Plane R-curves II

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Let Γ_R be the class of plane, oriented, rectifiable curves γ, such that, for almost every $x \in \gamma$, the part of γ preceding x is outside the open disk of radius R, centered in $x + R\mathbf{t}_x$, where $\mathbf{t}_x$ is the unit tangent vector at x. In [1] the present authors have obtained bounds for the length and the detour for C^1 regular curves in Γ_R. These bounds are proved here for all curves in Γ_R.

1. Introduction

Let $R > 0$. Let Γ_R be the class of the plane oriented local rectifiable curves γ satisfying the following property: for every $x \in \gamma$, let γ_x be the part of γ between the starting point and x and almost everywhere let $\mathbf{t}$ be the tangent vector to γ at x, then γ_x is not contained in the open circle centered at $x + R\mathbf{t}$ and of radius R. These curves have been studied in [1] and have been called *R-curves*.

Γ_R is a generalization of the class Γ introduced in [2, 3] and studied in $\mathbb{R}^n$ [5, 9]: $\gamma \in \Gamma$ if for every $x \in \gamma$ the arc γ_x is contained in the half plane bounded by the line through x ortogonal to $\mathbf{t}$. The class Γ has also been recently studied in many other spaces: Riemannian manifolds [6], finite-dimension normed spaces [8].

The steepest descent curves of quasi convex functions are curves of Γ [2, 4, 9]; the interest in the R-curves is that they are the steepest descent lines of functions whose level sets have reach greater than R, see [1];

In all previous papers an important goal is to get the apriori global rectifiability (boundeness of the length) of $\gamma \in \Gamma$. Here this result is obtained for the bounded planar curves $\gamma \in \Gamma_R$, Theorem 5.6.

In the previous definition of the curves of Γ it is assumed that γ is local rectifiable; it has been proved that the defining property is equivalent to the so called self-expanding property [10] (or self-contracted property [4] when opposite order is used for γ) for a continuous curve $t \to \gamma(t)$, with t not necessarily the parameter length:

$$|\gamma(t) - \gamma(t_1)| \geq |\gamma(t_2) - \gamma(t_1)| \quad \text{for} \quad 0 \leq t_1 \leq t_2 \leq t. \tag{1}$$

Another definition of the class Γ which comes out immediately from the geometric meaning involves $co(\gamma_x)$, the convex hull of γ_x. Then, $\gamma \in \Gamma$ if for almost every $x \in \gamma$ the tangent vector $\mathbf{t}$ lies in the normal cone to $co(\gamma_x)$ at x, that is:

$$\langle x - y, \mathbf{t} \rangle \geq 0, \quad \forall y \in \gamma_x. \tag{2}$$

Metric and geometric relaxation of both previous two definitions have been introduced in [11] and it was proved that bounded planar curves γ satisfying them have bounded length; counterexamples for not planar curves are also showned.

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On the other hand, the curves $\gamma \in \Gamma_R$, with length parameter s, satisfy the following two properties [1]:

$$|\gamma(s) - \gamma(s_1)| \geq |\gamma(s_2) - \gamma(s_1)| e^{(s_2 - s)/(2R)} \quad \text{for} \quad 0 \leq s_1 \leq s_2 \leq s; \tag{3}$$

$$\langle x - y, \mathbf{t} \rangle \geq -\frac{|x-y|^2}{2R}, \quad \forall y \in \gamma_x. \tag{4}$$

These properties are a generalization of 1 and of 2, respectively. In [1] a priori bounds for the length and the detour of a bounded $\gamma \in \Gamma_R$ have been proved under the assumption that γ is a C^1 curve.

In this work the same bounds are obtained if one assumes that γ is merely rectifiable so the defining property holds a.e., only; first natural or easily obtained properties for C^1 R-curves are extended to arbitrary rectifiable plane R-curves: Theorem 4.5 and Corollary 4.6. Then these properties are used to prove the R-*angle estimate* property of γ: Theorem 5.2. The main bounds for the length and the detour of γ: Theorem 5.5 and Theorem 5.6 then easily are obtained.

The plan of this work follows.

Some of the basic properties for $\gamma \in \Gamma_R$ are recalled in §3.

In §4 properties of tangent sets to $\gamma \in \Gamma_R$ are stated and proved. Theorem 4.5 proves that any unit vector of the tangent set at x to $cl(\gamma \setminus \gamma_x)$ is the inner normal at x of a disk excluding γ_x; in Corollary 4.6 it is proved that, at each point $x \in \gamma$, the tangent sets at x to γ_x and to $cl(\gamma \setminus \gamma_x)$ do not contain directions forming acute angles. These geometrical properties are obvious for a C^1 curve but they have to be proved when γ is rectifiable R-curve, only.

In §5 rectifiable R-curves contained in a disk of arbitrary fixed radius are studied. If γ_x is contained in an open disk of radius R, the R-hull of γ_x (defined in (6)) is considered. In Theorem 5.2 it is proved that the amplitude of the normal cone at x to the R-hull of γ_x, is greater or equal to $\pi/2$. This is an extension to the planar R-curves of the so called *angle estimate* of [2], which plays a fundamental role in order to get the bound for the length of γ in different situations even in CAT(0)-spaces [12]. As a consequence, in the same way as in [1], if γ is contained in a smaller disk, a bound of its length is obtained (Theorem 5.5). Moreover if γ is contained in a disk of arbitrary radius τ, bounds for the detour of γ and its length, depending on R and τ, are proved (Theorem 5.6) as in [1].

In our opinion, the geometrical properties contained in Theorem 4.5 and Theorem 5.2 could be interesting independently of their application in this work.

2. Definitions and preliminaries

Let $K \subset \mathbb{R}^2$, $Int(K)$ will be the interior of K, ∂K the boundary of K, $cl(K)$ the closure of K, $K^c = \mathbb{R}^2 \setminus K$. For every set $S \subset \mathbb{R}^2$, $co(S)$ is the convex hull of S. Let $B(z, \rho) = \{x \in \mathbb{R}^2 : |x - z| < \rho\}, S^1 = \partial B(0,1)$ and let $D(z, \rho) = cl(B(z, \rho))$. The notations $B_\rho(x), D_\rho(x)$ will also be used for open, closed disks of radius ρ centered at x. The usual scalar product between vectors $u, v \in \mathbb{R}^2$ will be denoted by $\langle u, v \rangle$.

