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Third Pisa Colloquium in Logic, Language and Epistemology

Essays in Honour of Mauro Mariani and Carlo Marletti

Edited by
Luca Bellotti, Luca Gili, Enrico Moriconi, Giacomo Turbanti

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PREFACE

ENRICO MORICONI

In the fall of 1969, I moved to Pisa to start my undergraduate studies and there I met Mauro Mariani and Carlo Marletti. They were in their second year of university and we were all enrolled in the Scuola Normale Superiore. The atmosphere of the Scuola is special in that students live in residences and spend most of their time together, thereby learning from each other and forming lasting friendships. Carlo and Mauro made an immediate impression on me. Already then Carlo was insightful and brilliant and Mauro was a bibliophile, I daresay he was a bookworm. Despite their capabilities and broad knowledge, they were down to earth and eager to help those who approached them with a philosophical question.

Mauro and Carlo were studying logic, epistemology and philosophy of language and they were finding their research paths in these fields. At the beginning of my second university year, when I was looking for a study topic in the same broad domain of all things logical, I naturally spent more time with them, benefitting from their insights and suggestions. Thanks to their inputs, I was prompted to widen my research interests and they provided me with answers to the many doubts I had while I was studying logic, philosophy of mathematics and, more generally, philosophy. At that time, they were focusing on W. V. O. Quine's philosophy. Later, Carlo developed an interest in *nominalism* and Mauro in *modal logics*. They eventually broadened their research topics to include Aristotle's logic, philosophy of language, linguistics, and Kripke's semantics for modal logics.

Years passing, thanks to the special atmosphere of the Scuola Normale, our friendship became ever deeper and together with Lello Frascolla, Ernesto Napoli, and the late Paolo Casalegno we formed a close group that shared a common research agenda. In the 1980s, Carlo, Mauro and I landed jobs at the Department of Philosophy of the University of Pisa, where our mentors Francesco Barone and Vittorio Sainati were the already established scholars working on logic, philosophy of science, and Aristotle. More recently, we were joined by the much younger Luca Bellotti, who is co-editing this volume.

Carlo and Mauro were excellent teachers and their classes included innovative approaches that went beyond the traditional syllabus. Yes, the students had to overcome some difficulties of communication, and not only those raised by the complexities of the philosophical topics treated: Mauro's teaching style was circuitous and Carlo's was concise, at times elliptical. But they were effective and many of their former students have since secured academic positions all over the world.

Two of their former students, Luca Gili and Giacomo Turbanti, together with Luca Bellotti and me, are editing this volume in honor of Carlo and Mauro. It is our pleasure to present this collection of essays in this year 2019 as Carlo and Mauro are turning 70. We thank friends and former students who contributed papers on the favourite research topics of the two *honorandi*. This volume contains essays originally written for this celebration, and eleven of them are by former students of Carlo and Mauro.

I thank all the people who enthusiastically contributed to the project. I thank Valentina Morotti for her precious help in drafting Carlo's and Mauro's bibliographies and Laura Tesconi for editing and type-setting the volume. This *Festschrift* is a token of friendship and gratitude from us all.

Cari Carlo e Mauro, buon compleanno!

PARADOXES AND SET EXISTENCE

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Abstract: We present some observations on paradoxes from the point of view of set existence principles, and in connection with self-reference and unfoundedness.

Keywords: Paradox, self-reference, truth, set existence, second order logic, subsystems of second order arithmetic.

1 *Self-reference vs. unfoundedness*

Self-reference plays a crucial role in the whole matter of paradoxes. Nevertheless we share the view of (Sorensen, 1998):

> Self-reference is deeply intertwined with logical and foundational aspects of mathematics, but the notion itself is still surprisingly barely understood...

So it may be of interest to inquire to what extent self-reference is essential or it can be eliminated in favor of alternatives.

Traditionally, some form of self-reference has been regarded as *necessary* to paradoxes; but in recent times there has been an attempt to *challenge the traditional view*, by stressing that there are paradoxes not based upon self-reference, but upon unfoundedness and even in hierarchical formalisms.

