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## Plane R-curves II

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# Plane $R$-curves II 

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Let $\Gamma_{R}$ be the class of plane, oriented, rectifiable curves $\gamma$, such that, for almost every $x \in \gamma$, the part of $\gamma$ preceding $x$ is outside the open disk of radius $R$, centered in $x+R \mathbf{t}_{x}$, where $\mathbf{t}_{x}$ is the unit tangent vector at $x$. In [1] the present authors have obtained bounds for the length and the detour for $C^{1}$ regular curves in $\Gamma_{R}$. These bounds are proved here for all curves in $\Gamma_{R}$.

## 1. Introduction

Let $R>0$. Let $\Gamma_{R}$ be the class of the plane oriented local rectifiable curves $\gamma$ satisfying the following property: for every $x \in \gamma$, let $\gamma_{x}$ be the part of $\gamma$ between the starting point and $x$ and almost everywhere let $\mathbf{t}$ be the tangent vector to $\gamma$ at $x$, then $\gamma_{x}$ is not contained in the open circle centered at $x+R \mathrm{t}$ and of radius $R$. These curves have been studied in 1] and have been called $R$-curves.
$\Gamma_{R}$ is a generalization of the class $\Gamma$ introduced in [2, 3] and studied in $\mathbb{R}^{n}$ [5, 9]: $\gamma \in \Gamma$ if for every $x \in \gamma$ the arc $\gamma_{x}$ is contained in the half plane bounded by the line through $x$ ortogonal to $\mathbf{t}$. The class $\Gamma$ has also been recently studied in many other spaces: Riemannian manifolds [6], finite-dimension normed spaces [8].

The steepest descent curves of quasi convex functions are curves of $\Gamma$ [2, 4, 9] the interest in the $R$-curves is that they are the steepest descent lines of functions whose level sets have reach greater than $R$, see [1;

In all previous papers an important goal is to get the apriori global rectifiability (boundeness of the length) of $\gamma \in \Gamma$. Here this result is obtained for the bounded planar curves $\gamma \in \Gamma_{R}$, Theorem 5.6

In the previous definition of the curves of $\Gamma$ it is assumed that $\gamma$ is local rectifiable; it has been proved that the defining property is equivalent to the so called self-expanding property [10] (or self-contracted property [4] when opposite order is used for $\gamma$ ) for a continuous curve $t \rightarrow \gamma(t)$, with $t$ not necessarily the parameter length:

$$
\begin{equation*}
\left|\gamma(t)-\gamma\left(t_{1}\right)\right| \geq\left|\gamma\left(t_{2}\right)-\gamma\left(t_{1}\right)\right| \quad \text { for } \quad 0 \leq t_{1} \leq t_{2} \leq t \tag{1}
\end{equation*}
$$

Another definition of the class $\Gamma$ which comes out immediately from the geometric meaning involves $\operatorname{co}\left(\gamma_{x}\right)$, the convex hull of $\gamma_{x}$. Then, $\gamma \in \Gamma$ if for almost every $x \in \gamma$ the tangent vector $\mathbf{t}$ lies in the normal cone to $\operatorname{co}\left(\gamma_{x}\right)$ at $x$, that is:

$$
\begin{equation*}
\langle x-y, \mathbf{t}\rangle \geq 0, \quad \forall y \in \gamma_{x} . \tag{2}
\end{equation*}
$$

Metric and geometric relaxation of both previous two definitions have been introduced in [11] and it was proved that bounded planar curves $\gamma$ satisfying them have bounded length; counterexamples for not planar curves are also showned.

[^0]On the other hand, the curves $\gamma \in \Gamma_{R}$, with length parameter $s$, satisfy the following two properties [1]:

$$
\begin{align*}
\left|\gamma(s)-\gamma\left(s_{1}\right)\right| & \geq\left|\gamma\left(s_{2}\right)-\gamma\left(s_{1}\right)\right| e^{\left(s_{2}-s\right) /(2 R)} \quad \text { for } \quad 0 \leq s_{1} \leq s_{2} \leq s  \tag{3}\\
\langle x-y, \mathbf{t}\rangle & \geq-\frac{|x-y|^{2}}{2 R}, \quad \forall y \in \gamma_{x} \tag{4}
\end{align*}
$$

These properties are a generalization of 1 and of 2 respectively. In 1 a priori bounds for the length and the detour of a bounded $\gamma \in \Gamma_{R}$ have been proved under the assumption that $\gamma$ is a $C^{1}$ curve.

In this work the same bounds are obtained if one assumes that $\gamma$ is merely rectifiable so the defining property holds a.e., only; first natural or easily obtained properties for $C^{1}$
$R$-curves are extended to arbitrary rectifiable plane $R$-curves: Theorem 4.5 and Corollary
4.6 Then these properties are used to prove the $R$-angle estimate property of $\gamma$ : Theorem
5.2 The main bounds for the length and the detour of $\gamma$ : Theorem 5.5 and Theorem 5.6 then easily are obtained.

The plan of this work follows.
Some of the basic properties for $\gamma \in \Gamma_{R}$ are recalled in 43
In $\$ 4$ properties of tangent sets to $\gamma \in \Gamma_{R}$ are stated and proved. Theorem 4.5 proves that any unit vector of the tangent set at $x$ to $\operatorname{cl}\left(\gamma \backslash \gamma_{x}\right)$ is the inner normal at $x$ of a disk excluding $\gamma_{x}$; in Corollary 4.6 it is proved that, at each point $x \in \gamma$, the tangent sets at $x$ to $\gamma_{x}$ and to $\operatorname{cl}\left(\gamma \backslash \gamma_{x}\right)$ do not contain directions forming acute angles. These geometrical properties are obvious for a $C^{1}$ curve but they have to be proved when $\gamma$ is rectifiable $R$-curve, only.

In $\$ 5$ rectifiable $R$-curves contained in a disk of arbitrary fixed radius are studied. If $\gamma_{x}$ is contained in an open disk of radius $R$, the $R$-hull of $\gamma_{x}$ (defined in $\sqrt{6}$ ) is considered. In Theorem 5.2 it is proved that the amplitude of the normal cone at $x$ to the $R$-hull of $\gamma_{x}$, is greater or equal to $\pi / 2$. This is an extension to the planar $R$-curves of the so called angle estimate of [2], which plays a fundamental role in order to get the bound for the length of $\gamma$ in different situations even in CAT(0)-spaces [12. As a consequence, in the same way as in [1], if $\gamma$ is contained in a smaller disk, a bound of its length is obtained (Theorem 5.5). Moreover if $\gamma$ is contained in a disk of arbitrary radius $\tau$, bounds for the detour of $\gamma$ and its length, depending on $R$ and $\tau$, are proved (Theorem 5.6) as in [1.

In our opinion, the geometrical properties contained in Theorem 4.5 and Theorem 5.2 could be interesting independently of their application in this work.

## 2. Definitions and preliminaries

Let $K \subset \mathbb{R}^{2}, \operatorname{Int}(K)$ will be the interior of $K, \partial K$ the boundary of $K, \operatorname{cl}(K)$ the closure of $K, K^{c}=\mathbb{R}^{2} \backslash K$. For every set $S \subset \mathbb{R}^{2}, c o(S)$ is the convex hull of $S$. Let
$B(z, \rho)=\left\{x \in \mathbb{R}^{2}:|x-z|<\rho\right\}, S^{1}=\partial B(0,1)$ and let $D(z, \rho)=\operatorname{cl}(B(z, \rho))$. The notations $B_{\rho}(x), D_{\rho}(x)$ will also be used for open, closed disks of radius $\rho$ centered at $x$. The usual scalar product between vectors $u, v \in \mathbb{R}^{2}$ will be denoted by $\langle u, v\rangle$.

Let $K$ be a non empty closed set. Let $q \in K$; the tangent cone of $K$ at $q$ is defined in 13 as:

$$
\operatorname{Tan}_{K}(q)=\left\{v \in \mathbb{R}^{2}: \forall \varepsilon>0 \exists x \in K \cap B_{\varepsilon}(q) \exists r>0 \text { s.t. }|r(x-q)-v|<\varepsilon\right\} .
$$

Let us recall that if $\operatorname{Tan}_{K}(q) \neq\{0\}$ then

$$
S^{1} \cap \operatorname{Tan}_{K}(q)=\bigcap_{\varepsilon>0} c l\left(\left\{\frac{x-q}{|x-q|}, q \neq x \in K \cap B(q, \varepsilon)\right\}\right) .
$$

The normal cone at $q$ to $K$ is the non empty closed convex cone, given by:

$$
\begin{equation*}
\operatorname{Nor}_{K}(q)=\left\{u \in \mathbb{R}^{2}:\langle u, v\rangle \leq 0 \quad \forall v \in \operatorname{Tan}_{K}(q)\right\} \tag{5}
\end{equation*}
$$

The dual cone of a set $K$ is $K^{\star}=\left\{y \in \mathbb{R}^{2}:\langle y, x\rangle \geq 0 \quad \forall x \in K\right\}$. Thus $\operatorname{Nor}_{K}(q)=-\left\{\operatorname{Tan}_{K}(q)\right\}^{\star}$.

