Finite groups whose non-linear irreducible characters of the same degree are Galois conjugate

FINITE GROUPS WHOSE NON-LINEAR IRREDUCIBLE CHARACTERS OF THE SAME DEGREE ARE GALOIS CONJUGATE

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ABSTRACT. We classify the finite groups whose non-linear irreducible characters that are not conjugate under the natural Galois action have distinct degrees, therefore extending the results in Berkovich et al. [Proc. Amer. Math. Soc. **115** (1992), 955-959] and Dolfi et al. [Israel J. Math. **198** (2013), 283-331].

1. INTRODUCTION

In 1992, Berkovich, Chillag and Herzog [BCH] classified the finite groups whose non-linear irreducible characters all have distinct degrees. Since Galois groups of suitable cyclotomic fields act in a natural degree-preserving way (see below) on the set $\mathrm{Irr}(G)$ of the irreducible characters of a finite group G, it seems natural to weaken the above mentioned condition by asking that there exists just one orbit on $\mathrm{Irr}(G)$ for every given irreducible character degree $\neq 1$. While the condition in [BCH] forces all non-linear characters in $\mathrm{Irr}(G)$ to be rational valued, we are now just imposing a minimality condition on the multiplicities of the degrees of the irreducible characters, without setting restrictions on their fields of values.

Let G be a finite group, n a multiple of $|G|$ and let $\mathcal{G}_n = \mathrm{Gal}(\mathbb{Q}_n|\mathbb{Q})$ be the Galois group of the n-th cyclotomic extension. Then $\mathcal{G}_n$ acts on the set $\mathrm{Irr}(G)$ as follows: for $\alpha \in \mathcal{G}_n$, $\chi \in \mathrm{Irr}(G)$ and $g \in G$, we define

$$\chi^\alpha(g) = \chi(g)^\alpha \ .$$

For $\chi, \psi \in \mathrm{Irr}(G)$, if there exists a Galois automorphism $\alpha \in \mathcal{G}_n$ such that $\chi^\alpha = \psi$, then we say that χ and ψ are *Galois conjugate* (in $\mathcal{G}_n$). This is clearly an equivalence relation on $\mathrm{Irr}(G)$. Characters in the same equivalence class have the same kernel, center, field of values and degree.

In this paper, we weaken the condition of [BCH], and prove the following result.

Theorem A. *Let G be a finite group. Every two non-linear irreducible characters of the same degree of G are Galois conjugate if and only if G is either abelian or one of the following.*

- **(a):** *G is a p-group (p a prime), $|G'| = p$ and $\mathbf{Z}(G)$ is cyclic;*
- **(b):** *G is a Frobenius group with prime power order kernel K and complement L, with L cyclic or $L \cong Q_8$. Moreover:*
  - **(b1):** *$L \cong Q_8$ and $|K| = 3^2$; or*

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(b2): *K is elementary abelian, $|K| = q^n$ (q prime), L is cyclic and $|L| = (q^n - 1)/d$, where d divides $q-1$ and $(d, n) = 1$; or*

(b3): *K is a Suzuki 2-group with $|K| = |K'|^2$ and L is cyclic of order $|K'| - 1$.*

(c): *G is non-solvable and either*

$$G \in \{\mathsf{A}_5, \mathsf{Sz}(8), \mathsf{J}_2, \mathsf{J}_3, \mathsf{L}_3(2), \mathsf{M}_{22}, \mathsf{Ru}, \mathsf{Th}, {}^3\mathsf{D}_4(2)\}$$

or

$$G \in \{\mathsf{A}_5 \times \mathsf{Sz}(8), \mathsf{A}_5 \times \mathsf{Th}, \mathsf{L}_3(2) \times \mathsf{Sz}(8)\} .$$

As a consequence of Theorem A, we get a new proof of the main result of [BCH].

Corollary B. *Let G be a finite group. Then, for every non-linear $\chi, \psi \in \mathrm{Irr}(G)$, $\chi \neq \psi$ implies $\chi(1) \neq \psi(1)$ if and only if G is either abelian or one of the following groups:*

(a): *extraspecial 2-groups;*

(b): *$G = KL$ is a Frobenius group with elementary abelian kernel K, $|K| = q^n$ for a prime q, and either*

(b1): *$L \cong Q_8$ and $q^n = 3^2$; or*

(b2): *L is cyclic of order $q^n - 1$.*

In [DNT] the finite groups such that all *non-principal* irreducible characters of the same degree are Galois conjugate are classified. We remark that for non-solvable groups, the class of groups studied in [DNT] and the class we are considering here in fact coincide by Theorem 3.9. However, the two classes differ significantly in the case of nilpotent groups: while the nilpotent groups in [DNT] are just groups of prime order [DNT], or the trivial group, here we have p-groups with cyclic center and commutator subgroup of prime order, or abelian groups (see Corollary 3.2).

Finally, we remark that by quoting Theorem 4.1 of [DNT] (see Theorem 3.7) our work depends on the Classification of Finite Simple Groups.

2. Preliminaries

In the following, by "group" we always mean "finite group". We use standard notation in character theory, as in [I]. Given a character $\chi \in \mathrm{Irr}(G)$, we define $\mathbb{Q}(\chi) = \mathbb{Q}[\{\chi(g) \mid g \in G\}]$, the field generated by the values of χ; $\mathbb{Q}(\chi)$ is called the *field of values* of χ. We stress here that two characters in $\mathrm{Irr}(G)$ are Galois conjugate in some Galois group $\mathcal{G}_n$ if and only if they are Galois conjugate in $\mathrm{Gal}(\mathbb{Q}(\chi)|\mathbb{Q})$ (see Lemma 2.2(a)). Therefore, we omit the explicit reference to a specific Galois extension of $\mathbb{Q}$, and we simply say "Galois conjugate".

Definition 2.1. *We say that a finite group G is a $\mathbf{GC}^*$-group (or $G \in \mathbf{GC}^*$) if any two* non-linear *irreducible characters of G are Galois conjugate whenever they have the same degree.*

The following lemma collects some basic facts, often used without explicit reference. In particular, part (d) shows that the class $\mathbf{GC}^*$ is stable by taking factor groups.

Lemma 2.2. *Let G be a finite group. Then the following hold true.*

- **(a):** *Let $\chi, \psi \in \mathrm{Irr}(G)$, and let $E = \mathbb{Q}_n$ be any cyclotomic field such that $\mathbb{Q}(\chi) \subseteq E$. Then χ and ψ are Galois conjugate in E if and only if they are Galois conjugate in $\mathbb{Q}(\chi)$.*
- **(b):** *If $\chi, \psi \in \mathrm{Irr}(G)$ are Galois conjugate, then $\chi(1) = \psi(1)$, $\mathbb{Q}(\chi) = \mathbb{Q}(\psi)$, $\ker(\chi) = \ker(\psi)$ and $\mathbf{Z}(\chi) = \mathbf{Z}(\psi)$.*
- **(c):** *Let $G = A \times B$, with A non-abelian. If $G \in \mathbf{GC}^*$, then $B = B'$.*
- **(d):** *Let N be a normal subgroup of G. If $G \in \mathbf{GC}^*$, then $G/N \in \mathbf{GC}^*$.*

Proof. (a) Since $\mathrm{Gal}(E|\mathbb{Q})$ is abelian, $\mathbb{Q}(\chi)|\mathbb{Q}$ is a normal extension and the claim follows by extending (resp. restricting) $\mathbb{Q}$-automorphisms to E (resp. to $\mathbb{Q}(\chi)$).