Historically, this fact is not novel, since it is well-known that there are genuine paradoxes arising from unfoundedness or ungroundedness, e.g., see (Mirimanoff, 1917a; Montague, 1955).[1] For instance, one can derive a semantical contradiction in presence of unfounded chains (Yablo's paradox). Or even in a *typed* theory of truth *à la* Tarski with a hierarchy of countably many truth predicates $T_0, T_1, \ldots$, provided the hierarchy is *ill-founded*, that is, each truth predicate T_i of level i applies to sentences with truth predicates T_k with higher level $k > i$, e.g., (Visser, 1989; Halbach, 2016).

For the reader's sake, let us recall the paradox in (Yablo, 1993).

Assume that there are infinitely many people a_0, a_1, a_2, ... and each a_i says the same sentence: "everybody following me is lying". Then, if p_i is the statement made by a_i, there is *no classical two-valued assignment* to the p_i's. Indeed,

[1] For an attempt at a comprehensive view, see (Cantini, 2009).

assume that p_0 is true. Then p_1 is false and there is some $j > 1$ with p_j true. By assumption, p_j is false as well, since every sentence below p_0 must be false. Hence by contradiction *we have shown that p_0 is false*. But that p_0 is false implies that some p_j with $j > 0$ *is true*. Hence, there exists $n > j$ such that p_n is false, which implies that p_k *is true*, for some $k > n$. By transitivity of the ordering relation, $k > j$, whence p_k *must be false*: contradiction! Since the previous argument for $i = 0$ is uniform in i, p_i is true iff p_i is false, for every i.

At a closer look, it may be questioned whether this is actually not a *paradox without self-reference*; for instance, if we try to formalize the very definition of the sequence $\{a_i\}_{i\in\omega}$ in the language of Peano arithmetic with a truth predicate T, it is natural to proceed self-referentially, selecting with the second recursion theorem (for primitive recursive indexes PRI) a PRI-index fixed point a, such that[2]

$$[a](n) = \overbrace{\ulcorner(\forall k)(k > n \to \neg T([a](k)))\urcorner}^{\cdot} \tag{1}$$

Thus we are driven to the *difficult problem* of defining the very notion of self-reference, which is presently actively investigated by a number of scholars and in different directions. But we won't deal with it in the present note, we only send the reader to samples of significant work, e.g., (Leitgeb, 2002; Cook, 2014; Bolander, 2017), or the most recent (Picollo, 2018).

For the rest of the paper, we consider the following questions:

1. Which grounds does self-reference hinge upon?
2. Is it possible to give criteria for self-reference in a proper sense, or for inferring structural properties, such as circularity, infinite descending chains, fixed point properties?

Incidentally, all this can be handled with topological ideas, as nicely documented by (Bernardi, 2001, 2009), from which we depart in Subsection 2.3 for dealing with question 1.

2 *Set existence and paradoxes*

As a prelude to problem 1, we may wonder whether there is some general structural mechanism underlying self-reference.

[2] $[a]$ represents the a-th primitive recursive function in a given enumeration. We here assume familiarity with the standard arithmetization; $\ulcorner - \urcorner$ represents a Gödelnumbering, while the dot notation refers to an arithmetized substitution operation.

The simplest answer is that there is indeed a standard *geometric* source, the *diagonalization operation*, which has nothing to do in principle with syntactical or computational tricks. Indeed, given a binary relation R or a binary function F, define:

$$\Delta R := \{x|R(x,x)\} \text{ and } (\Delta F)(x) := F(x,x)$$

Then considering basic binary relations and operations with logical significance – such as membership, predication, functional application, reference – we get by diagonalization *self-membership*, *self-predication*, *self-application*, *self-reference* or, more generally, some form of *circularity*.

Let us now quickly review three paradoxes. We argue informally in second order logic with monadic and dyadic predicate variables; we use interchangeably $\in$ and predicate application (that is, Rxy as well as $\langle x,y\rangle \in R$, $y \in X$ for $X(y)$, etc. ...).

2.1 *Cantor-Russell-Zermelo in predicative second-order logic*

It is folklore that the intended result can be rephrased as a pure logical fact.