Let K be a non empty closed set. Let $q \in K$; the *tangent cone* of K at q is defined in [13] as:

$$\text{Tan}_K(q) = \{v \in \mathbb{R}^2 : \forall \varepsilon > 0 \, \exists x \in K \cap B_\varepsilon(q) \, \exists r > 0 \, \text{s.t.} \, |r(x - q) - v| < \varepsilon\}.$$

Let us recall that if $\text{Tan}_K(q) \neq \{0\}$ then

$$S^1 \cap \text{Tan}_K(q) = \bigcap_{\varepsilon > 0} cl(\{\frac{x - q}{|x - q|}, q \neq x \in K \cap B(q, \varepsilon)\}).$$

The *normal cone* at q to K is the non empty closed convex cone, given by:

$$\text{Nor}_K(q) = \{u \in \mathbb{R}^2 : \langle u, v \rangle \leq 0 \quad \forall v \in \text{Tan}_K(q)\}. \tag{5}$$

The *dual cone* of a set K is $K^\star = \{y \in \mathbb{R}^2 : \langle y, x \rangle \geq 0 \quad \forall x \in K\}$. Thus $\text{Nor}_K(q) = -\{\text{Tan}_K(q)\}^\star$.

In the following definitions A will be a closed set. If $a \in A$, then $reach(A, a)$ is the supremum of all numbers ρ such that for every $x \in B(a, \rho)$ there exists a unique point $b \in A$ satisfying $|b - x| = \text{dist}\,(x, A)$, see [13]. Also:

$$reach(A) := \inf\{reach(A, a) : a \in A\}.$$

Let us define $co_R(A)$, the *R-hull* of A, as the closed set containing A, such that
(i) $co_R(A)$ has reach greater or equal to R;
(ii) if a set $B \supseteq A$ and $reach(B) \geq R$, then $B \supseteq co_R(A)$.
See [14, pp.105-107] for the properties of R-hull. It can be shown that

$$co_R(A) = \cap\{(B_R(z))^c : B_R(z) \cap A = \emptyset\}. \tag{6}$$

The R-hull of a closed set A may not exist, see [14, Remark 4.9]. However

Proposition 2.1 *[14, Theorem 4.8] If A is a plane closed connected subset of an open disk of radius R, then A has R-hull.*

3. Properties of R-curves

In this paper a *curve* in $\mathbb{R}^2$ is the image of a continuous function on an interval, valued into $\mathbb{R}^2$. Let $\gamma \subset \mathbb{R}^2$ be an oriented rectifiable curve and let $x(\cdot)$ be its parametric representation with respect to the arc length parameter $s \in [0, L]$. If $x_1 = x(s_1), x_2 = x(s_2) \in \gamma$ with $s_1 \leq s_2$, the notation $x_1 \preceq x_2$ will be used. Let us denote $x(s) = x$,

$$\gamma_x = \{y \in \gamma : y \preceq x\};\ \gamma_{x_1,x_2} = \{y \in \gamma : x_1 \preceq y \preceq x_2\}.$$

Definition 1 Let R be a fixed positive number. An R-curve $\gamma \subset \mathbb{R}^2$ is a rectifiable oriented curve with arc length parameter $s \in [0, L]$, tangent vector $\mathbf{t}(s) = x'(s)$ such that the inequality

$$|x(s_1) - x(s) - R\ \mathbf{t}(s)| \geq R \tag{7}$$

holds for almost all s and for $0 \leq s_1 \leq s \leq L$. Γ_R will denote the class of R-curves in $\mathbb{R}^2$.

The geometric meaning of (7) is that for every point $x = x(s) \in \gamma$, with tangent vector $\mathbf{t}(s)$, the set γ_x is outside of the open disk of radius R through x centered at $x + R\ \mathbf{t}(s)$.
Let us notice the following equivalent formulations of (7) for $0 \leq s_1 < s \leq L$:

$$|x(s_1) - x(s)|^2 - 2R\ \langle x(s_1) - x(s)), \mathbf{t}(s)\rangle \geq 0; \tag{8}$$

$$\langle x(s) - x(s_1), \mathbf{t}(s)\rangle \geq -\frac{|x(s_1) - x(s)|^2}{2R}. \tag{9}$$

Proposition 3.1 *[1, Lemma 3.1, Corollary 3.2] An R-curve does not intersect itself.*

Proposition 3.2 *[1, Theorem 3.3] Let $\gamma \in \Gamma_R$. For every $s \in (0, L)$, $x = x(s)$, $\gamma_x \subsetneq \gamma$, the following two subsets of S^1:*

$$U_x^+ = \{u \in S^1 : \exists s^{(k)} \geq s,\ \lim_{s^{(k)} \to s} x'(s^{(k)}) = u\}, \tag{10}$$

$$U_x^- = \{u \in S^1 : \exists s^{(k)} \leq s,\ \lim_{s^{(k)} \to s} x'(s^{(k)}) = u\} \tag{11}$$

are non empty. Moreover the following properties hold.
(i) if $x(\cdot)$ is differentiable at s, then $x'(s) \in U_x^+ \cap U_x^-$;
(ii) if $u \in U_x^+ \cup U_x^-$ then

$$|x(s_1) - x(s)|^2 - 2R\ \langle x(s_1) - x(s), u\rangle \geq 0 \quad \textit{for} \quad 0 \leq s_1 < s < L; \tag{12}$$

(iii) let $B^0 = B_R(x + Ru)$, $u \in S^1$ so that $B^0 \cap \gamma = \emptyset$, then

$$\exists u^+ \in U_x^+ : \langle u^+, u\rangle \leq 0, \qquad \exists u^- \in U_x^- : \langle u^-, u\rangle \geq 0; \tag{13}$$

(iv) if there exist $S^1 \ni u_k \to u$, $s^{(k)} \to s$, $s^{(k)} < s$, with $x(s^{(k)}) \in \partial B_R(x + Ru_k)$, then

$$\exists u^- \in U_x^- : \langle u^-, u\rangle \leq 0. \tag{14}$$

Proposition 3.3 *[1, Theorem 4.1]Let $x \in \gamma \in \Gamma_R$, γ contained in an open circle of radius R. Let*

$$W_x = \{u \in S^1 : (B_R(x + Ru))^c \supset \gamma_x\}. \tag{15}$$

Then

$$U_x^+ \cup U_x^- \subset W_x = \mathrm{Nor}_{\,co_R(\gamma_x)}(x) \cap S^1. \tag{16}$$

4. Tangent sets to rectifiable R-curves

The main theorem of this section is Theorem 4.5; Corollary 4.6 proves that the tangent vectors at x to γ_x and to $cl(\gamma \setminus \gamma_x)$ make an angle at least $\pi/2$ (if the curve γ is C^1 these tangent sets are opposite half lines).

In this section let us assume that γ is a plane R-curve of length $|\gamma| = L$ contained in an open disk of radius R. According to Proposition 2.1, for every $x \in \gamma$, γ_x has R-hull $co_R(\gamma_x)$.

For a vector $u = (a, b)$, let $u^\perp = (-b, a)$.

