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# LOCAL BOUNDEDNESS FOR MINIMIZERS OF SOME POLYCONVEX INTEGRALS

GIOVANNI CUPINI - FRANCESCO LEONETTI - ELVIRA MASCOLO

ABSTRACT. We give a regularity result for local minimizers  $u : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$  of a special class of polyconvex functionals. Under some structure assumptions on the energy density, we prove that local minimizers  $u$  are locally bounded. For each component  $u^\alpha$  of  $u$ , we first prove a Caccioppoli's inequality and then apply De Giorgi's iteration method to get the boundedness of  $u^\alpha$ . Our result can be applied to the polyconvex integral

$$\int_{\Omega} \left( \sum_{\alpha=1}^3 |Du^\alpha|^p + |\text{adj}_2 Du|^q + |\det Du|^r \right) dx$$

with suitable  $p, q, r > 1$ .

## 1. INTRODUCTION

We study the regularity of local minimizers for a special class of variational integrals

$$I(u, \Omega) = \int_{\Omega} f(Du) dx \tag{1.1}$$

where  $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a vector-valued map and  $Du$  is the  $m \times n$  Jacobian matrix of its partial derivatives

$$u \equiv (u^1, u^2, \dots, u^m), \quad Du = \left( \frac{\partial u^\alpha}{\partial x_i} \right)_{\substack{\alpha=1,2,\dots,m \\ i=1,2,\dots,n}}.$$

A function  $u \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^m)$  is a local minimizer of  $I$  if  $f(Du) \in L_{\text{loc}}^1(\Omega)$  and

$$I(u, \text{supp } \varphi) \leq I(u + \varphi, \text{supp } \varphi),$$

for all  $\varphi \in W^{1,1}(\Omega, \mathbb{R}^m)$  with  $\text{supp } \varphi \Subset \Omega$ .

Motivated by the applications to nonlinear elasticity, J. Ball in 1977 pointed out in [1] that convexity of  $f$  with respect to  $Du$  is unrealistic in the vectorial case. Indeed, it conflicts, for instance, with the natural requirement that the elastic energy is frame-indifferent. The convexity must be replaced by different and more general assumptions, as the so called *quasiconvexity* and *polyconvexity*, already introduced by Morrey in [36] in an abstract setting.

In particular, we are interested in the polyconvexity condition, which takes into account the constitutive hypothesis that the energy is invariant under the transformation  $g \mapsto g + \varphi$ , for every null Lagrangian  $\varphi$ .

A function  $f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ ,  $f = f(\xi)$ , is said *polyconvex* if there exists a convex function  $g : \mathbb{R}^{\tau(m,n)} \rightarrow \mathbb{R}$  such that

$$f(\xi) = g(T(\xi)), \tag{1.2}$$

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where

$$\tau(m, n) := \sum_{i=1}^{\min(m, n)} \binom{n}{i} \binom{m}{i}$$

and  $T(\xi)$  is the vector defined as follows:

$$T(\xi) := \left( \xi, \text{adj}_2 \xi, \dots, \text{adj}_i \xi, \dots, \text{adj}_{\min\{m, n\}} \xi \right).$$

Here  $\text{adj}_i \xi$  denotes the adjugate matrix of order  $i$ . In particular, if  $m = n$  then  $\text{adj}_n \xi = \det \xi$ .

The polyconvexity assumption is commonly used as a structural assumption in mathematical models of elasticity, since, if  $n = m = 3$ ,  $\xi$ ,  $\text{adj}_2 \xi$  and  $\det \xi$  govern the deformations of line, surface and volume, respectively.

Our main purpose is to illustrate some ideas and methods which lead to local boundedness for local minimizers of some polyconvex functionals.

In our paper  $n = m = 3$  and we assume that there exist  $F_\alpha : \mathbb{R}^3 \rightarrow [0, +\infty)$ ,  $G_\alpha : \mathbb{R}^3 \rightarrow [0, +\infty)$ , with  $\alpha \in \{1, 2, 3\}$ , and  $H : \mathbb{R} \rightarrow [0, +\infty)$  convex functions such that

$$f(\xi) := \sum_{\alpha=1}^3 \{F_\alpha(\xi^\alpha) + G_\alpha((\text{adj}_2 \xi)^\alpha)\} + H(\det \xi), \quad (1.3)$$

where

$$\xi = \begin{pmatrix} \xi_1^1 & \xi_2^1 & \xi_3^1 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 \\ \xi_1^3 & \xi_2^3 & \xi_3^3 \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix}, \quad \xi^\alpha \in \mathbb{R}^3 \text{ for } \alpha \in \{1, 2, 3\}.$$

and  $\text{adj}_2 \xi \in \mathbb{R}^{3 \times 3}$  denotes the adjugate matrix of order 2 whose components are

$$(\text{adj}_2 \xi)_{\gamma i} = (-1)^{\gamma+i} \det \begin{pmatrix} \xi_k^\alpha & \xi_l^\alpha \\ \xi_k^\beta & \xi_l^\beta \end{pmatrix} \quad \gamma, i \in \{1, 2, 3\},$$

where  $\alpha, \beta \in \{1, 2, 3\} \setminus \{\gamma\}$ ,  $\alpha < \beta$ , and  $k, l \in \{1, 2, 3\} \setminus \{i\}$ ,  $k < l$ . Moreover,

$$(\text{adj}_2 \xi)^\alpha = ((\text{adj}_2 \xi)_{\alpha 1}, (\text{adj}_2 \xi)_{\alpha 2}, (\text{adj}_2 \xi)_{\alpha 3}).$$

Integrands (1.3) occur as stored energy densities for certain models from nonlinear elasticity (see Ball [1],[2], Ogden [39]) and, by the results by Ball [1],[2] and Muller [38], see also the monograph by Dacorogna [8], the corresponding minimization problems have a solution.

To have regular local minimizers some growth conditions have to be considered. We assume that  $F_\alpha(\xi^\alpha)$  grows like  $|\xi^\alpha|^p$ ,  $G_\alpha((\text{adj}_2 \xi)^\alpha)$  grows like  $|(\text{adj}_2 \xi)^\alpha|^q$  and  $H(\det \xi)$  grows like  $|\det \xi|^r$ .

In this paper, under suitable assumptions on the exponents  $p, q, r$  (see condition (2.5)) we prove that the local minimizers of  $I$  are locally bounded in  $\Omega$ , see Theorem 2.1. We note that  $F_\alpha, G_\alpha, H$  may depend on  $x$  too:  $F_\alpha(x, \xi^\alpha), G_\alpha(x, (\text{adj}_2 \xi)^\alpha), H(x, \det \xi)$ ; see Theorem 2.1.

As an application of Theorem 2.1, let us consider the functional (1.1) with

$$f(Du) := \sum_{\alpha=1}^3 \left( |Du^\alpha|^{14/5} + |\text{adj}_2 Du^\alpha|^2 \right) + |\det Du|^{3/2}.$$

By Theorem 2.1 every local minimizer  $u : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$  of  $I$  is locally bounded. Note that the existence of a minimizer of  $I$  in  $\bar{u} + W_0^{1, \frac{14}{5}}(\Omega)$ , with  $\bar{u} \in W^{1, \frac{14}{5}}(\Omega)$ , comes from Remark 8.32 in [8], see Theorem 3.1 below.

