Generalized spline spaces over T-meshes: Dimension formula and locally refined generalized B-splines

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Abstract

Univariate generalized splines are smooth piecewise functions with sections in certain extended Tchebycheff spaces. They are a natural extension of univariate (algebraic) polynomial splines, and enjoy the same structural properties as their polynomial counterparts. In this paper, we consider generalized spline spaces over planar T-meshes, and we deepen their parallelism with polynomial spline spaces over the same partitions. First, we extend the homological approach from polynomial to generalized splines. This provides some new insights into the dimension problem of a generalized spline space defined on a prescribed T-mesh for a given degree and smoothness. Second, we extend the construction of LR-splines to the generalized spline context.

Keywords: Generalized splines; T-meshes; LR-meshes; Dimension formula

1. Introduction

Generalized splines are smooth piecewise functions with sections in spaces of the form (see [8])

$$\mathbb{P}_p^{U,V} := \langle 1, t, \ldots, t^{p-2}, U(t), V(t) \rangle, \quad t \in [a, b], \quad 2 \le p \in \mathbb{N}. \tag{1}$$

Classical polynomial splines are obtained by taking the functions U, V equal to t^{p-1}, t^p. In such a case, the space $\mathbb{P}_p^{U,V}$ is the space of algebraic polynomials of degree p, denoted by $\mathbb{P}_p$. Other interesting examples are trigonometric or exponential generalized splines for which U, V are taken as $\cos(\alpha t), \sin(\alpha t)$, or $\cosh(\alpha t), \sinh(\alpha t)$, respectively.

Under suitable conditions on U, V, the space (1) has the same structural properties as $\mathbb{P}_p$. Similarly, generalized splines possess all the desirable properties of polynomial splines. In particular, they admit a representation in terms of basis functions that are a natural extension of the polynomial B-splines. Moreover, classical algorithms (like degree elevation, knot insertion, differentiation formulas, etc.) can be explicitly rephrased for them. Such basis functions are referred to as generalized B-splines (GB-splines).

Generalized splines are popular tools in the computer aided geometric design (CAGD) community. Besides their theoretical interest, generalized spline spaces offer the possibility of

controlling the shape of their elements by means of some shape parameters (the value α in the case of trigonometric and exponential generalized splines mentioned above), see [7, 17, 18, 28]. Moreover, they are an interesting alternative to non-uniform rational B-splines (NURBS), see [6, 22, 36, 37] and references therein. In particular, trigonometric and exponential generalized splines allow for an exact representation of conic sections as well as some transcendental curves (helix, cycloid, etc.) and are attractive from the geometrical point of view. Indeed, in contrast with NURBS, they are able to provide parameterizations of conic sections with respect to the arc length so that equally spaced points in the parameter domain correspond to equally spaced points on the described curve. It is also worth mentioning that, contrarily to NURBS, trigonometric and exponential generalized B-splines behave completely similar to polynomial B-splines with respect to differentiation and integration.

Thanks to the above properties, tensor-products of generalized B-splines are also an interesting problem-dependent alternative to tensor-product (polynomial) B-splines and NURBS in isogeometric analysis (IgA), see [9, 23, 24, 25]. Introduced nearly a decade ago in a seminal paper by Hughes *et al.* [15], IgA is nowadays a well-established paradigm for the analysis of problems governed by partial differential equations (PDEs), see, e.g., [10] and references therein. It aims at improving the connection between numerical simulation and computer aided design (CAD) systems. The main idea of IgA is to use the functions adopted in CAD systems not only to describe the domain geometry, but also to represent the numerical solution of the differential problem, within an isoparametric framework.

Adaptive local refinement is fundamental in geometric modeling and is a crucial ingredient for obtaining, in an efficient way, an accurate solution of partial differential equations. Any tensor-product structure lacks adequate local refinement. The introduction and the success of the IgA paradigm triggered the interest in alternative structures that support local refinements. Confining the discussion to local tensor-product structures, we mention T-splines [20, 31, 32], hierarchical splines [13, 14, 35], and locally refined (LR-) splines [12, 16].

T-splines, hierarchical splines and LR-splines can be seen as special instances of splines over T-meshes, see [29, 30]. A complete understanding of these spline spaces requires the knowledge of the dimension of the spline space defined on a prescribed T-mesh for a given degree and smoothness, see [11, 19, 29] and references therein. Among the various techniques to tackle this difficult problem, one can use the homological approach proposed in [26], where the technique presented in [1] for splines on triangulations has been fine-tuned for splines on planar T-meshes. The resulting dimension formula is a key ingredient in the analysis of the properties of LR-splines, see [12].

As mentioned above, generalized splines enjoy the fundamental properties of polynomial splines, including the behavior with respect to local refinement. In particular, GB-splines support (locally refined) hierarchical structures in the same way as (polynomial) B-splines, see [25] (and also [14, 34]). T-spline structures based on trigonometric GB-splines have been addressed in [3]. Results on the dimension of generalized spline spaces over T-meshes have been provided in [5] by extending the approach based on so-called determining sets, see [29].

In this paper, we deepen the parallelism between polynomial splines and generalized splines over planar T-meshes. More precisely,

- we extend the homological approach of [26] to generalized splines, in order to address the problem of determining the dimension of a generalized spline space on a prescribed T-mesh for a given degree and smoothness;

- we extend the construction of LR-splines presented in [12] to generalized splines.

The remaining of the paper is divided into four sections. In Section 2 we give the definition of generalized spline spaces over T-meshes. Section 3 is devoted to the determination of the dimension of such spaces by means of the homological approach. Generalized LR-splines are described in Section 4. Finally, we end in Section 5 with some concluding remarks.

2. Generalized spline spaces over T-meshes

In this section we formulate the definitions of the meshes and of the spaces we deal with. We consider a region $\Omega \subset \mathbb{R}^2$ which is a finite union of closed axis-aligned rectangles, called *cells*, with pairwise disjoint interiors. We assume that Ω is simply connected and its interior Ω^o is connected; see Figure 1 for an illustration. The smallest rectangle containing Ω is denoted by $[a_h, b_h] \times [a_v, b_v]$.

Next we define a T-mesh on Ω using the notation and definition given in [26].

Definition 1 (T-mesh). *A T-mesh $\mathcal{T} := (\mathcal{T}_2, \mathcal{T}_1, \mathcal{T}_0)$ on Ω is defined as:*

- *$\mathcal{T}_2$ is the collection of cells in Ω;*

- *$\mathcal{T}_1 = \mathcal{T}_1^h \cup \mathcal{T}_1^v$ is a finite set of closed axis-aligned horizontal and vertical segments in $\bigcup_{\sigma \in \mathcal{T}_2} \partial\sigma$, called edges;*

- *$\mathcal{T}_0 := \bigcup_{\tau \in \mathcal{T}_1} \partial\tau$ is a finite set of points, called vertices;*

such that

- *for each $\sigma \in \mathcal{T}_2$, $\partial\sigma$ is a finite union of elements of $\mathcal{T}_1$;*

- *for $\sigma, \sigma' \in \mathcal{T}_2$ with $\sigma \neq \sigma'$, $\sigma \cap \sigma' = \partial\sigma \cap \partial\sigma'$ is a finite union of elements of $\mathcal{T}_1 \cup \mathcal{T}_0$;*

- *for $\tau, \tau' \in \mathcal{T}_1$ with $\tau \neq \tau'$, $\tau \cap \tau' = \partial\tau \cap \partial\tau' \subset \mathcal{T}_0$;*

- *for each $\gamma \in \mathcal{T}_0$, $\gamma = \tau_h \cap \tau_v$ where τ_h is a horizontal edge and τ_v is a vertical edge.*