(b) These assertions follow directly from the definitions.

(c) Let $\alpha \in \mathrm{Irr}(A)$ with $\alpha(1) > 1$ and let $\beta \in \mathrm{Irr}(B)$ with $\beta(1) = 1$. Then $\chi = \alpha \times 1_B$ and $\psi = \alpha \times \beta$ are non-linear irreducible characters of G and they have the same degree. It follows that $B \leq \ker(\chi) = \ker(\psi)$, so $\beta = 1_B$. Hence, $B' = B$.

(d) Let $\chi, \psi \in \mathrm{Irr}(G/N)$ be non-linear characters of the same degree. Then the same is true for their inflations $\chi_0, \psi_0 \in \mathrm{Irr}(G)$; so they are Galois conjugate and (observing that $\mathbb{Q}(\chi) = \mathbb{Q}(\chi_0)$) the claim follows. □

Let N be a normal subgroup of G and let $\lambda \in \mathrm{Irr}(N)$. We denote by $\mathrm{Irr}(G|\lambda) = \{\chi \in \mathrm{Irr}(G) \mid [\chi_N, \lambda] \neq 0\}$ the set of the irreducible characters of G lying above λ. If λ is invariant in G and $|\mathrm{Irr}(G|\lambda)| = 1$ we say that λ is *fully ramified* in G/N. In this case, if $\chi \in \mathrm{Irr}(G)$ is the (only) character lying above λ, then $\chi(g) = 0$ for all $g \in G \setminus N$ and $|G/N| = (\chi(1)/\lambda(1))^2$ (see [I, Problem 6.3]).

Lemma 2.3. *Let P be a p-group such that $|P'| = p$, where p is a prime. Let $Z = \mathbf{Z}(P)$. Then the following statements hold true.*

- **(a):** *Every non-linear irreducible character of P is a faithful character of degree $\sqrt{|P : Z|}$;*
- **(b):** *Every non-trivial character $\lambda \in \Lambda := \{\lambda \in \mathrm{Irr}(Z) \mid P' \not\leq \ker(\lambda)\}$ is fully ramified in P/Z. The map*

$$f : \Lambda \to \{\chi \in \mathrm{Irr}(P) \mid \chi(1) > 1\}$$

such that $f(\lambda) = \chi_\lambda$, where χ_λ is the unique irreducible character of G lying over λ, is a bijection.

Proof. This follows from Theorem 7.5 of [H1]. □

We also need a classical result on irreducible modules for abelian groups.

Lemma 2.4. *Let V be a faithful irreducible A-module, $|V| = q^n$ (q prime), for an abelian group A. Then A is cyclic and the semidirect product $V \rtimes A$ is isomorphic to a subgroup of the affine group $\mathrm{GF}(q^n)^+ \rtimes \mathrm{GF}(q^n)^\times$. Moreover, if U is any other faithful irreducible A-module of characteristic q, then $|U| = |V|$.*

Proof. It follows from [H, II.3.10] (and its proof). □

Finally, we give a result that will be used in pinning down the structure of nilpotent residuals (which will turn out to be Frobenius kernels) of $\mathbf{GC}^*$-groups.

Lemma 2.5. *Let $G \in \mathbf{GC}^*$ be a Frobenius group, with Frobenius kernel K a q-group (q prime). Let $N \leq K$ be normal in G and let $\lambda \in \mathrm{Irr}(N)$ be a non-principal K-invariant character. Then the following statements hold true.*

- **(a):** *If $q = 2$ and $\theta_1, \theta_2 \in \mathrm{Irr}(K|\lambda)$ are characters of the same degree, then there exists a Galois automorphism $\alpha \in Gal(\mathbb{Q}_{q^k}|\mathbb{Q})$, where $q^k = \exp(K)$, such that $\theta_1^\alpha = \theta_2$.*
- **(b):** *If K/N is abelian and $\exp(K) = q$, then $|\mathrm{Irr}(K|\lambda)| = 1$ (i.e. λ is fully ramified in K).*

Proof. Write $G = KL$ with L Frobenius complement. Let $\theta_1, \theta_2 \in \mathrm{Irr}(K|\lambda)$ be such that $\theta_1(1) = \theta_2(1)$. As G is a Frobenius group, θ_1^G and θ_2^G are non-linear irreducible characters of the same degree of G. Hence, as $G \in \mathbf{GC}^*$ and $\mathbb{Q}(\theta_i^G) \subseteq \mathbb{Q}(\theta_i) \subseteq \mathbb{Q}_{q^k}$, where $q^k = \exp(K)$, there exists a Galois automorphism $\alpha \in Gal(\mathbb{Q}_{q^k}|\mathbb{Q})$ such that $(\theta_1^G)^\alpha = \theta_2^G$. By Clifford theory, there exists an element $x \in L$ such that

$$\theta_1^\alpha = \theta_2^x \ . \tag{1}$$

Thus, by restricting to N, we get that $\lambda^\alpha = \lambda^x$. Now, for every positive integer m, $\lambda^{\alpha^m} = \lambda^{x^m}$ because Galois conjugation and group conjugation commute. As any non-trivial element of L fixes only the trivial character of N, we deduce that $o(x)$ divides $o(\alpha)$, so $o(x)$ divides $q^{k-1}(q-1)$. Since $|L|$ is coprime to q, we conclude that $o(x)$ divides $q - 1$.

(a): As $o(x) \mid (q-1)$, if $q = 2$ then by (1) we get $\theta_1^\alpha = \theta_2$ and (a) is proved.

(b): Assume now that $\exp(K) = q$ is prime and that K/N is abelian. By [MW, Lemma 12.6], there exists a (unique) subgroup U with $N \leq U \leq K$ such that every $\varphi \in \mathrm{Irr}(U|\lambda)$ extends λ and is fully ramified in K/U. It follows that $|\mathrm{Irr}(K|\lambda)| = |U/N|$ and that all characters in $\mathrm{Irr}(K|\lambda)$ have the same degree. By (1) we deduce that the action of $\mathcal{G} \times L$ on $\mathrm{Irr}(K|\lambda)$ (defined, for $\theta \in \mathrm{Irr}(K|\lambda)$ and $(\alpha, x) \in \mathcal{G} \times L$, by $\theta^{(\alpha,x)} = (\theta^\alpha)^x = (\theta^x)^\alpha$) is transitive on $\mathrm{Irr}(K|\lambda)$. Since $|\mathrm{Irr}(K|\lambda)| = |U/N|$ is a power of q and $\mathcal{G} \times L$ is a q'-group, it follows that $|\mathrm{Irr}(K|\lambda)| = 1$. □

3. $\mathbf{GC}^*$-groups

Theorem 3.1. *Let P be a non-abelian p-group, p a prime. Then P is a $\mathbf{GC}^*$-group if and only if P has cyclic center and commutator subgroup of prime order.*