Partial regularity results i.e. the regularity of solutions up to a set  $\Omega_0$  and the study of the properties of the singular set, see for example section 4.2 in [33] and section 1 in [34], are contained in [17], [15], [14], [40], [12], [22], [7]. For the polyconvex case, only few everywhere regularity results

are available; we mention those by Fusco and Hutchinson in [18], where the everywhere continuity is proved in the case  $n = m = 2$ , Fuchs and Seregin [16], where Hölder continuity for extremals is dealt with. Global pointwise bounds are in [24], [9], [28], [29], [26], [27]. Interesting results are contained in [42], [3], [4], [5], [6]; see also [25].

The main novelty of our result is the technique used to obtain the regularity result. Indeed we prove the local boundedness of vector valued minimizers  $u = (u^1, u^2, u^3)$  by using De Giorgi's iteration method, used until now only in the scalar case. Indeed, we first show that each component  $u^\alpha$  satisfies a Caccioppoli's inequality (see Proposition 2.3); then we apply De Giorgi's procedure, separately, to each  $u^\alpha$ .

The special structure on  $f$  in (1.3) is in some sense necessary to treat this type of functionals since in the vectorial framework minimizers can be unbounded, in view of some counterexamples, see [10], [43], section 3 in [33] and the recent [35].

The integrals considered can be inserted in the class of functionals with  $p, q$ -growth. The mathematical literature on the regularity under  $p, q$ -growth is very rich; energy functionals with anisotropic, non-standard or general growth have been studied by many authors and in different settings of applicability. Under  $p, q$ -growth it is now well known, as in our result, that a restriction between  $p$  and  $q$  must be imposed due to the counterexamples in [20], [30], [31], [32], [23], [11], [13]; we refer to [33] for a detailed survey on the subject.

Our paper is organized as follows. In the next section we present the precise statement of our local boundedness result (Theorem 2.1) and we describe our strategy for proving it; eventually we provide the proof. In Section 3 we recall an existence result for a suitable class of polyconvex functionals (Theorem 3.1); using this result and Theorem 2.1 we obtain the existence of locally bounded minimizers for a class of functionals satisfying the assumptions of these two results (Theorem 3.2). Section 4 contains the Appendix, devoted to two technical results used to prove Theorem 2.1.

## 2. LOCAL BOUNDEDNESS

We consider  $\Omega \subseteq \mathbb{R}^3$  open set, a function  $f : \Omega \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ , and the functional

$$I(u) := \int_{\Omega} f(x, Du(x)) dx$$

where  $u : \Omega \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,

$$Du := \begin{pmatrix} Du^1 \\ Du^2 \\ Du^3 \end{pmatrix} = \begin{pmatrix} u_{x_1}^1 & u_{x_2}^1 & u_{x_3}^1 \\ u_{x_1}^2 & u_{x_2}^2 & u_{x_3}^2 \\ u_{x_1}^3 & u_{x_2}^3 & u_{x_3}^3 \end{pmatrix}.$$

We assume that there exist Carathéodory functions  $F_\alpha : \Omega \times \mathbb{R}^3 \rightarrow [0, +\infty)$ ,  $G_\alpha : \Omega \times \mathbb{R}^3 \rightarrow [0, +\infty)$ ,  $\alpha \in \{1, 2, 3\}$ , and  $H : \Omega \times \mathbb{R} \rightarrow [0, +\infty)$ , such that  $\lambda \rightarrow F_\alpha(x, \lambda)$ ,  $\lambda \rightarrow G_\alpha(x, \lambda)$ ,  $t \rightarrow H(x, t)$  are convex, with

$$f(x, \xi) := \sum_{\alpha=1}^3 \{F_\alpha(x, \xi^\alpha) + G_\alpha(x, (\text{adj}_2 \xi)^\alpha)\} + H(x, \det \xi). \quad (2.1)$$

Here

$$\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix}, \quad \xi^\alpha \in \mathbb{R}^3 \text{ for } \alpha \in \{1, 2, 3\}$$

and  $\text{adj}_2 \xi \in \mathbb{R}^{3 \times 3}$  denotes the adjugate matrix of order 2 whose components are

$$(\text{adj}_2 \xi)_{\gamma i} = (-1)^{\gamma+i} \det \begin{pmatrix} \xi_k^\alpha & \xi_l^\alpha \\ \xi_k^\beta & \xi_l^\beta \end{pmatrix} \quad \gamma, i \in \{1, 2, 3\},$$

where  $\alpha, \beta \in \{1, 2, 3\} \setminus \{\gamma\}$ ,  $\alpha < \beta$ , and  $k, l \in \{1, 2, 3\} \setminus \{i\}$ ,  $k < l$ . Moreover,

$$(\text{adj}_2 \xi)^\alpha = ((\text{adj}_2 \xi)_{\alpha 1}, (\text{adj}_2 \xi)_{\alpha 2}, (\text{adj}_2 \xi)_{\alpha 3}).$$

We assume that there exist exponents  $1 < p \leq 3$ ,  $1 < q$ ,  $1 \leq r$ , constants  $k_1, k_3 > 0$ ,  $k_2 \geq 0$  and functions  $a, b, c : \Omega \rightarrow [0, +\infty)$  such that, for all  $\alpha \in \{1, 2, 3\}$ ,

$$k_1 |\lambda|^p - k_2 \leq F_\alpha(x, \lambda) \leq k_3 (|\lambda|^p + 1) + a(x) \quad \forall \lambda \in \mathbb{R}^3 \quad (2.2)$$

$$k_1 |\lambda|^q - k_2 \leq G_\alpha(x, \lambda) \leq k_3 (|\lambda|^q + 1) + b(x) \quad \forall \lambda \in \mathbb{R}^3 \quad (2.3)$$

$$0 \leq H(t) \leq k_3 (|t|^r + 1) + c(x) \quad \forall t \in \mathbb{R} \quad (2.4)$$

where  $a, b, c \in L^\sigma(\Omega)$ ,  $\sigma > 1$ .

Our main result is the following.

**Theorem 2.1.** *Let  $f$  satisfy (2.1) and growth conditions (2.2), (2.3), (2.4), with  $1 \leq r < q < p \leq 3$ . Assume*

$$\frac{p}{p^*} < \min \left\{ 1 - \frac{qp^*}{p(p^* - q)}, 1 - \frac{rp^*}{q(p^* - r)}, 1 - \frac{1}{\sigma} \right\}, \quad (2.5)$$

where  $p^* = \frac{3p}{3-p}$ , if  $p < 3$ , and, if  $p = 3$ , then  $p^*$  is any  $\nu > 3$ .

Then all the local minimizers  $u \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^3)$  of  $I$  are locally bounded.

**Remark 2.2.** *If  $\sigma = \infty$  then  $\frac{1}{\sigma}$  must be read as 0. Moreover, we remark that if  $p = 3$ , then  $p^*$  can be chosen large enough so that (2.5) is implied by the assumptions  $1 \leq r < q < p$  and  $\sigma > 1$ .*

For the sake of simplicity we prove the theorem in the case with no dependence on  $x$ , that is,  $f(\xi)$ ,  $F_\alpha(\xi^\alpha)$ ,  $G_\alpha((\text{adj}_2 \xi)^\alpha)$ ,  $H(t)$ , with  $a(x) = b(x) = c(x) = 0$  in the growth conditions. See also Remark 2.6.