A *segment* of $\mathcal{T}$ is a connected union of edges of $\mathcal{T}$ belonging to the same straight line. We denote by $\mathcal{T}_1^o$ the set of *interior edges*, i.e., the edges intersecting the interior of Ω. Analogously, $\mathcal{T}_0^o$ represents the set of the vertices in Ω^o, called *interior vertices*. The elements of the sets $\mathcal{T}_1 \setminus \mathcal{T}_1^o$ and $\mathcal{T}_0 \setminus \mathcal{T}_0^o$ are the *boundary edges* and the *boundary vertices*, respectively. We say that an interior vertex is a *crossing vertex* if it belongs to 4 distinct edges; it is a *T-vertex* if it belongs to exactly 3 edges. Moreover, $\mathcal{T}_1^{o,h}$ and $\mathcal{T}_1^{o,v}$ indicate the sets of the horizontal and vertical interior edges of $\mathcal{T}$, respectively, and we set $\mathcal{T}_1^o := \mathcal{T}_1^{o,h} \cup \mathcal{T}_1^{o,v}$. Then, the interior T-mesh is given by $\mathcal{T}^o := (\mathcal{T}_2, \mathcal{T}_1^o, \mathcal{T}_0^o)$. Finally, we denote by f_2 the number of rectangles, by f_1^h and f_1^v the number of horizontal and vertical interior edges, respectively, and by f_0 the number of interior vertices of $\mathcal{T}$.

Example 1. *Consider the T-mesh $\mathcal{T}$ depicted in Figure 2. In this case, we have*

- *$\mathcal{T}_2 = \{\sigma_1, \sigma_2, \sigma_3\}$, $f_2 = 3$;*

- *$\mathcal{T}_1^{o,h} = \{\tau_3^h\}$, $f_1^h = 1$;*

- *$\mathcal{T}_1^{o,v} = \{\tau_3^v, \tau_4^v\}$, $f_1^v = 2$;*

- *$\mathcal{T}_0^o = \{\gamma_5\}$, $f_0 = 1$.*

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Example 2 (Tensor-mesh). *Let $\Omega := [a_h, b_h] \times [a_v, b_v]$ be a rectangle in $\mathbb{R}^2$. Given $l, m \in \mathbb{N}$, $a_h = x_0 < \cdots < x_{l+1} = b_h$ and $a_v = y_0 < \cdots < y_{m+1} = b_v$. Then $\mathcal{T} = (\mathcal{T}_2, \mathcal{T}_1, \mathcal{T}_0)$, where*

$$\mathcal{T}_2 := \{[x_i, x_{i+1}] \times [y_j, y_{j+1}] : i = 0, \ldots, l, \ j = 0, \ldots, m\},$$
$$\mathcal{T}_1 := \{([x_i, x_{i+1}], y_j) : 0 \leq i \leq l, \ 0 \leq j \leq m+1\} \cup \{(x_i, [y_j, y_{j+1}]) : 0 \leq i \leq l+1, \ 0 \leq j \leq m\},$$
$$\mathcal{T}_0 := \{(x_i, y_j) : i = 0, \ldots, l+1, \ j = 0, \ldots, m+1\},$$

is called a tensor-mesh on Ω. In this case, we have $f_2 = (l+1)(m+1)$, $f_1^h = (l+1)m$, $f_1^v = l(m+1)$, $f_0 = lm$.

We will now define a generalized spline space over a T-mesh, where the smoothness of the elements of the space across the edges of the T-mesh is given. To this end, we first define what we mean by smoothness.

Definition 2 (Smoothness). *With each edge $\tau \in \mathcal{T}_1^o$, we associate an integer $r(\tau) \geq -1$. We say that $f \in C^{r(\tau)}(\tau)$ if the partial derivatives of f up to order $r(\tau)$ are continuous across the edge τ. We assume that $r(\tau) = r(\tau')$ for all τ, τ' lying on the same straight line, and we refer to this as the constant smoothness (along lines) assumption. Letting*

$$\boldsymbol{r} := \{\, r(\tau), \forall \tau \in \mathcal{T}_1^o \,\},$$

we call $\boldsymbol{r}$ a smoothness distribution on $\mathcal{T}$. We define the following class of smooth functions on Ω:

$$C^{\boldsymbol{r}}(\mathcal{T}) := \{\, f : \Omega \to \mathbb{R} : f \in C^{r(\tau)}(\tau), \ \forall \tau \in \mathcal{T}_1^o \,\}.$$

Given a smoothness distribution $\boldsymbol{r}$ on $\mathcal{T}$, with each vertex $\gamma \in \mathcal{T}_0^o$, we associate two integers $r_h(\gamma), r_v(\gamma)$, where $r_h(\gamma) := r(\tau_v)$ and $r_v(\gamma) := r(\tau_h)$ such that $\gamma = \tau_h \cap \tau_v$ and $\tau_h \in \mathcal{T}_1^{o,h}$, $\tau_v \in \mathcal{T}_1^{o,v}$. Note that the integers $r_h(\gamma), r_v(\gamma)$ are well defined by the constant smoothness (along lines) assumption.

Definition 3 (Extended T-mesh). *An extended T-mesh $(\mathcal{T}, \boldsymbol{r})$ is a T-mesh with a smoothness distribution $\boldsymbol{r}$. If $\mathcal{T}$ is a tensor-mesh (see Example 2), then a corresponding extended T-mesh is called an extended tensor-mesh.*

In the following, we denote by ℓ either h or v. We consider the spaces $\mathbb{P}_{p_\ell}^{U_\ell, V_\ell}$ as in (1), namely

$$\mathbb{P}_{p_\ell}^{U_\ell, V_\ell} := \langle 1, t, \ldots, t^{p_\ell - 2}, U_\ell(t), V_\ell(t) \rangle, \quad t \in [a_\ell, b_\ell], \quad 2 \leq p_\ell \in \mathbb{N}.$$

We assume that $U_\ell, V_\ell \in C^{p_\ell}([a_\ell, b_\ell])$, and

$$\dim\big(\mathbb{P}_{p_\ell}^{U_\ell, V_\ell}\big) = p_\ell + 1,$$

and for any element $\psi \in \mathbb{P}_{p_\ell}^{U_\ell, V_\ell}$,

- if $D_t^{p_\ell - 1}\psi(t_1) = D_t^{p_\ell - 1}\psi(t_2) = 0$, $t_1, t_2 \in [a_\ell, b_\ell]$, $t_1 \neq t_2$, then $D_t^{p_\ell - 1}\psi(t) = 0$, $t \in [a_\ell, b_\ell]$;

- if $D_t^{p_\ell - 1}\psi(t_1) = D_t^{p_\ell}\psi(t_1) = 0$, $t_1 \in (a_\ell, b_\ell)$, then $D_t^{p_\ell - 1}\psi(t) = 0$, $t \in [a_\ell, b_\ell]$.

We notice that these conditions imply that the space $\langle D_t^{p_\ell - 1}U_\ell, D_t^{p_\ell - 1}V_\ell \rangle$ is a Tchebycheff space on $[a_\ell, b_\ell]$ and an extended Tchebycheff space on (a_ℓ, b_ℓ). Note that this assures that the same holds for the space $\mathbb{P}_{p_\ell}^{U_\ell, V_\ell}$, see [8, 28].

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Let $p_h, p_v \in \mathbb{N}$ with $p_h, p_v \geq 2$. Then, we define the space

$$\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}} := \{\, q_h(x)q_v(y) \,:\, q_h \in \mathbb{P}_{p_h}^{U_h,V_h}([a_h, b_h]),\, q_v \in \mathbb{P}_{p_v}^{U_v,V_v}([a_v, b_v]) \,\}, \tag{2}$$

where $\boldsymbol{U} := (U_h, U_v)$, $\boldsymbol{V} := (V_h, V_v)$ and $\boldsymbol{p} := (p_h, p_v)$. If the space (2) is the space of bivariate algebraic polynomials of bi-degree $\boldsymbol{p}$, then it will be denoted by $\mathbb{P}_{\boldsymbol{p}}$. We are now ready to define the generalized spline space over a T-mesh.