Proof. Assume first that $|P'| = p$ and that $Z = \mathbf{Z}(P)$ is cyclic. By Lemma 2.3 the map f from the set $\Lambda = \{\lambda \in \mathrm{Irr}(Z) | P' \not\leq \ker(\lambda)\}$ onto $\{\chi \in \mathrm{Irr}(P) | \chi(1) > 1\}$ such that $f(\lambda) = \chi_\lambda$, where $\mathrm{Irr}(G|\lambda) = \{\chi_\lambda\}$, is a bijection. Note that Λ is also the set of the faithful characters of Z, as Z is cyclic. Moreover $\mathbb{Q}(\chi_\lambda) = \mathbb{Q}(\lambda)$, as $(\chi_\lambda)_Z$ is a multiple of λ and $\chi(x) = 0$ for all $x \in P \setminus Z$. Let $|Z| = p^a$. If $\lambda \in \Lambda$, then $\mathbb{Q}(\lambda) = \mathbb{Q}_{p^a}$ and hence, writing $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(\lambda)|\mathbb{Q}) = \mathrm{Gal}(\mathbb{Q}_{p^a}|\mathbb{Q})$, we have that $|\mathcal{G}| = |\Lambda|$. Since any element of Λ is stabilized only by the trivial automorphism of $\mathcal{G}$, it follows that $\mathcal{G}$ acts transitively on Λ. Let now $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ be non-linear characters; then $\chi_1 = f(\lambda_1)$ and $\chi_2 = f(\lambda_2)$ for suitable $\lambda_1, \lambda_2 \in \Lambda$. Now, there exists a Galois automorphism $\alpha \in \mathcal{G}$ such that $\lambda_1^\alpha = \lambda_2$. As $(\chi_{\lambda_1})^\alpha$ lies over λ_1^α, we have that $\chi_2 = f(\lambda_2) = f(\lambda_1^\alpha) = (\chi_{\lambda_1})^\alpha$. Hence $P \in \mathbf{GC}^*$.

Conversely, we show that if P is a $\mathbf{GC}^*$-group, then $\mathbf{Z}(P)$ is cyclic and $|P'| = p$. Let P be a counterexample of minimal order; hence either $\mathbf{Z}(P)$ is not cyclic, or $|P'| > p$.

First, we suppose that $P' \leq \mathbf{Z}(P)$ (i.e. that P has nilpotency class 2). To begin with, we also assume that $Z := \mathbf{Z}(P)$ is cyclic. So, $|P'| > p$ and, by minimality of P, we have that $|P'| = p^2$. Then by [BBC, Theorem 2.1], P is a central product of 2-generated subgroups with cyclic center and a (possibly trivial) cyclic subgroup. So, again by minimality, we have that P is 2-generated; write $P = \langle x, y \rangle$. P being a class 2 p-group, it follows that $\exp(P/Z) = \exp(P')$. Since $P' \leq Z$ and Z is cyclic, we have that P' is a cyclic group of order p^2, and therefore $\exp(P/Z) = \exp(P') = p^2$. Thus, again P being a class 2 p-group, we can choose generators x and y of P such that $o(xZ) = o(yZ) = p^2$ in P/Z. Now, if N is the (unique) subgroup of order p of P', then both x^pN and y^pN belong to $M/N := \mathbf{Z}(P/N)$. In fact, $P' = \langle [x, y] \rangle$ and $[x^p, y] = [x, y]^p = [x, y^p]$, so $[x^p, y]$ as well as $[y^p, x]$ belongs to N as $\exp(P') = p^2$. Now, by minimality, $\mathbf{Z}(P/N)$ is cyclic, and therefore M/Z is cyclic. On the other hand, since both x^p and y^p lie in M, M/Z cannot be cyclic. This contradiction shows that Z cannot be cyclic.

So, we assume that Z is not cyclic. Let N be a subgroup of order p of Z such that $N \neq P'$ (such a subgroup certainly exists as Z has more than one subgroup of order p). By minimality, $\mathbf{Z}(P/N)$ is cyclic and $|P'N/N| = p$. If $N \cap P' = 1$, then $|P'| = p$ and by Lemma 2.3 the irreducible characters of G lying over $1_N \times \lambda$ and $\mu \times \lambda$, where $\mu \in \mathrm{Irr}(N)$ and $\lambda \in \mathrm{Irr}(P')$ are non-principal characters, are non-linear characters of the same degree, but with distinct kernels, against $P \in \mathbf{GC}^*$.

Thus, $N \leq P'$, $|P'| = p^2$ and $Z = N \times Z_0$, where Z_0 is a non-trivial cyclic group. Let $M \leq Z_0$ with $|M| = p$. By minimality $|(P/M)'| = p$ and $\mathbf{Z}(P/M)$ is cyclic; this yields that $M = Z_0$ and hence that $Z = N \times M = P'$ is elementary abelian of order p^2. As P has class two, $\exp(P/Z) = p$. Let $U/N = \mathbf{Z}(P/N)$ and $W/M = \mathbf{Z}(P/M)$; by the minimality of P, they are cyclic groups. As $\exp(P/Z) = p$, this yields $|U/Z|, |W/Z| \leq p$. We claim that $|U/Z| = |W/Z|$. Contrarily assume that $|U/Z| \neq |W/Z|$. Then, by symmetry, let $W = Z$ and $|U/Z| = p$. So P/M is an extraspecial group and hence $|P/Z| = p^{2n}$ for some positive integer n. Then $|P/U| = p^{2n-1}$ is not a square and, as P/N has commutator subgroup of prime order, this gives a contradiction by Lemma 2.3. This proves our claim. Hence, $|P/N : \mathbf{Z}(P/N)| = |P/M : \mathbf{Z}(P/M)|$ and by Lemma 2.3 there are characters $\chi, \psi \in \mathrm{Irr}(P)$ such that $\chi(1) = \psi(1)$ with $\ker(\chi) = N$ and $\ker(\psi) = M$, so they cannot be Galois conjugate.

Therefore, we can assume that $P' \not\leq Z$; hence, in particular, $|P'| \geq p^2$. Let M be a normal subgroup of P such that $|M| = p$. Since P/M is a non-abelian $\mathbf{GC}^*$-group, minimality of P yields that $\mathbf{Z}(G/M)$ is cyclic and that $(P/M)' = P'M/M$ has prime order. Thus $M \leq P'$ and, as both Z/M and P'/M are subgroups of the cyclic group $\mathbf{Z}(P/M)$, and P'/M is a group of prime order which is not contained in Z/M, we deduce that $Z = M$. So, we conclude that Z is the only normal subgroup of order p of P, $Z < P'$ and that $|P'| = p^2$.

Now, $\mathbf{Z}(P/Z) = \mathbf{Z}_2(P)/Z$ is a cyclic group of exponent dividing $\exp(Z) = p$ ([H, III.2.13]). As $P'/Z \leq \mathbf{Z}(P/Z)$, we conclude that $\mathbf{Z}_2(P) = P'$ is a group of order p^2. Hence, P/Z is an extraspecial group; set $|(P/Z)/(P/Z)'| = |P/P'| = p^{2n}$.

We remark that $|P| \neq p^4$. In fact, otherwise P/Z is an extraspecial group of order p^3, so there exists a character $\chi \in \mathrm{Irr}(P)$ with $\ker(\chi) = Z$ and $\chi(1) = p$. As $|Z| = p$, there also exists a faithful character $\psi \in \mathrm{Irr}(P)$ ([I, (2.32)]). Since $\psi(1)^2$ divides $|P/Z|$ ([I, (2.30)]), it follows that $\psi(1) = p = \chi(1)$, a contradiction.

We also observe that $P' \cong C_p \times C_p$. In fact, if $P' = \mathbf{Z}_2(P)$ is cyclic then P has a cyclic subgroup C of index 2 by [H, III.7.7]. Thus $|C/Z| = 4$ (as P/Z is extraspecial). Then $|P/Z| = 2^3$, and hence $|P| = 16$, which is not possible.