Definition 2 Let $u_1, u_2 \in S^1$, $u_1 \neq u_2$ and $u_i = (\cos\theta_i, \sin\theta_i), i = 1, 2$, $\theta_1 < \theta_2 < \theta_1 + 2\pi$. Let $\mathcal{A}(u_1, u_2)$ be the closed counterclockwise oriented cone centered in O, with sides $\{\lambda u_i, \lambda \geq 0\}, i = 1, 2$.

Definition 3 When x, y are points on a circumference ∂B of radius R, with $|x - y| < 2R$, let us denote with $arc_{\partial B}(x, y)$ the shorter arc on ∂B from x to y. When no ambiguity arises, let us denote $arc(x, y) = arc_{\partial B}(x, y)$.

Definition 4 Let x, z be given points in $\mathbb{R}^2$, $x \neq z$, $|x - z| < 2R$. Let

$$w_x^+(z) = \frac{x - z}{2R} - \sqrt{1 - |\frac{x - z}{2R}|^2}\,\frac{(x - z)^\perp}{|x - z|}, \tag{17}$$

$$w_x^-(z) = \frac{x - z}{2R} + \sqrt{1 - |\frac{x - z}{2R}|^2}\,\frac{(x - z)^\perp}{|x - z|}. \tag{18}$$

Their geometrical meaning follows. Let $B^+_{x,z}, B^-_{x,z}$ the two disks of radius R through z and x such that their $arc(z, x)$ is clockwise oriented, counterclockwise oriented respectively. Then $w_x^+(z)$, $w_x^-(z)$ are the unit interior normals at z to $\partial B^+_{x,z}, \partial B^-_{x,z}$ respectively.

Remark 1 Obviously $B_R(z + Rv)$ is a disk through z not containing x iff

$$v \in \mathcal{A}(w_x^-(z), w_x^+(z)). \tag{19}$$

Let us notice that the cone $\mathcal{A}(w_x^-(z), w_x^+(z))$ is not convex.

Definition 5 Let $D :\equiv D_R(y_0)$ a given closed disk and $z \notin D$. Let dist $(z, D) < 2R$. Let $\partial B^-_{D,z}$, $\partial B^+_{D,z}$ the circumferences, with radius R, through z tangent to D at z^-, z^+ respectively. Let z^- such that $arc(z^-, z)$ on $\partial B^-_{D,z}$ is clockwise oriented, let z^+ such that $arc_{\partial B^+}(z^+, z)$ on $\partial B^+_{D,z}$ is counterclockwise oriented. Let $v_D^-(z)$, $v_D^+(z)$ the unit interior normals at z to $\partial B^-_{D,z}$, $\partial B^+_{D,z}$ respectively.

Then $v_D^-(z)$, $v_D^+(z)$ are the two unit vectors solutions to the equation

$$|z + Rv - y_0|^2 = 4R^2.$$

That is

$$\langle v, z - y_0\rangle = \frac{3R^2 - |z - y_0|^2}{2R} = |z - y_0| \cos\alpha, \tag{20}$$

with $\alpha \in (0, \pi)$. Thus

$$v_D^+(z) = \frac{z - y_0}{|z - y_0|}\cos\alpha + \frac{(z - y_0)^\perp}{|z - y_0|}\sin\alpha, \tag{21}$$

$$v_D^-(z) = \frac{z - y_0}{|z - y_0|}\cos\alpha - \frac{(z - y_0)^\perp}{|z - y_0|}\sin\alpha, \tag{22}$$

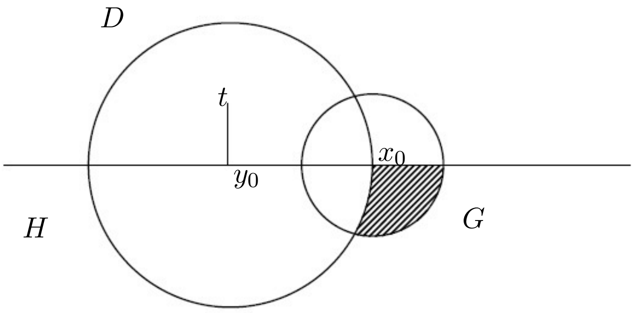

Figure 1. Constraints for z.

Remark 2 Let us notice that, if dist $(z, D) \le (\sqrt{3}-1)R$, the cone $\mathcal{A}(v_D^-(z), v_D^+(z))$ is convex; moreover if $x_0 \in \partial D$

$$\lim_{z\to x_0} v_D^+(z) = \lim_{z\to x_0} v_D^-(z) = \frac{x_0 - y_0}{|x_0 - y_0|}. \tag{23}$$

Proof. As $z \to x_0 \in \partial D$, then $|z - y_0| \to R$ and from (20) $\cos\alpha \to 1$. Thus from (21), (22) the thesis follows. □

LEMMA 4.1 *Under the same assumptions of the previous definitions, $B_R(z + Rv)$ is a disk through z not intersecting D iff*

$$v \in \mathcal{A}(v_D^-(z), v_D^+(z)) \cap S^1. \tag{24}$$

Moreover, if $x \in \partial D$ then, the following inclusion

$$\mathcal{A}(v_D^-(z), v_D^+(z)) \subset \mathcal{A}(w_x^-(z), w_x^+(z)) \tag{25}$$

holds.

Proof. The proof of (24) is obvious. Let us prove (25). If $v \in \mathcal{A}(v_D^-(z), v_D^+(z))$ then $B_R(z + Rv)$ does not intersect D, therefore $x \notin B_R(z + Rv)$. This implies that $v \in \mathcal{A}(w_x^-(z), w_x^+(z))$.□

LEMMA 4.2 *Let $D = D_R(y_0)$ be a given closed disk. Let H be the closed half plane with $y_0 \in \partial H$ and outer normal $t \in S^1$. Let $x_0 = y_0 - Rt^\perp \in \partial D$ and $G = H \cap D^c \cap B(x_0, R/2)$, see Fig.1. If*

$$z \in G, v \in S^1 \quad \textit{and} \quad B(z + Rv) \cap D \cap H = \emptyset, \tag{26}$$

then

$$v \in \mathcal{A}(v_D^-(z), w_{x_0}^+(z)) \tag{27}$$

and

$$\limsup_{z\to x_0,\, z\in G} \mathcal{A}(v_D^-(z), w_{x_0}^+(z)) \cap S^1 = \mathcal{A}(-t^\perp, t) \cap S^1. \tag{28}$$

For $0 < r \le R/2$, let

$$x_r = x_0 - rt^\perp, \quad G_r = G \cap (B_{D,x_r}^-)^c \cap B(x_0, r). \tag{29}$$

If $z \in G_r$, then the following inclusions between convex cones

$$\mathcal{A}(v_D^-(z), w_{x_0}^+(z)) \subset \mathcal{A}(v_D^-(x_r), w_{x_0}^+(x_r)) \subset \mathcal{A}(v_D^-(x_{R/2}), w_{x_0}^+(x_{R/2})) \tag{30}$$

and

$$\lim_{r\to 0^+} v_D^-(x_r) = -t^\perp, \quad \lim_{r\to 0^+} w_{x_0}^+(x_r) = t. \tag{31}$$

hold.

