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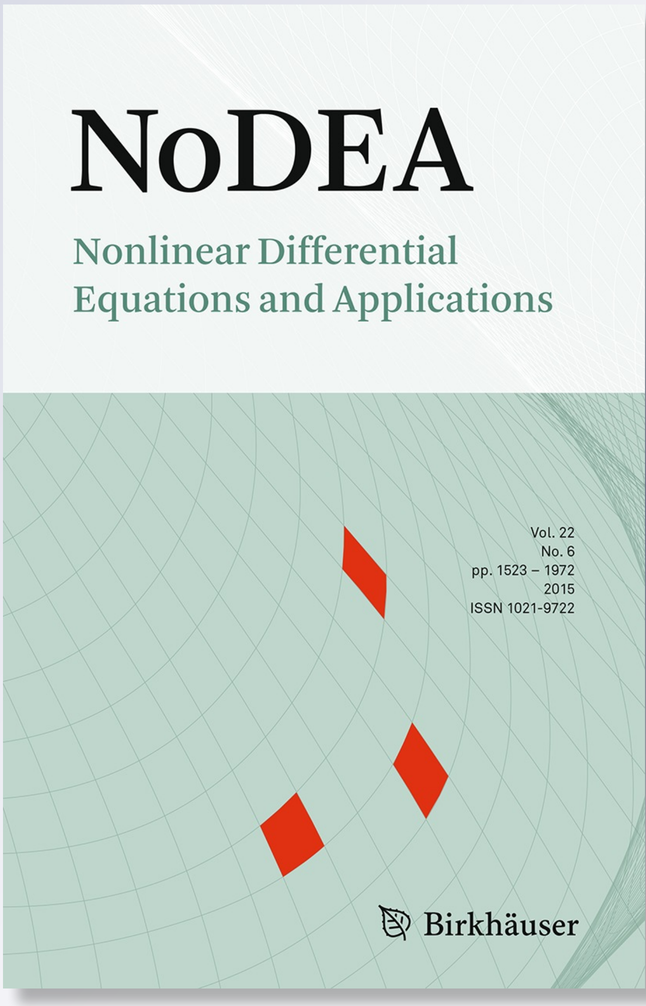

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Abstract. We investigate the structure of the set of periodic solutions of a time-dependent generalized version of the sunflower equation (in fact of the delayed Liénard equation), where the coefficients can vary periodically, thus allowing for environmental oscillations. Our result stems from a more general analysis, based on fixed point index and degree-theoretic methods, of the set of T-periodic solutions of T-periodically perturbed coupled delay differential equations on differentiable manifolds.

Mathematics Subject Classification. 34K13, 34C25, 34C40.

Keywords. Sunflower equation, Coupled differential equations, Branches of periodic solutions, Fixed point index.

1. Introduction

Physical and biological phenomena involving memory effects (such as viscosity in solids, hysteresis in ferroelectrics, latency of diseases, delay feedback, fading effect for hormones concentration, etc.) are described commonly by differential equations involving time delay (see, e.g., [2,11,17,27]). Indeed, in recent years there has been a growing interest around the dynamical behavior of such equations. In particular, a certain amount of research has been dedicated to the sunflower equation, i.e.

$$\ddot{y}(t) = -\frac{\alpha}{r}\dot{y}(t) - \frac{\beta}{r}\sin\big(y(t-r)\big), \tag{1}$$

where $r > 0$ is a finite time delay, α and β are experimental parameters (see [16], also [9] for the derivation). This scalar equation is a mathematical model used to describe the helical movements of the tip of a sunflower plant. The top of the stem of the sunflower performs a rotating movement and $y(t)$ is the angle of the plant with respect to the vertical direction, the lag time r corresponds to a delayed reaction in response to the effect due to accumulation of the growth hormone (auxin) alternatively on both side of the plant. Roughly speaking, the parameters α and β control, respectively, the fading of the effect of auxin's past concentration and the growth speed. Somolinos in [26] showed the existence of periodic solutions to (1) for a certain range of values for the involved parameters α, β and r. This existence result covers both the cases of small and large amplitude limit cycles generated by Hopf bifurcation. More recently, Liu and Kalmár-Nagy [20] computed limit cycle amplitudes and frequencies for (1). Other meaningful results related to this equation can be found e.g., in [2,8,10,19,22,31].

Our investigation follows a different path: We extend Eq. (1) introducing a more general dependence on the past and allowing time variation of the coefficients to account, e.g., for environmentally induced changes in the response of the plant to hormone concentration and "memory" fading. The resulting model is obtained by assuming that the coefficients $-\alpha/r$ of and $-\beta/r$ are actually T-periodic functions, $T > 0$ given, and replacing the sinus function in the second term in the left-hand side of (1) with a generic function depending on the present and past status. We also control the magnitude of this second term by prepending a parameter $\lambda \geq 0$ to it. The parametrized equation under consideration then, reads as follows:

$$\ddot{y}(t) = a(t)\dot{y}(t) + \lambda b(t)\phi\big(y(t), y(t-r)\big), \quad \lambda \geq 0, \tag{2}$$

where a, b and ϕ are continuous, a and b are T-periodic with average

$$\not{a} := \frac{1}{T}\int_0^T a(t)\,dt \neq 0$$

and $b(t) \neq 0$ for all t. The assumption $\not{a} \neq 0$ serves to generalize the constant coefficient of $\dot{y}(t)$ in (1): Ideally, $a(t)$ can be thought as a perturbation of the constant term $-\alpha/r$, namely $a(t) = -\alpha/r + \epsilon(t)$ where $\epsilon(t)$ is continuous, T-periodic and sufficiently small so that $\not{a} = -\alpha/r + \not{\epsilon} \neq 0$. Similar considerations hold for b.

We point out that, with respect to (1), an extended version of the second addendum as in (2) makes sense also in view of different applications. For instance, in [11, Ch. 4 §2] equations of the form (2) are considered (with constant experimental parameters) in connection with a mathematical description of sleep disorders.

What we investigate of (2) is the structure of the set of T-periodic solutions as $\lambda \geq 0$ varies. Informally, in our main result (Theorem 1.1) we find a topological condition on the function $p \mapsto \phi(p,p)$ which implies that there exists a connected set of 'nontrivial' T-periodic solutions of (2) whose closure is not compact and intersects the set of constant solutions of (2) for $\lambda = 0$. More precisely, let $C^1_T(\mathbb{R})$ be the Banach space of the T-periodic $\mathbb{R}$-valued functions (with the uniform topology). A pair $(\lambda, y) \in [0,\infty) \times C^1_T(\mathbb{R})$ is called a *T-pair* for (2) if y is a T-periodic solution of (2), also (λ, y) is called *trivial* if y is constant and $\lambda = 0$. For any $p \in \mathbb{R}$ let $\widehat{p}$ denote the constant function $\widehat{p} \equiv p$. Our main result is the following:

Theorem 1.1. *Let ϕ and a be as in (2) and let $\Omega \subseteq [0,\infty) \times C^1_T(\mathbb{R})$ be open. Take $W_\Omega := \big\{p \in \mathbb{R} : (0,\widehat{p}) \in \Omega\big\}$ and let $w(q) := \phi(q,q)$. Assume that* $\deg(w, W_\Omega)$ *is well-defined and nonzero. Then, there exists a connected set of nontrivial*

T-pairs for (2)*, whose closure meets the set* $\{(0,\widehat{p}) \in \Omega : w(p) = 0\}$ *and is not compact.*

This theorem will be proved by rewriting (2) as a first-order coupled system in the Liénard plane (as e.g., in [2]) and then applying a result about such systems in a very general setting. Actually, we provide a generalization of [28] (compare also [18]) based on [7] (Theorem 4.1), which deals with the structure of the set of harmonic solutions of periodically perturbed coupled ODEs on manifolds. Indeed, Theorem 4.1 is of some independent interest because it partially bridges the gap between [3] and [7,15] in the sense that the main results of those papers can be deduced from it (see Remark 4.3 below).