Let $N = \mathbf{C}_P(P')$. Since P/N is isomorphic to a non-trivial subgroup of $\mathrm{GL}_2(p)$, then $|P/N| = p$. Also, $N' = P'$. In fact, if $N' < P'$, then $N' \leq Z$, as Z is the only proper non-trivial subgroup of P' which is normal in P . Hence, N/Z is an abelian subgroup of index p in the extraspecial group P/Z. This implies $|P| = p^4$, which is again not possible.

Let T be a subgroup of order p of P', with $T \neq Z$. Let $\overline{N} = N/T$ and $\overline{W} = \mathbf{Z}(\overline{N})$. Since $\overline{N}' = \overline{P'}$ has order p, Lemma 2.3 yields that every $\lambda \in \mathrm{Irr}(\overline{W})$ such that $\overline{P'} \not\leq \ker(\lambda)$ is fully ramified in $\overline{N}/\overline{W}$; so, if $\varphi \in \mathrm{Irr}(N|\lambda)$, then $\varphi(1) = p^a$, where $|N/W| = p^{2a}$ and $\mathbb{Q}(\varphi) = \mathbb{Q}(\lambda)$; note that $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}_{p^2}$, as $\exp(W/T)$ divides p^2 (since P/P' has exponent p because P/Z is extraspecial). We also observe that φ is not P-invariant, as $\ker(\varphi) \cap P' = T$ is not normal in P. Hence, as $|P : N| = p$, φ^P is an irreducible character of P ([I, (6.19)]); since $Z \not\leq \ker(\varphi^P)$, then $\ker(\varphi^P) = 1$. So $\varphi^P \in \mathrm{Irr}(P)$ is a faithful character of degree p^{a+1}.

Let now $\mu \in \mathrm{Irr}(P')$ with $\ker(\mu) = T$. As W/T is abelian, it follows that μ has $|W/P'|$ extensions to W. Let $\lambda_1, \lambda_2 \in \mathrm{Irr}(W|\mu)$ be any extensions of μ, $\varphi_i \in \mathrm{Irr}(N)$ the (unique) character lying over λ_i and $\chi_i = \varphi_i^P \in \mathrm{Irr}(P)$, for $i = 1, 2$.

As $\chi_1(1) = \chi_2(1) > 1$ and $P \in \mathbf{GC}^*$, there exists a Galois automorphism $\alpha \in \mathrm{Gal}(\mathbb{Q}_{|P|}|\mathbb{Q})$ such that $\chi_1^\alpha = \chi_2$. Let $\{x_1 = 1, x_2, \ldots, x_p\}$ be a transversal for N in P. Then $(\chi_i)_N = \varphi_i^{x_1} + \varphi_i^{x_2} + \cdots + \varphi_i^{x_p}$, for $i = 1, 2$. Hence, $\varphi_1^\alpha = \varphi_2^{x_j}$ for some $j \in \{1, 2, \ldots, p\}$. We have

$$T = \ker(\varphi_1) \cap P' = \ker(\varphi_1^\alpha) \cap P' = \ker(\varphi_2^{x_j}) \cap P' = (\ker(\varphi_2) \cap P')^{x_j} = T^{x_j}.$$

As all conjugates T^{x_i} are distinct (because $|P : N| = p$ and T is not normal in P), it follows that $j = 1$ and that $\varphi_1^\alpha = \varphi_2$. Therefore, recalling that $(\varphi_i)_W = \lambda_i$ for $i = 1, 2$, we conclude that $\lambda_1^\alpha = \lambda_2$. So, by Lemma 2.2(a) there exists a $\beta \in \mathcal{H} = \mathrm{Gal}(\mathbb{Q}(\lambda_1)|\mathbb{Q})$ such that $\lambda_1^\beta = \lambda_2$. Hence, $\mathcal{H}$ acts transitively on the set $\mathrm{Irr}(W|\mu)$ of the extensions of μ to W. As $\mathbb{Q}(\lambda_1) \subseteq \mathbb{Q}_{p^2}$, we see that $|\mathcal{H}|$ divides $p(p-1)$. Since $|\mathrm{Irr}(W|\mu)| = |W/P'|$ is a power of p, we conclude that $|W/P'|$ divides p. Now, $P' < W$, as otherwise $\overline{N}$ would be an extraspecial group against $|\overline{N}| = |P/P'| = p^{2n}$. We conclude that $|W/P'| = p$ and hence that $p^{2a} = |N/W| = p^{2n-2}$.

Therefore, if $\chi \in \mathrm{Irr}(P)$ lies over μ, then $\chi(1) = p^{a+1} = p^n$ and, as observed above, $\ker(\chi) = 1$. But p^n is also the degree of any non-linear irreducible character of the extraspecial group P/Z; this is a contradiction, as characters with different kernels cannot be Galois conjugate. This is the final contradiction. □

As a consequence, we get a characterization of nilpotent groups in $\mathbf{GC}^*$.

Corollary 3.2. *Let $G \in \mathbf{GC}^*$ be a nilpotent group. Then either G is abelian or G is a group of prime power order with cyclic center and commutator subgroup of prime order.*

Proof. Assume that G is non-abelian and let P be a non-abelian Sylow p-subgroup of G, where p is a suitable prime. So $G = P \times K$ and hence $K = 1$ by part (c) of Lemma 2.2. □

Given a finite group G, we denote by G_∞ the nilpotent residual of G, that is the smallest term of the lower central series of G.

Lemma 3.3. *Let $G \in \mathbf{GC}^*$ be a solvable group and let $K = G_\infty > 1$ (i.e. G is not nilpotent). Let $N < K$ be such that either $N = 1$ or N is the unique minimal normal subgroup of G contained in K. If $(|G/K|, |K/N|) = 1$, then G is a Frobenius group with Frobenius kernel K.*

Proof. Let $\varphi \in \mathrm{Irr}(K)$ with $\varphi \neq 1_K$. Then $\varphi(1)$ divides $|K/N|$, as N is a normal abelian subgroup of K; so $\varphi(1)$ is coprime to $|G/K|$.

We claim that the determinantal order $o(\varphi)$ is also coprime to $|G/K|$. Recall that $o(\varphi)$ divides $|K/K'|$; so we are done if $N \leq K'$. But if $N \not\leq K'$, then the assumption that N is the unique minimal normal subgroup of G contained in K implies that $K' = 1$ and hence K is a q-group, for some prime q. So, $o(\varphi)$ is a power of q and q divides $|K/N|$, and the claim follows.

Thus, $(o(\varphi)\varphi(1), |G/K|) = 1$ and hence (by [I, (6.28)]) there exists a unique extension α of φ to $I_G(\varphi)$ such that $o(\alpha) = o(\varphi)$.

Now let β be any extension of φ to $I = I_G(\varphi)$ and let $\chi = \alpha^G$ and $\psi = \beta^G$. So, $\chi, \psi \in \mathrm{Irr}(G)$ are irreducible characters of the same degree. Observe that they are non-linear, otherwise their kernels would contain K, as $K = G_\infty \leq G'$, while they lie over $\varphi \neq 1_K$. So, as $G \in \mathbf{GC}^*$, there exists a $\sigma \in Gal(\mathbb{Q}(\chi)|\mathbb{Q})$ such that $\chi^\sigma = \psi$. Recalling that Galois conjugation commutes with character induction, Clifford correspondence yields that $\alpha^\sigma = \beta$. In particular $o(\beta) = o(\alpha) = o(\varphi)$ and hence (by the uniqueness of α) we conclude that $\beta = \alpha$. Whence, there exists a unique extension of φ to I. Now Gallagher's theorem ([I, 6.17]) yields $(I/K)' = I/K$ and, being I/K solvable, this implies $I = K$.