In the following definitions $A$ will be a closed set. If $a \in A$, then $\operatorname{reach}(A, a)$ is the supremum of all numbers $\rho$ such that for every $x \in B(a, \rho)$ there exists a unique point $b \in A$ satisfying $|b-x|=\operatorname{dist}(x, A)$, see [13]. Also:

$$
\operatorname{reach}(A):=\inf \{\operatorname{reach}(A, a): a \in A\}
$$

Let us define $\operatorname{co}_{R}(A)$, the $R$-hull of $A$, as the closed set containing $A$, such that
(i) $\operatorname{co}_{R}(A)$ has reach greater or equal to $R$;
(ii) if a set $B \supseteq A$ and $\operatorname{reach}(B) \geq R$, then $B \supseteq \operatorname{co}_{R}(A)$.

See [14 pp.105-107] for the properties of $R$-hull. It can be shown that

$$
\begin{equation*}
\operatorname{co}_{R}(A)=\cap\left\{\left(B_{R}(z)\right)^{c}: B_{R}(z) \cap A=\emptyset\right\} \tag{6}
\end{equation*}
$$

The $R$-hull of a closed set $A$ may not exist, see [14 Remark 4.9]. However
Proposition 2.1 14, Theorem 4.8] If $A$ is a plane closed connected subset of an open disk of radius $R$, then $A$ has $R$-hull.

## 3. Properties of $R$-curves

In this paper a curve in $\mathbb{R}^{2}$ is the image of a continuous function on an interval, valued into $\mathbb{R}^{2}$. Let $\gamma \subset \mathbb{R}^{2}$ be an oriented rectifiable curve and let $x(\cdot)$ be its parametric representation with respect to the arc length parameter $s \in[0, L]$. If $x_{1}=x\left(s_{1}\right), x_{2}=x\left(s_{2}\right) \in \gamma$ with $s_{1} \leq s_{2}$, the notation $x_{1} \preceq x_{2}$ will be used. Let us denote $x(s)=x$,

$$
\gamma_{x}=\{y \in \gamma: y \preceq x\} ; \gamma_{x_{1}, x_{2}}=\left\{y \in \gamma: x_{1} \preceq y \preceq x_{2}\right\} .
$$

Definition 1 Let $R$ be a fixed positive number. An $R$-curve $\gamma \subset \mathbb{R}^{2}$ is a rectifiable oriented curve with arc length parameter $s \in[0, L]$, tangent vector $\mathbf{t}(s)=x^{\prime}(s)$ such that the inequality

$$
\begin{equation*}
\left|x\left(s_{1}\right)-x(s)-R \mathbf{t}(s)\right| \geq R \tag{7}
\end{equation*}
$$

holds for almost all $s$ and for $0 \leq s_{1} \leq s \leq L . \Gamma_{R}$ will denote the class of $R$-curves in $\mathbb{R}^{2}$.
The geometric meaning of $\sqrt{7}$ is that for every point $x=x(s) \in \gamma$, with tangent vector
$\mathbf{t}(s)$, the set $\gamma_{x}$ is outside of the open disk of radius $R$ through $x$ centered at $x+R \mathbf{t}(s)$. Let us notice the following equivalent formulations of 7 for $0 \leq s_{1}<s \leq L$ :

$$
\begin{array}{r}
\left.\left|x\left(s_{1}\right)-x(s)\right|^{2}-2 R\left\langle x\left(s_{1}\right)-x(s)\right), \mathbf{t}(s)\right\rangle \geq 0 \\
\left\langle x(s)-x\left(s_{1}\right), \mathbf{t}(s)\right\rangle \geq-\frac{\left|x\left(s_{1}\right)-x(s)\right|^{2}}{2 R} . \tag{9}
\end{array}
$$

Proposition 3.1 [1, Lemma 3.1, Corollary 3.2] An $R$-curve does not intersect itself.
Proposition 3.2 [1, Theorem 3.3] Let $\gamma \in \Gamma_{R}$. For every $s \in(0, L), x=x(s), \gamma_{x} \subsetneq \gamma$, the following two subsets of $S^{1}$ :

$$
\begin{align*}
& U_{x}^{+}=\left\{u \in S^{1}: \exists s^{(k)} \geq s, \lim _{s(k) \rightarrow s} x^{\prime}\left(s^{(k)}\right)=u\right\}  \tag{10}\\
& U_{x}^{-}=\left\{u \in S^{1}: \exists s^{(k)} \leq s, \lim _{s^{(k)} \rightarrow s} x^{\prime}\left(s^{(k)}\right)=u\right\} \tag{11}
\end{align*}
$$

are non empty. Moreover the following properties hold.
(i) if $x(\cdot)$ is differentiable at $s$, then $x^{\prime}(s) \in U_{x}^{+} \cap U_{x}^{-}$;
(ii) if $u \in U_{x}^{+} \cup U_{x}^{-}$then

$$
\begin{equation*}
\left|x\left(s_{1}\right)-x(s)\right|^{2}-2 R\left\langle x\left(s_{1}\right)-x(s), u\right\rangle \geq 0 \quad \text { for } \quad 0 \leq s_{1}<s<L \tag{12}
\end{equation*}
$$

(iii) let $B^{0}=B_{R}(x+R u), u \in S^{1}$ so that $B^{0} \cap \gamma=\emptyset$, then

$$
\begin{equation*}
\exists u^{+} \in U_{x}^{+}:\left\langle u^{+}, u\right\rangle \leq 0, \quad \exists u^{-} \in U_{x}^{-}:\left\langle u^{-}, u\right\rangle \geq 0 \tag{13}
\end{equation*}
$$

(iv) if there exist $S^{1} \ni u_{k} \rightarrow u, s^{(k)} \rightarrow s$, $s^{(k)}<s$, with $x\left(s^{(k)}\right) \in \partial B_{R}\left(x+R u_{k}\right)$, then

$$
\begin{equation*}
\exists u^{-} \in U_{x}^{-}:\left\langle u^{-}, u\right\rangle \leq 0 . \tag{14}
\end{equation*}
$$

Proposition 3.3 [1, Theorem 4.1]Let $x \in \gamma \in \Gamma_{R}, \gamma$ contained in an open circle of radius $R$. Let

$$
\begin{equation*}
W_{x}=\left\{u \in S^{1}:\left(B_{R}(x+R u)\right)^{c} \supset \gamma_{x}\right\} . \tag{15}
\end{equation*}
$$

$$
\begin{equation*}
U_{x}^{+} \cup U_{x}^{-} \subset W_{x}=\operatorname{Nor}_{c o}^{R}\left(\gamma_{x}\right)(x) \cap S^{1} \tag{16}
\end{equation*}
$$

## 4. Tangent sets to rectifiable $R$-curves

The main theorem of this section is Theorem 4.5 Corollary 4.6 proves that the tangent vectors at $x$ to $\gamma_{x}$ and to $\operatorname{cl}\left(\gamma \backslash \gamma_{x}\right)$ make an angle at least $\pi / 2$ (if the curve $\gamma$ is $C^{1}$ these tangent sets are opposite half lines).

In this section let us assume that $\gamma$ is a plane $R$-curve of length $|\gamma|=L$ contained in an open disk of radius $R$. According to Proposition 2.1. for every $x \in \gamma, \gamma_{x}$ has $R$-hull $\cos _{R}\left(\gamma_{x}\right)$. For a vector $u=(a, b)$, let $u^{\perp}=(-b, a)$.

Definition 2 Let $u_{1}, u_{2} \in S^{1}, u_{1} \neq u_{2}$ and $u_{i}=\left(\cos \theta_{i}, \sin \theta_{i}\right), i=1,2, \theta_{1}<\theta_{2}<\theta_{1}+2 \pi$. Let $\mathcal{A}\left(u_{1}, u_{2}\right)$ be the closed counterclockwise oriented cone centered in $O$, with sides $\left\{\lambda u_{i}, \lambda \geq 0\right\}, i=1,2$.