Definition 4 (Generalized spline space). *Let $(\mathcal{T}, \boldsymbol{r})$ be an extended T-mesh, and let $p_h, p_v \in \mathbb{N}$ with $p_h, p_v \geq 2$. We define the space of generalized splines over the T-mesh $\mathcal{T}$, denoted by $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})$, as the space of functions in $C^{\boldsymbol{r}}(\mathcal{T})$ such that, restricted to each cell $\sigma \in \mathcal{T}_2$, they belong to $\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}$, i.e.,*

$$\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}) := \{\, s \in C^{\boldsymbol{r}}(\mathcal{T}) \,:\, s_{|\sigma} \in \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}},\, \sigma \in \mathcal{T}_2 \,\}.$$

In particular, in the case of bivariate algebraic polynomials,

$$\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{r}}(\mathcal{T}) := \{\, s \in C^{\boldsymbol{r}}(\mathcal{T}) \,:\, s_{|\sigma} \in \mathbb{P}_{\boldsymbol{p}},\, \sigma \in \mathcal{T}_2 \,\}.$$

Note that if the smoothness $r(\tau_v) \geq p_h$ associated with a vertical edge $\tau_v \in \mathcal{T}_1^{o,v}$ then for any two cells σ, σ' adjacent to τ_v we have

$$s_{|\sigma} = s_{|\sigma'}, \quad s \in \mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}).$$

This follows from the assumption that $\mathbb{P}_{p_h}^{U_h,V_h}$ is an extended Tchebycheff space on (a_h, b_h). A similar property holds for horizontal edges. Therefore, in the following we assume

$$r(\tau_v) < p_h, \quad \forall \tau_v \in \mathcal{T}_1^{o,v}, \quad r(\tau_h) < p_v, \quad \forall \tau_h \in \mathcal{T}_1^{o,h}.$$

3. Dimension of the generalized spline space

In this section we study the dimension of $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})$. Our arguments are based on homological techniques similar to the ones used in [26] for investigating the dimension of the space $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{r}}(\mathcal{T})$.

3.1. Properties of the section space $\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}$

First of all, we define and analyze some subspaces of $\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}$ that will be used in an alternative characterization of the spline space $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})$ as the kernel of a suitable linear map. This characterization will play a role in the analysis of the dimension of the spline space.

For any vertical edge τ of $\mathcal{T}$, we consider the following subspace of $\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}$:

$$\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\tau) := \{ q \in \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}} \,:\, D_x^i q(\bar{x}, y) \equiv 0, \forall y \in [a_v, b_v], i = 0, \ldots, r(\tau) \}, \tag{3}$$

where $\bar{x}$ is the abscissa of any point of τ. Analogously, for any horizontal edge τ we set

$$\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\tau) := \{ q \in \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}} \,:\, D_y^j q(x, \bar{y}) \equiv 0, \forall x \in [a_h, b_h], j = 0, \ldots, r(\tau) \}, \tag{4}$$

where $\bar{y}$ is the ordinate of any point of τ. Moreover, for any vertex $\gamma = (\bar{x}, \bar{y})$ we define the subspace

$$\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\gamma) := \{ q \in \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}} \,:\, D_x^i D_y^j q(\bar{x}, \bar{y}) \equiv 0, i = 0, \ldots, r_h(\gamma), j = 0, \ldots, r_v(\gamma) \}. \tag{5}$$

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Lemma 1. *The following dimension formulas hold:*

1. $\dim\left(\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}\right) = (p_h + 1) \times (p_v + 1)$;

2. $\dim\left(\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\tau)\right) = \begin{cases} (p_h + 1) \times (r(\tau) + 1), & \tau \;\; horizontal \\ (r(\tau) + 1) \times (p_v + 1), & \tau \;\; vertical \end{cases}$;

3. $\dim\left(\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\gamma)\right) = (r_h(\gamma) + 1) \times (r_v(\gamma) + 1)$.

Proof. Proving the first formula is trivial. In order to prove the second one, let us write a general element of $\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}$ in the form

$$q(x,y) = \sum_{i=0}^{p_h-2}\sum_{j=0}^{p_v-2} a_{i,j}x^iy^j + \sum_{i=0}^{p_h-2} b_i x^i U_v(y) + \sum_{i=0}^{p_h-2} c_i x^i V_v(y) + \sum_{j=0}^{p_v-2} d_j U_h(x)y^j + \sum_{j=0}^{p_v-2} e_j V_h(x)y^j$$
$$+ \alpha_{11}U_h(x)U_v(y) + \alpha_{12}U_h(x)V_v(y) + \alpha_{21}V_h(x)U_v(y) + \alpha_{22}V_h(x)V_v(y).$$

Suppose that τ is vertical (the proof for τ horizontal is analogous). An element belonging to $\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\tau)$ must then satisfy, by definition, the conditions

$$\sum_{i=l}^{p_h-2} a_{i,j}i(i-1)\cdots(i-l+1)\bar{x}^{i-l} + d_j D_x^l U_h(\bar{x}) + e_j D_x^l V_h(\bar{x}) = 0, \quad j = 0,\ldots,p_v-2,$$

$$\sum_{i=l}^{p_h-2} b_i i(i-1)\cdots(i-l+1)\bar{x}^{i-l} + \alpha_{11}D_x^l U_h(\bar{x}) + \alpha_{21}D_x^l V_h(\bar{x}) = 0,$$

$$\sum_{i=l}^{p_h-2} c_i i(i-1)\cdots(i-l+1)\bar{x}^{i-l} + \alpha_{12}D_x^l U_h(\bar{x}) + \alpha_{22}D_x^l V_h(\bar{x}) = 0,$$

for $l = 0,\ldots,r(\tau)$, where a sum is assumed to be empty whenever the lower index exceeds the upper one. Note that $\dim\left(\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\tau)\right)$ coincides with the rank of the matrix of this linear system. A suitable re-ordering of these equations allows us to show that the rank is $(r(\tau) + 1) \times (p_v + 1)$, by using the fact that the submatrix

$$\begin{bmatrix} D_x^{p_h-1}U_h(\bar{x}) & D_x^{p_h-1}V_h(\bar{x}) \\ D_x^{p_h}U_h(\bar{x}) & D_x^{p_h}V_h(\bar{x}) \end{bmatrix}$$

is non-singular (see, e.g., [8]). Similarly, to prove the third item of the lemma, we write an element of $\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}$ in the form

$$q(x,y) = \sum_{i=0}^{p_h-2}\sum_{j=0}^{p_v-2} a_{i,j}(x-\bar{x})^i(y-\bar{y})^j + \sum_{i=0}^{p_h-2} b_i(x-\bar{x})^i U_v(y) + \sum_{i=0}^{p_h-2} c_i(x-\bar{x})^i V_v(y)$$
$$+ \sum_{j=0}^{p_v-2} d_j U_h(x)(y-\bar{y})^j + \sum_{j=0}^{p_v-2} e_j V_h(x)(y-\bar{y})^j$$
$$+ \alpha_{11}U_h(x)U_v(y) + \alpha_{12}U_h(x)V_v(y) + \alpha_{21}V_h(x)U_v(y) + \alpha_{22}V_h(x)V_v(y).$$

An element belonging to $\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\gamma)$ must satisfy, by definition, the following conditions

$$\mu\eta a_{lk} + \mu b_l D_y^k U_v(\bar{y}) + \mu c_l D_y^k V_v(\bar{y}) + \eta d_k D_x^l U_h(\bar{x}) + \eta e_k D_x^l V_h(\bar{x}) + \alpha_{11}D_x^l U_h(\bar{x})D_y^k U_v(\bar{y})$$
$$+ \alpha_{12}D_x^l U_h(\bar{x})D_y^k V_v(\bar{y}) + \alpha_{21}D_x^l V_h(\bar{x})D_y^k U_v(\bar{y}) + \alpha_{22}D_x^l V_h(\bar{x})D_y^k V_v(\bar{y}) = 0,$$

for $0 \leq l \leq r_h(\gamma)$, $0 \leq k \leq r_v(\gamma)$, where

$$\mu = \begin{cases} l!, & l \leq p_h - 2 \\ 0, & l > p_h - 2 \end{cases}, \qquad \eta = \begin{cases} k!, & k \leq p_v - 2 \\ 0, & k > p_v - 2 \end{cases}.$$