Definition 2.1 *Given a structure $\langle X,R\rangle$, a subset Y of X is R-representable if and only if there exists an element $e \in X$ such that $Y = R_e = \{u \in X|uRe\}$.*

Cantor's theorem then becomes:

Proposition 2.2 *No binary relation R exists such that every subset of X is R-representable. In logical form:*

$$\neg\exists R\forall X\exists z\forall x(R(z,x) \leftrightarrow X(x))$$

Proof. The complement relative to X of the diagonal set $\Delta(R) = \{u \in X|uRu\}$ is not representable. □

The argument is constructive and by contradiction, and we stress that it is enough to have *much less* than predicative second-order logic: the comprehension schema is applied only to *quantifier free* formulas and the algebra of sets must be closed under complementation.

2.2 *Burali-Forti in impredicative second-order logic*

Definition 2.3 *Given a structure $\langle X,R\rangle$, let $Field(R)$ – the field of R – be the subset of those elements of X which are R-related to some element of X. Then:*

1. *$WF(R)$ is the set of all elements of X which are R-founded:*

$$WF(R) = \{x \in X | (\forall Y \subseteq X)(Progr_R(Y) \to x \in Y)\},$$

where $Progr_R(Y) := (\forall x \in Field(R))(\forall y(yRx \to y \in Y) \to x \in Y)$ (= Y is progressive with respect to R);

2. *A prenex formula A is Π^1_1 if it has the form $A := \forall Y_1 \ldots, Y_n.B$ where B has no occurrence of second order quantifiers.*

$WF(R)$ is the so-called *well-founded part* modulo R of the set X, and it is a familiar notion from proof-theoretic and set-theoretic studies.

Lemma 2.4 *The following formulas are provable in second order logic with comprehension restricted to Π^1_1-instances:*

(i) Closure: $Progr_R(WF(R))$;

(ii) Transfinite induction on R: if B is an arbitrary formula,

$$Progr_R(B) \to \forall x(x \in WF(R) \to B(x))$$

Remark 2.5 *Of course, if we assume (a form of) the axiom of dependent choice, we can derive in a suitable fragment of set theory or second-order arithmetic:*

$$WF(R) = \{x \in X | \textit{no infinite R-descending sequence starting with x exists}\}$$

Proposition 2.6 *There is no $w \in X$ such that $WF(R) = \{x \in X | xRw\}$*

Proof. Clearly R must be irreflexive on $WF(R)$ (use transfinite induction). But if there were an element $w \in X$, such that $\forall v(vRw \to v \in WF(R))$, then $w \in WF(R)$ by R-progressiveness, and hence wRw: contradiction! $\square$

In analogy with 2.1, the proposition states that the well-founded part of a set under a given R cannot be represented by means of an initial R-segment. But this is nothing but an abstract version of the Burali-Forti antinomy: if the collection of ordinals were a set, a single ordinal could represent the whole class of ordinals. On the other hand, the proposition can be regarded as *a logical version of the paradox of grounded sets*, see (Mirimanoff, 1917a, p. 43), and later work by Yuting (1953) and Montague (1955).

Lastly, the argument of proposition 2.6 is fully analogous to 2.1, but by contrast, we here need prima facie *impredicative second order logic*: the comprehension schema is applied to Π^1_1-formulas.

2.3 *Mirimanoff: "descente infinie"...*

Following (Mirimanoff, 1917a, p. 42), consider the $\in$-descending sequence of sets

$$\ldots E_{n+1} \in E_n \in \ldots \in E_1 \in E_0$$

where its underlying general term is given by $E^n = \{e, E^{n+1}\}$.[3] In general, if we replace the set theoretic expression $E^n = \{e, E^{n+1}\}$ by an arbitrary unary function f mapping a set X into itself, we are naturally led to consider unfounded chains for f, i.e., infinite countable sequences $\{x_n\}$ such that $x_n = f(x_{n+1})$.