Definition 3 When $x, y$ are points on a circumference $\partial B$ of radius $R$, with $|x-y|<2 R$, let us denote with $\operatorname{arc}_{\partial B}(x, y)$ the shorter arc on $\partial B$ from $x$ to $y$. When no ambiguity arises, let us denote $\operatorname{arc}(x, y)=$ $\operatorname{arc}_{\partial B}(x, y)$.
Definition 4 Let $x, z$ be given points in $\mathbb{R}^{2}, x \neq z,|x-z|<2 R$. Let

$$
\begin{align*}
& w_{x}^{+}(z)=\frac{x-z}{2 R}-\sqrt{1-\left|\frac{x-z}{2 R}\right|^{2}} \frac{(x-z)^{\perp}}{|x-z|}  \tag{17}\\
& w_{x}^{-}(z)=\frac{x-z}{2 R}+\sqrt{1-\left|\frac{x-z}{2 R}\right|^{2}} \frac{(x-z)^{\perp}}{|x-z|} \tag{18}
\end{align*}
$$

Their geometrical meaning follows. Let $B_{x, z}^{+}, B_{x, z}^{-}$the two disks of radius $R$ through $z$ and $x$ such that their $\operatorname{arc}(z, x)$ is clockwise oriented, counterclockwise oriented respectively. Then $w_{x}^{+}(z), w_{x}^{-}(z)$ are the unit interior normals at $z$ to $\partial B_{x, z}^{+}, \partial B_{x, z}^{-}$respectively.

Remark 1 Obviously $B_{R}(z+R v)$ is a disk through $z$ not containing $x$ iff

$$
\begin{equation*}
v \in \mathcal{A}\left(w_{x}^{-}(z), w_{x}^{+}(z)\right) \tag{19}
\end{equation*}
$$

Let us notice that the cone $\mathcal{A}\left(w_{x}^{-}(z), w_{x}^{+}(z)\right)$ is not convex.
Definition 5 Let $D: \equiv D_{R}\left(y_{0}\right)$ a given closed disk and $z \notin D$. Let dist $(z, D)<2 R$. Let $\partial B_{D, z}^{-}, \partial B_{D, z}^{+}$the circumferences, with radius $R$, through $z$ tangent to $D$ at $z^{-}, z^{+}$respectively. Let $z^{-}$such that $\operatorname{arc}\left(z^{-}, z\right)$ on $\partial B_{D, z}^{-}$is clockwise oriented, let $z^{+}$such that $\operatorname{arc}_{\partial B^{+}}\left(z^{+}, z\right)$ on $\partial B_{D, z}^{+}$is counterclockwise oriented. Let $v_{D}^{-}(z), v_{D}^{+}(z)$ the unit interior normals at $z$ to $\partial B_{D, z}^{-}, \partial B_{D, z}^{+}$respectively.

Then $v_{D}^{-}(z), v_{D}^{+}(z)$ are the two unit vectors solutions to the equation

$$
\left|z+R v-y_{0}\right|^{2}=4 R^{2}
$$

That is

$$
\begin{equation*}
\left\langle v, z-y_{0}\right\rangle=\frac{3 R^{2}-\left|z-y_{0}\right|^{2}}{2 R}=\left|z-y_{0}\right| \cos \alpha \tag{20}
\end{equation*}
$$

with $\alpha \in(0, \pi)$. Thus

$$
\begin{align*}
& v_{D}^{+}(z)=\frac{z-y_{0}}{\left|z-y_{0}\right|} \cos \alpha+\frac{\left(z-y_{0}\right)^{\perp}}{\left|z-y_{0}\right|} \sin \alpha  \tag{21}\\
& v_{D}^{-}(z)=\frac{z-y_{0}}{\left|z-y_{0}\right|} \cos \alpha-\frac{\left(z-y_{0}\right)^{\perp}}{\left|z-y_{0}\right|} \sin \alpha \tag{22}
\end{align*}
$$



Figure 1. Constraints for $z$.

Remark 2 Let us notice that, if dist $(z, D) \leq(\sqrt{3}-1) R$, the cone $\mathcal{A}\left(v_{D}^{-}(z), v_{D}^{+}(z)\right)$ is convex; moreover if $x_{0} \in \partial D$

$$
\begin{equation*}
\lim _{z \rightarrow x_{0}} v_{D}^{+}(z)=\lim _{z \rightarrow x_{0}} v_{D}^{-}(z)=\frac{x_{0}-y_{0}}{\left|x_{0}-y_{0}\right|} \tag{23}
\end{equation*}
$$

Proof. As $z \rightarrow x_{0} \in \partial D$, then $\left|z-y_{0}\right| \rightarrow R$ and from $\cos \alpha \rightarrow 1$. Thus from 21), 222 the thesis follows. $\square$

Lemma 4.1 Under the same assumptions of the previous definitions, $B_{R}(z+R v)$ is a disk through $z$ not intersecting $D$ iff

$$
\begin{equation*}
v \in \mathcal{A}\left(v_{D}^{-}(z), v_{D}^{+}(z)\right) \cap S^{1} \tag{24}
\end{equation*}
$$

Moreover, if $x \in \partial D$ then, the following inclusion

$$
\begin{equation*}
\mathcal{A}\left(v_{D}^{-}(z), v_{D}^{+}(z)\right) \subset \mathcal{A}\left(w_{x}^{-}(z), w_{x}^{+}(z)\right) \tag{25}
\end{equation*}
$$

holds.
Proof. The proof of 24 is obvious. Let us prove 25. If $v \in \mathcal{A}\left(v_{D}^{-}(z), v_{D}^{+}(z)\right)$ then $B_{R}(z+R v)$ does not intersect $D$, therefore $x \notin B_{R}(z+R v)$. This implies that $v \in \mathcal{A}\left(w_{x}^{-}(z), w_{x}^{+}(z)\right) . \square$

Lemma 4.2 Let $D=D_{R}\left(y_{0}\right)$ be a given closed disk. Let $H$ be the closed half plane with $y_{0} \in \partial H$ and outer normal $t \in S^{1}$. Let $x_{0}=y_{0}-R t^{\perp} \in \partial D$ and $G=H \cap D^{c} \cap B\left(x_{0}, R / 2\right)$, see Fig.1. If

$$
\begin{equation*}
z \in G, v \in S^{1} \quad \text { and } \quad B(z+R v) \cap D \cap H=\emptyset \tag{26}
\end{equation*}
$$

then

$$
\begin{equation*}
v \in \mathcal{A}\left(v_{D}^{-}(z), w_{x_{0}}^{+}(z)\right) \tag{27}
\end{equation*}
$$

and

$$
\begin{equation*}
\limsup _{z \rightarrow x_{0}, z \in G} \mathcal{A}\left(v_{D}^{-}(z), w_{x_{0}}^{+}(z)\right) \cap S^{1}=\mathcal{A}\left(-t^{\perp}, t\right) \cap S^{1} \tag{28}
\end{equation*}
$$

For $0<r \leq R / 2$, let

$$
\begin{equation*}
x_{r}=x_{0}-r t^{\perp}, \quad G_{r}=G \cap\left(B_{D, x_{r}}^{-}\right)^{c} \cap B\left(x_{0}, r\right) . \tag{29}
\end{equation*}
$$

If $z \in G_{r}$, then the following inclusions between convex cones

$$
\begin{equation*}
\mathcal{A}\left(v_{D}^{-}(z), w_{x_{0}}^{+}(z)\right) \subset \mathcal{A}\left(v_{D}^{-}\left(x_{r}\right), w_{x_{0}}^{+}\left(x_{r}\right)\right) \subset \mathcal{A}\left(v_{D}^{-}\left(x_{R / 2}\right), w_{x_{0}}^{+}\left(x_{R / 2}\right)\right) \tag{30}
\end{equation*}
$$

$$
\begin{equation*}
\lim _{r \rightarrow 0^{+}} v_{D}^{-}\left(x_{r}\right)=-t^{\perp}, \quad \lim _{r \rightarrow 0^{+}} w_{x_{0}}^{+}\left(x_{r}\right)=t \tag{31}
\end{equation*}
$$

hold
Proof. The constraint 26 implies that $x_{0} \notin B(z+R v)$. By Remark 1

$$
v \in \mathcal{A}\left(w_{x_{0}}^{-}(z), w_{x_{0}}^{+}(z)\right)
$$

As $z \in G, \partial B(z+R v)$ is tangent to $D \cap H$ when $v=v_{D}^{-}(z)$. So the bound $w_{x_{0}}^{-}(z)$ has to be changed with $v_{D}^{-}(z)$ and 27) follows.