Proof. The constraint (26) implies that $x_0 \notin B(z+Rv)$. By Remark 1

$$v \in \mathcal{A}(w_{x_0}^-(z), w_{x_0}^+(z)).$$

As $z \in G$, $\partial B(z+Rv)$ is tangent to $D \cap H$ when $v = v_D^-(z)$. So the bound $w_{x_0}^-(z)$ has to be changed with $v_D^-(z)$ and (27) follows.

Let us choose for $z \in G$ a polar coordinate system at x_0, with axis $-t^\perp$. That is

$$z = x_0 + \rho(\cos\theta, \sin\theta),\ 0 < \rho \le R/2,\ -\arccos(-\frac{\rho}{2R}) \le \theta \le 0.$$

For $\rho \in (0, r]$ fixed, by using (17) of Definition 4, it is not difficult to see that the largest amplitude of the angle between $w_{x_0}^+(z)$ and $-t^\perp$ is reached when $\theta = 0$, to say at $z = x_\rho :\equiv x_0 - \rho t^\perp$. Moreover for $\rho \in (0, r]$ the largest amplitude of the angle between $w_{x_0}^+(x_\rho)$ and $-t^\perp$ is reached when $\rho = r$, that is at x_r. Then

$$z \in G \Longrightarrow \mathcal{A}(v_D^-(z), w_{x_0}^+(z)) \subset \mathcal{A}(v_D^-(z), w_{x_0}^+(x_r)). \tag{32}$$

For all $z \in G_r$, let us consider the arc of the circumference $\partial B_{D,z}^-$, tangent to ∂D at z^-, which intersects ∂H in a point $x_{\rho(z)}$ between x_0 and x_r. All points $z \in arc(z^-, x_{\rho(z)})$ on $\partial B_{D,z}^-$ have the same $B_{D,z}^-$, and the angle between $v_D^-(z)$ and $-t^\perp$ is maximum at $x_{\rho(z)}$ by construction. Moreover the amplitude of $\mathcal{A}(v_D^-(x_\rho), -t^\perp)$ is increasing for $\rho \in (0, r]$, $0 < r \le R/2$. From this property and (32), the inclusions (30) are proved. The cone $\mathcal{A}(v_D^-(x_{R/2}), w_{x_0}^+(x_{R/2}))$ is an half plane, as by (22) and by (17)

$$v_D^-(x_{R/2}) = -\frac{1}{4}t^\perp - \frac{\sqrt{15}}{4}t, \quad w_{x_0}^+(x_{R/2}) = \frac{1}{4}t^\perp + \frac{\sqrt{15}}{4}t.$$

From (23) with $x_0 = y_0 - Rt^\perp$, x_r in place of z, the first limit in (31) follows; the second limit follows by (17) with x_0 in place of x. The proof of (28) follows from (31) and (30). ☐

LEMMA 4.3 *Let $\gamma \in \Gamma_R$ with arc length parametrization $[0, L] \ni s \to x(s)$. Let $0 < s_0 < L$, $x_0 = x(s_0)$. Let $\mathcal{A}(u_1, u_2)$ convex. If $x'(s) \in \mathcal{A}(u_1, u_2)$, for $s_0 < s < L$ a.e., then*

$$x(s) - x_0 \in \mathcal{A}(u_1, u_2), \quad s_0 < s < L. \tag{33}$$

Proof. Let w the direction of the bisector vector to $\mathcal{A}(u_1, u_2)$. Then

$$\mathcal{A}(u_1, u_2) \setminus \{0\} = \{u : \langle \frac{u}{|u|}, w\rangle \ge \cos\alpha\}, \tag{34}$$

with $\cos\alpha \ge 0$. Therefore if $x'(s) \in \mathcal{A}(u_1, u_2)$ for $s_0 < s < L$, then

$$\langle \frac{x(s) - x_0}{|x(s) - x_0|}, w\rangle = \frac{1}{|x(s) - x_0|}\int_{s_0}^{s} \langle x'(\sigma), w\rangle d\sigma \ge \frac{(s - s_0)\cos\alpha}{|x(s) - x_0|} \ge \cos\alpha.$$

Thus

$$\frac{x(s) - x_0}{|x(s) - x_0|} \in \mathcal{A}(u_1, u_2)$$

and (33) is proved. ☐

LEMMA 4.4 *Let $\gamma \in \Gamma_R$ with arc length parametrization $[0, L] \ni s \to x(s)$. Let $0 < s_0 < L$, $x_0 = x(s_0)$. Let $U_{x_0}^+$ the set defined by (10). Let us assume that there exist $w \in S^1$, $\alpha \in (0, \pi/2]$ and a sequence*

$s_n \to s_0, s_0 < s_n$ *satisfying*

$$\langle \frac{x(s_n) - x_0}{|x(s_n) - x_0|}, w \rangle < \cos \alpha. \tag{35}$$

Then there exists $u \in U^+_{x_0}$ *satisfying*

$$\langle u, w \rangle \leq \cos \alpha. \tag{36}$$

Proof. Let

$$u_i = w \cos \alpha + (-1)^i w^\perp \sin \alpha, \quad (i = 1, 2).$$

It follows that $w = \frac{u_1+u_2}{|u_1+u_2|}$ is the bisector vector to the cone

$$\mathcal{A}(u_1, u_2) = \{u : \langle \frac{u}{|u|}, w \rangle \geq \cos \alpha\}.$$

Then (35) implies that

$$x(s_n) - x_0 \notin \mathcal{A}(u_1, u_2).$$

Thus (33) of Lemma 4.3 does not hold; then $\exists\, \tau_n \to s_0^+$ with $\langle x'(\tau_n), w \rangle < \cos \alpha$. By possibly passing to a subsequence, we get $x'(\tau_n) \to u \in U^+_{x_0}$, with u satisfying (36). □

Let us recall that $co_R(\gamma_{x_0})$ is by (6)

$$co_R(\gamma_{x_0}) = \cap\{(B_R(z))^c : B_R(z) \cap \gamma_{x_0} = \emptyset\}.$$

THEOREM 4.5 *Let* $\gamma \in \Gamma_R$, γ *contained in an open disk of radius R. Let* $0 < s_0 < L$, $x_0 = x(s_0)$. *Then*

$$\mathrm{Tan}_{\,cl(\gamma \setminus \gamma_{x_0})}(x_0) \subset \mathrm{Nor}_{\,co_R(\gamma_{x_0})}(x_0). \tag{37}$$