**Sketch of the proof.** We now provide a sketch of the proof of Theorem 2.1. For a local minimizer  $u = (u^1, u^2, u^3)$  we will prove that each component is locally bounded. In the following we consider the first component  $u^1$ . We can argue similarly for the other components  $u^2, u^3$ .

STEP 1. Caccioppoli inequality for  $u^1$ . We use the minimality condition with a suitable test function; such a test function and the particular structure (2.1) of the density  $f$  guarantee a Caccioppoli inequality for  $u^1$  on every superlevel set  $\{u^1 > k\}$ . More precisely, fixed  $x_0 \in \Omega$  and a ball  $B_{R_0}(x_0) \Subset \Omega$  (we will not write the center  $x_0$  if no confusion may arise) we have that there exists  $c > 0$  such that for all  $s, t > 0$ ,  $s < t \leq R_0$ ,

$$\int_{\{u^1 > k\} \cap B_s} |Du^1|^p dx \leq c \int_{\{u^1 > k\} \cap B_t} \left( \frac{u^1 - k}{t - s} \right)^{p^*} dx + c |\{u^1 > k\} \cap B_t|^\vartheta \quad (2.6)$$

with a suitable  $\vartheta > 0$ . The Caccioppoli inequality (2.6) permits to apply the classical methods to get the regularity in the scalar case. Observe that on the right hand side of (2.6) we do not get the same exponent  $p$  as in the left hand side, but the larger  $p^*$ ; it still allows us to prove the local boundedness of  $u^1$ , see also [19] and [37].

STEP 2. Decay of the “excess” on superlevel sets. For a suitable radius  $R < R_0$  and a suitable level  $d$ , we define a sequence  $\rho_h$  of radii starting from  $R$  and decreasing to  $\frac{R}{2}$ , another sequence  $k_h$  of levels starting from  $\frac{d}{2}$  and increasing to  $d$ . We define the “excess” on the superlevel set as follows

$$J_h := \int_{\{u^1 > k_h\} \cap B_{\rho_h}} (u^1 - k_h)^{p^*} dx. \quad (2.7)$$

Note that  $J_h$  is a decreasing sequence. Using Sobolev inequality and Caccioppoli estimate (2.6) we are able to show that

$$J_{h+1} \leq cQ^h J_h^{\vartheta p^*/p} \quad (2.8)$$

for some constants  $c, Q > 1$ .

STEP 3. Iteration. In the right hand side of (2.8) there is competition between the increasing  $Q^h$  and the decreasing  $J_h^{\vartheta p^*/p}$ ; if  $\vartheta p^*/p > 1$  and the initial value  $J_0$  is small, then

$$J_h \leq Q^{\frac{-h}{\vartheta p^*/p - 1}} J_0, \quad (2.9)$$

so that

$$\lim_{h \rightarrow +\infty} J_h = 0, \quad (2.10)$$

which implies

$$u^1 \leq d \quad \text{a. e. in } B_{R/2}. \quad (2.11)$$

Since assumption (2.5) guarantees  $\vartheta p^*/p > 1$  we get (2.11). Lower bounds for  $u^1$  can be obtained by showing that  $-u$  is a minimizer for a similar functional.

To accomplish this program, we will use two technical lemmas; their statements and proofs can be found in the Appendix.

**STEP 1. Caccioppoli inequality.** The particular structure (2.1) of the density  $f$  guarantees a Caccioppoli inequality for any component  $u^\alpha$  of  $u$  on every superlevel set  $\{u^\alpha > k\}$ . In the next proposition we state this result in the case of the first component  $u^1$ .

**Proposition 2.3.** *Let  $f$  be as in (2.1), satisfying the growth conditions (2.2), (2.3), (2.4), with*

$$q < \frac{p^* p}{p^* + p} \quad \text{and} \quad r < \frac{p^* q}{p^* + q}. \quad (2.12)$$

Let  $u \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^3)$  be a local minimizer of  $I$ .

Let  $B_R(x_0) \Subset \Omega$ ,  $|B_R| < 1$ ,  $R < 1$ , and, fixed  $k \in \mathbb{R}$ , denote

$$A_{k,\tau}^1 := \{x \in B_\tau(x_0) : u^1(x) > k\} \quad 0 < \tau \leq R.$$

Then there exists  $c > 0$ , independent of  $k$ , such that for every  $0 < s < t \leq R$ .

$$\begin{aligned} \int_{A_{k,s}^1} |Du^1|^p dx &\leq c \int_{A_{k,t}^1} \left( \frac{u^1 - k}{t - s} \right)^{p^*} dx \\ &+ c \left\{ 1 + \left( \int_{B_R} (|Du^2| + |Du^3|)^p dx \right)^{\frac{qp^*}{(p^*-q)p}} + \left( \int_{B_R} |(\text{adj}_2 Du)^1|^q dx \right)^{\frac{rp^*}{(p^*-r)q}} \right\} |A_{k,t}^1|^\vartheta \end{aligned} \quad (2.13)$$

where  $\vartheta := \min\{1 - \frac{qp^*}{p(p^*-q)}, 1 - \frac{rp^*}{q(p^*-r)}, 1 - \frac{1}{\sigma}\}$ .

*Proof.* For the sake of simplicity we will give a proof assuming that the integrand function  $f$  is independent on  $x$ , and, consequently, that  $a, b, c$  in (2.2), (2.3), (2.4) are equal to 0.

Let  $B_R(x_0) \Subset \Omega$ ,  $|B_R| < 1$ ,  $R < 1$ . Let  $s, t$  be such that  $s < t \leq R$ . Consider a cut-off function  $\eta \in C_0^\infty(B_t)$  satisfying the following assumptions:

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_s(x_0), \quad |D\eta| \leq \frac{2}{t-s}. \quad (2.14)$$

Fixed  $k \in \mathbb{R}$ , define  $w \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^3)$ ,

$$w^1 := \max(u^1 - k, 0), \quad w^2 := 0, \quad w^3 := 0,$$

and, for  $\mu \geq p^*$ ,

$$\varphi := -\eta^\mu w.$$

For a.e.  $x$  in  $\Omega \setminus (\{\eta > 0\} \cap \{u^1 > k\})$  we have  $\varphi = 0$ , thus

$$f(Du + D\varphi) = f(Du) \quad \text{a.e. in } \Omega \setminus (\{\eta > 0\} \cap \{u^1 > k\}). \quad (2.15)$$

For a.e.  $x$  in  $\{\eta > 0\} \cap \{u^1 > k\}$  denote

$$A := \begin{pmatrix} \mu\eta^{-1}(k - u^1)D\eta \\ Du^2 \\ Du^3 \end{pmatrix}. \quad (2.16)$$

We notice that

$$Du + D\varphi = \begin{pmatrix} (1 - \eta^\mu)Du^1 + \mu\eta^{\mu-1}(k - u^1)D\eta \\ Du^2 \\ Du^3 \end{pmatrix} = (1 - \eta^\mu)Du + \eta^\mu A.$$

Since

$$\det(Du + D\varphi) = (1 - \eta^\mu) \det Du + \eta^\mu \det A$$

and

$$\text{adj}_2(Du + D\varphi) = (1 - \eta^\mu) \text{adj}_2 Du + \eta^\mu \text{adj}_2 A$$

and using that  $f$  is polyconvex, we get

$$f(Du + D\varphi) \leq (1 - \eta^\mu)f(Du) + \eta^\mu f(A) \quad \text{a.e. in } \{\eta > 0\} \cap \{u^1 > k\}. \quad (2.17)$$

By the minimality of  $u$ ,  $f(Du) \in L_{\text{loc}}^1(\Omega)$ . Lemma 4.2 in Appendix ensures that

$$\eta^\mu f(A) \in L^1(\{u^1 > k\} \cap \{\eta > 0\}).$$

Therefore (2.15) and (2.17) imply  $f(Du + D\varphi) \in L_{\text{loc}}^1(\Omega)$ .

