat{x}_0) - a_1 b [g_1, j_1] \cdot \alpha(\widehat{x}_0) - \delta x \cdot (a_1 g_1 + b j_1 + a_2 g_2) \cdot \alpha(\widehat{x}_0) \\ &= J_1''[a_1, b, a_2]^2 > 0, \end{aligned}$$

a contradiction. We can thus conclude that

$$\det(L_0) \det(L_1) > 0, \quad \det(L_1) \det(L_1^2) > 0.$$

Analogously one can show that

$$\det(L_0) \det(L_2) > 0, \quad \det(L_2) \det(L_2^2) > 0.$$

Such inequalities also imply $\det(L_1^2) \det(L_2^2) > 0$ so that all the determinants have the same sign.

7.2.3. Case when $d\tau_1|_{T_{\hat{\ell}_0}\Lambda_0}$ and $d\tau_2|_{T_{\hat{\ell}_0}\Lambda_0}$ do not coincide

In order to apply Theorem 4.1 of [6], we now show that there exists a point in the image of $\pi_* \mathcal{H}_*^{\max}$ whose preimage is a singleton.

Let $(\bar{\delta}t, \bar{\delta}\ell) \in C_0 \cap (\mathbb{R} \times T_{\hat{\ell}_0}\Lambda_0)$ such that $\langle d\tau_1, \bar{\delta}\ell \rangle = \langle d\tau_2, \bar{\delta}\ell \rangle$, i.e.

$$\bar{\delta}t < \langle d\tau_1, \delta\ell \rangle = \langle d\tau_2, \bar{\delta}\ell \rangle = \langle d\tau_1^2, \bar{\delta}\ell \rangle = \langle d\tau_2^2, \bar{\delta}\ell \rangle$$

and let $\bar{\delta}y := \pi_* \mathcal{H}_*^{\max}(\bar{\delta}t, \bar{\delta}\ell) = L_0(\bar{\delta}t, \bar{\delta}\ell)$.

With the same computation of section 7.2.1 one can prove that there is no $(\delta t, \delta\ell)$ in the set $(C_1 \cup C_2) \cap (\mathbb{R} \times T_{\hat{\ell}_0}\Lambda_0)$ such that $\bar{\delta}y = \pi_* \mathcal{H}_*^{\max}(\delta t, \delta\ell)$. It now remains to prove that no element of $(C_1^2 \cup C_2^2) \cap (\mathbb{R} \times T_{\hat{\ell}_0}\Lambda_0)$ can be mapped to $\bar{\delta}y$. We present the proof for $C_1^2 \cap (\mathbb{R} \times T_{\hat{\ell}_0}\Lambda_0)$. Similar considerations work for $C_2^2 \cap (\mathbb{R} \times T_{\hat{\ell}_0}\Lambda_0)$. Assume, by contradiction,

$$\bar{\delta}y = \pi_* \mathcal{H}_*^{\max}(\delta t_1^2, \delta\ell_1^2) \quad \text{for some } (\delta t_1^2, \delta\ell_1^2) \in C_1^2 \cap (\mathbb{R} \times T_{\hat{\ell}_0}\Lambda_0)$$

so that, by (6.7a) and (6.7b),

$$\begin{aligned} (\delta t_1^2 - \langle d\tau_1^2, \delta\ell_1^2 \rangle) g_2(\hat{x}_0) + \langle d(\tau_1^2 - \tau_1)(\hat{\ell}_0), \delta\ell_1^2 \rangle j_1(\hat{x}_0) + \\ + \langle d\tau_1, \delta\ell_1^2 \rangle g_1(\hat{x}_0) + \pi_* \delta\ell_1^2 = \bar{\delta}t g_1(\hat{x}_0) + \pi_* \bar{\delta}\ell \end{aligned} \quad (7.13)$$

Set

$$\begin{aligned} \delta\ell := \delta\ell_1^2 - \bar{\delta}\ell, \quad \delta x := \pi_* \delta\ell \quad a_1 := \langle d\tau_1, \delta\ell_1^2 \rangle - \bar{\delta}t, \\ b := \langle d(\tau_1^2 - \tau_1)(\hat{\ell}_0), \delta\ell_1^2 \rangle, \quad a_2 := \delta t_1^2 - \langle d\tau_1^2, \delta\ell_1^2 \rangle \end{aligned} \quad (7.14)$$

and notice that a_2 is nonnegative while $b + a_2$ is positive. Equation (7.13) reads

$$\delta x + a_1 g_1(\hat{x}_0) + b j_1(\hat{x}_0) + a_2 g_2(\hat{x}_0) = 0, \quad \delta x \in \pi_* T_{\hat{\ell}_0}\Lambda_0 = T_{\hat{x}_0}M_0 \quad (7.15)$$

