PERIODIC PERTURBATIONS OF CONSTRAINED MOTION PROBLEMS ON A CLASS OF IMPLICITLY DEFINED MANIFOLDS

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Abstract. We study forced oscillations on differentiable manifolds which are globally defined as the zero set of appropriate smooth maps in some Euclidean spaces. Given a T-periodic perturbative forcing field, we consider the two different scenarios of a nontrivial unperturbed force field and of perturbation of the zero field. We provide simple, degree-theoretic conditions for the existence of branches of T-periodic solutions. We apply our construction to a class of second order Differential-Algebraic Equations.

1. Introduction

In this paper we study T-periodic solutions of some parametrized families of T-periodic constrained second order Ordinary Differential Equations (ODEs). In other words, we study forced oscillations on a differentiable submanifold of some Euclidean space. As a physical interpretation, the equations we consider here represent the motion equations of a constrained system, the manifold being the constraint. We work under the assumption that such a manifold is globally defined as the zero set of a C^∞-smooth map and, given a T-periodic perturbative forcing field, we consider two different scenarios according whether there exists a nontrivial unperturbed force field. Namely, let $g\colon U \subseteq \mathbb{R}^m \times \mathbb{R}^s \to \mathbb{R}^s$ be a C^∞ map such that the Jacobian matrix of g with respect to the last s variables, $\partial_2 g(x,y)$, is nonsingular for all $(x,y) \in U$. This implies that $0 \in \mathbb{R}^s$ is a regular value of g so that $M = g^{-1}(0)$ is a C^∞-smooth submanifold of $\mathbb{R}^k = \mathbb{R}^m \times \mathbb{R}^s$ of dimension m (and codimension s).

For second order ODEs on differentiable manifolds we adopt the notation of e.g. [4]. We consider parametrized second order ODEs on M that, with this notation, assume the following forms:

$$\ddot{\xi}_\pi = f(\xi,\dot{\xi}) + \lambda h(t,\xi,\dot{\xi}) \tag{1.1a}$$

and

$$\ddot{\xi}_\pi = \lambda h(t,\xi,\dot{\xi}), \tag{1.1b}$$

where $\lambda \geq 0$ is a parameter, $\xi(t) \in M$, $\ddot{\xi}_\pi$ stands for the tangential part of the acceleration, $h\colon \mathbb{R} \times TM \to \mathbb{R}^k$ and $f\colon TM \to \mathbb{R}^k$ are continuous maps with the property that $f(\xi,\eta)$ and $h(t,\xi,\eta)$ belong to $T_\xi M$ for any $(t,\xi,\eta) \in \mathbb{R} \times TM$, and h is T-periodic in the first variable.

Notice that the assumptions on M and g imply that, locally, the manifold M can be represented as graph of some map from an open subset of $\mathbb{R}^m$ to $\mathbb{R}^s$. Thus, writing $\xi(t) = \big(x(t), y(t)\big)$, Equations (1.1) can be locally simplified via the implicit function theorem. In view of this fact one might think that it is possible to reduce Equations (1.1) to ordinary differential equations in $\mathbb{R}^m$. It is not so. In fact,

2000 *Mathematics Subject Classification.* 34C40; 34A09, 34C25.

Key words and phrases. Ordinary differential equations on manifolds, differential algebraic equations, degree of a vector field, periodic solution.

globally, M may not be the graph of a map from an open subset of $\mathbb{R}^m$ to $\mathbb{R}^s$ as, for instance, when $U = \mathbb{R}^3$ and $g\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$g(x,y) = g(x; y_1, y_2) = \big(e^{y_1}\cos y_2 - x, e^{y_1}\sin y_2 - x\big).$$

In this case, although $\det \partial_2 g(x,y) \neq 0$, one clearly has that the 1-dimensional manifold $M = g^{-1}(0)$ is not the graph of a function $x \mapsto \big(y_1(x), y_2(x)\big)$. In fact, M consists of infinitely many connected components each lying in a plane $y_2 = \frac{\pi}{4} + l\pi$ for $l \in \mathbb{Z}$.

Observe also that even when M is a (global) graph of some map Γ, the expression of Γ might be too complicated to use or impossible to determine analytically so that, the simplified versions of Equations (1.1) may be too difficult to use. A simple example of this fact is obtained by taking $m = s = 1$, $U = \mathbb{R} \times \mathbb{R}$ and $g(x,y) = y^7 + y - x^2 + x^5$.

A pair (λ, ξ), with $\lambda \geq 0$ and $\xi\colon \mathbb{R} \to M$ a T-periodic solution of (1.1a) (resp., of (1.1b)) corresponding to λ, is called a *solution pair* of (1.1a) (resp., of (1.1b)). The set of solution pairs is regarded as a subset of $[0, +\infty) \times C^1_T(M)$, where $C^1_T(M)$ is the metric subspace of the Banach space $C^1_T(\mathbb{R}^k)$ of the T-periodic C^1 maps from $\mathbb{R}$ to M. In this paper we investigate the structure of the set of solution pairs of Equations (1.1). Namely, we will prove global continuation results for solution pairs of (1.1).

Our results here stem from a combination of those of [5, 10] about branches of T-periodic solutions of constrained second order equations, with the formula contained in [2] for the computation of the degree of a tangent vector field in terms of the Brouwer degree of an appropriate map. In the situation considered in this paper (namely, for manifolds which are of the form $M = g^{-1}(0)$ with g as above), our results are complementary and, in some sense, improve those of [5, 10] since the Brouwer degree of the considered maps, having a simpler nature than the degree of a tangent vector field, is in principle easier to compute. In fact, the degree of a tangent vector field can be seen as an extension of the notion of Brouwer degree (a different one being the notion of degree of maps between oriented manifolds). Actually, since vector fields in Euclidean spaces can be regarded as maps and vice versa, the degree of a vector field is essentially the Brouwer degree, with respect to 0, of the field seen as a map.

A remarkable application of our results, which justifies our interest in the particular differentiable manifolds considered in this paper, is the content of the last section where we study periodic perturbations of a particular class of second order Differential-Algebraic Equations (DAEs) in semi-explicit form.

Recently, DAEs have received increasing interest due, in particular, to applications in engineering and have been the subject of extensive study (see e.g. [14] for a comprehensive treatment) aimed mostly (but not only) to numerical methods. Our approach here, as in our recent works [1, 2, 18], is directed towards qualitative theory of some particular DAEs which are studied by means of topological methods, making use of the equivalence of the given equations and suitable ODEs on manifolds. In particular, in this paper we investigate particular DAEs of second order, whereas the papers [1, 2, 18] were devoted to the first order case. The type of equations that we study here may be used to represent some nontrivial physical systems, as, for instance, constrained systems (see e.g. [17]). As we will prove, such equations are equivalent to second order ODEs on precisely the kind of manifold that we consider in the first part of the paper. This fact enables us to take advantage of the continuation results in [5, 10] and to find some surprisingly simple formulas useful for the study of connected sets of forced oscillations when a periodic forcing term is introduced. Some applications of our results are shown in a few illustrative examples.

We observe that our results, besides their intrinsic interest, can be useful in order to establish existence theorems in presence of a priori bounds (for a general reference on continuation methods, see e.g. [16]). In addition, they can be used to get some 'topological' multiplicity theorems for forced oscillations as in [7]. We will not pursue these lines here, though.

Finally, we wish to point out that in this paper we make use of deep results which are based on the fixed point index, but our techniques require just the notion of the well-known Brouwer degree. Our effort has been to make the paper accessible also to researchers which are not particularly familiar with topological methods.

2. Preliminaries

2.1. **Tangent vector fields and the notion of degree.** We now recall some basic notions about tangent vector fields on manifolds as well as the notion of degree of an admissible tangent vector field.