By the local minimality of  $u$ , (2.15) and (2.17) we have

$$\int_{A_{k,t}^1 \cap \{\eta > 0\}} f(Du) dx \leq \int_{A_{k,t}^1 \cap \{\eta > 0\}} \{(1 - \eta^\mu)f(Du) + \eta^\mu f(A)\} dx.$$

The inequality above implies

$$\int_{A_{k,t}^1 \cap \{\eta > 0\}} \eta^\mu f(Du) dx \leq \int_{A_{k,t}^1 \cap \{\eta > 0\}} \eta^\mu f(A) dx. \quad (2.18)$$

Taking into account (2.16) and the particular structure of  $f$ , see (2.1), we obtain

$$F_2(A^2) = F_2(Du^2), \quad F_3(A^3) = F_3(Du^3), \quad G_1((\text{adj}_2 A)^1) = G_1((\text{adj}_2 Du)^1), \quad (2.19)$$

then, by (2.18),

$$\begin{aligned} & \int_{A_{k,t}^1 \cap \{\eta > 0\}} \eta^\mu \left\{ F_1(Du^1) + \sum_{\alpha=2}^3 G_\alpha((\text{adj}_2 Du)^\alpha) + H(\det Du) \right\} dx \\ & \leq \int_{A_{k,t}^1 \cap \{\eta > 0\}} \eta^\mu \left\{ F_1(\mu\eta^{-1}(k - u^1)D\eta) + \sum_{\alpha=2}^3 G_\alpha((\text{adj}_2 A)^\alpha) + H(\det A) \right\} dx. \end{aligned} \quad (2.20)$$

By the growth assumption (2.2)

$$\eta^\mu F_1(\mu\eta^{-1}(k - u^1)D\eta) \leq c\eta^\mu + c\mu^p \eta^{\mu-p} \left( \frac{u^1 - k}{t - s} \right)^p \quad \text{a.e. in } A_{k,t}^1 \cap \{\eta > 0\}.$$

Therefore, recalling  $\mu > p$  and the inequality  $z^p \leq z^{p^*} + 1$  if  $z \geq 0$ , we obtain

$$\int_{A_{k,t}^1 \cap \{\eta > 0\}} \eta^\mu F_1(\mu\eta^{-1}(k - u^1)D\eta) dx \leq c \int_{A_{k,t}^1} \left\{ 1 + \left( \frac{u^1 - k}{t - s} \right)^{p^*} \right\} dx. \quad (2.21)$$

Moreover, by (2.3) and Lemma 4.1-(c) in Appendix

$$\begin{aligned} \eta^\mu \sum_{\alpha=2}^3 G_\alpha((\text{adj}_2 A)^\alpha) & \leq \eta^\mu k_3 \sum_{\alpha=2}^3 (|(\text{adj}_2 A)^\alpha|^q + 1) \\ & \leq c\eta^\mu + c\mu^q \eta^{\mu-q} \left( \frac{u^1 - k}{t - s} \right)^q (|Du^2| + |Du^3|)^q. \end{aligned}$$

The first inequality in (2.12) implies  $q < p^*$ . Using the Young inequality with exponents  $\frac{p^*}{q}$  and  $\frac{p^*}{p^*-q}$  we get that, a.e. in  $A_{k,t}^1 \cap \{\eta > 0\}$ ,

$$c\mu^q \eta^{\mu-q} \left( \frac{u^1 - k}{t - s} \right)^q (|Du^2| + |Du^3|)^q \leq c \left( \frac{u^1 - k}{t - s} \right)^{p^*} + c (|Du^2| + |Du^3|)^{\frac{qp^*}{p^*-q}}.$$

We have so proved that

$$\int_{A_{k,t}^1 \cap \{\eta > 0\}} \eta^\mu \sum_{\alpha=2}^3 G_\alpha((\text{adj}_2 A)^\alpha) dx \leq c \int_{A_{k,t}^1} \left\{ 1 + \left( \frac{u^1 - k}{t - s} \right)^{p^*} + (|Du^2| + |Du^3|)^{\frac{qp^*}{p^*-q}} \right\} dx. \quad (2.22)$$

By (2.4) and computing  $\det(A)$  with respect to the first row (see Lemma 4.1-(b))

$$\eta^\mu H(\det A) \leq c\eta^\mu + c\mu^r \eta^{\mu-r} \left( \frac{u^1 - k}{t - s} \right)^r \left| (\text{adj}_2 Du)^1 \right|^r.$$

Notice that, by (2.12),  $r < p^*$ . By the Young inequality with exponents  $\frac{p^*}{r}$  and  $\frac{p^*}{p^*-r}$  we get

$$c\mu^r \eta^{\mu-r} \left( \frac{u^1 - k}{t - s} \right)^r \left| (\text{adj}_2 Du)^1 \right|^r \leq c \left( \frac{u^1 - k}{t - s} \right)^{p^*} + \left| (\text{adj}_2 Du)^1 \right|^{\frac{rp^*}{p^*-r}}.$$

Therefore

$$\int_{A_{k,t}^1 \cap \{\eta > 0\}} \eta^\mu H(\det A) dx \leq c \int_{A_{k,t}^1} \left\{ 1 + \left( \frac{u^1 - k}{t - s} \right)^{p^*} + \left| (\text{adj}_2 Du)^1 \right|^{\frac{rp^*}{p^*-r}} \right\} dx. \quad (2.23)$$



Taking into account that the inequalities in (2.12) are equivalent to  $\frac{qp^*}{p^*-q} < p$  and  $\frac{rp^*}{p^*-r} < q$ , by the Hölder inequality we obtain

$$\begin{aligned} & \int_{A_{k,t}^1} \left\{ (|Du^2| + |Du^3|)^{\frac{qp^*}{p^*-q}} + \left| (\text{adj}_2 Du)^1 \right|^{\frac{rp^*}{p^*-r}} \right\} dx \\ & \leq \left( \int_{B_R} (|Du^2| + |Du^3|)^p dx \right)^{\frac{qp^*}{(p^*-q)p}} |A_{k,t}^1|^{1 - \frac{qp^*}{p(p^*-q)}} \\ & \quad + \left( \int_{B_R} \left| (\text{adj}_2 Du)^1 \right|^q dx \right)^{\frac{rp^*}{(p^*-r)q}} |A_{k,t}^1|^{1 - \frac{rp^*}{q(p^*-r)}} \end{aligned} \quad (2.24)$$

Since  $|A_{k,t}^1| \leq |B_R| \leq 1$ , by (2.20), (2.21), (2.22), (2.23) and (2.24) and we get

$$\begin{aligned} & \int_{A_{k,s}^1} \left\{ F_1(Du^1) + \sum_{\alpha=2}^3 G_\alpha((\text{adj}_2 Du)^\alpha) + H(\det Du) \right\} dx \\ & \leq c \int_{A_{k,t}^1} \left( \frac{u^1 - k}{t - s} \right)^{p^*} dx \\ & \quad + c \left\{ 1 + \left( \int_{B_R} (|Du^2| + |Du^3|)^p dx \right)^{\frac{qp^*}{(p^*-q)p}} + \left( \int_{B_R} \left| (\text{adj}_2 Du)^1 \right|^q dx \right)^{\frac{rp^*}{(p^*-r)q}} \right\} |A_{k,t}^1|^\vartheta, \end{aligned} \quad (2.25)$$

where

$$\vartheta := \min \left\{ 1 - \frac{qp^*}{p(p^*-q)}, 1 - \frac{rp^*}{q(p^*-r)} \right\}.$$

Since  $G_2, G_3, H \geq 0$  and

$$F_1(Du^1) \geq k_1 |Du^1|^p - k_2,$$

then (2.25) implies (2.13). □

### STEP 2: Decay of the “excess” on superlevel sets.

In this step we consider a *scalar* Sobolev function  $v : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $n \geq 2$ .