Therefore, we have shown that $I_G(\varphi) = K$ for every $\varphi \in \mathrm{Irr}(K)$, $\varphi \neq 1_K$. By Brauer Permutation Lemma ([I, 6.32]), we conclude that no non-trivial conjugacy class of K is fixed by any non-trivial element of G/K. Thus, $\mathbf{C}_G(x) \leq K$ for every non-trivial element $x \in K$, so G is a Frobenius group with kernel K. □

Proposition 3.4. *Let G be a solvable, non-nilpotent group, with $G \in \mathbf{GC}^*$. Then $G = KL$ is a Frobenius group, where $K = G_\infty$ is the Frobenius kernel, L is the Frobenius complement and either L is cyclic or $L \cong Q_8$.*

Proof. We first observe that it is enough to show that G is a Frobenius group with kernel $K = G_\infty$. In fact, G/K is a nilpotent group in $\mathbf{GC}^*$, so Corollary 3.2 and the structure of Frobenius complements then yield that G/K is either a cyclic group or it is a quaternion group Q_8, since Q_{2^n} has commutator subgroup of order 2^{n-2}.

We work by induction on $|G|$. Let $K = G_\infty$. Assume first that K is minimal normal in G; hence K is an abelian q-group for some prime q.

If $K < G'$, then $G/K \in \mathbf{GC}^*$ is a non-abelian nilpotent group and by Corollary 3.2 G/K is a p-group, for a prime $p \neq q$. Hence, we are done by Lemma 3.3 (with $N = 1$). If $K = G'$, let Q be a Sylow q-subgroup of G. So $K \leq Q$ and Q is normal in G. Thus, as $1 \neq \mathbf{Z}(Q) \cap K \triangleleft G$, we see that $K \leq \mathbf{Z}(Q)$. Let L be a q-complement of G. Since $[L, Q]$ is a subgroup of $K = G'$, $[L, Q] \triangleleft G$. So $[L, Q] \neq 1$, as otherwise $G = L \times Q$ would be nilpotent, and we deduce that $[L, Q] = K$. By coprime action, $Q = [L, Q] \times \mathbf{C}_Q(L) = K \times \mathbf{C}_Q(L)$ and hence $G = LK \times \mathbf{C}_Q(L)$. Recalling that LK is non-abelian, Lemma 2.2 (c) yields that $\mathbf{C}_Q(L) = 1$. Therefore, $Q = K$ and we can again apply Lemma 3.3 (with $N = 1$).

Thus, we can assume that K is not minimal normal in G. If $N_1, N_2 \leq K$ are distinct minimal normal subgroup of G, then by induction G/N_i is a Frobenius group with Frobenius kernel K/N_i, for $i = 1, 2$, as $K/N_i = (G/N_i)_\infty$. In particular, $(|G/K|, |K/N_i|) = 1$ for $i = 1, 2$ and hence $(|G/K|, |K|) = 1$; again, we conclude using Lemma 3.3. So, we can reduce to the case that there is an unique minimal normal subgroup N of G such that $N < K$. We conclude by using induction and Lemma 3.3. □

We now start working towards a finer description of the solvable (non-nilpotent) $\mathbf{GC}^*$-groups. In order to motivate the next result, we mention that the affine group $\mathrm{GF}(5^2)^+ \rtimes \mathrm{GF}(5^2)^\times$ is a $\mathbf{GC}^*$-group, but its subgroup of index 2 is not a $\mathbf{GC}^*$-group, since it has two *rational* characters of degree 12.

Given an abelian group K, we denote by $\mathrm{Irr}(K)^\#$ the set of non-principal irreducible characters of K.

Theorem 3.5. *Let $G = KL$ be a Frobenius group, with Frobenius kernel K and complement L. Assume that K is abelian. Then $G \in \mathbf{GC}^*$ if and only if K is minimal normal in G, $|K| = q^n$ (for a prime q) and*

- **(a):** *either $G \cong (C_3 \times C_3) \rtimes Q_8$ or*
- **(b):** *L is cyclic and $|L| = (q^n - 1)/d$, where d is a divisor of $q - 1$ and d is coprime to n.*

Proof. Assume $G \in \mathbf{GC}^*$. Let $\varphi, \psi \in \mathrm{Irr}(K)$ be non-principal characters; then φ^G and ψ^G are irreducible characters and they have the same degree $|L|$, as K is abelian. Then there exists a Galois automorphism α such that $\psi^G = (\varphi^G)^\alpha = (\varphi^\alpha)^G$ and hence, by Clifford theory, there is an element $x \in L$ such that $\varphi^\alpha = \psi^x$. In particular, $\ker(\varphi) = \ker(\varphi^\alpha) = \ker(\psi)^x$. As every subgroup $N < K$ with cyclic factor group K/N, is the kernel of a suitable non-principal irreducible character of K, it follows that L acts transitively on the set of such subgroups. Therefore, K is elementary abelian, as otherwise it has non-trivial cyclic factor groups of distinct orders. Also K is an irreducible L-module, because given a proper non-trivial L-submodule H of K there exist M_1, M_2 maximal subgroups of K such that $H \leq M_1$ and $H \not\leq M_2$. Hence, K is minimal normal in G. Write $|K| = q^n$, where q is a prime.

Viewing K as a $\mathrm{GF}(q)$-vector space, L acts transitively on the hyperplanes of K so $(q^n - 1)/(q - 1)$ divides $|L|$. Also, as L acts fixed point freely on K, we have that $|L|$ divides $q^n - 1$.

So, if $L \cong Q_8$, then $(q, n) \in \{(3, 2), (7, 2)\}$. Assuming $q = 7$, $n = 2$, then $G \cong (C_7 \times C_7) \rtimes Q_8$ has six irreducible characters of degree 8, and their fields of values have degree 3 over $\mathbb{Q}$, so $G \notin \mathbf{GC}^*$. Hence, if $L \cong Q_8$, then $G \cong (C_3 \times C_3) \rtimes Q_8$. We notice here that the group $(C_3 \times C_3) \rtimes Q_8$ has exactly two non-linear irreducible characters, one of degree 2 and one of degree 8, so it is a $\mathbf{GC}^*$-group.

By Proposition 3.4, we can now assume that L is cyclic. So, by Lemma 2.4 we can assume that L is a subgroup of the multiplicative group L_0 of the field $\mathrm{GF}(q^n)$ acting on $K = \mathrm{GF}(q^n)^+$. We denote by U_0 the subgroup of order $q-1$ of L_0 and set $U = U_0 \cap L$. Write $d = [L_0 : L]$, where d divides $q-1$.

We first show that $L_0 = LU_0$ if and only if $(d, (q^n-1)/(q-1)) = 1$. In fact, assuming $L_0 = LU_0$, we have that $[U_0 : U] = [L_0 : L] = d$ and hence $|U| = (q-1)/d$. Since L_0 is cyclic, we also have that $|U| = (|U_0|, |L|) = |U|((q-1)/|U|, [L : U])$. Hence, $(d, (q^n-1)/(q-1)) = ((q-1)/|U|, [L : U]) = 1$. Conversely, as $[L_0 : L] = d$ and $[L_0 : U_0] = (q^n-1)/(q-1)$, if $(d, (q^n-1)/(q-1)) = 1$, then U_0 and L are subgroups of coprime indices in L_0 and hence $L_0 = LU_0$.

So, observing that (as d is a divisor of $q-1$) $(d, (q^n-1)/(q-1)) = (d, n)$, in order to complete the proof of the theorem it is enough to show that $G \in \mathbf{GC}^*$ if and only if $L_0 = LU_0$.

