Existence and non-existence of limit cycles for Liénard prescribed curvature equations

Jean Mawhin[a,*], Gabriele Villari[b], Fabio Zanolin[c]

[a] *Institut de Recherche en Mathématique et Physique*
Université Catholique de Louvain
chemin du Cyclotron, 2, 1348 Louvain-la-Neuve, Belgium

[b] *Dipartimento di Matematica e informatica "Ulisse Dini"*
Università degli Studi di Firenze
viale Morgagni, 67/A, 50137 Firenze, Italy

[c] *Dipartimento di Scienze Matematiche, Informatiche e Fisiche*
Università degli Studi di Udine
via delle Scienze 206, 33100 Udine, Italy

Abstract

We study the curvature Liénard equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \lambda f(x)\dot{x} + g(x) = 0, \quad \lambda > 0$$

under different conditions on $f(x)$ and $g(x)$ and prove results of existence and non-existence of limit cycles. Applications are given to the Van der Pol type curvature equations.

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1. Introduction

Since the pioneering work of Liénard [11] in 1928, the literature devoted to the study of periodic solutions of Liénard differential equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

* Corresponding author

Email addresses: jean.mawhin@uclouvain.be (Jean Mawhin), villari@math.unifi.it (Gabriele Villari), fabio.zanolin@uniud.it (Fabio Zanolin)

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under various conditions upon the continuous functions f and g is enormous.
In recent years there has been a growing interest toward the study of generalizations of the equation, such as

$$\frac{d}{dt}\phi(\dot{x}) + f(x)\dot{x} + g(x) = 0$$

in which a nonlinear differential operator is involved. In particular, when Newton's acceleration $\ddot{x}$ is replaced by relativistic one, namely for the equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + f(x)\dot{x} + g(x) = 0,$$

the problem has been recently considered in [16] and [13], where existence conditions for nontrivial periodic solutions have been given.

In this paper, we consider the similar looking prescribed curvature equation of Liénard type

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \lambda f(x)\dot{x} + g(x) = 0 \tag{1.1}$$

with $xg(x) > 0$ for $x \neq 0$ and $\lambda > 0$. Moreover, like in Van der Pol equation, $f(x)$ is negative for $\alpha < 0 < \beta$ and positive for $x < \alpha$ and $x > \beta$.

In Section 2 equation (1.1) is reduced to a planar system whose main feature is to be defined only on the strip $\mathbb{R}\times\,]-1,1[$. To control this fact is actually the main problem of the whole paper. The local existence and uniqueness of the Cauchy problem, and the continuous dependence with respect to λ, is proved in the same strip in Lemma 2.1, when f is continuous and g locally Lipschitzian.

The associated Duffing system ($f \equiv 0$) is studied in Section 3, where the various shapes of the level curves for the energy integral are analyzed, depending upon the asymptotic behavior of the indefinite integral of g. A feature of the energy integral is to be defined on the closed strip $\mathbb{R}\times[-1,1]$. This provides new insights on generalized solutions of bounded variation type for differential equations with prescribed curvature, already considered, in other situations and through other approaches, following [1, 6, 12], in [2, 3, 5, 8, 10, 14] for example. However, dealing with (1.1), we restrict ourselves to the search of *classical periodic solutions*, namely closed curves in the strip-domain $\mathbb{R}\times\,]-1,1[$. This constraint leads to non-existence results which do not appear in the study of the Liénard equation in the whole plane.

Section 4 considers equation (1.1) when the limits of G at $\pm\infty$, supposed to be equal, are greater than one. Theorem 4.1 gives conditions under which the equation has no nontrivial periodic solution. A consequence is the non-existence of closed orbits for the k-Van der Pol prescribed curvature equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \lambda(x^2-k^2)\dot{x} + g(x) = 0,$$

when $k^2 \geq 2$ in Corollary 5.1.

The same equation is considered in Section 5 where, in Theorem 5.1, the existence of a closed orbit is proved when k is sufficiently small. The Van der Pol prescribed curvature equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \lambda(x^2-1)\dot{x} + g(x) = 0,$$

is shown to have no nontrivial periodic solution when $\lambda > 2/3$, and numerical results suggest that the results holds true also for the smaller positive values of λ.

We return to equation (1.1) in Section 6, which deals with the case where $G(\pm\infty) < 1$. In Theorem 6.1, conditions are given for the existence as well as for the non-existence of a closed orbit.