Let us choose for $z \in G$ a polar coordinate system at $x_{0}$, with axis $-t^{\perp}$. That is

$$
z=x_{0}+\rho(\cos \theta, \sin \theta), 0<\rho \leq R / 2,-\arccos \left(-\frac{\rho}{2 R}\right) \leq \theta \leq 0
$$

For $\rho \in(0, r]$ fixed, by using 17 of Definition 4 it is not difficult to see that the largest amplitude of the angle between $w_{x_{0}}^{+}(z)$ and $-t^{\perp}$ is reached when $\theta=0$, to say at $z=x_{\rho}: \equiv x_{0}-\rho t^{\perp}$. Moreover for $\rho \in(0, r]$ the largest amplitude of the angle between $w_{x_{0}}^{+}\left(x_{\rho}\right)$ and $-t^{\perp}$ is reached when $\rho=r$, that is at $x_{r}$. Then

$$
\begin{equation*}
z \in G \Longrightarrow \mathcal{A}\left(v_{D}^{-}(z), w_{x_{0}}^{+}(z)\right) \subset \mathcal{A}\left(v_{D}^{-}(z), w_{x_{0}}^{+}\left(x_{r}\right)\right) \tag{32}
\end{equation*}
$$

For all $z \in G_{r}$, let us consider the arc of the circumference $\partial B_{D, z}^{-}$, tangent to $\partial D$ at $z^{-}$, which intersects $\partial H$ in a point $x_{\rho(z)}$ between $x_{0}$ and $x_{r}$. All points $z \in \operatorname{arc}\left(z^{-}, x_{\rho(z)}\right)$ on $\partial B_{D, z}^{-}$have the same $B_{D, z}^{-}$, and the angle between $v_{D}^{-}(z)$ and $-t^{\perp}$ is maximum at $x_{\rho(z)}$ by construction. Moreover the amplitude of $\mathcal{A}\left(v_{D}^{-}\left(x_{\rho}\right),-t^{\perp}\right)$ is increasing for $\rho \in(0, r], 0<r \leq R / 2$. From this property and 32, the inclusions 30 are proved. The cone $\mathcal{A}\left(v_{D}^{-}\left(x_{R / 2}\right), w_{x_{0}}^{+}\left(x_{R / 2}\right)\right)$ is an half plane, as by 22 and by 17

$$
v_{D}^{-}\left(x_{R / 2}\right)=-\frac{1}{4} t^{\perp}-\frac{\sqrt{15}}{4} t, \quad w_{x_{0}}^{+}\left(x_{R / 2}\right)=\frac{1}{4} t^{\perp}+\frac{\sqrt{15}}{4} t .
$$

From (23) with $x_{0}=y_{0}-R t^{\perp}, x_{r}$ in place of $z$, the first limit in (31) follows; the second limit follows by 17) with $x_{0}$ in place of $x$. The proof of 28 follows from 31) and 30. $\square$

Lemma 4.3 Let $\gamma \in \Gamma_{R}$ with arc length parametrization $[0, L] \ni s \rightarrow x(s)$. Let $0<s_{0}<L$, $x_{0}=x\left(s_{0}\right)$. Let $\mathcal{A}\left(u_{1}, u_{2}\right)$ convex. If $x^{\prime}(s) \in \mathcal{A}\left(u_{1}, u_{2}\right)$, for $s_{0}<s<L$ a.e., then

$$
\begin{equation*}
x(s)-x_{0} \in \mathcal{A}\left(u_{1}, u_{2}\right), \quad s_{0}<s<L \tag{33}
\end{equation*}
$$

Proof. Let $w$ the direction of the bisector vector to $\mathcal{A}\left(u_{1}, u_{2}\right)$. Then

$$
\begin{equation*}
\mathcal{A}\left(u_{1}, u_{2}\right) \backslash\{0\}=\left\{u:\left\langle\frac{u}{|u|}, w\right\rangle \geq \cos \alpha\right\} \tag{34}
\end{equation*}
$$

with $\cos \alpha \geq 0$. Therefore if $x^{\prime}(s) \in \mathcal{A}\left(u_{1}, u_{2}\right)$ for $s_{0}<s<L$, then

$$
\left\langle\frac{x(s)-x_{0}}{\left|x(s)-x_{0}\right|}, w\right\rangle=\frac{1}{\left|x(s)-x_{0}\right|} \int_{s_{0}}^{s}\left\langle x^{\prime}(\sigma), w\right\rangle d \sigma \geq \frac{\left(s-s_{0}\right) \cos \alpha}{\left|x(s)-x_{0}\right|} \geq \cos \alpha
$$

Thus

$$
\frac{x(s)-x_{0}}{\left|x(s)-x_{0}\right|} \in \mathcal{A}\left(u_{1}, u_{2}\right)
$$

and $\sqrt{33}$ is proved. $\square$
Lemma 4.4 Let $\gamma \in \Gamma_{R}$ with arc length parametrization $[0, L] \ni s \rightarrow x(s)$. Let $0<s_{0}<L$, $x_{0}=x\left(s_{0}\right)$. Let $U_{x_{0}}^{+}$the set defined by 10 . Let us assume that there exist $w \in S^{1}, \alpha \in(0, \pi / 2]$ and a sequence
$s_{n} \rightarrow s_{0}, s_{0}<s_{n}$ satisfying

$$
\begin{equation*}
\left\langle\frac{x\left(s_{n}\right)-x_{0}}{\left|x\left(s_{n}\right)-x_{0}\right|}, w\right\rangle<\cos \alpha \tag{35}
\end{equation*}
$$

Then there exists $u \in U_{x_{0}}^{+}$satisfying

$$
\begin{equation*}
\langle u, w\rangle \leq \cos \alpha \tag{36}
\end{equation*}
$$

Proof. Let

$$
u_{i}=w \cos \alpha+(-1)^{i} w^{\perp} \sin \alpha, \quad(i=1,2) .
$$

It follows that $w=\frac{u_{1}+u_{2}}{\left|u_{1}+u_{2}\right|}$ is the bisector vector to the cone

$$
\mathcal{A}\left(u_{1}, u_{2}\right)=\left\{u:\left\langle\frac{u}{|u|}, w\right\rangle \geq \cos \alpha\right\}
$$

Then 35 implies that

$$
x\left(s_{n}\right)-x_{0} \notin \mathcal{A}\left(u_{1}, u_{2}\right) .
$$

Thus 33) of Lemma 4.3 does not hold; then $\exists \tau_{n} \rightarrow s_{0}^{+}$with $\left\langle x^{\prime}\left(\tau_{n}\right), w\right\rangle<\cos \alpha$. By possibly passing to a subsequence, we get $x^{\prime}\left(\tau_{n}\right) \rightarrow u \in U_{x_{0}}^{+}$, with $u$ satisfying 36. प

Let us recall that $\operatorname{co}_{R}\left(\gamma_{x_{0}}\right)$ is by

$$
\operatorname{co}_{R}\left(\gamma_{x_{0}}\right)=\cap\left\{\left(B_{R}(z)\right)^{c}: B_{R}(z) \cap \gamma_{x_{0}}=\emptyset\right\}
$$

Theorem 4.5 Let $\gamma \in \Gamma_{R}, \gamma$ contained in an open disk of radius $R$. Let $0<s_{0}<L, x_{0}=x\left(s_{0}\right)$. Then

$$
\begin{equation*}
\operatorname{Tan}_{c l\left(\gamma \backslash \gamma_{x_{0}}\right)}\left(x_{0}\right) \subset \operatorname{Nor}_{c o}^{R}\left(\gamma_{x_{0}}\right)\left(x_{0}\right) . \tag{37}
\end{equation*}
$$

Proof. Let $t \in S^{1}$ be the bisector vector to $\operatorname{Nor}_{c o s_{R}\left(\gamma_{x_{0}}\right)}\left(x_{0}\right)$ namely

$$
\begin{equation*}
\operatorname{Nor}_{\operatorname{co}_{R}\left(\gamma_{x_{0}}\right)}\left(x_{0}\right)=\left\{\lambda v: \lambda \geq 0, v \in S^{1},\langle v, t\rangle \geq l \geq 0\right\} . \tag{38}
\end{equation*}
$$

Let us split the proof in two cases:
I) $l>0$, i.e. $\operatorname{Nor}_{\operatorname{co}_{R}\left(\gamma_{x_{0}}\right)}\left(x_{0}\right)$ is strictly convex;
II) $l=0$, i.e. Nor $\operatorname{co}_{R}\left(\gamma_{x_{0}}\right)\left(x_{0}\right)$ is an half plane.