Let $M \subseteq \mathbb{R}^k$ be a manifold. Let w be a tangent vector field on M, that is, a continuous map $w\colon M \to \mathbb{R}^k$ with the property that $w(\xi) \in T_\xi M$ for any $\xi \in M$. If w is (Fréchet) differentiable at $\xi \in M$ and $w(\xi) = 0$, then the differential $dw_\xi\colon T_\xi M \to \mathbb{R}^k$ maps $T_\xi M$ into itself (see e.g. [15]), so that the determinant $\det dw_\xi$ of dw_ξ is defined. If, in addition, ξ is a nondegenerate zero (i.e. $dw_\xi\colon T_\xi M \to \mathbb{R}^k$ is injective) then ξ is an isolated zero and $\det dw_\xi \neq 0$.

Let W be an open subset of M in which we assume w *admissible* (for the degree); that is, the set $w^{-1}(0) \cap W$ is compact. Then, one can associate to the pair (w, W) an integer, $\deg(w, W)$, called the *degree (or characteristic) of the vector field w in W*, which, in a sense, counts (algebraically) the zeros of w in W (see e.g. [8, 13, 15] and references therein). In fact, when the zeros of w are all nondegenerate, then the set $w^{-1}(0) \cap W$ is finite and

$$\deg(w, W) = \sum_{\xi \in w^{-1}(0) \cap W} \operatorname{sign} \det dw_\xi. \tag{2.1}$$

Observe that in the flat case, i.e. when $M = \mathbb{R}^k$, $\deg(w, W)$ is just the classical Brouwer degree with respect to zero, $\deg_B(w, V, 0)$, where V is any bounded open neighborhood of $w^{-1}(0) \cap W$ whose closure is contained in W.

The notion of degree of an admissible tangent vector field plays a crucial role throughout this paper. All the standard properties of the Brouwer degree for continuous maps on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, still hold in this more general context (see e.g. [8]).

Remark 2.1. *The Excision Property allows the introduction of the notion of index of an isolated zero of a tangent vector field. Indeed, let $\xi \in M$ be an isolated zero of w. Clearly,* $\deg(w, V)$ *is well defined for each open $V \subseteq M$ such that $V \cap w^{-1}(0) = \{\xi\}$. By the Excision Property* $\deg(w, V)$ *is constant with respect to such V's. This common value of* $\deg(w, V)$ *is, by definition, the* index of w at *ξ, and is denoted by* $\mathrm{i}\,(w, \xi)$. *Using this notation, if (w, W) is admissible, by the Additivity Property we get that, if all the zeros in W of w are isolated, then*

$$\deg(w, W) = \sum_{\xi \in w^{-1}(0) \cap W} \mathrm{i}\,(w, \xi). \tag{2.2}$$

By formula (2.1) *we have that, if ξ is a nondegenerate zero of w, then*

$$\mathrm{i}\,(w, \xi) = \operatorname{sign} \det dw_\xi.$$

Notice that (2.1) *and* (2.2) *differ in the fact that, in the latter, the zeros of w are not necessarily nondegenerate as they have to be in the former. In fact, in* (2.2)*, w need not be differentiable at its zeros.*

2.2. **Tangent vector fields on implicitly defined manifolds.** Let $\Psi\colon \mathbb{R}\times M \to \mathbb{R}^k$ be a (time-dependent) tangent vector field on $M \subseteq \mathbb{R}^k$ that is, a continuous map with the property that $\Psi(t,\xi) \in T_\xi M$ for each $(t,\xi) \in \mathbb{R}\times M$. Assume that there is a connected open subset U of $\mathbb{R}^k$ and a C^∞ map $g\colon U \to \mathbb{R}^s$ with the property that $M = g^{-1}(0)$. Suppose that up to change of coordinates, writing $\mathbb{R}^k = \mathbb{R}^m \times \mathbb{R}^s$, $m = k - s$, one has that the partial derivative of g with respect to the second variable, $\partial_2 g(x,y)$, is invertible for each $(x,y) \in U$.

According to the above decomposition of $\mathbb{R}^k$, we can write, for any $\xi \in \mathbb{R}^k$, $\xi = (x,y)$ and, with a slight abuse of notation, for any $t \in \mathbb{R}$

$$\Psi(t,\xi) = \Psi(t,x,y) = \big(\Psi_1(t,x,y), \Psi_2(t,x,y)\big).$$

Notice that one must have

$$\Psi_2(t,x,y) = -(\partial_2 g(x,y))^{-1}\partial_1 g(x,y)\Psi_1(t,x,y). \tag{2.3}$$

In fact, $\Psi(t,\xi) \in T_\xi M$ being equivalent to $\Psi(t,\xi) \in \ker g'(x,y)$, one has for each $(t,x,y) \in \mathbb{R}\times M$ that

$$0 = g'(x,y)\Psi(t,x,y) = \partial_1 g(x,y)\Psi_1(t,x,y) + \partial_2 g(x,y)\Psi_2(t,x,y),$$

which implies (2.3); here $g'(x,y)$ denotes the Fréchet differential of g at (x,y).

We now give a formula for the degree of tangents vector fields on M in terms of (potentially easier to compute) degree of appropriate vector fields on U.

Remark 2.2. *Assume that $\psi\colon M \to \mathbb{R}^k$ is a tangent vector field on M. Since $M = g^{-1}(0)$ is a closed subset of the metric space U, the well-known Tietze's Theorem (see e.g.* [3]*) implies that there exists an extension $\widetilde{\psi}\colon U \to \mathbb{R}^k$ of ψ.*

Remark 2.2 shows that it is not restrictive to assume, as we sometimes do, that the given tangent vector fields are actually defined on a convenient neighborhood of the manifold M. In fact, although an arbitrary extension of ψ may have many zeros outside M, we are interested in the degree of ψ on M which only takes into account those zeros of ψ that lie on M.

The following result (see [2], [18]) gives a formula for the degree of tangents vector fields on M in terms of degree of appropriate vector fields on U.

Theorem 2.3. *Let $U \subseteq \mathbb{R}^m \times \mathbb{R}^s$ be open and connected, let $g\colon U \to \mathbb{R}^s$ be a C^∞ function such that $\partial_2 g(x,y)$ is nonsingular for any $(x,y) \in U$ and let $M = g^{-1}(0)$. Assume that $\psi\colon M \to \mathbb{R}^m \times \mathbb{R}^s$ is tangent to M and let $\widetilde{\psi}_1$ be the projection on $\mathbb{R}^k$ of an arbitrary continuous extension $\widetilde{\psi}$ of ψ to U. Define $\mathcal{F}\colon U \to \mathbb{R}^m \times \mathbb{R}^s$ by $\mathcal{F}(x,y) = \big(\widetilde{\psi}_1(x,y), g(x,y)\big)$. Then, $\mathcal{F}$ is admissible in U if and only if so is ψ in M, and*

$$\deg(\psi, M) = \mathfrak{s}\deg(\mathcal{F}, U), \tag{2.4}$$

where $\mathfrak{s}$ is the sign of $\det\partial_2 g(x,y)$ which is clealy constant for all (x,y) in the connected set U.

3. Second order equations on manifolds

As in the previous section, let M be a manifold in $\mathbb{R}^k$. A continuous map $\varphi\colon \mathbb{R}\times TM \to \mathbb{R}^k$ is said to be *tangent to M* if $\varphi(t,\xi,\eta) \in T_\xi M$ for any $(t,\xi,\eta) \in \mathbb{R}\times TM$.