Let us assume that  $\Omega$  is an open set in  $\mathbb{R}^n$  and  $v$  is a *scalar* function  $v \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R})$ ,  $p \geq 1$ . Fix  $B_{R_0}(x_0) \Subset \Omega$ , with  $R_0 < 1$  small enough so that

$$|B_{R_0}(x_0)| < 1 \quad \text{and} \quad \int_{B_{R_0}} |v|^{p^*} dx < 1. \quad (2.26)$$

Here  $p^* = \frac{np}{n-p}$ , if  $p < n$ , and  $p^*$  is any  $\nu > p$ , else.

For every  $R \in (0, R_0]$  we define the decreasing sequences

$$\rho_h := \frac{R}{2} + \frac{R}{2^{h+1}} = \frac{R}{2} \left( 1 + \frac{1}{2^h} \right), \quad \bar{\rho}_h := \frac{\rho_h + \rho_{h+1}}{2} = \frac{R}{2} \left( 1 + \frac{3}{4 \cdot 2^h} \right).$$

Fixed a positive constant  $d \geq 1$ , define the increasing sequence of positive real numbers

$$k_h := d \left( 1 - \frac{1}{2^{h+1}} \right), \quad h \in \mathbb{N}.$$

Moreover, define the sequence  $(J_{v,h})$ ,

$$J_{v,h} := \int_{A_{k_h, \rho_h}} (v - k_h)^{p^*} dx,$$

where  $A_{k,\rho} = \{v > k\} \cap B_\rho$ . The following result holds.

**Proposition 2.4.** *Let  $v \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R})$ ,  $p \geq 1$ . Fix  $B_{R_0}(x_0) \Subset \Omega$ , with  $R_0 < 1$  small enough such that (2.26) holds. If there exists  $0 \leq \vartheta \leq 1$  and  $c_0 > 0$  such that for every  $0 < s < t \leq R_0$  and for every  $k \in \mathbb{R}$*

$$\int_{A_{k,s}} |Dv|^p dx \leq c_0 \left\{ \int_{A_{k,t}} \left( \frac{v-k}{t-s} \right)^{p^*} dx + |A_{k,t}|^\vartheta \right\}, \quad (2.27)$$

then, for every  $R \in (0, R_0]$ ,

$$J_{v,h+1} \leq c(\vartheta, R) \left( 2^{\frac{p^* p^*}{p}} \right)^h J_{v,h}^{\frac{\vartheta p^*}{p}},$$

with the positive constant  $c$  independent of  $h$ .

*Proof.* In the following we write  $J_h$  in place of  $J_{v,h}$ .

Notice that  $(J_h)$  is a decreasing sequence, since the following chain of inequalities holds:

$$J_{h+1} \leq \int_{A_{k_{h+1}, \rho_h}} (v - k_{h+1})^{p^*} dx \leq \int_{A_{k_h, \rho_h}} (v - k_h)^{p^*} dx \leq J_h \quad \forall h. \quad (2.28)$$

Let now define a sequence  $(\zeta_h)$  of cut-off functions in  $C_c^\infty(B_{\bar{\rho}_h}(x_0))$ , such that  $0 \leq \zeta_h \leq 1$ ,  $\zeta_h \equiv 1$  in  $B_{\rho_{h+1}}$ ,  $|D\zeta_h| \leq \frac{2^{h+4}}{R}$ .

If we denote  $(v - k_{h+1})_+ = \max\{v - k_{h+1}, 0\}$  we get

$$\begin{aligned} J_{h+1} &= \int_{A_{k_{h+1}, \rho_{h+1}}} (v - k_{h+1})^{p^*} \zeta_h^{p^*} dx \leq \int_{A_{k_{h+1}, \bar{\rho}_h}} (v - k_{h+1})^{p^*} \zeta_h^{p^*} dx \\ &= \int_{B_R} (\zeta_h (v - k_{h+1})_+)^{p^*} dx. \end{aligned} \quad (2.29)$$

By the Sobolev embedding Theorem and the properties of  $\zeta_h$  we get

$$\begin{aligned} &\int_{B_R} (\zeta_h (v - k_{h+1})_+)^{p^*} dx \\ &\leq c \left( \int_{B_R} |D(\zeta_h (v - k_{h+1})_+)|^p dx \right)^{\frac{p^*}{p}} \\ &\leq c \left\{ \left( \int_{B_R} |Dv \zeta_h|^p \chi_{\{v > k_{h+1}\}} dx \right)^{\frac{1}{p}} + \left( \int_{B_R} |(v - k_{h+1})_+ D\zeta_h|^p dx \right)^{\frac{1}{p}} \right\}^{p^*} \\ &\leq c \left\{ \left( \int_{A_{k_{h+1}, \bar{\rho}_h}} |Dv|^p dx \right)^{\frac{1}{p}} + \left( \left( \frac{2^h}{R} \right)^p \int_{A_{k_{h+1}, \rho_h}} (v - k_{h+1})^p dx \right)^{\frac{1}{p}} \right\}^{p^*}. \end{aligned} \quad (2.30)$$

Using (2.27) with  $k = k_{h+1}$ ,  $t = \rho_h$ ,  $s = \bar{\rho}_h$  we obtain

$$\int_{A_{k_{h+1}, \bar{\rho}_h}} |Dv|^p dx \leq c \left\{ \left( \frac{2^h}{R} \right)^{p^*} \int_{A_{k_{h+1}, \rho_h}} |v - k_{h+1}|^{p^*} dx + |A_{k_{h+1}, \rho_h}|^\vartheta \right\}. \quad (2.31)$$

Collecting (2.29), (2.30), (2.31), we obtain

$$J_{h+1} \leq c \left\{ \left( \frac{2^h}{R} \right)^{p^*} \int_{A_{k_{h+1}, \rho_h}} (v - k_{h+1})^{p^*} dx + |A_{k_{h+1}, \rho_h}|^\vartheta + \left( \frac{2^h}{R} \right)^p \int_{A_{k_{h+1}, \rho_h}} (v - k_{h+1})^p dx \right\}^{\frac{p^*}{p}}. \quad (2.32)$$