Case I).
Assume by contradiction that there exists

$$
\begin{equation*}
\theta \in S^{1} \cap \operatorname{Tan}_{c l\left(\gamma \backslash \gamma_{x_{0}}\right)}\left(x_{0}\right), \quad \theta \notin N_{c o o_{R}\left(\gamma_{x_{0}}\right)}\left(x_{0}\right) \tag{39}
\end{equation*}
$$

Then there exists a sequence $s_{n} \rightarrow s_{0}^{+}$, satisfying

$$
\langle t, \theta\rangle=\lim _{n \rightarrow \infty}\left\langle\frac{x\left(s_{n}\right)-x_{0}}{\left|x\left(s_{n}\right)-x_{0}\right|}, t\right\rangle<l
$$

As $l>0$ there exists a positive $\delta<l$, for which, for $n$ sufficiently large, the inequalities

$$
\left\langle\frac{x\left(s_{n}\right)-x_{0}}{\left|x\left(s_{n}\right)-x_{0}\right|}, t\right\rangle<l-\delta
$$

hold. Let us apply Lemma 4.4 with $\cos \alpha=l-\delta>0$. Therefore, there exists $u \in U_{x_{0}}^{+}$, satisfying $\langle u, t\rangle \leq l-\delta$. This is impossible as $U_{x_{0}}^{+} \subset$ Nor $_{c o o_{R}\left(\gamma_{x_{0}}\right)}\left(x_{0}\right)$, see 16.

Case II).
In this case $x_{0}-\operatorname{Nor}_{\operatorname{co}_{R}\left(\gamma_{x_{0}}\right)}\left(x_{0}\right)$ is a closed half plane $H$ with outer normal $t$ at $x_{0} \in \partial H$ and

$$
\operatorname{Tan}_{\operatorname{co}_{R}\left(\gamma_{x_{0}}\right)}\left(x_{0}\right) \cap S^{1}=\operatorname{Tan}_{\gamma_{x_{0}}}\left(x_{0}\right) \cap S^{1}=\{-t\}
$$

Let $\bar{x} \in \gamma_{x_{0}}, 0<\bar{r}=\left|\bar{x}-x_{0}\right| \leq R / 2$ such that $\gamma_{\bar{x}, x_{0}} \backslash\{\bar{x}\} \subset B_{\bar{r}}\left(x_{0}\right)$. Then the open set $\operatorname{Int}(H) \cap B_{\bar{r}}\left(x_{0}\right) \backslash \gamma_{x_{0}}$ is disconnected in two open sets $Q^{+}, Q^{-}$where $x_{0}-\bar{r} t^{\perp} \in \partial Q^{+}$.
Let us consider now the cases: $\left.\left.I I_{a}\right), I I_{b}\right) ; I I_{c}$ ).
$\left.I I_{a}\right)$ : there exists $0<r<\bar{r}$ such that

$$
c l\left(\gamma \backslash \gamma_{x_{0}}\right) \cap B_{r}\left(x_{0}\right) \subset \operatorname{cl}\left(H^{c}\right)
$$

Then

$$
\operatorname{Tan}_{c l\left(\gamma \backslash \gamma_{x_{0}}\right)}\left(x_{0}\right) \subset x_{0}-H=\text { Nor }_{c o s_{R}\left(\gamma_{x_{0}}\right)}\left(x_{0}\right)
$$

and the thesis holds in this case.
$\left.I I_{b}\right)$ : there exists $0<r_{1}<\bar{r}$ such that

$$
c l\left(\gamma \backslash \gamma_{x_{0}}\right) \cap D_{r_{1}}\left(x_{0}\right) \cap H=\gamma_{x_{0}, x_{1}}, \quad x_{1}=x\left(s_{1}\right), s_{1}>s_{0}
$$

Let

$$
\begin{aligned}
& M^{+}:=\left\{x: x=x_{0}-\lambda t^{\perp}, \lambda>0\right\} \\
& M^{-}:=\left\{x: x=x_{0}+\lambda t^{\perp}, \lambda>0\right\} .
\end{aligned}
$$

As $\gamma_{\bar{x}, x_{0}}$ does not intersect $\gamma_{x_{0}, x_{1}}$ (except at $x_{0}$ ) then either $\gamma_{x_{0}, x_{1}} \subset Q^{+} \cup M^{+}$or $\gamma_{x_{0}, x_{1}} \subset Q^{-} \cup M^{-}$. Up to a reflection we can assume that $\gamma_{x_{0}, x_{1}} \subset Q^{+} \cup M^{+}$. Therefore

$$
\gamma_{x_{0}, x_{1}} \subset\left(B_{R}\left(x_{0}+R t^{\perp}\right)\right)^{c}, \quad \gamma_{x_{0}, x_{1}} \subset H \cap D_{r_{1}}\left(x_{0}\right) .
$$

Let $D=D_{R}\left(x_{0}+R t^{\perp}\right)$.
Claim 1: if $x^{\prime}(s)$ exists for $s_{0}<s<s_{1}$, then $B_{R}\left(x(s)+R x^{\prime}(s)\right)$ does not meet the half disk $D \cap H$.
By contradiction, let us assume that there exists $y \in B_{R}\left(x(s)+R x^{\prime}(s)\right) \cap D \cap H$, then $y \in Q^{-}$. As $x(s) \in Q^{+} \cup M^{+}$then, on the segment $y x(s)$, there should exist a point of $\gamma_{\bar{x}, x_{0}} \cap B_{R}\left(x(s)+R x^{\prime}(s)\right)$. This is impossible as $\gamma$ is an $R$-curve. Let $x_{r_{1}}=x_{0}-r_{1} t^{\perp}$ and $s\left(r_{1}\right)<s_{1}$ such that $\gamma_{x_{0}, x\left(s\left(r_{1}\right)\right)} \subset$ $\left(B_{D, x_{r_{1}}}^{-}\right)^{c} \cap B\left(x_{0}, r_{1}\right)$. By previous claim, the assumptions of Lemma 4.2 are satisfied with

$$
y_{0}=x_{0}+R t^{\perp}, z=x(s), v=x^{\prime}(s), r=r_{1} .
$$

Then $x^{\prime}(s)$ satisfies the constraint 27) and by 30):

$$
x^{\prime}(s) \in \mathcal{A}\left(v_{D}^{-}\left(x_{r_{1}}\right), w_{x_{0}}^{+}\left(x_{r_{1}}\right)\right) \quad \text { a.e. } \quad s_{0}<s<s\left(r_{1}\right) .
$$

From Lemma 4.3 as $\mathcal{A}\left(v_{D}^{-}\left(x_{r_{1}}\right), w_{x_{0}}^{+}\left(x_{r_{1}}\right)\right)$ is convex, it follows that

$$
\begin{equation*}
\frac{x(s)-x_{0}}{\left|x(s)-x_{0}\right|} \in \mathcal{A}\left(v_{D}^{-}\left(x_{r_{1}}\right), w_{x_{0}}^{+}\left(x_{r_{1}}\right)\right), \quad s_{0}<s<s_{1} \tag{40}
\end{equation*}
$$

Therefore

$$
\operatorname{Tan}_{c l\left(\gamma \backslash \gamma_{x_{0}}\right)}\left(x_{0}\right) \subset \mathcal{A}\left(v_{D}^{-}\left(x_{r_{1}}\right), w_{x_{0}}^{+}\left(x_{r_{1}}\right)\right) .
$$

Moreover if case $I I_{b}$ occurs for some $r_{1}$ then, for all $0<r<r_{1}$, there exists $x_{r}=x\left(s_{r}\right)$ such that

$$
c l\left(\gamma \backslash \gamma_{x_{0}}\right) \cap D_{r}\left(x_{0}\right) \cap H=\gamma_{x_{0}, x_{r}}
$$

and (40) holds with $r$ in place of $r_{1}, s_{r}$ in place of $s_{1}$, for all $0<r<r_{1}$. Thus

$$
\operatorname{Tan}_{c l\left(\gamma \backslash \gamma_{x_{0}}\right)}\left(x_{0}\right) \subset \mathcal{A}\left(v_{D}^{-}\left(x_{r}\right), w_{x_{0}}^{+}\left(x_{r}\right)\right), \forall 0<r<r_{1} .
$$