It is known (compare [4]) that the *motion equation* associated with the *force* φ can be written in the form

$$\ddot{\xi}_\pi = \varphi(t, \xi, \dot{\xi}), \tag{3.1}$$

where $\ddot{\xi}_\pi$ stands for the *parallel component of the acceleration* $\ddot{\xi} \in \mathbb{R}^k$ at the point ξ. Namely, $\ddot{\xi}_\pi$ denotes the orthogonal projection of $\ddot{\xi}$ onto $T_\xi M$. A *solution* of (3.1) is a C^2 map $\xi\colon J \to M$, defined on a nontrivial interval J, such that $\ddot{\xi}_\pi(t) = \varphi(t, \xi(t), \dot{\xi}(t))$ for all $t \in J$.

Further, Equation (3.1) can be written in an equivalent way as a first order equation on the tangent bundle TM in the form

$$\dot{\zeta} = \Psi(t, \zeta), \tag{3.2}$$

where, for $\zeta = (\xi, \eta)$,

$$\Psi(t; \xi, \eta) = \big(\eta, r(\xi, \eta) + \varphi(t; \xi, \eta)\big), \tag{3.3}$$

and the map $r\colon TM \to \mathbb{R}^k$ is smooth, quadratic in the second variable $\eta \in T_\xi M$ for any $\xi \in M$, and with values in $(T_\xi M)^\perp$. Such a map r is strictly related to the second fundamental form on M and may be interpreted as the *reactive force* due to the constraint M. Actually $r(\xi, \eta)$ is the unique vector in $\mathbb{R}^k$ which makes $\big(\eta, r(\xi, \eta)\big)$ tangent to TM at (ξ, η). It is well known that Ψ is a (time-dependent) tangent vector field on TM. Hence (3.2) is actually a first order equation on TM. Using the function r we can rewrite Equation (3.1) also as the following second order ODE (see [4]):

$$\ddot{\xi} = r(\xi, \dot{\xi}) + \varphi(t, \xi, \dot{\xi}), \tag{3.4}$$

We now focus on the computation of the map r.

Let $\xi\colon J \subseteq \mathbb{R} \to M$ be a local solution of (3.1) taking values in a neighborhood $V \subseteq M$ of $\xi_0 := \xi(t_0)$ for some $t_0 \in J$. Restricting V if necessary, we can assume that there exists $U \subseteq \mathbb{R}^k$ and a C^∞ function $G\colon U \to \mathbb{R}^s$ such that $U \cap M = V = G^{-1}(0)$. Differentiating twice at t_0 the relation $G(\xi(t)) = 0$, we get

$$G''(\xi_0)\big(\dot{\xi}_0, \dot{\xi}_0\big) + G'(\xi_0)\ddot{\xi}(t_0) = 0, \tag{3.5}$$

where, for the sake of simplicity, we write $\dot{\xi}_0$ instead of $\dot{\xi}(t_0)$. According to Equation (3.4) we have

$$\ddot{\xi}(t_0) = \varphi\big(t_0, \xi_0, \dot{\xi}_0\big) + r\big(\xi_0, \dot{\xi}_0\big). \tag{3.6}$$

Since $\ddot{\xi}_\pi(t_0) = \varphi\big(t_0, \xi_0, \dot{\xi}_0\big) \in T_{\xi_0}M$, one has $G'(\xi_0)\varphi\big(t_0, \xi_0, \dot{\xi}_0\big) = 0$. Hence (3.5) yields

$$G'(\xi_0) r\big(\xi_0, \dot{\xi}_0\big) = -G''(\xi_0)\big(\dot{\xi}_0, \dot{\xi}_0\big).$$

And, finally,

$$r\big(\xi_0, \dot{\xi}_0\big) = -\Big(G'(\xi_0)|_{(T_{\xi_0}M)^\perp}\Big)^{-1} G''(\xi_0)\big(\dot{\xi}_0, \dot{\xi}_0\big). \tag{3.7}$$

Recall, in fact, that $G'(\xi_0)|_{(T_{\xi_0}M)^\perp}$ is invertible because $T_{\xi_0}M$ is the kernel of $G'(\xi_0)$.

Observe that equations (3.1)–(3.6) have another immediate interesting consequence. Namely, that if $t \mapsto \zeta(t)$ is an M-valued C^2 curve then

$$\mathcal{P}^\perp_{\zeta(t)}\ddot{\zeta}(t) = r\big(\zeta(t), \dot{\zeta}(t)\big), \tag{3.8}$$

where $\mathcal{P}^\perp_{\zeta(t)}$ denotes the orthogonal projection of $\mathbb{R}^k$ onto the orthogonal complement to $T_{\zeta(t)}M$.

When M is of the form considered in Section 2.2 we are able to get a more explicit expression for the inertial reaction r. The expression that we are going to obtain, although strictly speaking not necessary for the carrying out of our arguments, can be useful for understanding our geometric setting. The reader may skip the remaining part of this section at a first reading.

Assume, as in Section 2.2, that $M = g^{-1}(0)$ where $g\colon U \to \mathbb{R}^s$ is a C^∞ map with $\partial_2 g(x,y)$ is invertible for each $(x,y) \in U$. Let $\xi\colon J \subseteq \mathbb{R} \to M$ be a local solution of (3.1) and write $\xi = (x,y)$. Setting $\xi_0 = (x_0,y_0)$ and $\dot\xi_0 = (\dot x_0, \dot y_0)$, Equation (3.5) becomes

$$g''(x_0,y_0)\big((\dot x_0,\dot y_0),(\dot x_0,\dot y_0)\big) + \partial_1 g(x_0,y_0)\ddot x(t_0) + \partial_2 g(x_0,y_0)\ddot y(t_0) = 0. \tag{3.9}$$

Thus, $r(x_0,y_0;\dot x_0,\dot y_0)$ is the (unique) solution lying in $\big(T_{(x_0,y_0)}M\big)^\perp$ of the linear equation

$$g'(x_0,y_0)r(x_0,y_0;\dot x_0,\dot y_0) = -g''(x_0,y_0)\big((\dot x_0,\dot y_0),(\dot x_0,\dot y_0)\big). \tag{3.10}$$

Now, to find the explicit expression of r we need two elementary linear algebra lemmas. The proof of the first is straightforward and is left to the reader.

Lemma 3.1. *Let A be an $s \times m$ matrix and B be an $s \times s$ matrix. Let $W = \{(u,v) \in \mathbb{R}^m \times \mathbb{R}^s : Au + Bv = 0\}$. Assume B invertible. Then, $W^\perp = \{(u,v) \in \mathbb{R}^m \times \mathbb{R}^s : u = A^T(B^{-1})^T v\}$.*

Lemma 3.2. *Let A be an $s \times m$ matrix and B be an $s \times s$ matrix. Assume B invertible. Then, the $s \times s$ matrix $C = AA^T(B^{-1})^T + B$ is invertible too.*

Proof. To prove that C is invertible we will show that so is $B^{-1}C$. To see this, we prove that $B^{-1}C$ is symmetric and positive definite. Notice that

$$B^{-1}C = B^{-1}AA^T(B^{-1})^T + I = B^{-1}A\big(B^{-1}A\big)^T + I,$$

thus $B^{-1}C$ is of the form $XX^T + I$ with X a $s \times m$ matrix, I being the identity. In particular, $B^{-1}C$ is symmetric. Now, observe that XX^T is always symmetric and positive semidefinite and thus $B^{-1}C$ is positive definite, being the sum of a positive definite matrix (the identity) and a positive semidefinite matrix. ☐

Set now $A = \partial_1 g(x_0,y_0)$, $B = \partial_2 g(x_0,y_0)$, and $C = AA^T(B^{-1})^T + B$ (recall that B is an invertible $s \times s$ matrix by assumption). Further, let $r = (u,v)$ and $\sigma = -g''(x_0,y_0)\big((\dot x_0,\dot y_0),(\dot x_0,\dot y_0)\big)$, so that Equation (3.10) can be written as

$$Au + Bv = \sigma. \tag{3.11}$$

Taking into account the above lemmas, simple calculations show that Equation (3.11) has a unique solution $(u,v) \in \mathbb{R}^m \times \mathbb{R}^s$ that lies in $\big(T_{(x_0,y_0)}M\big)^\perp$ and is given by

$$\begin{cases} u = A^T(B^{-1})^T C^{-1}\sigma, \\ v = C^{-1}\sigma. \end{cases}$$

Thus, $r(x_0,y_0;\dot x_0,\dot y_0)$ has the following expression:

$$r(x_0,y_0;\dot x_0,\dot y_0) = \begin{pmatrix} A^T(B^{-1})^T C^{-1}\sigma \\ C^{-1}\sigma \end{pmatrix}, \tag{3.12}$$

where $\sigma = -g''(x_0,y_0)\big((\dot x_0,\dot y_0),(\dot x_0,\dot y_0)\big)$.