so that $J_1''[\delta x, a_1, b, a_2] > 0$ by the coercivity Assumption 4.1. Hence, by (4.3) using (7.15),

$$\begin{aligned} 0 < (a_1 g_1 + b j_1 + a_2 g_2)^2 \cdot \alpha(\hat{x}_0) - a_1 b [g_1, j_1] \cdot \alpha(\hat{x}_0) - a_1 a_2 [g_1, g_2] \cdot \alpha(\hat{x}_0) - b a_2 [j_1, g_2] \cdot \alpha(\hat{x}_0) \\ = -a_1 b [g_1, j_1] \cdot \alpha(\hat{x}_0) - \delta x \cdot (b j_1 + a_2 g_2) \cdot \alpha(\hat{x}_0) - a_1 a_2 [g_1, g_2] \cdot \alpha(\hat{x}_0) - b a_2 [j_1, g_2] \cdot \alpha(\hat{x}_0) \end{aligned}$$

in the first addendum we replace a_1 as in (7.14), whereas in the second one we substitute $-\delta x \cdot j_1 \cdot \alpha(\hat{x}_0)$ with $[g_1, j_1] \cdot \alpha(\hat{x}_0) \langle d\tau_1, \delta\ell \rangle$ as in (6.5)

$$\begin{aligned} = b (\bar{\delta}t - \langle d\tau_1, \delta\ell_1^2 \rangle) [g_1, j_1] \cdot \alpha(\hat{x}_0) + b [g_1, j_1] \cdot \alpha(\hat{x}_0) \langle d\tau_1, \delta\ell \rangle - a_2 \delta x \cdot g_2 \cdot \alpha(\hat{x}_0) - a_1 a_2 [g_1, g_2] \cdot \alpha(\hat{x}_0) \\ - b a_2 [j_1, g_2] \cdot \alpha(\hat{x}_0) \end{aligned}$$

$$= b (\overline{\delta t} - \langle d\tau_1, \overline{\delta \ell} \rangle) [g_1, j_1] \cdot \alpha(\widehat{x}_0) - a_2 (\delta x \cdot g_2 \cdot \alpha(\widehat{x}_0) + a_1 [g_1, g_2] \cdot \alpha(\widehat{x}_0) + b [j_1, g_2] \cdot \alpha(\widehat{x}_0))$$

we can write $[g_1, g_2] \cdot \alpha(\widehat{x}_0) = [g_1, j_2] \cdot \alpha(\widehat{x}_0) + [g_1, g_2 - j_2] \cdot \alpha(\widehat{x}_0) = [g_1, j_2] \cdot \alpha(\widehat{x}_0) + [g_1, j_1] \cdot \alpha(\widehat{x}_0)$ obtaining

$$= b (\overline{\delta t} - \langle d\tau_1, \overline{\delta \ell} \rangle) [g_1, j_1] \cdot \alpha(\widehat{x}_0) - a_2 \left\{ \delta x \cdot g_2 \cdot \alpha(\widehat{x}_0) + a_1 [g_1, j_2] \cdot \alpha(\widehat{x}_0) + a_1 [g_1, j_1] \cdot \alpha(\widehat{x}_0) + b [j_1, g_2] \cdot \alpha(\widehat{x}_0) \right\}$$

in the second addendum within the curly brackets we replace a_1 as in (7.14) and in the last addendum we replace b as in (7.14). Taking into account that $\langle d(\tau_1^2 - \tau_1)(\widehat{\ell}_0), \overline{\delta \ell} \rangle = 0$ we obtain

$$= b (\overline{\delta t} - \langle d\tau_1, \overline{\delta \ell} \rangle) [g_1, j_1] \cdot \alpha(\widehat{x}_0) - a_2 \left\{ \delta x \cdot g_2 \cdot \alpha(\widehat{x}_0) - (\overline{\delta t} - \langle d\tau_1, \delta \ell_1^2 \mp \overline{\delta \ell} \rangle) [g_1, j_2] \cdot \alpha(\widehat{x}_0) + a_1 [g_1, j_1] \cdot \alpha(\widehat{x}_0) + \langle d(\tau_1^2 - \tau_1)(\widehat{\ell}_0), \delta \ell_1^2 \mp \overline{\delta \ell} \rangle [j_1, g_2] \cdot \alpha(\widehat{x}_0) \right\}$$