Since  $z^p \leq z^{p^*} + 1$  for every  $z \geq 0$ , then

$$\left( \frac{2^h}{R} \right)^p \int_{A_{k_{h+1}, \rho_h}} (v - k_{h+1})^p dx \leq \left( \frac{2^h}{R} \right)^{p^*} \int_{A_{k_{h+1}, \rho_h}} (v - k_{h+1})^{p^*} dx + |A_{k_{h+1}, \rho_h}|,$$

so obtaining

$$J_{h+1} \leq c \left\{ \left( \frac{2^h}{R} \right)^{p^*} \int_{A_{k_{h+1}, \rho_h}} (v - k_{h+1})^{p^*} dx + |A_{k_{h+1}, \rho_h}|^\vartheta + |A_{k_{h+1}, \rho_h}| \right\}^{\frac{p^*}{p}}. \quad (2.33)$$

Since

$$|A_{k_{h+1}, \rho_h}| (k_{h+1} - k_h)^{p^*} \leq \int_{A_{k_{h+1}, \rho_h}} (v - k_h)^{p^*} dx \leq J_h,$$

then

$$|A_{k_{h+1}, \rho_h}| \leq \frac{J_h}{(k_{h+1} - k_h)^{p^*}} = \left( \frac{2^{h+2}}{d} \right)^{p^*} J_h.$$

Taking also into account that

$$\int_{A_{k_{h+1}, \rho_h}} (v - k_{h+1})^{p^*} dx \leq \int_{A_{k_{h+1}, \rho_h}} (v - k_h)^{p^*} dx \leq J_h,$$

the inequality (2.33) gives

$$J_{h+1} \leq c \left\{ \left( \frac{2^h}{R} \right)^{p^*} J_h + \left( \frac{2^h}{d} \right)^{\vartheta p^*} J_h^\vartheta + \left( \frac{2^h}{d} \right)^{p^*} J_h \right\}^{\frac{p^*}{p}}. \quad (2.34)$$

Since  $J_h \leq 1$  for every  $h$  and recalling that  $d \geq 1 > R_0 \geq R$ , we get

$$\left( \frac{2^h}{R} \right)^{p^*} J_h + \left( \frac{2^h}{d} \right)^{\vartheta p^*} J_h^\vartheta + \left( \frac{2^h}{d} \right)^{p^*} J_h \leq \left\{ 2 \frac{2^{hp^*}}{R^{p^*}} + \frac{2^{h\vartheta p^*}}{R^{\vartheta p^*}} \right\} J_h^\vartheta \leq \left( \frac{2}{R^{p^*}} + \frac{1}{R^{\vartheta p^*}} \right) 2^{hp^*} J_h^\vartheta.$$

By (2.34) it follows

$$J_{h+1} \leq c \left\{ \left( \frac{2}{R^{p^*}} + \frac{1}{R^{\vartheta p^*}} \right) 2^{hp^*} J_h^\vartheta \right\}^{\frac{p^*}{p}} \leq c(\vartheta, R) \left( 2^{\frac{p^* p^*}{p}} \right)^h J_h^{\vartheta \frac{p^*}{p}}.$$

□

### STEP 3: Iteration and proof of Theorem 2.1.

We now resume the proof of Theorem 2.1. As in the proof of Proposition 2.3, we will consider an integrand function  $f$  independent on  $x$ ; consequently,  $a, b, c$  in (2.2), (2.3), (2.4) have to be considered 0.

We need the following classical result, see e.g. [21].

**Lemma 2.5.** *Let  $\gamma > 0$  and let  $(J_h)$  be a sequence of real positive numbers, such that*

$$J_{h+1} \leq A \lambda^h J_h^{1+\gamma} \quad \forall h \in \mathbb{N} \cup \{0\}, \quad (2.35)$$

with  $A > 0$  and  $\lambda > 1$ . If  $J_0 \leq A^{-\frac{1}{\gamma}} \lambda^{-\frac{1}{\gamma^2}}$ , then  $J_h \leq \lambda^{-\frac{h}{\gamma}} J_0$  and  $\lim_{h \rightarrow \infty} J_h = 0$ .

Fix  $B_{R_0}(x_0) \Subset \Omega$ , with  $R_0 < 1$  small enough such that  $|B_{R_0}(x_0)| < 1$  and  $\int_{B_{R_0}} |u|^{p^*} dx \leq 1$ . By Proposition 2.3 we have that  $u^1$  satisfies, for every  $0 < s < t \leq R_0$  and every  $k \in \mathbb{R}$ ,

$$\begin{aligned} \int_{A_{k,s}^1} |Du^1|^p dx &\leq c \int_{A_{k,t}^1} \left( \frac{u^1 - k}{t - s} \right)^{p^*} dx \\ &+ c \left\{ 1 + \left( \int_{B_{R_0}} (|Du^2| + |Du^3|)^p dx \right)^{\frac{qp^*}{(p^*-q)p}} + \left( \int_{B_{R_0}} |(\text{adj}_2 Du)^1|^q dx \right)^{\frac{rp^*}{(p^*-r)q}} \right\} |A_{k,t}^1|^\vartheta, \end{aligned} \quad (2.36)$$

where  $c > 0$  is independent of  $s, t, k$  and  $\vartheta := \min\{1 - \frac{qp^*}{p(p^*-q)}, 1 - \frac{rp^*}{q(p^*-r)}\}$ .

Therefore the scalar function  $u^1$  satisfies (2.27) of Proposition 2.4 with constant  $c_0$  depending on

$$\int_{B_{R_0}} (|Du^2| + |Du^3|)^p dx \quad \text{and} \quad \int_{B_{R_0}} |(\text{adj}_2 Du)^1|^q dx.$$

Note that these integrals are finite by (2.2) and (2.3). Moreover, they are independent of  $u^1$ ; indeed  $(\text{adj}_2 Du)^1$  is depending only on  $u^2$  and  $u^3$ .

As above, let us define

$$k_h := d \left( 1 - \frac{1}{2^{h+1}} \right), \quad h \in \mathbb{N}$$

with  $d \geq 1$  ( $d$  will be fixed later) and, for every  $R \in (0, R_0]$ , define

$$\rho_h := \frac{R}{2} + \frac{R}{2^{h+1}} = \frac{R}{2} \left( 1 + \frac{1}{2^h} \right), \quad \bar{\rho}_h := \frac{\rho_h + \rho_{h+1}}{2} = \frac{R}{2} \left( 1 + \frac{3}{4 \cdot 2^h} \right)$$

and

$$J_{u^1, h} := \int_{A_{k_h, \rho_h}^1} (u^1 - k_h)^{p^*} dx.$$

Proposition 2.4, applied to  $u^1$ , gives

$$J_{u^1, h+1} \leq c(\vartheta, R) \left( 2^{\frac{p^* p^*}{p}} \right)^h J_{u^1, h}^{\vartheta \frac{p^*}{p}}, \quad (2.37)$$

with the positive constant  $c$  independent of  $h$  and, by (2.5), with the exponent  $\vartheta \frac{p^*}{p}$  greater than 1.

Indeed, since

$$J_{u^1, 0} = \int_{A_{\frac{d}{2}, R}^1} \left( u^1 - \frac{d}{2} \right)^{p^*} dx \rightarrow_{d \rightarrow +\infty} 0,$$

we can choose  $d \geq 1$  large enough, so that

$$J_{u^1, 0} < c(\vartheta, R)^{-\frac{1}{\vartheta \frac{p^*}{p} - 1}} \left( 2^{\frac{p^* p^*}{p}} \right)^{-\frac{1}{(\vartheta \frac{p^*}{p} - 1)^2}}.$$

Therefore, by Lemma 2.5,  $\lim_{h \rightarrow \infty} J_{u^1, h} = 0$ . Thus,  $u^1 \leq d$  a.e. in  $B_{\frac{R_0}{2}}$ . We have so proved that  $u^1$  is locally bounded from above.