Clearly not every f has an unfounded chain, e.g., if X is the set N of natural numbers and f is the successor function, then the sequence $a_n = a_{n+1} + 1$ stops after finitely many steps. Moreover, some paradoxes show that the existence of unfounded chains yields a contradiction. Hence it is natural to consider the following

Problem 2.7 *Given X, f, find conditions ensuring unfounded chains for f exist.*

Here we like to recall a theorem of Bernardi (2009). First of all, an easy observation:

Lemma 2.8 *If $f : X \longrightarrow X$ is surjective, f has an unfounded chain.*

Theorem 2.9 *Let X be a sequentially compact Hausdorff space.[4] If $f : X \longrightarrow X$ is continuous, then f has an unfounded chain.*

Proof. The argument follows an iterative pattern. Let us only sketch the idea. For $x \in X$, define $f^n(x) = f(\ldots f(x) \ldots)$ (n-times) and consider the sequence $\{f^n(x)\}$. Then by sequential compactness of X, we can extract a convergent subsequence $\{f^{n_i}(x)\}$, and hence there exists (a unique limit) $x_0 = \lim f^{n_i}(x)$. Now consider the sequence $\{f^{n_i-1}(x)\}$ (possibly omitting the first element); we can again extract a convergent subsequence $\{f^{m_i-1}(x)\}$ with $x_1 = \lim f^{m_i-1}(x)$. Hence by continuity

$$f(x_1) = f(\lim f^{m_i-1}(x)) = \lim f(f^{m_i-1}(x)) = \lim f^{m_i}(x) = x_0 \ldots$$

$\square$

We are now in the position to fix an upper bound to the set existence principles which are needed for 2.9. Indeed, we can apply results from the so-called

[3] e is any given individual object.

[4] For definitions of these standard topological notions, see a standard reference in topology or simply (Simpson, 1999).

Reverse Mathematics, and in particular a theorem of Friedman – see (Simpson, 1999, p. 107) – according to which sequential compactness for the Cantor space 2^N is provable in the fragment ACA_0 of second order arithmetic. As a consequence, for $X := 2^N$, it turns out that

Corollary 2.10 *Theorem* 2.9 is provable in the subsystem ACA_0 of second order arithmetic based on comprehension for arithmetical formulas, which is conservative over first-order Peano Arithmetic PA.

3 *Yablo in a second order framework*

Fix the Cantor space 2^N of all binary countable sequences. If $a \in 2^N$, we interpret $a(i)$ as the truth value of the proposition p_i in the Yablo sequence. Then the Yablo function is the operation Y, such that, if $a \in 2^N$, $n \in N$:

- $Y(a)(n) = 1$ iff $a(i) = 0$, for every $i > n$;
- $Y(a)(n) = 0$, otherwise.

Does a solution $a \in 2^N$ to $Y(a) = a$ exist? Clearly by *the Yablo paradox*, the answer is NO.

In order to classify the set existence principe underlying Yablo's paradox, we develop an alternative formalization. We consider a weak fragment WID_1 of ID_1 – the so-called theory of elementary inductive definition – but *with no arithmetic* in the background.

Definition 3.1

1. $\mathscr{L}_2$ is the second order language, which includes (i) countably many variables of two sorts (individual and predicate variables), standard logical connectives and quantifiers for both individual and predicate variables; (ii) an individual constant c, two function symbols *Suc* (successor), *Pair* (binary), *Nat* (unary predicate) and *Less* (binary). Individual terms are inductively generated from c and individual variables x, y, ... by application of *Suc*, *Pair*. Atomic formulas have the form $t = s$, $X(t)$, $Nat(t)$, $Less(t,s)$. $\mathscr{L}_2$-formulas are then inductively generated by logical operations and quantifiers.

2. $\mathscr{L}_1$ is the first-order fragment of $\mathscr{L}_2$ without second order quantifiers $\forall X$, $\exists Y$. A first-order formula $A(u,X)$ of $\mathscr{L}_1$ (with the free variables shown)

is an *elementary operator* if all subformulas of A of the form $t \in X$ occur *positively*.[5]

3. $\mathscr{L}_{ind}$ is $\mathscr{L}_1$ expanded by an additional distinct predicate constant I_A, for each elementary operator A of $\mathscr{L}_1$.[6] Hence $\mathscr{L}_{ind}$-formulas also include atomic formulas of the form $I_A(t)$.

For the sake of clarity, we henceforth write 0 for c, $t+1$ for $suc(t)$, and (t,s) for $Pair(t,s)$; $t \in N$, $t < s$, $t \in I_A$, instead of $Nat(s)$, $Less(t,s)$, $I_A(t)$.