By (31) it follows that

$$
\operatorname{Tan}_{c l\left(\gamma \backslash \gamma_{x_{0}}\right)}\left(x_{0}\right) \subset \mathcal{A}\left(-t^{\perp}, t\right) \subset \operatorname{Nor}_{c^{\prime}\left(\gamma_{x_{0}}\right)}\left(x_{0}\right)
$$

and the thesis holds in this case too.
$I I_{c}$ ): let us assume now that the cases $I I_{a}$ and $I I_{b}$ do not occur.
Let $x(s) \in\left(\gamma \backslash \gamma_{x_{0}}\right) \cap B_{r}\left(x_{0}\right) \cap \operatorname{Int}(H)$ with $r<\bar{r}$. Either $x(s) \in Q^{+}$or $x(s) \in Q^{-}$. Let $x(s) \in Q^{+}$. Let $x\left(s_{+}^{\prime}\right), x\left(s_{+}^{\prime \prime}\right), s_{0}<s_{+}^{\prime}<s<s_{+}^{\prime \prime}$, the end points of the maximal connected component in $Q^{+}$of $\gamma \backslash \gamma_{x_{0}}$ containing $x(s)$, with $x\left(s_{+}^{\prime}\right) \in M^{+}$. With no restriction we can assume $\gamma_{x\left(s_{+}^{\prime}\right), x\left(s_{+}^{\prime \prime}\right)} \subset\left(B_{D}^{-}\left(x_{r}\right)\right)^{c}$. The same arguments of case $I I_{b}$ prove that

$$
\begin{equation*}
x^{\prime}(s) \in \mathcal{A}\left(v_{D}^{-}\left(x_{r}\right), w_{x_{0}}^{+}\left(x_{r}\right)\right), \quad s_{+}^{\prime}<s<s_{+}^{\prime \prime} \tag{41}
\end{equation*}
$$

Let $\gamma^{+}=x_{0} x\left(s_{+}^{\prime}\right) \cup \gamma_{x\left(s_{+}^{\prime}\right), x\left(s_{+}^{\prime \prime}\right)}$ the curve obtained by joining the segment $x_{0} x\left(s_{+}^{\prime}\right)$ with $\gamma_{x\left(s_{+}^{\prime}\right), x\left(s_{+}^{\prime \prime}\right)}$. Obviously the unit tangent vector to the points of $\gamma^{+}$a.e. satisfies the constraint 41 since $-t^{\perp} \in$ $\mathcal{A}\left(v_{D}^{-}\left(x_{r}\right), w_{x_{0}}^{+}\left(x_{r}\right)\right)$

Then, by Lemma 4.3 it follows that

$$
\begin{equation*}
\frac{x(s)-x_{0}}{\left|x(s)-x_{0}\right|} \in \mathcal{A}\left(v_{D}^{-}\left(x_{r}\right), w_{x_{0}}^{+}\left(x_{r}\right)\right), \quad s_{+}^{\prime}<s<s_{+}^{\prime \prime} . \tag{42}
\end{equation*}
$$

Let us argue now by contradiction: if does not hold then there exists $q \in S^{1}$ so that

$$
q \in \operatorname{Tan}_{c l\left(\gamma \backslash \gamma_{x_{0}}\right)}\left(x_{0}\right), \quad\langle q, t\rangle<0
$$

and

$$
q=\lim _{n \rightarrow \infty} \frac{x\left(s_{n}\right)-x_{0}}{\left|x\left(s_{n}\right)-x_{0}\right|},
$$

with $x\left(s_{n}\right) \in Q^{+} \cup Q^{-}$. By passing to a subsequence, we assume (up to a reflection) that $x\left(s_{n}\right) \in Q^{+}$. Since by 31)

$$
\lim _{r \rightarrow 0+} \mathcal{A}\left(v_{D}^{-}\left(x_{r}\right), w_{x_{0}}^{+}\left(x_{r}\right)\right)=\mathcal{A}\left(-t^{\perp}, t\right)
$$

then passing to the limit in $\sqrt[42]{ }$, with $x\left(s_{n}\right)$ in place of $x(s)$, the vector $q \in \mathcal{A}\left(-t^{\perp}, t\right)$. This is in contradiction with $\langle q, t\rangle<0$. $\square$

Corollary 4.6 Let $\gamma \in \Gamma_{R}$ and let $x \in \gamma$ be not the end point of $\gamma$. Let

$$
w \in \operatorname{Tan}_{\gamma_{x}}(x), \quad v \in \operatorname{Tan}_{c l\left(\gamma \backslash \gamma_{x}\right)}(x)
$$

Then

$$
\begin{equation*}
\langle w, v\rangle \leq 0 . \tag{43}
\end{equation*}
$$

Proof. As $\gamma_{x} \subset \operatorname{co}_{R}\left(\gamma_{x}\right)$ then

$$
\operatorname{Nor}_{c o s_{R}\left(\gamma_{x}\right)}(x) \subset \operatorname{Nor}_{\gamma_{x}}(x) .
$$

By 37 the inequality 43 follows. $\square$


Figure 2. Curved angles

## 5. Bounds for the length of rectifiable $R$-curves

The aim of this section is to extend to rectifiable $R$-curves $\gamma$, not necessarily $C^{1}$, the results on the length and the detour obtained in 1 .

Let us recall the following geometric definition and a comparison result:
Definition 6 Let $b, c \in \mathbb{R}^{2}, 0<|b-c|<2 R$. Let $B_{R}(b)$ and $B_{R}(c)$ two open disks, of radius $R$ and center $b, c$ respectively. Let $x \in \partial B_{R}(b) \cap \partial B_{R}(c)$. Let $l$ be the line through $b$ and $c$, let $H$ be the half plane with boundary $l$ containing $x$.

The unbounded region $a \check{n} g(b x c): \equiv B_{R}(b)^{c} \cap B_{R}(c)^{c} \cap H$ will be called curved angle. Moreover

$$
\operatorname{meas}(a \check{n} g(b x c)):=\operatorname{meas}\left(\operatorname{Tan}_{a \check{n} g(b x c)}(x) \cap S^{1}\right)
$$

is the measure of the angle between the half tangent lines at $x$ to the boundary of $a \check{n} g(b x c)$.
Proposition 5.1 [1, Lemma 4.2]Let $x, x_{2} \in \mathbb{R}^{2},\left|x-x_{2}\right|<R$. Let $B^{2}=B_{R}(b)$ with $\partial B^{2} \supset\left\{x, x_{2}\right\}$. Let $B^{*}=B_{R}\left(c_{*}\right)$ the disk of radius $R$, with $\partial B^{*}$ orthogonal at $x_{2}$ to $\partial B^{2}$ and $x \in B^{*}$, see Fig.2. Let us assume that there exists $x_{1} \in\left(B^{*} \cup B^{2}\right)^{c}$ with the properties:
(i) $\left|x_{1}-x\right|<R,\left|x_{2}-x_{1}\right|<R$;
(ii) $x_{1}$ lies in the half plane with boundary the line through $x$ and $x_{2}$ not containing $b$;
(iii) there exists $B^{1}=B_{R}\left(c_{1}\right)$ with $\left\{x_{1}, x\right\} \subset \partial B^{1}$, with arc $\left(x, x_{1}\right) \subset\left(B^{2}\right)^{c}$, such that the line through $x$ and $x_{1}$ separates $c_{1}$ and $x_{2}$.

Then the measure of the curved angle an̆ $g\left(b x c_{1}\right)$ is less than $\pi / 2$.

Theorem 5.2 ( $R$-angle estimate) Let $\gamma \in \Gamma_{R}$. Assume that for every $x \in \gamma, \gamma_{x}$ is contained in an open disk $B_{R}(x)$. Then, the measure of $\operatorname{Nor}_{\operatorname{co}_{R}\left(\gamma_{x}\right)}(x) \cap S^{1}$ is greater or equal to $\pi / 2$.