Proof. Let $t \in S^1$ be the bisector vector to $\mathrm{Nor}_{\,co_R(\gamma_{x_0})}(x_0)$ namely

$$\mathrm{Nor}_{\,co_R(\gamma_{x_0})}(x_0) = \{\lambda v : \lambda \geq 0, v \in S^1, \langle v, t \rangle \geq l \geq 0\}. \tag{38}$$

Let us split the proof in two cases:
I) $l > 0$, i.e. $\mathrm{Nor}_{\,co_R(\gamma_{x_0})}(x_0)$ is strictly convex;
II) $l = 0$, i.e. $\mathrm{Nor}_{\,co_R(\gamma_{x_0})}(x_0)$ is an half plane.
Case I).
Assume by contradiction that there exists

$$\theta \in S^1 \cap \mathrm{Tan}_{\,cl(\gamma \setminus \gamma_{x_0})}(x_0), \quad \theta \notin N_{co_R(\gamma_{x_0})}(x_0). \tag{39}$$

Then there exists a sequence $s_n \to s_0^+$, satisfying

$$\langle t, \theta \rangle = \lim_{n \to \infty} \langle \frac{x(s_n) - x_0}{|x(s_n) - x_0|}, t \rangle < l.$$

As $l > 0$ there exists a positive $\delta < l$, for which, for n sufficiently large, the inequalities

$$\langle \frac{x(s_n) - x_0}{|x(s_n) - x_0|}, t \rangle < l - \delta$$

hold. Let us apply Lemma 4.4 with $\cos \alpha = l - \delta > 0$. Therefore, there exists $u \in U^+_{x_0}$, satisfying $\langle u, t \rangle \leq l - \delta$. This is impossible as $U^+_{x_0} \subset \mathrm{Nor}_{\,co_R(\gamma_{x_0})}(x_0)$, see (16).

Case II).
In this case $x_0 - \mathrm{Nor}_{co_R(\gamma_{x_0})}(x_0)$ is a closed half plane H with outer normal t at $x_0 \in \partial H$ and

$$\mathrm{Tan}_{co_R(\gamma_{x_0})}(x_0) \cap S^1 = \mathrm{Tan}_{\gamma_{x_0}}(x_0) \cap S^1 = \{-t\}.$$

Let $\overline{x} \in \gamma_{x_0}, 0 < \overline{r} = |\overline{x} - x_0| \leq R/2$ such that $\gamma_{\overline{x},x_0} \setminus \{\overline{x}\} \subset B_{\overline{r}}(x_0)$. Then the open set $Int(H) \cap B_{\overline{r}}(x_0) \setminus \gamma_{x_0}$ is disconnected in two open sets Q^+, Q^- where $x_0 - \overline{r}t^\perp \in \partial Q^+$.
Let us consider now the cases: II_a), II_b); II_c).

II_a): there exists $0 < r < \overline{r}$ such that

$$cl(\gamma \setminus \gamma_{x_0}) \cap B_r(x_0) \subset cl(H^c).$$

Then

$$\mathrm{Tan}_{cl(\gamma \setminus \gamma_{x_0})}(x_0) \subset x_0 - H = \mathrm{Nor}_{co_R(\gamma_{x_0})}(x_0)$$

and the thesis holds in this case.

II_b): there exists $0 < r_1 < \overline{r}$ such that

$$cl(\gamma \setminus \gamma_{x_0}) \cap D_{r_1}(x_0) \cap H = \gamma_{x_0,x_1}, \quad x_1 = x(s_1), s_1 > s_0.$$

Let

$$M^+ := \{x : x = x_0 - \lambda t^\perp, \lambda > 0\},$$

$$M^- := \{x : x = x_0 + \lambda t^\perp, \lambda > 0\}.$$

As $\gamma_{\overline{x},x_0}$ does not intersect γ_{x_0,x_1} (except at x_0) then either $\gamma_{x_0,x_1} \subset Q^+ \cup M^+$ or $\gamma_{x_0,x_1} \subset Q^- \cup M^-$. Up to a reflection we can assume that $\gamma_{x_0,x_1} \subset Q^+ \cup M^+$. Therefore

$$\gamma_{x_0,x_1} \subset (B_R(x_0 + Rt^\perp))^c, \quad \gamma_{x_0,x_1} \subset H \cap D_{r_1}(x_0).$$

Let $D = D_R(x_0 + Rt^\perp)$.
Claim 1: if $x'(s)$ exists for $s_0 < s < s_1$, then $B_R(x(s) + Rx'(s))$ does not meet the half disk $D \cap H$.
By contradiction, let us assume that there exists $y \in B_R(x(s) + Rx'(s)) \cap D \cap H$, then $y \in Q^-$. As $x(s) \in Q^+ \cup M^+$ then, on the segment $yx(s)$, there should exist a point of $\gamma_{\overline{x},x_0} \cap B_R(x(s) + Rx'(s))$. This is impossible as γ is an R-curve. Let $x_{r_1} = x_0 - r_1 t^\perp$ and $s(r_1) < s_1$ such that $\gamma_{x_0,x(s(r_1))} \subset (B^-_{D,x_{r_1}})^c \cap B(x_0, r_1)$. By previous claim, the assumptions of Lemma 4.2 are satisfied with

$$y_0 = x_0 + Rt^\perp, z = x(s), v = x'(s), r = r_1.$$

Then $x'(s)$ satisfies the constraint (27) and by (30):

$$x'(s) \in \mathcal{A}(v^-_D(x_{r_1}), w^+_{x_0}(x_{r_1})) \quad \text{a.e.} \quad s_0 < s < s(r_1).$$

From Lemma 4.3, as $\mathcal{A}(v^-_D(x_{r_1}), w^+_{x_0}(x_{r_1}))$ is convex, it follows that

$$\frac{x(s) - x_0}{|x(s) - x_0|} \in \mathcal{A}(v^-_D(x_{r_1}), w^+_{x_0}(x_{r_1})), \quad s_0 < s < s_1. \tag{40}$$

Therefore

$$\mathrm{Tan}_{cl(\gamma \setminus \gamma_{x_0})}(x_0) \subset \mathcal{A}(v^-_D(x_{r_1}), w^+_{x_0}(x_{r_1})).$$

Moreover if case II_b occurs for some r_1 then, for all $0 < r < r_1$, there exists $x_r = x(s_r)$ such that

$$cl(\gamma \setminus \gamma_{x_0}) \cap D_r(x_0) \cap H = \gamma_{x_0,x_r}$$

and (40) holds with r in place of r_1, s_r in place of s_1 , for all $0 < r < r_1$. Thus

$$\mathrm{Tan}_{\,cl(\gamma\setminus\gamma_{x_0})}(x_0) \subset \mathcal{A}(v_D^-(x_r), w_{x_0}^+(x_r)), \forall\, 0 < r < r_1.$$

By (31) it follows that

$$\mathrm{Tan}_{\,cl(\gamma\setminus\gamma_{x_0})}(x_0) \subset \mathcal{A}(-t^\perp, t) \subset \mathrm{Nor}_{\,co_R(\gamma_{x_0})}(x_0)$$

and the thesis holds in this case too.