Recent results about the existence and non-existence of limit cycles have been obtained in [7] for Liénard type equations of the form

$$\frac{d}{dt}(\phi(\dot{x})) + f(x)\phi(\dot{x}) + g(x) = 0$$

involving the curvature operator. Clearly, the structure of this equation and that of (1.1) differs considerably and therefore one cannot expect to obtain the same kind of results.

2. Equivalent planar systems

As already mentioned in the Introduction, equation (1.1) is of the form

$$\big(\phi(\dot{x})\big)' + \lambda f(x)\dot{x} + g(x) = 0,$$

with $\phi : I \to J$ an increasing homeomorphism of an open interval I onto an open interval J. In our case $I = \mathbb{R}$ and $J =]-1,1[$. Setting $y = \phi(\dot{x})$, so that $\dot{x} := \phi^{-1}(y)$, we can write equation (1.1) as an equivalent planar system of the form

$$\dot{x} = \phi^{-1}(y), \quad \dot{y} = -\lambda f(x)\phi^{-1}(y) - g(x).$$

In our case,

$$y = \frac{\dot{x}}{\sqrt{1+\dot{x}^2}}, \qquad \dot{x} = \frac{y}{\sqrt{1-y^2}} \tag{2.1}$$

and therefore we get the following modified Liénard system

$$\dot{x} = \frac{y}{\sqrt{1-y^2}}, \qquad \dot{y} = -\lambda f(x)\frac{y}{\sqrt{1-y^2}} - g(x). \tag{2.2}$$

Another approach, inspired by the use of the Liénard plane in the classical case, and adopted in the recent paper [13] is to write equation (1.1) in the form

$$\frac{d}{dt}\left[\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \lambda F(x)\right] + g(x) = 0,$$

where $F : \mathbb{R} \to \mathbb{R}$ is defined by

$$F(x) := \int_0^x f(s)\,ds,$$

and make the change of variable

$$z = \frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \lambda F(x),$$

which is equivalent to

$$\dot{x} = \frac{z - \lambda F(x)}{\sqrt{1 - (z - \lambda F(x))^2}}, \quad \text{with } |z - \lambda F(x)| < 1.$$

Hence, equation (1.1) can be written as the equivalent system

$$\dot{x} = \frac{z - \lambda F(x)}{\sqrt{1 - (z - \lambda F(x))^2}}, \quad \dot{z} = -g(x). \tag{2.3}$$

In this manner, we can provide a result similar to [13, Lemma 1], namely

Lemma 2.1. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitzian, the Cauchy problem for equation* (1.1)*, or for systems* (2.2) *or* (2.3) *is locally uniquely solvable at points where $|z - \lambda F(x)| < 1$ for* (2.3) *and where $|y| < 1$ for* (2.2)*. Moreover, the solutions depend continuously upon the parameter λ.*

Proof. It is sufficient to notice that F is of class C^1 and apply standard local existence, uniqueness and continuous dependence results for ODEs with locally Lipschitz vector fields to system (2.3).

3. The associated Duffing prescribed curvature system

To prove the existence of limit cycles for system (2.2) it is useful to perform a preliminary study of the phase portrait of the associated curvature-Duffing system

$$\dot{x} = \frac{y}{\sqrt{1 - y^2}}, \quad \dot{y} = -g(x), \tag{3.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitzian. We observe that both equations (2.2) and (3.1) have similarities with the classical case. The most important feature is that actually we are able to define an energy first integral $H(x, y)$ for system (3.1), which is of Hamiltonian type

$$\dot{x} = \frac{\partial H}{\partial y}(x, y), \quad \dot{y} = -\frac{\partial H}{\partial x}(x, y),$$

with

$$H(x, y) = 1 - \sqrt{1 - y^2} + G(x),$$

where $G : \mathbb{R} \to \mathbb{R}$ is defined by

$$G(x) := \int_0^x g(s)\, ds.$$

The relevant difference in comparison with the classical case lies in the fact that both systems (2.2) and (3.1) are defined only in the strip $|y| < 1$ and this will be crucial for what follows.

We have added the constant 1 to $-\sqrt{1-y^2}$ in order that, for $|y|$ small, $1-\sqrt{1-y^2}$ is close to the classical expression $y^2/2$. Moreover, in this manner, if we also assume the natural condition

$$(*) \qquad g(x)x > 0 \;\text{ for }\; x \neq 0,$$

we have also that

$$H(x,y) > H(0,0) = 0, \quad \forall\,(x,y) \in \mathbb{R}\times\,]-1,1[\,\setminus\{(0,0)\},$$

that is the origin, which is the unique equilibrium point of (3.1) and also the minimum point of the above introduced energy function H, is associated to the level zero of energy.