Example 3.3. *Let $k = 2$, $m = 1$ and $g(x,y) = \frac{1}{2}x^2 - y - 2$. We wish to compute the map r in this case. Clearly $M = g^{-1}(0)$ is a parabola. With the above notation, we have*

$$A = \partial_1 g(x,y) = x, \quad B = \partial_2 g(x,y) = -1, \quad C = AA^T(B^{-1})^T + B = 1 - x^2.$$

Also,

$$\sigma = -g''(x,y)\left(\left(\begin{smallmatrix}\dot x\\ \dot y\end{smallmatrix}\right),\left(\begin{smallmatrix}\dot x\\ \dot y\end{smallmatrix}\right)\right) = -\left(\begin{smallmatrix}\dot x & \dot y\end{smallmatrix}\right)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\left(\begin{smallmatrix}\dot x\\ \dot y\end{smallmatrix}\right) = -\dot x^2.$$

Therefore,

$$r(x,y;\dot x,\dot y)=\begin{pmatrix}A^T(B^{-1})^TC^{-1}\sigma\\ C^{-1}\sigma\end{pmatrix}=\begin{pmatrix}\frac{x\dot x^2}{1-x^2}\\ \frac{\dot x^2}{x^2-1}\end{pmatrix}.$$

Example 3.4. *Let $k=3$, $m=2$ and $g(x,y)=z-x^2-y^2$. We wish to compute the map r in this case. Clearly $M=g^{-1}(0)$ is a paraboloid. With the above notation, we have*

$$A=\partial_1 g(x,y)=\begin{pmatrix}-2x & -2y\end{pmatrix},\quad B=\partial_2 g(x,y)=1,$$
$$C=AA^T(B^{-1})^T+B=4x^2+4y^2+1.$$

Also,

$$\sigma=-g''(x,y,z)\left(\begin{pmatrix}\dot x\\ \dot y\\ \dot z\end{pmatrix},\begin{pmatrix}\dot x\\ \dot y\\ \dot z\end{pmatrix}\right)=-\begin{pmatrix}\dot x & \dot y & \dot z\end{pmatrix}\begin{pmatrix}-2&0&0\\0&-2&0\\0&0&0\end{pmatrix}\begin{pmatrix}\dot x\\ \dot y\\ \dot z\end{pmatrix}=2\dot x^2+2\dot y^2.$$

Therefore,

$$r(x,y,z;\dot x,\dot y,\dot z)=\begin{pmatrix}A^T(B^{-1})^TC^{-1}\sigma\\ C^{-1}\sigma\end{pmatrix}=\begin{pmatrix}\dfrac{-4x(\dot x^2+\dot y^2)}{1+4x^2+4y^2}\\ \dfrac{-4y(\dot x^2+\dot y^2)}{1+4x^2+4y^2}\\ 2\dot x^2+2\dot y^2\end{pmatrix}$$

The examples above, in conjunction with Equations (3.2) and (3.3), show that even for simple manifolds second order equations can be rather complicated. The following example confirms this fact, strengthening the case for the indirect methods investigated in this paper.

Example 3.5. *Let $k=3$, $m=1$ and $g(x,y,z)=\big(z^3+z-x,z-y+x^2\big)$. We wish to compute the map r in this case. Here $M=g^{-1}(0)$ is a smooth curve. With the above notation, we have*

$$A=\partial_1 g(x,y)=\begin{pmatrix}-1\\2x\end{pmatrix},\quad B=\partial_2 g(x,y)=\begin{pmatrix}0&3z^2+1\\-1&1\end{pmatrix}.$$

Also,

$$\sigma=-g''(x,y,z)\left(\begin{pmatrix}\dot x\\ \dot y\\ \dot z\end{pmatrix},\begin{pmatrix}\dot x\\ \dot y\\ \dot z\end{pmatrix}\right)=-\begin{pmatrix}6\dot z^2\\2\dot x^2\end{pmatrix}.$$

Lengthy but straightforward computations yield

$$r(x,y,y;\dot x,\dot y,\dot z)=\frac{2(1+4x^2)}{5+20x^2+16x^4+18z^4+36z^4x^2+12z^2+24z^2x^2}\cdot$$
$$\cdot\begin{pmatrix}(18xz^4+3z^2+12xz^2+8x^3+4x+1)\dot x^2-(18xz^2+6x+12x^2+9)\dot z^2\\ +9\dot z^2z^2+3\dot z^2-2\dot x^2-4\dot x^2x^2-9\dot x^2z^4-6\dot x^2z^2\\ 18\dot z^2z^2+6\dot z^2+36\dot z^2z^2x^2+12\dot z^2x^2+\dot x^2+4\dot x^2x^2\end{pmatrix}.$$

3.1. **Branches of harmonic solutions.** In what follows, if N is a differentiable manifold embedded in some $\mathbb{R}^k$, we will denote by $C^1_T(N)$, the metric subspace of the Banach space $\big(C^1_T(\mathbb{R}^k)\,,|\cdot|_1\big)$ of all the T-periodic C^1 maps $\xi\colon\mathbb{R}\to N$ with the usual C^1 norm.

In this section we recall two known results from [5] and [10] about the sets of *solution pairs* of (1.1a) and of (1.1b), i.e. of those pairs $(\lambda,\xi)\in[0,\infty)\times C^1_T(N)$ with ξ a T-periodic solution of (1.1a) and of (1.1b), respectively. Recall that a solution pair (λ,ξ) of (1.1a) or of (1.1b) is said to be *trivial* if $\lambda=0$ and ξ is constant.

For the sake of simplicity we make some conventions. We will regard every space as its image in the following diagram of natural inclusions

$$\begin{array}{ccc} N & \longrightarrow & C^1_T(N) \\ \big\downarrow & & \big\downarrow \\ [0,\infty)\times N & \longrightarrow & [0,\infty)\times C^1_T(N) \end{array} \tag{3.13}$$

In particular, we will identify N with its image in $C^1_T(N)$ under the embedding which associates to any $\xi_0 \in N$ the map $\hat{\xi}_0 \in C^1_T(N)$ constantly equal to ξ_0. Moreover we will regard N as the slice $\{0\}\times N \subset [0,\infty)\times N$ and, analogously, $C^1_T(N)$ as $\{0\}\times C^1_T(N)$. We point out that the images of the above inclusions are closed. According to these identifications, if Ω is an open subset of $[0,\infty)\times C^1_T(N)$, by $N\cap\Omega$ we mean the open subset of N given by all $\xi_0 \in N$ such that the pair $(0,\hat{\xi}_0)$ belongs to Ω. If $\mathcal{O}$ is an open subset of $[0,\infty)\times N$, then $\mathcal{O}\cap N$ represents the open set $\big\{\xi_0 \in N : (0,\xi_0)\in\mathcal{O}\big\}$.