We notice that, identifying U_0 with the group of non-zero residue classes $\overline{a}$ mod q, if $\overline{a} \in U_0$ and $\lambda \in \mathrm{Irr}(K)^\#$, then $\lambda^{\overline{a}} = \lambda^{\alpha_a}$, where $\alpha_a \in \mathcal{G}_q = \mathrm{Gal}(\mathbb{Q}_q|\mathbb{Q})$ is the Galois automorphism that takes q-th roots of unity to their b-th power, where b is the inverse of a mod q.

Assume first that $L_0 = LU_0$ and take non-linear characters $\chi, \psi \in \mathrm{Irr}(G)$. Then $\chi = \lambda^G$ and $\psi = \mu^G$ for some $\lambda, \mu \in \mathrm{Irr}(K)^\#$. As $L_0 = LU_0$ acts transitively on $\mathrm{Irr}(K)^\#$, there exists a Galois automorphism $\alpha \in \mathcal{G}_q$ and an element $x \in L$ such that $\lambda = (\mu^x)^\alpha$. Hence,

$$\chi = \lambda^G = ((\mu^x)^\alpha)^G = ((\mu^x)^G)^\alpha = \psi^\alpha$$

because $(\mu^x)^G = \mu^G$ as $x \in L$. So G is a $\mathbf{GC}^*$-group.

Assume now that $G \in \mathbf{GC}^*$. Note that G has exactly d non-linear irreducible characters, all of degree $|L|$, because $|\mathrm{Irr}(K)^\#|/|L| = d$ and $G = KL$ is a Frobenius group. Hence, considering a non-linear $\chi \in \mathrm{Irr}(G)$, we have that d divides $|\mathrm{Gal}(\mathbb{Q}(\chi)|\mathbb{Q})| = [\mathbb{Q}(\chi) : \mathbb{Q}]$. Writing $\chi = \lambda^G$ for a suitable $\lambda \in \mathrm{Irr}(K)^\#$, one observes that $\chi(g) = 0$ for every $g \in G \setminus K$. For $x \in K$, taking a transversal T of U in L, one has

$$\chi(x) = \sum_{y \in L} \lambda^y(x) = \sum_{t \in T} \sum_{a \in U} \lambda^{at}(x) = \sum_{t \in T} \sum_{a \in U} \lambda^a(x^{t^{-1}}) = \sum_{t \in T} \sum_{\alpha \in \mathcal{H}} (\lambda(x^{t^{-1}}))^\alpha$$

where $\mathcal{H}$ is a subgroup of $\mathcal{G}_q$ such that $[\mathcal{G}_q : \mathcal{H}] = [U_0 : U]$. Hence $\sum_{\alpha \in \mathcal{H}} (\lambda(x^{t^{-1}}))^\alpha$ is an element of $E = \mathrm{Fix}(\mathcal{H})$. We conclude that $\chi(x) \in E$ for every $x \in K$ and then $\mathbb{Q}(\chi) \subseteq E$. Therefore, d divides $[E : Q] = [U_0 : U] = [LU_0 : L]$ and hence $L_0 = LU_0$. □

A non-abelian 2-group K is a *Suzuki 2-group* if K has more than one involution and there exists a soluble group of automorphisms of K which is transitive on the set of the involutions of K (see [HB, Definition 7.1]). If K is a Suzuki 2-group, then $K' = \mathbf{Z}(K) = \Phi(K) = \{x \in K \mid x^2 = 1\}$ and either $|K| = |K'|^2$ or $|K| = |K'|^3$ ([HB, Theorem 7.9]).

Theorem 3.6. *Let $G = KL \in \mathbf{GC}^*$ be a solvable Frobenius group with kernel $K = G_\infty$ and complement L. Then either K is an elementary abelian q-group, q a prime, or K is a Suzuki 2-group such that $|K| = |K'|^2$, L is cyclic and $|L| = |K'| - 1$.*

Proof. By Proposition 3.4 we know that L is cyclic or $L \cong Q_8$. So, if K is abelian, we conclude by applying Theorem 3.5.

We assume now that K is non-abelian. Thus L is cyclic (as Frobenius groups with complements of even order have abelian kernel). By applying Theorem 3.5 to the $\mathbf{GC}^*$-group G/K', we get that K/K' is an irreducible L-module; write $|K/K'| = q^n$, for a prime q. Since K is nilpotent, this implies that K is a q-group. Also, $|L| = (q^n - 1)/d$, where d divides $q - 1$ and $(d, n) = 1$.

We will first show that $q = 2$. By taking a suitable factor group, we can assume that K' is minimal normal in G. Hence, $\exp(K) \in \{q, q^2\}$ and $K' = \mathbf{Z}(K)$. Moreover, by Lemma 2.4 we have that $|K'| = |K/K'|$. If $\exp(K) = q$, then by Lemma 2.5 we see that every non-principal irreducible character of K' is fully ramified in K/K'. Hence, by the second orthogonality relation, for any $x \in K \setminus K'$ we have $|K/K'| = |\mathbf{C}_{K/K'}(xK')| = |\mathbf{C}_K(x)| \geq |\langle x, K'\rangle| > |K'|$, a contradiction. So, $\exp(K) = q^2$.

Assume, working by contradiction, that $q \neq 2$. So K (having class 2) is a regular q-group and $K' = \Omega_1(K) = \{x \in K \mid x^q = 1\}$. Looking at the action of L on K', by Lemma 2.4 we can identify K' with the additive group of the field $\mathbb{F} = \mathrm{GF}(q^n)$ and L with a subgroup of $M = \mathbb{F}^\times$. Let $U = \mathrm{GF}(q)^\times \leq M$. As $|L| = (q^n - 1)/d$, with d a divisor of $q - 1$ and d coprime to n, we have that $LU = M$; in fact, $(|M : L|, |M : U|) = (d, (q^n - 1)/(q - 1)) = (q, n) = 1$. It follows that L acts transitively on the subgroups of order q of K'. As all elements in $K \setminus K'$ have order q^2, we conclude L acts transitively on the subgroups of order q of K and hence Shult's theorem [S] yields that K is abelian, a contradiction. Hence, $q = 2$ and then $|L| = 2^n - 1$.

We are going to show that K' is minimal normal in G. Assume,working by contradiction, that there exists a non-trivial subgroup N of K' such that K'/N is a chief factor of G. Taking a suitable factor group, we can also assume that N is minimal normal in G. So, $N \leq \mathbf{Z}(K)$ and hence N is an irreducible L-module. By Lemma 2.4, then K/K', K'/N and N are all faithful irreducible L-modules of the same order 2^n, for a positive integer n . Moreover, K' is abelian, as $[K, K', K] = [K', K, K] = 1$ implies $[K', K'] = 1$ by the Three Subgroups Lemma. Finally, we note that by induction K/N is a Suzuki 2-group with $|K/N| = |K'/N|^2$.

Now, $\exp(K') \in \{2, 4\}$. If $\exp(K') = 4$, then $N \setminus \{1\}$ is the set of all involutions of K' (as K'/N is an irreducible L-module). As in the Suzuki group K/N all elements not belonging to K'/N have order 4, we see that $N \setminus \{1\}$ is the set of all involutions of K. Observing that L acts transitively on $N \setminus \{1\}$, we conclude that K is a Suzuki 2-group. So, in particular $|K'| \leq |K|^{1/2}$, a contradiction as $|K'| = |K|^{2/3}$. Thus, K' is elementary abelian and then $\exp(K) = 4$.