The matrix of such a system has rank $\big(r_h(\gamma) + 1\big) \times \big(r_v(\gamma) + 1\big)$, see also [5], which completes the proof. $\qquad\qquad\square$

3.2. Topological chain complexes

As done in [26] in the algebraic polynomial case, we define the following complexes, see also [1] and [33]:

$$
\begin{array}{ccccccccc}
& & & & 0 & & 0 & & \\
& & & & \downarrow & & \downarrow & & \\
\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o): & & 0 & \xrightarrow{\hat{\partial}_2} & \displaystyle\bigoplus_{\tau \in \mathcal{T}_1^o} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\tau) & \xrightarrow{\hat{\partial}_1} & \displaystyle\bigoplus_{\gamma \in \mathcal{T}_0^o} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\gamma) & \xrightarrow{\hat{\partial}_0} & 0 \\
& & & & \downarrow & & \downarrow & & \\
\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}(\mathcal{T}^o): & 0 \xrightarrow{\partial_3} \displaystyle\bigoplus_{\sigma \in \mathcal{T}_2} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}} & \xrightarrow{\partial_2} & & \displaystyle\bigoplus_{\tau \in \mathcal{T}_1^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}} & \xrightarrow{\partial_1} & \displaystyle\bigoplus_{\gamma \in \mathcal{T}_0^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}} & \xrightarrow{\partial_0} & 0 \\
& & & & \downarrow & & \downarrow & & \\
\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o): & 0 \xrightarrow{\bar{\partial}_3} \displaystyle\bigoplus_{\sigma \in \mathcal{T}_2} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}} & \xrightarrow{\bar{\partial}_2} & & \displaystyle\bigoplus_{\tau \in \mathcal{T}_1^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\tau) & \xrightarrow{\bar{\partial}_1} & \displaystyle\bigoplus_{\gamma \in \mathcal{T}_0^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\gamma) & \xrightarrow{\bar{\partial}_0} & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

$$\tag{6}$$

The maps of the complex $\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}(\mathcal{T}^o)$ are induced by the usual boundary maps, so they are defined as follows. We consider all the edges $\tau \in \mathcal{T}_1$ oriented, and we represent them with the notation $\tau = [\gamma_1\gamma_2]$, where $\gamma_1, \gamma_2 \in \mathcal{T}_0$. The opposite edge is represented by $[\gamma_2\gamma_1]$, and by convention we set $[\gamma_1\gamma_2] = -[\gamma_2\gamma_1]$.

- The map ∂_3 is the identity map.

- The map $\partial_2 : \bigoplus_{\sigma \in \mathcal{T}_2} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}} \to \bigoplus_{\tau \in \mathcal{T}_1^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}$ is given by

$$\partial_2(q) = \bigoplus_{\tau \in \mathcal{T}_1^o} \sum_{\sigma \in S(\tau)} q_\sigma, \qquad q \in \bigoplus_{\sigma \in \mathcal{T}_2} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}},$$

  where, for any $\tau \in \mathcal{T}_1^o$, $S(\tau)$ is the set of the cells in $\mathcal{T}_2$ which contain τ, and for each cell $\sigma \in \mathcal{T}_2$, whose counter-clockwise boundary is formed by the edges $\tau_1 = [\gamma_1\gamma_2], \ldots, \tau_l = [\gamma_l\gamma_1]$, q_σ is the component of q associated with the cell σ if the boundary of σ contains τ and its opposite if the boundary of σ contains the opposite of τ.

- The map $\partial_1 : \bigoplus_{\tau \in \mathcal{T}_1^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}} \to \bigoplus_{\gamma \in \mathcal{T}_0^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}$ is given by

$$\partial_1(q) = \bigoplus_{\gamma \in \mathcal{T}_0^o} \sum_{\tau \in S(\gamma)} q_\tau, \qquad q \in \bigoplus_{\tau \in \mathcal{T}_1^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}},$$

  where, for any $\gamma \in \mathcal{T}_0^o$, $S(\gamma)$ is the set of the edges in $\mathcal{T}_1^o$ which have γ as one of the endpoints, and, for each oriented edge $\tau = [\gamma_1\gamma_2] \in \mathcal{T}_1^o$, q_τ is the component of q associated with τ if $\gamma = \gamma_2$ and its opposite if $\gamma = \gamma_1$.

- For any $q \in \bigoplus_{\gamma \in \mathcal{T}_0^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}$, $\partial_0(q) = 0$.

The maps of the complex $\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o)$, denoted by $\hat{\partial}_2$, $\hat{\partial}_1$ and $\hat{\partial}_0$, are obtained from ∂_2, ∂_1 and ∂_0 by restriction. Indeed, for any $\tau \in \mathcal{T}_1^o$ and $\gamma \in \mathcal{T}_0^o$, an element q_τ of $\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\tau)$ also belongs to $\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\gamma)$, provided that the edge τ has an endpoint in γ, and therefore $\sum_{\tau \in S(\gamma)} q_\tau$ belongs to $\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\gamma)$ as well. As a consequence, the image of the restriction of ∂_1 to $\bigoplus_{\tau \in \mathcal{T}_1^o} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\tau)$ is included in $\bigoplus_{\gamma \in \mathcal{T}_0^o} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\gamma)$.

The maps of $\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o)$, denoted by $\bar{\partial}_2$, $\bar{\partial}_1$ and $\bar{\partial}_0$, are naturally induced since the considered vector spaces are quotients of the ones of $\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}(\mathcal{T}^o)$.

Note that, by construction, we have $\hat{\partial}_i \circ \hat{\partial}_{i+1} = 0$, $\partial_i \circ \partial_{i+1} = 0$, $\bar{\partial}_i \circ \bar{\partial}_{i+1} = 0$, $i = 0, 1$.

The vertical maps in each column of the diagram in (6) are the inclusion and the quotient map, respectively.

We will now study the homology of the complexes. First, we recall the definition of homology.

Definition 5. *Given a complex*

$$\mathcal{A} : \cdots \to A_{i+1} \overset{\delta_{i+1}}{\to} A_i \overset{\delta_i}{\to} A_{i-1} \cdots$$

the i-homology is defined as $H_i(\mathcal{A}) := \ker \delta_i / \operatorname{im} \delta_{i+1}$.

Our interest in the homology of the complexes in (6) is motivated by the fact that the homology of the cells in $\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o)$ is related to the space $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})$. More precisely, the following result holds.

Proposition 1. *It holds*

$$H_2(\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o)) = \ker \bar{\partial}_2 = \mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}). \tag{7}$$

Proof. The proof is a straightforward extension of the proof of Proposition 2.9 in [26], see also Theorems 2.4 and 3.2 in [1]. Nevertheless, for the sake of completeness, we detail the short and simple argument.