The nontrivial level curves for (3.1) are defined by

$$1-\sqrt{1-y^2}+G(x) = K, \quad \text{for } K > 0.$$

In particular, one can see that a level curve intersects the y-axis when $\sqrt{1-y^2} = 1-K$, and hence this happens if and only if $0 < K < 1$, in which case the coordinates of the intersection are $(0, \pm\sqrt{K(2-K)})$, with $\pm\sqrt{K(2-K)}$ being clearly the maximal and minimal ordinates of the level curve.

A level curve intersects the x-axis at some $x > 0$ (respectively at some $x < 0$) if there exists a value $x > 0$ (respectively, $x < 0$) such that $G(x) = K$. By the assumption $(*)$, $G(x) > 0$ for $x \neq 0$, and G is decreasing for negative x and increasing for positive x. Therefore, G has limits $G(-\infty)$, $G(+\infty)$, possibly infinite, at $-\infty$ and $+\infty$ respectively, and

$$0 < G(x) < G(+\infty), \forall\, x > 0, \;\; 0 < G(x) < G(-\infty), \forall\, x < 0. \tag{3.2}$$

For the simplicity of the discussion, let us assume that

$$G(-\infty) = G(+\infty) := G(\infty).$$

Hence the origin of system (3.1) is always a local center. If $G(\pm\infty) < 1$, it will not be a global center in the strip $\mathbb{R}\times\,]-1,1[$ because, for $G(\pm\infty) \leq K < 1$, the corresponding level curves do not intersect the x-axis and go to $\pm\infty$ when $x \to \pm\infty$. The center is global in the strip $\mathbb{R}\times\,]-1,1[$ if $G(\pm\infty) = 1$.

Even if the systems (2.2) and (3.1) are not defined for $y = \pm 1$, nonetheless, the energy function H is defined on the closed strip $\mathbb{R}\times[-1,1]$, Therefore, it will be convenient to speak of level curves passing through a point of the form $(x_0, \pm 1)$. In this setting, one can say that the level energy curve "intersects" the line $y = \pm 1$ at the given point. Following this argument, we have that the level curve intersects the point $(0, \pm 1)$ for $K = 1$, while intersects the point $P(x_0, \pm 1)$ for $K = 1 + G(x_0)$ (see Figure 1).

On the other hand, it follows immediately from (3.1) that the orbits of its solutions are solutions of the differential equation

$$y'(x) = -\frac{g(x)\sqrt{1-y^2}}{y}. \tag{3.3}$$

When $K = 1$, the corresponding level curve passing through $(0, \pm 1)$ can be associated, using relation (2.1), to a function $x(t)$, which we call a generalized solution of (E), having a vertical slope ($\dot{x} = \pm\infty$) for two values of t.

When $K \geq 1 + G(x_0) > 1$, for some $x_0 > 0$, any solution of (3.3) passing through $(x_0, 1)$ or through $(x_0, -1)$ is such that

$$\lim_{x \to x_0} y'(x) = y'(x_0) = 0,$$

so that the level curves of H which cross the lines $y = \pm 1$, arrive tangentially to these lines.

For $K = 1 + G(x_0) > 1$, the level curve passing through $P(x_0, 1)$ crosses the x-axis at some point $P(x_+, 0)$ and the line $y = -1$ at $P(x_0, -1)$. It then crosses again the x-axis at a point $P(x_-, 0)$, with $x_- < 0 < x_+$ and the line $y = 1$ at some $P(x_1, 1)$, with $x_1 < 0 < x_0$. So this level curve of H consists in the arc $\widehat{P(x_0,1)P(x_0,-1)}$ passing through $P(x_+, 0)$, the segment $\overline{P(x_0,-1)P(x_1,-1)}$ of the line $y = -1$, the arc $\widehat{P(x_1,-1)P(x_1,1)}$ passing through $P(x_-, 0)$ and the segment $\overline{P(x_1,1)P(x_0,1)}$ of the line $y = 1$. The corresponding generalized periodic solution $x(t)$ jumps from x_0 to x_1 at some time t_1 and from x_1 to x_0 at some time t_2.