We need to introduce some further notation. Given $f\colon TN \to \mathbb{R}^k$ tangent to the manifold N, we define the tangent vector field $f|_N\colon N \to \mathbb{R}^k$ given by $f|_N(\xi) = f(\xi,0)$ for any $\xi \in N$. The following results of [5] and of [10] play a central role:

Theorem 3.6 ([10] Th. 4.2)**.** *Let $N \subseteq \mathbb{R}^k$ be a boundaryless manifold, and let $f\colon TN \to \mathbb{R}^k$ and $h\colon \mathbb{R}\times TN \to \mathbb{R}^k$ be tangent to N. Assume that h is T-periodic in the first variable. Let Ω be an open subset of $[0,\infty)\times C^1_T(N)$. Assume that* $\deg\big(f|_N, N\cap\Omega\big)$ *is well defined and nonzero. Then, Ω contains a connected set Γ of nontrivial solution pairs for* (1.1a) *whose closure meets $\big(f|_N\big)^{-1}(0)\cap\Omega$ and is not contained in any compact subset of Ω. In particular, if N is closed in $\mathbb{R}^k$ and $\Omega = [0,\infty)\times C^1_T(N)$, then Γ is unbounded.*

Given $h\colon \mathbb{R}\times TN \to \mathbb{R}^k$ tangent to the manifold N, we define the *mean value* tangent vector field $w_h\colon N \to \mathbb{R}^k$ given by

$$w_h(\xi) = \frac{1}{T}\int_0^T h(t,\xi,0)dt. \tag{3.14}$$

Theorem 3.7 ([5] Th. 2.2)**.** *Let $N \subseteq \mathbb{R}^k$ be a boundaryless manifold, and let $h\colon \mathbb{R}\times TN \to \mathbb{R}^k$ be tangent to N and T-periodic in the first variable. Let Ω be an open subset of $[0,\infty)\times C^1_T(N)$. Assume that* $\deg\big(w_h, N\cap\Omega\big)$ *is well defined and nonzero. Then, Ω contains a connected set Γ of nontrivial solution pairs for* (1.1b) *whose closure meets $w_h^{-1}(0)\cap\Omega$ and is not contained in any compact subset of Ω. In particular, if N is closed in $\mathbb{R}^k$ and $\Omega = [0,\infty)\times C^1_T(N)$, then Γ is unbounded.*

4. Main results

Throughout this section U will be an open and connected subset of $\mathbb{R}^m\times\mathbb{R}^s$. From now on, we will denote by Pr_1 the projection onto the first factor in the Cartesian product $\mathbb{R}^m\times\mathbb{R}^s$.

We will always assume that $g\colon U \to \mathbb{R}^s$ is a C^∞ function such that $\partial_2 g(x,y)$ is nonsingular for any $(x,y)\in U$, and that $M = g^{-1}(0)$, so that M is a boundaryless manifold in $\mathbb{R}^s$. Moreover we will write $\xi = (x,y)\in U$. It will also be convenient, given a continuous tangent vector field, $f\colon TM \to \mathbb{R}^m\times\mathbb{R}^s$, to denote by $\widetilde{f}$ an arbitrary extension of the map $f|_M\colon (x,y)\mapsto f(x,y,0,0)$ to U (see Remark 2.2) and to let $\mathrm{Pr}_1\widetilde{f}(x,y)$ be the projection of $\widetilde{f}(x,y)$ on $\mathbb{R}^m$ for any $(x,y)\in U$.

Theorem 4.1. *Let $f\colon TM \to \mathbb{R}^m\times\mathbb{R}^s$ and $h\colon \mathbb{R}\times TM \to \mathbb{R}^m\times\mathbb{R}^s$ be continuous tangent vector fields, with h of a given period $T>0$ in the first variable. Define $\mathcal{F}\colon U \to \mathbb{R}^m\times\mathbb{R}^s$ by $\mathcal{F}(x,y) = \big(\mathrm{Pr}_1\widetilde{f}(x,y), g(x,y)\big)$ for any $\xi = (x,y)\in U$. Given* *an open set* $\Omega \subseteq [0,\infty) \times C^1_T(M)$*, let* $\mathcal{O} \subseteq \mathbb{R}^m \times \mathbb{R}^s$ *be open with the property that* $\mathcal{O} \cap M = M \cap \Omega$*. Assume that* $\deg(\mathcal{F}, \mathcal{O})$ *is well defined and nonzero. Then there exists a connected set* Γ *of nontrivial solution pairs for* (1.1a) *in* Ω *whose closure in* Ω *meets* $(f|_M)^{-1}(0,0) \cap \Omega$ *and is not compact. In particular, if* M *is closed in* $\mathbb{R}^m \times \mathbb{R}^s$ *and* $\Omega = [0,\infty) \times C^1_T(M)$*, then* Γ *is unbounded.*

Proof. By Theorem 2.3 we have

$$|\deg(f|_M, M \cap \Omega)| = |\deg(f|_M, \mathcal{O} \cap M)| = |\deg(\mathcal{F}, \mathcal{O})|.$$

Thus, $\deg(f, M \cap \Omega) \neq 0$ and the assertion follows from Theorem 3.6 with $M = N$. □

The following result concerns Equation (1.1b). As in the previous section, given $h\colon \mathbb{R} \times TM \to \mathbb{R}^k$ tangent to M, we define the tangent vector field w_h on M by (3.14). Moreover by $\mathrm{Pr}_1\, \widetilde{w}_h(x,y)$ we denote the projection on $\mathbb{R}^m$ of an arbitrary extension $\widetilde{w}_h(x,y)$ of the map $w_h(x,y)$ to U.

Theorem 4.2. *Let* $h\colon \mathbb{R} \times TM \to \mathbb{R}^m \times \mathbb{R}^s$ *be a continuous tangent vector field, of a given period* $T > 0$ *in the first variable. Define* $\Phi\colon U \to \mathbb{R}^m \times \mathbb{R}^s$ *by* $\Phi(x,y) = \big(\mathrm{Pr}_1\, \widetilde{w}_h(x,y), g(x,y)\big)$ *for any* $\xi = (x,y) \in U$*. Given an open set* $\Omega \subseteq [0,\infty) \times C^1_T(M)$*, let* $\mathcal{O} \subset \mathbb{R}^m \times \mathbb{R}^s$ *be an open subset with the property that* $\mathcal{O} \cap M = M \cap \Omega$*. Assume that* $\deg(\Phi, \mathcal{O})$ *is well defined and nonzero. Then there exists a connected set* Γ *of nontrivial solution pairs for* (1.1b) *in* Ω *whose closure in* Ω *is not compact and meets the set* $w_h^{-1}(0) \cap \Omega$*. In particular, if* M *is closed in* $\mathbb{R}^m \times \mathbb{R}^s$ *and* $\Omega = [0,\infty) \times C^1_T(M)$*, then* Γ *is unbounded.*

Proof. By Theorem 2.3 we have

$$|\deg(w_h, M \cap \Omega)| = |\deg(w_h, \mathcal{O} \cap M)| = |\deg(\Phi, \mathcal{O})|.$$

Thus, $\deg(w_h, M \cap \Omega) \neq 0$ and the assertion follows from Theorem 3.7 with $M = N$. □

Remark 4.3. *It is worth noticing that, in Theorem 4.1, a point* $(x,y) \in \mathbb{R}^m \times \mathbb{R}^s$ *belongs to* $(f|_M)^{-1}(0,0)$ *if and only if* $\mathcal{F}(x,y) = 0$*. Similarly, in Theorem 4.2,* $(x,y) \in w_h^{-1}(0)$ *if and only if* $\Phi(x,y) = 0$*.*