A feature of this paper is the combined use of topological methods (degree-theoretic and fixed point index) along with a classical Liénard-plane analysis. The former are used in Sects. 2, 3 and 4 to investigate coupled delay equations on differentiable manifold. Then, in Sect. 5, the obtained results are exploited to approach the particular case of Eq. (2).

2. Coupled delay differential equations

Let us describe more precisely our setting. Let $M \subseteq \mathbb{R}^k$ and $N \subseteq \mathbb{R}^s$ be boundaryless smooth manifolds, let $\mathfrak{f}\colon \mathbb{R} \times (M \times N)^2 \to \mathbb{R}^k$ be tangent to M, and let $\mathfrak{g}\colon M \times N \to \mathbb{R}^s$ and $\mathfrak{h}\colon \mathbb{R} \times (M \times N)^2 \to \mathbb{R}^s$ be tangent to N: This means that, for any $(t,p,q,v,w) \in \mathbb{R} \times (M \times N)^2$, then $\mathfrak{g}(p,q,v,w)$ and $\mathfrak{h}(t,p,q,v,w)$ belong to the tangent space T_qN, and $\mathfrak{f}(t,p,q,v,w)$ is in T_pM, respectively. Let also $a\colon \mathbb{R} \to \mathbb{R}$ be continuous. Given $T > 0$, we assume that $\mathfrak{f}$, $\mathfrak{h}$ and a are T-periodic in the t variable. Consider the following system of delay differential equations for $\lambda \geq 0$:

$$\begin{cases} \dot{x}(t) = \lambda \mathfrak{f}\big(t, x(t), y(t), x(t-r), y(t-r)\big), \\ \dot{y}(t) = a(t)\mathfrak{g}\big(x(t), y(t)\big) + \lambda \mathfrak{h}\big(t, x(t), y(t), x(t-r), y(t-r)\big), \end{cases} \tag{3}$$

where the time lag $r > 0$ is given. This system is equivalent to a single parameter-dependent delay differential equation on the product manifold $M \times N \subseteq \mathbb{R}^{k+s}$.

Denote by $C_T(M)$ and $C_T(N)$ the spaces of T-periodic continuous functions from $\mathbb{R}$ to M and N, respectively, with the topology of uniform convergence. We investigate the properties of the set of the T-periodic triples (or briefly T-triples) of (3), i.e. of those triples $(\mu, x, y) \in [0,\infty) \times C_T(M) \times C_T(N)$, where (x, y) is a solution to (3) when $\lambda = \mu$. In particular, we shall give conditions for the existence of a noncompact connected component of *nontrivial T-triples* (which we call a "*branch*") emanating from the set $\nu^{-1}(0)$, where $\nu : M \times N \to \mathbb{R}^{k+s}$ is the vector field, tangent to $M \times N \subseteq \mathbb{R}^{k+s}$, given by

$$\nu(p,q) = \big(w_{\mathfrak{f}}(p,q),\, \mathfrak{g}(p,q)\big), \tag{4}$$

with $w_{\mathfrak{f}}(p,q) := \frac{1}{T}\int_0^T \mathfrak{f}(t,p,q,p,q)\,dt$. In the present setting, a T-triple (λ, x, y) of (3) is said to be *trivial* if $\lambda = 0$ and (x, y) is constant.

It is tempting to try to achieve the desired generalization of [28] by simply using a time-transformation as in [6,29] to get rid of the factor $a(t)$ in (3) and then adapt the argument of [3, 15] to the present case. Nevertheless, this simple procedure does not work because the transformed perturbing term would result in a form inappropriate for our methods. In fact, the time-transformation used in [29] does not preserve the fixed-delay structure. Instead, to prove our result, we follow the lead of [7] and combine the techniques of [29] and [28].

Consider the system of equations (3). We are interested in its T-periodic solutions. Without loss of generality, as suggested in [12], we will assume that $T \geq r$. In fact, for $n \in \mathbb{N}$, the system (3) and

$$\begin{cases} \dot{x}(t) = \lambda\mathfrak{f}\Big(t, x(t), y(t), x\big(t-(r-nT)\big), y\big(t-(r-nT)\big)\Big) \\ \dot{y}(t) = a(t)\mathfrak{g}\big(x(t), y(t)\big) + \lambda\mathfrak{h}\Big(t, x(t), y(t), x\big(t-(r-nT)\big), y\big(t-(r-nT)\big)\Big), \end{cases}$$

have the same T-periodic solutions. Thus, if necessary, one can replace r with $r - nT$, where $n \in \mathbb{N}$ is such that $0 < r - nT \leq T$.

Let us now introduce some notation. Given any $X \subseteq \mathbb{R}^k$, $\widetilde{X}$ denotes the metric space $C\big([-r, 0], X\big)$ with the distance inherited from the Banach space $\widetilde{\mathbb{R}}^k = C([-r, 0], \mathbb{R}^k)$ with the usual supremum norm.

Given any $(p, q) \in M \times N$, denote by $p^\# \in \widetilde{M}$ and $q^\# \in \widetilde{N}$ the constant functions $p^\#(t) \equiv p$ and $q^\#(t) \equiv q$, $t \in [-r, 0]$, respectively. Thus, $(p^\#, q^\#) \in \widetilde{M \times N} \simeq \widetilde{M} \times \widetilde{N}$. For any $U \subseteq M \times N$, define $U^\# = \big\{(p^\#, q^\#) \in \widetilde{M \times N} : (p, q) \in U\big\}$. Also, given $W \subseteq \widetilde{M \times N}$, we put $W_\# = \big\{(p, q) \in M \times N : (p^\#, q^\#) \in W\big\}$. Finally, we will denote by $C_T(X)$ the metric subspace of the Banach space $\big(C_T(\mathbb{R}^k), \|\cdot\|\big)$ of all the T-periodic continuous maps $x : \mathbb{R} \to X$ (as above, with the usual C^0 norm). Observe that $C_T(X)$ is complete if and only if X is complete (or, equivalently, closed as a subset of $\mathbb{R}^k$). Nevertheless, since M and N are locally compact, $C_T(M \times N) \simeq C_T(M) \times C_T(N)$ is always locally complete.