Definition 3.2 WID_1 is the theory in the language $\mathscr{L}_{ind}$, which contains standard classical logic with identity and in addition:

- $\forall u \forall v(u < v \to u \in N \wedge v \in N)$
- $\forall x(\neg x < x) \wedge \forall x \forall y \forall z(x < y \wedge y < z \to x < z)$
- $\forall x(x < x+1 \wedge 0 \leq x)$
- $\forall u \forall v \forall x \forall y((u,v) = (x,y) \to u = x \wedge v = y)$
- $0 \in N \wedge \forall x(x \in N \to (x+1) \in N)$
- if $A(u,X)$ describes a positive operator,

$$Clos_A(I_A)$$
$$Clos_A(\{x|B(x)\}) \to \forall x(x \in I_A \to B(x))$$

  Here $Clos_A(\{x|B(x)\})$ formalizes that $\{x|B(x)\}$ is closed under the operator defined by A, i.e., $\forall x(A(x,\{x|B(x)\}) \to B(x))$, while B is any formula of the *full language* $\mathscr{L}_{ind}$ (hence also with predicates I_C, C being an operator); the second schema corresponds to the minimality of I_A and hence to transfinite induction.

It is *not required* that N *satisfies induction*, nor that $x+1$ *is the least* $z > x$. Of course, by irreflexivity, $x \neq x+1$ and hence $<$ is serial (i.e., $\forall x \exists y(x < y \wedge x \neq y)$) and $1 \neq 0$, for $1 := 0+1 := suc(c)$.

---

[5] For shortness, this means that every subformula of A of the form $t \in X$ occurs unnegated in the negation normal form of A. By classical logic the negation normal form of A is obtained by rewriting A using only $\wedge$,$\vee$, $\forall$, $\exists$, and $\neg$ applied only to atomic formulas, and by erasing double negations.

[6] Intuitively, I_A names the set inductively defined by the monotone operator defined by A.

Note that no syntax coding is used, nor standard *self-referential* tricks. This should be contrasted with the proposal by (Halbach and Zhang, 2017), where the language contains *names* for sentences and a ternary *satisfaction predicate*. In our case the ontological commitment reduces to the mere existence of fixed points of elementary operators in the language $\mathscr{L}_1$.

In particular, $(n,i) \in X$ stands for *the n-th sentence has value i under the assignement X*. Then the *Yablo*-function can be described by the formula $A(u,X)$:

$$\begin{aligned}&\exists i \exists k\{i \in \{0,1\} \wedge k \in N \wedge u = (k,i) \wedge \\ &\wedge[(i = 1 \wedge \forall y > k.(y,0) \in X) \vee \\ &\vee(i = 0 \wedge \exists y > k.(y,1) \in X)]\}\end{aligned}$$

By inspection $A(u,X)$ is positive in X and hence there exists the least fixed point $Y^\infty = I_A$ of the monotone operator defined by A

Lemma 3.3 *For $a \in N$, i, $j \in \mathbf{2} = \{0,1\}$, then*

$$(a,i) \in Y^\infty \wedge (a,j) \in Y^\infty \Rightarrow i = j.$$

Proof. By transfinite induction on the definition of Y^∞. □

Proposition 3.4 (Yablo's paradox) $Y^\infty = \emptyset$

Proof. Were $(x,0) \in Y^\infty$ (with $x \in N$), then $(y,1) \in Y^\infty$, for some $y \in N$ such that $y > x$. Hence $\forall z > y.(z,0) \in Y^\infty$ whence $(y+1,0) \in Y^\infty$, which implies $(z,1) \in Y^\infty$, for some $z > y+1$. But by transitivity $z > y$, so $(z,0) \in Y^\infty$, which implies by uniqueness $0 = 1$: contradiction. If $(x,1) \in Y^\infty$, then $\forall y > x.(y,0) \in Y^\infty$, hence $(x+1,0) \in Y^\infty$, i.e., for some $z > x+1$, and hence $(z,1) \in Y^\infty$. But $z > x$ and by assumption $(z,0) \in Y^\infty$, whence again contradiction. □