We now write $\langle d\tau_1, \delta \ell \rangle$ as in (6.5) so that

$$= b (\overline{\delta t} - \langle d\tau_1, \overline{\delta \ell} \rangle) [g_1, j_1] \cdot \alpha(\widehat{x}_0) - a_2 \left\{ \delta x \cdot g_2 \cdot \alpha(\widehat{x}_0) - (\overline{\delta t} - \langle d\tau_1, \overline{\delta \ell} \rangle) [g_1, j_2] \cdot \alpha(\widehat{x}_0) + \frac{-\delta x \cdot j_1 \cdot \alpha(\widehat{x}_0)}{[g_1, j_1] \cdot \alpha(\widehat{x}_0)} [g_1, j_2] \cdot \alpha(\widehat{x}_0) + a_1 [g_1, j_1] \cdot \alpha(\widehat{x}_0) + \langle d(\tau_1^2 - \tau_1)(\widehat{\ell}_0), \delta \ell \rangle [j_1, g_2] \cdot \alpha(\widehat{x}_0) \right\}.$$

In the last addendum we write $\langle d(\tau_1^2 - \tau_1)(\widehat{\ell}_0), \delta \ell \rangle$ as in (6.6) and simplify

$$= (\overline{\delta t} - \langle d\tau_1, \overline{\delta \ell} \rangle) (b [g_1, j_1] \cdot \alpha(\widehat{x}_0) + a_2 [g_1, j_2] \cdot \alpha(\widehat{x}_0)) + a_2 \left(-a_1 [g_1, j_1] \cdot \alpha(\widehat{x}_0) - \delta x \cdot (g_2 - j_2) \cdot \alpha(\widehat{x}_0) \right).$$

We now use the relation $g_2 - j_2 = j_1 - g_1$ in $\delta x \cdot (g_2 - j_2) \cdot \alpha(\widehat{x}_0)$, the definition of a_1 as in (7.14) and the fact that $\delta x \cdot g_1 \cdot \alpha(\widehat{x}_0) = 0$ since $\delta x \in T_{\widehat{x}_0} M_0$ yielding

$$= (\overline{\delta t} - \langle d\tau_1, \overline{\delta \ell} \rangle) \left(b [g_1, j_1] \cdot \alpha(\widehat{x}_0) + a_2 [g_1, j_2] \cdot \alpha(\widehat{x}_0) \right) + a_2 \left((\overline{\delta t} - \langle d\tau_1, \delta \ell_1^2 \mp \overline{\delta \ell} \rangle) [g_1, j_1] \cdot \alpha(\widehat{x}_0) - \delta x \cdot j_1 \cdot \alpha(\widehat{x}_0) \right).$$

Finally, we compute $\langle d\tau_1, \delta \ell \rangle$ as in (6.5) and simplify with the last addendum obtaining

$$= (\overline{\delta t} - \langle d\tau_1, \overline{\delta \ell} \rangle) \left(b [g_1, j_1] \cdot \alpha(\widehat{x}_0) + a_2 [g_1, j_2] \cdot \alpha(\widehat{x}_0) \right) + a_2 \left((\overline{\delta t} - \langle d\tau_1, \overline{\delta \ell} \rangle) [g_1, j_1] \cdot \alpha(\widehat{x}_0) \right) = (\overline{\delta t} - \langle d\tau_1, \overline{\delta \ell} \rangle) \left((b + a_2) [g_1, j_1] \cdot \alpha(\widehat{x}_0) + a_2 [g_1, j_2] \cdot \alpha(\widehat{x}_0) \right).$$

The first parenthesis is negative since $(\overline{\delta t}, \overline{\delta \ell}) \in C_0 \cap (\mathbb{R} \times T_{\widehat{\ell}_0} \Lambda_0)$ while the second one is positive by (7.14). Thus the product is negative, which contradicts $J_1''[\delta x, a_1, b, a_2]^2 > 0$.

7.2.4. Case when $d\tau_1|_{T_{\widehat{\ell}_0} \Lambda_0}$ and $d\tau_2|_{T_{\widehat{\ell}_0} \Lambda_0}$ coincide

In the case when $d\tau_1|_{T_{\widehat{\ell}_0} \Lambda_0} \equiv d\tau_2|_{T_{\widehat{\ell}_0} \Lambda_0}$ Theorem 4.1 of [6] does not apply as the interiors of \overline{C}_1 and \overline{C}_2 are empty. Thus we prove the invertibility of the projected maximised flow by Clarke's inverse function theorem.