Example 4.4. *Consider the simple case when* $m = s = 1$*,* $U = \mathbb{R} \times \mathbb{R}$ *and* $g(x,y) = \frac{1}{3}(x^2+1)y + \frac{1}{27}y^3 + x$*. Assume that a point* $\boldsymbol{P}$ *of mass* $\boldsymbol{m}$ *is constrained without friction to the curve* $M = g^{-1}(0)$ *lying on a vertical plane (see figure 1) and acted by the (constant) gravitational force* $\varphi(x,y) = (0, -\boldsymbol{mg})^T$*. Here,* $\boldsymbol{g}$ *denotes the gravitational constant. The motion of* $\boldsymbol{P}$ *is determined by the following second order equation on* M*:*

$$\boldsymbol{m}\ddot{\xi}_\pi = \mathcal{P}_\xi \varphi(\xi)$$

where $\mathcal{P}_\xi$ *denotes the orthogonal projection of* $\mathbb{R}^2$ *onto* $T_\xi M$*. In order to apply Theorems 4.1 or 4.2 it is necessary to compute* $\mathcal{P}_\xi \varphi(\xi)$*. This is readily done, in fact,*

$$f(\xi) := \mathcal{P}_\xi \varphi(\xi) = \varphi(\xi) - \frac{\langle \varphi(\xi), \nabla g(\xi) \rangle}{\|\nabla g(\xi)\|^2} \nabla g(\xi),$$

where '$\langle \cdot, \cdot \rangle$*' denotes the scalar product. In coordinates, we get the rather unappealing formula below:*

$$f(x,y) = \frac{3\boldsymbol{mg}}{(6xy+9)^2 + (y^2+3x^2+3)^2} \begin{pmatrix} (y^2+3x^2+3)(2xy+3) \\ -3(2xy+3) \end{pmatrix}.$$

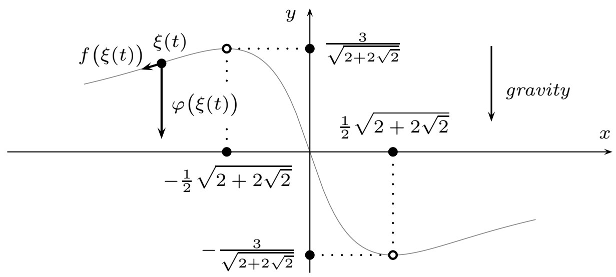

FIGURE 1. The setting of Example 4.4.

Simple considerations show that there are exactly two zeros of $f(x, y)$ *in* U *corresponding to the two real roots of the polynomial* $4x^4 - 4x^2 - 1$, *that is* $x = \pm\frac{1}{2}\sqrt{2+2\sqrt{2}}$. *Now, let* $\mathcal{O} = (-\infty, 0) \times \mathbb{R}$, $\Omega = [0, \infty) \times C^1_T(\mathcal{O})$, *and*

$$\mathcal{F}(x, y) = \left(\frac{3\boldsymbol{mg}(y^2 + 3x^2 + 3)(2xy + 3)}{(6xy + 9)^2 + (y^2 + 3x^2 + 3)^2}, \frac{1}{3}(x^2 + 1)y + \frac{1}{27}y^3 + x \right).$$

A lengthy but straightforward computation shows that $\deg(\mathcal{F}, \mathcal{O}) = 1$. *Hence, Theorem 4.1 yields that if any* T*-periodic force* $\lambda h(t, \xi, \dot{\xi})$, $\lambda \geq 0$, *is added to* f *there exists an unbounded connected set* Γ *of nontrivial solution pairs for* (1.1a) *in* Ω *whose closure in* Ω *meets* $f^{-1}(0) \cap \Omega$.

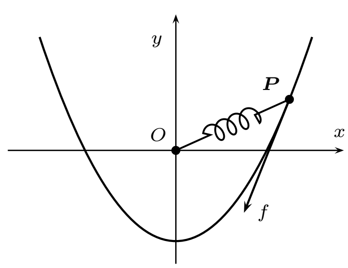

FIGURE 2. The setting of Example 4.5.

Example 4.5. *Consider a unit mass point* $\boldsymbol{P}$ *constrained without friction to the parabola* M *of Example 3.3, of equation* $\frac{1}{2}x^2 - y - 2 = 0$ *lying on an horizontal plane (so that the weight has no effect on the motion). Assume that the extremities of a spring with elastic coefficient* $\boldsymbol{k}$ *and negligible mass are attached one to* $\boldsymbol{P}$ *and the other to the origin* O. *(See Figure 2.) It is easy to verify that the active force acting on* $\boldsymbol{P}$ *is given by*

$$f(x, y) = -\frac{\boldsymbol{k}(y + 1)}{x^2 + 1} \begin{pmatrix} x \\ x^2 \end{pmatrix}.$$

Clearly $f|_M$ *is a vector field tangent to* M, *and the motion of* $\boldsymbol{P}$ *is described by the equation* $\ddot{\xi}_\pi = f(\xi)$ *on* M. *Here again* $m = s = 1$ *and* $U = \mathbb{R} \times \mathbb{R}$. *Let*

$$\mathcal{F}(x, y) = \left(-\frac{\boldsymbol{k}x(y + 1)}{x^2 + 1}, \frac{1}{2}x^2 - y - 2 \right),$$

which is clearly defined on U, *and let* $\Omega = [0, \infty) \times C^1_T(M)$. *Since* $\deg(\mathcal{F}, U) = 1$, *Theorem 4.1 shows that if any* T*-periodic continuous force* $\lambda h(t, \xi, \dot{\xi})$, $\lambda \geq 0$, *is* *added to f, then there exists an unbounded connected set Γ of nontrivial solution pairs for* (1.1) *in Ω whose closure in Ω meets*

$$f^{-1}(0)\cap\Omega=\{(-\sqrt{2},-1),(0,-2),(\sqrt{2},-1)\}.$$

5. Applications to a class of Differential-Algebraic Equations

Let us now consider applications to semi-explicit DAEs of the following forms on an open set $U\subseteq\mathbb{R}^m\times\mathbb{R}^s$:

$$\begin{cases}\ddot{x}=\mathfrak{f}(x,y,\dot{x},\dot{y})+\lambda\mathfrak{h}(t,x,y,\dot{x},\dot{y}),\\ \mathfrak{g}(x,y)=0,\end{cases}\tag{5.1}$$

and

$$\begin{cases}\ddot{x}=\lambda\mathfrak{h}(t,x,y,\dot{x},\dot{y}),\\ \mathfrak{g}(x,y)=0,\end{cases}\tag{5.2}$$

where we assume that $\mathfrak{f}\colon TU\to\mathbb{R}^m\times\mathbb{R}^s$ and $\mathfrak{h}\colon\mathbb{R}\times TU\to\mathbb{R}^m\times\mathbb{R}^s$ are continuous maps, $\mathfrak{h}$ is T-periodic in the first variable, and $\mathfrak{g}\colon U\to\mathbb{R}^s$ is C^∞ and such that $\partial_2\mathfrak{g}(x,y)$ is invertible for all $(x,y)\in U$.

Let us discuss briefly the notion of solution of DAEs as (5.1) and (5.2). In general, let $\mathcal{E}\colon\mathbb{R}\times TU\to\mathbb{R}^m$ be a continuous map and let $\mathfrak{g}$ be as in (5.1) and (5.2). Consider the equation

$$\begin{cases}\ddot{x}=\mathcal{E}(t,x,y,\dot{x},\dot{y}),\\ \mathfrak{g}(x,y)=0.\end{cases}\tag{5.3}$$

By a *solution* of (5.3) we mean a pair of C^1 functions $x\colon J\to\mathbb{R}^k$ and $y\colon J\to\mathbb{R}^s$, J an interval, with the property that $\ddot{x}(t)=\mathcal{E}(t,x(t),y(t),\dot{x}(t),\dot{y}(t))$ and $\mathfrak{g}\big(x(t),y(t)\big)=0$ for all $t\in J$. Differential-Algebraic Equations as (5.3) are closely related to second order differential equations on manifolds as described in Section 3. It is now our aim to clarify such a connection.