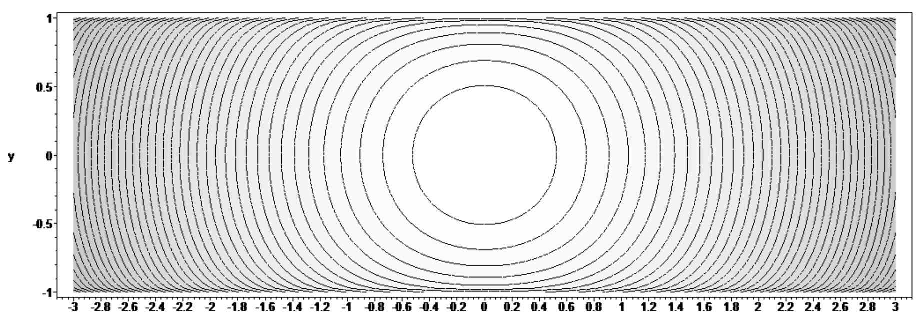


Figure 1: Level lines of the Hamiltonian function $H(x, y) = 1 - \sqrt{1 - y^2} + G(x)$, for $G(x) = x^2/2$. Note that the level lines H = constant when hit the lines $y = 1$ and $y = -1$, they touch them tangentially. This phase portrait will play a crucial role in the following.

Notice also that, given $(x_0, 1)$, there are infinitely many generalized orbits passing through this point, because not only $\{(x, 1) : x \geq x_0\}$ is a positive generalized orbit, but also the union of $[x_0, x_1] \times \{1\}$ with an arbitrary $x_1 > x_0$, followed by the piece of energy level curve joining $(x_1, 1)$ to $(x_1, -1)$. This is reminiscent of the Peano's non-uniqueness phenomenon for the Cauchy problem associated to non-Lipschitzian vector fields, and as the partial derivative with respect to y of the right-hand member of (3.3) is given by

$$\frac{g(x)}{\sqrt{1 - y^2}} + \frac{g(x)\sqrt{1 - y^2}}{y^2},$$

we see that it is not locally Lipschitzian on the lines $y = \pm 1$. This situation was already noticed (from a slightly different point of view) in [14, p. 672]. Following

[14] and due to the conservative nature of the equation, it seems natural to choose, among all the possible continuations of a solution arc, the one with the same energy.

4. Liénard prescribed curvature equation : the case where $G(\infty) > 1$

At this point, we come back to system (2.2). Following again the approach in [13], we take the energy function H as a Lyapunov function for (2.2). For a discussion about the use of the energy function method for the classical Liénard system, see also [4].

If we evaluate $\dot{H}(x, y)$ along the trajectories of system (2.2), we obtain

$$\begin{aligned} \dot{H}(x,y) &= \frac{\partial H}{\partial x}(x,y)\dot{x} + \frac{\partial H}{\partial y}(x,y)\dot{y} \\ &= \frac{y}{\sqrt{1-y^2}}\left(-\lambda f(x)\frac{y}{\sqrt{1-y^2}} - g(x)\right) + g(x)\frac{y}{\sqrt{1-y^2}} \\ &= -\lambda f(x)\frac{y^2}{1-y^2}. \end{aligned}$$

In the classical case one had $\dot{H}(x, y) = -\lambda f(x)y^2$ but clearly y^2 and $\frac{y^2}{1-y^2}$ play the same role. Therefore, when $f(x)$ is positive, the trajectories of system (2.2) enter trajectories of system (3.1), while, when $f(x)$ is negative, the trajectories of system (2.2) exit trajectories of system (3.1). This means that, if we assume $f(0) < 0$, the origin of system (2.2) is a source.

The slope of the trajectories of system (2.2) is given by

$$y'(x) = -\lambda f(x) - g(x)\frac{\sqrt{1-y^2}}{y}$$

and the 0-isocline, namely the curve in which $\dot{y} = 0$, is given by

$$\frac{-g(x)}{\lambda f(x)} = \frac{y}{\sqrt{1-y^2}}$$

Let us consider the case when $G(\infty) > 1$ *(possibly infinite).*

As observed above, the level curve

$$1 - \sqrt{1-y^2} + G(x) = 1$$

intersects the x-axis at two points $x_1 < 0 < x_2$, which are the two solutions of $G(x) = 1$. We just notice that if $g(x) = x$, then $G(x) = \frac{1}{2}x^2$ and therefore $x_1 = -\sqrt{2}$, $x_2 = \sqrt{2}$. In the light of the above presented comparison properties, we have the following non-existence result.