Case when $d\tau_1|_{T_{\hat{\ell}_0}\Lambda_0} \equiv d\tau_2|_{T_{\hat{\ell}_0}\Lambda_0} \equiv \mathbf{0}$. Let $a_0 \geq 0$, $a_\nu, \alpha_\nu^2 \geq 0$, $\nu = 1, 2$, be such that $a_0 + a_1 + a_1^2 + a_2 + a_2^2 = 1$. Then, for any $(\delta t, \delta \ell) = (\delta t, d\alpha_* \delta x) \in \mathbb{R} \times T_{\hat{\ell}_0}\Lambda_0$, by equations (6.7), we get

$$\begin{aligned} L_0(\delta t, \delta \ell) &= \delta t g_1(\hat{x}_0) + \delta x, & L_1(\delta t, \delta \ell) &= \delta t j_1(\hat{x}_0) + \delta x, \\ L_2(\delta t, \delta \ell) &= \delta t j_2(\hat{x}_0) + \delta x, & L_1^2(\delta t, \delta \ell) &= L_2^2(\delta t, \delta \ell) = \delta t g_2(\hat{x}_0) + \delta x. \end{aligned}$$

By Proposition 7.3 we have

- $j_1(\hat{x}_0) = c_1 g_1(\hat{x}_0) + \delta x_0^1$ for some $c_1 > 0$ and $\delta x_0^1 \in T_{\hat{x}_0}M_0$;
- $j_2(\hat{x}_0) = c_2 g_1(\hat{x}_0) + \delta x_0^2$ for some $c_2 > 0$ and $\delta x_0^2 \in T_{\hat{x}_0}M_0$;
- $g_2(\hat{x}_0) = c_1^2 j_1(\hat{x}_0) + \delta x_0^{1,2}$ for some $c_1^2 > 0$ and $\delta x_0^{1,2} \in T_{\hat{x}_0}M_0$.

Thus

$$g_2(\hat{x}_0) = c_1^2 (c_1 g_1(\hat{x}_0) + \delta x_0^1) + \delta x_0^{1,2} = c_3 g_1(\hat{x}_0) + \delta x_0^3$$

where $c_3 := c_1^2 c_1 > 0$ and $\delta x_0^3 := c_1^2 \delta x_0^1 + \delta x_0^{1,2} \in T_{\hat{x}_0}M_0$.

Thus $(a_0 L_0 + a_1 L_1 + a_2 L_2 + a_1^2 L_1^2 + a_2^2 L_2^2)(\delta t, \delta \ell) = 0$ if and only if

$$\delta x + \delta t \left(a_1 \delta x_0^1 + (a_1^2 + a_2^2) \delta x_0^3 + a_2 \delta x_0^2 \right) + \delta t \left(a_0 + a_1 c_1 + (a_1^2 + a_2^2) c_3 + a_2 c_2 \right) g_1(\hat{x}_0) = 0.$$

As $g_1(\hat{x}_0) \notin T_{\hat{x}_0}M_0$ this equality yields

$$\begin{aligned} \delta x + \delta t \left(a_1 \delta x_0^1 + (a_1^2 + a_2^2) \delta x_0^3 + a_2 \delta x_0^2 \right) &= 0, \\ \delta t \left(a_0 + a_1 c_1 + (a_1^2 + a_2^2) c_3 + a_2 c_2 \right) &= 0. \end{aligned}$$

Since $a_0 + a_1 c_1 + (a_1^2 + a_2^2) c_3 + a_2 c_2 > 0$ we get $\delta t = 0$ and $\delta x = 0$. We have thus proved that any convex combination of the five linear approximations of $\pi_* \mathcal{H}_*^{\max}$ at $(\hat{\tau}, \hat{\ell}_0)$ is invertible, and Clarke's inverse function theorem applies.

Case when $d\tau_1|_{T_{\hat{\ell}_0}\Lambda_0} \equiv d\tau_2|_{T_{\hat{\ell}_0}\Lambda_0} \neq \mathbf{0}$. In this case $\ker d\tau_1|_{T_{\hat{\ell}_0}\Lambda_0} \equiv \ker d\tau_2|_{T_{\hat{\ell}_0}\Lambda_0}$ is a $(n-2)$ -dimensional linear space and

$$d\tau_1^2 \equiv d\tau_2^2 \equiv d\tau_2 \equiv d\tau_1 \text{ on } T_{\hat{\ell}_0}\Lambda_0.$$