From Definition 5 and from (6) we have $H_2(\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o)) = \ker \bar{\partial}_2$. An element in $\ker \bar{\partial}_2$ is a collection of functions $(q_\sigma)_{\sigma \in \mathcal{T}_2}$ where $q_\sigma \in \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}$ and $q_\sigma - q_{\sigma'} \in \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\tau)$ if σ, σ' share the internal edge τ. By (3)–(4), this implies that the piecewise function which is q_σ on σ and $q_{\sigma'}$ on σ' belongs to $C^{r(\tau)}(\tau)$ (see Definition 2). As this is true for all the interior edges, any piecewise function $(q_\sigma)_{\sigma \in \mathcal{T}_2} \in \ker \bar{\partial}_2$ is of class $C^{\boldsymbol{r}}(\mathcal{T})$, that is an element of $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})$. $\square$

The next results address the exactness of $\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}(\mathcal{T}^o)$, and so they are of interest for determining a dimension formula for $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})$. They can be proved with the same line of arguments as considered in [26] to prove Propositions D.1–D.3 for the algebraic polynomial case. Indeed, their proofs are just based on general properties of complexes and on the topological features of the T-mesh. For this reason, we omit the corresponding technical proofs.

Proposition 2. *It holds*

$$H_0(\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}(\mathcal{T}^o)) = 0;$$

$$H_1(\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}(\mathcal{T}^o)) = 0;$$

$$H_2(\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}(\mathcal{T}^o)) = \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}.$$

Finally, the next proposition extends Lemma 2.2 and Proposition 2.7 in [26] to the generalized B-spline case. Their proofs are again based on general properties of complexes and on the topological features of the T-mesh, just like in [26]; therefore we omit them.

Proposition 3. *It holds*

$$H_0(\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}(\mathcal{T}^o)) = H_0(\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o)) = 0; \tag{8}$$

$$H_1(\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o)) = H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o)). \tag{9}$$

3.3. Dimension of the spline space $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})$

By using the above results, we are finally able to give a dimension formula for the generalized spline space over an extended T-mesh $(\mathcal{T},\boldsymbol{r})$.

Theorem 1. *Given an extended T-mesh* $(\mathcal{T},\boldsymbol{r})$, *we have*

$$\dim\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})\big) = \sum_{\sigma \in \mathcal{T}_2} (p_h + 1)(p_v + 1)$$

$$- \sum_{\tau \in \mathcal{T}_1^{o,h}} (p_h + 1)(r(\tau) + 1) - \sum_{\tau \in \mathcal{T}_1^{o,v}} (r(\tau) + 1)(p_v + 1)$$

$$+ \sum_{\gamma \in \mathcal{T}_0^o} (r_h(\gamma) + 1)(r_v(\gamma) + 1) + \dim\big(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o))\big). \tag{10}$$

Proof. If we consider the Euler characteristic of the complex

$$\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o): \ 0 \xrightarrow{\bar{\partial}_3} \bigoplus_{\sigma \in \mathcal{T}_2} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}} \xrightarrow{\bar{\partial}_2} \bigoplus_{\tau \in \mathcal{T}_1^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\tau) \xrightarrow{\bar{\partial}_1} \bigoplus_{\gamma \in \mathcal{T}_0^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\gamma) \xrightarrow{\bar{\partial}_0} 0,$$

we get the relation

$$\dim\big(\bigoplus_{\sigma \in \mathcal{T}_2} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}\big) - \dim\big(\bigoplus_{\tau \in \mathcal{T}_1^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\tau)\big) + \dim\big(\bigoplus_{\gamma \in \mathcal{T}_0^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\gamma)\big)$$

$$= \dim\big(H_2(\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o))\big) - \dim\big(H_1(\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o))\big) + \dim\big(H_0(\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o))\big).$$

By combining (7), (8), (9) with the above equality, we arrive at the formula (10). $\qquad\square$

Remark 1. *Since* $\dim\big(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o))\big) \geq 0$, *from Theorem 1 we easily get the same lower bound for* $\dim\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})\big)$ *as in the algebraic polynomial case, see [26, Section 3],*

$$\dim\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})\big) \geq \sum_{\sigma \in \mathcal{T}_2} (p_h + 1)(p_v + 1)$$

$$- \sum_{\tau \in \mathcal{T}_1^{o,h}} (p_h + 1)(r(\tau) + 1) - \sum_{\tau \in \mathcal{T}_1^{o,v}} (r(\tau) + 1)(p_v + 1)$$

$$+ \sum_{\gamma \in \mathcal{T}_0^o} (r_h(\gamma) + 1)(r_v(\gamma) + 1). \tag{11}$$

We say that a smoothness distribution $\boldsymbol{r}$ on $\mathcal{T}$ is *constant* if there exist two integers ϱ_h, ϱ_v such that

$$r(\tau_v) = \varrho_h, \quad \forall \tau_v \in \mathcal{T}_1^{o,v}, \quad r(\tau_h) = \varrho_v, \quad \forall \tau_h \in \mathcal{T}_1^{o,h}. \tag{12}$$

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Example 3. *Given an extended T-mesh $(\mathcal{T}, \boldsymbol{r})$ with a constant smoothness distribution $\boldsymbol{r}$ as in (12), the dimension formula (10) simplifies to*

$$\dim\!\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})\big) = (p_h+1)(p_v+1)f_2 - (p_h+1)(\varrho_v+1)f_1^h$$
$$- (\varrho_h+1)(p_v+1)f_1^v + (\varrho_h+1)(\varrho_v+1)f_0$$
$$+ \dim\!\big(H_0(\mathfrak{J}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}^o))\big). \tag{13}$$

The formula (13) corresponds to the formula found in [26] for the algebraic polynomial spline space defined on the same extended T-mesh.

Under certain conditions on the T-mesh and/or the generalized spline space, the homology term in the dimension formula (10) is zero, so that the dimension of $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})$ agrees with the lower bound in (11). In the following we discuss some examples.

Example 4. *For an extended tensor-mesh $(\mathcal{T}, \boldsymbol{r})$, one can easily check that the dimension of $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})$ agrees with the lower bound in (11). Taking a tensor-mesh $\mathcal{T}$ as in Example 2 and a constant smoothness distribution $\boldsymbol{r}$ as in (12), the dimension formula (13) simplifies to*

$$\dim\!\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})\big) = \big((p_h+1)(l+1) - l(\varrho_h+1)\big)\big((p_v+1)(m+1) - m(\varrho_v+1)\big).$$

In the algebraic polynomial context, it is known (see [26, 29]) that if the degree is large enough with respect to the smoothness, then the dimension of $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{r}}(\mathcal{T})$ agrees with the lower bound in (11). This extends to the generalized spline setting, as stated in Corollary 1. First, we recall from [29] the concept of a cycle (see Figure 3 for an illustration).

Definition 6 (Cycle). *A segment of a T-mesh is called a composite edge if all the vertices lying in its interior are T-vertices and if it cannot be extended to a longer segment with the same property. A sequence $\gamma_1, \ldots, \gamma_n$ of T-vertices in a T-mesh $\mathcal{T}$ is said to form a cycle if γ_i lies in the interior of a composite edge of $\mathcal{T}$ having one of its endpoints at γ_{i+1} (we assume $\gamma_{n+1} = \gamma_1$).*

A relevant class of T-meshes without cycles are the so-called LR-meshes, see the next section for their definition. Note that they are called hierarchical T-meshes in [26, Section 4.1].

Corollary 1. *Let $\mathcal{T}$ be a T-mesh without cycles, and let $\boldsymbol{r}$ be a constant smoothness distribution on $\mathcal{T}$ as in (12). Then, if we assume that $p_h \geq 2\varrho_h + 1$ and $p_v \geq 2\varrho_v + 1$, we have $\dim\!\big(H_0(\mathfrak{J}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}))\big) = 0$.*

Proof. From Theorem 3.7 in [5] we know that $\dim\!\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})\big)$ equals the lower bound in (11), so $\dim\!\big(H_0(\mathfrak{J}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T}))\big) = 0$. $\square$

Example 5. *Consider the T-mesh $\mathcal{T}$ in Figure 2, and the constant smoothness distribution $\boldsymbol{r}$ on $\mathcal{T}$ as in (12) with $\varrho_h = \varrho_v = 1$. The space $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})$ with $\boldsymbol{p} = (3,3)$ has dimension 28. This immediately follows from the mesh numbers in Example 1, the dimension formula in Example 3 and Corollary 1.*

Example 4 and Corollary 1 show that for a large class of (generalized) spline spaces we have

$$\dim\!\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V},\boldsymbol{r}}(\mathcal{T})\big) = \dim\!\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{r}}(\mathcal{T})\big). \tag{14}$$

We conjecture that (14) is true for all generalized spline spaces on T-meshes, at least generically. The latter means that if for a given space $\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{U},\boldsymbol{V}}$ and an extended T-mesh $(\mathcal{T}, \boldsymbol{r})$ the equality in (14) does not hold, then there exists an arbitrarily small perturbation of the vertices of $\mathcal{T}$ making the equality true. This is also inspired by the concept of generic embeddings, used in [1] in the context of dimensions of polynomial spline spaces on triangulations.