II_c): let us assume now that the cases II_a and II_b do not occur.

Let $x(s) \in (\gamma\setminus\gamma_{x_0}) \cap B_r(x_0) \cap Int(H)$ with $r < \bar{r}$. Either $x(s) \in Q^+$ or $x(s) \in Q^-$. Let $x(s) \in Q^+$. Let $x(s'_+), x(s''_+), s_0 < s'_+ < s < s''_+$, the end points of the maximal connected component in Q^+ of $\gamma\setminus\gamma_{x_0}$ containing $x(s)$, with $x(s'_+) \in M^+$. With no restriction we can assume $\gamma_{x(s'_+),x(s''_+)} \subset (B_D^-(x_r))^c$. The same arguments of case II_b prove that

$$x'(s) \in \mathcal{A}(v_D^-(x_r), w_{x_0}^+(x_r)), \quad s'_+ < s < s''_+. \tag{41}$$

Let $\gamma^+ = x_0x(s'_+) \cup \gamma_{x(s'_+),x(s''_+)}$ the curve obtained by joining the segment $x_0x(s'_+)$ with $\gamma_{x(s'_+),x(s''_+)}$. Obviously the unit tangent vector to the points of γ^+ a.e. satisfies the constraint (41) since $-t^\perp \in \mathcal{A}(v_D^-(x_r), w_{x_0}^+(x_r))$.

Then, by Lemma 4.3, it follows that

$$\frac{x(s)-x_0}{|x(s)-x_0|} \in \mathcal{A}(v_D^-(x_r), w_{x_0}^+(x_r)), \quad s'_+ < s < s''_+. \tag{42}$$

Let us argue now by contradiction: if (37) does not hold then there exists $q \in S^1$ so that

$$q \in \mathrm{Tan}_{\,cl(\gamma\setminus\gamma_{x_0})}(x_0), \quad \langle q, t\rangle < 0$$

and

$$q = \lim_{n\to\infty} \frac{x(s_n)-x_0}{|x(s_n)-x_0|},$$

with $x(s_n) \in Q^+ \cup Q^-$. By passing to a subsequence, we assume (up to a reflection) that $x(s_n) \in Q^+$. Since by (31)

$$\lim_{r\to 0+} \mathcal{A}(v_D^-(x_r), w_{x_0}^+(x_r)) = \mathcal{A}(-t^\perp, t),$$

then passing to the limit in (42), with $x(s_n)$ in place of $x(s)$, the vector $q \in \mathcal{A}(-t^\perp, t)$. This is in contradiction with $\langle q, t\rangle < 0$. □

Corollary 4.6 *Let $\gamma \in \Gamma_R$ and let $x \in \gamma$ be not the end point of γ. Let*

$$w \in \mathrm{Tan}_{\,\gamma_x}(x), \quad v \in \mathrm{Tan}_{\,cl(\gamma\setminus\gamma_x)}(x).$$

Then

$$\langle w, v\rangle \le 0. \tag{43}$$

Proof. As $\gamma_x \subset co_R(\gamma_x)$ then

$$\mathrm{Nor}_{\,co_R(\gamma_x)}(x) \subset \mathrm{Nor}_{\,\gamma_x}(x).$$

By (37) the inequality (43) follows. □

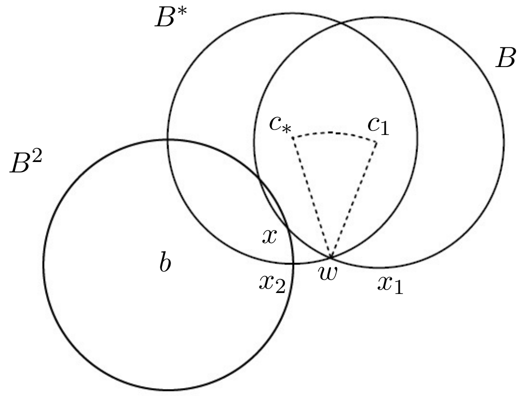

Figure 2. Curved angles

5. Bounds for the length of rectifiable R-curves

The aim of this section is to extend to rectifiable R-curves γ, not necessarily C^1, the results on the length and the detour obtained in [1].

Let us recall the following geometric definition and a comparison result:

Definition 6 Let $b, c \in \mathbb{R}^2$, $0 < |b-c| < 2R$. Let $B_R(b)$ and $B_R(c)$ two open disks, of radius R and center b, c respectively. Let $x \in \partial B_R(b) \cap \partial B_R(c)$. Let l be the line through b and c, let H be the half plane with boundary l containing x.

The unbounded region $a\check{n}g(bxc) :\equiv B_R(b)^c \cap B_R(c)^c \cap H$ will be called *curved angle*. Moreover

$$\text{meas}(a\check{n}g(bxc)) := \text{meas}(\text{Tan}_{\,a\check{n}g(bxc)}(x) \cap S^1)$$

is the measure of the angle between the half tangent lines at x to the boundary of $a\check{n}g(bxc)$.

Proposition 5.1 *[1, Lemma 4.2]Let $x, x_2 \in \mathbb{R}^2$, $|x - x_2| < R$. Let $B^2 = B_R(b)$ with $\partial B^2 \supset \{x, x_2\}$. Let $B^* = B_R(c_*)$ the disk of radius R, with ∂B^* orthogonal at x_2 to ∂B^2 and $x \in B^*$, see Fig.2. Let us assume that there exists $x_1 \in (B^* \cup B^2)^c$ with the properties:*
(i) $|x_1 - x| < R, |x_2 - x_1| < R$;
(ii) x_1 lies in the half plane with boundary the line through x and x_2 not containing b;
(iii) there exists $B^1 = B_R(c_1)$ with $\{x_1, x\} \subset \partial B^1$, with $arc(x, x_1) \subset (B^2)^c$, such that the line through x and x_1 separates c_1 and x_2.
Then the measure of the curved angle $a\check{n}g(bxc_1)$ is less than $\pi/2$.