Let (x,y) be a solution of (5.3). Differentiating twice the relation $\mathfrak{g}\big(x(t),y(t)\big)=0$ with respect to t, we get

$$\partial_1\mathfrak{g}\big(x(t),y(t)\big)\ddot{x}(t)+\partial_2\mathfrak{g}\big(x(t),y(t)\big)\ddot{y}(t)+\mathfrak{g}''\big(x(t),y(t)\big)\big(\dot{x}(t),\dot{y}(t)\big)=0.$$

Thus, recalling that $\ddot{x}(t)=\mathcal{E}\big(t,x(t),y(t),\dot{x}(t),\dot{y}(t)\big)$, we get

$$\ddot{y}(t)=-\big[\partial_2\mathfrak{g}\big(x(t),y(t)\big)\big]^{-1}\Big(\partial_1\mathfrak{g}\big(x(t),y(t)\big)\mathcal{E}\big(t,x(t),y(t),\dot{x}(t),\dot{y}(t)\big)\\ +\mathfrak{g}''\big(x(t),y(t)\big)\big(\dot{x}(t),\dot{y}(t)\big)\Big).$$

For $t\in\mathbb{R}$, $p,u\in\mathbb{R}^m$ and $q,v\in\mathbb{R}^s$ (hence $(p,q,u,v)\in TU$), let

$$\widetilde{\mathcal{E}}(t,p,q,u,v)=\begin{pmatrix}\mathcal{E}(t,p,q,u,v)\\ -\big[\partial_2\mathfrak{g}(p,q)\big]^{-1}\Big(\partial_1\mathfrak{g}(p,q)\mathcal{E}\big(t,p,q,u,v\big)+\mathfrak{g}''(p,q)\big(u,v\big)\Big)\end{pmatrix}$$

The following proposition shows that DAEs as (5.3) can be regarded as second order ODEs on differential manifolds as discussed in Section 3.

Proposition 5.1. *Equation* (5.3) *is equivalent to the following second order ODE on $M=\mathfrak{g}^{-1}(0)$:*

$$\ddot{\xi}_\pi=\mathcal{P}_\xi\widetilde{\mathcal{E}}(t,\xi,\dot{\xi}),\tag{5.4}$$

in the sense that $\big(x(t),y(t)\big)$ is a solution of (5.3) *if and only if $\xi(t)=\big(x(t),y(t)\big)$ is a solution of* (5.4). *Here $\mathcal{P}_{\xi(t)}$ denotes the orthogonal projection of $\mathbb{R}^m\times\mathbb{R}^s$ onto $T_{\xi(t)}M$.*

Proof. Clearly any solution of (5.3) that meets M satisfies also (5.4).

Let us prove the converse. According to Equation (3.8) one has that for any M-valued C^2 curve $t \mapsto \xi(t)$ one has that, for any t,

$$\mathcal{P}^{\perp}_{\xi(t)}\ddot{\xi}(t) = r\big(\xi(t), \dot{\xi}(t)\big)$$

where $\mathcal{P}^{\perp}_{\xi(t)}$ denotes the projection of the ambient space $\mathbb{R}^k = \mathbb{R}^m \times \mathbb{R}^s$ onto the orthogonal complement $(T_{\xi(t)}M)^{\perp}$ of $T_{\xi(t)}M$, and $r\colon TM \to \mathbb{R}^k$ is the map given by

$$r\big(\xi, \eta\big) = -\left(\mathfrak{g}'(\xi)|_{T_\xi M^{\perp}}\right)^{-1} \mathfrak{g}''(\xi)(\eta, \eta) \in (T_\xi M)^{\perp}$$

for all $\xi \in M$ and $\eta \in T_\xi M$. Let us now see what relation exists between r and $\widetilde{\mathcal{E}}$. Observe that, given any M-valued curve ξ for which $\ddot{\xi}(t) \equiv \widetilde{\mathcal{E}}\big(t, \xi(t), \dot{\xi}(t)\big)$, applying $\mathcal{P}^{\perp}_{\xi(t)}$ on both sides of this equality we get

$$r\big(\xi(t), \dot{\xi}(t)\big) = \mathcal{P}^{\perp}_{\xi(t)}\ddot{\xi} = \mathcal{P}^{\perp}_{\xi(t)}\widetilde{\mathcal{E}}\big(t, \xi(t), \dot{\xi}(t)\big),$$

whenever defined. The arbitrariness of ξ shows that

$$r(\xi, \eta) = \mathcal{P}^{\perp}_{\xi}\widetilde{\mathcal{E}}(t, \xi, \eta)$$

for all $\xi \in M$, $\eta \in T_\xi M$ and $t \in \mathbb{R}$.

Let now $\xi\colon J \subseteq \mathbb{R} \to U$ be a solution of (5.4). From equation (3.6) we get that, whenever defined,

$$\ddot{\xi}(t) = \mathcal{P}_{\xi(t)}\widetilde{\mathcal{E}}\big(t, \xi(t), \dot{\xi}(t)\big) + r\big(\xi(t), \dot{\xi}(t)\big)$$

Hence,

$$\ddot{\xi}(t) = \mathcal{P}_{\xi(t)}\widetilde{\mathcal{E}}\big(t, \xi(t), \dot{\xi}(t)\big) + \mathcal{P}_{\xi(t)}\widetilde{\mathcal{E}}\big(t, \xi(t), \dot{\xi}(t)\big) = \widetilde{\mathcal{E}}\big(t, \xi(t), \dot{\xi}(t)\big),$$

and the assertion is proved. □

We observe that the equivalence of (5.3) and (5.4) could be proven also using the explicit expression for r found in Section 3 (see formula (3.12)).

The equivalence shown by Proposition 5.1 allows us to apply the results of the previous sections to Equations (5.1) and (5.2) by means of the corresponding ODEs on M.

Let $\mathfrak{f}$, $\mathfrak{g}$, and $\mathfrak{h}$ be as above. Define the maps $\Psi\colon T(U) \to \mathbb{R}^m \times \mathbb{R}^s$ and $\Upsilon\colon \mathbb{R} \times T(U) \to \mathbb{R}^m \times \mathbb{R}^s$ by

$$\Psi(p, q, u, v) = \\ \Big(\mathfrak{f}(p, q, u, v), -\big[\partial_2\mathfrak{g}(p, q)\big]^{-1}\Big(\partial_1\mathfrak{g}(p, q)\mathfrak{f}\big(p, q, u, v\big) + \mathfrak{g}''(p, q)\big(u, v\big)\Big)\Big),$$

and

$$\Upsilon(t, p, q, u, v) = \\ \Big(\mathfrak{h}(t, p, q, u, v), -\big[\partial_2\mathfrak{g}(p, q)\big]^{-1}\Big(\partial_1\mathfrak{g}(p, q)\mathfrak{h}\big(t, p, q, u, v\big) + \mathfrak{g}''(p, q)\big(u, v\big)\Big)\Big).$$

We have that (5.1) and (5.2) are equivalent, respectively, to

$$\ddot{\xi}_\pi = \mathcal{P}_\xi\Psi(\xi, \dot{\xi}) + \lambda\mathcal{P}_\xi\Upsilon(t, \xi, \dot{\xi}), \tag{5.5}$$

and

$$\ddot{\xi}_\pi = \lambda\mathcal{P}_\xi\Upsilon(t, \xi, \dot{\xi}), \tag{5.6}$$

where $\xi = (x, y)$.

We now use Equations (5.5) and (5.6) to derive, by means of Theorems 4.1 and 4.2, results concerning the sets of solution pairs for Equations (5.1) and (5.2).