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4. Locally refined generalized splines

In the context of algebraic polynomial splines, the well-known knot insertion process of tensor-product B-splines gives rise to LR B-splines [12]. In this section we extend this construction to the generalized spline setting.

4.1. Generalized B-splines

Let Ξ be a sequence of knots over the interval $[a, b]$,

$$\Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{m+p+1}\}, \quad m, p \in \mathbb{N}. \tag{15}$$

It is well known that it is possible to construct B-spline-like functions with sections in spaces $\mathbb{P}_p^{U,V}$ as in (1), see [18, 23, 37] and references therein. The so-called *generalized B-splines* *(GB-splines) of degree p*, defined over the knot sequence (15), will be denoted by

$$B_{[\xi_i,\ldots,\xi_{i+p+1}]}^{(p)}.$$

They can be defined by means of the following recurrence relation:

$$B_{[\xi_i,\xi_{i+1},\xi_{i+2}]}^{(1)}(t) := \begin{cases} \frac{V^{(p-1)}(t)}{V^{(p-1)}(\xi_{i+1})}, & \text{if } t \in [\xi_i, \xi_{i+1}), \\ \frac{U^{(p-1)}(t)}{U^{(p-1)}(\xi_{i+1})}, & \text{if } t \in [\xi_{i+1}, \xi_{i+2}), \\ 0, & \text{elsewhere}, \end{cases}$$

and

$$B_{[\xi_i,\ldots,\xi_{i+p+1}]}^{(p)}(t) := d_{i,\Xi}^{(p-1)} \int_{-\infty}^{t} B_{[\xi_i,\ldots,\xi_{i+p}]}^{(p-1)}(s)\mathrm{d}s - d_{i+1,\Xi}^{(p-1)} \int_{-\infty}^{t} B_{[\xi_{i+1},\ldots,\xi_{i+p+1}]}^{(p-1)}(s)\mathrm{d}s, \quad p \geq 2,$$

where

$$d_{i,\Xi}^{(p)} := \frac{1}{\int_{-\infty}^{+\infty} B_{[\xi_i,\ldots,\xi_{i+p+1}]}^{(p)}(s)\mathrm{d}s},$$

and fractions with zero denominators are considered to be zero. The knot sequence (15) allows us to define m GB-splines of degree p, namely

$$B_{[\xi_1,\ldots,\xi_{p+2}]}^{(p)}, \ldots, B_{[\xi_m,\ldots,\xi_{m+p+1}]}^{(p)}.$$

GB-splines possess all desirable properties of classical polynomial B-splines [2, 18]. We collect them in the following proposition.

Proposition 4. *Let* $B_{[\xi_i,\ldots,\xi_{i+p+1}]}^{(p)}$, $i = 1, \ldots, m$ *be GB-splines of degree* $p \geq 2$ *associated with the knot sequence (15). Then, the following properties hold:*

- *piecewise structure:* $B_{[\xi_i,\ldots,\xi_{i+p+1}]}(t) \in \mathbb{P}_p^{U,V}$, $t \in [\xi_j, \xi_{j+1})$;

- *positivity:* $B_{[\xi_i,\ldots,\xi_{i+p+1}]}^{(p)}(t) \geq 0$;

- *partition of unity:* $\sum_{i=1}^{m} B_{[\xi_i,\ldots,\xi_{i+p+1}]}^{(p)}(t) = 1$, $t \in [\xi_{p+1}, \xi_{m+1})$;

- *compact support:* $B_{[\xi_i,\ldots,\xi_{i+p+1}]}^{(p)}(t) = 0$, $t \notin [\xi_i, \xi_{i+p+1}]$;

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- smoothness: $B^{(p)}_{[\xi_i,\dots,\xi_{i+p+1}]}(t)$ is $p - \mu_j$ times continuously differentiable at ξ_j being μ_j the multiplicity of ξ_j in the knot sequence;

- local linear independence: $B^{(p)}_{[\xi_{i-p},\dots,\xi_{i+1}]}(t),\dots,B_{[\xi_i,\dots,\xi_{i+p+1}]}(t)$ are linearly independent on $[\xi_i, \xi_{i+1})$.

The following formula for inserting one knot follows from Theorem 5.5 in [21]:

Proposition 5. *Suppose for some integers* i, m *with* $i \leq m \leq i + p$ *that* $\xi \in (\xi_i, \xi_{i+p+1}) \cap [\xi_m, \xi_{m+1})$, *and let* $\eta_i, \dots, \eta_{i+p+2}$ *be the numbers* $\xi, \xi_i, \dots, \xi_{i+p+1}$ *sorted in nondecreasing order. Then, we have*

$$
B_{[\xi_i,\dots,\xi_{i+p+1}]} = \begin{cases} B_{[\eta_i,\dots,\eta_{i+p+1}]}, & i \leq m - p, \\ \lambda_i B_{[\eta_i,\dots,\eta_{i+p+1}]} + (1 - \lambda_{i+1}) B_{[\eta_{i+1},\dots,\eta_{i+p+2}]}, & m - p + 1 \leq i \leq m, \\ B_{[\eta_{i+1}\dots,\eta_{i+p+2}]}, & i > m, \end{cases}
$$

where $0 \leq \lambda_j \leq 1$ *for* $j = i, \dots, i + p$. *In particular,* $\lambda_{m-p+1} = 1$ *and* $\lambda_{m+1} = 0$.

Explicit formulas for the λ_j's are given in [21]. For alternative, less general, but easier to compute formulas we refer to [37]. The spaces (1) also support a degree-raising process.

Multivariate versions of GB-splines can be obtained straightforwardly by the usual tensor-product approach.

4.2. Extended LR-mesh

An LR-mesh is a T-mesh constructed from a tensor-mesh by applying successive refinements. In each refinement at least one cell is split in two by an axis-aligned segment, see [12]. Since we use continuity across knot lines instead of multiplicities we repeat some definitions from [12].

Definition 7 (Split). *Let* $\mathcal{T} = (\mathcal{T}_2, \mathcal{T}_1, \mathcal{T}_0)$ *be a T-mesh and let* ε *be an axis-aligned segment. We say that* ε *splits* $\sigma \in \mathcal{T}_2$ *if* $\sigma \setminus \varepsilon$ *is not connected. We say that* ε *is a minimal split of* σ *if it splits* σ *and* $\varepsilon \subseteq \sigma$. *We say that* ε *splits* $\mathcal{T}$ *if* $\varepsilon = \tau_1 \cup \cdots \cup \tau_n$, *a finite union, with each* τ_i *either a minimal split of a cell in* $\mathcal{T}_2$ *or an edge in* $\mathcal{T}_1$.