Consider now a non-principal character $\lambda \in \mathrm{Irr}(N)$. Let $\mu \in \mathrm{Irr}(K'|\lambda)$ and let $T = I_K(\mu)$. By [MW, Lemma 12.6] there exists a (uniquely determined) subgroup $U = U_\mu$, with $K' \leq U \leq T$, such that every $\nu \in \mathrm{Irr}(U|\mu)$ extends μ and is fully ramified in T/U (so, in particular, $|T/U|$ is a square). By Clifford correspondence, it follows that all characters $\theta \in \mathrm{Irr}(K|\mu)$ have the same degree (depending only on $|U/K'|$) and that $|\mathrm{Irr}(K|\mu)| = |U/K'|$. By Lemma 2.5 it follows that $|U/K'| \leq 2$.

For $\mu_0 \in \mathrm{Irr}(K'|\lambda)$, we have $\mu_0 = \mu\epsilon$, for some $\epsilon \in \mathrm{Irr}(K'/N)$ and K'/N is central in K/N, so $I_K(\mu_0) = T$. Recalling that $|T : U_{\mu_0}|$ is a square and that both $|U/K'|$ and $|U_{\mu_0}/K'|$ are at most 2, we also get that $|U| = |U_{\mu_0}|$. Hence, all characters $\theta \in \mathrm{Irr}(K|\lambda)$ have the same degree. So, again using Lemma 2.5, we get that $|\mathrm{Irr}(K|\lambda)| \leq 2$.

Observing that $|\mathrm{Irr}(K'|\lambda)| = |K'/N| = |K/K'|$, we deduce that the number m of orbits in the conjugation action of K/K' on $\mathrm{Irr}(K'|\lambda)$ is $|T/K'|$. But, by Clifford theorem, $m \leq |\mathrm{Irr}(K|\lambda)|$, so we get that $|T/K'| \leq 2$. As $|T : U|$ is a square and $K' \leq U \leq T$, we conclude that $T = U$. Assume $|T/K'| = 2$ and let $\mu_1, \mu_2 \in \mathrm{Irr}(K'|\lambda)$ be representatives of the orbits of K/K' on $\mathrm{Irr}(K'|\lambda)$. As observed before, $|\mathrm{Irr}(K|\mu_i)| = |U/K'| = 2$. But by Clifford theorem $\mathrm{Irr}(K|\mu_1) \cap \mathrm{Irr}(K|\mu_2) = \emptyset$, and $\mathrm{Irr}(K|\lambda) = \mathrm{Irr}(K|\mu_1) \cup \mathrm{Irr}(K|\mu_2)$, hence $|\mathrm{Irr}(K|\lambda)| = 4$, a contradiction.

It follows that $T = K'$ and hence $\mu^G = \theta \in \mathrm{Irr}(K)$ for every $\mu \in \mathrm{Irr}(K'|\lambda)$. Since K/K' is transitive on $\mathrm{Irr}(K'|\lambda)$, then $|\mathrm{Irr}(K|\lambda)| = 1$ and we conclude that every non-principal $\lambda \in \mathrm{Irr}(N)$ is fully ramified with respect to K/N. Therefore, we have that $\chi(x) = 0$ for all $\chi \in \mathrm{Irr}(K) \setminus \mathrm{Irr}(K/N)$ and $x \in K \setminus N$. Now, K/N has exponent greater than 2 (as K/N is non-abelian) and hence there exists an element $y \in K \setminus N$ such that $x := y^2 \notin N$. Since both K/K' and K'/N are elementary abelian, it follows that $x \in K'$ and that $y \in K \setminus K'$. By the second orthogonality relation, $|\mathbf{C}_K(x)| = |\mathbf{C}_{K/N}(xN)|$. But $|\mathbf{C}_{K/N}(xN)| = |K/N| = |N|^2 = |K'|$, while both y and K' centralize x so $|\mathbf{C}_K(x)| > |K'|$, a contradiction.

Hence, we have that K' is minimal normal in G. So, $K' \leq \mathbf{Z}(K)$ and both K/K' and K' are irreducible L-module. In particular, $K' = \mathbf{Z}(K)$ and $|K'| = |K/K'| = 2^n$. Recalling that $|L| = 2^n - 1$, we see that L acts transitively on the non-identity elements of both $Z = \mathbf{Z}(K)$ and K/Z. As all elements in a coset xZ, with $x \in K \setminus Z$, have the same order, we deduce that all elements $x \in K \setminus Z$ have order 4. Therefore, $Z \setminus \{1\}$ is the set of the involutions of K. As L acts transitively on it, we conclude that K is a Suzuki 2-group with $|K| = |K'|^2$. The proof is complete. □

As defined in [DNT] a finite group G is a $\mathbf{GC}$-group (or $G \in \mathbf{GC}$, for short) if every two *non-principal* irreducible characters of G are Galois conjugate whenever they have the same degree. Clearly, $\mathbf{GC}$ is a subclass of $\mathbf{GC}^*$ and, for a perfect group G, $G \in \mathbf{GC}^*$ if and only if $G \in \mathbf{GC}$. We are going to show that a non-solvable $\mathbf{GC}^*$-group is perfect, and then we apply the classification of non-solvable $\mathbf{GC}$-groups given in [DNT].

Let us consider the following list of simple groups:

$$\mathcal{S} = \{\mathsf{A}_5, \mathsf{Sz}(8), \mathsf{J}_2, \mathsf{J}_3, \mathsf{L}_3(2), \mathsf{M}_{22}, \mathsf{Ru}, \mathsf{Th}, {}^3\mathsf{D}_4(2)\}\ .$$

As proved in [DNT], all groups in $\mathcal{S}$ are $\mathbf{GC}$-groups and hence (being perfect) are $\mathbf{GC}^*$-groups.

We will make use of the following result from [DNT]:

Theorem 3.7 ([DNT, Theorem 4.1]). *Let S be a non-abelian simple group. Then either $G \in \mathcal{S}$ or for every almost simple group A with socle S there exist two non*

Galois conjugate characters $\chi_1, \chi_2 \in \mathrm{Irr}(A)$ *such that* $\chi_1(1) = \chi_2(1) > 1$ *and* $(\chi_i)_S \in \mathrm{Irr}(S)$ *for* $i = 1, 2$.

The proof of the following result mimics the proof of [DNT, Theorem 3.2]; we have anyway decided to sketch it for completeness.

Lemma 3.8. *If* $G \in \mathbf{GC}^*$ *and* S *is a non-abelian composition factor of* G, *then* $S \in \mathcal{S}$.

Proof. Let S be a non-abelian composition factor of the $\mathbf{GC}^*$-group G. Then G has a chief factor $N/M \cong S^n$, for some positive integer n. By replacing G with a suitable factor group, we can assume that $N = S_1 \times S_2 \times \cdots \times S_n$, with $S_i \cong S$, is a minimal normal subgroup of G and that $\mathbf{C}_G(N) = 1$. Set $S = S_1$ and write $T = \mathbf{N}_G(S)$, $C = \mathbf{C}_G(S)$; so T/C is an almost simple group with socle (isomorphic to) S.

Assume, working by contradiction, that $S \notin \mathcal{S}$; so by Theorem 3.7 there exist two non Galois conjugate characters $\theta_1, \theta_2 \in \mathrm{Irr}(T/C)$ such that $\alpha_i = (\theta_i)_S \in \mathrm{Irr}(S)$, for $i = 1, 2$.