Theorem 4.1. *Consider the Liénard prescribed curvature equation* (1.1) *with the following assumptions.*

1) *$\lambda > 0$, $f : \mathbb{R} \to \mathbb{R}$ is continuous and $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitzian.*

2) *$xg(x) > 0$ for $x \neq 0$, $G(\infty) > 1$.*

3) *There are $\alpha < 0 < \beta$ such that $f(x) < 0$ for $\alpha < x < \beta$ and $f(x) > 0$ for $x < \alpha$ and $x > \beta$.*

If $\alpha \le x_1 < 0 < x_2 \le \beta$, then, for every $\lambda > 0$, equation (1.1) *has no nontrivial periodic solution.*

Proof. It suffices to consider system (2.2) and a point $P(x, y)$ on the level curve

$$H(x,y) = -\sqrt{1-y^2} + 1 + G(x) = 1.$$

In virtue of the comparison result, the negative semi-trajectory $\Gamma^-(P)$ tends to the origin, while the positive semi-trajectory $\Gamma^+(P)$ is bounded away from the level curve and therefore is forced to the line $y = 1$ or the line $y = -1$. Therefore system (2.2) has no limit cycles. (see Figure 2). □

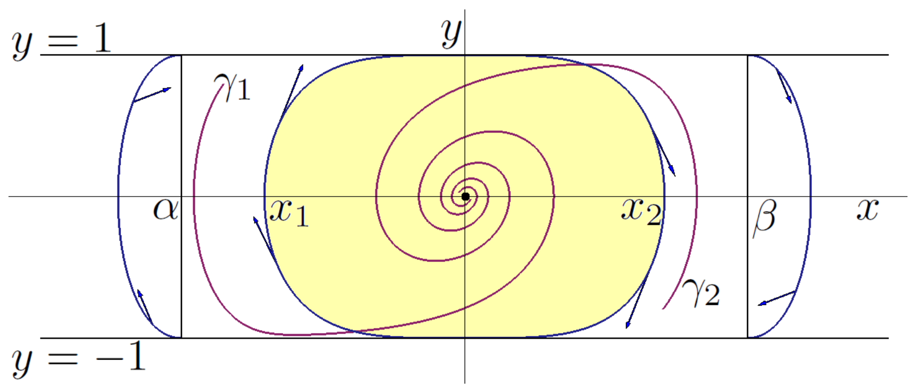


Figure 2: Qualitative phase-portrait of (2.2) in the framework of Theorem 4.1. The shadowed area represents the region $H(x, y) \le 1$. We show two semi-trajectories γ_1 and γ_2 which depart from points near the origin and are going to hit the line $y = 1$ or the line $y = -1$ in finite time.

5. Van der Pol prescribed curvature equation : existence and non-existence of limit cycles

A consequence of Theorem 4.1, for the case of the k-Van der Pol prescribed curvature equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \lambda(x^2 - k^2)\dot{x} + x = 0, \tag{5.1}$$

is the following non-existence result.

Corollary 5.1. *For every $\lambda > 0$, equation* (5.1) *has no nontrivial periodic solutions, provided that $k^2 \ge 2$.*

Proof. As mentioned above, in this case, because $G(x) = \frac{1}{2}x^2$, we have $x_1 = -\sqrt{2}$ and $x_2 = \sqrt{2}$. Moreover $\alpha = -k$ and $\beta = k$. We apply Theorem 4.1 and get immediately the thesis. □

We now state and prove an existence result for the k-van der Pol prescribed curvature equation (5.1).

Theorem 5.1. *For every $\lambda > 0$, there exists a $\hat{k} = \hat{k}(\lambda) > 0$ such that for every $0 < k < \hat{k}$, equation (5.1) has at least a nontrivial periodic solution.*

Proof. Working in the phase-plane the proof is straightforward, because we consider the Hopf bifurcation from the origin, which produces a small amplitude limit cycle. Indeed, easy computations show that equation (5.1) can be written as

$$\ddot{x} + \lambda(1+\dot{x}^2)^{3/2}(x^2-k^2)\dot{x} + (1+\dot{x}^2)^{3/2}x = 0,$$

and is therefore equivalent to the planar system in the standard phase place (x, v)

$$\dot{x} = v,\ \ \dot{v} = -\lambda(1+v^2)^{3/2}(x^2-k^2)v - (1+v^2)^{3/2}x,$$

whose linearization near $(0,0)$ is

$$\dot{x} = v,\ \ \dot{v} = \lambda k^2 v - x.$$

The eigenvalues of the corresponding matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & \lambda k^2 \end{pmatrix}$$

are $\mu_{\pm}(k) = \frac{\lambda k^2 \pm \sqrt{\lambda^2 k^4 - 4}}{2}$, so that $\mu_{pm}(0) = \pm i$ and the conditions for Hopf bifurcation of an unstable focus to a small amplitude limit cycle are satisfied (see e.g. [9], p. 270). □

At this point we consider the case $k = 1$, namely the Van der Pol prescribed curvature equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \lambda(x^2-1)\dot{x} + x = 0 \tag{5.2}$$

Corollary 5.1 cannot be applied and therefore we investigate the problem of existence of nontrivial periodic solutions.