3. Poincaré-type translation operator

Assume now, unless differently stated, that a, $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ are C^1. Consider the map H with domain $\mathcal{D}_H \subseteq \mathbb{R} \times \widetilde{M} \times \widetilde{N} \times \mathbb{R}$ in $\widetilde{M} \times \widetilde{N}$ defined by

$$H(\lambda, \varphi, \psi, \mu)(\theta) = \Big(x_{\lambda,\mu}\big(\varphi, \psi, T + \theta\big), y_{\lambda,\mu}\big(\varphi, \psi, T + \theta\big)\Big), \quad \theta \in [-r, 0],$$

where $t \mapsto \big(x_{\lambda,\mu}(\varphi, \psi, t), y_{\lambda,\mu}(\varphi, \psi, t)\big)$ denotes the unique maximal solution of the initial-value problem

$$\begin{cases} \dot{x}(t) = \lambda\left[\mu\mathfrak{f}\left(t, x(t), y(t), x(t-r), y(t-r)\right) + (1-\mu)\frac{a(t)}{\bar{a}}w_{\mathfrak{f}}\left(x(t), y(t)\right)\right], & t > 0, \\ \dot{y}(t) = a(t)\mathfrak{g}\left(x(t), y(t)\right) + \lambda\mu\mathfrak{h}\left(t, x(t), y(t), x(t-r), y(t-r)\right), & \\ x(t) = \varphi(t),\ y(t) = \psi(t), & t \in [-r, 0]. \end{cases}$$

Well known properties of differential equations imply that $\mathcal{D}_H$ is an open subset of $\mathbb{R} \times \widetilde{M} \times \widetilde{N} \times \mathbb{R}$. A similar argument shows that the set $\mathcal{D}' := \big\{(\varphi, \psi) \in \widetilde{M} \times \widetilde{N} : (0, \varphi, \psi, 1) \in \mathcal{D}_H\big\}$ is open as well. Also, since we are assuming $T \geq r$

(see above), the Theorem of Ascoli-Arzelà implies that H is a locally compact map (compare, e.g., [24] or [5]).

Remark 3.1. Consider the following equation:

$$\begin{cases} \dot{x}(t) = 0, \\ \dot{y}(t) = a(t)\mathfrak{g}\left(x(t), y(t)\right). \end{cases} \tag{5}$$

Given $V \subseteq \widetilde{M} \times \widetilde{N}$ such that $\overline{V} \subseteq \mathcal{D}'$ we have that all solutions of (5) starting at time $t = 0$ from $\overline{V_\#}$ are defined (at least) for $t \in [0, T]$. An argument similar to, e.g., [29, Remark 2.3] or [7, Remark 2.1]) shows that the same assertion holds for (5) when $a(t)$ is replaced with its average $\not{a}$:

$$\begin{cases} \dot{x}(t) = 0, \\ \dot{y}(t) = \not{a}\mathfrak{g}\big(x(t), y(t)\big). \end{cases} \tag{6}$$

In fact, one could prove that solutions of (5) and of (6), leaving at time $t = 0$ from the same point, coincide at time $t = T$. Thus, T-periodic orbits (images of solutions) of (5) and (6) must coincide. More precisely, let $\{\Phi_t\}_{t\in\mathbb{R}}$ be the local flow associated to (6). That is, $\Phi\colon U \to M \times N$ is defined on an open subset U of $\mathbb{R} \times M \times N$, containing $\{0\} \times M \times N$, with the property that for any $(p, q) \in M \times N$ the curve $t \mapsto \Phi_t(p, q)$ is the maximal solution of (6) given the initial condition $\Phi_0(p, q) = (p, q)$. Then, given $\tau \in \mathbb{R}$, the domain of Φ_τ is the open set consisting of those points $(p, q) \in M \times N$ for which the maximal solution of (6) starting from (p, q) at $t = 0$ is defined up to τ. (We are interested, in particular, to the case $\tau = T$.) Let $\{\Psi_t\}_{t\in\mathbb{R}}$ be the anologous local flow associated to (5). The argument of the above cited remarks show that $\Psi_T(p, q) = \Phi_T(p, q)$ whenever this relation makes sense, in particular for all $(p, q) \in \overline{V_\#}$.

The following definition is convenient:

Definition 3.2. We say that $V \subseteq \widetilde{M} \times \widetilde{N}$ has the *constant periodic property for* (5) if any T-periodic solution (x, y) of Eq. (5) that intersects $\partial V_\#$ is constant.

We have the following result:

Lemma 3.3. *Let $V \subseteq \widetilde{M} \times \widetilde{N}$ be open and such that*

$$Z_V := \left\{(p^\#, q^\#) \in V : \nu(p, q) = 0\right\}$$

is compact. Then, there exists an open neighborhood $W \subseteq V$ of Z_V and $\varepsilon > 0$ s.t. $[0, \varepsilon] \times \overline{W} \times [0, 1] \subseteq \mathcal{D}_H$ and $H\big([0, \varepsilon] \times \overline{W} \times [0, 1]\big)$ is compact.

Assume in addition that $V_\#$ is relatively compact, $\overline{V} \subseteq \mathcal{D}'$ and that V has the constant periodic property for (5) *(Definition 3.2). Then W can be taken in such a way that it has the constant periodic property as well. That is, if (x, y) is a T-periodic solution of* (5) *intersecting $\partial W_\#$, then (x, y) is constant.*

Proof. One immediately checks that the set Z_V consists of T-periodic solutions of (5). Thus, we have that $Z_V \subseteq \mathcal{D}'$ and the first part of the lemma follows from the local compactness of H.

Let us now prove the second part of the assertion. Let $\{\Phi_t\}_{t\in\mathbb{R}}$ be the local flow associated to (6) as in Remark 3.1. The map $(t, p, q) \mapsto \Phi_t(p, q)$ is continuous and, therefore, the "attainable set" $\mathcal{A}_T := \Phi_{[0,T]}\big(\overline{V_\#}\big)$ is compact. Thus, the union $\mathcal{O}_T$ of all T-periodic orbits of (6) starting from points of $\overline{V_\#}$, being closed in $\mathcal{A}_T$, is compact as well. Clearly, since (6) is autonomous, $\mathcal{O}_T$ is actually the set of all T-periodic orbits of (6) that intersect $\overline{V_\#}$.

Remark 3.1 shows that $\mathcal{O}_T$ consists indeed of all T-periodic orbits of (5) that intersect $\overline{V_\#}$. Let us denote by K the union of Z_V with this set. Clearly K is contained in $\mathcal{D}'$. The local compactness of H implies the existence of an open neighborhood $W \subseteq V$ of K and a positive ε with the property that $[0,\varepsilon]\times\overline{W}\times[0,1]\subseteq\mathcal{D}_H$ and $H\big([0,\varepsilon]\times\overline{W}\times[0,1]\big)$ is compact. The second part of the claim follows now from the fact a T-periodic solution of (5) whose image intersects the boundary $\partial W_\#$, of the set W just constructed, necessarily intersects $\partial V_\#$ and thus must be constant. □

It is convenient to set

$$Q_T^\lambda = H(\lambda,\cdot,\cdot,1), \quad \text{and} \quad \widetilde{Q}_T^\lambda = H(\lambda,\cdot,\cdot,0).$$

We will denote the domain of $H(\cdot,\cdot,\cdot,1)$ by the letter $\mathcal{D}$.