Theorem 5.2 (R-angle estimate) *Let $\gamma \in \Gamma_R$. Assume that for every $x \in \gamma$, γ_x is contained in an open disk $B_R(x)$. Then, the measure of* Nor$_{\,co_R(\gamma_x)}(x) \cap S^1$ *is greater or equal to $\pi/2$.*

Proof. Let x be a fixed point of γ. Let $u_1, u_2 \in S^1$ such that, see (16)

$$W_x = \text{Nor}_{\,co_R(\gamma_x)}(x) \cap S^1 = \mathcal{A}(u_1, u_2) \cap S^1.$$

$\mathcal{A}(u_1, u_2)$ is a convex cone counterclockwise oriented by definition. Let $\Delta_i = (B_R(x + Ru_i))^c, i = 1, 2$. If $u_1 = -u_2$ then Nor$_{\,co_R(\gamma_x)}(x)$ is an half plane and the thesis holds. Let $u_1 \neq -u_2$. Let $c_i = x+Ru_i, i = 1, 2$. If $u_1 \neq u_2$ let us consider the curved angle $a\check{n}g(c_1xc_2)$. Let us consider $\mathcal{Z} :\equiv a\check{n}g(c_1xc_2) \cap D(x, R)$; its boundary is splitted in three circular arcs $arc_{\partial\Delta_1}(x, z_1), arc_{\partial\Delta_2}(x, z_2), arc_{\partial D(x,R)}(z_1, z_2)$. In case $u_1 = u_2$ then $\Delta_1 = \Delta_2 = \Delta$, let $\mathcal{Z} = \Delta \cap D(x, R)$. Again $\partial\mathcal{Z}$ is splitted in three arcs with z_1, x, z_2 on the same circumference of radius R (z_1, z_2 on the opposite sides with respect to x). By construction $co_R(\gamma_x) \subset \mathcal{Z}$.

There are three possible cases:
(a_1) at least one of the two sets $\gamma_x \cap arc_{\partial\Delta_i}(x, z_i) \setminus \{x\}$, i=1,2, is empty;
(a_2) for $i = 1$ or $i = 2$ the point x is an accumulation point of $\gamma_x \cap \partial\Delta_i$;
(b) there exist two points $x_i \in \gamma_x \cap arc_{\partial\Delta_i}(x, z_i), x_i \neq x$ (i=1,2), such that

$$\gamma_{x_i,x} \cap \partial\Delta_i = \{x_i, x\}, i = 1, 2.$$

Case (a_1): with no loss of generality one can assume that

$$\gamma_x \cap arc_{\partial\Delta_1}(x, z_1) \setminus \{x\} = \emptyset.$$

Let $u_1 = (\cos\alpha_1, \sin\alpha_1)$, $0 \le \alpha_1 < 2\pi$ and let, for $\delta > 0$,

$$u^\delta := (\cos(\alpha_1 - \delta), \sin(\alpha_1 - \delta)).$$

Since the vector u_1 bounds W_x, by (15), for δ sufficiently small, one has

$$\gamma_x \cap B_R(x + Ru^\delta) \neq \emptyset.$$

This means that, for $\delta = \frac{1}{k} > 0$ and k sufficiently large, there exists a sequence $s^{(k)} \to s^-$, such that $x(s^{(k)}) \to x$, $u^{\frac{1}{k}} \to u_1$ and

$$x(s^{(k)}) \in \partial B_R(x + Ru^{\frac{1}{k}}) \setminus D_R(x + Ru_1).$$

Then by (iv) of Proposition 3.2, there exists $u^- \in U_x^-$ so that $\langle u^-, u_1\rangle \le 0$. As $U_x^- \subset W_x$ by (16), the thesis follows.
Case (a_2): with no loss of generality one can assume that $i = 1$; then exists a sequence $s^{(k)} \to s^-$, such that $x(s^{(k)}) \in \partial\Delta_1$. Then by (iv) of Proposition 3.2, with $u^{\frac{1}{k}} = u_1$, there exists $u^- \in U_x^-$ so that $\langle u^-, u_1\rangle \le 0$; as in Case (a_1) the thesis follows.
Case (b): Let $x_i = x(s_i), i = 1, 2$ with $s_1 < s_2 < s$. Let $\tilde{u_2}$ so that $x + u_2R = x_2 + \tilde{u_2}R$. Let

$$B^2 := B_R(x + u_2R) = B_R(x_2 + \tilde{u_2}R), \quad B^1 := B_R(x + u_1R).$$

Let us notice that, by construction, $\gamma_x \subset co_R(\gamma_x) \subset \mathcal{Z} \subset (B^2)^c$. Let Q be the closed region of $\mathcal{Z}$ bounded by $arc_{\partial\Delta_2}(x_2, x), arc_{\partial\Delta_1}(x, x_1), \gamma_{x_1,x_2}$; up to a reflection (with respect to the line $\{x+\lambda(u_1+u_2), \lambda \in \mathbb{R}\}$) we can assume that the points x_1, x_2, x are in the clockwise order of ∂Q.

Let $\mathcal{A}^- = \mathcal{A}(\tilde{u_2}^\perp, -\tilde{u_2}), \mathcal{A}^+ = \mathcal{A}(-\tilde{u_2}, -\tilde{u_2}^\perp)$. By the definition of Q

$$\operatorname{Tan}_Q(x_2) \subset \mathcal{A}^- \cup \mathcal{A}^+ \quad \text{and} \quad -\tilde{u_2}^\perp \in \operatorname{Tan}_Q(x_2).$$

By the non intersection property, $\gamma_{x_2,x}$ is contained in $Q \setminus \gamma_{x_1,x_2}$, then the inclusions

$$\operatorname{Tan}_{\gamma_{x_2,x}}(x_2) \subset \operatorname{Tan}_Q(x_2) \subset \mathcal{A}^- \cup \mathcal{A}^+ \tag{44}$$

hold.
Claim: $\operatorname{Tan}_{\gamma_{x_2,x}}(x_2)$ cannot have a vector $v \in \mathcal{A}^-$.

By contradiction, let us assume that $v \in \operatorname{Tan}_{\gamma_{x_2,x}}(x_2) \cap \mathcal{A}^- \cap S^1$; then, by Theorem 4.5, $v \in Nor_{co_R(\gamma_{x_1,x_2})}(x_2)$ and $\gamma_{x_1,x_2} \subset (B_R(x_2 + Rv))^c$. Furthermore by construction $\gamma_{x_1,x_2} \subset \mathcal{Z}$.

There are two possibilities: $v \neq -\tilde{u_2}$ or $v = -\tilde{u_2}$.

Let $v \neq -\tilde{u_2}$. Let $\mathcal{T}$ be the closed connected component of $(B^2)^c \cap (B_R(x_2 + Rv))^c \cap D_R(x_2)$ containing $arc_{\partial\Delta_2}(x_2, x)$. As

$$arc_{\partial\Delta_2}(x_2, x) \cup \gamma_{x_1,x_2} \subset \mathcal{T} \quad \text{and} \quad \gamma_{x_1,x_2} \cap arc_{\partial\Delta_1}(x, x_1) = \{x_1\},$$

then $arc_{\partial\Delta_1}(x, x_1) \subset \mathcal{T}$. Therefore $\partial Q \subset \mathcal{T}$, so $Q \subset \mathcal{T}$. Then

$$\operatorname{Tan}_Q(x_2) \subset \operatorname{Tan}_{\mathcal{T}}(x_2) = \mathcal{A}(v^\perp, -\tilde{u_2}^\perp) \subset \mathcal{A}^+. \tag{45}$$

By (44) it follows that $v \in \mathcal{A}^+ \setminus \{-\tilde{u_2}\}$, contradiction.