In this case system (2.2) becomes

$$\dot{x} = \frac{y}{\sqrt{1-y^2}}, \qquad \dot{y} = -\lambda(x^2-1)\frac{y}{\sqrt{1-y^2}} - x. \tag{5.3}$$

The 0-isocline satisfies the equation

$$\frac{y}{\sqrt{1-y^2}} = \frac{-x}{\lambda(x^2-1)}$$

and we observe that it can obtained by the curve of the 0-isocline in the classical case, namely

$$y = \frac{-x}{\lambda(x^2-1)}.$$

In fact, we consider such a curve for $y \in (-\infty, +\infty)$ and then we "shrink" the curve into the strip $|y| < 1$ (see Figure 3).

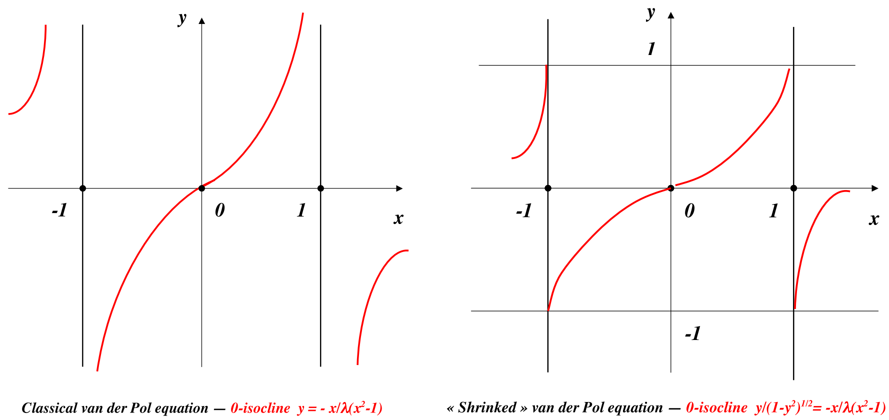


Figure 3: The 0-isocline of the Van der Pol equation in the phase-plane (left) and the corresponding "shrinked" deformation to the strip $\mathbb{R} \times]-1, 1[$ which gives the 0-isocline for the Van der Pol prescribed curvature equation (right).

However, the idea of using this curve in order to produce a trajectory which comes from 'infinity' without intersecting the x-axis before intersecting the line $x = 1$ (as was done in [13]), actually does not work, because such trajectory can "blow up" to the line $y = 1$ (see Figure 4). This is precisely the problem mentioned in the Introduction.

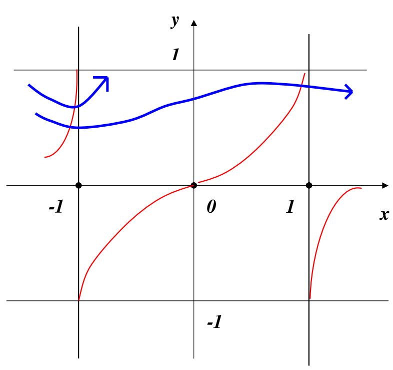


Figure 4: Example of a hypothetical trajectory which meets the line $y = 1$ after crossing the 0-isocline.

As expected, for λ large there are no limit cycles. This comes from the following proposition.

Proposition 5.1. *If $\lambda > \frac{3}{2}$, equation (5.2) has no closed orbit.*

Proof. Consider the point $P(-1,0)$ which lies on the level curve

$$H(x,y) = -\sqrt{1-y^2} + 1 + G(x) = \frac{1}{2},$$

for $G(x) = x^2/2$. Clearly, for every $\lambda > 0$, $\Gamma^-(P)$ tends to the origin. If $\Gamma^+(P)$ intersects the positive y-axis at a point $Q(0, y)$ we evaluate

$$\begin{aligned} y_Q - y_P &= \int_{-1}^{0} y'(x)dx \\ &= \int_{-1}^{0} \left(-\lambda(x^2-1) - \frac{x\sqrt{1-y^2}}{y}\right) dx > -\lambda \int_{-1}^{0} (x^2-1)dx. \end{aligned}$$