The following is the main result of this section (cf. [7,15]). It relates the fixed point index of Q_T^λ for small $\lambda>0$ (see, e.g., [21,23] for an introduction) with the degree of the tangent vector field ν. Recall that this notion, roughly speaking, counts (algebraically) the zeros of a vector field; for an exposition of this topic we refer, e.g., to [21] or [14].

Theorem 3.4. *Given $V\subseteq\widetilde{M}\times\widetilde{N}$ open and such that*

- (i) *$V_\#$ is relatively compact;*
- (ii) *There exists $s>0$ such that $[0,s]\times\overline{V}\subseteq\mathcal{D}$;*
- (iii) *Z_V is compact;*
- (iv) *If (x,y) is a T-periodic solution of (5) whose image intersects $\partial V_\#$, then (x,y) is constant.*

Then there exists $\lambda_*\in(0,s]$ such that, for $\lambda\in(0,\lambda_*)$, $\operatorname{ind}(Q_T^\lambda,V)$ is well defined and

$$\operatorname{ind}(Q_T^\lambda,V)=\operatorname{sign}(\phi)^{\dim N}\deg\big(-\nu,V_\#\big).$$

The symbol "$\operatorname{ind}(Q_T^\lambda,V)$" in the above formula denotes the fixed point index of Q_T^λ in the open set V, whereas "$\deg\big(-\nu,V_\#\big)$" denotes the degree of the tangent vector field $-\nu$ in the open subset $V_\#$ of $M\times N$.

Proof of Theorem 3.4. Let W and ε be as in Lemma 3.3. Consider the sets

$$\mathcal{S}=\big\{(\lambda,\varphi,\psi)\in[0,\varepsilon]\times\overline{W} : H(\lambda,\varphi,\psi,1)=(\varphi,\psi)\big\},$$
$$\mathcal{S}_0=\mathcal{S}\cap\big(\{0\}\times\widetilde{M}\times\widetilde{N}\big).$$

Clearly, $\mathcal{S}$ is compact being a closed subset of the compact set $[0,\varepsilon]\times H\big([0,\varepsilon]\times\overline{W}\times[0,1]\big)$. Thus $\mathcal{S}_0$ is compact as well. Using the definition of Q_T^λ, we will prove the following fact:

Claim 1. There exists $\lambda_0\in(0,\min\{\varepsilon,s\}]$ such that if $(\varphi,\psi)\in V$ is a fixed point of Q_T^λ with $\lambda\in(0,\lambda_0]$ then $(\varphi,\psi)\in W$. That is, Q_T^λ has no fixed points in $\overline{V}\backslash W$ for $\lambda\in(0,\lambda_0]$.

To prove this claim we proceed by contradiction. If the claim is false there exist sequences $\{\lambda_n\} \subseteq (0, \lambda_0]$, and $\{(\varphi_n, \psi_n)\} \subseteq \overline{V} \backslash W$, with $\lambda_n \to 0$ and $(\lambda_n, \varphi_n, \psi_n) \in \mathcal{S}$. By the compactness of $\mathcal{S}_0 \cap (\overline{V} \backslash W)$ we can assume that $(\varphi_n, \psi_n) \to (\varphi_0, \psi_0) \in \mathcal{S}_0 \cap (\overline{V} \backslash W)$. The continuous dependence on data shows that the solution of (5) with initial data (φ_0, ψ_0) is T-periodic. Assumption (iv) shows that there exists $p_0 \in M$ and $q_0 \in N$ such that $(\varphi_0, \psi_0) = (p_0^\#, q_0^\#)$. Clearly, one has $\mathfrak{g}(p_0, q_0) = 0$. Let (x_n, y_n) be the unique maximal solution of

$$\begin{cases} \dot{x}(t) = \lambda_n \mathfrak{f}(t, x(t), y(t), x(t-r), y(t-r)), & t > 0, \\ \dot{y}(t) = a(t)\mathfrak{g}(x(t), y(t)) + \lambda_n \mathfrak{h}(t, x(t), y(t), x(t-r), y(t-r)), & \\ x(t) = \varphi_n(t), \quad y(t) = \psi_n(t), & t \in [-r, 0]. \end{cases}$$

Then,

$$0 = x_n(T) - x_n(0) = \lambda_n \int_0^T \mathfrak{f}(t, x_n(t), y_n(t), x_n(t-r), y_n(t-r))\, \mathrm{d}t.$$

So that, in particular,

$$0 = \int_0^T \mathfrak{f}(t, x_n(t), y_n(t), x_n(t-r), y_n(t-r))\, \mathrm{d}t$$

and, passing to the limit, we get

$$0 = \int_0^T \mathfrak{f}(t, p_0, q_0, p_0, q_0)\, \mathrm{d}t = w_{\mathfrak{f}}(p_0, q_0).$$

Hence, $\nu(p_0, q_0) = \big(w_{\mathfrak{f}}(p_0, q_0), \mathfrak{g}(p_0, q_0)\big) = 0$. This contradicts the choice of W and completes the proof of Claim 1.

Claim 1 shows that, for $\lambda \in (0, \lambda_0]$, the set of the fixed points of Q_T^λ that lie in V is, in fact, contained in W. Hence, by the compactness of $\mathcal{S}$, it is compact too. As a consequence, $\operatorname{ind}(Q_T^\lambda, V)$ and $\operatorname{ind}(Q_T^\lambda, W)$ are well-defined and, by the excision property,

$$\operatorname{ind}(Q_T^\lambda, V) = \operatorname{ind}(Q_T^\lambda, W), \quad \text{for } \lambda \in (0, \lambda_0]. \tag{7}$$

In fact, when λ is sufficiently small, something more can be obtained:

Claim 2. There exists $\lambda_* \in (0, \lambda_0]$, such that the homotopy $H_\lambda : \overline{W} \times [0,1] \to \widetilde{M} \times \widetilde{N}$ given by $H_\lambda(\varphi, \psi, \mu) = H(\lambda, \varphi, \psi, \mu)$, is admissible for each $\lambda \in (0, \lambda_*]$.

To prove the claim we ought to show that for each $\lambda \in (0, \lambda_*]$, $\lambda_* > 0$ sufficiently small, the set of fixed points

$$\mathcal{F}_\lambda = \left\{(\varphi, \psi) \in \overline{W} : H(\lambda, \varphi, \psi, \mu) = (\varphi, \psi), \quad \text{for some } \mu \in [0,1]\right\},$$

which is compact being a closed subset of $H\big([0, \varepsilon] \times \overline{W} \times [0,1]\big)$, is contained in W. Suppose by contradiction that this is not the case, that is, that such a choice of λ_* cannot be done. Then there are sequences $\{\lambda_n\} \subseteq (0, \lambda_0]$, $\{\mu_n\} \subseteq [0,1]$ and $\{(\varphi_n, \psi_n)\} \subseteq \partial W$ with $\lambda_n \to 0$ and

$$H(\lambda_n, \varphi_n, \psi_n, \mu_n) = (\varphi_n, \psi_n). \tag{8}$$

As in the proof of Claim 1, by the compactness of $H\big([0, \varepsilon] \times \overline{W} \times [0,1]\big)$ we can assume that $(\varphi_n, \psi_n) \to (\varphi_0, \psi_0) \in \partial W$. The continuous dependence on data shows that the solution of (5) with initial data (φ_0, ψ_0) is T-periodic.