Let us consider first Equation (5.1). We define $\mathfrak{f}_0(p,q) := \mathfrak{f}(p,q,0,0)$ and

$$\Psi_0(p,q) := \Psi(p,q,0,0) = \Big(\mathfrak{f}_0(p,q), -\big[\partial_2\mathfrak{g}(p,q)\big]^{-1}\partial_1\mathfrak{g}(p,q)\mathfrak{f}_0(p,q)\Big).$$

Observe that Ψ_0 is tangent to M in the sense that $\Psi_0(p,q) \in T_{(p,q)}M$ for any $(t,p,q) \in \mathbb{R}\times M$. Hence,

$$\mathrm{Pr}_1\,\mathcal{P}_{(p,q)}\Psi_0(p,q) = \mathfrak{f}_0(p,q).$$

Then from Theorem 4.1 and Remark 4.3 we get the following.

Corollary 5.2. *Let $\mathfrak{f}$, $\mathfrak{h}$, $\mathfrak{g}$ and U be as above. For any $(x,y)\in U$, define $\mathcal{F}\colon U \to \mathbb{R}^m\times\mathbb{R}^s$ by*

$$\mathcal{F}(x,y) = \big(\mathfrak{f}_0(x,y), \mathfrak{g}(x,y)\big).$$

Let $\Omega \subseteq [0,\infty)\times C^1_T(\mathbb{R}^m\times\mathbb{R}^s)$ be open. Assume that $\deg(\mathcal{F}, U\cap\Omega)$ is well defined and nonzero. Then there exists a connected set Γ of nontrivial solution pairs for (5.1) in Ω whose closure in Ω meets $\mathcal{F}^{-1}(0)\cap\Omega$ and is not compact. In particular, if $\mathfrak{g}^{-1}(0)$ is a closed subset of $\mathbb{R}^m\times\mathbb{R}^s$ and $\Omega = [0,\infty)\times C^1_T(\mathbb{R}^m\times\mathbb{R}^s)$, then Γ is unbounded.

A similar argument allows us to deduce the following consequence of Theorem 4.2. We focus on Equation (5.2). We define $\mathfrak{h}_0(t,p,q) := \mathfrak{h}(t,p,q,0,0)$ and the tangent vector field $w_{\mathfrak{h}_0}$ on M as in (3.14) with (p,q) in place of ξ.

Since, clearly, $w_{\mathcal{P}_{(p,q)}\Upsilon} \in T_{(p,q)}M$ for any $(p,q)\in U$, one has that

$$\mathrm{Pr}_1\, w_{\mathcal{P}_{(p,q)}\Upsilon}(p,q) = w_{\mathfrak{h}_0}(p,q).$$

Hence from Theorem 4.2 and Remark 4.3 we get

Corollary 5.3. *Let $\mathfrak{h}$ and $\mathfrak{g}$ be as above. For any $(x,y)\in U$, define $\mathcal{F}\colon U\to\mathbb{R}^m\times\mathbb{R}^s$ by*

$$\mathcal{F}(x,y) = \big(w_{\mathfrak{h}_0}(x,y), \mathfrak{g}(x,y)\big).$$

Let $\Omega \subseteq [0,\infty)\times C^1_T(\mathbb{R}^m\times\mathbb{R}^s)$ be open. Assume that $\deg(\mathcal{F}, U\cap\Omega)$ is well defined and nonzero. Then there exists a connected set Γ of nontrivial solution pairs for (5.2) in Ω whose closure in Ω meets $\mathcal{F}^{-1}(0)\cap\Omega$ and is not compact. In particular, if $\mathfrak{g}^{-1}(0)$ is a closed subset of $\mathbb{R}^m\times\mathbb{R}^s$ and $\Omega = [0,\infty)\times C^1_T(\mathbb{R}^m\times\mathbb{R}^s)$, then Γ is unbounded.

We conclude this section with a few illustrative examples.

Example 5.4. *Consider the setting of Example 4.5. The horizontal acceleration experienced by $\boldsymbol{P}$ is $-\boldsymbol{k}\frac{(y+1)x}{x^2+1}$. Hence, the dynamics of Example 4.5 is described also by the following DAE:*

$$\begin{cases} \ddot{x} = -\boldsymbol{k}\frac{(y+1)x}{x^2+1}, \\ \frac{1}{2}x^2 - y - 2 = 0. \end{cases}$$

Let us perturb this equation as follows

$$\begin{cases} \ddot{x} = -\boldsymbol{k}\frac{(y+1)x}{x^2+1} + \lambda\mathfrak{h}(t,x,y,\dot{x},\dot{y}), \\ \frac{1}{2}x^2 - y - 2 = 0, \end{cases}$$

where $\mathfrak{h}\colon \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$ is a continuous T-periodic function. Let $\Omega = [0,\infty)\times C^1_T(\mathbb{R}^m\times\mathbb{R}^s)$ and put

$$\mathcal{F}(x,y) = \left(-\boldsymbol{k}\frac{(y+1)x}{x^2+1}, \frac{1}{2}x^2 - y - 2\right),$$

as in Example 4.5. Since $\deg(\mathcal{F}, U\cap\Omega) = 1 \neq 0$*, Corollary 5.2 imply that in* Ω *there exists an unbounded connected set* Γ *of nontrivial solution pairs for the perturbed equation whose closure meets*

$$\mathcal{F}^{-1}(0)\cap\Omega = \Big\{\big(-\sqrt{2},-1\big),(0,-2),\big(\sqrt{2},-1\big)\Big\}.$$

The result is exactly the same as in Example 4.5, but now we do not need the second component of the active force, nor we need to bother about second order differential equations on manifolds. Notice also that the result holds true for any choice of the function $\mathfrak{h}$*.*

Example 5.5. *Let* $m = s = 1$ *and consider the following DAE (the constraint is that of Example 3.3):*

$$\begin{cases} \ddot{x} = \lambda\big(x+\sin(t)y\big), \\ \frac{1}{2}x^2 - y^3 - y = 0. \end{cases}$$

Let $\Omega = [0,\infty)\times C^1_T(\mathbb{R}^m\times\mathbb{R}^s)$ *and put*

$$\mathcal{F}(x,y) = \left(\frac{1}{2\pi}\int_0^{2\pi}\big(x+\sin(t)y\big)\,\mathrm{d}t\,,\ \frac{1}{2}x^2-y^3-y\right) = \left(x\,,\ \frac{1}{2}x^2-y^3-y\right)$$

Since $\deg(\mathcal{F}, U\cap\Omega) = -1$*, by Corollary 5.3, we get that in* Ω *there exists an unbounded connected set* Γ *of nontrivial solution pairs for this equation whose closure in* Ω *meets* $\{(0,0)\}$ *(regarded as a a solution pair).*

Example 5.6. *Let* $m = 1$*,* $k = 3$ *and* $U = \mathbb{R}^3$*, and consider the following DAE (the constraint is that of Example 3.5):*

$$\begin{cases} \ddot{x} = x - 2y + \lambda\mathfrak{h}(t,x,y,z,\dot{x},\dot{y},\dot{z}), \\ z^3 + z - x = 0, \\ z - y + x^2 = 0, \end{cases}$$

where $\mathfrak{h}\colon \mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ *is any continuous* T*-periodic perturbing function. Let* $\Omega = [0,\infty)\times C^1_T(\mathbb{R}^1\times\mathbb{R}^2)$ *and put*

$$\mathcal{F}(x,y) = \big(x-2y, z^3+z-x, z-y+x^2\big).$$

As it is readily checked $\mathcal{F}^{-1}(0,0,0) = \{(0,0,0)\}$ *and* $\deg(\mathcal{F}, U\cap\Omega) = -1$*. Hence, for any 'perturbing' function* $\mathfrak{h}$*, Corollary 5.3 yields an unbounded connected set* Γ *in* Ω *of nontrivial solution pairs for this equation whose closure in* Ω *meets the point* $\{(0,0,0)\}$ *(regarded as a solution pair). Notice also, as in Example 5.4, that the above statement holds true for any choice of the function* $\mathfrak{h}$*.*

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