Let v_1, \dots, v_{n-1} be a basis of $T_{\hat{x}_0}M_0$ such that $\langle d\tau_1, d\alpha_* v_1 \rangle = 1$, $\langle d\tau_1, d\alpha_* v_s \rangle = 0$, for any $s = 2, \dots, n-1$. As a basis for $\mathbb{R} \times T_{\hat{\ell}_0}\Lambda_0$ choose $(1, 0)$, $(1, d\alpha_* v_1)$, $(0, d\alpha_* v_s)$, $s = 2, \dots, n-1$ and, as a basis for $\mathbb{R}^n = T_{\hat{x}_0}\mathbb{R}^n$, choose $g_1(\hat{x}_0)$, v_s , $s = 1, 2, \dots, n-1$. By equations (6.7) we get

$$\begin{aligned} L_0(1, 0) &= g_1(\hat{x}_0), & L_1(1, 0) &= j_1(\hat{x}_0), & L_2(1, 0) &= j_2(\hat{x}_0), & L_1^2(1, 0) &= L_2^2(1, 0) = g_2(\hat{x}_0), \\ L_0(1, d\alpha_* v_1) &= L_1(1, d\alpha_* v_1) = L_2(1, d\alpha_* v_1) = L_1^2(1, d\alpha_* v_1) = L_2^2(1, d\alpha_* v_1) = g_1(\hat{x}_0) + v_1, \\ L_0(0, d\alpha_* v_s) &= L_1(0, d\alpha_* v_s) = L_2(0, d\alpha_* v_s) = L_1^2(0, d\alpha_* v_s) = L_2^2(0, d\alpha_* v_s) = v_s \quad \forall s = 2, \dots, n-1. \end{aligned}$$

In order to write the matrix representation following these bases we need to compute $j_1(\hat{x}_0)$, $j_2(\hat{x}_0)$ and $g_2(\hat{x}_0)$ in terms of the basis $g_1(\hat{x}_0)$, v_1, \dots, v_{n-1} .

Observe that, by Proposition 7.3(1), we have $k_\nu(\hat{x}_d) = c_\nu h_1(\hat{x}_d) + \delta x_\nu$ where $\delta x_\nu \in T_{\hat{x}_d}M_\nu$ and $c_\nu > 0$. The differential at $\hat{\ell}_0$ of the surjective map $\ell \in \Lambda_0 \mapsto \pi \mathcal{H}_{\tau_\nu}^{\max}(\ell) \in M_\nu$ operates as follows: $\delta \ell_0 \in T_{\hat{\ell}_0}\Lambda_0 \mapsto \hat{S}_{\hat{\tau}_*} \pi_* \delta \ell_0 + \langle d\tau_\nu, \delta \ell_0 \rangle h_1(\hat{x}_d)$. Therefore there exists $\delta x_{0\nu} \in T_{\hat{x}_0}M_0$ such that $\delta x_\nu = \hat{S}_{\hat{\tau}_*} \delta x_{0\nu} + \langle d\tau_\nu, d\alpha_* \delta x_{0\nu} \rangle h_1(\hat{x}_d)$. Thus $k_\nu(\hat{x}_d) = c_\nu h_1(\hat{x}_d) + \hat{S}_{\hat{\tau}_*} \delta x_{0\nu} + \langle d\tau_\nu, d\alpha_* \delta x_{0\nu} \rangle h_1(\hat{x}_d)$, equivalently,

$$j_\nu(\hat{x}_0) = \left(c_\nu + \langle d\tau_\nu, d\alpha_* \delta x_{0\nu} \rangle \right) g_1(\hat{x}_0) + \delta x_{0\nu} = (c_\nu + \gamma_{\nu 1}) g_1(\hat{x}_0) + \sum_{s=1}^{n-1} \gamma_{\nu s} v_s, \quad \nu = 1, 2$$

where $\delta x_{0\nu} = \sum_{s=1}^{n-1} \gamma_{\nu s} v_s$.

Analogously, by Proposition 7.3(2), one can show that

$$g_2(\hat{x}_0) = \left(\gamma_{\nu 1}^2 + c_\nu^2 (c_\nu + \gamma_{\nu 1}) \right) g_1(\hat{x}_0) + \sum_{s=1}^{n-1} (c_\nu^2 \gamma_{\nu s} + \gamma_{\nu s}^2) v_s, \quad \nu = 1, 2$$

for appropriate numbers $\gamma_{\nu s}, \gamma_{\nu s}^2, s = 1, \dots, n-1$.