Assumption (iv) shows that there exists $p_0 \in M$ and $q_0 \in N$ such that $(\varphi_0, \psi_0) = (p_0^{\#}, q_0^{\#})$. Clearly, we get $\mathfrak{g}(p_0, q_0) = 0$. From (8) it follows that if (x_n, y_n) is the solution of

$$\begin{cases} \dot{x}(t) = \lambda_n \Big[\mu_n \mathfrak{f}\big(t, x(t), y(t), x(t-r), y(t-r)\big) \\ \qquad\quad + (1-\mu_n) \frac{a(t)}{\phi} w_{\mathfrak{f}}\big(x(t), y(t)\big) \Big], & t > 0, \\ \dot{y}(t) = a(t)\mathfrak{g}\left(x(t), y(t)\right) \\ \qquad\quad + \lambda_n \mu_n \mathfrak{h}\Big(t, x(t), y(t), x(t-r), y(t-r) \Big), \\ x(t) = \varphi_n(t), \quad y(t) = \psi_n(t), & t \in [-r, 0]. \end{cases}$$

Then,

$$\begin{aligned} 0 = x_n(T) - x_n(0) = \lambda_n \int_0^T \mu_n \mathfrak{f}\Big(t, x_n(t), y_n(t), x_n(t-r), y_n(t-r) \Big)\, \mathrm{d}t \\ + \lambda_n \int_0^T (1-\mu_n) \frac{a(t)}{\phi} w_{\mathfrak{f}}\big(x_n(t), y_n(t)\big)\, \mathrm{d}t. \end{aligned}$$

So that

$$\begin{aligned} 0 = \mu_n \int_0^T \mathfrak{f}\left(t, x_n(t), y_n(t), x_n(t-r), y_n(t-r)\right)\, \mathrm{d}t \\ + (1-\mu_n) \int_0^T \frac{a(t)}{\phi} w_{\mathfrak{f}}\big(x_n(t), y_n(t)\big)\, \mathrm{d}t. \end{aligned}$$

Passing to the limit we get

$$\begin{aligned} 0 &= \mu_0 \int_0^T \mathfrak{f}\big(t, p_0, q_0, p_0, q_0\big)\, \mathrm{d}t + (1-\mu_0) \int_0^T \frac{a(t)}{\phi} w_{\mathfrak{f}}\big(p_0, q_0\big)\, \mathrm{d}t \\ &= \mu_0 w_{\mathfrak{f}}\big(p_0, q_0\big) + (1-\mu_0) w_{\mathfrak{f}}\big(p_0, q_0\big) = w_{\mathfrak{f}}(p_0, q_0), \end{aligned}$$

which contradicts the choice of W and proves Claim 2.

Claim 2, along with the homotopy invariance property, imply that for $\lambda \in (0, \lambda_*]$

$$\operatorname{ind}(Q_T^\lambda, W) = \operatorname{ind}\big(H(\lambda, \cdot, \cdot, 1), W\big) = \operatorname{ind}\big(H(\lambda, \cdot, \cdot, 0), W\big) = \operatorname{ind}(\widetilde{Q}_T^\lambda, W). \tag{9}$$

Consider the tangent vector field v_λ on $M \times N$ given by

$$v_\lambda(p, q) := \left(\frac{\lambda}{\phi} w_{\mathfrak{f}}(p, q), \lambda \mathfrak{g}(p, q) \right).$$

Theorem 3.2 of [7] imply that, for each fixed $\lambda \in (0, \lambda_*]$

$$\operatorname{ind}(\widetilde{Q}_T^\lambda, W) = \operatorname{sign}(\phi)^{\dim(M \times N)} \deg(-v_\lambda, W_{\#}). \tag{10}$$

Since $\lambda > 0$, a well known property of the degree yields

$$\deg(-v_\lambda, W_{\#}) = \deg(-v_1, W_{\#}). \tag{11}$$

Lemma 1 of [28] shows that

$$\deg(-v_1, W_\#) = \operatorname{sign}(\phi)^{\dim M} \deg(-\nu, W_\#), \tag{12}$$

hence, by equalities (9)–(12), taking into account that $\dim(M \times N) = \dim M + \dim N$, we get

$$\begin{aligned} \operatorname{ind}(Q_T^\lambda, W) &= \operatorname{ind}(\widetilde{Q}_T^\lambda, W) = \operatorname{sign}(\phi)^{\dim(M\times N)} \deg(-v_\lambda, W_\#) \\ &= \operatorname{sign}(\phi)^{\dim(M\times N)} \deg(-v_1, W_\#) \\ &= \operatorname{sign}(\phi)^{\dim(M\times N)} \operatorname{sign}(\phi)^{\dim(M)} \deg(-\nu, W_\#) \\ &= \operatorname{sign}(\phi)^{2\dim M + \dim N} \deg(-\nu, W_\#) \\ &= \operatorname{sign}(\phi)^{\dim N} \deg(-\nu, W_\#). \end{aligned} \tag{13}$$

Finally, by (7), (13) and the excision property of the degree, we get

$$\begin{aligned} \operatorname{ind}(Q_T^\lambda, V) = \operatorname{ind}(Q_T^\lambda, W) \\ &= \operatorname{sign}(\phi)^{\dim N} \deg(-\nu, W_\#) \\ &= \operatorname{sign}(\phi)^{\dim N} \deg(-\nu, V_\#), \end{aligned}$$

which proves the assertion. □

4. Branches of T-periodic solutions

Let $T > 0$ be given, by $C_T(\mathbb{R}^d)$ we mean the Banach space of all the continuous T-periodic functions $\zeta\colon \mathbb{R} \to \mathbb{R}^d$ whereas $C_T(X)$ denotes the metric subspace of $C_T(\mathbb{R}^d)$ consisting of all those $\zeta \in C_T(\mathbb{R}^d)$ that take values in X. It is not difficult to prove that $C_T(X)$ is complete if and only if X is closed in $\mathbb{R}^d$.

It is also convenient to introduce the following notation: Given (p, q) in $M \times N$, let $(\widehat{p}, \widehat{q}) \in C_T(M \times N) = C_T(M \times N)$ be the constant maps $\big(\widehat{p}(t), \widehat{q}(t)\big) \equiv (p, q)$, $t \in \mathbb{R}$.

We are now in a position to state our result concerning the "branches" of T-triples of (3). Its proof follows closely the one of Theorem 5.1 in [15] (see also [7, 13]), for this reason we only provide a sketch for the sake of completeness.