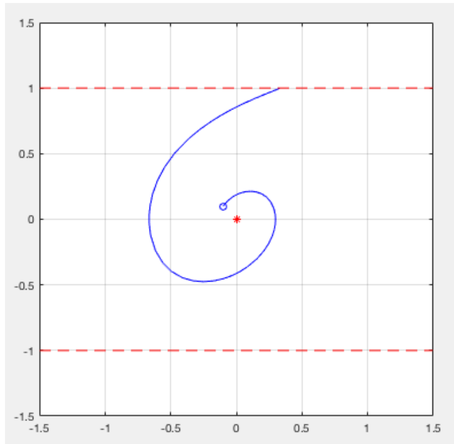
Therefore

$$y_Q = \int_{-1}^{0} y'(x)dx > -\int_{-1}^{0} \lambda(x^2-1)dx = -\lambda\left[\frac{x^2}{3} - x\right]_{-1}^{0} = \frac{2}{3}\lambda.$$

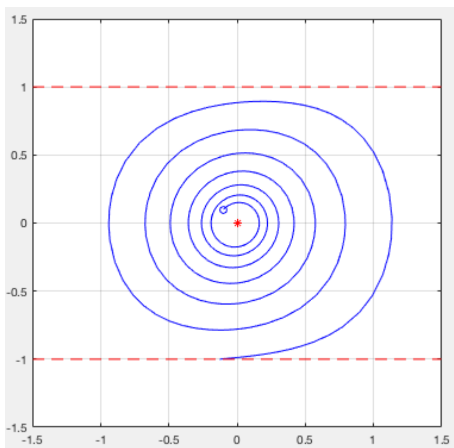
This implies that if $\lambda > \frac{3}{2}$ then $\Gamma^+(P)$ does not intersect the y-axis and "blows up" to the line $y = 1$. □

It remains to investigate the case in which λ is small.

There is numerical evidence (see the computations for $\lambda = 0.5$, $\lambda = 0.1$ in Figure 5 and for $\lambda = 0.01$ in Figure 6) that there are no limit cycles.



(a) The case of (5.3) for $\lambda = 0.5$.



(b) The case of (5.3) for $\lambda = 0.1$.

Figure 5: A numerical simulation for system (5.3) for a case of non-existence of limit cycles with different values of λ. We see the solutions that unwind away from the origin and hit the line $y = 1$ or the line $y = -1$ after some time.

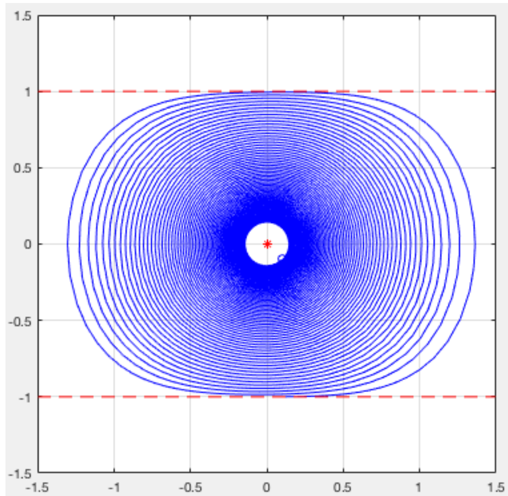


Figure 6: A numerical simulation for system (5.3) with $\lambda = 0.01$.

This suggests that actually there are no limit cycles for every positive λ, but at the moment we do not have a proof. Therefore we can just make the following

Conjecture. *The Van der Pol prescribed curvature equation (5.2) has no limit cycle for every* $\lambda > 0$.

6. Liénard prescribed curvature equation : the case where $G(\infty) < 1$

Now we investigate the case where $0 < G(\infty) < 1$. Here the situation is similar to the one of the previous paper [13] and we have an existence result.

Theorem 6.1. *Assume that the following assumptions hold.*

1) $f : \mathbb{R} \to \mathbb{R}$ *is continuous and* $g : \mathbb{R} \to \mathbb{R}$ *is locally Lipschitzian.*
2) $\lambda > 0$, $xg(x) > 0$ *for* $x \neq 0$, $0 < G(\infty) < 1$.
3) *There are* $\alpha < 0 < \beta$ *such that* $f(x) < 0$ *for* $\alpha < x < \beta$ *and* $f(x) > 0$ *for* $x < \alpha$ *and* $x > \beta$.
4) $F(x) \to +\infty$ *when* $x \to \pm\infty$.