Thus, the matrices associated to the five mappings in these bases are

$$A_0 = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & & I_{n-2} \\ 0 & 0 & & & \end{pmatrix}, \quad A_1 = \begin{pmatrix} c_1 + \gamma_{11} & 1 & 0 & \dots & 0 \\ \gamma_{11} & 1 & 0 & \dots & 0 \\ \gamma_{12} & 0 & & & \\ \vdots & \vdots & & & I_{n-2} \\ \gamma_{1,n-1} & 0 & & & \end{pmatrix},$$

$$A_2 = \begin{pmatrix} c_2 + \gamma_{21} & 1 & 0 & \dots & 0 \\ \gamma_{21} & 1 & 0 & \dots & 0 \\ \gamma_{22} & 0 & & & \\ \vdots & \vdots & & & I_{n-2} \\ \gamma_{2,n-1} & 0 & & & \end{pmatrix}, \quad A_1^2 = A_2^2 = \begin{pmatrix} \gamma_{11}^2 + c_1^2 (c_1 + \gamma_{11}) & 1 & 0 & \dots & 0 \\ c_1^2 \gamma_{11} + \gamma_{11}^2 & 1 & 0 & \dots & 0 \\ c_1^2 \gamma_{12} + \gamma_{12}^2 & 0 & & & \\ \vdots & \vdots & & & I_{n-2} \\ c_1^2 \gamma_{1,n-1} + \gamma_{1,n-1}^2 & 0 & & & \end{pmatrix}$$

i.e. the five matrices differ only in the first column. Thus

$$\begin{aligned} \det(a_0 A_0 + a_1 A_1 + a_2 A_2 + a_1^2 A_1^2 + a_2^2 A_2^2) \\ = a_0 \det A_0 + a_1 \det A_1 + a_2 \det A_2 + a_1^2 \det A_1^2 + a_2^2 \det A_2^2 \end{aligned}$$

and is positive as all the determinants have the same sign and $\det A_0 = 1$. Thus Clarke's inverse function theorem applies.

Remark 7.1. For $t \in (\hat{\tau}, \hat{T}]$ we have

$$\pi_* \mathcal{H}_*^{\max}(\delta t, \delta \ell) = \begin{cases} \exp(t - \hat{\tau}) h_{2*} L_1^2(\delta t, \delta \ell) & \text{if } \langle d\tau_1, \delta \ell \rangle < \langle d\tau_2, \delta \ell \rangle, \\ \exp(t - \hat{\tau}) h_{2*} L_2^2(\delta t, \delta \ell) & \text{if } \langle d\tau_2, \delta \ell \rangle < \langle d\tau_1, \delta \ell \rangle. \end{cases}$$

In Section 7.2.2 we have shown that $\det(L_1^2) \det(L_2^2) > 0$ so that $\pi_* \mathcal{H}_*^{\max}$ is one-to-one also for any $t \in (\hat{\tau}, \hat{T}]$ and $\pi \mathcal{H}^{\max}$ is locally invertible at any $(t, \hat{\ell}_0), t \in (\hat{\tau}, \hat{T}]$.

8. PROOF OF STATE-LOCAL OPTIMALITY

We can now complete the proof of Theorem 4.2. Let us go back to inequality (5.4). By the coercivity of $J''_\nu, \nu = 1, 2$, (7.2) and by continuity, the quantity $h_1 \cdot h_1 \cdot \alpha(\bar{q})$ in (5.4) is positive so that $p_0(T - \hat{T}) \geq C(\xi(T)) - C(\hat{x}_f)$. Let

$$\tilde{C}: y \in \mathcal{O}(\hat{x}_f) \mapsto C(y) + \beta(y) = p_0(\psi^{\mathbb{R}}(y)) + \alpha(\pi \psi^{\Lambda_0}(y)) + \beta(y) \in \mathbb{R}.$$

Then $\tilde{C}|_{\mathcal{O}(\hat{x}_f) \cap N_f} \equiv C|_{\mathcal{O}(\hat{x}_f) \cap N_f}$. Let \mathcal{V} be the neighborhood of $\hat{\xi}([0, \hat{T}])$ defined in (5.1).

For $y \in \mathcal{V}$ consider $\psi := (\pi \mathcal{H}^{\max})^{-1}$:

$$\psi: y \in \mathcal{V} \mapsto (\psi^{\mathbb{R}}(y), \psi^{\Lambda_0}(y)) = (t(y), \ell(y)) \in (-\varepsilon, \hat{T} + \varepsilon) \times \Lambda_0.$$

By the invertibility of $\pi \mathcal{H}^{\max}$ and $\pi_* \mathcal{H}_*^{\max}$ there exists a neighborhood $\mathcal{O}(\hat{x}_f) \subset \mathcal{V}$ such that, for any $y \in \mathcal{O}(\hat{x}_f)$ and any $\delta y \in T_y N_f$ there exists a unique couple $(\delta t, \delta \ell) \in T_{\psi(y)}(\mathbb{R} \times \Lambda_0)$ and $\nu \in \{1, 2\}$ such that (see equations (6.7))

$$\begin{aligned} \delta y &= \langle d\psi^{\mathbb{R}}(y), \delta y \rangle h_2(y) + \pi_* \mathcal{H}_{t(y)}^{\max} \delta \ell \\ &= \widehat{S}_{\hat{T}*} \left(\pi_* \delta \ell + \langle d\tau_\nu, \delta \ell \rangle g_1(\pi\psi(y)) + \langle d(\tau_\nu^2 - \tau_\nu), \delta \ell \rangle j_\nu(\pi\psi(y)) + (\delta t - \langle d\tau_\nu^2, \delta \ell \rangle) g_2(\pi\psi(y)) \right). \end{aligned}$$