Theorem 4.1. *Let Ω be an open subset of $[0,\infty) \times C_T(M \times N)$, and let $\Omega_{M\times N} := \{(p, q) \in M \times N : (0, \widehat{p}, \widehat{q}) \in \Omega\}$. Assume that $\deg\big(\nu, \Omega_{M\times N}\big)$ is well-defined and nonzero. Then there exists a connected set Γ of nontrivial T-triples for (3) in Ω whose closure in $[0,\infty) \times C_T(M \times N)$ meets $\nu^{-1}(0) \cap \Omega_{M\times N}$ and is not contained in any compact subset of Ω. In particular, if $M \times N$ is closed in $\mathbb{R}^{k+s}$ and $\Omega = [0,\infty) \times C_T(M \times N)$, then Γ is unbounded.*

The proof of this theorem is based on the following global connection result (see [13]), which will also be needed later.

Lemma 4.2. *Let Y be a locally compact metric space and let Z be a compact subset of Y. Assume that any compact subset of Y containing Z has nonempty boundary. Then $Y \backslash Z$ contains a connected set whose closure (in Y) intersects Z and is not compact.*

We are now ready to sketch the proof of the theorem.

Sketch of the proof of Theorem 4.1. This proof can be roughly divided into three steps:

Step 1. We assume first that the maps a, $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ are C^1, so that uniqueness of solutions holds for (3). Consider the following notion:

A triple $(\lambda, \varphi, \psi) \in [0, \infty) \times \widetilde{M} \times \widetilde{N}$ is said to be a *starting triple* for (3) if the following initial value problem has a T-periodic solution:

$$\begin{cases} \dot{x}(t) = \lambda \mathfrak{f}\big(t, x(t), y(t), x(t-r), y(t-r)\big), & \\ \dot{y}(t) = a(t)\mathfrak{g}\big(x(t), y(t)\big) + \lambda \mathfrak{h}\big(t, x(t), y(t), x(t-r), y(t-r)\big), & t > 0, \\ x(t) = \varphi(t), & \\ y(t) = \psi(t), & t \in [-r, 0]. \end{cases} \tag{14}$$

A triple of the type $(0, p^\#, q^\#)$ with $g(p, q) = 0$ is clearly a starting triple and will be called a *trivial starting triple*. The set of all starting triples for (3) will be denoted by S. By known continuous dependence properties of delay differential equations the set $\mathcal{V} \subseteq [0, \infty) \times \widetilde{M} \times \widetilde{N}$ of all triples (λ, φ, ψ) such that the unique solution of (14) is defined at least up to T is open (compare it to the set $\mathcal{D}$ defined in Sect. 3). Clearly $\mathcal{V}$ contains the set S of all starting triples for (3).

Given an open set W of $[0, \infty) \times \widetilde{M} \times \widetilde{N}$, let

$$W^0_\# := \Big(W \cap (\{0\} \times \widetilde{M} \times \widetilde{N})\Big)_\# = \big\{(p, q) \in M \times N : (0, p^\#, q^\#) \in W\big\}.$$

Our first step consists of proving that, if $\deg\big(\nu, W^0_\#\big)$ is well-defined and nonzero, then there exists in $S \cap W$ a connected set $\mathcal{G}$ of nontrivial starting triples whose closure in $S \cap W$ meets $\big\{(0, p^\#, q^\#) \in W \cap \mathcal{V} : g(p, q) = 0\big\}$ and is not compact.

The proof of this fact follows closely the one of [15, Prop. 4.1] using Theorem 3.4 in place of [15, Theorem 3.2]. Loosely speaking, this proof uses the properties of the fixed point index and of the degree of a tangent vector field to obtain a contradiction with Lemma 4.2 (Compare also [7, Theorem 4.1]).

Step 2. As in Step 1 we assume that the maps a, $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ are C^1. Denote by X the set of T-periodic triples of (3) and by S the set of starting triples of the same equation, as above. Define the map $\Pi : X \to S$ by

$$\Pi(\lambda, x, y) = \big(\lambda, x|_{[-r,0]}, y|_{[-r,0]}\big)$$

and observe that Π is continuous, onto and, since $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ are smooth, it is also one to one. Furthermore, by the continuous dependence on data, $\Pi^{-1} : S \to X$ is continuous as well. Take

$$S_\Omega = \{(\lambda, \varphi, \psi) \in S : \text{the solution of (14) is contained in } \Omega\},$$

so that $X \cap \Omega$ and S_Ω correspond under the homeomorphism $\Pi : X \to S$. Thus, S_Ω is an open subset of S and, consequently, we can find an open subset W of $[0, \infty) \times \widetilde{M} \times \widetilde{N}$ such that $S \cap W = S_\Omega$. This implies, as in [15, Theorem 5.1], that

$$\big\{(p, q) \in W^0_\# : g(p, q) = 0\big\} = \{(p, q) \in \Omega_{M \times N} : g(p, q) = 0\}\,.$$

The excision property of the degree of tangent vector fields yields

$$\deg\big(g, W^0_\#\big) = \deg\big(g, \Omega_{M\times N}\big) \neq 0.$$

By Step 1 we deduce the existence of a connected set

$$\Sigma \subseteq (S\cap W)\setminus\big\{(0,p^\#,q^\#)\in W : g(p,q)=0\big\}$$

whose closure in $S\cap W$ meets $\big\{(0,p^\#,q^\#)\in W : g(p,q)=0\big\}$ and is not compact. Clearly, $\Gamma = \Pi^{-1}(\Sigma)$ satisfies the assertion.

Step 3. We now only need to remove the C^1-regularity assumption on the maps a, $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ replacing it with continuity. This is done by an approximation procedure that follows closely the one used in [15, Theorem 5.1]. For this reason we skip the details. □

Remark 4.3. One can easily check that Theorem 4.1 implies both [3, Lemma 4.5] and [4, Lemma 3.5], albeit in the less general case of boundaryless manifolds, which are valid for a single differential equation of the form

$$\dot{x}(t) = \lambda\mathfrak{f}\big(t, x(t), x(t-r)\big),$$

where $\mathfrak{f}\colon \mathbb{R}\times M\times M \to \mathbb{R}^k$ is tangent to M. At the same time, Theorem 4.1 extends [15, Thm. 5.1] (see also Theorem 4.1 in [7]) that applies to the differential equations of the following type:

$$\dot{y}(t) = a(t)\mathfrak{g}(y) + \lambda\mathfrak{h}\big(t, y(t), y(t-r)\big).$$

where $\mathfrak{g}\colon M\to\mathbb{R}^k$ and $\mathfrak{h}\colon \mathbb{R}\times M\times M\to\mathbb{R}^k$ are tangent to M.

5. Sunflower-like equation

In order to prove our main theorem, we look at how the results of the previous section apply to Eq. (2). Let us now recall the notion of T-periodic pair (or T-pair for brevity) for this equation and some related facts. A pair $(\lambda, y) \in [0,\infty)\times C^1_T(\mathbb{R})$ is called a *T-pair* for (2) if y is a T-periodic solution of (2). We say that a T-pair (λ, y) is *trivial* if y is constant and $\lambda = 0$.

To study Eq. (2), we introduce a transformation that allows us to rewrite this model in an equivalent but easier to handle form. We need the following technical lemma whose proof is a standard ODE argument which we provide for the sake of completeness.