If ε *splits* $\sigma \in \mathcal{T}_2$ *then* $\sigma \setminus \varepsilon$ *has two connected components* σ_1 *and* σ_2 *with closures* $\overline{\sigma}_1, \overline{\sigma}_2$. *Moreover, denote by* $\mathcal{T}_2(\varepsilon)$ *the set of cells in* $\mathcal{T}_2$ *that are split by* ε. *We define*

$$
\mathcal{T} + \varepsilon := (\mathcal{T}_2 + \varepsilon, \mathcal{T}_1 + \varepsilon, \mathcal{T}_0 + \varepsilon), \quad \mathcal{T}_2 + \varepsilon := (\mathcal{T}_2 \setminus \mathcal{T}_2(\varepsilon)) \cup \left(\bigcup_{\sigma \in \mathcal{T}_2(\varepsilon)} \{\overline{\sigma}_1, \overline{\sigma}_2\} \right),
$$

and where $\mathcal{T}_1 + \varepsilon$, $\mathcal{T}_0 + \varepsilon$ *are the set of edges and vertices obtained after* ε *has been introduced in the mesh. (See [12] for a more precise definition of* $\mathcal{T}_1 + \varepsilon$, $\mathcal{T}_0 + \varepsilon$). *Similarly, we define the sets* $\mathcal{T}_1^o + \varepsilon$, $\mathcal{T}_0^o + \varepsilon$.

Definition 8 (Extended split). *Let* ε *be a split of an extended T-mesh* $(\mathcal{T}, \boldsymbol{r})$ *and let* $r^\varepsilon \geq -1$ *be an integer defining the smoothness of the edges of* $\mathcal{T} + \varepsilon$ *belonging to* ε. *We associate with each edge* τ *of* $\mathcal{T} + \varepsilon$ *an integer* $r_\varepsilon(\tau)$ *given by*

$$
r_\varepsilon(\tau) := \begin{cases} r(\tau'), & \text{if } \tau \subseteq \tau' \in \mathcal{T}_1^o, \\ r^\varepsilon, & \text{if } \tau \in (\mathcal{T}_1^o + \varepsilon) \setminus \mathcal{T}_1^o. \end{cases}
$$

In addition, we have to assume that r^ε is such that the constant smoothness (along lines) assumption is satisfied, see Definition 2. We call $(\mathcal{T} + \varepsilon, \boldsymbol{r}_\varepsilon)$ an extended split of $(\mathcal{T}, \boldsymbol{r})$, where

$$\boldsymbol{r}_\varepsilon := \{\, r_\varepsilon(\tau) : \tau \in \mathcal{T}_1^o + \varepsilon \,\}.$$

Definition 9 (Extended LR-mesh). *An extended LR-mesh $(\mathcal{T}, \boldsymbol{r})$ is an extended T-mesh, where either*

1. *it is an extended tensor-mesh, or*
2. *it is an extended split of an extended LR-mesh.*

The corresponding mesh $\mathcal{T}$ is called an LR-mesh.

The T-mesh in Figure 3 is not an LR-mesh. Indeed, starting with the boundary of the rectangle there is no way we can insert one of the line segments so that it splits the rectangle in two elements (it has a cycle, see Definition 6). Figure 4 shows the construction of an LR-mesh obtained by successive insertion of splits.

4.3. LR GB-splines

LR GB-splines are a collection of minimally supported functions on an LR-mesh. Each of them is a tensor-product GB-spline defined on a local tensor-mesh which is part of the LR-mesh. A formal definition is given in the following.

Definition 10 (Tensor-submesh). *Given $x_0 < \cdots < x_{k+1}$ and $y_0 < \cdots < y_{n+1}$, we consider the tensor-mesh $\mathcal{S}$ with vertices (x_i, y_j), $i = 0, \ldots, k+1$, $j = 0, \ldots, n+1$. If the edges of this tensor-mesh are completely covered by edges belonging to some T-mesh $\mathcal{T}$, we say that $\mathcal{S}$ is a tensor-submesh of $\mathcal{T}$. It is a maximal tensor-submesh if no other segment of $\mathcal{T}$ completely crosses the domain defined by $\mathcal{S}$, namely the rectangle $[x_0, x_{k+1}] \times [y_0, y_{n+1}]$.*

Definition 11 (Extended tensor-submesh). *Given an extended T-mesh $(\mathcal{T}, \boldsymbol{r})$, degrees p_h, p_v, and a set of integers $\bar{\boldsymbol{r}} := \{\, -1 \leq \bar{r}_{h,i} < p_h, \ i = 0, \ldots, k+1 \,\} \cup \{\, -1 \leq \bar{r}_{v,j} < p_v, \ j = 0, \ldots, n+1 \,\}$, we say that $(\mathcal{S}, \bar{\boldsymbol{r}})$ is an extended tensor-submesh of $(\mathcal{T}, \boldsymbol{r})$ if*

$$\bar{r}_{h,i} = r(\tau), \ \text{if } \tau \in \mathcal{T}_1^o, \tau \text{ belonging to the line } x = x_i;$$
$$\bar{r}_{v,j} = r(\tau), \ \text{if } \tau \in \mathcal{T}_1^o, \tau \text{ belonging to the line } y = y_j.$$

We say that an extended tensor-submesh has minimal support with respect to $\bar{\boldsymbol{r}}, p_h, p_v$ if it is maximal and

$$\sum_{i=0}^{k+1} (p_h - \bar{r}_{h,i}) = p_h + 2, \quad \sum_{j=0}^{n+1} (p_v - \bar{r}_{v,j}) = p_v + 2.$$

Definition 12 (Minimal support). *Let $(\mathcal{T}, \boldsymbol{r})$ be an extended T-mesh in $\mathbb{R}^2$. A tensor-product GB-spline $B : \mathbb{R}^2 \to \mathbb{R}$ has minimal support on $(\mathcal{T}, \boldsymbol{r})$ if it is defined on an extended tensor-submesh $(\mathcal{S}, \bar{\boldsymbol{r}})$ of minimal support.*

The function B in Definition 12 is given by

$$B(x, y) = B_{[\xi_0, \ldots, \xi_{p_h+1}]}^{(p_h)}(x) B_{[\zeta_0, \ldots, \zeta_{p_v+1}]}^{(p_v)}(y),$$

where

$$\xi_0, \ldots, \xi_{p_h+1} = \overbrace{x_0, \ldots, x_0}^{p_h - \bar{r}_{h,0}}, \ldots, \overbrace{x_{k+1}, \ldots, x_{k+1}}^{p_h - \bar{r}_{h,k+1}},$$

$$\zeta_0, \ldots, \zeta_{p_v+1} = \overbrace{y_0, \ldots, y_0}^{p_v - \bar{r}_{v,0}}, \ldots, \overbrace{y_{n+1}, \ldots, y_{n+1}}^{p_v - \bar{r}_{v,n+1}}.$$

As a consequence, the support of the GB-spline B is the rectangle $[x_0, x_{k+1}] \times [y_0, y_{n+1}]$.

Definition 13 (LR GB-splines). *If $(\mathcal{T}, \boldsymbol{r})$ is an extended LR-mesh, then the tensor-product GB-splines of minimal support on $(\mathcal{T}, \boldsymbol{r})$ are called LR GB-splines.*

Given an extended LR-mesh $(\mathcal{T}, \boldsymbol{r})$ and bi-degree $\boldsymbol{p} := (p_h, p_v)$, we now construct a collection $\mathcal{B}$ of LR GB-splines of degree $\boldsymbol{p}$ on $(\mathcal{T}, \boldsymbol{r})$. We recall that $(\mathcal{T}, \boldsymbol{r})$ is defined as a sequence of extended LR-meshes $(\mathcal{M}_1, \boldsymbol{r}_1), \ldots, (\mathcal{M}_n, \boldsymbol{r}_n)$ where $(\mathcal{M}_1, \boldsymbol{r}_1)$ is an extended tensor-mesh and $(\mathcal{T}, \boldsymbol{r}) = (\mathcal{M}_n, \boldsymbol{r}_n)$. We start with the complete collection $\mathcal{B}_1$ of tensor-product GB-splines of degree $\boldsymbol{p}$ on $(\mathcal{M}_1, \boldsymbol{r}_1)$. Suppose we have constructed the collection of LR GB-splines $\mathcal{B}_i$ on $(\mathcal{M}_i, \boldsymbol{r}_i)$ for some $1 \leq i < n$. To form the new mesh $\mathcal{M}_{i+1}$, an axis-aligned segment ε_i is inserted which we assume is long enough to split the support of at least one GB-spline in $\mathcal{B}_i$. Following [12], we construct the new collection of LR GB-splines $\mathcal{B}_{i+1}$ on $(\mathcal{M}_{i+1}, \boldsymbol{r}_{i+1})$ by a sequence of updates.