Applying the one-form $\mathcal{H}_{t(y)}^{\max}(\ell(y)) \in T_y^* \mathbb{R}^n$ we get

$$\begin{aligned} \langle \mathcal{H}_{t(y)}^{\max}(\ell(y)), \delta y \rangle &= p_0 \langle d\psi^{\mathbb{R}}(y), \delta y \rangle + \langle \mathcal{H}_{t(y)}^{\max}(\ell(y)), \pi_* \mathcal{H}_{t(y)}^{\max} \delta \ell \rangle \\ &= p_0 \langle d\psi^{\mathbb{R}}(y), \delta y \rangle + \langle \ell(y), \pi_* \delta \ell \rangle = p_0 \langle d\psi^{\mathbb{R}}(y), \delta y \rangle + \langle d\alpha(\pi\ell(y)), \pi_* \delta \ell \rangle \end{aligned}$$

i.e. $dC(y) = \mathcal{H}_{t(y)}^{\max}(\ell(y))$ and $d\tilde{C}(y) = \mathcal{H}_{t(y)}^{\max}(\ell(y)) + d\beta(y)$ for any $y \in \mathcal{O}(\hat{x}_f)$. In particular, choosing $y = \hat{x}_f$, by the transversality condition (3.3) in PMP we get

$$\langle d\tilde{C}(\hat{x}_f), \delta y \rangle = \langle dC(\hat{x}_f), \delta y \rangle - \langle \hat{\ell}_f, \delta y \rangle = \langle \hat{\ell}_f, \delta y \rangle - \langle \hat{\ell}_f, \delta y \rangle = 0 \quad \forall \delta y \in T_{\hat{x}_f} \mathbb{R}^n.$$

Differentiating again, and taking into account that σ is invariant with respect to the flow of $\widehat{\mathcal{H}}_{\hat{T}*}$ we get

$$\begin{aligned} D^2 \tilde{C}(\hat{x}_f)[\delta y]^2 &= \langle \mathcal{H}_*^{\max}(\delta t, \delta \ell), \delta y \rangle + D^2 \beta[\delta y]^2 = \langle \mathcal{H}_*^{\max} \psi_* \delta y, \delta y \rangle + D^2 \beta[\delta y]^2 = \sigma((\mathcal{H}^{\max} \circ \psi)_* \delta y, d(-\beta)_* \delta y) \\ &= \sigma(\mathcal{H}_*^{\max}(\delta t, \delta \ell), d(-\beta)_* \pi_* \mathcal{H}_*^{\max}(\delta t, \delta \ell)) = \sigma\left(\widehat{\mathcal{H}}_{\hat{T}*}^{-1} \mathcal{H}_*^{\max}(\delta t, \delta \ell), d(-\beta)_* \widehat{S}_{\hat{T}*}^{-1} \pi_* \mathcal{H}_*^{\max}(\delta t, \delta \ell)\right). \end{aligned}$$

By equations (6.7), the definition of σ , equation (2.2), and recalling that $d\alpha(\hat{x}_0) = \widehat{\ell}_0 = -d\widehat{\beta}(\hat{x}_0)$ the last expression reads