Lemma 5.1. *Let $a\colon \mathbb{R}\to\mathbb{R}$ be as in (2) and such that $\bar{a}\neq 0$. Then, there exists a unique T-periodic C^1 function $\sigma\colon\mathbb{R}\to\mathbb{R}$ for which*

$$a(t) = \frac{\dot{\sigma}(t)}{\sigma(t)} - \sigma(t), \quad \textit{for all} \quad t\in\mathbb{R}. \tag{15}$$

Clearly, σ has constant sign so that, in particular $\bar{\sigma}\neq 0$.

As a direct consequence, we have that Eq. (2) can be rewritten as

$$\ddot{y}(t) = \left(\frac{\dot{\sigma}(t)}{\sigma(t)} - \sigma(t)\right)\dot{y}(t) + \lambda b(t)\phi\big(y(t), y(t-r)\big), \quad \lambda\geq 0, \tag{16}$$

with σ chosen as in Lemma 5.1.

Proof of Lemma 5.1. It is easy to verify by inspection that, for any $c \in \mathbb{R}$,

$$\zeta(t) := e^{-\int_0^t a(s)ds}\left(c - \int_0^t e^{\int_0^s a(\ell)d\ell} ds\right),$$

is a solution of equation

$$\dot{\zeta}(t) = -\zeta(t)a(t) - 1, \tag{17}$$

which corresponds to (15) under the transformation $\zeta(t) = 1/\sigma(t)$. Clearly, $\zeta(0) = c$. Taking

$$c = \frac{\int_0^T e^{\int_0^s a(\ell)d\ell} ds}{e^{-T\not{a}} - 1} e^{-T\not{a}},$$

(recall that $\not{a} \neq 0$) we get $\zeta(0) = \zeta(T)$. Since the right-hand-side of (17) is T-periodic we obtain that ζ is T-periodic as well. In fact, the above is the only choice of c for which $\zeta(0) = \zeta(T)$; thus (17) has a unique T-periodic solution.

We need to prove that the function $t \mapsto 1/\zeta(t)$ is a T-periodic solution of (15). It is sufficient to show that $\zeta(t) \neq 0$ for all $t \in \mathbb{R}$. We consider the two possibilities $\not{a} > 0$ and $\not{a} < 0$ separately:

Case $\not{a} > 0$. Clearly, $e^{-T\not{a}} < 1$ so that, since $e^{-T\not{a}} \int_0^t e^{\int_0^s a(\ell)d\ell} ds > 0$ we have $c < 0$. Now, being $e^{-\int_0^t a(s)ds} > 0$ for all $t \in \mathbb{R}$, we get $\zeta(t) < 0$ for all t.

Case $\not{a} < 0$. In this case one has $e^{-T\not{a}} > 1$, thus

$$1 < \frac{e^{-T\not{a}}}{e^{-T\not{a}} - 1}.$$

Since $t \mapsto e^{-\int_0^t a(s)ds}$ is a positive function we have:

$$\int_0^t e^{\int_0^s a(\ell)d\ell} ds < \int_0^T e^{\int_0^s a(\ell)d\ell} ds < \frac{e^{-T\not{a}}}{e^{T\not{a}} - 1} \int_0^T e^{\int_0^s a(\ell)d\ell} ds,$$

so that $\zeta(t) > 0$.

Thus, in both cases, we find a T-periodic solution of (15). The uniqueness, follows from the fact that if $t \mapsto \sigma(t)$ is a T-periodic solution of (15), hence defined for all $t \in \mathbb{R}$, then $\sigma(t) \neq 0$. Then, $t \mapsto 1/\sigma(t)$ is a T-periodic solution of (17) which, as discussed above, is unique. □

Remark 5.2. From the proof of Lemma 5.1 it follows that σ has (constantly) the opposite sign of the average $\not{a}$. This fact has the obvious consequence that the signs of the averages of σ and of $1/\sigma$ coincide with $-\operatorname{sign}(\not{a})$. Furthermore, if $b \colon \mathbb{R} \to \mathbb{R}$ is as in (2), one has that b/σ has nonzero average and

$$\operatorname{sign}\left(\frac{1}{T}\int_0^T \frac{b(t)}{\sigma(t)}\,\mathrm{d}t\right) = -\operatorname{sign}(\not{a})\operatorname{sign}\big(b(0)\big).$$

For the average $\not{\sigma}$ of σ we can actually prove that $\not{\sigma} = -\not{a}$ by the following simple argument that only uses Eq. (15):

$$\not{a} = \frac{1}{T}\int_0^T a(t)\,\mathrm{d}t = \frac{1}{T}\int_0^T \frac{\dot{\sigma}(t)}{\sigma(t)}\,\mathrm{d}t - \frac{1}{T}\int_0^T \sigma(t)\,\mathrm{d}t = -\not{\sigma}.$$

In fact,

$$\int_0^T \frac{\dot{\sigma}(t)}{\sigma(t)}\,\mathrm{d}t = \ln\big(\sigma(T)\big) - \ln\big(\sigma(0)\big) = 0$$

because of the T-periodicity of σ.

To investigate Eq. (2) or, equivalently, (16) we follow the approach used in [28, §5]. Along this path we find convenient to treat a more general class of equations, i.e.

$$\ddot{y}(t) = \left(\frac{\dot{\gamma}(t)}{\gamma(t)} - \gamma(t)g\big(y(t)\big)\right)\dot{y}(t) + \lambda f\big(t, y(t), y(t-r)\big), \quad \lambda \geq 0, \tag{18}$$

where $f\colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ is continuous and T-periodic in t, $\gamma\colon \mathbb{R} \to \mathbb{R}$ is T-periodic and nonzero, and $g\colon \mathbb{R} \to \mathbb{R}$ is C^1.

Introducing a new variable x, Eq. (18) can be equivalently rewritten in $\mathbb{R}^2$ (as in the so-called Liénard plane technique) as follows:

$$\begin{cases} \dot{x}(t) = \lambda f\big(t, y(t), y(t-r)\big)\gamma^{-1}(t), \\ \dot{y}(t) = \big(x - G(y)\big)\gamma(t), \end{cases} \qquad \lambda \geq 0, \tag{19}$$

where $G(y)$ is a primitive of $g(y)$ and γ plays the role of σ in Lemma 5.1. Indeed, taking the derivative of the second equation in (19), we have

$$\begin{aligned} \ddot{y}(t) &= \dot{\gamma}(t)\Big(x(t) - G\big(y(t)\big)\Big) + \gamma(t)\dot{x}(t) - g\big(y(t)\big)\dot{y}(t) \\ &= \tfrac{\dot{\gamma}(t)}{\gamma(t)}\dot{y}(t) - g\big(y(t)\big)\dot{y}(t) + \lambda f\big(t, y(t), y(t-r)\big). \end{aligned}$$

By this relation, one can easily see that (19) is equivalent to (18). Because of this equivalence, Theorem 4.1 can be applied to (18):