1. As long as there is a $B \in \mathcal{B}_i$ that does not have minimal support on $(\mathcal{M}_{i+1}, \boldsymbol{r}_{i+1})$, there must be at least a horizontal and/or vertical segment which is a union of edges of $\mathcal{M}_{i+1}$ that splits the support of B. Then, we proceed as follows. Suppose ε_i is such a vertical segment that splits the support of B at a point $x = \xi$. If $B(x, y) = B_1(x)B_2(y)$ then we insert ξ in the univariate GB-spline B_1 using Proposition 5 and get two univariate GB-splines $B_{1,1}$ and $B_{1,2}$, and two tensor-product GB-splines obtained from B by replacing B_1 by $B_{1,1}$ and $B_{1,2}$, respectively. We update $\mathcal{B}_j$ by removing B and adding the two new tensor-product GB-splines. We also remove duplicate GB-splines if necessary. A horizontal segment is handled analogously by inserting a knot in the univariate GB-spline B_2.

2. When all $B \in \mathcal{B}_i$ have minimal support we set $\mathcal{B}_{i+1} = \mathcal{B}_i$.

We refer to [12, Section 3.2] for a bilinear example.

It has been shown in [12, Theorem 3.4] that the final collection of LR B-splines $\mathcal{B} = \mathcal{B}_n$ only depends on the final mesh $\mathcal{T}$, and its proof generalizes to our LR GB-splines. With a proper scaling, LR B-splines form a positive partition of unity, see [12, Theorem 7.2], and the same also holds for our LR GB-splines.

Under certain conditions, the LR GB-splines span the full space of generalized splines on an extended LR-mesh. For example, we have the following result. Its proof follows the same line of arguments as in [12, Theorem 5.2].

Theorem 2. *Let $(\mathcal{M}_1, \boldsymbol{r}_1), \ldots, (\mathcal{M}_n, \boldsymbol{r}_n)$ be a sequence of extended LR-meshes with corresponding collections of LR GB-splines $\mathcal{B}_1, \ldots, \mathcal{B}_n$ of degree $\boldsymbol{p}$. If*

$$\dim\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{r}_i}(\mathcal{M}_i)\big) = \dim\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{r}_{i-1}}(\mathcal{M}_{i-1})\big) + 1, \quad i = 2, \ldots, n, \tag{16}$$

then

$$\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{r}_i}(\mathcal{M}_i) = \langle B \in \mathcal{B}_i \rangle, \quad i = 1, \ldots, n.$$

In general, LR GB-splines are not always linearly independent, as already known for polynomial LR-splines, see, e.g., [12, Example 6.4]. The dimension theory for GB-splines on T-meshes (described in Section 3) applies to LR GB-splines, and can be used to check their linear independence. Moreover, the dimension results can be of help in the design of (local) refinement algorithms ensuring collections of linearly independent LR GB-splines.

Example 6. *It is clear that the collection of LR GB-splines defined over an extended tensor-mesh are linearly independent as they are nothing else than tensor-product GB-splines.*

Example 7. *Consider a sequence of extended LR-meshes $(\mathcal{M}_1, \boldsymbol{r}_1), \ldots, (\mathcal{M}_n, \boldsymbol{r}_n)$ satisfying (16). From Theorem 2 it follows that linear independence is ensured if the number of LR GB-splines increases by one going from $(\mathcal{M}_{i-1}, \boldsymbol{r}_{i-1})$ to $(\mathcal{M}_i, \boldsymbol{r}_i)$, $i = 2, \ldots, n$.*

There are also other criteria that are sufficient for ensuring that a collection of LR GB-splines is linearly independent. For example, the so-called *peeling strategy* presented in [12, Section 6] can be used in the GB-spline context as well.

5. Conclusions

In this paper we have considered generalized spline spaces over planar T-meshes, and we have shown that they share several structural properties with polynomial spline spaces over the same partitions. First, we have provided some new insights into the problem of determining the dimension of a generalized spline space defined on a prescribed T-mesh for a given degree and smoothness, by extending the homological approach of [26] to generalized splines. Second, we have shown that the construction of LR-splines presented in [12] can be easily extended to the generalized spline context and we have outlined this construction.

It is worth emphasizing that, besides the structural similarities presented here, many other properties can be carried over from polynomial to generalized spline spaces. In particular, it has already been shown that the construction and properties of (truncated) hierarchical B-splines and analysis-suitable T-splines also extend to the generalized spline context, see [14, 34] and [3, 4] respectively.

Finally, we remark that generalized splines are a special case of the wider and very interesting class of Tchebycheff splines [21, 27, 28]. Tchebycheff splines are smooth piecewise functions with sections in Tchebycheff spaces, and admit a B-spline-like basis in the same way as generalized splines. Therefore, it is reasonable to believe that the results presented in this paper are also valid for Tchebycheff splines.

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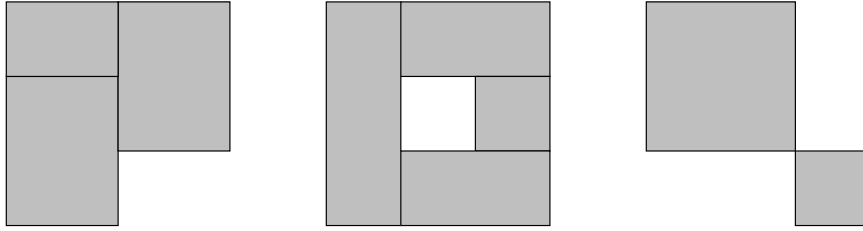
Figure 1: Left: a simply connected region with connected interior. Center: a region which is not simply connected. Right: a region where the interior is not connected.
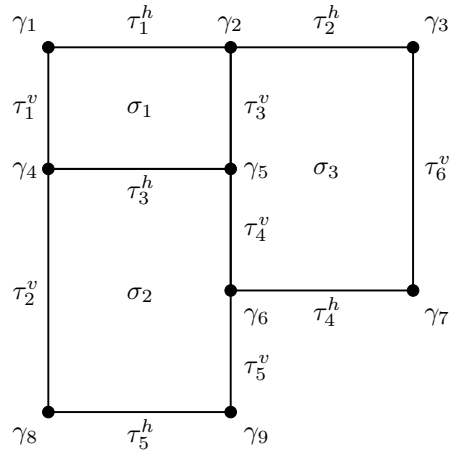

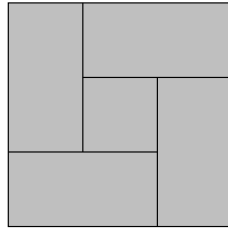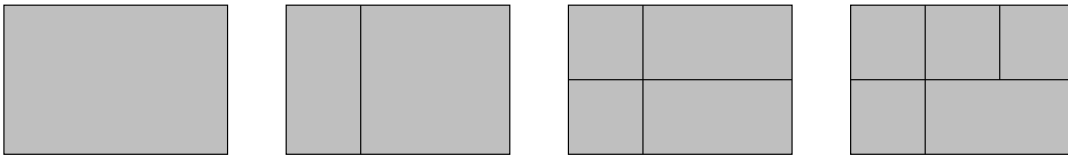Figure 2: Example of a T-mesh.

Figure 3: A T-mesh with a cycle, so it is not an LR-mesh.

Figure 4: A sequence of LR-meshes.