Then equation (1.1) *has at least a nontrivial periodic solution for* λ *small enough, while it has no nontrivial periodic solutions for* λ *large enough.*

Proof. We work with (2.2) and denote with Δ the level curve

$$H(x,y) = 1 - \sqrt{1-y^2} + G(x) = G(\infty)$$

Such a curve does not intersect the x-axis, while intersects the positive y-axis at a point $P(0,K)$, with $0 < K < 1$. Consider a point $Q(x,y)$ on Δ with $x < \alpha$ and follow $\Gamma^+(Q)$. We know that for $x < \alpha$ the trajectories of (2.2) enter Δ, and therefore $\Gamma^-(Q)$ comes from 'infinity' without having intersected the x-axis before. Then it intersects Δ at Q and, subsequently, $\Gamma^+(Q)$ intersects the vertical line $x = \alpha$ at a point below Δ. Now, using a standard continuity argument, for λ sufficiently small $\Gamma^+(Q)$ remains close to Δ and hence it intersects the vertical line $x = \beta$. Now such a trajectory enters the level curves of $H(x,y)$ and eventually intersects the x-axis at some point $(\hat{x},0)$ with $\hat{x} > \beta$. This happens because there are no horizontal asymptotes.

Indeed if an horizontal asymptote occurs then we have that $y'(x)$ goes to 0, but being

$$y'(x) = -f(x) - \frac{\sqrt{1-y^2}}{y} g(x)$$

the assumption on $F(x)$ gives the desired contradiction, by integration of the above formula.

After having intersected the x-axis, $\Gamma^+(Q)$ remains bounded by some level curve until it reaches $x = \beta$. At this point, using again the continuity argument, we know that for λ sufficiently small $\Gamma^+(Q)$ intersects the the line $x = \alpha$ and then, as was proved for x positive, it intersects the x-axis at some point $(\bar{x},0)$ with $\bar{x} < \alpha$. Therefore $\Gamma^+(Q)$ is winding. The origin is a source, and consequently we can apply the Poincaré-Bendixson Theorem in order to get the existence of at least a stable limit cycle, that is the desired nontrivial periodic solution.

On the other hand, using the same argument of the Van der Pol case for the non-existence of limit cycles, we know that for λ large enough $\Gamma^+(P(-1,0))$ does not intersect the y-axis and "blows up" to the line $y = 1$. This proves the theorem. □

As an example, let us consider the special case of a Liénard prescribed curvature equation with a Van der Pol dissipation function.

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \lambda(x^2-1)\dot{x} + g(x) = 0 \tag{6.1}$$

with

$$g(x) = \frac{1}{2}x \ \text{ for } \ |x| < 1 \ \text{ and } \ g(x) = \frac{1}{2x^3} \ \text{ for } \ |x| \geq 1. \tag{6.2}$$

Clearly $G(x)$ goes to $\frac{1}{2}$ for $x \to \pm\infty$. Therefore equation (6.1) has at least a nontrivial periodic solution for λ small enough, while has no nontrivial periodic solutions for $\lambda > \frac{3}{2}$.

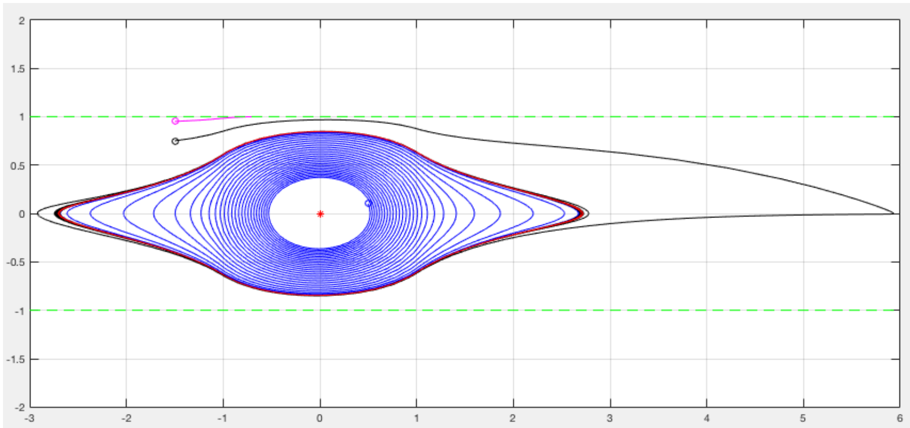


Figure 7: A numerical simulation for system (6.1) with $g(x)$ as in (6.2) and with $\lambda = 0.01$.

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