Proof. Let $x$ be a fixed point of $\gamma$. Let $u_{1}, u_{2} \in S^{1}$ such that, see 16

$$
W_{x}=\operatorname{Nor}_{c o s_{R}\left(\gamma_{x}\right)}(x) \cap S^{1}=\mathcal{A}\left(u_{1}, u_{2}\right) \cap S^{1}
$$

$\mathcal{A}\left(u_{1}, u_{2}\right)$ is a convex cone counterclockwise oriented by definition. Let $\Delta_{i}=\left(B_{R}\left(x+R u_{i}\right)\right)^{c}, i=1,2$. If $u_{1}=-u_{2}$ then $\operatorname{Nor}_{c o s_{R}\left(\gamma_{x}\right)}(x)$ is an half plane and the thesis holds. Let $u_{1} \neq-u_{2}$. Let $c_{i}=x+R u_{i}, i=1,2$. If $u_{1} \neq u_{2}$ let us consider the curved angle $a \check{n} g\left(c_{1} x c_{2}\right)$. Let us consider $\mathcal{Z}: \equiv a \check{n} g\left(c_{1} x c_{2}\right) \cap D(x, R)$; its boundary is splitted in three circular $\operatorname{arcs} \operatorname{arc} c_{\partial \Delta_{1}}\left(x, z_{1}\right), \operatorname{arc}_{\partial \Delta_{2}}\left(x, z_{2}\right), \operatorname{arc} c_{\partial D(x, R)}\left(z_{1}, z_{2}\right)$. In case $u_{1}=u_{2}$ then $\Delta_{1}=\Delta_{2}=\Delta$, let $\mathcal{Z}=\Delta \cap D(x, R)$. Again $\partial \mathcal{Z}$ is splitted in three arcs with $z_{1}, x, z_{2}$ on the same circumference of radius $R\left(z_{1}, z_{2}\right.$ on the opposite sides with respect to $\left.x\right)$. By construction $\operatorname{co}_{R}\left(\gamma_{x}\right) \subset \mathcal{Z}$.

There are three possible cases:
$\left(a_{1}\right)$ at least one of the two sets $\gamma_{x} \cap \operatorname{arc}_{\partial \Delta_{i}}\left(x, z_{i}\right) \backslash\{x\}$, $\mathrm{i}=1,2$, is empty;
$\left(a_{2}\right)$ for $i=1$ or $i=2$ the point $x$ is an accumulation point of $\gamma_{x} \cap \partial \Delta_{i}$;
(b) there exist two points $x_{i} \in \gamma_{x} \cap \operatorname{arc}_{\partial \Delta_{i}}\left(x, z_{i}\right), x_{i} \neq x(\mathrm{i}=1,2)$, such that

$$
\gamma_{x_{i}, x} \cap \partial \Delta_{i}=\left\{x_{i}, x\right\}, i=1,2
$$

Case $\left(a_{1}\right)$ : with no loss of generality one can assume that

$$
\gamma_{x} \cap \operatorname{arc}_{\partial \Delta_{1}}\left(x, z_{1}\right) \backslash\{x\}=\emptyset
$$

Let $u_{1}=\left(\cos \alpha_{1}, \sin \alpha_{1}\right), 0 \leq \alpha_{1}<2 \pi$ and let, for $\delta>0$,

$$
u^{\delta}:=\left(\cos \left(\alpha_{1}-\delta\right), \sin \left(\alpha_{1}-\delta\right)\right) .
$$

Since the vector $u_{1}$ bounds $W_{x}$, by , for $\delta$ sufficiently small, one has

$$
\gamma_{x} \cap B_{R}\left(x+R u^{\delta}\right) \neq \emptyset
$$

This means that, for $\delta=\frac{1}{k}>0$ and $k$ sufficiently large, there exists a sequence $s^{(k)} \rightarrow s^{-}$, such that $x\left(s^{(k)}\right) \rightarrow x, u^{\frac{1}{k}} \rightarrow u_{1}$ and

$$
x\left(s^{(k)}\right) \in \partial B_{R}\left(x+R u^{\frac{1}{k}}\right) \backslash D_{R}\left(x+R u_{1}\right) .
$$

Then by (iv) of Proposition 3.2 there exists $u^{-} \in U_{x}^{-}$so that $\left\langle u^{-}, u_{1}\right\rangle \leq 0$. As $U_{x}^{-} \subset W_{x}$ by 16, the thesis follows.
Case $\left(a_{2}\right)$ : with no loss of generality one can assume that $i=1$; then exists a sequence $s^{(k)} \rightarrow s^{-}$, such that $x\left(s^{(k)}\right) \in \partial \Delta_{1}$. Then by (iv) of Proposition 3.2 with $u^{\frac{1}{k}}=u_{1}$, there exists $u^{-} \in U_{x}^{-}$so that $\left\langle u^{-}, u_{1}\right\rangle \leq 0$; as in Case $\left(a_{1}\right)$ the thesis follows.
Case (b): Let $x_{i}=x\left(s_{i}\right), i=1,2$ with $s_{1}<s_{2}<s$. Let $\tilde{u_{2}}$ so that $x+u_{2} R=x_{2}+\tilde{u_{2}} R$. Let

$$
B^{2}:=B_{R}\left(x+u_{2} R\right)=B_{R}\left(x_{2}+\tilde{u_{2}} R\right), \quad B^{1}:=B_{R}\left(x+u_{1} R\right) .
$$

Let us notice that, by construction, $\gamma_{x} \subset \operatorname{co}_{R}\left(\gamma_{x}\right) \subset \mathcal{Z} \subset\left(B^{2}\right)^{c}$. Let $Q$ be the closed region of $\mathcal{Z}$ bounded by $\operatorname{arc}_{\partial \Delta_{2}}\left(x_{2}, x\right), \operatorname{arc}_{\partial \Delta_{1}}\left(x, x_{1}\right), \gamma_{x_{1}, x_{2}}$; up to a reflection (with respect to the line $\left\{x+\lambda\left(u_{1}+u_{2}\right), \lambda \in \mathbb{R}\right\}$ ) we can assume that the points $x_{1}, x_{2}, x$ are in the clockwise order of $\partial Q$.

Let $\mathcal{A}^{-}=\mathcal{A}\left(\tilde{u_{2}}{ }^{\perp},-\tilde{u_{2}}\right), \mathcal{A}^{+}=\mathcal{A}\left(-\tilde{u_{2}},-\tilde{u_{2}} \perp\right)$. By the definition of $Q$

$$
\operatorname{Tan}_{Q}\left(x_{2}\right) \subset \mathcal{A}^{-} \cup \mathcal{A}^{+} \quad \text { and } \quad-\tilde{u_{2}}{ }^{\perp} \in \operatorname{Tan}_{Q}\left(x_{2}\right)
$$

By the non intersection property, $\gamma_{x_{2}, x}$ is contained in $Q \backslash \gamma_{x_{1}, x_{2}}$, then the inclusions

$$
\begin{equation*}
\operatorname{Tan}_{\gamma_{x_{2}, x}}\left(x_{2}\right) \subset \operatorname{Tan}_{Q}\left(x_{2}\right) \subset \mathcal{A}^{-} \cup \mathcal{A}^{+} \tag{44}
\end{equation*}
$$

hold.
Claim: $\operatorname{Tan}_{\gamma_{x_{2}, x}}\left(x_{2}\right)$ cannot have a vector $v \in \mathcal{A}^{-}$.
By contradiction, let us assume that $v \in \operatorname{Tan}_{\gamma_{x_{2}, x}}\left(x_{2}\right) \cap \mathcal{A}^{-} \cap S^{1}$; then, by Theorem 4.5 $v \in$ $N o r_{c o s_{R}\left(\gamma_{x_{1}, x_{2}}\right)}\left(x_{2}\right)$ and $\gamma_{x_{1}, x_{2}} \subset\left(B_{R}\left(x_{2}+R v\right)\right)^{c}$. Furthermore by construction $\gamma_{x_{1}, x_{2}} \subset \mathcal{Z}$.

There are two possibilities: $v \neq-\tilde{u_{2}}$ or $v=-\tilde{u_{2}}$.
Let $v \neq-\tilde{u_{2}}$. Let $\mathcal{T}$ be the closed connected component of $\left(B^{2}\right)^{c} \cap\left(B_{R}\left(x_{2}+R v\right)\right)^{c} \cap D_{R}\left(x_{2}\right)$ containing $\operatorname{arc}_{\partial \Delta_{2}}\left(x_{2}, x\right)$. As

$$
\operatorname{arc}_{\partial \Delta_{2}}\left(x_{2}, x\right) \cup \gamma_{x_{1}, x_{2}} \subset \mathcal{T} \quad \text { and } \quad \gamma_{x_{1}, x_{2}} \cap \operatorname{arc}_{\partial \Delta_{1}}\left(x, x_{1}\right)=\left\{x_{1}\right\}
$$

then $\operatorname{arc}_{\partial \Delta_{1}}\left(x, x_{1}\right) \subset \mathcal{T}$. Therefore $\partial Q \subset \mathcal{T}$, so $Q \subset \mathcal{T}$. Then

$$
\begin{equation*}
\operatorname{Tan}_{Q}\left(x_{2}\right) \subset \operatorname{Tan} \mathcal{T}\left(x_{2}\right)=\mathcal{A}\left(v^{\perp},-\tilde{u_{2}}{ }^{\perp}\right) \subset \mathcal{A}^{+} \tag{45}
\end{equation*}
$$