For $i = 1, 2$, let

$$\beta_i = \alpha_i \times 1_{S_2} \times \cdots \times 1_{S_n} \in \mathrm{Irr}(N)$$

and observe that $I_G(\beta_i) = T$.

Considering now $\theta_i \in \mathrm{Irr}(T)$ by inflation, we have that $(\theta_i)_N = \beta_i$. Hence, by Clifford correspondence $\chi_i = (\theta_i)^G \in \mathrm{Irr}(G)$, for $i = 1, 2$. As $\chi_1(1) = \chi_2(1) > 1$, there exists a Galois automorphism $\sigma \in \mathcal{G}_{|G|}$ such that $\chi_1^\sigma = \chi_2$.

But, for $i = 1, 2$ $(\chi_i)_N = \sum_{j=1}^n \beta_i^{x_j}$, where $\{x_1 = 1, x_2, \ldots, x_n\}$ is a transversal of T in G. As Galois conjugation commutes with induction and restriction, we get that $\beta_1^\sigma = \beta_2^{x_j}$ for some j and, as $\ker(\beta_1^\sigma) = \ker(\beta_1) = S_2 \times \cdots \times S_n \neq \ker(\beta_2^{x_j})$ for $j > 1$, we conclude that $\beta_1^\sigma = \beta_2$. So, $\theta_1^\sigma, \theta_2 \in \mathrm{Irr}(T|\beta_2)$ and hence Clifford correspondence yields that $\theta_1^\sigma = \theta_2$, against the fact that θ_1 and θ_2 are not Galois conjugate. □

Theorem 3.9. *Let* G *be a non-solvable group with* $G \in \mathbf{GC}^*$. *Then* $G = G'$

Proof. We work by induction on $|G|$. Assume that G is not a simple group and let N be a minimal normal subgroup of G.

Suppose first that G/N is solvable. Then N is non-solvable and hence $N = S^k$ for some $S \in \mathcal{S}$ by Proposition 3.8. Then S has a non-principal $\mathrm{Aut}(S)$-invariant rational character α of odd degree (see [DNT, page 299] or [Atlas]). Then $\psi = \alpha \times \alpha \times \cdots \times \alpha \in \mathrm{Irr}(N)$ is a G-invariant rational character of odd degree with $o(\psi) = 1$ (as $N = N'$). By [NT, Theorem 2.3] it follows that ψ extends to a rational character $\chi \in \mathrm{Irr}(G)$. As we are assuming that G/N is a non-trivial solvable group, there exists a non-principal linear character $\lambda \in \mathrm{Irr}(G)$. So, $\lambda\chi \in \mathrm{Irr}(G)$ is a non-linear character of the same degree as χ, and $\lambda\chi \neq \chi$ by Gallagher's theorem. Being G a $\mathbf{GC}^*$-group, this gives a contradiction, as χ is rational, so it is fixed by Galois conjugation.

Hence, G/N is non-solvable. By induction, G/N is perfect and hence $G'N = G$. Assuming that N is not contained in G', we have $N \cap G' = 1$ and hence $G = G' \times N$. So N is a non-trivial abelian group. As G' is non-abelian (since G is non-solvable), Lemma 2.2 (c) yields that $N = N' = 1$, a contradiction. Therefore, $N \leq G'$ and hence $G = G'N = G'$. □

We are now ready to prove Theorem A.

Proof of Theorem A. Assume first that G is a non-abelian solvable $\mathbf{GC}^*$-group. If G is nilpotent, then we get type (a). Assume that $K = G_\infty < G$. Then by Proposition 3.4 G is a Frobenius group with Frobenius kernel K and complement L, with L cyclic or $L \cong Q_8$ (so G is of type (b)). If K is abelian, then we get either type $(b1)$ or $(b2)$ by Theorem 3.5. If K is non-abelian, then we get type $(b3)$ by Theorem 3.6.

Assume now that G is a non-solvable $\mathbf{GC}^*$-group. Then Theorem 3.9 yields that $G = G'$ and hence G is a $\mathbf{GC}$-group. Now (c) follows by Theorem A of [DNT].

Conversely, we will now show that any group of type $(a) - (c)$ is a $\mathbf{GC}^*$-group. Groups of type (a) are $\mathbf{GC}^*$-groups by Theorem 3.1.

Let now G be a group of type (b). If it is of type $(b1)$, then G has only two non-linear irreducible characters, one of degree 2 and the other of degree 8, so it is a $\mathbf{GC}^*$-group. If G is of type $(b2)$, then G is a $\mathbf{GC}^*$-group by Theorem 3.5.

So we assume that G is of type $(b3)$. We note that all non-linear characters $\chi \in \mathrm{Irr}(G)$ of odd order have the same degree $|L|$ and that they are Galois conjugate. In fact, $\ker(\chi) = K'$ and G/K' is a $\mathbf{GC}^*$-group by Theorem 3.5. Now, denoting by Δ the set of all non-linear irreducible characters of K, by Theorem 7.9 of [HB] and Lemma 2.9 of [DNT] we have that $|\Delta| = 2|L|$ and that every $\theta \in \Delta$ is not rational valued. Hence, no $\theta \in \Delta$ is real valued, as $\theta_{K'}$ is rational valued and every element in $K \setminus K'$ has order 4. Considering that by Brauer Permutation Lemma L acts fixed point freely on $\mathrm{Irr}(K) \setminus \{1_K\}$, we see that L has exactly two orbits O_1 and O_2 on Δ. Write $\chi = \theta_1^G$, $\psi = \theta_2^G$, where $\theta_1, \theta_2 \in \mathrm{Irr}(K)$ non-linear characters, and $\theta_1 \in O_1$ and $\theta_2 \in O_2$. Now, $\overline{\theta_1} = \theta_2^x$ for some $x \in L$, because complex conjugation does not stabilize the orbit O_i, for $i = 1, 2$ (otherwise, as $|O_i|$ is odd, there would be some real character in O_i). Hence,

$$\overline{\chi} = \overline{\theta_1^G} = (\overline{\theta_1})^G = (\theta_2^x)^G = \psi\ .$$

Thus G is a $\mathbf{GC}^*$-group.

(We also remark that in this case $(b3)$ G has exactly three non-linear irreducible characters, one of odd degree $|L|$ and χ and ψ above.)

Finally, by Theorem A of [DNT] the groups listed in (c) are $\mathbf{GC}$-groups. So, being perfect groups, they are also $\mathbf{GC}^*$-groups. □

To conclude, we prove Corollary B, the Berkovich-Chillag-Herzog classification of groups with distinct non-linear degrees.

Proof of Corollary B. It is enough to check which of the groups listed in Theorem A have distinct non-linear degrees.

For a p-group G of type (a), Lemma 2.3 yields that G has exactly $(p-1)|\mathbf{Z}(G) : G'|$ non-linear irreducible characters of the same degree. Hence, G has distinct non-linear degrees if and only if $p = 2$ and G is extraspecial.

Assume now that G is of type (b), so $G = KL$ is a Frobenius group. First, we recall that $(C_3 \times C_3) \rtimes Q_8$ has distinct non-linear degrees (exactly one irreducible character of degree 2 and of degree 8). Next, we observe that the groups listed in type $(b2)$ have exactly d non-linear characters, all of degree $|L|$. Hence they are groups with distinct non-linear degrees if and only if $d = 1$. Also, by the remark at the end of the proof of Theorem A, the groups of type ($b3$) have two non-linear irreducible characters of the same degree.

Finally, it is readily checked (see [Atlas]) that the groups of type (d) also have two non-linear irreducible characters of the same degree. □

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