If $v = -\tilde{u_2}$, then $(B^2)^c \cap (B_R(x_2 + Rv))^c \cap D_R(x_2)$ consists of two curvilinear triangles symmetric with respect to x_2. Let $\mathcal{T}$ that one containing $arc_{\partial\Delta_2}(x_2, x)$. When $u_1 = u_2$ then $x \in arc_{\partial\Delta_2}(x_2, x_1)$, then $x_1 \in \mathcal{T}$. When $u_1 \neq u_2$ then $x_1 \in \mathcal{T}$ otherwise $\mathcal{Z}$ cannot be defined. In both cases $x_1 \in \mathcal{T}$. Again $Q \subset \mathcal{T}$ and

$$\operatorname{Tan}_Q(x_2) \subset \operatorname{Tan}_{\mathcal{T}}(x_2) = \{-\lambda\tilde{u_2}^\perp \ \lambda \ge 0\}.$$

It follows that $v \in \mathcal{A}^+ \setminus \{-\tilde{u_2}\}$, contradiction. The claim is proved.

Then

$$\mathrm{Tan}_{\gamma_{x_2,x}}(x_2) \subset \mathcal{A}(v^-, -\tilde{u_2}^\perp)$$

with $v^- \in \mathrm{Tan}_{\gamma_{x_2,x}}(x_2)$, $v^- \in \mathcal{A}^+, v^- \neq -\tilde{u_2}$.

By Theorem 4.5, $v^- \in Nor_{co_R(\gamma_{x_1,x_2})}(x_2)$ and by construction $\tilde{u_2} \in \mathrm{Nor}_{co_R(\gamma_{x_1,x_2})}(x_2) \cap S^1$ too. As $-\tilde{u_2}^\perp \in arc_{S^1}(v^-, \tilde{u_2})$ then, by convexity,

$$-\tilde{u_2}^\perp \in Nor_{co_R(\gamma_{x_1,x_2})}(x_2). \tag{46}$$

Therefore

$$x_1 \in \gamma_{x_1,x_2} \subset (B_R(x_2 - R\tilde{u_2}^\perp))^c.$$

In particular, this fact implies that $u_1 \neq u_2$. As $x \in B_R(x_2 - R\tilde{u_2}^\perp)$, the assumptions of Proposition 5.1 are satisfied with

$$B^* = B_R(x_2 - R\tilde{u_2}^\perp),\ b = x_2 + R\tilde{u_2},\ c_* = x_2 - R\tilde{u_2}^\perp,\ c_1 = x + Ru_1.$$

Then

$$\mathrm{meas}(a\breve{n}g(bxc_1)) = \pi - \mathrm{meas}(\mathrm{Nor}_{co_R(\gamma_x)}(x) \cap S^1) \leq \pi/2.$$

Then the measure of $\mathrm{Nor}_{co_R(\gamma_x)}(x) \cap S^1$ is greater or equal to $\pi/2$ and the thesis follows.□

The following corollary is a consequence of Theorems 4.5 and 5.2.

Corollary 5.3 *For every $x \in \gamma \in \Gamma_R$, x not end point of γ, γ contained in an open disk of radius R, the following*

$$(-\mathrm{Tan}_{co_R(\gamma_x)}(x) \cup \mathrm{Tan}_{cl(\gamma\setminus\gamma_x)}(x)) \cap S^1 \subset W_x \tag{47}$$

holds.

Proof. From Theorem 5.2, $\mathrm{Tan}_{co_R(\gamma_x)}(x) \cap S^1$ has measure less than $\pi/2$. Then $\forall v, w, \in \mathrm{Tan}_{co_R(\gamma_x)}(x)$

$$\langle v, w\rangle \geq 0.$$

Thus $-v \in \mathrm{Nor}_{co_R(\gamma_x)}(x)$, this implies, by Proposition 3.3, that $-\mathrm{Tan}_{co_R(\gamma_x)}(x) \subset \mathrm{Nor}_{co_R(\gamma_x)}(x) = W_x$. The inclusion $\mathrm{Tan}_{cl(\gamma\setminus\gamma_x)}(x)) \cap S^1 \subset W_x$ follows by Theorem 4.5.□

Next theorems are related to the tangent cone of the classic convex hull $co(\gamma_x)$.

Theorem 5.4 *Let $\gamma \in \Gamma_R$. Assume that for every $x \in \gamma$, γ_x is contained in $D(x, R/N)$, $N > 1$. Then for every $x \in \gamma$*

$$\mathrm{meas}(\mathrm{Tan}_{co(\gamma_x)}(x)) \leq \pi/2 + 2\arcsin\frac{1}{2N}$$

holds.

Proof. With the same arguments of the proof of [1, Theorem 4.1], the thesis holds by replacing [1, Lemma 4.5] with Theorem 5.2.□

Let $|\gamma|$ be the length of γ, and $p(s)$ the perimeter of $co(\gamma(x(s)))$.

Theorem 5.5 *Let $N > 1$. Let z_0 be a fixed point in the plane. If $\gamma \in \Gamma_R$ and $\gamma \subset D(z_0, R/(2N))$ then*

$$|\gamma| \leq \frac{\pi}{N-1}R, \quad p'(s) \geq 1 - \frac{1}{N} \quad a.e. \quad s \in [0, |\gamma|].$$

Proof. With the same arguments of the proof of [1, Theorem 4.2], the thesis holds by using Theorem 5.4.□

Let us recall that the detour of $\gamma_{x_1,x}$ is $\frac{|\gamma_{x_1,x}|}{|x-x_1|}$.

Theorem 5.6 *Let $w_0 \in \mathbb{R}^2$, $\tau > 0$. Let $\gamma \in \Gamma_R$, $\gamma \subset D(w_0, \tau)$. Then there exists a positive constant $c(R,\tau)$, depending only on R and τ such that*

$$|\gamma| \leq c(R,\tau), \tag{48}$$

where
(i) $c(R,\tau) \leq 4\pi\tau \leq \pi R$ if $\tau \leq \frac{R}{4}$;
(ii) $c(R,\tau) \leq (1 + (16\sqrt{2}e^{\pi/2})^2(\frac{\tau}{R})^2)\pi R$ if $\tau > \frac{R}{4}$.

Moreover the detour of γ is bounded:

$$\frac{|\gamma_{x_1,x}|}{|x - x_1|} < 3\frac{c(R,\tau)}{R} \quad \forall x_1 \prec x \in \gamma. \tag{49}$$

Proof. By Theorem 5.5, with the same arguments of the proof of [1, Theorem 5.1,Theorem 5.2], the inequalities (48), (49) are obtained. □

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