By 44 it follows that $v \in \mathcal{A}^{+} \backslash\left\{-\tilde{u_{2}}\right\}$, contradiction.
If $v=-\tilde{u_{2}}$, then $\left(B^{2}\right)^{c} \cap\left(B_{R}\left(x_{2}+R v\right)\right)^{c} \cap D_{R}\left(x_{2}\right)$ consists of two curvilinear triangles symmetric with respect to $x_{2}$. Let $\mathcal{T}$ that one containing $\operatorname{arc}_{\partial \Delta_{2}}\left(x_{2}, x\right)$. When $u_{1}=u_{2}$ then $x \in \operatorname{arc} c_{\partial \Delta_{2}}\left(x_{2}, x_{1}\right)$, then $x_{1} \in \mathcal{T}$. When $u_{1} \neq u_{2}$ then $x_{1} \in \mathcal{T}$ otherwise $\mathcal{Z}$ cannot be defined. In both cases $x_{1} \in \mathcal{T}$. Again $Q \subset \mathcal{T}$ and

$$
\operatorname{Tan}_{Q}\left(x_{2}\right) \subset \operatorname{Tan}_{\mathcal{T}}\left(x_{2}\right)=\left\{-\lambda \tilde{u_{2}}{ }^{\perp} \lambda \geq 0\right\} .
$$

It follows that $v \in \mathcal{A}^{+} \backslash\left\{-\tilde{u_{2}}\right\}$, contradiction. The claim is proved.

Then

$$
\operatorname{Tan}_{\gamma_{x_{2}, x}}\left(x_{2}\right) \subset \mathcal{A}\left(v^{-},-\tilde{u_{2}} \perp\right)
$$

with $v^{-} \in \operatorname{Tan}_{\gamma_{x_{2}, x}}\left(x_{2}\right), v^{-} \in \mathcal{A}^{+}, v^{-} \neq-\tilde{u_{2}}$.
By Theorem 4.5 $v^{-} \in \operatorname{Nor} \operatorname{co}_{R}\left(\gamma_{\left.x_{1}, x_{2}\right)}\left(x_{2}\right)\right.$ and by construction $\tilde{u_{2}} \in \operatorname{Nor}_{c o o_{R}\left(\gamma_{x_{1}, x_{2}}\right)}\left(x_{2}\right) \cap S^{1}$ too. As $-\tilde{u_{2}}{ }^{\perp} \in \operatorname{arc}_{S^{1}}\left(v^{-}, \tilde{u_{2}}\right)$ then, by convexity,

$$
\begin{equation*}
-\tilde{u_{2}}{ }^{\perp} \in N o r_{c o_{R}\left(\gamma_{x_{1}, x_{2}}\right)}\left(x_{2}\right) \tag{46}
\end{equation*}
$$

Therefore

$$
x_{1} \in \gamma_{x_{1}, x_{2}} \subset\left(B_{R}\left(x_{2}-R{\tilde{u_{2}}}^{\perp}\right)\right)^{c} .
$$

In particular, this fact implies that $u_{1} \neq u_{2}$. As $x \in B_{R}\left(x_{2}-R \tilde{u_{2}}{ }^{\perp}\right)$, the assumptions of Proposition 5.1 are satisfied with

$$
B^{*}=B_{R}\left(x_{2}-R \tilde{u_{2}}{ }^{\perp}\right), b=x_{2}+R \tilde{u_{2}}, c_{*}=x_{2}-R \tilde{u_{2}}{ }^{\perp}, c_{1}=x+R u_{1}
$$

Then

$$
\operatorname{meas}\left(a \check{n} g\left(b x c_{1}\right)\right)=\pi-\operatorname{meas}\left(\operatorname{Nor}_{c o s_{R}\left(\gamma_{x}\right)}(x) \cap S^{1}\right) \leq \pi / 2
$$

Then the measure of $\operatorname{Nor}_{\operatorname{co}_{R}\left(\gamma_{x}\right)}(x) \cap S^{1}$ is greater or equal to $\pi / 2$ and the thesis follows. $\square$
The following corollary is a consequence of Theorems 4.5 and 5.2
Corollary 5.3 For every $x \in \gamma \in \Gamma_{R}, x$ not end point of $\gamma, \gamma$ contained in an open disk of radius $R$, the following

$$
\begin{equation*}
\left(-\operatorname{Tan}_{c o o_{R}\left(\gamma_{x}\right)}(x) \cup \operatorname{Tan}_{c l\left(\gamma \backslash \gamma_{x}\right)}(x)\right) \cap S^{1} \subset W_{x} \tag{47}
\end{equation*}
$$

holds.
Proof. From Theorem 5.2. Tan $\operatorname{co}_{R}\left(\gamma_{x}\right)(x) \cap S^{1}$ has measure less than $\pi / 2$. Then $\forall v, w, \in \operatorname{Tan}_{c o s_{R}\left(\gamma_{x}\right)}(x)$

$$
\langle v, w\rangle \geq 0 .
$$

Thus $-v \in \operatorname{Nor}_{c o}{ }^{\left(\gamma_{R}\right)}(x)$, this implies, by Proposition 3.3 that $-\operatorname{Tan}_{c o}{ }_{R}\left(\gamma_{x}\right)(x) \subset \operatorname{Nor}_{c o_{R}\left(\gamma_{x}\right)}(x)=W_{x}$. The inclusion $\left.\operatorname{Tan}_{c l\left(\gamma \backslash \gamma_{x}\right)}(x)\right) \cap S^{1} \subset W_{x}$ follows by Theorem 4.5

Next theorems are related to the tangent cone of the classic convex hull $\operatorname{co}\left(\gamma_{x}\right)$.
Theorem 5.4 Let $\gamma \in \Gamma_{R}$. Assume that for every $x \in \gamma, \gamma_{x}$ is contained in $D(x, R / N), N>1$. Then for every $x \in \gamma$

$$
\operatorname{meas}\left(\operatorname{Tan}_{\operatorname{co}\left(\gamma_{x}\right)}(x)\right) \leq \pi / 2+2 \arcsin \frac{1}{2 N}
$$

holds.
Proof. With the same arguments of the proof of [1. Theorem 4.1], the thesis holds by replacing [1 Lemma 4.5] with Theorem 5.2

Let $|\gamma|$ be the length of $\gamma$, and $p(s)$ the perimeter of $\operatorname{co}(\gamma(x(s)))$.
Theorem 5.5 Let $N>1$. Let $z_{0}$ be a fixed point in the plane. If $\gamma \in \Gamma_{R}$ and $\gamma \subset D\left(z_{0}, R /(2 N)\right)$ then

$$
|\gamma| \leq \frac{\pi}{N-1} R, \quad p^{\prime}(s) \geq 1-\frac{1}{N} \quad \text { a.e. } \quad s \in[0,|\gamma|]
$$

Proof. With the same arguments of the proof of 11 Theorem 4.2], the thesis holds by using Theorem $5.4 \square$
Let us recall that the detour of $\gamma_{x_{1}, x}$ is $\frac{\left|\gamma_{x_{1}, x}\right|}{\left|x-x_{1}\right|}$.

Theorem 5.6 Let $w_{0} \in \mathbb{R}^{2}, \tau>0$. Let $\gamma \in \Gamma_{R}, \gamma \subset D\left(w_{0}, \tau\right)$. Then there exists a positive constant $c(R, \tau)$, depending only on $R$ and $\tau$ such that

$$
\begin{equation*}
|\gamma| \leq c(R, \tau) \tag{48}
\end{equation*}
$$

where
(i) $c(R, \tau) \leq 4 \pi \tau \leq \pi R$ if $\tau \leq \frac{R}{4}$;
(ii) $c(R, \tau) \leq\left(1+\left(16 \sqrt{2} e^{\pi / 2}\right)^{2}\left(\frac{\tau}{R}\right)^{2}\right) \pi R$ if $\tau>\frac{R}{4}$.

Moreover the detour of $\gamma$ is bounded:

$$
\begin{equation*}
\frac{\left|\gamma_{x_{1}, x}\right|}{\left|x-x_{1}\right|}<3 \frac{c(R, \tau)}{R} \quad \forall x_{1} \prec x \in \gamma . \tag{49}
\end{equation*}
$$

Proof. By Theorem 5.5 with the same arguments of the proof of 1 Theorem 5.1,Theorem 5.2], the inequalities 48, 49 are obtained. $\square$

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