4 *Yablo in Frege structures*

The definition (1) can be smoothly recovered in an abstract theory of self-referential truth, based on extended combinatory logic; see (Aczel, 1980; Cantini, 2016). Let $\mathsf{KF}\lceil$ comprise an extension of combinatory logic with numbers and the fixed point axiom ($\mathscr{T}$)

$$\forall x(\mathscr{T}(x,T) \leftrightarrow T(x))$$

for abstract truth, according to the Kleene strong three-valued logic. $\mathscr{T}(x,T)$ formalizes the closure conditions:

$$\frac{a=b}{T[a=b]} \qquad \frac{\neg(a=b)}{T[\neg(a=b)]} \qquad \frac{N(a)}{T[N(a)]} \qquad \frac{\neg N(a)}{T[\neg N(a)]}$$

for the basic atomic formulas with $=$ and N. Further, the following additional clauses for the compound formulas:

$$\frac{T(a)}{T(\dot{\neg}\dot{\neg}a)} \qquad \frac{T(a) \quad T(b)}{T(a\dot{\wedge}b)} \qquad \frac{T(\dot{\neg}a)\ [\ or\, T(\dot{\neg}b)]}{T(\dot{\neg}(a\dot{\wedge}b))}$$

$$\frac{\forall x T(ax)}{T(\dot{\forall}a)} \qquad \frac{\exists x T(\dot{\neg}ax)}{T(\dot{\neg}\dot{\forall}a)}$$

In addition, $\mathsf{KF}\lceil$ includes:

1. Consistency axiom: $\neg(T(x) \wedge T(\dot{\neg}x))$;
2. the axiom of induction on natural numbers N for propositional functions f with determinate truth values (i.e., such that $\forall x(T(fx) \vee T(\dot{\neg}(fx)))$;
3. self-application is built in the system.

Define $A \mapsto [A]$ with $FV(A) = FV([A])$, $[A]$ being a term representing the propositional function defined by A; e.g., $[\forall xA] := \dot{\forall}(\lambda x.[A])$, where $\dot{\forall}$ represents the universal quantifier), $FV(--)$ is the set of free variables in a given term or formula.

Theorem 4.1 $\mathsf{KF}\lceil$ *proves the existence of a function Y such that*

$$\begin{aligned}
&\forall x \in N.Yx = [\forall k(k > x \to \neg T(Yk))] \\
&\forall x \in N.\neg T(Yx) \\
&\forall x \in N.\neg T\dot{\neg}(Yx) \\
&\neg[\forall x \in N.(T(Yx) \leftrightarrow \forall y > x.\neg T(Yy))]
\end{aligned}$$

Hence no sentence in the Yablo sequence is *determinate* as to its truth value.

Proof. Y exists by fixed point theorem. Since N is determinate, we verify by contradiction for any $x \in N$, $\neg T(Yx)$ and hence we assume $T(Yx)$ for $x \in N$. Then:

1. $\forall k(k \in N \wedge k > x \to T(\dot{\neg}(Yk))$;
2. hence with $k := x+1 > x$, $T(\dot{\neg}(Y(x+1)))$;
3. by definition of Y, logic and T-consistency, for some $i \in N$ with $i > x+1 > x$, $T(Yi)$
4. hence by (1) $T(\dot{\neg}Yi)$, against T-consistency. Therefore $\neg T(Yx)$.

The other statements are verified by similar arguments. $\square$

5 *Conclusions*

Usually, the issue of paradoxes leads to a revisionist attitude against features of (classical or constructive) standard logical systems. By contrast, in the preceding sections we have preserved the level of plain logic untouched and we have searched for a minimal frame, possibly not so much dependent on powerful metatheoretical tools, but emphasizing the place of higher order logic. This is made clear by projecting in second order logic a generalized version of the antinomy of Burali-Forti due to Mirimanoff, where a crucial role is played by infinite descending sequences as sources of paradox instead of self-reference. Then we have seen that Mirimanoff's approach finds a natural complement in the topological analysis by Bernardi. In turn, we have here observed that Bernardi's analysis sends us back again to the connection with arithmetic, as embedded in second order logic, and to problems related to the so-called reverse mathematics, which emphasizes exactly set existence. The final section indeed shows that Yablo's paradox with its seemingly non-logical features can be tamed up to a certain extent, so as to be integrated in a logical frame. The appendix is an exercise in testing how classical Frege's structures fare well with Yablo's paradox.

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