To prove that  $u^1$  is locally bounded from below, we notice that  $-u$  is a local minimizer of  $\int_{\Omega} \tilde{f}(Dv) dx$  where

$$\tilde{f}(\xi) := \sum_{\alpha=1}^3 \{F_{\alpha}(-\xi^{\alpha}) + G_{\alpha}((\text{adj}_2 \xi)^{\alpha})\} + H(-\det \xi),$$

If we denote

$$\tilde{F}_{\alpha}(\lambda) := F_{\alpha}(-\lambda), \quad \tilde{H}(t) = H(-t), \quad \lambda \in \mathbb{R}^3, t \in \mathbb{R},$$

the functions  $\tilde{F}_{\alpha}, \tilde{H}_{\alpha}$  are convex and satisfy (2.2) and (2.4).

The function  $\tilde{f}$  satisfies the assumptions of Theorem 2.1; so we obtain that there exists  $d'$  such that  $-u^1 \leq d'$  a.e. in  $B_{\frac{R_0}{2}}$ . We have so proved that  $u^1 \in L^{\infty}(B_{\frac{R_0}{2}}(x_0))$ . Due to the arbitrariness of  $x_0$  and  $R_0$ , we get  $u^1 \in L_{\text{loc}}^{\infty}(\Omega)$ .

The symmetric structure of the energy density  $f$  allows to obtain an analogous statement to Proposition 2.3 also for  $u^2$  and  $u^3$ . Therefore, reasoning as for  $u^1$  (see also Remark 4.3), we obtain that  $u^2, u^3 \in L_{\text{loc}}^{\infty}(\Omega)$ , too.

**Remark 2.6.** *We proved Theorem 2.1 by assuming that the integrand function is independent of  $x$ . In the general case,  $f$  depending on  $x$  and satisfying the general growth conditions (2.2)-(2.4), with  $a, b, c$  belonging to  $L^{\sigma}$ ,  $\sigma > 1$ , the proof goes in a similar way, with the additional condition  $1 - \frac{1}{\sigma} > \frac{p}{p^*}$ .*

### 3. EXISTENCE AND REGULARITY

Consider an open, bounded set  $\Omega \subseteq \mathbb{R}^3$  and a Caratheodory function  $f : \Omega \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ ,  $f(x, \xi) := g(x, T(\xi))$  with

$$T(\xi) := (\xi, \text{adj}_2 \xi, \det \xi) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \times \mathbb{R}$$

where  $z \mapsto g(x, z)$  is convex.

Let  $\bar{u} \in W^{1,p}(\Omega; \mathbb{R}^3)$  be a function such that  $\int_{\Omega} f(x, D\bar{u}(x)) dx < +\infty$ . Consider the minimization problem

$$\min \left\{ I(u) := \int_{\Omega} f(x, Du(x)) dx : u \in \bar{u} + W_0^{1,p}(\Omega; \mathbb{R}^3) \right\}. \quad (\text{P})$$

We suppose that there exist constants  $c_1 > 0$ ,  $c_2 \geq 0$  and real exponents  $p, q, \tilde{r} > 1$  such that

$$c_1 (|\xi|^p + |\text{adj}_2 \xi|^q + |\det \xi|^{\tilde{r}}) - c_2 \leq f(x, \xi). \quad (3.1)$$

The following existence result holds (see Remark 8.32 (iii) in [8]).

**Theorem 3.1.** *Consider the variational problem (P). If  $f$  satisfies (3.1) with exponents*

$$2 \leq p < +\infty, \quad \frac{p}{p-1} \leq q < +\infty, \quad 1 < \tilde{r} < +\infty,$$

then (P) has a solution.

As a consequence of Theorem 3.1 and Theorem 2.1 we have the following.

**Theorem 3.2.** *Consider the variational problem (P), where  $\Omega$  is an open bounded set in  $\mathbb{R}^3$  and  $f$  satisfies (2.1), the growth conditions (2.2), (2.3), (2.4) and*

$$k_4 |\det \xi|^{\tilde{r}} - k_5 \leq H(\det \xi) \quad (3.2)$$

with  $p \in \left( \frac{3+\sqrt{45}}{4}, 3 \right)$ ,  $\frac{p}{p-1} \leq q < \frac{p^*(p^*-p)p}{(p^*)^2+(p^*-p)p}$ ,  $1 < \tilde{r} \leq r < \frac{p^*(p^*-p)q}{(p^*)^2+(p^*-p)q}$ ,  $k_4 > 0$ ,  $k_5 \geq 0$  and  $\sigma > \frac{3}{p}$ .

Then there exists a minimizer  $u$  of (P), with  $u \in L_{\text{loc}}^{\infty}(\Omega, \mathbb{R}^3)$ .

*Proof.* It suffices to notice that  $\frac{p}{p^*} < 1 - \frac{qp^*}{p(p^*-q)}$  is equivalent to  $q < \frac{p^*(p^*-p)p}{(p^*)^2+(p^*-p)p}$  and that  $p \in \left(\frac{3+\sqrt{45}}{4}, 3\right)$  implies that  $\frac{p}{p-1} < \frac{p^*(p^*-p)p}{(p^*)^2+(p^*-p)p}$ . The thesis immediately follows by Theorem 3.1 and, taking into account that a minimizer is also a local minimizer, by Theorem 2.1.  $\square$

#### 4. APPENDIX

Given a vector  $v = (v_1, \dots, v_n) \in \mathbb{R}^n$  we write  $|v| := \sqrt{\sum_{i=1}^n v_i^2}$ . Analogously, given a matrix  $A = (a_{ij})$ ,  $i, j \in \{1, \dots, n\}$ ,  $A^i$  is its  $i$ -th row and  $|A| := \sqrt{\sum_{i,j=1}^n a_{ij}^2}$ .

**Lemma 4.1.** *Consider the matrices  $A, B \in \mathbb{R}^{3 \times 3}$*

$$A = \begin{pmatrix} A^1 \\ B^2 \\ B^3 \end{pmatrix}, \quad B = \begin{pmatrix} B^1 \\ B^2 \\ B^3 \end{pmatrix}.$$

*Then the following estimates hold:*

- (a)  $|A| \leq |A^1| + |B^2| + |B^3|$ ,
- (b)  $|\det A| \leq |A^1| |(\operatorname{adj}_2 B)^1|$ ,
- (c)  $|(\operatorname{adj}_2 A)_{2j}| \leq |A^1| |B^3|$  and  $|(\operatorname{adj}_2 A)_{3j}| \leq |A^1| |B^2|$ , for all  $j \in \{1, 2, 3\}$ .