Proposition 5.3. *Let f, g and γ be as in (18), and let $\Omega \subseteq [0,\infty) \times C^1_T(\mathbb{R})$ be open. Define the open subset of $C_T(\mathbb{R}\times\mathbb{R})$*

$$\widehat{\Omega} := \big\{(\lambda, \varphi, \psi) \in [0,\infty) \times C_T(\mathbb{R}\times\mathbb{R}) : (\lambda, \varphi) \in \Omega\big\},$$

and, according to the notation of Theorem 4.1,

$$\widehat{\Omega}_{\mathbb{R}^2} = \Big\{(\lambda, p, q) \in [0,\infty) \times \mathbb{R} \times \mathbb{R} : (\lambda, \widehat{p}, \widehat{q}) \in \widehat{\Omega}\Big\}.$$

Consider the vector field ν in $\mathbb{R}^2$, given by

$$\nu(p, q) := \big(\bar{w}(q),\, p - G(q)\big),$$

with $\bar{w}(q) := \frac{1}{T}\int_0^T f(t, q, q)\gamma^{-1}(t)\,\mathrm{d}t$. Assume that ν is admissible in $\Omega_{\mathbb{R}^2}$ for the degree and that $\deg(\nu, \Omega_{\mathbb{R}^2}) \neq 0$. Then, there exists a connected set of nontrivial T-pairs for (18) whose closure meets the set $\big\{(0, \widehat{p}) \in \Omega : \bar{w}(p) = 0\big\}$ and, is not compact.

Proof. By Theorem 4.1, there exists a connected set Γ of nontrivial T-triples for (19) whose closure meets the set

$$\Big\{(0, \widehat{p}, \widehat{q}) \in \widehat{\Omega} : \bar{w}(q) = 0,\ p = G(q)\Big\}$$

and is not compact.

Observe that to any $(\lambda, y, z) \in \Gamma$ one can associate the nontrivial T-pair (λ, y) for (18). In this way, one gets a connected set of nontrivial T-pairs for (18) whose closure meets the set $\{(0, \widehat{p}) \in \Omega : \bar{w}(p) = 0\}$ and is not compact. □

Example 5.4. Consider Eq. (18) with $\gamma(t) = \sin(t) + 2$ and $g(y) \equiv 1$; that is:

$$\ddot{x}(t) = \left(\frac{\cos(t)}{\sin(t)+2} - (\sin(t)+2)\right)\dot{x}(t) + \lambda x(t-r). \tag{20}$$

Take $T = 2\pi$. Clearly, the average $\not{\gamma} = 2$ and, for any $q \in \mathbb{R}$,

$$\bar{w}(q) = \frac{1}{2\pi}\int_0^{2\pi} \frac{q}{\sin(t)+2}dt = \frac{q}{\sqrt{3}}.$$

Let $\Omega = [0, \infty) \times C^1_T(\mathbb{R})$. The vector field $\nu(p, q) = \big(q/\sqrt{3}, p - q\big)$ is clearly admissible in $\widehat{\Omega}_{\mathbb{R}^2} = \mathbb{R}^2$ and has degree 1. Then, by Proposition 5.3, there exists a connected set of nontrivial 2π-pairs for (20) whose closure meets the set

$$\big\{(0, \widehat{p}) \in [0, \infty) \times C^1_T(\mathbb{R}) : \bar{w}(p) = 0\big\}$$

and is not compact.

Remark 5.5. When $\gamma(t) \equiv 1$, the system of Eq. (19) reduces to

$$\begin{cases} \dot{x}(t) = \lambda f\big(t, y(t), y(t-r)\big), \\ \dot{y}(t) = \big(x - G(y)\big), \end{cases} \qquad \lambda \geq 0,$$

which is equivalent to the equation

$$\ddot{y}(t) = -g\big(y(t)\big)\dot{y}(t) + \lambda f\big(t, y(t), y(t-r)\big). \tag{21}$$

in the particular case when $f\big(t, y(t), y(t-r)\big) = f\big(y(t-r)\big)$, Eq. (21) gives a so-called delayed Liénard equation (or Liénard sunflower-type equation) see, e.g., [1,2,25,30,32–34]. Clearly, Proposition 5.3 applies (for the non-delayed case, see [28]).

When f is of the form $f\big(t, y(t), y(t-r)\big) = b(t)\phi\big(y(t-r)\big)$ with b and ϕ as in (2), Proposition 5.3 combined with Lemma 5.1 implies the main result of the paper, Theorem 1.1, concerning Eq. (2), which we are now ready to prove.

Proof of Theorem 1.1. By Lemma 5.1 there exists a unique T-periodic function of constant sign $\sigma\colon \mathbb{R} \to \mathbb{R}$ such that $a(t) = \dot{\sigma}(t)/\sigma(t) - \sigma(t)$. Therefore, (2) can be written in the form (16).

Take $G(y) = y$. Then the maps $\bar{w}$ and ν of Proposition 5.3 become, respectively

$$\bar{w}(q) = \frac{1}{T}\int_0^T \frac{b(t)\phi(q,q)}{\sigma(t)}\,\mathrm{d}t = \phi(q,q)\,\frac{1}{T}\int_0^T \frac{b(t)}{\sigma(t)}\,\mathrm{d}t,$$

and

$$\nu(p, q) = \big(\bar{w}(q), p - q\big).$$

Let $\widehat{\Omega}_{\mathbb{R}^2}$ as in Proposition 5.3, with $\gamma = \sigma$. Since the average of b/σ is nonzero (see Remark 5.2), one easily checks that

$$\left|\deg\left(\nu, \widehat{\Omega}_{\mathbb{R}^2}\right)\right| = \left|-\operatorname{sign}\left(\frac{1}{T}\int_0^T \frac{b(t)\,\mathrm{d}t}{\sigma(t)}\right)\deg(w, W_\Omega)\right| = |\deg(w, W_\Omega)|\,.$$

Thus, $\deg(w, W_\Omega) \neq 0$ implies $\deg\left(\nu, \widehat{\Omega}_{\mathbb{R}^2}\right) \neq 0$. The assertion now follows from Proposition 5.3. $\square$

In the following example we consider the case of Eq. (2) when the perturbing term $\phi\big(y(t), y(t-r)\big) = \sin\big(y(t-r)\big)$, namely of the original sunflower equation but with time-periodic coefficients.

Example 5.6. (Sunflower-like equation) Consider the following scalar equation:

$$\ddot{x}(t) = a(t)\dot{x}(t) + \lambda b(t)\sin\big(x(t-r)\big). \tag{22}$$

where $a, b\colon \mathbb{R} \to \mathbb{R}$ are as in (2). Let Ω be the open subset of $[0,\infty) \times C^1_T(\mathbb{R})$ given by $\Omega = [0,\infty) \times C^1_T\big((-1,1)\big)$, and let W_Ω be as in Theorem 1.1. Let $T = 2\pi$. One immediately checks that $\deg(w, W_\Omega) = 1$, where

$$w(p) = \frac{1}{2\pi}\int_0^{2\pi} \sin(p)\,\mathrm{d}t = \sin(p).$$

By Theorem 1.1 there exists a connected set of nontrivial T-pairs for (2), whose closure meets the set $\{(0,\widehat{p}) \in \Omega : w(p) = 0\}$ and is not compact.

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