$$\begin{aligned} D^2 \tilde{C}(\hat{x}_f)[\delta y]^2 &= \left\langle \delta \ell + \langle d\tau_\nu, \delta \ell \rangle \overrightarrow{G}_1(\widehat{\ell}_0) + \langle d(\tau_\nu^2 - \tau_\nu), \delta \ell \rangle \overrightarrow{J}_\nu(\widehat{\ell}_0) + (\delta t - \langle d\tau_\nu^2, \delta \ell \rangle) \overrightarrow{G}_2(\widehat{\ell}_0), \right. \\ &\quad \left. \pi_* \delta \ell + \langle d\tau_\nu, \delta \ell \rangle g_1(\hat{x}_0) + \langle d(\tau_\nu^2 - \tau_\nu), \delta \ell \rangle j_\nu(\hat{x}_0) + (\delta t - \langle d\tau_\nu^2, \delta \ell \rangle) g_2(\hat{x}_0) \right\rangle \\ &\quad + D^2 \widehat{\beta}(\hat{x}_0) \left[\pi_* \delta \ell + \langle d\tau_\nu, \delta \ell \rangle g_1 + \langle d(\tau_\nu^2 - \tau_\nu), \delta \ell \rangle j_\nu + (\delta t - \langle d\tau_\nu^2, \delta \ell \rangle) g_2 \right]^2 \\ &= D^2(\alpha + \widehat{\beta})(\hat{x}_0) \left(\pi_* \delta \ell, \pi_* \delta \ell + \langle d\tau_\nu, \delta \ell \rangle g_1 + \langle d(\tau_\nu^2 - \tau_\nu), \delta \ell \rangle j_\nu + (\delta t - \langle d\tau_\nu^2, \delta \ell \rangle) g_2 \right) \\ &\quad + \left(\pi_* \delta \ell + \langle d\tau_\nu, \delta \ell \rangle g_1 + \langle d(\tau_\nu^2 - \tau_\nu), \delta \ell \rangle j_\nu + (\delta t - \langle d\tau_\nu^2, \delta \ell \rangle) g_2 \right) \\ &\quad \cdot \left(\langle d\tau_\nu, \delta \ell \rangle g_1 + \langle d(\tau_\nu^2 - \tau_\nu), \delta \ell \rangle j_\nu + (\delta t - \langle d\tau_\nu^2, \delta \ell \rangle) g_2 \right) \cdot \widehat{\beta}(\hat{x}_0) \end{aligned} \quad (8.1)$$

which is positive by (7.8).

This proves that if $p_0 = 1$, then $\hat{\xi}$ is a state-local optimal trajectory. Let us now show that the minimum is strict. In particular this fact implies that when $p_0 = 0$ the trajectory $\hat{\xi}$ is isolated among the admissible ones.

If $p_0(T - \hat{T}) = 0$, then by (5.4), (7.2) and (8.1) we get $\xi(T) = \hat{x}_f$ and $t_0 = 0$ i.e. $\xi(0) \in M_0$ and, by the expression for I_1 in Section 5,

$$\langle \mathcal{H}^{\max}(\psi(\xi(t))), \dot{\xi}(t) \rangle = p_0 \quad \text{a.e. } t \in [0, T]. \quad (8.2)$$

As $\xi(T) = \widehat{x}_f$, by the regularity assumption along the bang arcs, Assumption 3.2, equation (8.2) implies $\dot{\xi}(t) = h_2(\xi(t))$ as long as $\mathcal{H}^{\max}(\psi(\xi(t))) \in \{\ell: H^{\max}(\ell) = H_2(\ell)\}$ so that $\xi(t) = \widehat{\xi}(t-T+\widehat{T})$ for any $t \in [\widehat{\tau}+T-\widehat{T}, T]$. In particular $\xi(\widehat{\tau}+T-\widehat{T}) = \widehat{x}_d$. Proposition 7.3 implies that any solution through \widehat{x}_d when run backwards in time cannot access the interior of the regions $\pi\{\ell \in T^*\mathbb{R}^n: H^{\max}(\ell) = K_1(\ell)\}$ and $\pi\{\ell \in T^*\mathbb{R}^n: H^{\max}(\ell) = K_2(\ell)\}$ for times t close to $\widehat{\tau}+T-\widehat{T}$. Further one can exclude that the solution sticks to the manifold M_ν by observing that, by Proposition 7.3 any convex combination of h_1 and k_ν points inside $\pi\{\ell \in T^*\mathbb{R}^n: H^{\max}(\ell) = K_\nu(\ell)\}$. Analogously one can exclude that the solution sticks to the manifold M_ν^2 by observing that, by Proposition 7.3 any convex combination of h_2 and k_ν points outside $\pi\{\ell \in T^*\mathbb{R}^n: H^{\max}(\ell) = K_\nu(\ell)\}$. Thus the solution ξ , run backwards in time enters the interior of the region $\pi\{\ell \in T^*\mathbb{R}^n: H^{\max}(\ell) = H_1(\ell)\}$. Hence $\xi(t) = \widehat{\xi}(t-T+\widehat{T})$ for any $t \in [0, T]$. If $T = \widehat{T}$ this immediately yields $\xi \equiv \widehat{\xi}$.

Since $\xi(T-\widehat{T}) = \widehat{x}_0 \in M_0$ then, if $p_0 = 0$, and h_1 is not tangent to M_0 in a neighborhood of \widehat{x}_0 , then, possibly restricting \mathcal{V} , we get that ξ can cross M_0 only once. Hence $T = \widehat{T}$, i.e. also in this case $\xi \equiv \widehat{\xi}$.

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