*Proof.* The first estimate is trivial, because

$$|A| = \sqrt{|A^1|^2 + |B^2|^2 + |B^3|^2} \leq |A^1| + |B^2| + |B^3|.$$

To prove the second one, notice that

$$|\det A| \leq \sum_{j=1}^3 |A_{1j}| |(\operatorname{adj}_2 A)_{1j}|.$$

Since the second and third rows of  $A$  and  $B$  coincide,

$$(\operatorname{adj}_2 A)_{1j} = (\operatorname{adj}_2 B)_{1j} \quad j \in \{1, 2, 3\};$$

and, moreover,

$$\left| (\operatorname{adj}_2 B)^1 \right| = |((\operatorname{adj}_2 B)_{11}, (\operatorname{adj}_2 B)_{12}, (\operatorname{adj}_2 B)_{13})|$$

we have

$$\sum_{j=1}^3 |A_{1j}| |(\operatorname{adj}_2 A)_{1j}| = \sum_{j=1}^3 |A_{1j}| |(\operatorname{adj}_2 B)_{1j}| \leq |A^1| |(\operatorname{adj}_2 B)^1|$$

and we conclude.

To prove (c) notice that, fixed  $j \in \{1, 2, 3\}$ ,

$$|(\operatorname{adj}_2 A)_{2j}| \leq |A_{1i}| |B_{3k}| + |A_{1k}| |B_{3i}| \quad i, k \in \{1, 2, 3\} \setminus \{j\}, \quad i \neq k;$$

so the first inequality in (c) follows. Analogously the second inequality follows.  $\square$

**Lemma 4.2.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^3$ . Consider a Caratheodory function  $f : \Omega \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ . Assume that there exists  $c_1, c_3 > 0$  and  $c_2 \geq 0$  such that for every  $\xi \in \mathbb{R}^9$*

$$c_1 (|\xi|^p + |\operatorname{adj}_2 \xi|^q) - c_2 \leq f(x, \xi) \leq c_3 (|\xi|^p + |\operatorname{adj}_2 \xi|^q + |\det \xi|^r + 1 + \omega(x)). \quad (4.1)$$

*with  $1 \leq p, 1 \leq q, 1 \leq r, \omega(x) \geq 0$ .*

Let  $u \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^3)$  be such that  $x \rightarrow f(x, Du(x)) \in L_{\text{loc}}^1(\Omega)$ . Fixed  $\eta \in C_c^1(\Omega)$ ,  $\eta \geq 0$ , and  $k \in \mathbb{R}$ , denote, for a.e.  $x \in \{u^1 > k\} \cap \{\eta > 0\}$ ,

$$A := \begin{pmatrix} \mu\eta^{-1}(k - u^1)D\eta \\ Du^2 \\ Du^3 \end{pmatrix}.$$

For the sake of simplicity, we write  $f(A)$  instead of  $f(x, A)$  and  $f(Du)$  instead of  $f(x, Du(x))$ . If (2.12) holds and  $\omega \in L_{\text{loc}}^1(\Omega)$ , then

$$\eta^t f(A) \in L^1(\{u^1 > k\} \cap \{\eta > 0\}) \quad \forall t \geq p^*.$$

*Proof.* Denote  $\hat{u} := (u^2, u^3)$  and

$$D\hat{u} := \begin{pmatrix} Du^2 \\ Du^3 \end{pmatrix}.$$

By the growth condition (4.1) and Lemma 4.1 we have, a.e. in  $\{u^1 > k\} \cap \{\eta > 0\}$ ,

$$\begin{aligned} f(A) &\leq c \{ (\mu\eta^{-1}(u^1 - k)|D\eta|)^p + |D\hat{u}|^p + |(\text{adj}_2 Du)^1|^q + 1 + \omega \} \\ &\quad + c \{ \mu\eta^{-1}(u^1 - k)|D\eta||D\hat{u}| \}^q + c \{ \mu\eta^{-1}(u^1 - k)|D\eta| |(\text{adj}_2 Du)^1| \}^r. \end{aligned} \quad (4.2)$$

Since (2.12) holds,  $q < p$  and  $r < q$ ; thus there exist  $\alpha > 1$  and  $\beta > 1$  such that

$$q\alpha < p^*, \quad q\alpha' = p, \quad \text{and} \quad r\beta < p^*, \quad r\beta' = q.$$

Therefore, by Young inequality, there exists  $c > 0$  such that, a.e. in  $\{u^1 > k\} \cap \{\eta > 0\}$ ,

$$\begin{aligned} &\{ \mu\eta^{-1}(u^1 - k)|D\eta||D\hat{u}| \}^q + \{ \mu\eta^{-1}(u^1 - k)|D\eta| |(\text{adj}_2 Du)^1| \}^r \\ &\leq c (\mu\eta^{-1}(u^1 - k)|D\eta|)^{q\alpha} + c (\mu\eta^{-1}(u^1 - k)|D\eta|)^{r\beta} + c (|D\hat{u}|^p + |(\text{adj}_2 Du)^1|^q) \end{aligned} \quad (4.3)$$

Denote  $\tilde{q} := \max\{p, q\alpha, r\beta\}$ . We have

$$(\mu\eta^{-1}(u^1 - k)|D\eta|)^p + (\mu\eta^{-1}(u^1 - k)|D\eta|)^{q\alpha} + (\mu\eta^{-1}(u^1 - k)|D\eta|)^{r\beta} \leq \mu^{\tilde{q}}\eta^{-\tilde{q}}(u^1 - k)^{\tilde{q}}|D\eta|^{\tilde{q}} + 3. \quad (4.4)$$

Therefore, by (4.2), (4.3) and (4.4), a.e. in  $\{u^1 > k\} \cap \{\eta > 0\}$  we have

$$\eta^t f(A) \leq c \{ \mu^{\tilde{q}}\eta^{t-\tilde{q}}(u^1 - k)^{\tilde{q}}|D\eta|^{\tilde{q}} + \eta^t |D\hat{u}|^p + \eta^t |(\text{adj}_2 Du)^1|^q + \eta^t + \eta^t \omega \}$$

By (4.1) and  $f(Du) \in L_{\text{loc}}^1(\Omega)$  we obtain

$$\eta^t |D\hat{u}|^p + \eta^t |(\text{adj}_2 Du)^1|^q \leq \eta^t (|Du|^p + |\text{adj}_2 Du|^q) \leq \frac{\eta^t}{c_1} (f(Du) + c_2) \in L^1(\Omega).$$

Since  $u \in L_{\text{loc}}^{p^*}(\Omega; \mathbb{R}^3)$  and  $t - \tilde{q} > t - p^* \geq 0$ , then  $\eta^{t-\tilde{q}}(u^1 - k)^{\tilde{q}}|D\eta|^{\tilde{q}} \in L^1(\{u^1 > k\} \cap \{\eta > 0\})$ . We have so proved that  $\eta^t f(A) \in L^1(\{u^1 > k\} \cap \{\eta > 0\})$  for all  $t \geq p^*$ .  $\square$

**Remark 4.3.** Analogous inequalities to those in Lemma 4.1 hold true if

$$A = \begin{pmatrix} B^1 \\ A^2 \\ B^3 \end{pmatrix}, \quad \text{or} \quad A = \begin{pmatrix} B^1 \\ B^2 \\ A^3 \end{pmatrix}.$$

Therefore, a statement similar to Lemma 4.2 can be given for  $u^2$  and  $u^3$ , with

$$A := \begin{pmatrix} Du^1 \\ \mu\eta^{-1}(k - u^2)D\eta \\ Du^3 \end{pmatrix} \quad \text{and} \quad A := \begin{pmatrix} Du^1 \\ Du^2 \\ \mu\eta^{-1}(k - u^3)D\eta \end{pmatrix},$$

respectively.

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